# YSC2229: Introductory Data Structures and Algorithms

Week 04: Advanced Sorting Techniques

# Merge sort

- <u>Idea</u>: split the array to be sorted into two equal (±1) parts, sort these arrays by *recursive* calls, and then *merge* them, preserving the ordering.

```
MergeSort(A[0 … n-1]) {
 if (n = 1) {
   return A;             // 1-element array, nothing to sort
 } else {
   m := n/2;
   L := A[0, …, m-1];   // split the array into Left and Right half (by copying)
   R := A[m, …, n-1];
   Merge(MergeSort(L), MergeSort(R), A)    // in-place merge results into A
   return A;
}
```

# Merge sort by example

Recursive descent: *splitting* the array

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 12 | 8 | 18 | 3 | 4 | 6 |

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 2 | 12 | 8 |

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| 18 | 3 | 4 | 6 |

| 0 | 1 |
|---|---|
| 1 | 2 |

| 0 | 1 |
|---|---|
| 12 | 8 |

| 0 | 1 |
|---|---|
| 18 | 3 |

| 0 | 1 |
|---|---|
| 4 | 6 |

| 0 |
|---|
| 1 |

| 0 |
|---|
| 2 |

| 0 |
|---|
| 12 |

| 0 |
|---|
| 8 |

| 0 |
|---|
| 18 |

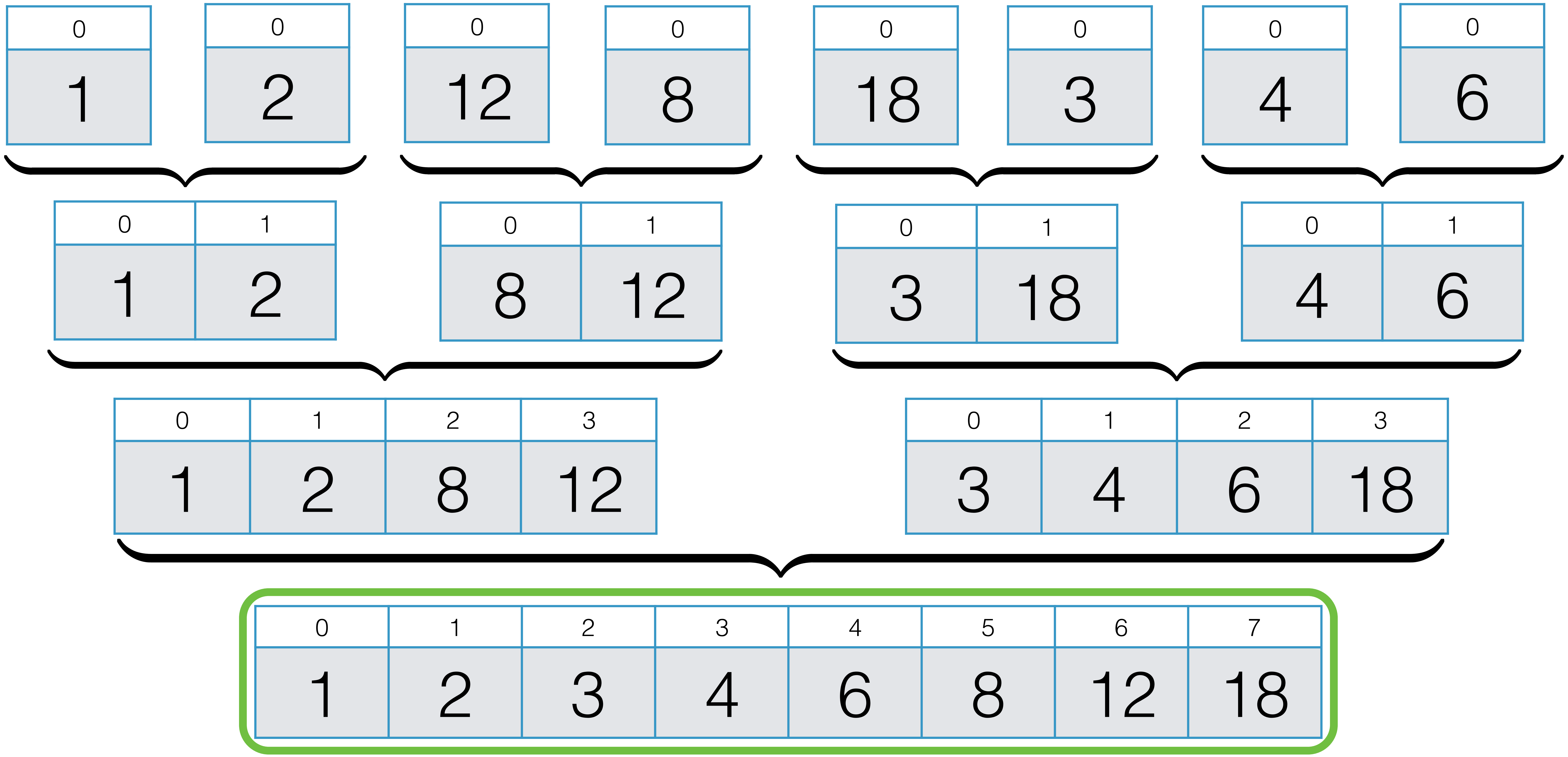
| 0 |
|---|
| 3 |

| 0 |
|---|
