Not every partition is generating (generating partition)

3️⃣ Example — logistic map $x_{n+1} = 4x_n(1 - x_n)$

This map is **two-to-one**: each value of x_{n+1} has two possible preimages.
To describe it uniquely, you need to know which *branch* of the map the current point is on — the left or the right of $x = 0.5$.

Hence, the **generating partition** is:

$$A_0 = [0, 0.5], \quad A_1 = (0.5, 1].$$

If you instead divided [0, 1] into three equal intervals

$$[0, 1/3], [1/3, 2/3], [2/3, 1],$$

you would *lose information*: several distinct points could follow the same sequence of interval visits, because the partition does not correspond to the natural branches of the map.

show how a non-generating partition (If you instead divided [0, 1] into three equal intervals) fails to distinguish two different trajectories in logistic map

Using a non-generating partition for a chaotic map like the **Logistic Map at** $r = 4$ (i.e., $f(x) = 4x(1-x)$) immediately causes **different initial conditions to produce the same symbol sequence**, thus failing to distinguish their unique trajectories.

The key to a generating partition lies in making the partition boundary coincide with the map's **critical point** (where the derivative is zero, i.e., the peak of the parabola). For $f(x) = 4x(1-x)$, the critical point is $x_c = 0.5$.

The Generating Partition (2 Intervals)

The correct generating partition is $\mathcal{P}_2 = \{A_0, A_1\}$:

- $A_0 = [0, \mathbf{0.5}]$
- $A_1 = (\mathbf{0.5}, 1]$

This partition works because the map is **monotone** (either increasing or decreasing) over each interval.

How a 3-Interval Partition Fails

If you divide the phase space $[0, 1]$ into three equal intervals, $\mathcal{P}_3 = \{A_0, A_1, A_2\}$, the central interval A_1 crosses the critical point $x_c = 0.5$.

- $A_0 = [0, 1/3]$ (Symbol: 0)
- $A_1 = (1/3, 2/3)$ (Symbol: 1) $\leftarrow$ **Contains** $x_c = 0.5$
- $A_2 = [2/3, 1]$ (Symbol: 2)

The Failure Example

Consider two different initial conditions, x_A and x_B, chosen to lie in the central interval A_1, but on opposite sides of the critical point $x_c = 0.5$:

1. **Initial Point A (x_A):** Choose x_A to be just *below* the critical point, say $x_A = 0.4$.
   - $x_A \in A_1 \implies$ **Symbol sequence starts with:** $i_0 = 1$
   - $x_1 = f(x_A) = 4(0.4)(1 - 0.4) = 4(0.4)(0.6) = 0.96.$
   - $x_1 = 0.96 \in A_2 \implies$ **Next symbol is:** $i_1 = 2$
2. **Initial Point B (x_B):** Choose x_B to be just *above* the critical point, say $x_B = 0.6$.
   - $x_B \in A_1 \implies$ **Symbol sequence starts with:** $i_0 = 1$
   - $x_1' = f(x_B) = 4(0.6)(1 - 0.6) = 4(0.6)(0.4) = 0.96.$
   - $x_1' = 0.96 \in A_2 \implies$ **Next symbol is:** $i_1 = 2$

Trajectory	Initial Point x_0	i_0 (Cell of x_0)	$x_1 = f(x_0)$	i_1 (Cell of x_1)	Symbolic Sequence
A	$x_A = 0.4$	$\mathbf{1}$	0.96	$\mathbf{2}$	$1, 2, \dots$
B	$x_B = 0.6$	$\mathbf{1}$	0.96	$\mathbf{2}$	$1, 2, \dots$

The Failure of Uniqueness

After two steps, the symbolic dynamics have already failed:

- The two initial points are clearly **different**: $x_A \neq x_B$.
- Their symbol sequences start with the **same finite string**: $(i_0, i_1) = (1, 2)$.
- Since $x_1 = x'_1 = 0.96$, the entire subsequent history of both trajectories will be **identical** from $n = 1$ onward: $x_n = x'_n$ for all $n \geq 1$. Therefore, the rest of their symbol sequences will also be identical: $i_n = i'_n$ for all $n \geq 1$.

Conclusion: Both different initial conditions, x_A and x_B, generate the exact **same infinite symbol sequence**, violating the condition that the symbol sequence must **uniquely determine** the initial value x_0.

This failure occurs because the central interval A_1 contains the critical point, meaning the map $f(x)$ is **not one-to-one** over that interval. Two different points in A_1 can map to the exact same point x_1, collapsing two distinct initial histories into one future, which a generating partition must prevent.

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2. The Necessary Condition for a Generating Partition

For a partition to be a generating partition for a map f, the intervals defined by the set of all points that produce a finite sequence of length N, called $\bigcap_{n=0}^{N-1} f^{-n}(A_{i_n})$, must have a length that shrinks to zero as $N \to \infty$.

$$\lim_{N\to\infty} \operatorname{diam}\left(\bigcap_{n=0}^{N-1} f^{-n}(A_{i_n})\right) = 0$$

In the chaotic maps like the **Tent Map** and the **Logistic Map** at $r = 4$, this condition is met by a simple partition (like $[0, 0.5]$ and $(0.5, 1]$) because the map's stretching and folding action ensures that:

1. **Stretching:** The map *expands* the distance between nearby points.
2. **Partitions:** This expansion, when combined with the partition, causes the region of initial conditions that share the same first N symbols to shrink exponentially fast.

