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A time varying approach to the stock return–inflation puzzle

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Summary. In the large literature on the stock return–inflation puzzle, existing works have used constant coefficient linear regression models or change point analysis with abrupt change points. Motivated by the time varying stock return–inflation relationship and the drawbacks of change point analysis, we propose to use the recently emerged locally stationary models to model stock return and inflation. Although the model exhibits non-parametric time varying dependence structure over a long time span, it has local stationarity within each small time interval. Detailed empirical analysis is conducted and comparisons are made between various approaches. We find that the stock return–inflation correlation is negative during early sample periods and turns positive during late sample periods, but the turning time point is different for the total inflation rate and core inflation rate.

Keywords: Locally stationary models; Non-parametric inferences; Stock return–inflation puzzle; Time varying models

1. Introduction

One main goal of financial market research is to study how stock returns are affected by different factors. Among many other important factors such as unemployment and government's monetary policy, inflation plays an essential role in consumer spending, which in turn can affect stock markets. The well-known Fisher hypothesis or Fisher effect states that nominal asset returns, including interest rates, move one to one with inflation rates and thus stock returns should be positively correlated with the inflation rate. The practical implication is that one can buy stocks to hedge against inflation.

Surprisingly, numerous post-war empirical studies (Bodie, 1976; Nelson, 1976; Jaffe and Mandelker, 1976; Gultekin, 1983; Cochran and Defina, 1993; Caporale and Jung, 1997; Sharpe, 2002; Kim and In, 2005; Li *et al.*, 2010) have suggested that there is a significant negative correlation, or at least a lack of positive correlation, between stock returns and inflation, which contradicts the Fisher hypothesis. This is often called the stock return–inflation puzzle. In a seminal work, Fama (1981) argued that, as inflation increases, real variables (business activities) slow down, thus leading to a decline in stock returns. More discussions on the stock return–inflation puzzle and its possible explanation can be found in, for example, Stulz (1986), Kaul (1987), Marshall (1992), Balduzzi (1995), Hess and Lee (1999), Gallagher and Taylor (2002) and Lee (2010).

In the large literature on studying how stock returns are related to inflation rates and historical returns, most existing works used some linear regression model of the form

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$$X_t = \alpha + \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{j=1}^{q} \beta_j Z_{t+1-j} + \varepsilon_t, \qquad t = 1, \ldots, T, \tag{1.1}$$

where X_t and Z_t are respectively the stock return and inflation rate during period t. This linear model assumes a time invariant model structure, which is an assumption that can hardly be justified for data that are collected over a long time span. When studying the predictability of stock returns from the interest rate (which is often used as a prediction of future inflation), Campbell (1987) pointed out that, if the interest rate is non-stationary, then the asymptotic theory behind his statistical inference may be problematic. Similarly, in the context of the stock return–inflation puzzle, empirical analysis based on stationary model (1.1) may be invalid if the inflation rate has unknown non-stationarity.

To model time varying effects, one approach is the piecewise constant version of model (1.1):

$$X_t = \alpha^{(k)} + \sum_{i=1}^{p} \phi_i^{(k)} X_{t-i} + \sum_{j=1}^{q} \beta_j^{(k)} Z_{t+1-j} + \varepsilon_t, \qquad t_{k-1} \leqslant t \leqslant t_k - 1, \quad k = 1, \ldots, K, \tag{1.2}$$

where $t_1 < t_2 < \ldots < t_{K-1}$ ($t_0 = 1$ and $t_K = T + 1$) are some change points, and K is the number of change points. Change point analysis enables a different treatment for the post-change-point period (Csörgő and Horváth, 1997). In the context of the stock return–inflation puzzle, Bodie (1976) and Jaffe and Mandelker (1976) fitted different models for each of the four 5-year subperiods during 1953–1972. Change point analysis relies on abrupt changes due to a sudden policy change or catastrophic events and does not work for gradual changes over time. For example, consider the model $X_t = f(t/T) + \varepsilon_t$; if $f(\cdot)$ is continuous, then there is no change point although the trend changes over time. Furthermore, change point analysis requires segmentation of the data into subperiods, and a shift in the change points may substantially change the analysis, as demonstrated in our numerical analysis in Section 4.

Motivated by the possibly time varying stock return–inflation relationship and the drawbacks of change point analysis, we propose to use the recently emerged locally stationary models (Dahlhaus, 1997, 2012; Dahlhaus and Subba Rao, 2006; Subba Rao, 2006; Wu and Zhou, 2011; Vogt, 2012; Zhao, 2015) to model stock return and inflation. By allowing the model dynamics to change smoothly in time, locally stationary models can provide flexible and tractable alternatives over stationary models. Specifically, we let the coefficients α, ϕ_i and β_j in model (1.1) be non-parametric functional coefficients of the rescaled time t/T, without imposing parametric assumptions on the functional coefficients. This feature is especially useful when the data exhibit a complicated time varying pattern that can hardly be described by using any simple parametric (such as constant or linear) curves. Our time varying approach reveals that the stock return–inflation correlation is negative during early sample periods and turns positive during late sample periods, but the turning time point is different for the total inflation rate and core inflation rate.

2. Model and methodology

2.1. Time varying exogenous auto-regressive model and local approximation

To model the time varying stock return–inflation relationship, we propose a generalization of model (1.1):

$$X_t = \alpha(t/T) + \sum_{i=1}^{p} \phi_i(t/T) X_{t-i} + \sum_{j=1}^{q} \beta_j(t/T) Z_{t+1-j} + \sigma(t/T)\varepsilon_t, \qquad t = 1, \ldots, T, \tag{2.1}$$

where X_t and Z_t are triangular arrays depending on the sample size T, but for ease of presentation we omit such dependence, and $\alpha(z), \phi_1(z), \ldots, \phi_p(z), \beta_1(z), \ldots, \beta_q(z), \sigma(z)$ are functions in $z \in [0, 1]$. We call model (2.1) the time-varying-coefficient auto-regressive (AR) model with exogenous inputs, or TV-ARX(p, q).

Model (2.1) has some nice features over models (1.1) and (1.2). First, model (2.1) allows the coefficients to vary over time, and thus it can model time varying phenomena. Second, we do not impose structural assumptions on the functions $(\alpha(z), \phi_1(z), \ldots, \phi_p(z), \beta_1(z), \ldots, \beta_q(z), \sigma(z))$. Third, if the functions involved are continuous, then they can be approximated by constants within each local time window, and consequently model (2.1) is a locally stationary process. Local stationarity enables us to use local data to address model estimation and inference. Furthermore, by incorporating the exogenous inputs $\{Z_t\}_t$, model (2.1) is more flexible than the locally stationary AR model in Subba Rao (1970) and the piecewise stationary AR model in Davis *et al.* (2006). By incorporating an indicator function, change point analysis enables a different treatment for the post-change-point period due to some sudden policy change or catastrophic events (Duan *et al.*, 2012). Model (2.1) excludes the case of indicator functions and relies on a smooth and gradual change as opposed to the abrupt change in the change point model (1.2).

As shown in the literature (see Dahlhaus (2012) for a survey), under appropriate conditions, a locally stationary process can be approximated by a stationary process within each local time window. For model (2.1), if all the functions involved are continuous, when $t/T \approx z$ for a given z, $\alpha(t/T) \approx \alpha(z)$, $\phi_i(t/T) \approx \phi_i(z)$, $\beta_j(t/T) \approx \beta_j(z)$ and $\sigma(t/T) \approx \sigma(z)$, then intuitively the non-stationary process $\{X_t\}_{t\in\mathbb{Z}}$ has the local stationary approximation $X_t \approx X_t(z)$ within local time window $t/T \approx z$, where

$$X_t(z) = \alpha(z) + \sum_{i=1}^{p} \phi_i(z) X_{t-i}(z) + \sum_{j=1}^{q} \beta_j(z) Z_{t+1-j}(z) + \sigma(z)\varepsilon_t, \qquad t = 1, \ldots, T. \tag{2.2}$$

Here $Z_t(z)$ is defined in assumption 1 in supplementary on-line material where more details can be found. For any fixed z, $\{X_t(z)\}_{t\in\mathbb{Z}}$ is a stationary model of the form (1.1).

2.2. *Non-parametric estimation*

We can rewrite model (2.1) as

$$X_t = U_t^{\mathrm{T}} \theta(t/T) + \sigma(t/T)\varepsilon_t, \qquad t = 1, \ldots, T, \tag{2.3}$$

where

$$U_t = (1, X_{t-1}, \ldots, X_{t-p}, Z_t, \ldots, Z_{t+1-q})^{\mathrm{T}}, \tag{2.4}$$

$$\theta(\cdot) = (\alpha(\cdot), \phi_1(\cdot), \ldots, \phi_p(\cdot), \beta_1(\cdot), \ldots, \beta_q(\cdot))^{\mathrm{T}}. \tag{2.5}$$

We can view U_t and $\theta(\cdot)$ as the covariates and time varying non-parametric functional coefficients respectively. Therefore, model (2.3) can be viewed as a varying-coefficient model (see Fan and Zhang (2008) for a review). Following section 2.1.1 in Fan and Zhang (2008), we estimate $\theta(z)$ and its derivative $\theta'(z)$ by a kernel-type local linear smoothing method:

$$(\hat{\theta}(z), \hat{\theta}'(z)) = \underset{(\theta,\vartheta)}{\arg\min} \sum_{t=1}^{T} \{X_t - U_t^{\mathrm{T}}\theta - (t/T - z)U_t^{\mathrm{T}}\vartheta\}^2 K_t(z), \tag{2.6}$$

where $K_t(z) = K\{(t/T - z)/b_T\}$ for a kernel function $K(\cdot)$ and a bandwidth $b_T > 0$.