| 4 |

| 0 |
|---|
| 6 |

# Merge sort by example

Merging the *sorted* sub-arrays



# Merge sort complexity

| | |
|---|---|
| M(n) | `MergeSort(A[0 … n-1]) {` |
| | `if (n = 1) {` |
| M(1) = 0 | `return A;` |
| | `} else {` |
| | `m := n/2;` |
| copying: n/2 | `L := A[0, …, m-1];` |
| copying: n/2 | `R := A[m, …, n-1];` |
| merging: n comparisons | `Merge(` |
| M(n/2) | `MergeSort(L),` |
| M(n/2) | `MergeSort(R), A)` |
| | `return A;` |
| | `}` |

- The complexity does *not* depend on the input *properties*, just its *size* ⇒ *worst-case = average case*.

# Merge sort complexity

$$\boxed{\begin{array}{l} M(n) = 2\,M(n/2) + 2n, \text{ if } n > 1 \\ M(1) = 0 \end{array}}$$

Change variable $n \mapsto 2^k$: $M(n) = h(k) = 2\,h(k-1) + 2\cdot 2^k$

Change of function: $h(k) = 2^k\,g(k)$

$h(0) = g(0) = M(1) = 0$

By substituting $h$: $2^k\,g(k) = 2\cdot 2^{k-1}\,g(k-1) + 2\cdot 2^k$

$g(k) = g(k-1) + 2$

By method of differences: $g(k) = 2k + M(1)$

# Merge sort complexity

$$\boxed{\begin{array}{l} M(n) = 2\,M(n/2) + 2n, \text{ if } n > 1 \\ M(1) = 0 \end{array}}$$

$M(n) = h(k) = 2\,h(k-1) + 2 \cdot 2^k$ $\quad$ $h(k) = 2^k\,g(k)$ $\quad$ $g(k) = 2k + M(1)$