A **non-generating** partition fails to capture this rapid shrinking, making many initial conditions map to the same symbolic sequence and breaking the one-to-one correspondence.

III. The Necessary Condition for a Generating Partition

For a partition to be a generating partition, the intervals that produce a specific finite sequence of length N must shrink to zero as $N \to \infty$.

A. Mathematical Condition

The set of initial points x_0 that generate a specific finite symbol sequence $(i_0, i_1, ..., i_{N-1})$ is defined as:

$$\mathbf{I_{i_0 i_1 \ldots i_{N-1}}} = \bigcap_{n=0}^{N-1} f^{-n}(A_{i_n})$$

- A_{i_n}: A cell (interval) of the partition, indexed by the symbol i_n.
- $\mathbf{f^{-n}(A_{i_n})}$: The set of all initial points x_0 that land in cell A_{i_n} after n iterations.
- $\bigcap$ **(Intersection)**: Represents the "AND" condition, meaning x_0 must satisfy *all* conditions simultaneously (i.e., its 0-th through $(N-1)$-th iterates must land in the specified cells).

B. The Shrinking Condition

The diameter (length) of this set must tend to zero as the length of the symbolic sequence (N) goes to infinity:

$$\lim_{N \to \infty} \operatorname{diam} \left(\bigcap_{n=0}^{N-1} f^{-n}(A_{i_n}) \right) = 0$$

In chaotic maps, the map's stretching and folding action, when combined with a correct generating partition, ensures this region of initial conditions shrinks exponentially fast. A non-generating partition fails to capture this rapid shrinking, breaking the one-to-one correspondence.

Mathematical Meaning of the Set

The expression $\left(\bigcap_{n=0}^{N-1} f^{-n}(A_{i_n})\right)$ defines a specific subset of the phase space, which we can call $I_{i_0 i_1 \dots i_{N-1}}$.

Symbol	Meaning
$\bigcap$	The **intersection** symbol (representing "AND")
$n = 0$ to $N - 1$	The time steps from the initial time ($n = 0$) up to the time before N (a sequence of length N)
A_{i_n}	A cell (interval) of the partition, indexed by the symbol i_n
$f^{-n}(\cdot)$	The n**-th preimage** of the map f

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1. The Preimage $f^{-n}(A_{i_n})$

The term $f^{-n}(A_{i_n})$ represents **all initial points** x_0 that, after n iterations of the map f, land in the partition cell A_{i_n}.

$$f^{-n}(A_{i_n}) = \{x_0 \in \text{Phase Space} \mid f^n(x_0) \in A_{i_n}\}$$

- For $n = 0$: $f^{-0}(A_{i_0}) = A_{i_0}$. This is simply the set of initial points that start in the cell A_{i_0}.
- For $n = 1$: $f^{-1}(A_{i_1})$. This is the set of initial points x_0 such that $x_1 = f(x_0)$ lands in A_{i_1}.

2. The Intersection $\bigcap_{n=0}^{N-1}$

The intersection symbol means that we are looking for the points x_0 that satisfy **all** the conditions simultaneously.

The full expression $\bigcap_{n=0}^{N-1} f^{-n}(A_{i_n})$ is the set of initial points x_0 that generate the specific, finite symbol sequence $(i_0, i_1, \dots, i_{N-1})$:

2️⃣ Why arbitrary partitions usually fail

If your partition boundaries are placed *in the wrong places*, it's possible that two distinct points always fall into the **same sequence of cells** under iteration.

That happens especially when:

- The **map** is not monotonic over the cells.
- Or the **partition boundaries** do not align with the map's **critical points** (folds, turning points, or discontinuities).

In those cases, parts of the space that evolve differently under the map may still be *symbolically indistinguishable* — meaning the partition **does not generate** unique symbolic sequences.

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Hence, the **generating partition** is:

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you would *lose information*: several distinct points could follow the same sequence of interval visits, because the partition does not correspond to the natural branches of the map.

4️⃣ Why the boundary at the critical point matters

The **critical point** (where the map changes monotonic branch) determines the fundamental "fold" in the dynamics.
A generating partition must distinguish the **monotonic pieces** of the map — otherwise, the inverse mapping is ambiguous.

For 1D maps that are *piecewise monotonic*, the **set of monotonicity intervals** (branches) gives a natural generating partition.

6️⃣ Conceptual rule of thumb

> A generating partition in 1D divides the phase space along the **natural monotonicity domains** of the map, so that each symbolic sequence uniquely traces an orbit through the inverse branches.

So your statement

> “Then every division of the 1D phase space into several intervals will be generating”
> is **false**, because only *special partitions* — those aligned with the structure of the map (its critical points or monotonic intervals) — ensure the symbolic coding is one-to-one.