Recall the local stationary approximation processes $\{X_t(z)\}_{t\in\mathbb{Z}}$ in model (2.2) and $\{Z_t(z)\}_{t\in\mathbb{Z}}$ in assumption 1 (in the supplementary on-line material). Define the local stationary approximation of U_t as

$$U_t(z) = (1, X_{t-1}(z), \ldots, X_{t-p}(z), Z_t(z), \ldots, Z_{t+1-q}(z))^{\mathrm{T}}. \tag{2.7}$$

Theorem 1. Suppose that assumptions 1–4 in the supplementary on-line material hold. For any $z \in (0,1)$,

$$\surd(Tb_T)\left\{\hat{\theta}(z) - \theta(z) - b_T^2\mu_K \frac{\theta''(z)}{2}\right\} \Rightarrow N\left(0, \sigma^2(z)\, \mathbb{E}[U_0(z)U_0(z)^{\mathrm{T}}]^{-1} \int_{\mathbb{R}} K^2(u)\mathrm{d}u\right), \tag{2.8}$$

where $\mu_K = \int_{\mathbb{R}} u^2 K(u)\mathrm{d}u$ and $U_t(z)$ is defined as in equation (2.7).

2.3. Bandwidth selection and order selection

2.3.1. Bandwidth selection

Bandwidth selection is a challenging issue in non-parametric regression. For local linear regression the automatic plug-in bandwidth selector of Ruppert *et al.* (1995) works for independent data. For dependent data, the optimal bandwidth $b_T^* = \nu b_T^{\mathrm{o}}$, where b_T^{o} is the optimal bandwidth when ignoring dependence and ν is some variance correction factor due to dependence. However, it is generally difficult to estimate ν (Altman, 1990; Wu and Zhao, 2007).

Because of dependence and non-stationarity, bandwidth selection becomes even more challenging for TV-ARX. We propose an *ad hoc* bandwidth selection here. Similarly to equation (2.6), the mean function $g(z) := \mathbb{E}[X_t(z)]$ can be estimated by local linear regression (for a bandwidth h_T):

$$(\hat{g}(z), \hat{g}'(z)) = \underset{(g,g')}{\arg\min} \sum_{t=1}^{T} \{X_t - g - (t/T - z)g'\}^2 K\left(\frac{t/T - z}{h_T}\right). \tag{2.9}$$

Since $g(z)$ is a function of $\theta(z)$, a good bandwidth h_T for estimating $g(z)$ could provide some information about the bandwidth choice for estimating $\theta(z)$. We propose the following procedure.

(a) In equation (2.9), use the automatic plug-in bandwidth selector of Ruppert *et al.* (1995) (implemented by the R command `dpill`) to obtain the optimal bandwidth h_T^* for estimating the mean trend function $g(z)$. This optimal bandwidth h_T^* ignores the dependence.
(b) Let $b_T = \nu h_T^*$, where ν is the variance correction factor due to dependence. Our simulation studies in Section 3 show that $\nu = 2$ usually performs quite well.

2.3.2. Order selection

For a parametric model with k parameters, Bayesian information criterion (BIC) order selection is achieved by minimizing $T\log(\text{noise variance estimator}) + k\log(T)$. For model (2.1), we choose (p,q) to minimize

$$\mathrm{BIC}(p,q) = \sum_{t=1}^{T} \log\{\hat{\sigma}^2(t/T)\} + (p+q+1)\log(T), \tag{2.10}$$

where the estimate $\hat{\sigma}^2(z)$ of $\sigma^2(z)$ is

$$\hat{\sigma}^2(z) = \sum_{t=1}^{T} \{X_t - U_t^{\mathrm{T}}\hat{\theta}(t/T)\}^2 K_t(z) \Big/ \sum_{t=1}^{T} K_t(z).$$

2.4. *Wild bootstrap inference*

The presence of several unknown quantities makes it challenging to use theorem 1 to address statistical inference. Also, because of the relatively small local sample size caused by local stationarity, some asymptotically negligible terms may be non-negligible in finite sample statistical inference. To address these issues, we adopt a similar wild bootstrap procedure to that in Liu (1998) and Gonçalves and Kilian (2004). For model (2.1), the procedure works as follows.

(a) Compute residuals
$$e_t = X_t - \left\{ \hat{\alpha}(t/T) + \sum_{i=1}^{p} \hat{\phi}_i(t/T) X_{t-i} + \sum_{j=1}^{q} \hat{\beta}_j(t/T) Z_{t+1-j} \right\}.$$

(b) For independent and identically distributed $\{r_t\}_{t=1}^{T}$ with $\mathbb{P}(r_t = 1) = \mathbb{P}(r_t = -1) = \frac{1}{2}$, recursively compute $(X_1^{\mathrm{b}} = \ldots = X_p^{\mathrm{b}} = 0)$
$$X_t^{\mathrm{b}} = \hat{\alpha}(t/T) + \sum_{i=1}^{p} \hat{\phi}_i(t/T) X_{t-i}^{\mathrm{b}} + \sum_{j=1}^{q} \hat{\beta}_j(t/T) Z_{t+1-j} + e_t^{\mathrm{b}}, \qquad e_t^{\mathrm{b}} = e_t r_t, \tag{2.11}$$
where the superscript b denotes bootstrap data.

In step (a), from model (2.1) and the consistency $\hat{\theta}(\cdot) \approx \theta(\cdot)$, we see that $e_t \approx \sigma(t/T)\varepsilon_t$. In model (2.11), $e_t^{\mathrm{b}} = e_t r_t \approx \sigma(t/T)(\varepsilon_t r_t)$. Thus, the time varying scale function $\sigma(t/T)$ is preserved in the bootstrap error e_t^{b} with the original noise ε_t replaced by the new noise $\varepsilon_t r_t$. We refer the reader to Davidson and Flachaire (2008) for more discussion on appealing features of this wild bootstrap approach.

We discuss the wild bootstrap approach in three specific contexts:

(a) pointwise confidence intervals (PCIs),
(b) simultaneous confidence band (SCB) construction and
(c) testing the significance of exogenous variables.

2.4.1. *Pointwise confidence intervals*

Let $\theta_k(\cdot)$ be the kth component of $\theta(\cdot)$. A PCI for $\theta_k(z)$ at a given time point z can help us to address inference at a particular time point. Let $\hat{\theta}_k(z)$ be the kth component of $\hat{\theta}(z)$ in equation (2.6). With the bootstrap data $\{(X_t^{\mathrm{b}}, Z_t)\}_{t=1}^{T}$ in model (2.11), we use equation (2.6) to obtain $\hat{\theta}^{\mathrm{b}}(z)$ first and then calculate $W_T^{\mathrm{b}}(z) = \surd(Tb_T)|\hat{\theta}_k^{\mathrm{b}}(z) - \hat{\theta}_k(z)|$. Repeat the procedure to obtain many realizations of $W_T^{\mathrm{b}}(z)$ and let $q_{1-\alpha}^{\mathrm{b}}(z)$ be the $1-\alpha$ sample quantile of the realizations $W_T^{\mathrm{b}}(z)$. Then the $(1-\alpha)$-bootstrap PCI for $\theta_k(z)$ is $(\hat{\theta}_k(z) - q_{1-\alpha}^{\mathrm{b}}(z)/\surd(Tb_T), \hat{\theta}_k(z) + q_{1-\alpha}^{\mathrm{b}}(z)/\surd(Tb_T))$.

2.4.2. *Simultaneous confidence band*

Compared with a PCI, an SCB can capture the overall variability over time. The construction of SCBs involves calculating the maximal deviation over a time set (say $\mathcal{Z}$); see the pioneering work in Bickel and Rosenblatt (1973) for independent data and recent work in Zhao and Wu (2008) for time series data. Using the notation in the PCI case above, we calculate
$$W_T^{\mathrm{b}} = \max_{z \in \mathcal{Z}} \surd(Tb_T)|\hat{\theta}_k^{\mathrm{b}}(z) - \hat{\theta}_k(z)|. \tag{2.12}$$
Then the $(1-\alpha)$-bootstrap SCB for $\theta_k(\cdot)$ over a time set $\mathcal{Z}$ is
$$(\hat{\theta}_k(z) - q_{1-\alpha}^{\mathrm{b}}/\surd(Tb_T), \quad \hat{\theta}_k(z) + q_{1-\alpha}^{\mathrm{b}}/\surd(Tb_T)), \qquad z \in \mathcal{Z}. \tag{2.13}$$
Here $q_{1-\alpha}^{\mathrm{b}}$ is the $1-\alpha$ sample quantile of the realizations W_T^{b}. To test the null hypothesis $\theta_k(\cdot) = 0$, we can check whether the horizontal zero line is entirely contained within the constructed SCB.

2.4.3. Testing the significance of exogenous variables

In model (2.1), it is of interest to test whether the exogenous inputs $\{Z_t\}$ have a significant contribution to the main time series $\{X_t\}$. Specifically, we consider testing

$$H_0:\beta_1(\cdot)=\ldots=\beta_q(\cdot)\equiv 0.$$

Denote by $\hat{\beta}(\cdot)$ the $(\beta_1(\cdot),\ldots,\beta_q(\cdot))$ components of $\hat{\theta}(\cdot)$ in equation (2.6). Then a test can be formed. A global measure of the deviation of $\hat{\beta}(\cdot)$ from zero on a set $\mathcal{Z}$ is

$$D_T=\max_{z\in\mathcal{Z}}\hat{\beta}(z)^{\mathrm{T}}\hat{\beta}(z). \tag{2.14}$$

Similarly, we consider the bootstrap version $D_T^{\mathrm{b}}=\max_{z\in\mathcal{Z}}(\hat{\beta}^{\mathrm{b}}(z)-\hat{\beta}(z))^{\mathrm{T}}(\hat{\beta}^{\mathrm{b}}(z)-\hat{\beta}(z))$. Here $\hat{\beta}^{\mathrm{b}}(z)$ has the same meaning as $\hat{\beta}(z)$ but using bootstrap data. At level of significance α, the null hypothesis is rejected if D_T exceeds the $1-\alpha$ sample quantile of these realizations of D_T^{b}.

