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TRANSLATOR'S PREFACE

In the interest of speed and economy the notation of the original text has been retained so that the cross product of two vectors **A** and **B** is denoted by [**AB**], the dot product by (**AB**), the Laplacian operator by Δ, the curl by rot, etc. It might also be worth pointing out that the temperature is frequently expressed in energy units in the Soviet literature so that the Boltzmann constant will be missing in various familiar expressions. In matters of terminology, whenever possible several forms are used when a term is first introduced, e.g., magnetoacoustic and magnetosonic waves, "probkotron" and mirror machine, etc. It is hoped in this way to help the reader to relate the terms used here with those in existing translations and with the conventional nomenclature. In general the system of literature citation used in the bibliographies follows that of the American Institute of Physics "Soviet Physics" series; when a translated version of a given citation is available only the English translation is cited, unless reference is made to a specific portion of the Russian version. Except for the correction of some obvious misprints the text is that of the original.

We wish to express our gratitude to Academician Leontovich for kindly providing the latest corrections and additions to the Russian text, and especially for some new material, which appears for the first time in the American edition.

CONTENTS

ELECTROMAGNETIC WAVES IN A PLASMA

V. D. Shafranov

Introduction

In the present review we shall attempt to give a systematic presentation of the basic features of the theory of linear oscillations in a homogeneous plasma.

The bulk of the material (§§ 1-12) has appeared earlier in the form of a report.* Some minor modifications have been made in the preparation of the present review: problems have been added in §§ 7 and 9, and in writing the dielectric tensor, we have replaced the function $W(x)$ by the more compact form $Z(x) = -i\sqrt{\pi}\,xW(x)$, etc. In addition, in the present version we have added three new sections (§§ 13-15) and Appendix III.

The oscillations of a plasma are described by the self-consistent equations of motion of the particles (or the appropriate kinetic equations) together with Maxwell's equations [1-12]. However, if the oscillations are linear this entire system of equations can be reduced to a single Maxwell equation in which the charge density and current density are expressed as linear functions of the electric field by means of the dielectric tensor $\varepsilon_{\alpha\beta}(\mathbf{k},\omega)$; this tensor is computed beforehand from the equations of motion [13]. This approach, which was first used in the investigation of oscillations in a "cold" plasma (in which the velocity associated with the thermal motion of the charges is small compared with the phase velocity of the electromagnetic waves), is now widely used in the analysis of high-temperature plasmas [14-24]. It has also been used successfully to analyze oscillations in an inhomogeneous plasma (cf. the review by A.B. Mikhailovskii in the present volume).

Using Maxwell's equations, but with "microscopic currents," it is also possible to describe the fluctuation (thermal) fields in addition to the mean fields [25-28]. By taking account of the transverse and longitudinal components of the electric field in appropriate fashion, it is possible, in a relatively

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* V. D. Shafranov, Electromagnetic Waves in a Plasma, Report No. 194, Institute for Atomic Energy, 1960.

simple way, to analyze effects associated with the Coulomb interaction between charges (retardation and scattering of a charge moving in a plasma, fluctuations in charge density, etc.). In describing fluctuation effects, in addition to making use of the dielectric tensor, one frequently makes use of the correlation function for the microcurrents $G_{\alpha\beta}(\mathbf{k}, \omega)$. This function can be computed quite easily in a plasma which is not too dense, the case that is usually considered. In a plasma in thermodynamic equilibrium (i.e., one characterized by a Maxwellian distribution of velocities appropriate to a single temperature) the tensor $G_{\alpha\beta}(\mathbf{k}, \omega)$ is uniquely related to the anti-Hermitian part of the dielectric tensor $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega)$. This method of describing fluctuations, which was developed by Leontovich and Rytov, has the advantage that the calculation of various kinds of fluctuations reduces to solving the familiar equations of Maxwell. In thermodynamic-equilibrium cases, the results, expressed in terms of the tensor $\varepsilon_{\alpha\beta}$, are not confined to application in a plasma but have a much wider application (cf. § 90 in [25]). The equivalence between this method of describing fluctuations in a plasma and the method based on the equations for the correlation functions [29-35] is discussed in Appendix III.

When the correlation function for the microcurrents is used, it is a simple matter to compute the intensity of the radiation (as well as the radiation friction, the electromagnetic energy, etc.) for a single charge. The essence of using the correlation function for the microcurrents to determine averaged macroscopic characteristics lies in replacing the time average by an average over phase (i.e., an average over an ensemble of noninteracting particles).

The author has not attempted to develop a complete and rigorous description of the material in the present review. The principle task the author has undertaken has been to furnish a rapid introduction to the theory of electromagnetic oscillations in a plasma and to prepare the reader for studying the more difficult problems that arise in the analysis of instabilities in an inhomogeneous plasma and the effect of these instabilities on the plasma. Similarly, the list of references is far from complete. Useful background material on the theory of electromagnetic waves in a plasma and on general problems of plasma electrodynamics are available in the monographs by Ginzburg [36], Silin and Rukhadze [37], and in a recently published book by Stix [38]; these works will also introduce the reader to the general literature in the field.

