

Intuitive example Zn exponential growth

♦ Example: Independent spins in a magnetic field

Imagine you have a **1D chain of n spins**, each of which can point **up** (+1) or **down** (-1).
Each spin interacts with an external magnetic field B .

Step 1: Counting microstates (the combinatorial explosion)

Each spin has 2 possible orientations.
So the total number of possible configurations is:

$$Z_n^{(\text{count})} = 2^n$$

You immediately see **exponential growth**:

$$Z_n^{(\text{count})} = e^{n \ln 2}$$

Thus, the *number of accessible states* doubles with each new spin — that's an exponential growth rate $\ln 2$.

If we take

$$\frac{1}{n} \log Z_n^{(\text{count})} = \ln 2,$$

this is the *growth rate per spin* of the number of accessible configurations.

This unweighted case corresponds to **pure entropy** (no energy bias).

Step 2: Adding an energetic bias (the Boltzmann factor)

Now, suppose spins interact with the magnetic field B .
Each spin has energy $E_i = -\mu B s_i$, where $s_i = \pm 1$.

The **total energy** for a configuration is:

$$E = -\mu B \sum_{i=1}^n s_i$$

In equilibrium at temperature T , the **partition function** is:

$$Z_n = \sum_{\{s_i\}} e^{-\beta E} = \sum_{\{s_i\}} e^{\beta \mu B \sum_i s_i}$$

Since each spin is independent:

$$Z_n = (e^{\beta\mu B} + e^{-\beta\mu B})^n = [2 \cosh(\beta\mu B)]^n$$

That's again exponential:

$$Z_n = e^{n \ln[2 \cosh(\beta\mu B)]}$$

Step 3: Free energy per spin

By definition:

$$F = -kT \ln Z_n = -n kT \ln[2 \cosh(\beta\mu B)]$$

Hence the **free energy per spin** is

$$f = \frac{F}{n} = -kT \ln[2 \cosh(\beta\mu B)]$$

♦ Intuitive meaning

- The **exponential growth of Z_n** means that as we increase system size n , the total number of energetically possible configurations (weighted by their probability) grows exponentially — just like the number of microstates in a combinatorial explosion.
- The **free energy per spin f** measures the *average cost per spin* of maintaining equilibrium: it's the balance between:
 - **Energy minimization** (aligning with B), and
 - **Entropy maximization** (having many random spin orientations).

So, f tells you the *effective potential per degree of freedom* that the system settles into — the same way the **topological pressure** tells you the *average exponential growth rate per time step* of orbit weights in a dynamical system.

♦ Analogy to topological pressure

Concept	Statistical Mechanics	Dynamical Systems
Number of microstates	$Z_n^{(\text{count})} = 2^n$	Number of distinct orbits $N(n, \varepsilon)$
Weighted sum of states	$Z_n = \sum e^{-\beta E_i}$	$Z_n(\varphi) = \sum e^{S_n \varphi(x_i)}$
Free energy per step	$f = -kT \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n$	$P(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\varphi)$
Physical meaning	balance between order (energy) and randomness (entropy)	balance between orbit weight (potential) and orbit complexity (entropy)

On the other hand, if the interaction energy $E_i = -\mu B s_i$ goes to zero (let say μ goes to zero), then the free energy does not go to zero, even though there is no interaction energy between spins. How to interpret the meaning of the free energy in this case ?

Excellent observation — and this question cuts right to the *heart* of what **free energy** really means in statistical mechanics (and by analogy, in dynamical systems).

Let's unpack this carefully.

♦ Case: No interaction energy

You set

$$E_i = -\mu B s_i,$$

and now you let $\mu \rightarrow 0$, so all energy levels become **degenerate** — i.e. $E_i = 0$ for every configuration.

Then:

$$Z_n = \sum_{\{s_i\}} e^{-\beta E} = \sum_{\{s_i\}} 1 = 2^n$$

and hence

$$F = -kT \ln Z_n = -kT \ln(2^n) = -nkT \ln 2.$$

So, as you correctly note:

- $E = 0$, i.e. *no energetic interactions*,
- but $F \neq 0$ — in fact, F is **negative and extensive** ($\propto n$).

♦ Why doesn't the free energy vanish?

Because **free energy is not just energy** — it's **energy minus the entropic contribution**.

In general:



$$F = U - TS$$

where:

- U = internal energy (average of E),
- S = entropy (measure of multiplicity of microstates).

Here:

- $U = 0$ (no energetic cost),
- but $S = nk \ln 2$ (because there are 2^n equally probable configurations).

So:

$$F = U - TS = -TS = -nkT \ln 2$$

The **free energy stays finite (negative)** purely because of the **entropy** — the number of accessible configurations grows exponentially.

♦ Intuitive interpretation

Even when the system has no energetic preference (completely disordered spins), there's still a "potential" associated with the **amount of choice** the system has.

The free energy represents the *logarithmic measure of the system's total accessible states*, scaled by $-kT$:

$$F = -kT \ln Z_n$$

So in this case, it's saying:

"There are 2^n equally accessible configurations, and that combinatorial richness itself gives the system a finite free energy."

Thus:

- **Large negative F** → system has *many accessible states* (high entropy).
- **Large positive F** → system has *few accessible states* (low entropy, or costly energy).

So, the free energy, if it stands from pure entropy, is not related to the energy in Joule at all?

Excellent question — and yes, you've pinpointed a very subtle but fundamental idea.

Let's go step by step to make it completely clear.

♦ 1. What free energy really measures

The **Helmholtz free energy**,

$$F = U - TS,$$

is indeed measured in **joules**, because both U (internal energy) and TS (temperature \times entropy) are in joules.