3. Monte Carlo studies

In numerical analysis we use the standard normal kernel. Consider TV-ARX(2,2):

$$\begin{aligned}X_t&=\alpha(t/T)+\phi_1(t/T)X_{t-1}+\phi_2(t/T)X_{t-2}+\beta_1(t/T)Z_t+\beta_2(t/T)Z_{t-1}+\sigma(t/T)\varepsilon_t,\\ Z_t&=t/T+0.2\sum_{j=0}^{3}(t/T)^j\eta_{t-j}.\end{aligned} \tag{3.1}$$

We use $T=600$, $\alpha(z)=0.5z$, $\phi_1(z)=-0.4z$, $\phi_2(z)=0.4\cos(2\pi z)$, $\beta_1(z)=0.5\exp(z)$, $\beta_2(z)=-0.6z$, $\sigma(z)=0.2\surd(1+z^2)$ and standard normal distributions for ε_t and η_t. Throughout, we evaluate the estimation and inference methods on the set of 19 evenly spaced points $\mathcal{Z}=\{0.05,0.10,\ldots,0.95\}$. This choice of 19 points is for computation time in simulations (the bootstrap approach is time consuming), and we use 99 evenly spaced points $\mathcal{Z}=\{0.01,0.02,\ldots,0.99\}$ in the real data analysis.

3.1. Bandwidth selection

For $\hat{\theta}(z)$ in equation (2.6), its mean integrated squared error (MISE) on the set $\mathcal{Z}$ is

$$\mathrm{MISE}(\hat{\theta})=\mathbb{E}[\mathrm{ISE}(\hat{\theta})],\qquad \mathrm{ISE}(\hat{\theta})=0.05\sum_{z\in\mathcal{Z}}(\hat{\theta}(z)-\theta(z))^{\mathrm{T}}(\hat{\theta}(z)-\theta(z)). \tag{3.2}$$

The true optimal bandwidth b_T^* is the minimizer of $\mathrm{MISE}(\hat{\theta})$. By averaging 1000 realizations of $\mathrm{ISE}(\hat{\theta})$ in expression (3.2), we find that the true optimal bandwidth b_T^* is about 0.15.

Now we consider the bandwidth selection that was described in Section 2.3.1. For h_T in the mean function estimator (2.9), based on 1000 realizations, the median of these 1000 optimal bandwidths selected by the plug-in bandwidth selector of Ruppert *et al.* (1995) is about 0.07. This bandwidth tends to undersmooth the data because of ignorance of the dependence. As discussed in step (b) of Section 2.3.1, we consider $b_T=0.07\nu$ for some $\nu>1$. Specifically, we evaluate our estimation and inference methods by using different bandwidths $b_T=0.10,0.15,0.20,0.25$. For these four bandwidths, the corresponding $\mathrm{MISE}(\hat{\theta})$s are 0.035, 0.031, 0.038 and 0.053 respectively. We have also tried some smaller bandwidths $b_T=0.05,0.07$, but they led to clear undersmoothing.

To understand how the true optimal bandwidth b_T^* (by minimizing expression (3.2) based on 1000 realizations) is related to the bandwidth h_T^* that was selected by Ruppert *et al.* (1995), we consider

Table 1. Ratio $b_T^*/\text{median}(h_T^*)$†

Distribution of ε_t	*Results for the following values of* ρ *in equation (3.3):*						
	0.2	*0.4*	*0.6*	*0.8*	*1.0*	*1.2*	*1.4*
$N(0,1)$	1.8	1.8	1.8	1.9	2.2	2.4	2.0
Student $t_4/\surd 2$	2.4	1.7	2.1	2.0	2.1	2.3	1.8
Laplace/$\surd 2$	2.0	1.9	1.6	1.7	2.0	2.1	1.8

†b_T^* is the true bandwidth minimizing equation (3.2) based on 1000 realizations and median(h_T^*) is the median among 1000 realizations of the plug-in bandwidth h_T^* of Ruppert *et al.* (1995).

$$X_t = \rho\{\alpha(t/T) + \phi_1(t/T)X_{t-1} + \phi_2(t/T)X_{t-2} + \beta_1(t/T)Z_t + \beta_2(t/T)Z_{t-1}\} + \sigma(t/T)\varepsilon_t,$$
$$Z_t = t/T + 0.2\sum_{r=0}^{3}(t/T)^r\eta_{t-r}, \tag{3.3}$$

for $\rho = 0.2, 0.4, \ldots, 1.4$, and three choices of the distribution of ε_i: a standard normal, a $t_4/\surd 2$ (a Student t-distribution with 4 degrees of freedom, normalized to have unit variance) and a Laplace distribution divided by $\surd 2$ (normalized to have unit variance). Let median(h_T^*) be the median among 1000 realizations of h_T^*. From the ratio $b_T^*/\text{median}(h_T^*)$ that is presented in Table 1, we make the practical recommendation $b_T = 2h_T^*$.

3.2. Order selection

To examine the sensitivity of the order selection procedure in Section 2.3.2 to bandwidth choice, we consider the four choices of bandwidth $b_T = 0.10, 0.15, 0.20, 0.25$ and minimize the BIC target function (2.10) over $p, q = 0, 1, \ldots, 4$. Table 2 presents the counts of selected orders among 100 realizations. With $b_T = 0.10, 0.15, 0.20, 0.25$, the percentages of correct order selection $(2, 2)$ are 16%, 43%, 56% and 53% respectively. Since there are 25 combinations of (p, q), the percentage of correct order selection is reasonably good, especially for larger bandwidths $b_T = 0.15, 0.20, 0.25$.

3.3. Simultaneous confidence band

In SCB (2.13), the bootstrap quantile $q_{1-\alpha}^{\mathrm{b}}$ depends on a specific realization of $\{(X_t, Z_t)\}_{t=1}^T$. Since it is computationally expensive to compute $q_{1-\alpha}^{\mathrm{b}}$ for each of a large number of realizations, to facilitate computation we adopt the following approach:

(a) simulate 60 realizations of $\{(X_t, Z_t)\}_{t=1}^T$;
(b) for each realization, simulate 50 bootstrap realizations to compute 50 realizations of the bootstrap statistic W_T^{b} in equation (2.12);
(c) combine 60×50 realizations of W_T^{b} and use their $1-\alpha$ sample quantile as $q_{1-\alpha}^{\mathrm{b}}$.

Based on 1000 realizations, Table 3 presents the coverage probability of the bootstrap SCB for each of the five coefficients. For coefficients $\alpha(\cdot)$, $\phi_1(\cdot)$, $\beta_1(\cdot)$ and $\beta_2(\cdot)$, the SCB has coverage probability close to the nominal level, and the performance is relatively robust against bandwidth choice. For $\phi_2(\cdot)$, the SCB performs reasonably well for smaller $b_T = 0.10$ and $b_T = 0.15$ but

Table 2. BIC order selection for expression (3.1)†

b_T	Counts of the following selected orders (p,q) among 100 realizations:									
	(2,2)	(2,3)	(2,4)	(3,2)	(3,3)	(3,4)	(4,2)	(4,3)	(4,4)	Other
0.10	16	8	9	5	7	7	18	13	17	0
0.15	43	10	11	12	1	4	10	6	3	0
0.20	56	9	8	12	3	3	8	1	0	0
0.25	53	4	5	13	1	0	21	0	3	0

†The 'other' category includes all orders (p,q) with $p \leqslant 1$ or $q \leqslant 1$.

Table 3. Empirical coverage probabilities of the bootstrap SCB

b_T	Results for $1-\alpha=95\%$					Results for $1-\alpha=90\%$				
	$\alpha(\cdot)$	$\phi_1(\cdot)$	$\phi_2(\cdot)$	$\beta_1(\cdot)$	$\beta_2(\cdot)$	$\alpha(\cdot)$	$\phi_1(\cdot)$	$\phi_2(\cdot)$	$\beta_1(\cdot)$	$\beta_2(\cdot)$
0.10	91.6	94.2	94.4	93.2	92.2	85.7	89.0	87.4	86.8	86.9
0.15	94.2	93.8	87.8	94.3	93.8	88.2	89.8	76.5	89.2	87.2
0.20	96.0	95.8	58.4	94.1	94.2	92.9	92.2	34.8	88.6	88.8
0.25	96.5	95.1	34.9	93.6	92.8	91.5	90.3	1.4	87.9	84.8

Table 4. Empirical coverage probabilities of the bootstrap SCB, using the same setting as in Table 3 but $\phi_2(t)=0.4\cos(\pi t)$

b_T	Results for $1-\alpha=95\%$					Results for $1-\alpha=90\%$				
	$\alpha(\cdot)$	$\phi_1(\cdot)$	$\phi_2(\cdot)$	$\beta_1(\cdot)$	$\beta_2(\cdot)$	$\alpha(\cdot)$	$\phi_1(\cdot)$	$\phi_2(\cdot)$	$\beta_1(\cdot)$	$\beta_2(\cdot)$
0.10	93.9	95.2	93.4	92.4	93.9	88.2	90.4	87.9	87.8	88.9
0.15	94.1	94.8	95.8	94.5	94.8	87.2	88.9	91.5	88.6	90.2
0.20	91.7	96.1	94.7	92.8	94.2	87.5	91.3	89.3	87.3	88.2
0.25	93.0	94.3	94.4	94.6	95.2	87.9	88.5	88.8	89.0	89.8

poorly for larger $b_T=0.20$ and $b_T=0.25$. Since $\phi_2(\cdot)$ has the most curvature, a large b_T tends to oversmooth and leads to poor coverage.

To understand this better, we tried another less curvy function $\phi_2(t)=0.4\cos(\pi t)$ with a period 2 compared with the period 1 of $0.4\cos(2\pi t)$. The coverage probabilities that are presented in Table 4 are close to the nominal level for all coefficients and bandwidths that were considered. For curvy functions, a small local bandwidth is needed to capture such frequent local fluctuations. This presents a dilemma as a small local bandwidth leads to a small effective local sample size.