§ 1. Dispersion Equation (General Relations)

When the physical quantities that affect the state of a plasma do not differ greatly from their stationary values ($q = q_0 + q_1$, $q_1 \ll q_0$), the equations that describe the behavior of the plasma can be linearized; moreover, in a number of cases these equations can be solved. If the plasma is uniform in the stationary state ($\nabla q_0 = 0$), the linearized equations are equations with constant coefficients. Under these conditions they can be solved in general form, for example, by Fourier methods – that is, by expanding the quantity $q_1(\mathbf{r}, t)$ in space and time in Fourier series or integrals. When this approach is used one usually finds it sufficient to write the solutions in the form of a plane wave $q_1(\mathbf{k}, \omega) e^{i(\mathbf{kr} - \omega t)}$ (this question is discussed in greater detail in § 7). If the linearized equations are homogeneous, i.e., if there are no specified field sources, an algebraic system of homogeneous equations is obtained for the amplitudes $q_1(\mathbf{k}, \omega)$ and, as is well known, these equations have a nontrivial solution if the determinant of this system vanishes. The statement of this condition is called the dispersion equation. The dispersion equation relates the frequency ω and the wave vector $\mathbf{k}$. Depending on the formulation of a particular problem, either ω or $\mathbf{k}$ can be specified, in which case the dispersion equation then determines the function $\mathbf{k}(\omega)$ or the function $\omega(\mathbf{k})$, respectively; either of these quantities can be complex:

$$k = k_1(\omega) + ik_2(\omega); \tag{1.1}$$

$$\omega(k) = \omega_1(k) - i\omega_2(k). \tag{1.2}$$

Specifying a real value of ω implies a problem that treats propagation of waves generated at this frequency. In this case, the quantity k is frequently replaced by the refractive index N:

$$k = \frac{\omega}{c} N = \frac{\omega}{c}(p + iq). \tag{1.3}$$

Assume that the wave propagates along the z axis ($k_x = k_y = 0$, $k_z = k$). Then $\exp i(\mathbf{kr} - \omega t) = \exp i(\omega/c)p[z - (c/p)t] - (\omega/c)qz$. As is well known, the real part of the refractive index p determines the phase velocity of the wave $v_\Phi = c/p$, while the imaginary part q, depending on its sign, either implies damping or growth of the wave as a function of distance.

Specifying a real vector $\mathbf{k}$ corresponds to the problem of characteristic oscillations of the plasma. In this case $\exp i(kz - \omega t) = \exp i(kz - \omega_1 t) - \omega_2 t$. The real part of the frequency ω_1 determines the characteristic oscillation frequency with the specified spatial dependence, while the imaginary part ω_2, depending on its sign, determines the rate of damping or growth of the oscillations.

In order to obtain the dispersion equation, it is necessary to solve Maxwell's equations together with the equations of motion of the charges. It is frequently more convenient to find the mean current density from the equations of motion at the outset and then to relate this quantity to the electric field. Since this relation is linear, in the general case it can be written in the form

$$j_\alpha = \sigma_{\alpha\beta}(\mathbf{k},\ \omega) E_\beta, \tag{1.4}$$

where $\sigma_{\alpha\beta}(\mathbf{k}, \omega)$ is the complex conductivity tensor.

In the case at hand, in which we consider harmonic oscillations, it is convenient to include the current density in the electric induction:

$$\frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \equiv \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \tag{1.5}$$

or, taking account of the time dependence $e^{-i\omega t}$,

$$j_\alpha = -i\frac{\omega}{4\pi}(D_\alpha - E_\alpha) = -i\frac{\omega}{4\pi}(\varepsilon_{\alpha\beta} - \delta_{\alpha\beta}) E_\beta. \tag{1.6}$$

Here we have introduced the electrical conductivity tensor $\varepsilon_{\alpha\beta}$, which relates the Fourier components of **D** and **E**:

$$D_\alpha = \varepsilon_{\alpha\beta} E_\beta. \tag{1.7}$$

In accordance with Eqs. (1.4) and (1.6), the tensor $\varepsilon_{\alpha\beta}$ is related to the tensor $\sigma_{\alpha\beta}$ by the expression

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + i\frac{4\pi}{\omega}\sigma_{\alpha\beta}. \tag{1.8}$$

We also note that the term $\delta_{\alpha\beta}$ is related, as is evident from Eqs. (1.5) and (1.6), to the displacement current $(1/4\pi)(\partial \mathbf{E}/\partial t)$; in a longitudinal wave ($\operatorname{div} \mathbf{E} = 4\pi\rho \neq 0$) this quantity is related to the space charge. Thus, neglecting the electric charge in a plasma corresponds to the condition $|\varepsilon_{\alpha\beta}| \gg 1$.

The tensor $\varepsilon_{\alpha\beta}$ (or $\sigma_{\alpha\beta}$) determines completely the nature of the small oscillations of the medium since the dispersion equation depends only on $\varepsilon_{\alpha\beta}$. We note that the existence of absorption properties in a medium can be established without solving the dispersion equation, by using one form of the tensor $\varepsilon_{\alpha\beta}$. In the case of weak damping, in which the wave is essentially monochromatic, the electromagnetic energy absorbed in the medium is described by the mean (over a period of the oscillation) value of the scalar product of **j** and **E**:

$$Q = \overline{\operatorname{Re}\mathbf{j}\cdot\operatorname{Re}\mathbf{E}} = \frac{1}{2}\operatorname{Re}\mathbf{j}\cdot\mathbf{E}^* = \frac{1}{4}\left(j_\alpha E_\alpha^* + j_\alpha^* E_\alpha\right). \tag{1.9}$$

Substituting the expression for the current (1.4) and changing the subscripts in the second term, we have

$$Q = \frac{1}{4}\left(\sigma_{\alpha\beta}E_\alpha^* E_\beta + \sigma_{\beta\alpha}^* E_\beta E_\alpha^*\right) = \frac{1}{2}\,\sigma'_{\alpha\beta}E_\alpha^* E_\beta =$$

$$= \frac{1}{2}\,\frac{\omega}{4\pi}\,\varepsilon''_{\alpha\beta}E_\alpha^* E_\beta, \tag{1.10}$$

where $\sigma'_{\alpha\beta}$ and $i\varepsilon_{\alpha\beta}$ are the Hermitian and anti-Hermitian parts of the tensors $\sigma_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$, respectively:

$$\sigma'_{\alpha\beta} = \frac{\sigma_{\alpha\beta} + \sigma_{\beta\alpha}^*}{2}; \quad i\varepsilon''_{\alpha\beta} = \frac{\varepsilon_{\alpha\beta} - \varepsilon_{\beta\alpha}^*}{2}. \tag{1.11}$$

Thus, the absorption is to be associated with the anti-Hermitian part of the dielectric tensor.