$g(k) = 2k$

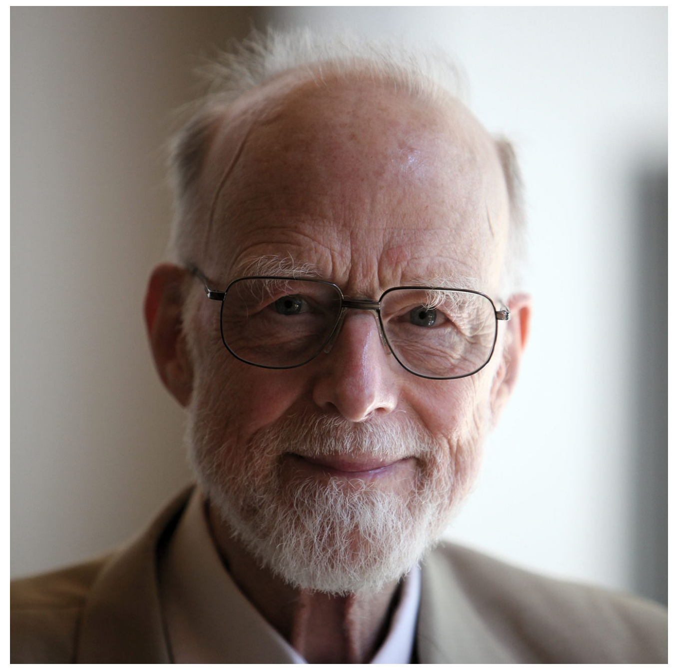
$h(k) = 2 \cdot 2^k \cdot k$

$M(n) = 2n \log_2 n \in O(n \log_2 n \mid \textit{n is a power of 2})$

Since $n \cdot \log n$ is non-decreasing for $n > 1$, and it is also *smooth*,

$$\boxed{M(n) \in O(n \log n)}$$

# Quicksort

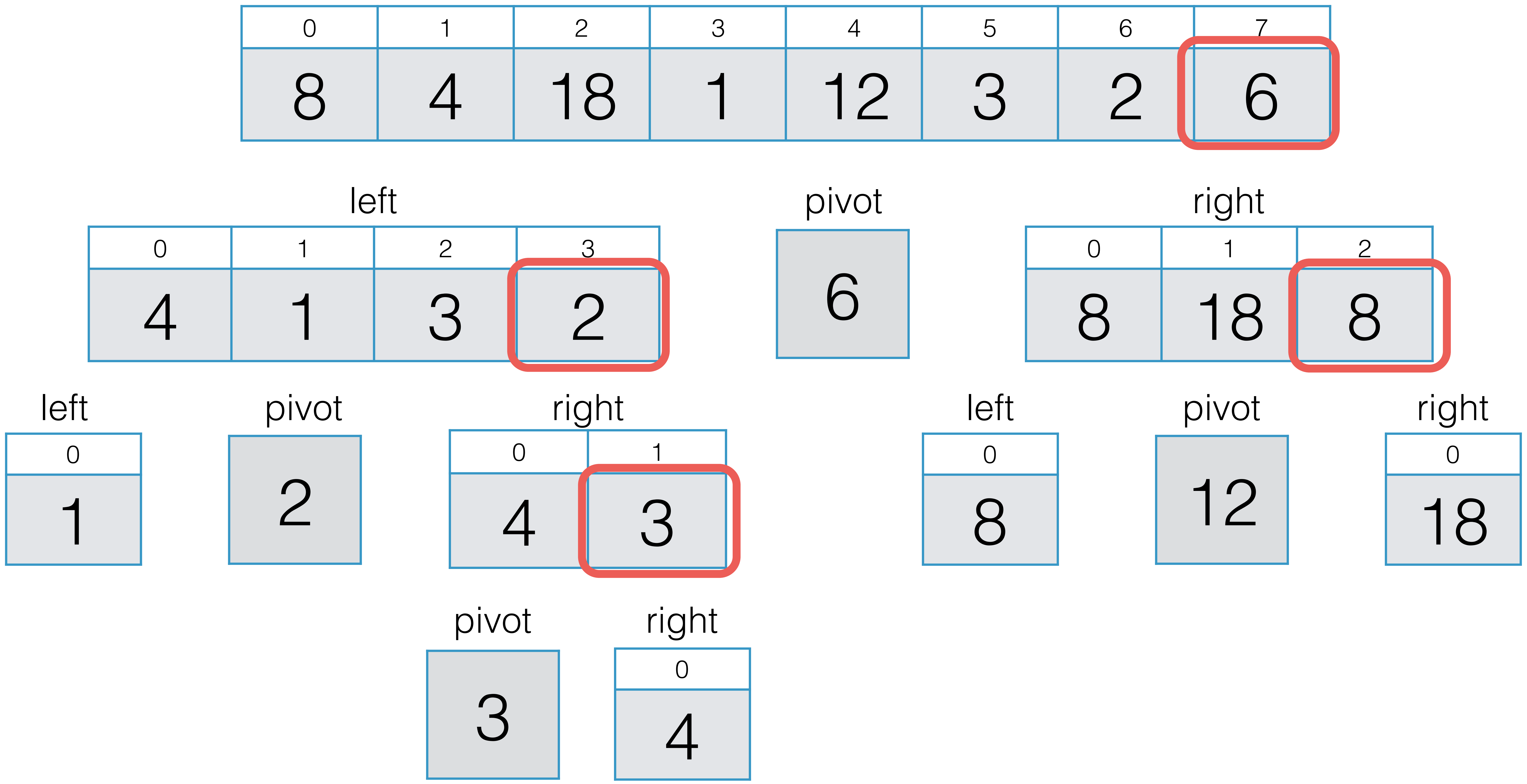


- Invented by Tony Hoare (the same as of Hoare triples) in 1961;
- Idea: divide-and-conquer with partially sorted sub-arrays;
- In practice, one of the *fastest* sorting algorithms as of today.

```
QuickSort (A[0 … n-1]) {
 if (n ≤ 1) { return A; }     // nothing to sort, return A
 else {
   l := 0; r := 0;
   pivot := A[n-1];           // take the last array element as a "pivot"
   for (i = 1 … n-1) {
    if (A[i] < pivot) then {
      L[l] := A[i];           // collect all elements of A smaller than pivot in
      l := l + 1;             // the "left" subarray L
    } else {
      R[r] := A[i];           // collect all elements of A greater or equal than pivot in
      r := r + 1;             // the "Right" subarray R
    }
   }
   Concat(QuickSort(L), pivot, QuickSort(R), A)  // run recursively on L, R, and then
   return A;                                        concatenate (L ++ [pivot] ++ R) into A
}
```