3.4. Test for significance of exogenous inputs

Using the same notation as in equation (3.1), we consider data from the model

$$X_t=\alpha(t/T)+\phi_1(t/T)X_{t-1}+\phi_2(t/T)X_{t-2}+\tilde{\beta}_1(t/T)Z_t+\tilde{\beta}_2(t/T)Z_{t-1}+\sigma(t/T)\varepsilon_t, \qquad (3.4)$$

Table 5. Empirical size ($\delta=0$) and power ($\delta=0.1, 0.2, 0.3, 0.4$)

b_T	*Null hypothesis*	*Results for 5% significance and the following values of* δ:					*Results for 10% significance and the following values of* δ:				
		0.0	*0.1*	*0.2*	*0.3*	*0.4*	*0.0*	*0.1*	*0.2*	*0.3*	*0.4*
0.10	$\tilde{\beta}_1=\tilde{\beta}_2=0$	0.049	0.165	0.422	0.854	0.999	0.110	0.255	0.629	0.937	1.000
	$\tilde{\beta}_1=0$	0.066	0.231	0.646	0.973	1.000	0.125	0.344	0.831	0.997	1.000
	$\tilde{\beta}_2=0$	0.050	0.072	0.133	0.189	0.389	0.112	0.141	0.223	0.316	0.543
0.15	$\tilde{\beta}_1=\tilde{\beta}_2=0$	0.081	0.148	0.534	0.887	0.996	0.119	0.231	0.668	0.946	1.000
	$\tilde{\beta}_1=0$	0.069	0.229	0.638	0.975	1.000	0.136	0.362	0.803	0.988	1.000
	$\tilde{\beta}_2=0$	0.055	0.076	0.185	0.293	0.404	0.116	0.128	0.269	0.409	0.578
0.20	$\tilde{\beta}_1=\tilde{\beta}_2=0$	0.047	0.163	0.511	0.840	0.997	0.122	0.255	0.683	0.933	1.000
	$\tilde{\beta}_1=0$	0.066	0.206	0.719	0.959	1.000	0.128	0.300	0.832	0.987	1.000
	$\tilde{\beta}_2=0$	0.070	0.098	0.138	0.268	0.422	0.131	0.163	0.236	0.393	0.594
0.25	$\tilde{\beta}_1=\tilde{\beta}_2=0$	0.103	0.121	0.564	0.922	0.999	0.150	0.216	0.697	0.965	0.999
	$\tilde{\beta}_1=0$	0.095	0.198	0.752	0.976	0.999	0.140	0.305	0.832	0.991	1.000
	$\tilde{\beta}_2=0$	0.076	0.046	0.178	0.314	0.415	0.162	0.098	0.273	0.461	0.561

where $\tilde{\beta}_j(z)=\delta\beta_j(z),\ j=1,2$, and the parameter δ controls the contribution of the exogenous inputs $\{Z_t\}$. We consider three null hypotheses:

$$H_0\ (\tilde{\beta}_1=\tilde{\beta}_2=0): X_t=\alpha(t/T)+\phi_1(t/T)X_{t-1}+\phi_2(t/T)X_{t-2}+\sigma(t/T)\varepsilon_t,$$
$$H_0\ (\tilde{\beta}_1=0): X_t=\alpha(t/T)+\phi_1(t/T)X_{t-1}+\phi_2(t/T)X_{t-2}+\tilde{\beta}_2(t/T)Z_{t-1}+\sigma(t/T)\varepsilon_t,$$
$$H_0\ (\tilde{\beta}_2=0): X_t=\alpha(t/T)+\phi_1(t/T)X_{t-1}+\phi_2(t/T)X_{t-2}+\tilde{\beta}_1(t/T)Z_t+\sigma(t/T)\varepsilon_t.$$

The wild bootstrap method in Section 2.4 is used to obtain the bootstrap distribution of the test D_T in equation (2.14); for the second and third hypotheses; this test is equivalent to checking whether the zero line is entirely contained within the bootstrap SCB. As in the SCB construction above, to facilitate computation we use 20×50 realizations (20 realizations of original data with 50 bootstrap realizations each) to obtain the bootstrap critical values of D_T. To examine the sensitivity of the bandwidth, we use four bandwidths $b_T=0.10, 0.15, 0.20, 0.25$. Based on 1000 realizations, Table 5 presents the empirical size ($\delta=0$) and power ($\delta=0.1, 0.2, 0.3, 0.4$) at levels of significance 5% and 10%. The empirical size is close to the level of significance, and the power quickly increases. as the model deviates from H_0, i.e. as δ moves away from 0. Also, the test is quite robust against the choice of bandwidth.

4. Analysis of the stock return–inflation puzzle

4.1. Data description

A commonly used measure of the inflation rate is the percentage change of the consumer price index (CPI), which is a statistical measure, over time, of the prices of goods and services in major expenditure groups. We consider two monthly CPI measures: the seasonally adjusted total CPI (available from `https://fred.stlouisfed.org/series/CPIAUCSL`; we thank a referee for pointing us to this and other links below), and the core CPI which excludes food and energy (available from `https://fred.stlouisfed.org/series/PCEPILFE`). The total CPI serves as a comprehensive measure of inflation, but the core CPI provides a more reliable alternative measure as food and energy prices are often very volatile. The inflation rate Z_t is the percentage change of CPI from month t to $t+1$:

$$Z_t = \frac{\mathrm{CPI}_{t+1}}{\mathrm{CPI}_t} - 1,$$

where CPI_t is the CPI at month t. Accordingly, we term the two inflation rates based on the total CPI and core CPI as the total inflation rate and core inflation rate respectively.

Fig. 1 plots total inflation rate (Fig. 1(a)) and core inflation rate (Fig. 1(b)) series $\{Z_t\}_{t=1}^{T}$ from January 1959 to July 2018, with a total of $T=715$ monthly observations for each series. The 60-month moving average trend (the thick full curve) in each plot shows a complicated time varying pattern that cannot be captured by a simple parametric curve. There is a steady upward trend before 1980 and a downward trend after 1980 with some fluctuations during different time periods. In fact, the stationarity test of Kwiatkowski *et al.* (1992) (R command `kpss.test`) has a nearly 0 p-value for each of the three series $\{Z_t\}$, $\{|Z_t|\}$ and $\{Z_t^2\}$, and for each of the two inflation rate series. Also, the more powerful second-order stationarity test in Cho (2016) (R command `unsys.station.test` in package `unsystation`) suggests rejection of stationarity at level of significance 0.05.

For stock returns, we consider monthly stock returns with dividends; see Kenneth French's website (`http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/f-f_factors.html`) on how the returns were constructed. Let X_t be the real stock return defined as

$$X_t = \text{inflation-adjusted stock return} = \text{nominal return} - \text{inflation rate } Z_t.$$

Fig. 2 plots X_t during January 1959–July 2018. To model the stock return–inflation relationship, given the clear non-stationarity in Fig. 1, TV-ARX (2.1) provides a reasonable framework.

We consider the total inflation rate and core inflation rate separately in Section 4.2 and Section 4.3 respectively. Throughout, all hypothesis tests are conducted at level of significance 5% and all coefficient functions are estimated at 99 evenly spaced points $\mathcal{Z} = \{0.01, 0.02, \ldots, 0.99\}$.

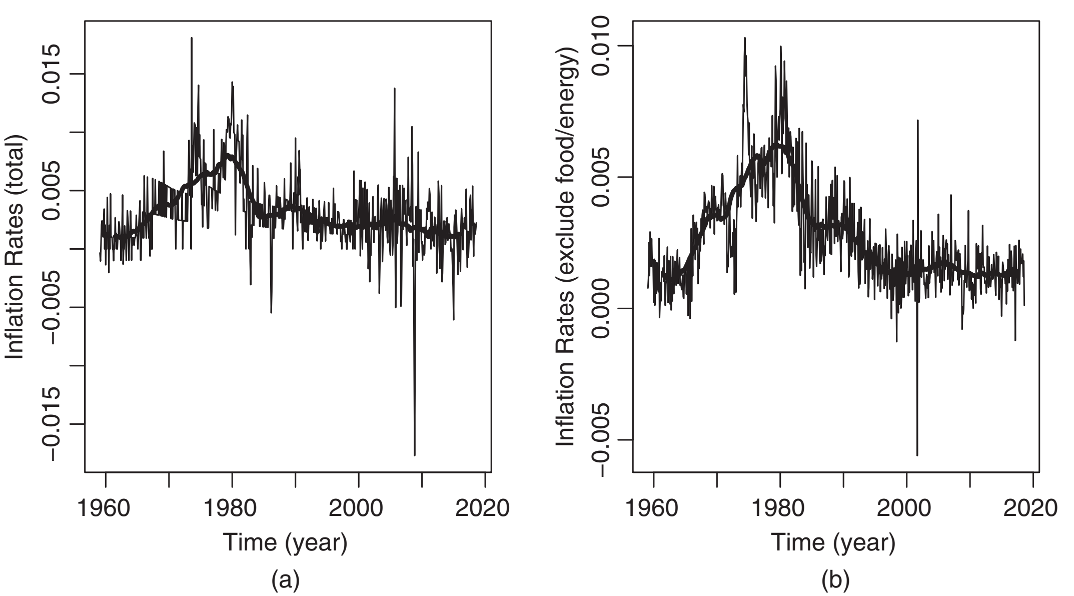

Fig. 1. CPI-based monthly inflation rates (a) total inflation rate and (b) core inflation rate, and 60-month moving average trend (———) during January 1959–July 2018

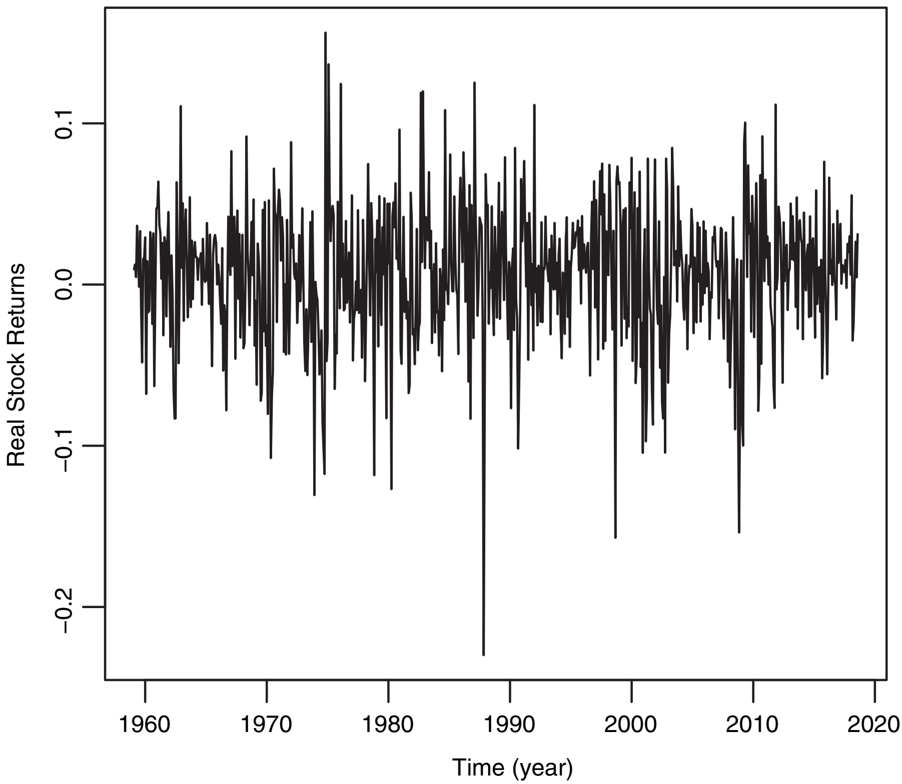

Fig. 2. Monthly real stock returns (nominal return minus inflation rate) during January 1959–July 2018

