The dumped-driven pendulum, even though it is the 2D system, can exhibit chaotic behaviour. Why ?

The damped-driven pendulum, despite being a 2D system, can exhibit chaotic behavior because its dynamics are governed by a **nonlinear second-order differential equation** that, when driven and damped, effectively gives it a **three-dimensional phase space** due to the time-dependence of the driving force.

Key Factors for Chaos

1. Nonlinearity

The equation of motion for a simple pendulum is:

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0^2\sin(\theta) = A\cos(\omega t)$$

Where:

- θ is the angular displacement.
- $\gamma\frac{d\theta}{dt}$ is the **damping** term (dissipation).
- $\omega_0^2\sin(\theta)$ is the **restoring force** term, which is the crucial **nonlinearity**. For small angles, $\sin(\theta) \approx \theta$, and the system is linear and non-chaotic (simple harmonic motion). However, the full $\sin(\theta)$ function allows for more complex, non-repeating trajectories.
- $A\cos(\omega t)$ is the **periodic driving force**.

2. Phase Space Dimensionality

Chaos, by definition (e.g., in the sense of a **strange attractor**), requires a phase space of at least **three dimensions**.

- A simple, undriven, undamped pendulum is a **2D system** in its phase space $(\theta, d\theta/dt)$.
- The **damped-driven** pendulum is described by the two variables $(\theta, d\theta/dt)$ and an **explicit time dependence** from the driving term $A\cos(\omega t)$.
- To analyze the autonomous flow, we introduce a third variable, $\phi = \omega t$, where $d\phi/dt = \omega$. The system can then be written as three first-order ODEs:
  - $d\theta/dt = \dot{\theta}$
  - $d\dot{\theta}/dt = -\gamma\dot{\theta} - \omega_0^2 \sin(\theta) + A\cos(\phi)$
  - $d\phi/dt = \omega$ (or ϕ is taken modulo 2π)
- This transformation effectively gives the system a **3D phase space** $(\theta, \dot{\theta}, \phi)$, which is sufficient to allow for the **stretching and folding** mechanism necessary for **sensitive dependence on initial conditions**—the hallmark of chaos.

3. Dissipation and Driving

- **Dissipation (Damping, γ):** This is necessary to keep the trajectories bounded, preventing the phase space volume from growing indefinitely. It allows for the formation of a **strange attractor**, where all trajectories settle.
- **Driving ($A\cos(\omega t)$):** This continuously injects energy into the system, counteracting the damping and providing the perturbation necessary to explore the complex phase space topology enabled by the nonlinearity.

In summary, the **nonlinearity** $(\sin\theta)$ provides the complex structure, the **driving force** lifts the system to the required 3D phase space, and the **damping** ensures the chaotic motion remains on a bounded strange attractor.