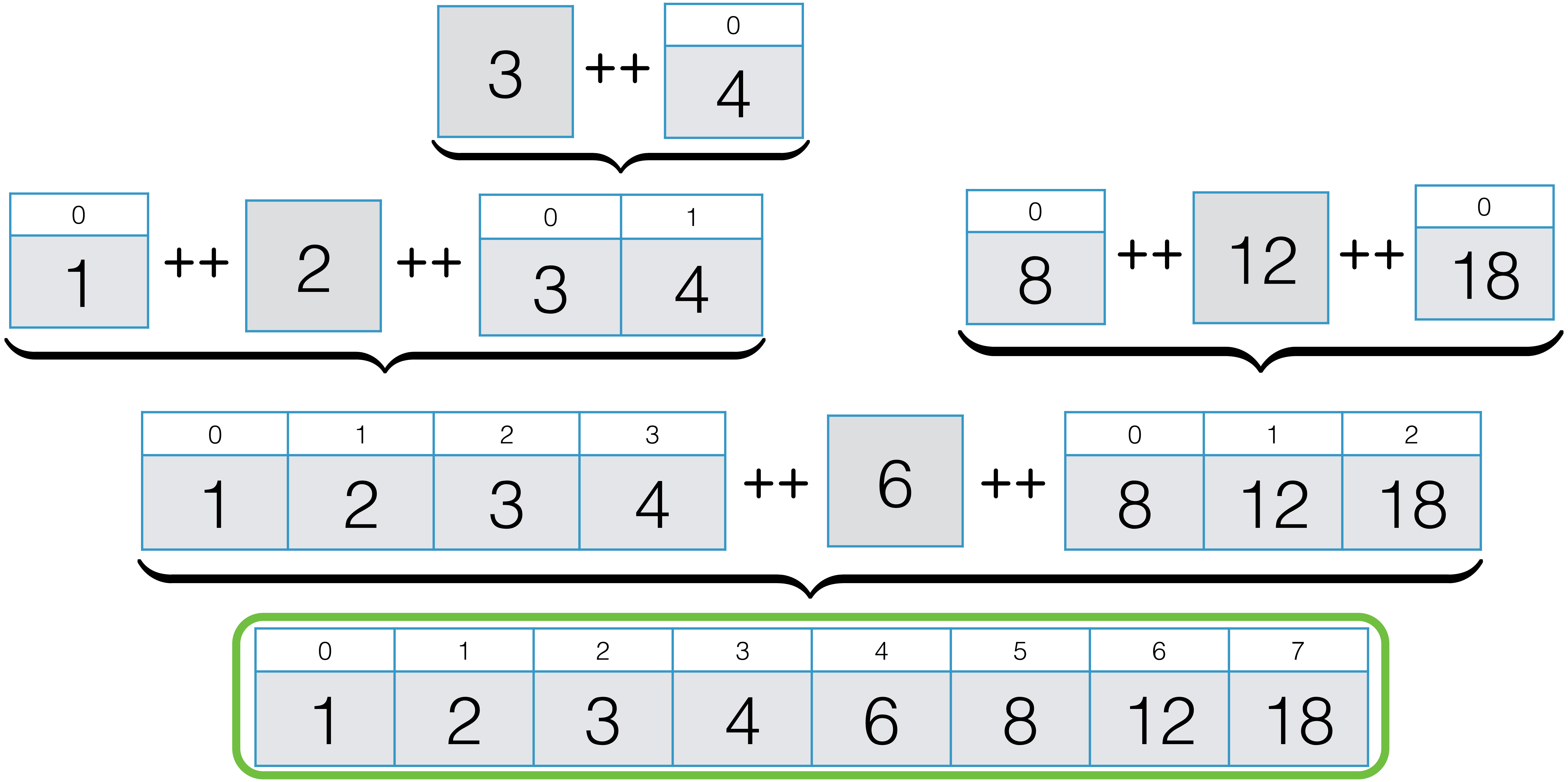
# Quicksort by example

Recursive descent: *choosing pivots* and constructing sub-arrays



# Quicksort by example

Combining *sorted* sub-arrays and pivots



# Quicksort vs. Merge sort

- *Quicksort* can be seen as a complement to *Merge sort* in distributing the computational complexity;
- In *Merge sort*, creating sub-arrays is simply *copying*, whereas in *Quicksort* it requires *rearranging* elements wrt. the *pivot*;
- In *Merge sort*, combining partial results is *merging* (complicated, requires comparisons), whereas in *Quicksort* they are *concatenated* (simple, no comparisons).