We can now write Maxwell's equations, introducing the space–time dependence in the form $e^{i(\mathbf{kr} - \omega t)}$:

$$[\mathbf{kB}] = -\frac{\omega}{c}\,\mathbf{D}; \tag{1.12}$$

$$[\mathbf{kE}] = \frac{\omega}{c}\,\mathbf{B}. \tag{1.13}$$

The equations $\operatorname{div}\mathbf{D} = 0$ and $\operatorname{div}\mathbf{B} = 0$ now represent the condition that the vectors $\mathbf{D}$ and $\mathbf{B}$ must be transverse to the vector $\mathbf{k}$ ($\mathbf{kD} = \mathbf{kB} = 0$) and are obviously satisfied as a consequence of these latter equations. Eliminating the magnetic field from Eqs. (1.12) and (1.13) we obtain the vector equation

$$k^2\mathbf{E} - \mathbf{k}(\mathbf{kE}) - \frac{\omega^2}{c^2}\,\mathbf{D} = 0. \tag{1.14}$$

Introducing the refractive index $N = kc/\omega$, we can now write the vector equation in the form $\mathbf{D} = N^2\mathbf{E}_\perp$, or

$$\mathbf{D}_\perp = N^2\mathbf{E}_\perp; \quad \mathbf{D}_\parallel = 0. \tag{1.14a}$$

The symbols $\perp$ and $\parallel$ refer to the components of the vector which are respectively perpendicular to the vector $\mathbf{k}$ and parallel to it.

Assume that the vector $\mathbf{k}$ is along the z axis. Expanding Eq. (1.14) in components, we then have

$$(N^2 - \varepsilon_{xx})\,E_x - \varepsilon_{xy}E_y - \varepsilon_{xz}E_z = 0; \tag{1.15}$$

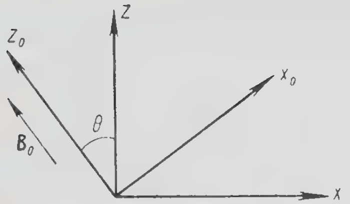

Fig. 1

$$-\varepsilon_{yx}E_x + (N^2 - \varepsilon_{yy})E_y - \varepsilon_{yz}E_z = 0;$$
$$-\varepsilon_{zx}E_x - \varepsilon_{zy}E_y - \varepsilon_{zz}E_z = 0. \quad (1.15)$$

We shall first consider this system of equations for the case of an isotropic plasma (no magnetic field). Under these conditions the only preferred direction is the direction of propagation of the wave **k**. Hence, the dielectric tensor assumes the form

$$\varepsilon_{\alpha\beta} = \varepsilon_\perp \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) + \varepsilon_\parallel \frac{k_\alpha k_\beta}{k^2}. \quad (1.16)$$

This tensor is diagonal in the coordinate system in which the z axis is parallel to the wave vector: $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_\perp$; $\varepsilon_{zz} = \varepsilon_\parallel$ and the remaining components vanish. Equation (1.15) can be written as follows:

$$(N^2 - \varepsilon_\perp)\mathbf{E}_\perp = 0; \quad (1.17)$$

$$\varepsilon_\parallel \mathbf{E}_\parallel = 0, \quad (1.18)$$

where $\mathbf{E}_\perp = \{E_x, E_y, 0\}$ is the component of the electric vector perpendicular to the wave vector, while $\mathbf{E}_\parallel$ is the parallel component. It is evident from these last equations that the longitudinal oscillations are independent of the transverse oscillations. The dispersion relation for the transverse waves is

$$N^2 = \varepsilon_\perp, \quad (1.19)$$

and the relation for the longitudinal waves is

$$\varepsilon_\parallel = 0 \quad (1.20)$$

(the dispersion relations will be studied in detail below after we have obtained actual expressions for $\varepsilon_{\alpha\beta}$).

Now let us consider the system in (1.15) for the case of a plasma in a magnetic field. The dielectric tensor is usually taken in a coordinate system x_0, y_0, z_0 in which the z_0 axis is along the magnetic field, while the x_0 axis lies in the plane of the vectors $\mathbf{B}_0$ and $\mathbf{k}$ (Fig.1). In this coordinate system the tensor $\varepsilon_{\alpha\beta}$ can be written (cf. § 9),

$$\varepsilon^0_{\alpha\beta} = \begin{pmatrix} \varepsilon_1 & ig & \xi \\ -ig & \varepsilon_2 & if \\ \xi & -if & \eta \end{pmatrix}, \quad (1.21)$$

where the quantities ε_1, ε_2, η, g, f, ξ are real numbers if there is no absorption. In a cold plasma (T = 0) the components ξ and f vanish (cf. § 3). In the coordinate system x, y, z (with z axis along **k**) in which the system in (1.15) is written, the tensor $\varepsilon_{\alpha\beta}$ can be expressed in terms of $\varepsilon^0_{\alpha\beta}$ by means of the usual tensor transformation formulas. Let θ be the angle between the vectors **B** and **k**. The components $\varepsilon_{\alpha\beta}$ are then given by

$$\left.\begin{aligned} &\varepsilon_{xx} = \varepsilon_1 \cos^2\theta + \eta \sin^2\theta - \xi \sin 2\theta; \\ &\varepsilon_{xy} = -\varepsilon_{yx} = i(g\cos\theta + f\sin\theta); \\ &\varepsilon_{xz} = \varepsilon_{zx} = (\varepsilon_1 - \eta)\sin\theta\cdot\cos\theta + \xi\cos 2\theta; \\ &\varepsilon_{yy} = \varepsilon_2; \\ &\varepsilon_{yz} = -\varepsilon_{zy} = i(f\cos\theta - g\sin\theta); \\ &\varepsilon_{zz} = \varepsilon_1 \sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta. \end{aligned}\right\} \tag{1.22}$$