4.2. *The case of total inflation rate*

4.2.1. *TV-ARX approach*

For bandwidth h_T in equation (2.9), the bandwidth of Ruppert *et al.* (1995) is 0.08. Thus, by the discussion in Section 2.3.1, we use $b_T = 0.16$. To examine the robustness of the selected order, we also consider two other bandwidths: $b_T = 0.12, 0.20$. We then minimize $\mathrm{BIC}(p,q)$ in equation (2.10) over $0 \leqslant p, q \leqslant 5$. The order selection is consistent across three bandwidths: for each bandwidth, the smallest $\mathrm{BIC}(p,q)$ is achieved at $(p,q) = (0,2)$, and the second smallest $\mathrm{BIC}(p,q)$ is achieved at $(p,q) = (1,2)$. To avoid missing important variables, we consider both TV-ARX$(0,2)$ and TV-ARX$(1,2)$.

4.2.1.1. TV-ARX(0,2) modelling. For TV-ARX(0,2) modelling, stock return X_t is modelled by using both contemporary inflation Z_t and lagged inflation Z_{t-1}. Using bandwidth $b_T = 0.16$, the full curves in Fig. 3 are the estimated functions $\hat{\alpha}(\cdot)$ (Fig. 3(a)), $\hat{\beta}_1(\cdot)$ (Fig. 3(b)) and $\hat{\beta}_2(\cdot)$ (Fig. 3(c)). To examine the sensitivity of the estimates to the choice of bandwidth, we also include the estimated curves by using bandwidth $b_T = 0.12$ (the thin dotted curve) and $b_T = 0.2$ (the thin broken curve). The estimated curves generally match well for different bandwidths. From Fig. 3, we see some interesting features.

(a) The coefficients $\hat{\beta}_1(\cdot)$ and $\hat{\beta}_2(\cdot)$ clearly vary with time in a non-linear non-quadratic manner, suggesting a complicated time varying relationship between stock return and inflation rate. This attests the appropriateness of our TV-ARX with non-parametric time varying coefficients.

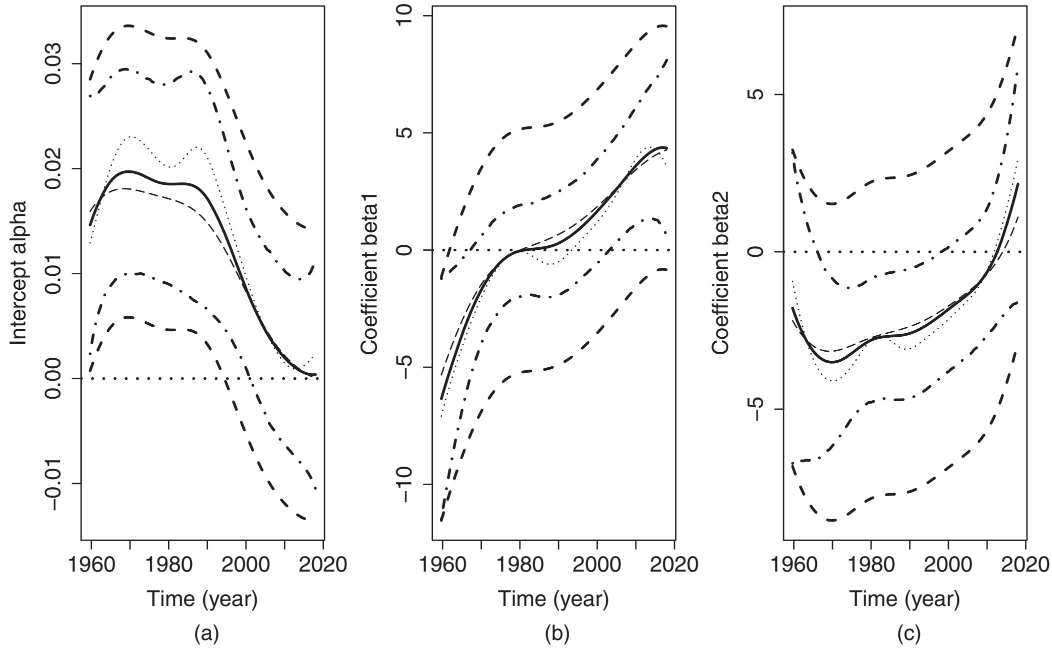

Fig. 3. TV-ARX(0, 2) modelling—estimated curves (———, $b_T = 0.16$; ·······, $b_T = 0.12$; – – – –, $b_T = 0.20$) and their 95% SCB (– – – –, $b_T = 0.16$) and 95% PCI (· — · —, $b_T = 0.16$) (for better interpretation, we linearly transform the domain $[0, 1]$ of the functional curves to the calendar period January 1959–July 2018): (a) intercept $\hat{\alpha}(\cdot)$; (b) coefficient $\hat{\beta}_1(\cdot)$; (c) coefficient $\hat{\beta}_2(\cdot)$

(b) $\hat{\beta}_1(\cdot)$ is negative before 1980 and positive after 1980. Before 1980, real stock return is negatively correlated with inflation (albeit decreasing in magnitude), which provides other empirical support for Fama's theory that stock returns are negatively correlated with inflation. However, after 1980, the stock return–inflation correlation turns positive and increases over time. Interestingly, as can be seen in Fig. 1, the year 1980 also marks the turning point for the trend of inflation rate: an upward trend for pre 1980 and a downward trend for post 1980. This seems to suggest that, during periods with upward inflation rate, a higher total inflation rate is associated with a smaller stock return, whereas the relationship is the opposite during periods with downward inflation.

(c) $\hat{\beta}_2(\cdot)$ is negative during the entire period except for the last 5 years. This suggests that real stock return X_t is mostly negatively correlated with the lagged inflation Z_{t-1}, with the exception of the last 5 years when the correlation is positive.

In Fig. 3 we also include the bootstrap 95% SCB (the broken curve; $b_T = 0.16$) for each of the three curves. Since the SCB for $\beta_1(\cdot)$ does not entirely cover the horizontal dotted zero line, we have strong evidence for rejecting the null hypothesis $\beta_1(\cdot) = 0$ and conclude that overall $\beta_1(\cdot)$ is significantly different from 0. However, we cannot rule out the null hypothesis $\beta_2(\cdot) = 0$ from the SCB, i.e., judged from the entire time period perspective, the stock return has significant correlation with contemporary inflation whereas its correlation with lagged inflation is non-significant.

Whereas the SCB can assess the overall effect during the entire time period, the PCI can examine the local effect during some specific time period. Even in the absence of an overall effect, a local effect may still exist, and to examine such a local effect we include the 95% PCI (the chain curve; $b_T=0.16$) in Fig. 3. For $\beta_1(\cdot)$, we can identify significant negative correlation before 1967 and positive correlation after 2003. For $\beta_2(\cdot)$, there is significant negative correlation between 1967 and 1997.

On the basis of 10000 bootstrap realizations, we have strong evidence for rejecting the joint null hypothesis $H_0:\beta_1(\cdot)=\beta_2(\cdot)=0$. Therefore, the inflation rates (Z_t, Z_{t-1}) are jointly significant in modelling the return X_t. This agrees with the conclusion from the above SCB analysis.

4.2.1.2. TV-ARX(1,2) modelling. For TV-ARX(1,2) modelling, stock return X_t is modelled by using the historical return X_{t-1} in addition to (Z_t, Z_{t-1}). Fig. 4 contains the same information as Fig. 3, with the addition of $\hat{\phi}_1(\cdot)$ for the new variable X_{t-1}. The estimated curves $\hat{\beta}_1(\cdot)$ and $\hat{\beta}_2(\cdot)$ exhibit similar patterns to those in Fig. 3. From the SCB for $\phi_1(\cdot)$, we have no evidence for rejecting the null hypothesis $H_0:\phi_1(\cdot)=0$, i.e. past stock return X_{t-1} is likely to have no significant overall effect on X_t. Furthermore, the PCI indicates the absence of any local effect. This, together with the fact that TV-ARX(0,2) has a smaller BIC, suggests that we probably should not include X_{t-1} in the model and TV-ARX(0,2) is preferred.

In summary, our TV-ARX modelling suggests that the stock return–inflation relationship varies with time and that stock returns are significantly (negatively pre 1980 and positively post 1980) correlated with contemporary total inflation. Although the 1-month lagged inflation Z_{t-1} has no overall effect, it does show some negative local effect. However, past returns carry no significant information about future return. One possible economic explanation for the turning point around year 1980 is the Federal Reserve's interest rate hike during 1980–1982. In an effort to bring down the accelerating inflation in the 1970s, Federal Reserve policy makers approved a new policy to hike the interest rate in the early 1980s, and such monetary policy probably changed the stock return–inflation relationship. In fact, Li and Zhao (2013) identified a change point in the autocovariance structure of interest rates around the same time period.

