The Emergence of the True Lyapunov Exponent

The true Lyapunov exponent — representing the sustained exponential divergence of nearby trajectories — becomes apparent only over long-time observation. While short-term dynamics are dominated by alternating phases of stretching, compression, and rotation, these local deformations fluctuate rapidly and obscure any clear trend. Only after many cycles does the cumulative deformation reveal the net divergence rate that defines the system's chaotic behavior.

In the right-top panel we focus on the local dynamics of a small circular phase space element that initially represents an infinitesimal neighborhood of the trajectory. This initial circular element (nondistorted reference shape) is shown as the **green dashed contour**.

The **blue contour** represents the instantaneous deformation over a single integration step. This is essentially the "local" action of the flow: at every moment, the circular element is stretched along one direction and compressed along another direction. The **green arrow** marks the instantaneous stretching direction vector, which highlights the axis along which divergence of nearby trajectories is strongest.

The **red contour** depicts the cumulative deformation of the **initially circular phase space element** that has accumulated from the initial time (t=0). As the system evolves, this element is continuously deformed by the nonlinear motion of the Duffing oscillator. With time, this shape elongates and compresses along certain directions, gradually losing its circular form. The overall transformation combines three distinct processes: stretching, compression, and rotation.

An important feature here is that while stretching and compression directions evolve relatively slowly, the red phase space element itself undergoes continuous **rotation around its center of mass**. This rotation constantly reorients different regions of the red element with respect to the stretching/compression directions. As a result, points within the neighborhood experience an alternating sequence: first being pulled apart, then pressed together, depending on their instantaneous orientation with respect to the stretching/compression directions.

This interplay of **slowly varying stretching directions** with **frequent rotational motion** generates a highly irregular pattern of local deformation. For two nearby trajectories, this means that **their separation is not monotonic**: they may approach each other, then diverge, then approach again, in a chaotic and unpredictable sequence.

Only when one observes the **cumulative deformation** (red contour) over long times does the true nature of chaos become visible. The phase space element is gradually smeared out and spread across the accessible region of phase space. This long-term divergence, quantified by the Lyapunov exponent, represents the genuine chaotic mixing of the system and cannot be inferred from instantaneous deformation alone.

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The **red contour** clearly shows the results of the accumulated deformation. While the instantaneous deformations are chaotic, the true signature of the chaotic system—the overall **exponential divergence** of nearby trajectories—only becomes evident through the **gradual, persistent elongation** of this cumulative contour over a longer time scale. This **gradual smearing** of the phase space element across the entire phase space is the visual manifestation of a positive **Lyapunov exponent**, which defines the system as chaotic. The increasing separation of the points (the spreading of the red contour) is the true, long-term dynamic, which eventually dominates the localized, rapid rotational motion.