More relevant examples

Excellent and very perceptive question.
Indeed, while the **logistic map** is a textbook starting point for chaos and entropy, it is *too simple* to show the full richness of **topological pressure phenomena** — especially things like **phase transitions**, **multifractal measures**, and **nontrivial equilibrium states**.

Let's look at systems that are more *interesting* and *informative* for exploring **topological pressure** and the **thermodynamic formalism**.

🔹 1. The Full Shift on k Symbols

Why: It's the simplest chaotic system where topological pressure can be computed *exactly* and used to build intuition.

- **Dynamics:**
  Sequences of symbols $(s_0, s_1, s_2, \dots)$ with $s_i \in \{1, 2, \dots, k\}$.
  The map is the **shift**: $\sigma(s_0, s_1, s_2, \dots) = (s_1, s_2, \dots)$.
- **Pressure:**

$$P(\phi) = \ln \lambda_{\max},$$

  where $\lambda_{\max}$ is the maximal eigenvalue of the **Ruelle transfer operator** $\mathcal{L}_\phi$.
  For constant $\phi = 0$, $P(0) = \ln k$.
- **Why interesting:** You can choose $\phi(s_0, s_1)$ to depend on transitions (e.g. "energies" of symbol pairs), producing *exactly solvable analogues* of spin models (Ising-like).
  Phase transitions appear naturally when potentials become long-range or non-Hölder.

🔹 2. The Smale Horseshoe Map

Why: It exhibits *hyperbolic chaos* with a fractal invariant set — the canonical example of a **uniformly hyperbolic system**.

- The topological entropy can be computed from the stretching rate and symbolic dynamics of the horseshoe.

- For potentials like $\phi(x) = -t \ln |Df(x)|$, the pressure function $P(t)$ gives access to:
  - Hausdorff dimension of the invariant Cantor set (where $P(t) = 0$)
  - Multifractal spectra of unstable Lyapunov exponents.

Why interesting: You can explicitly compute or approximate $P(t)$ and see how geometry, expansion rate, and entropy interplay.

🔹 3. The Hénon Map

$$x_{n+1} = 1 - ax_n^2 + y_n, \quad y_{n+1} = bx_n$$

Why: A 2D, dissipative, non-uniformly hyperbolic system with a rich attractor.

- Unlike the logistic map, the Hénon attractor has **both stable and unstable manifolds**, and **nonuniform hyperbolicity**.
- The topological pressure $P(\phi_t)$ with $\phi_t = -t \ln |Df_u(x)|$ (unstable Jacobian) can reveal:
  - Dimension spectrum of invariant measure.
  - Phase transitions in $P(t)$.
  - Relation between entropy and Lyapunov spectrum.

Why interesting: This is a realistic example where the **thermodynamic formalism meets nonuniform hyperbolicity**, and analytical computation is nontrivial but insightful numerically.

🔹 4. The Baker's Map

$$(x, y) \mapsto \begin{cases} (2x, y/2), & x < 1/2, \\ (2x - 1, (y+1)/2), & x \geq 1/2. \end{cases}$$

Why: It's a piecewise-linear, area-preserving, *exactly solvable chaotic map* with uniform hyperbolicity.

- Easy to compute Lyapunov exponents, entropy, and fractal dimensions.
- Good testbed for verifying $P(\phi_t)$ numerically.
- Natural link between symbolic coding and geometric structure.

🔹 5. Geodesic Flow on a Negatively Curved Surface

Why: This is the continuous-time prototype of **Anosov flows**, with rich thermodynamic formalism.

- For potential $\phi = -t \log J^u(x)$ (Jacobian along unstable direction), $P(\phi)$ governs the **dimension spectrum** and **decay of correlations.**
- The equilibrium state is the **Bowen–Margulis measure**, the natural invariant measure for chaotic geodesic flow.

Why interesting: It's the archetypal example connecting dynamical pressure to geometry and ergodic theory.

🔹 6. Intermittent (Nonuniformly Hyperbolic) Maps

Examples:

- Pomeau–Manneville map
  $f(x) = x + x^{1+\alpha} \mod 1$
- Logistic map at tangency parameter.

Why: These systems exhibit **phase transitions** in the pressure function $P(t)$ — that is, $P(t)$ becomes *non-analytic* at a critical t_c.
This mimics thermodynamic critical points.

- Below t_c: system dominated by expanding orbits.
- Above t_c: dominated by slow (intermittent) orbits.

Why interesting: These are perfect for understanding **how topological pressure encodes dynamical phase transitions**.

🔹 7. Repellers in Chaotic Scattering

For instance, the open baker's map or 3-disk scattering system.

- The invariant set (repeller) is fractal.
- The pressure function $P(t)$ yields:
  - Hausdorff dimension (from $P(t) = 0$)
  - Escape rate (from $P(1)$)
  - Dynamical zeta functions (via Ruelle formalism).

Why interesting: Connects chaos, fractals, and physical observables like transport or escape.

🔹 Summary Table

System	Dimension	Type	Why good for topological pressure
Full shift	symbolic	hyperbolic	exact computations, pedagogical
Smale horseshoe	2D	hyperbolic	geometric fractal structure
Baker's map	2D	hyperbolic	simple analytic form
Hénon map	2D	nonuniform hyperbolic	realistic chaos, multifractal
Geodesic flow	continuous	Anosov flow	links geometry ↔ dynamics
Pomeau–Manneville	1D	intermittent	phase transitions in $P(t)$
Open baker / repeller	2D	fractal chaotic scattering	escape rate ↔ pressure

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🔸 In short:

> If you want to **understand topological pressure as a true thermodynamic potential**,
> start from the **shift map** (exact solvability), move to the **horseshoe or baker's map** (geometry + symbolic dynamics),
> and finally explore **intermittent** or **Hénon-like systems** to see **phase transitions and multifractality**.