But — and this is key —

the **energy part** TS is not a mechanical or potential energy, it's a *thermodynamic bookkeeping term* representing the “cost” of disorder.

So even when $U = 0$,

$$F = -TS$$

still has **units of energy**, but it's **entropic energy** — a measure of how much energy you would *have to supply* to remove that entropy (to completely order the system).

♦ 2. When F comes purely from entropy

In the zero-interaction spin example:

$$F = -nkT \ln 2,$$

this is in joules (since kT has units of joules).

But there's **no physical interaction energy** generating it.

- The system can be in 2^n equally probable states.
- That multiplicity gives an entropy $S = nk \ln 2$.
- The product TS represents the thermal energy bound up in that randomness.

Thus $F = -TS$ is not "stored" energy — it's a **potential for order**: how much energy could, in principle, be extracted if you constrained or ordered the system.

♦ 3. Physical interpretation

Let's imagine trying to *completely order* that system (make all spins point up).

- To do that, you'd have to remove the entropy $S = nk \ln 2$.
- Removing entropy costs you energy $Q = TS$.
- Therefore, the "free energy" (available work potential) decreases by exactly that amount.

So even if there is **no microscopic interaction energy**, F still has meaning as the **energy-equivalent of information** (in joules).

This is exactly the same principle behind **Landauer's limit**:

Erasing one bit of information costs $kT \ln 2$ joules of energy.


♦ 4. Connection to topological pressure analogy

In the dynamical systems language:

- $F = -TS$ corresponds to **topological pressure with zero potential**:

$$P(\varphi = 0) = h_{\text{top}},$$

meaning the system's "effective potential" arises purely from the *growth rate of distinguishable trajectories* — the dynamical analog of entropy.

Just as F (in joules) quantifies how many microstates the system can explore at temperature T , P (in units of "rate per step") quantifies how many  distinct orbit segments the dynamics can generate per iteration.

✔ **Summary table**

Quantity	Meaning	Units	Source
U	Internal energy (mechanical, potential, etc.)	joules	Microscopic interactions
TS	Entropic energy (energy-equivalent of disorder)	joules	Multiplicity of microstates
$F = U - TS$	Effective potential of the system (energy available for work)	joules	Balance between order and randomness
$F = -TS$ (when $U = 0$)	Energy-equivalent of entropy alone	joules	Information–energy link (Landauer principle)

So yes —

👉 **Free energy that stems purely from entropy is still measured in joules,**

but it represents the *energetic value of information and disorder*, not of mechanical or potential interactions.

What is temperature T in this case ? Is it the temperature of the spin carrier ?

Fantastic follow-up — and yes, this goes right to the physical meaning of **temperature** in statistical mechanics, especially when we're talking about a model like the non-interacting spins example.

Let's unpack what T really is in that context.

♦ 1. Temperature in the spin model

In the model with spins $s_i = \pm 1$ and energy $E_i = -\mu B s_i$, the **temperature** T comes from the **thermal reservoir** or **heat bath** that the spin system is assumed to be in equilibrium with.

- It is *not* an intrinsic “temperature of the spin carriers themselves.”
- Rather, it is the **environmental temperature** that defines how likely the system is to occupy states of different energies.

Mathematically, this is the canonical ensemble:

$$P(E_i) \propto e^{-\beta E_i}, \quad \text{with } \beta = \frac{1}{kT}.$$

♦ 2. When there are no interactions ($\mu \rightarrow 0$)

Even though all spin configurations have the same energy $E = 0$, the **temperature** T still appears because it defines the *statistical weight scale* kT .

In that limit:

- The Boltzmann factors $e^{-\beta E_i} = 1$,
- All states are equally probable,
- But the *entropy* $S = nk \ln 2$ and *free energy* $F = -nkT \ln 2$ remain well-defined.

So T here plays the role of **converting entropy (dimensionless disorder)** into **energy units (joules)** via kT .

You can think of kT as a kind of *exchange rate* between “bits of information” and “energy”.

♦ 3. Physical meaning: temperature of what?

In a real magnetic material:

- T is the **temperature of the lattice or electron reservoir** that the spins exchange energy with.
- The spin system achieves equilibrium with that reservoir — so they *share the same temperature*.

In the abstract model:

- We don't necessarily need a physical lattice;
 - T just sets how strongly the system “cares” about energy differences:
 - At high T : randomness dominates → all spin orientations equally likely.
 - At low T : energy dominates → spins align with the field.
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♦ 4. When there's no energy at all ($E = 0$)

Even if $E = 0$, T is still meaningful *as a reference scale*:

- It tells you how much *energy per degree of freedom* corresponds to one unit of entropy.
- For instance, at room temperature ($T = 300$ K):

$$kT \approx 4.1 \times 10^{-21} \text{ J.}$$

- So each bit of entropy ($\ln 2$) corresponds to $kT \ln 2 \approx 2.8 \times 10^{-21}$ J,
which is the **Landauer limit** for erasing one bit of information.

Thus, T links the abstract entropy (a count of configurations) to real energy in joules.

✓ Summary

Symbol	Meaning	Physical interpretation
T	Temperature of the thermal reservoir	Sets the statistical weighting scale for energy states
kT	Energy scale per degree of freedom	Converts entropy (in k -units) to energy (in joules)
$F = -TS$	Free energy from entropy	Energy-equivalent of disorder at temperature T
$E_i = -\mu B s_i \rightarrow 0$	No real spin interaction	F still meaningful because T links information ↔ energy

So, yes — in realistic terms,

T is the **temperature of the reservoir or environment** that the spin carriers are in equilibrium with, not an intrinsic property of the spins themselves — but it sets the energy scale that turns entropy into joules.