It is evident from Eq. (1.15) that the transverse and longitudinal oscillations cannot be separated in the general case. In order for these modes to be separable the conditions $\varepsilon_{xz} = \varepsilon_{zx} = 0$ and $\varepsilon_{yz} = -\varepsilon_{zy} = 0$ must be satisfied. The only case in which these conditions are satisfied is in longitudinal propagation $\theta = 0$ (it will be shown below that $\xi = f = 0$ when $\theta = 0$). However, in an anisotropic plasma there are generally frequency regions in which the ratio of the longitudinal component of the electric field to the transverse component becomes very large, so that one can speak of longitudinal oscillations. Inspection of Eqs. (1.14a) and (1.15) shows that this situation arises at short wavelengths: $N^2 \to \infty$ when $\varepsilon_{zz} = 0$. Hence, the approximate dispersion relation for longitudinal waves is

$$\varepsilon_{zz} \equiv \varepsilon_1 \sin^2\theta + \eta\cos^2\theta + \xi\sin^2\theta = 0. \tag{1.23}$$

This expression can also be obtained directly from Maxwell's equations if one writes rot **E** = 0, i.e., if the Fourier components are given by $\mathbf{E} = (\mathbf{k}/k)E$. The condition $\mathbf{kD} = 0$ can then be written in the form of Eq. (1.18) with $\varepsilon_{\parallel} = (k_\alpha k_\beta / k^2)\,\varepsilon_{\alpha\beta} \equiv \varepsilon_{zz}$.

The general dispersion equation [obtained by setting the determinant of the system in (1.15) equal to zero] is of the form

$$AN^4 + BN^2 + C = 0. \tag{1.24}$$

The coefficients A, B, C are given by

$$\begin{aligned} &A = \varepsilon_{zz} = \varepsilon_1 \sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta; \\ &B = (g\sin\theta - f\cos\theta)^2 + \xi^2(\cos^2 2\theta + \sin^4\theta) - \\ &- \varepsilon_1\eta - \varepsilon_2(\varepsilon_1\sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta); \end{aligned} \tag{1.24a}$$

$$C = \varepsilon_1\varepsilon_2\eta - \eta g^2 - \varepsilon_1 f^2 - \varepsilon_2\xi^2 - 2gf\xi. \tag{1.24a}$$

The two formal solutions of Eq. (1.24) will be denoted by ε_l ($l = 1,2$)*

$$N^2 = \varepsilon_l \equiv -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \tag{1.25}$$

or, in somewhat different form,

$$\varepsilon_l = \frac{1}{2}(\eta_{xx} + \eta_{yy}) \pm \sqrt{\frac{(\eta_{xx} - \eta_{yy})^2}{4} + \eta_{xy}\eta_{yx}}\,. \tag{1.25a}$$

Here we have introduced the following quantities, in terms of which the components of the polarization vector can be expressed:

$$\left.\begin{aligned}
&\eta_{xx} = \varepsilon_{xx} - \frac{\varepsilon_{xz}\varepsilon_{zx}}{\varepsilon_{zz}} = \frac{\varepsilon_1\eta - \xi^2}{\varepsilon_1 \sin^2\theta + \eta\cos^2\theta + \xi \sin 2\theta};\\
&\eta_{xy} = -\eta_{yx} = \varepsilon_{xy} - \frac{\varepsilon_{xz}\varepsilon_{zy}}{\varepsilon_{zz}} =\\
&= i\,\frac{(g\eta + f\xi)\cos\theta + (g\xi + f\varepsilon_1)\sin\theta}{\varepsilon_1 \sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta};\\
&\eta_{yy} = \varepsilon_{yy} - \frac{\varepsilon_{yz}\varepsilon_{zy}}{\varepsilon_{zz}} =\\
&= \frac{\varepsilon_2(\varepsilon_1 \sin^2\theta + \eta\cos^2\theta) - g^2\sin^2\theta + (\varepsilon_2\xi + fg)\sin 2\theta - f^2\cos^2\theta}{\varepsilon_1\sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta}
\end{aligned}\right\} \tag{1.26}$$

If the tensor $\varepsilon_{\alpha\beta}$ is independent of **k** (in a plasma this means, as we shall see below, that the thermal motion of the particles is neglected), the quantity N^2 can be determined in explicit form from Eqs. (1.25) and (1.25a).

The two values of the square of the refractive index (1.25) refer to the two wave modes that propagate in an anisotropic medium. The first mode is called the ordinary wave, and the second is called the extraordinary wave. In order to examine the question of polarization for these waves we shall derive an expression for N^2 in a somewhat different manner.

First, we denote by $i\alpha_x$ and $i\alpha_z$ the ratios of the x and z components of the vector **E** to its y component:

$$\mathbf{E} = E_y\{i\alpha_x,\ 1,\ i\alpha_z\}. \tag{1.27}$$

Using the last equation in (1.15), we express α_z in terms of α_x:

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*"Formal" because the coefficients A, B, C depend on N^2.

$$i\alpha_z = i\alpha_x \varepsilon_{zx}/\varepsilon_{zz} - \varepsilon_{zy}/\varepsilon_{zz}. \tag{1.28}$$

Eliminating α_z from the first two equations in (1.15), we have

$$\left.\begin{aligned} &(N^2 - \eta_{xx})\, i\alpha_x - \eta_{xy} = 0; \\ &-\eta_{xy} i\alpha_x + N_2 - \eta_{yy} = 0. \end{aligned}\right\} \tag{1.29}$$