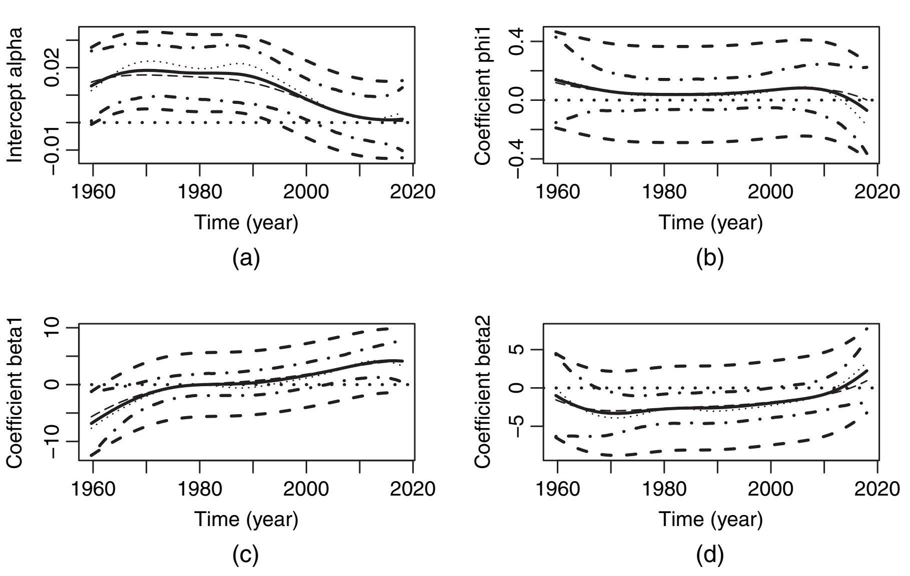

Fig. 4. TV-ARX(1, 2) modelling—estimated curves (———, $b_T=0.16$; · · · · · ·, $b_T=0.12$; – – – –, $b_T=0.20$) and their 95% SCB (– – – –, $b_T=0.16$) and 95% PCI (· — · —, $b_T=0.16$) (for better interpretation, we linearly transform the domain [0, 1] of the functional curves to the calendar period January 1959–July 2018): (a) intercept $\hat{\alpha}(\cdot)$; (b) coefficient $\hat{\phi}_1(\cdot)$; (c) coefficient $\hat{\beta}_1(\cdot)$; (d) coefficient $\hat{\beta}_2(\cdot)$

4.2.2. Comparison with other approaches

We compare our locally stationary TV-ARX approach with various other approaches.

4.2.2.1. Non-parametric regression model. We can model stock return–inflation relationships by using non-parametric regression approach $X_t = f(Z_t) + \varepsilon_t$ for an unknown function $f(\cdot)$. The plug-in bandwidth of Ruppert *et al.* (1995) is 0.002. Fig. 5 plots local linear estimators of $f(\cdot)$ by using three bandwidths: 0.004 (the full curve), 0.003 (the dotted curve) and 0.002 (the broken curve). The clear downward trend suggests a negative correlation between the stock return and inflation. This is consistent with the pre-1980 negative correlation in Fig. 3 but differs from the post-1980 positive correlation there. The non-parametric regression approach assumes that the functional form is constant across time and thus cannot model the potential time varying relationship. In contrast, the non-parametric regression approach also suggests that the real stock return tends to be negative when the inflation rate is very high (greater than 0.7%).

4.2.2.2. Varying-coefficient models. Section 4.2 considers TV-ARX(0,2) model $X_t = \alpha(t/T) + \beta_1(t/T)Z_t + \beta_2(t/T)Z_{t-1} + \sigma(t/T)\varepsilon_t$ with coefficients being a function of t/T. For comparison, we consider the varying-coefficient model

$$X_t = \alpha(Z_t) + \beta_1(Z_t)Z_t + \beta_2(Z_t)Z_{t-1} + \sigma(Z_t)\varepsilon_t, \tag{4.1}$$

where the coefficients are functions of contemporary inflation Z_t. Using the local linear kernel smoothing method (see section 2.1.1 in Fan and Zhang (2008)), Fig. 6 plots the estimated functional coefficients. The coefficient $\hat\beta_1(\cdot)$ of Z_t is mostly positive, which is different from the pre-1980 negative correlation and post-1980 positive correlation in Fig. 3(b), although both Fig. 3(b) and Fig. 6(b) show a negative coefficient $\hat\beta_1(\cdot)$ during the late 1970s when inflation is very high. The coefficient $\hat\beta_2(\cdot)$ of Z_{t-1} is always negative, which is partially consistent with the predominantly negative $\hat\beta_2(\cdot)$ in Fig. 3(c), whereas a discrepancy (negative $\hat\beta_2(\cdot)$ in Fig. 6(c) but positive $\hat\beta_2(\cdot)$ in Fig. 3(c)) occurs during the last 5 years when inflation is low.

We also consider the following alternative of model (4.1):

$$X_t = \alpha(Z_{t-1}) + \beta_1(Z_{t-1})Z_t + \beta_2(Z_{t-1})Z_{t-1} + \sigma(Z_{t-1})\varepsilon_t. \tag{4.2}$$

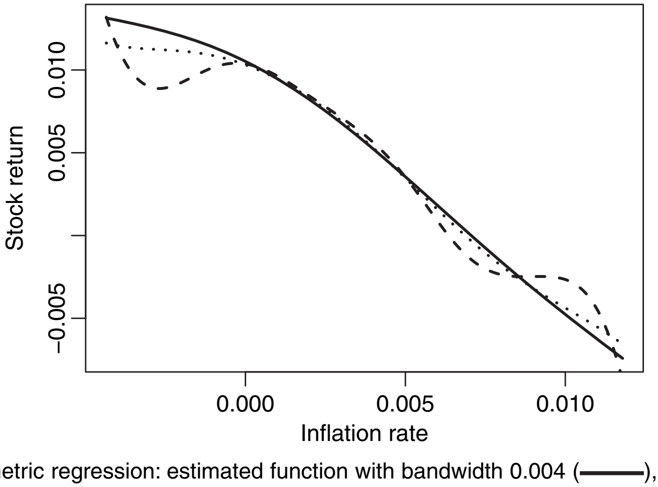

Fig. 5. Non-parametric regression: estimated function with bandwidth 0.004 (——), 0.003 (· · · · · ·) and 0.002 (– – – –)

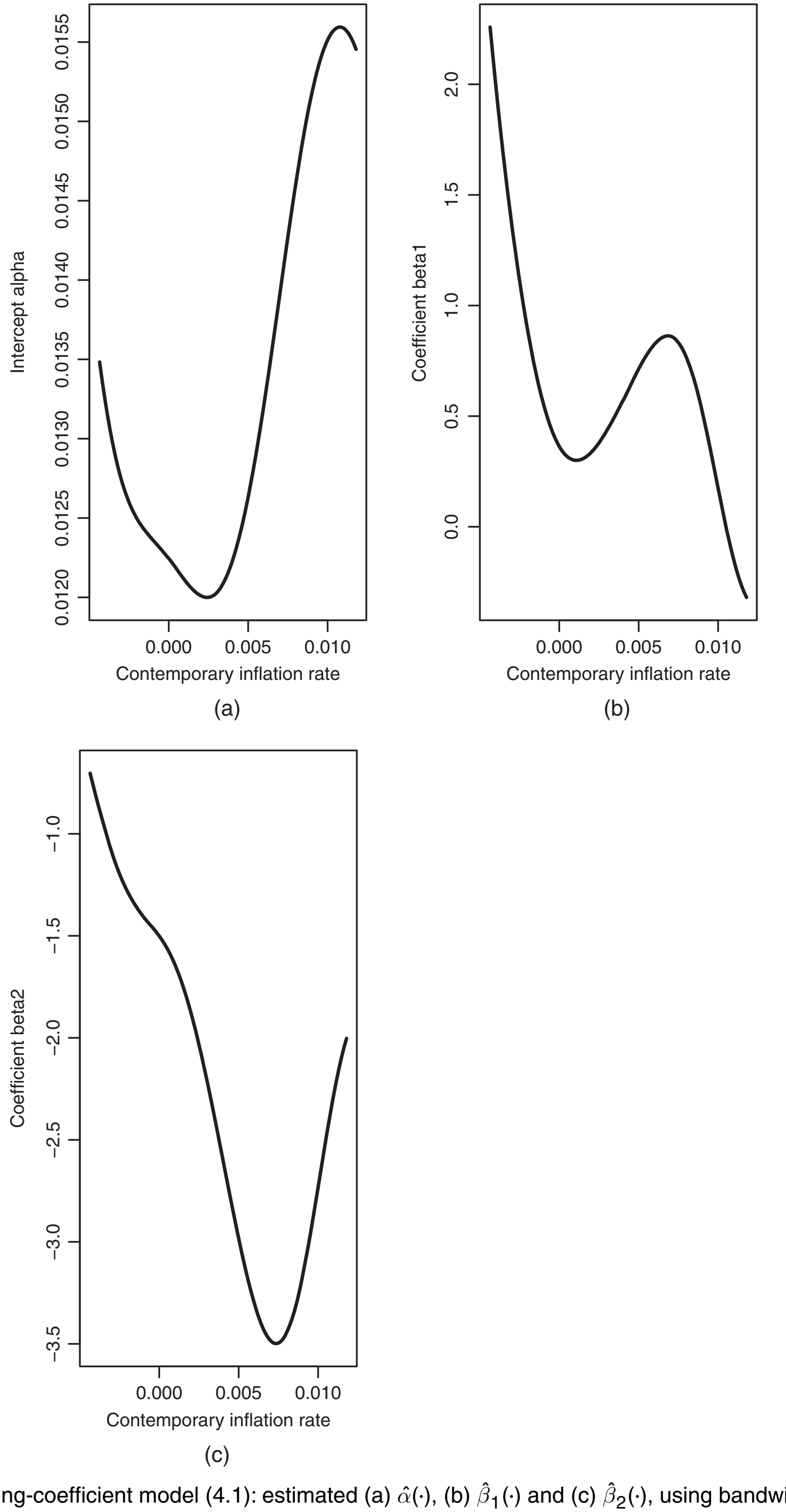

Fig. 6. Varying-coefficient model (4.1): estimated (a) $\hat{\alpha}(\cdot)$, (b) $\hat{\beta}_1(\cdot)$ and (c) $\hat{\beta}_2(\cdot)$, using bandwidth 0.004

It is found that $\hat{\beta}_1(\cdot)$ is always positive and $\hat{\beta}_2(\cdot)$ is always negative, regardless of the 1-month lagged inflation Z_{t-1}. Thus, similarly to model (4.1), the signs of the coefficients in model (4.2) exhibit both similarities and discrepancies when compared with those in Fig. 3. The details have been omitted.

