

# On the distribution of the expansion rates $E_N$

✚ The equation (16.1.4):

$$P(x_0^{(k)}) = \frac{1}{Z_N^{\text{top}}(\beta)} \exp[-\beta N E_N(x_0^{(k)})]$$

I interpret the equation 16.1.4 such that the local expansion rates are distributed according to the Gibbs-like distribution. But there is no reason for the local expansion rates  $E_N$  to be Gibbs-distributed.

Excellent — that's a **very deep and subtle observation**, and you're absolutely right to question it.

Let's unpack this carefully, because your intuition is correct:

**Equation (16.1.4) does *not* claim that the local expansion rates  $E_N(x_0^{(k)})$  are empirically distributed according to a Gibbs law.**

Rather, it **defines** such a distribution for theoretical (thermodynamic) purposes.

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✚ **Step 1 — What equation (16.1.4) *actually* says**

$$P(x_0^{(k)}) = \frac{1}{Z_N^{\text{top}}(\beta)} e^{-\beta N E_N(x_0^{(k)})}$$

is **not a measured probability distribution** of expansion rates across the phase space.

It is an **assigned weight function**, *constructed* by analogy with the Boltzmann–Gibbs distribution, to form a thermodynamic ensemble over trajectories.

That is, this is not an *empirical probability law*, but an *artificial (formal) one* chosen to make the dynamical system amenable to thermodynamic formalism.

## 🧠 Step 2 — Why it's *not* the true distribution of $E_N$

The actual (measurable) distribution of the finite-time expansion rates

$$E_N(x_0) = \frac{1}{N} \sum_{n=0}^{N-1} \ln |f'(x_n)|$$

depends on:

- the invariant measure of the map  $f$ ,
- correlations between  $f'(x_n)$  at different times,
- ergodic or multifractal properties of the attractor.

That distribution could, for instance, be broad, skewed, or non-Gaussian; in general, it is **not exponential (Gibbs-like)**.

So in reality,  $E_N(x_0)$  has some natural density  $p(E_N)$  that arises from the dynamics, but  $P(x_0^{(k)})$  in Eq. (16.1.4) is not that — it's an **imposed statistical weight**.

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## ⚙️ Step 3 — Why we introduce a Gibbs-like weighting anyway

The **thermodynamic formalism** (originating from Sinai, Ruelle, Bowen) does something deliberate and artificial:

It *postulates* a Gibbs-like ensemble over orbits, where each orbit's "energy" is identified with some dynamical quantity (here, the local expansion rate).

Then one defines the "partition function" and "free energy per step" as analogues of thermodynamic quantities:



$$Z_N^{\text{top}}(\beta) = \sum_k e^{-\beta N E_N(x_0^{(k)})}, \quad P_{\text{top}}(\beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_N^{\text{top}}(\beta)$$

The point is not that the system is thermal or that  $E_N$  is thermally distributed — but that this formal construction **encodes** dynamical complexity (number, stability, and weighting of orbits) in a mathematically analogous way to how free energy encodes the statistical structure of energies in physics.

So this “Gibbs-like distribution” is a **generating device**, not an empirical law.


#### Step 4 — Interpretation in words

- In thermodynamics: the Gibbs measure describes *how microstates actually occur* in equilibrium.
- In dynamical systems: the “Gibbs-like measure” is a **formal ensemble** that allows us to *probe the structure* of orbits depending on their expansion rate.

Hence:

The local expansion rates are not Gibbs-distributed — rather, we *assign* Gibbs weights to them to construct a topological pressure that captures the exponential growth of orbit weights as a function of  $\beta$ .

#### Step 5 — A physical analogy

In physics, you might introduce a *canonical ensemble* even for a system that’s not truly in equilibrium, because the ensemble provides a convenient way to  mpute quantities (via partition functions).

Similarly here: the Gibbs form is a **tool**, not a statement of fact.

It allows one to interpolate between counting all orbits equally ( $\beta = 0$ ) and weighting by their instability ( $\beta \neq 0$ ), to study how dynamical complexity depends on instability.

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### ✓ Summary

You are **100% correct**:

- There is *no physical reason* for the actual expansion rates  $E_N$  to be Gibbs-distributed.
- Equation (16.1.4) **defines** an *artificial ensemble* over trajectories with Gibbs-type weights.
- This ensemble is used to construct the **topological pressure**, a formal analog of free energy that encapsulates the growth rate of weighted orbit sums.

If you like, I can show you explicitly how this definition leads to the **Legendre-type relationship** between topological pressure and the large-deviation spectrum of local Lyapunov exponents — that's where the real meaning of this formal "Gibbs distribution" appears. Would you like that next?