In this way we obtain an expression for N^2 in terms of the ratio of the components of **E**:

$$N^2 = \frac{\eta_{xy}}{i\alpha_x} + \eta_{xx} = \eta_{yy} + i\alpha_x \eta_{yx}. \tag{1.30}$$

The quantity α_x, which characterizes the polarization of the wave, is governed by the quadratic equation

$$\alpha_x^2 + i\,\frac{\eta_{xx} - \eta_{yy}}{\eta_{yx}}\,\alpha_x + \frac{\eta_{xy}}{\eta_{yx}} = 0. \tag{1.31}$$

Correspondingly, for the two values of α_x given by this equation there are two waves, $\mathbf{E}_1$ and $\mathbf{E}_2$, each of which has its own refractive index, in accordance with Eq. (1.30). Since $\alpha_{x1} + \alpha_{x2} = (\eta_{xx} - \eta_{yy})/i\eta_{yx}$ and $\alpha_{x1}\alpha_{x2} = \eta_{xy}/\eta_{yx}$, using Eq. (1.30) for N^2 one finds easily that

$$N_1^2 + N_2^2 = \eta_{xx} + \eta_{yy};\ N_1^2 N_2^2 = \eta_{xx}\eta_{yy} - \eta_{xy}\eta_{yx}.$$

It follows that $N_{1,2}^2$ is given by Eq. (1.24), which is obtained directly from the dispersion equation.

We note, as is evident from Eq. (1.26), that the free term in Eq. (1.31) for α_x is equal to $\eta_{xy}/\eta_{yx} = -1$. Consequently,

$$\alpha_{x1}\alpha_{x2} = -1. \tag{1.32}$$

In the absence of absorption, η_{yx} is imaginary while η_{yy} and η_{xx} are both real. Consequently, both roots α_{x1} and α_{x2}, as well as α_{z1} and α_{z2} are real numbers. The electric vector of the wave with the specified polarization can be written in the form of a sum of two vectors: one is along the y axis and the other is perpendicular to the first and lies in the xz plane, being shifted in phase with respect to the first by $\pi/2$:

$$\left.\begin{aligned} &\mathbf{E} = \mathbf{E}^{(1)} + i\mathbf{E}^{(2)}; \\ &\mathbf{E}^{(1)} = E_y\{0,\ 1,\ 0\}; \qquad \mathbf{E}^{(2)} = E_y\{\alpha_x,\ 0,\ \alpha_z\}. \end{aligned}\right\} \tag{1.33}$$

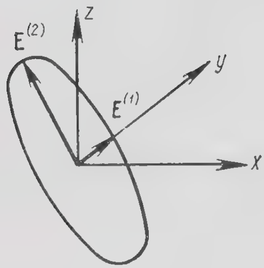


Fig. 2

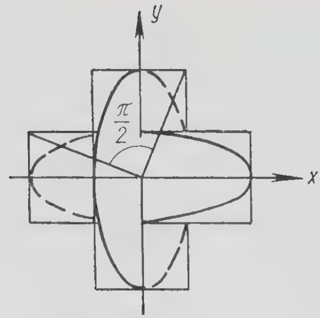


Fig. 3

These two vectors are evidently the semiaxes of an ellipse which is described by the end of the electric vector (Fig. 2). Thus, in general, the wave will exhibit elliptic polarization. The transverse component of **E** also evidently exhibits elliptic polarization and the semiaxes of the corresponding ellipse are given by

$$\mathbf{E}_{\perp}^{(1)} = E_y\{0,\ 1,\ 0\};\quad \mathbf{E}_{\perp}^{(2)} = E_y\{\alpha_x,\ 0,\ 0\}.$$

It is easily shown that the condition in (1.32) means that the diagonals of the rectangle in which these ellipses are described (corresponding to different polarization) are mutually orthogonal (Fig. 3).

In many cases it is of interest to determine the polarization vector in the coordinate system x_0, y_0, and z_0 (magnetic fields along the z_0 axis, the **k** vector in the x_0z_0 plane forming an angle θ with $\mathbf{B}_0$). In this coordinate system the components of the polarization vector are given by the usual formulas for coordinate transformation

$$\alpha_x^0 = \alpha_x \cos\theta + \alpha_x \sin\theta;\quad \alpha_y^0 = 1;\quad \alpha_z^0 = \alpha_z\cos\theta - \alpha_x \sin\theta. \tag{1.34}$$

When the values of $\varepsilon_{\alpha\beta}$, η_{xy}, and η_{xx} are substituted from Eqs. (1.22) and (1.26), one obtains the following expression for the components of the polarization vector (for $\alpha_y = \alpha_y^0 = 1$):

$$\alpha_x = \frac{(g\eta + f\xi)\cos\theta + (g\xi + f\varepsilon_1)\sin\theta}{N^2(\varepsilon_1\sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta) - \xi_1\eta + \xi^2}, \tag{1.35}$$

$$\alpha_z = \frac{(g\eta + f\xi)\sin\theta - (g\xi + f\varepsilon_1)\cos\theta - N^2(g\sin\theta - f\cos\theta)}{N^2(\varepsilon_1\sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta) - \varepsilon_1\eta + \xi^2}, \tag{1.36}$$

$$\alpha_x^0 = \frac{g\eta + f\xi - N^2(g\sin\theta - f\cos\theta)\sin\theta}{N^2(\varepsilon_1\sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta) - \varepsilon_1\eta + \xi^2}, \tag{1.37}$$

$$\alpha_z^0 = -\frac{g\xi + f\varepsilon_1 + N^2(g\sin\theta - f\cos\theta)\cos\theta}{N^2(\varepsilon_1\sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta) - \varepsilon_1\eta + \xi^2}. \quad (1.38)$$