4.2.2.3. Constant coefficient model (1.1). For the constant coefficient model (1.1), the BIC selects order $(p,q)=(0,0)$, which is a white noise model. For comparison with our locally stationary model TV-ARX(0,2) in Section 4.2.1, we also consider $(p,q)=(0,2)$. From the results in Table 6, the contemporary inflation Z_t has a non-significant positive coefficient whereas the lagged inflation Z_{t-1} has a significant negative coefficient, which differs from the conclusion from our TV-ARX(0,2) modelling in Section 4.2.1.

4.2.2.4. Piecewise constant coefficient model (1.2). If we believe that the stock return–inflation relationship is different during different periods, then we can use the piecewise constant approach; see Nelson (1976) and Balduzzi (1995). The change point approach requires dividing data into different subperiods and fitting a separate model with BIC order selection for each subperiod. In addition to a BIC-selected order, we also include the case $(p,q)=(0,2)$ for all segmentations that are considered in Table 6. Our goal is not to propose the best segmentation, but rather we aim to demonstrate the sensitivity of the change point analysis to different segmentations.

Table 6. The case of total inflation: stock return modelling via models (1.1) and (1.2)†

Period	*(p,q)*	*Fitted model*
Constant coefficient model (1.1)		
1959–2018	BIC: (0,0)	$X_t=0.0060+\varepsilon_t$
	(0,2)	$X_t=0.0103+0.7730Z_t-2.1849‡Z_{t-1}+\varepsilon_t$
Piecewise constant coefficient model (1.2); segmentation 1, 2 subperiods (pre 1980 and post 1980)		
Pre 1980	BIC: (0,1)	$X_t=0.0106-1.9457‡Z_t+\varepsilon_t$
	(0,2)	$X_t=0.0138-0.2229Z_t-2.5800‡Z_{t-1}+\varepsilon_t$
Post 1980	BIC: (0,0)	$X_t=0.0077+\varepsilon_t$
	(0,2)	$X_t=0.0085+1.5438Z_t-1.8622‡Z_{t-1}+\varepsilon_t$
Piecewise constant coefficient model (1.2), segmentation 2, 3 subperiods with 20 years each		
1959–1978	BIC: (0,1)	$X_t=0.0123-2.6827‡Z_t+\varepsilon_t$
	(0,2)	$X_t=0.0167-0.8863Z_t-3.0291‡Z_{t-1}+\varepsilon_t$
1979–1998	BIC: (0,0)	$X_t=0.0107+\varepsilon_t$
	(0,2)	$X_t=0.0205+0.0442Z_t-2.7346Z_{t-1}+\varepsilon_t$
1999–2018	BIC: (0,1)	$X_t=-0.0002+2.6924‡Z_t+\varepsilon_t$
	(0,2)	$X_t=0.0003+3.0060‡Z_t-0.6868Z_{t-1}+\varepsilon_t$
Piecewise constant coefficient model (1.2), segmentation 3, 4 subperiods with 15 years each		
1959–1973	BIC: (0,1)	$X_t=0.0111-2.7785‡Z_t+\varepsilon_t$
	(0,2)	$X_t=0.0161-1.5298Z_t-3.2075‡Z_{t-1}+\varepsilon_t$
1974–1988	BIC: (0,1)	$X_t=0.0191-2.4627‡Z_t+\varepsilon_t$
	(0,2)	$X_t=0.02270-0.4499Z_t-2.6857Z_{t-1}+\varepsilon_t$
1989–2003	BIC: (0,0)	$X_t=0.0081+\varepsilon_t$
	(0,2)	$X_t=0.0186+0.1691Z_t-4.7096‡Z_{t-1}+\varepsilon_t$
2004–2018	BIC: (0,1)	$X_t=0.0011+3.0778‡Z_t+\varepsilon_t$
	(0,2)	$X_t=0.0014+3.2533‡Z_t-0.3980Z_{t-1}+\varepsilon_t$

†For each period, we consider both BIC-selected order (p,q) and fixed order $(p,q)=(0,1)$.
‡Significant at the 5% level.

From Fig. 1, year 1980 marks the turning point for the trend of the inflation rate, and it also marks the turning point for the sign of the estimated coefficient $\hat{\beta}_1(\cdot)$ in Fig. 3. Thus, a natural segmentation is to divide data into two subperiods: pre 1980 and post 1980 (segmentation 1 in Table 6). For pre 1980, the BIC selects order $(p, q) = (0, 1)$ and the significant negative coefficient for Z_t is consistent with the pre-1980 negative value of the estimated curve $\hat{\beta}_1(\cdot)$ in Fig. 3. However, for post 1980, the BIC selects white noise model $(p, q) = (0, 0)$, failing to capture the post-1980 positive correlation that is identified in Fig. 3. When fitting the model with $(p, q) = (0, 2)$, the contemporary inflation Z_t is not significant in either period but the lagged inflation Z_{t-1} has a significant negative sign for both periods. The conclusion is different from our TV-ARX approach.

Table 6 also includes detailed model fitting for two other segmentations: segmentation 2 with three subperiods and segmentation 3 with four subperiods. Overall, the TV-ARX approach and the piecewise constant coefficient approach have agreement and also disagreement over some periods. For example, for segmentation 2, the BIC-selected model $(p, q) = (0, 1)$ can identify negative correlation with Z_t during 1959–1978 (part of pre 1980) and positive correlation with Z_t during 1999–2018 (part of post 1980) but fails to capture any correlation with Z_{t-1}.

Change point analysis is sensitive to different segmentation of the data. For example, segmentation 1 and segmentation 2 result in quite different conclusions for the post-1980 period. For segmentation 1, the BIC-selected model fits white noise for the entire post-1980 period; for segmentation 2, we have a white noise model for 1979–1998 and $(p, q) = (0, 1)$ for 1999–2018. If we use the same order $(p, q) = (0, 2)$, segmentation 1 identifies significant correlation with Z_{t-1} for post 1980; however, for segmentation 2, neither Z_t nor Z_{t-1} is significant during 1979–1998 and only Z_t is significant during 1999–2018. The significant variables identified are completely different under the two segmentations.

In summary, the piecewise constant coefficient model (1.2) can provide a better approach

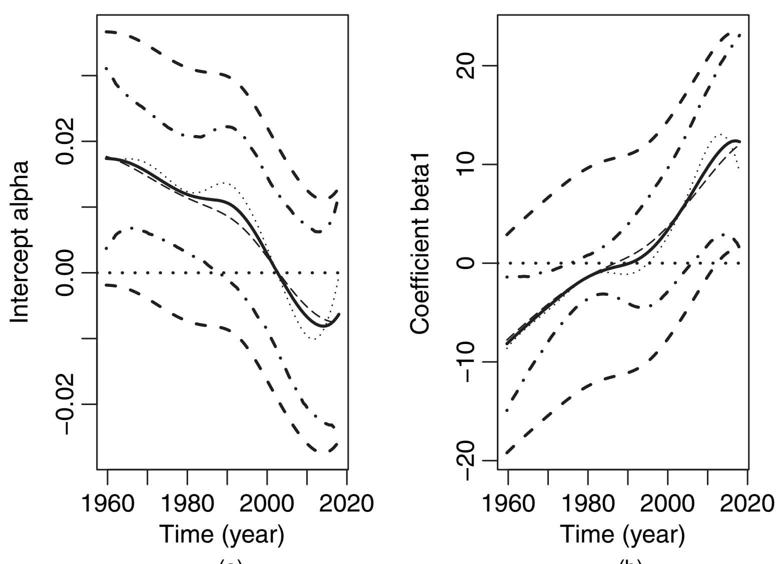

Fig. 7. TV-ARX(0,1) modelling—estimated curves (———, $b_T = 0.16$; · · · · · · ·, $b_T = 0.12$; – – – –, $b_T = 0.20$) and their 95% SCB (– – – –, $b_T = 0.16$) and 95% PCI (· — · —, $b_T = 0.16$) (for better interpretation, we linearly transform the domain $[0, 1]$ of the functional curves to the calendar period January 1959–July 2018): (a) intercept $\hat{\alpha}(\cdot)$; (b) coefficient $\hat{\beta}_1(\cdot)$

Table 7. The case of core inflation: stock return modelling via models (1.1) and (1.2)†

Period	*(p,q)*	*Fitted model*
Constant coefficient model (1.1)		
1959–2018	BIC: (0,0)	$X_t = 0.0064 + \varepsilon_t$
	(0,1)	$X_t = 0.0087 - 0.8661 Z_t + \varepsilon_t$
Piecewise constant coefficient model (1.2); segmentation 1, 2 subperiods (pre 1980 and post 1980)		
Pre 1980	BIC: (0,0)	$X_t = 0.0035 + \varepsilon_t$
	(0,1)	$X_t = 0.0113 - 2.3032 Z_t + \varepsilon_t$
Post 1980	BIC: (0,0)	$X_t = 0.0080 + \varepsilon_t$
	(0,1)	$X_t = 0.0061 + 0.8498 Z_t + \varepsilon_t$
Piecewise constant coefficient model (1.2), segmentation 2, 2 subperiods (pre 1990 and post 1990)		
Pre 1990	BIC: (0,0)	$X_t = 0.0056 + \varepsilon_t$
	(0,1)	$X_t = 0.0119 - 1.7269 Z_t + \varepsilon_t$
Post 1990	BIC: (0,0)	$X_t = 0.0073 + \varepsilon_t$
	(0,1)	$X_t = 0.0026 + 2.9385 Z_t + \varepsilon_t$
Piecewise constant coefficient model (1.2); segmentation 3: 3 subperiods with 20 years each		
1959–1978	BIC: (0,0)	$X_t = 0.0031 + \varepsilon_t$
	(0,1)	$X_t = 0.0128 - 3.0337‡ Z_t + \varepsilon_t$
1979–1998	BIC: (0,0)	$X_t = 0.0111 + \varepsilon_t$
	(0,1)	$X_t = 0.0150 - 1.1890 Z_t + \varepsilon_t$
1999–2018	BIC: (0,0)	$X_t = 0.0051 + \varepsilon_t$
	(0,1)	$X_t = -0.0033 + 5.9243‡ Z_t + \varepsilon_t$
Piecewise constant coefficient model (1.2); segmentation 4: 4 subperiods with 15 years each		
1959–1973	BIC: (0,0)	$X_t = 0.0040 + \varepsilon_t$
	(0,1)	$X_t = 0.0139 - 4.2049‡ Z_t + \varepsilon_t$
1974–1988	BIC: (0,0)	$X_t = 0.0064 + \varepsilon_t$
	(0,1)	$X_t = 0.0178 - 2.3257 Z_t + \varepsilon_t$
1989–2003	BIC: (0,0)	$X_t = 0.0086 + \varepsilon_t$
	(0,1)	$X_t = 0.0096 - 0.5440 Z_t + \varepsilon_t$
2004–2018	BIC: (0,1)	$X_t = -0.0107 + 12.1071‡ Z_t + \varepsilon_t$

†For each period, we consider both BIC-selected order (p,q) and fixed order $(p,q) = (0,1)$.
‡Significant at the 5% level.

than the constant coefficient model (1.1) by capturing some time varying stock return–inflation relationship, but it is sensitive to the choice of segmentation and the model dynamics still remain constant within each subperiod. By contrast, by letting the data speak for themselves in a fully non-parametric way, our method does not rely on such segmentation and thus the conclusion is expected to be more reliable. Also, our approach can identify interesting time varying relationships.