# Worst-case complexity of Quicksort

- Worst case is achieved when the arrays L and R are severely *imbalanced*;
- This happens, for instance, if the *pivot* is always the *smallest* element in the array.

Q(0) = 0
(no comparisons)

(n - 1) comparisons

Q(|L|) + Q(|R|)

```
QuickSort (A[0 … n-1]) {
 if (n ≤ 1) { return A; }
 else {
   l := 0; r := 0;
   pivot := A[n - 1];
   for (i = 1 … n-1) {
    if (A[i] < pivot) then {
       L[l] := A[i];
       l := l + 1;
     } else {
       R[r] := A[i];
       r := r + 1;
     }
   }
   Concat(QuickSort(L), pivot, QuickSort(R), A)
   return A;
}
```

# Worst-case complexity of Quicksort

- In the worst case, |L| = n - 1, so we obtain the following recurrence relation:

$$Q(1) = 0$$
$$Q(n) = \underbrace{Q(n - 1)}_{Q(|L|)} + n - 1, \text{ if } n > 1$$

By method of differences:

$$Q(n) = \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2} \in O(n^2)$$

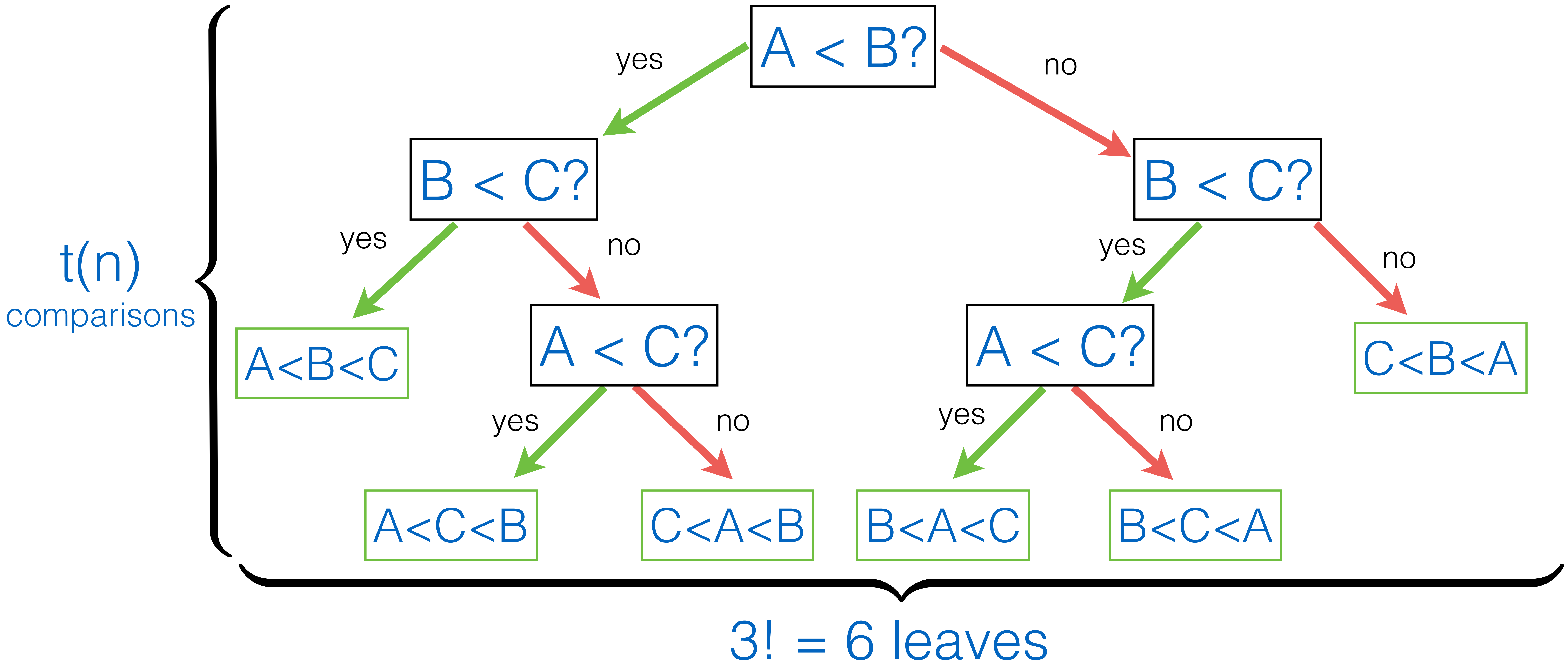
But for *Quicksort*, this worst case is *highly* improbable.