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Yes

Thought for 8s

Great — here's the standard large-deviations / Legendre-transform derivation that connects the *topological partition function* to the *spectrum* (rate function / multifractal entropy) of finite-time expansion rates  $E_N$ . I'll keep the argument compact and explicit.

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## 1) Rewrite the partition function as an integral over expansion rates

Start from the definition

$$Z_N^{\text{top}}(\beta) = \sum_{k=1}^{K(N)} \exp \left[ -\beta N E_N(x_0^{(k)}) \right],$$

where indices  $k$  label the representative initial conditions (or symbolic orbits). Introduce the empirical density (counting measure) of values of  $E_N$ :

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$$\rho_N(\lambda) = \#\{k : E_N(x_0^{(k)}) \in (\lambda, \lambda + d\lambda)\}.$$

Then formally

$$Z_N^{\text{top}}(\beta) = \int d\lambda \rho_N(\lambda) e^{-\beta N \lambda}.$$


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## 2) Assume a large-deviation (exponential) scaling for $\rho_N(\lambda)$

For many chaotic systems the number of orbits with a given finite-time expansion rate obeys an exponential law in  $N$ . Write

$$\rho_N(\lambda) \asymp \exp[N s(\lambda)],$$

where  $s(\lambda)$  is the *entropy density* (or multifractal spectrum / rate function up to sign) that counts orbits having expansion rate  $\lambda$ . (This is the hypothesis of a large-deviation form for the empirical distribution of  $E_N$ .)

Plugging into  $Z_N$ :

$$Z_N^{\text{top}}(\beta) \approx \int d\lambda \exp[N(s(\lambda) - \beta \lambda)].$$


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## 3) Saddle-point for large $N$

For large  $N$  the integral is dominated by the maximum of the exponent. By Laplace / saddle-point method,

$$\frac{1}{N} \ln Z_N^{\text{top}}(\beta) \xrightarrow{N \rightarrow \infty} \sup_{\lambda} \{ s(\lambda) - \beta \lambda \}.$$

Define the **topological pressure per step** (the “free energy”)

$$P_{\text{top}}(\beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_N^{\text{top}}(\beta).$$

Then the relation is

$$P_{\text{top}}(\beta) = \sup_{\lambda} \{ s(\lambda) - \beta \lambda \}.$$

This is exactly a **Legendre transform**:  $P(\beta)$  is the Legendre transform of  $s(\lambda)$ .

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#### 4) Inverse transform — obtaining the spectrum from the pressure

Because of the Legendre duality, under mild convexity/regularity conditions

$$s(\lambda) = \inf_{\beta} \{ P_{\text{top}}(\beta) + \beta \lambda \}.$$

So the spectrum  $s(\lambda)$  (which tells you how many orbits have expansion rate  $\lambda$ ) is recovered as the Legendre transform of the pressure.

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#### 5) Probabilistic and thermodynamic interpretations

- The saddle point  $\lambda^*(\beta)$  that achieves the supremum satisfies

$$\frac{d}{d\lambda} [s(\lambda) - \beta\lambda]_{\lambda=\lambda^*(\beta)} = 0 \quad \Rightarrow \quad s'(\lambda^*) = \beta.$$

Equivalently, differentiating the pressure gives

$$P'_{\text{top}}(\beta) = -\lambda^*(\beta).$$

So the mean (typical) expansion rate **under the Gibbs-like weights** at inverse temperature  $\beta$  is  $-P'(\beta)$ .

- Higher derivatives give fluctuation information:

$$P''_{\text{top}}(\beta) = -\frac{d\lambda^*}{d\beta} \quad (\text{related to variance of } E_N \text{ under the Gibbs measure}).$$

- In large-deviation language, the natural (unweighted) probability density of  $E_N$  behaves like

$$\mathbb{P}(E_N \approx \lambda) \asymp e^{Ns(\lambda)}.$$

The Gibbs-reweighting multiplies that by  $e^{-N\beta\lambda}$ , making the *Gibbs-typical* value  $\lambda^*(\beta)$  the one that maximizes  $s(\lambda) - \beta\lambda$ .

## 6) Summary (conceptual)

- The Gibbs-like measure in (16.1.4) is a *tool* that biases the ensemble of orbits according to their expansion rate.
- The topological partition sum  $Z_N^{\text{top}}(\beta)$  is then the Laplace transform of the orbit-counting function  $\rho_N(\lambda)$ .
- For large  $N$  the pressure  $P_{\text{top}}(\beta)$  is the Legendre transform of the orbit-counting exponent  $s(\lambda)$ . Equivalently,  $s(\lambda)$  is the Legendre transform of  $P_{\text{top}}(\beta)$ .
- This is precisely the connection between thermodynamic formalism and large deviations / multifractal spectra of finite-time Lyapunov exponents.