Problem. Show that for weak absorption the imaginary part of frequency ω_2, for real k, and the imaginary part of the wave number k_2, for real ω, are related to the projection of the group velocity $d\omega/dk$ in the direction of the wave vector by the following simple expression:

$$\omega_2 = \mathbf{k}_2\frac{d\omega}{d\mathbf{k}}. \quad (1)$$

Solution. We start with the dispersion equation

$$k^2c^2 = \omega^2\varepsilon(\mathbf{k},\ \omega). \quad (2)$$

Differentiating with respect to k, we have

$$\frac{\partial\omega}{\partial\mathbf{k}}\frac{\partial(\omega^2\varepsilon)}{\omega^2\partial\omega} = \frac{2\mathbf{k}c^2}{\omega^2} - \frac{\partial\varepsilon}{\partial\mathbf{k}}. \quad (3)$$

Writing $\varepsilon = \varepsilon_1 + i\varepsilon_2$ and $\mathbf{k} = \mathbf{k}_1 + i\mathbf{k}_2$, in the linear approximation in ε_2 and $\mathbf{k}_2$ (weak absorption) for a given ω:

$$\left(\frac{2\mathbf{k}c^2}{\omega^2} - \frac{\partial\varepsilon}{\partial\mathbf{k}}\right)\mathbf{k}_2 \doteq \varepsilon_2. \quad (4)$$

Similarly, for a given $\mathbf{k}$, writing $\omega = \omega_1 - i\omega_2$, we find

$$\frac{\partial(\omega^2\varepsilon)}{\partial\omega}\,\omega_2 = \omega^2\varepsilon_2. \quad (5)$$

Using Eqs. (3), (4), and (5) we then find the desired relation (1).

§ 2. Unmagnetized Plasma. Hydrodynamic Approximation

In many cases it is convenient to analyze plasma oscillations by using the so-called hydrodynamic description. A comparison with the kinetic analysis shows that the hydrodynamic approximation loses a number of important features of plasma oscillations; in particular, damping effects that derive from the thermal motion of the particles cannot be treated. However, if one is content with a relatively simple qualitative description of the relation between ω and $\mathbf{k}$, the hydrodynamic approximation suffices. In using the hydrodynamic approximation, one should understand that it is completely unjustified when the phase velocities of the various waves are so small as to become comparable with the thermal motion of the charged particles, and when the oscillations are damped (cf. §§ 5, 10, and 11).

For reasons of simplicity we shall neglect the friction between the various plasma components and assume that the oscillations are adiabatic. Under these conditions the equations that describe the motion of each species will be

$$mn\frac{d\mathbf{v}}{dt} = -\nabla p + en\mathbf{E}; \tag{2.1}$$

$$\frac{d}{dt}\left(\frac{p}{n^{\gamma}}\right) = 0; \quad p = nT; \tag{2.2}$$

$$\frac{dn}{dt} + n\,\mathrm{div}\,\mathbf{v} = 0. \tag{2.3}$$

We linearize these equations, assuming that $\mathbf{E}_0 = \mathbf{v}_0 = 0$, $p = p_0 + p^{(1)}$, $n = n_0 + n^{(1)}$ and that the quantities $\mathbf{E}$, $\mathbf{v}$, $p^{(1)}$, and $n^{(1)}$ are harmonic, being described by a relation of the form $e^{i(\mathbf{kr} - \omega t)}$.

From Eqs. (2.2) and (2.3), the pressure $p^{(1)}$ can be expressed in the following form:

$$-i\omega p^{(1)} = -\gamma p_0\,\mathrm{div}\,\mathbf{v} = -i\gamma(\mathbf{kv})\,p_0. \tag{2.4}$$

The pressure gradient is nonvanishing only in the direction of propagation of the wave, which we shall take to be the z axis. The corresponding projections of the equations of motion (2.1) are

$$-i\omega\mathbf{v}_{\perp} = \frac{e}{m}\mathbf{E}_{\perp}; \tag{2.5}$$

$$-i\omega v_z = -i\gamma\frac{k^2T}{m\omega}v_z + \frac{e}{m}E_z. \tag{2.6}$$

Thus we find that the current density $\mathbf{j} = \Sigma e n_0 \mathbf{v}$ (the summation is taken over particle species) is governed by

$$\mathbf{j}_{\perp} = i\sum\frac{e^2n_0}{m\omega}\mathbf{E}_{\perp}; \qquad j_z = i\sum\frac{e^2n_0\omega}{m\omega^2 - \gamma k^2T}E_z. \tag{2.7}$$

By definition, the quantities appearing in front of $\mathbf{E}_{\perp}$ and $\mathbf{E}_Z$ are, respectively, $\sigma_{\perp}$ and $\sigma_{\parallel}$. From Eq. (1.8), which relates $\varepsilon_{\alpha\beta}$ and $\sigma_{\alpha\beta}$, we then find

$$\varepsilon_{\perp} = 1 - \sum\frac{4\pi e^2n_0}{m\omega^2}; \tag{2.8}$$

$$\varepsilon_{\parallel} = 1 - \sum\frac{4\pi e^2n_0}{m\omega^2 - \gamma k^2T}. \tag{2.9}$$

Let us first consider the transverse oscillations. In the expression for $\varepsilon_\perp$ we can neglect terms that are small by virtue of the large mass of the ions (this procedure corresponds to neglecting the ion motion). In accordance with Eq. (1.19), the refractive index N is then given by

$$N^2 = 1 - \frac{\omega_{0e}^2}{\omega^2}, \tag{2.10}$$

where ω_{0e} is the Langmuir or plasma frequency

$$\omega_{0e} = \frac{4\pi e^2 n_e}{m_e}. \tag{2.11}$$

It is evident from the derivation of the expression for N^2 that the term ω_{0e}^2/ω^2 represents the ratio of the current produced by the charges in the plasma to the vacuum displacement current. When $\omega < \omega_{0e}$ the refractive index is pure imaginary: $p = 0$, $q = \sqrt{(\omega_{0e}^2/\omega^2) - 1}$, and the wave falls off exponentially as $\exp^{-(\omega/c)qz}$. The fact that the wave cannot propagate is explained by the shielding of the electromagnetic field by the current associated with the plasma charges,* which is similar to the shielding of the field by an induction current in the ordinary skin effect. However, in contrast with the usual skin effect, here the absorption and the energy flux in the plasma are both zero.