# Best worst time for comparison-based sorting

- *Quicksort*, *Insertion sort*, *Merge sort* are all *comparison-based* sorting algorithms: they compare elements *pairwise*;
- An "ideal" algorithm will *always* perform no more than $t(n)$ comparisons, where $n$ is the size of the array being sorted;
  - What is then $t(n)$?
- A number of *possible orderings* of $n$ elements is $n!$, and such an algorithm should find "the right one" by following a path in a *binary tree*, where each node corresponds to comparing just *two* elements.

# Decision tree of a comparison-based sorting

- **Example**: array [A, B, C] of three elements;
- All possible orderings between A, B, and C are possible.



# Best-worst case complexity analysis

- By making $t(n)$ steps in a *decision tree*, the algorithm should be able to say, which ordering it is;
- The number of reachable leaves in $t(n)$ steps is $2^{t(n)}$;
- The number of possible orderings is $n!$ is, therefore

$$2^{t(n)} \geq n!$$

# Best-worst case complexity analysis

$$2^{t(n)} \geq n!$$

$$t(n) \geq \log_2(n!)$$

Stirling's formula for large n: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$t(n) \approx n \log_e n$$
$$= (\log_e 2)\, n \log_2 n$$

$$\boxed{t(n) \in O(n \log n)}$$

# Can we do sorting better than in $O(n \log n)$?

Yes, if we don't base it on *comparisons*.

# Quiz

- We want to sort n integer numbers, all in the range 1…n;
- *No repetitions*, all numbers are present *exactly once*;
- What is the worst-case complexity?

**Answer**: O(n)

- We know that it has to be 1, 2, …, n-1, n, so just generate this sequence.

# Bucket sort

- We want to sort an array A of n records, whose *keys* are integer numbers;
- *All* keys in A are in the range 1…k;
- There *might be* repeated keys, some keys might be *absent*;
- **Idea**: allocate k "*buckets*" and put records into them, the "flush" the buckets in their order.

# Bucket sort

```
BucketSort (A[0 … n-1], k) {
 buckets := array of k empty lists;    // create k empty buckets

 for (i = 0..n-1) {
   key := A[i].key;    // get the next key
   bucket := buckets[key];    // find the bucket for the key
   buckets[key] := bucket ++ [A[i]];  // add the record into bucket
 }

 result = []
 for (j = 0..k-1) {   // concatenate all buckets
   result := result ++ buckets[j];
 }
 return result;
}
```

# Bucket Sort by Example

Keys are integer numbers, $k = 6$

A =

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 1 | 5 | 3 | 5 | 2 |

| [] | [] | [] | [] | [] | [] |
|---|---|---|---|---|---|

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|

# Bucket Sort by Example

Keys are integer numbers, $k = 6$

A =

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 1 | 5 | 3 | 5 | 2 |

[] [] [] [] [] [6]

1 2 3 4 5 6

# Bucket Sort by Example

Keys are integer numbers, $k = 6$

A =

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 1 | 5 | 3 | 5 | 2 |

[] [2] [] [] [] [6]

1 2 3 4 5 6

# Bucket Sort by Example

Keys are integer numbers, $k = 6$

A =

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 1 | 5 | 3 | 5 | 2 |

# Bucket Sort by Example

Keys are integer numbers, k = 6

A =

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 1 | 5 | 3 | 5 | 2 |

[1] [2] [3] [] [] [6]

1 2 3 4 5 6

# Bucket Sort by Example

Keys are integer numbers, $k = 6$

A =

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 1 | 5 | 3 | 5 | 2 |

[1] [2] [3] [] [5] [6]

1 2 3 4 5 6

# Bucket Sort by Example

Keys are integer numbers, k = 6

A =

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 1 | 5 | 3 | 5 | 2 |

[1] [2] [3, 3] [] [5] [6]

