# Topological Pressure

## 1. Motivation

Topological entropy $h_{\text{top}}(f)$ measures the exponential growth rate of distinguishable orbit segments in a dynamical system. Topological pressure $P(\phi)$ generalizes this concept by introducing a continuous *potential function* $\phi : X \to \mathbb{R}$, which allows different orbits to be weighted differently—analogous to weighting microstates by energy in statistical mechanics.

## 2. Definition

Let $f : X \to X$ be a continuous map on a compact metric space $X$, and $\phi : X \to \mathbb{R}$ be a continuous function. Define the $n$-step potential sum along an orbit:

$$S_n\phi(x) = \sum_{k=0}^{n-1} \phi(f^k(x)).$$

For given $n$ and $\varepsilon > 0$, let $E \subset X$ be a maximal $(n, \varepsilon)$-separated set, and define the corresponding partition function:

$$Z_n(\phi, \varepsilon) = \sum_{x \in E} \exp(S_n\phi(x)).$$

The *topological pressure* is then

$$P(\phi) = \lim_{\varepsilon \to 0} \left( \limsup_{n \to \infty} \frac{1}{n} \ln Z_n(\phi, \varepsilon) \right).$$

## 3. Interpretation

When $\phi = 0$, the weights are unity, $Z_n = N(n, \varepsilon)$, and

$$P(0) = h_{\text{top}}(f),$$

so topological pressure generalizes topological entropy. For nonzero $\phi$, orbit segments contribute with different weights $e^{S_n\phi(x)}$, and $P(\phi)$ measures the exponential growth rate of this weighted orbit complexity.

## 4. Variational Principle

The topological pressure satisfies the *variational principle*:

$$P(\phi) = \sup_{\mu \in \mathcal{M}_f} \left[ h_\mu(f) + \int \phi \, d\mu \right],$$

where $\mathcal{M}_f$ is the set of $f$-invariant probability measures and $h_\mu(f)$ is the measure-theoretic (Kolmogorov–Sinai) entropy. The measure $\mu_\phi$ that attains the supremum is called the *equilibrium state* for $\phi$.

## 5. Thermodynamic Analogy

The correspondence with thermodynamics is summarized below:

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\begin{array}{lcl}
\text{Energy} & \leftrightarrow & \text{Potential function } \phi, \\
\text{Temperature } (T = 1/\beta) & \leftrightarrow & \text{Scaling of } \phi, \\
\text{Partition function } Z = \sum e^{-\beta E} & \leftrightarrow & Z_n(\phi, \varepsilon) = \sum e^{S_n \phi(x)}, \\
\text{Free energy } F = -T \ln Z & \leftrightarrow & P(\phi) = \lim \frac{1}{n} \ln Z_n, \\
\text{Entropy } S = -\partial F / \partial T & \leftrightarrow & h_\mu(f) = P(\phi) - \int \phi \, d\mu.
\end{array}
$$

## 6. Example: Lyapunov Weights

For one-dimensional maps, choosing

$$\phi(x) = -t \ln |f'(x)|$$

leads to the *Bowen equation*

$$P(\phi_t) = 0,$$

whose solution in $t$ gives the Hausdorff dimension of the corresponding invariant set.

## 7. Summary

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|---|---|
| Topological entropy: | $h_{\text{top}} = P(0)$, |
| Topological pressure: | $P(\phi) =$ weighted exponential growth rate of orbit complexity, |
| Variational principle: | $P(\phi) = \sup_\mu \big(h_\mu + \int \phi \, d\mu\big)$. |