At low frequencies $\omega^2 \ll \omega_{0e}^2$, the depth of penetration of the field into the plasma $\delta = c/\omega q$ is independent of the frequency ω, being given by

$$\delta = \frac{c}{\omega_{0e}}. \tag{2.12}$$

We obtain this result without expanding the field in harmonics. It follows from Maxwell's equations that the transverse waves ($\operatorname{div} \mathbf{E} = 0$) obey an equation of the form

$$\Delta \mathbf{E} = \frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{2.13}$$

If the characteristic time in which the field changes t is relatively large ($t \gg 1/\omega_0$), the displacement current can be neglected. Expressing $\partial \mathbf{j}/\partial t$ using the equation of motion for the electrons

$$m\frac{\partial \mathbf{j}}{\partial t} = e^2 n_e \mathbf{E}, \tag{2.14}$$

*The polarization current introduces a phase shift between the vectors **E** and **B**. Propagation is possible if the phase shift is not equal to $\pi/2$. When $\omega < \omega_0$, the phase shift turns out to be equal to $\pi/2$.

we find

$$\Delta \mathbf{E} = \frac{4\pi e^2 n_e}{mc^2} \mathbf{E} \equiv \frac{\mathbf{E}}{\delta^2}. \tag{2.15}$$

In the case of a plane plasma layer, the solution of this equation is of the form $\mathbf{E} = \mathbf{E}_0 e^{-z/\delta}$, where δ is evidently given by Eq. (2.12).

When $\omega > \omega_{0e}$, the refractive index N is a real quantity ($0 < N < 1$) and transverse waves can propagate in the plasma. Characteristic plasma oscillations are possible in this frequency range, the frequency of these oscillations being related to the wave number by Eq. (2.10), where $N^2 = k^2c^2/\omega^2$:

$$\omega^2 = \omega_{0e}^2 + k^2c^2. \tag{2.16}$$

It is well known that the group velocity for these waves, i.e., the velocity with which energy is transported, is given by

$$v_{gr} = \frac{d\omega}{dk} = \frac{c^2}{v_\Phi}, \tag{2.17}$$

where $v_\Phi = \omega/k$ is the phase velocity.

The effect of electron friction on wave propagation can be estimated roughly by adding to the right side of the equation of motion (2.5) a term $\nu \mathbf{v}_\perp$, where ν is the mean number of collisions of electrons with ions. In this case, Eq. (2.10) is replaced by

$$N^2 = \varepsilon_\perp = 1 - \frac{\omega_{0e}^2}{\omega(\omega + i\nu)} = \varepsilon_1 + i\varepsilon_2, \tag{2.18}$$

where the real and imaginary parts of the square of the refractive index are given by

$$\varepsilon_1 = 1 - \frac{\omega_{0e}^2}{\omega^2 + \nu^2}; \quad \varepsilon_2 = \frac{\nu}{\omega} \frac{\omega_{0e}^2}{\omega^2 + \nu^2}. \tag{2.19}$$

The real and imaginary parts of the refractive index $N = p + iq$ can be expressed in terms of ε_1 and ε_2 by the relations

$$\left.\begin{aligned} p &= \frac{1}{\sqrt{2}} \sqrt{\varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2}}; \\ q &= \frac{1}{\sqrt{2}} \sqrt{-\varepsilon_1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2}}. \end{aligned}\right\} \tag{2.20}$$

Let us now consider the longitudinal oscillations of the plasma. The elastic force which is responsible for the longitudinal oscillations can be the longitudinal electric field caused by separation of the charges, or the pressure gradient. We limit ourselves to the case in which there are ions of only one species (mass m_i and charge ze). The dispersion equation for longitudinal waves is then obtained from Eqs. (1.20) and (2.9):

$$\varepsilon_{\parallel} = 1 - \frac{4\pi e^2 n_e}{m_e \omega^2 - \gamma_e k^2 T_e} - \frac{4\pi z^2 e^2 n_i}{m_i \omega^2 - \gamma_i k^2 T_i} = 0. \tag{2.21}$$

We now introduce the notation

$$\beta_e^2 = \frac{\gamma_e T_e}{m_e c^2}; \quad \beta_i^2 = \frac{\gamma_i T_i}{m_i c^2}; \quad \omega_{0i}^2 = \frac{4\pi z^2 e^2 n_i}{m_i}. \tag{2.22}$$

Then Eq. (2.21) for $N = kc/\omega$ can be written in the form

$$N^4 - \left\{ \frac{1 - \frac{\omega_{0e}^2}{\omega^2}}{\beta_e^2} + \frac{1 - \frac{\omega_{0i}^2}{\omega^2}}{\beta_i^2} \right\} N^2 + \frac{1 - \frac{\omega_{0e}^2 + \omega_{0i}^2}{\omega^2}}{\beta_e^2 \beta_i^2} = 0. \tag{2.23}$$

The solution of this equation is shown graphically in Fig. 4, from which it is evident that there are two oscillation branches, one of which may be called the ion branch and the other the electron branch.