1 2 3 4 5 6

# Bucket Sort by Example

Keys are integer numbers, $k = 6$

A =

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 1 | 5 | 3 | 5 | 2 |

[1] [2] [3, 3] [] [5, 5] [6]

1 2 3 4 5 6

# Bucket Sort by Example

Keys are integer numbers, $k = 6$

A =

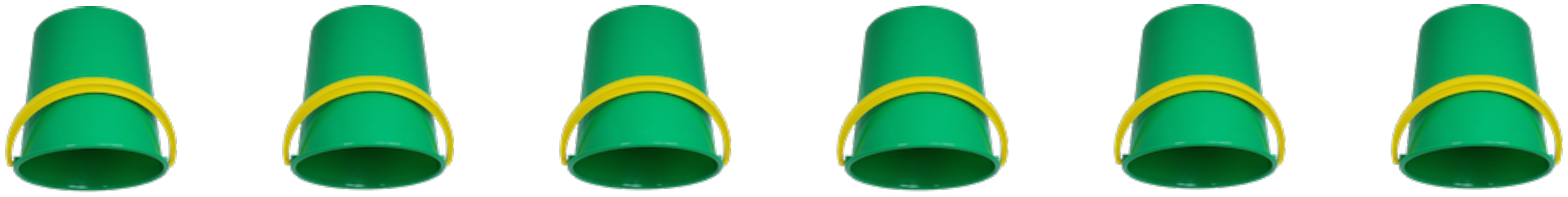
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 6 | 2 | 3 | 1 | 5 | 3 | 5 | 2 |

[1]  [2, 2]  [3, 3]  []  [5, 5]  [6]

1  2  3  4  5  6

# Bucket Sort by Example

1 2 3 4 5 6



[1] ++ [2, 2] ++ [3, 3] ++ [] ++ [5, 5] ++ [6]

result = $\boxed{[1, 2, 2, 3, 3, 5, 5, 6]}$

# Bucket Sort Worst-case Complexity

O(k) → `buckets := array of k empty lists;`

O(n) →

```
for (i = 0..n-1) {
  key := A[i].key;
  bucket := buckets[key];
  buckets[key] := bucket ++ [A[i]];
}
```

```
result = []
```

O(k) →

```
for (j = 0..k-1) {
  result := result ++ buckets[j];
}
return result;
```

Overall complexity: $O(n + k)$

# Remarks on Bucket Sort

- Bucket sort works for any sets of keys, known *in advance*;
- For instance, it can work with a *pre-defined set of strings*;
- But what if the *size* $k$ of the set of keys is *much larger* than $n$?
  - The complexity $O(n + k)$ is not so good in this case.

# Stability of Sorting Algorithms

A sorting algorithm is ***stable*** if, when two records in the original array have the same key, they stay in *their original order* in the sorted result.

- Is *Insertion sort* stable?
  - **Yes**
- What about *Bucket sort*?
  - **Yes**
- *Merge sort*?
  - **Maybe**. It depends on how we divide the list into two and how we merge them, resolving situations for elements with the same key.
- *Quicksort*?
  - **Maybe**. Depends on the implementation of the partition step.

# Radix sort

- An enhancement of the *Bucket sort*'s idea, for the case when the size of key set k in the array A is *very large*;
- **Idea**: partition each key using its decimal representation:
  - key = a + 10 b + 100 c + 1000 d + …
  - then, sort keys by each register of the decimal representation, *right-to-left*, using *Bucket sort*
  - For each internal bucket sort k = 10 (the base of decimal representation);
- Essentially:

```
RadixSort(A) {
  BucketSort A by a with k = 10;
  BucketSort A by b with k = 10;
  BucketSort A by c with k = 10;
  …
}
```

# Radix sort

(in very crude pseudocode)

```
RadixSort(A) {
  L := zip(A.keys, A);
  while (some key in L.fst is non zero) {
    L := BucketSort(L[keys mod 10], 10);   // sort by last register
    L.fst := L.fst / 10;     // shift L keys' representation to the next register
  }
  return L.snd;    // return sorted second component
}
```

# Radix sort by Example

A =

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 234 | 124 | 765 | 238 | 976 | 157 | 235 | 953 |