4.3. The case of core inflation rate

Since the procedure is similar to that of the total inflation rate case in Section 4.2, we outline only some main differences. For each choice of bandwidth $b_T = 0.12, 0.16, 0.20$, the smallest and second-smallest $\mathrm{BIC}(p,q)$ values over $0 \leqslant p,q \leqslant 5$ are achieved at $(p,q) = (0,1)$ and $(p,q) = (1,1)$ respectively. For TV-ARX(1,1), the SCB and PCI (the plot has been omitted) suggest that X_{t-1} has no significant overall or local effect. Thus, model TV-ARX(0,1) is preferred.

As in Fig. 3, Fig. 7 contains the estimated curve, SCB and PCI for $\hat{\alpha}(\cdot)$ and $\hat{\beta}_1(\cdot)$. Clearly, the stock return–inflation relationship is time varying: before 1990, the real stock return is negatively correlated with inflation, whereas for post 1990 there is a steadily increasing positive correlation.

Interestingly, the turning time point 1990 is different from the turning time 1980 in the total inflation rate case. Also, the PCI identifies significant negative correlation during 1959–1975 and significant positive correlation during 2005–2018.

For comparison, in Table 7 we report the results from the constant coefficient model (1.1) and the piecewise constant coefficient model (1.2) with different segmentations of the data. In Fig. 7, the turning point for the sign of $\hat{\beta}_1(\cdot)$ occurs at year 1990, so we also include the segmentation of the pre-1990 and post-1990 periods. The BIC selects a white noise model for all subperiods except 2004–2018 in segmentation 4. For 2004–2018, the BIC selects $(p,q)=(0,1)$ with a significant positive correlation, which is consistent with the positive value of $\hat{\beta}_1(\cdot)$ over 2005–2018 in Fig. 7. When using $(p,q)=(0,1)$, the significant negative coefficient of Z_t during 1959–1978 (segmentation 3) and 1959–1973 (segmentation 4) is consistent with the significant negative value of $\hat{\beta}_1(\cdot)$ over 1959–1975 in Fig. 7.

5. Supplementary on-line material

Technical conditions and proofs are presented in the supplementary on-line material.

The data that are analysed in the paper and the programs that were used to analyse them can be obtained from

https://rss.onlinelibrary.wiley.com/hub/journal/14679876/series-c-datasets

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References

Altman, N. S. (1990) Kernel smoothing of data with correlated errors. *J. Am. Statist. Ass.*, **85**, 749–759.
Balduzzi, P. (1995) Stock returns, inflation, and the 'proxy hypothesis': a new look at the data. *Econ. Lett.*, **48**, 47–55.
Bickel, P. J. and Rosenblatt, M. (1973) On some global measures of the deviations of density function estimates. *Ann. Statist.*, **1**, 1071–1095.
Bodie, Z. (1976) Common stocks as a hedge against inflation. *J. Finan.*, **31**, 459–470.
Campbell, J. Y. (1987) Stock returns and the term structure. *J. Finan. Econ.*, **18**, 373–399.
Caporale, T. and Jung, C. (1997) Inflation and real stock prices. *Appl. Finan. Econ.*, **7**, 265–266.
Cho, H. (2016) A test for second-order stationarity of time series based on unsystematic sub-samples. *Stat*, **5**, 262–277.
Cochran, S. J. and Defina, R. H. (1993) Inflation's negative effects on real stock prices: new evidence and a test of the proxy effect hypothesis. *Appl. Econ.*, **25**, 263–274.
Csörgő, M. and Horváth, L. (1997) *Limit Theorems in Change-point Analysis*. New York: Wiley.
Dahlhaus, R. (1997) Fitting time series models to nonstationary processes. *Ann. Statist.*, **25**, 1–37.
Dahlhaus, R. (2012) Locally stationary processes. In *Handbook of Statistics*, vol. 30, *Time Series Analysis: Methods and Applications* (eds T. Subba Rao, S. Subba Rao and C. R. Rao), pp. 351–413. Amsterdam: Elsevier.
Dahlhaus, R. and Subba Rao, S. (2006) Statistical inference for time-varying ARCH processes. *Ann. Statist.*, **34**, 1075–1114.
Davidson, R. and Flachaire, E. (2008) The wild bootstrap, tamed at last. *J. Econmetr.*, **146**, 162–169.
Davis, R. A., Lee, T. and Rodriguez-Yam, G. (2006) Structural break estimation for nonstationary time series models. *J. Am. Statist. Ass.*, **101**, 223–239.
Duan, J.-C., Sun, J. and Wang, T. (2012) Multiperiod corporate default prediction—a forward intensity approach. *J. Econmetr.*, **170**, 191–209.

Fama, E. (1981) Stock returns, real activity, inflation and money. *Am. Econ. Rev.*, **71**, 545–564.
Fan, J. and Zhang, W. (2008) Statistical methods with varying coefficient models. *Statist. Inf.*, **1**, 179–195.
Gallagher, L. A. and Taylor, M. P. (2002) The stock return-inflation puzzle revisited. *Econ. Lett.*, **75**, 147–156.
Gonçalves, S. and Kilian, L. (2004) Bootstrapping autoregressions in the presence of conditional heteroskedasticity of unknown form. *J. Econmetr.*, **123**, 89–120.
Gultekin, N. B. (1983) Stock market returns and inflation: evidence from other countries. *J. Finan.*, **38**, 49–65.
Hess, P. J. and Lee, B. S. (1999) Stock returns and inflation with supply and demand disturbances. *Rev. Finan. Stud.*, **12**, 1203–1218.
Jaffe, J. F. and Mandelker, G. (1976) The "Fisher effect" for risky assets: an empirical investigation. *J. Finan.*, **31**, 447–458.
Kaul, G. (1987) Stock returns and inflation: the role of the monetary sector. *J. Finan. Econ.*, **18**, 253–276.
Kim, S. and In, F. (2005) The relationship between stock returns and inflation: new evidence from wavelet analysis. *J. Empir. Finan.*, **12**, 435–444.
Kwiatkowski, D., Phillips, P. C. B., Schmidt, P. and Shin, Y. (1992) Testing the null hypothesis of stationarity against the alternative of a unit root. *J. Econmetr.*, **54**, 159–178.
Lee, B. S. (2010) Stock returns and inflation revisited: an evaluation of the inflation illusion hypothesis. *J. Bnkng Finan.*, **34**, 1257–1273.
Li, L., Narayan, P. K. and Zheng, X. (2010) An analysis of inflation and stock returns for the UK. *J. Int. Finan. Markts Inst. Mony*, **20**, 519–532.
Li, X. and Zhao, Z. (2013) Testing for changes in autocovariances of nonparametric time series models. *J. Statist. Planng Inf.*, **143**, 237–250.
Liu, R. Y. (1988) Bootstrap procedures under some non-I.I.D. models. *Ann. Statist.*, **16**, 1696–1708.
Marshall, D. A. (1992) Inflation and asset returns in a monetary economy. *J. Finan.*, **47**, 1315–1342.
Nelson, C. R. (1976) Inflation and rates of return on common stocks. *J. Finan.*, **31**, 471–483.
Ruppert, D., Sheather, S. J. and Wand, M. P. (1995) An effective bandwidth selector for local least squares regression. *J. Am. Statist. Ass.*, **90**, 1257–1270.
Sharpe, S. A. (2002) Reexamining stock valuation and inflation: the implications of analysts' earnings forecasts. *Rev. Econ. Statist.*, **84**, 632–648.
Stulz, R. M. (1986) Asset pricing and expected inflation. *J. Finan.*, **41**, 209–223.
Subba Rao, S. (2006) On some nonstationary, nonlinear random processes and their stationary approximations. *Adv. Appl. Probab.*, **38**, 1155–1172.
Subba Rao, T. (1970) The fitting of non-stationary time-series models with time-dependent parameters. *J. R. Statist. Soc.* B, **32**, 312–322.
Vogt, M. (2012) Nonparametric regression for locally stationary time series. *Ann. Statist.*, **40**, 2601–2633.
Wu, W. B. and Zhao, Z. (2007) Inference of trends in time series. *J. R. Statist. Soc.* B, **69**, 391–410.
Wu, W. B. and Zhou, Z. (2011) Gaussian approximations for non-stationary multiple time series. *Statist. Sin.*, **21**, 1397–1413.
Zhao, Z. (2015) Inference for local autocorrelation process in locally stationary models. *J. Bus. Econ. Statist.*, **33**, 296–306.
Zhao, Z. and Wu, W. B. (2008) Confidence bands in nonparametric time series regression. *Ann. Statist.*, **36**, 1854–1878.

Supporting information

Additional 'supporting information' may be found in the on-line version of this article:

'Technical assumptions and proofs for "A time-varying approach to the stock return-inflation puzzle"'.