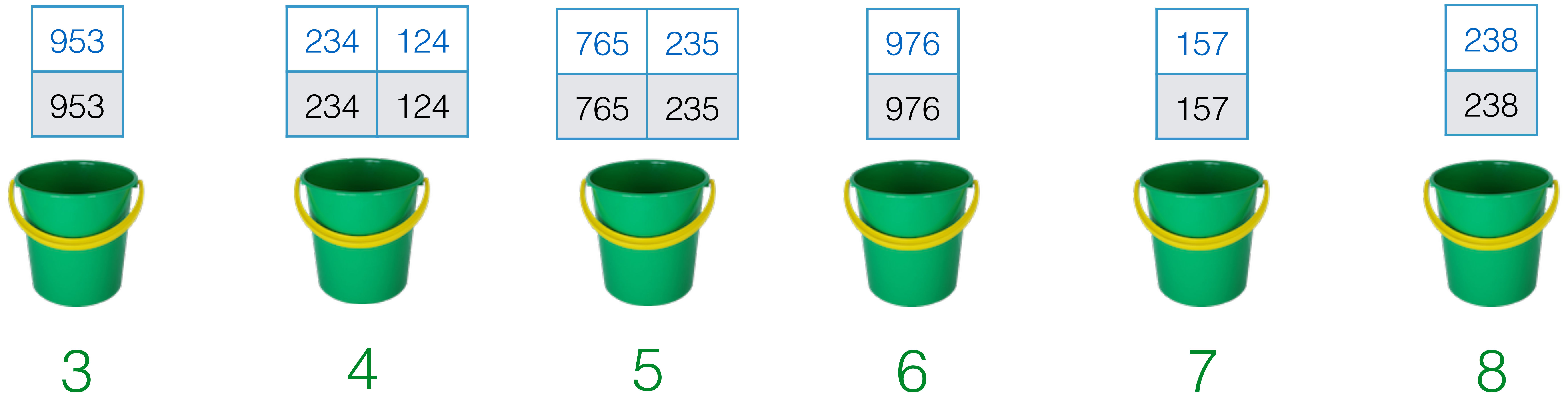
L =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 234 | 124 | 765 | 238 | 976 | 157 | 235 | 953 |
| 234 | 124 | 765 | 238 | 976 | 157 | 235 | 953 |

# Radix sort by Example

L =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 234 | 124 | 765 | 238 | 976 | 157 | 235 | 953 |
| 234 | 124 | 765 | 238 | 976 | 157 | 235 | 953 |



# Radix sort by Example

L =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 953 | 234 | 124 | 765 | 235 | 976 | 157 | 238 |
| 953 | 234 | 124 | 765 | 235 | 976 | 157 | 238 |

3 4 5 6 7 8

# Radix sort by Example

L =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 95 | 23 | 12 | 76 | 23 | 97 | 15 | 23 |
| 953 | 234 | 124 | 765 | 235 | 976 | 157 | 238 |

| |
|---|
| 12 |
| 124 |

| | | |
|---|---|---|
| 23 | 23 | 23 |
| 234 | 235 | 238 |

| | |
|---|---|
| 95 | 15 |
| 953 | 157 |

| |
|---|
| 76 |
| 765 |

| |
|---|
| 97 |
| 976 |

2 3 5 6 7

# Radix sort by Example

L =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 12 | 23 | 23 | 23 | 95 | 15 | 76 | 97 |
| 124 | 234 | 235 | 238 | 953 | 157 | 765 | 976 |

2 3 5 6 7

- Thanks to *stability* of Bucket sort, values within buckets remain *sorted* with respect to *lower* registers (e.g., for bucket 3).

# Radix sort by Example

L =

| 1 | 2 | 2 | 2 | 9 | 1 | 7 | 9 |
|---|---|---|---|---|---|---|---|
| 124 | 234 | 235 | 238 | 953 | 157 | 765 | 976 |

| 1 | 1 |
|---|---|
| 124 | 157 |

| 2 | 2 | 2 |
|---|---|---|
| 234 | 235 | 238 |

| 7 |
|---|
| 765 |

| 9 | 9 |
|---|---|
| 953 | 976 |

1 2 7 9

# Radix sort by Example

L =

| - | - | - | - | - | - | - | - |
|---|---|---|---|---|---|---|---|
| 124 | 157 | 234 | 235 | 238 | 765 | 953 | 976 |

1

2

7

9

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 124 | 157 | 234 | 235 | 238 | 765 | 953 | 976 |

# Complexity of Radix sort

O(n)

$O(\log_{10}k)$ iterations,
O(n) each

```
RadixSort(A) {
  L := zip(A.keys, A);
  while (some key in L.fst is non zero) {
    L := BucketSort(L[keys mod 10], 10);
    L.fst := L.fst / 10;
  }
  return L.snd;
}
```

Overall complexity: O(n log k)