We shall first be interested in the electron branch. At relatively high frequencies ($\omega^2 \gg \gamma_i k^2 T_i / m_i$, $\gamma_e k_2 T_e^2 / m_e$) the third term in Eq. (2.21), for $\varepsilon_{\parallel}$, which takes account of the ion oscillations, can be neglected. Under these conditions we find

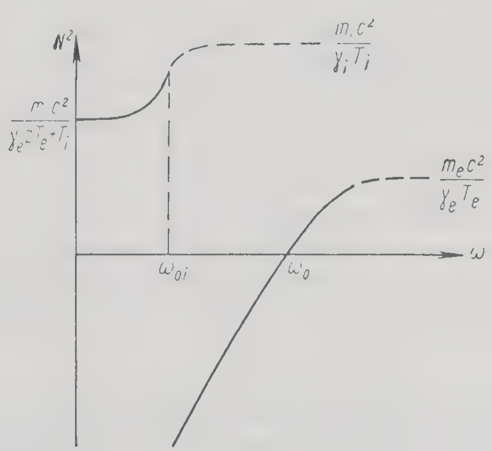


Fig. 4

$$\omega^2 = \omega_0^2 + \gamma_e \frac{k^2 T_e}{m_e}, \tag{2.24}$$

or, equivalently,

$$N^2 = \frac{1 - \dfrac{\omega_0^2}{\omega^2}}{\beta_e^2}. \tag{2.25}$$

We now consider the characteristic oscillations of a plasma as given by Eq. (2.24). The first term in the expression for ω^2 stems from the elastic force produced by the charge separation, while the second is due to the hydrodynamic pressure of the electron gas.

It will be shown in § 10 that similar relations are obtained from a kinetic analysis. When collisions are neglected, the coefficient γ_e, as determined by the kinetic analysis, is found to be equal to 3, which corresponds to the adiabaticity index (specific-heat ratio) for the case of one-dimensional motion (i.e., one degree of freedom). In the hydrodynamic approximation it is assumed that in the course of the motion the oscillation energy is distributed between all three degrees of freedom, so that the adiabaticity index $\gamma_e = 5/3$ does not agree with that given by the kinetic analysis. In actual cases the collision frequency, which characterizes the redistribution of the wave energy, is always smaller than the wave frequency. Hence, the appropriate value is $\gamma_e = 3$.*

Equation (2.24) shows that the frequency of the longitudinal oscillations is always close to the plasma frequency. Increasing the frequency by a factor of approximately 1.5 brings the frequency into the range of very short wavelengths ($k^2 \gtrsim 4\pi e^2 n_e / T_e$), comparable with the Debye length. In this region the phase velocity of the wave is of the same order of magnitude as the thermal velocity of the electrons and, as will be shown in the kinetic analysis, strong damping occurs. In Fig. 4 the corresponding part of the electron branch is shown by the dashed line. The dispersion equation written in the form (2.25) is convenient for problems involving the propagation of waves at a specified frequency. On the basis of the considerations given above with respect to the characteristic oscillations it is clear that longitudinal waves can only propagate when $\omega \geq \omega_0$, and even then, only in a very narrow frequency range. When

*We note that if one uses the hydrodynamic approach to the equation of motion it is necessary to retain the friction force $\mathbf{R} = -0.7 n_e \nabla T_e$. In this case the effective values of γ_e is found to be closer to the kinetic value $\gamma_{e\,\mathrm{eff}} = \gamma_e + 0.7(\gamma_e - 1) = 2.13$.

$\omega < \omega_0$, the square of the refractive index is negative: $N^2 = -q^2 < 0$ and a field applied to the plasma will decay exponentially in accordance with the factor $e^{-z/\delta}$. The depth of penetration $\delta = c/\omega q$ is

$$\delta^2 = \frac{c^2\beta_e^2}{\omega_{0e}^2} = \frac{\gamma_e T_e}{4\pi e^2 n_e}, \quad \omega_{0e}^2 \gg \omega^2 \gg \gamma \frac{k^2 T}{m_i}, \tag{2.26}$$

$$\delta^2 = \frac{c^2\beta_e^2}{\omega_{0e}^2} + \frac{c^2\beta_i^2}{\omega_{0i}^2} = \frac{\gamma_i T_i + \gamma_e z T_e}{4\pi z^2 e^2 n_{0i}}, \quad \omega^2 \ll \gamma \frac{k^2 T}{m_i}, \tag{2.27}$$

i.e., this depth is equal to the Debye length. The second formula is obtained from Eq. (2.21) by neglecting the terms $m_e\omega^2$ and $m_i\omega^2$. This procedure corresponds to neglecting the inertia terms in the equation of motion so that the expression in (2.27), which gives the depth of penetration, represents the hydrostatic equilibrium of the electrons and ions in the electric field.

We now consider the ion branch. This branch is obtained from Eq. (2.21) by omitting the term $m_e\omega^2$, a procedure corresponding to neglect of the electron inertia. In this case, the electron equation of motion becomes a statement of the equilibration of the gradient of the electron pressure which arises due to oscillations in the electrostatic field **E**:

$$en_e\mathbf{E} = -\nabla p_e. \tag{2.28}$$

Solving the dispersion equation (2.21) with respect to ω^2, and taking $zn_i = n_e$, we find

$$\omega^2 = \gamma_i \frac{k_2 T_i}{m_i} + \gamma_e \frac{k^2 z T_e}{\left(1 + \frac{\gamma_e k^2 T_e}{4\pi e^2 n_e}\right) m_i}. \tag{2.29}$$

At wavelengths greater than the Debye length,

$$\frac{\gamma_e k^2 T_e}{4\pi e^2 n_e} \ll 1, \tag{2.30}$$

Eq. (2.29) is an equation for the acoustic oscillations

$$\omega^2 = k^2 \frac{\gamma_i T_i + \gamma_e z T_e}{m_i}. \tag{2.31}$$

The condition in (2.30) corresponds to neglecting unity in Eq. (2.21) for $\varepsilon_{||}$, i.e., neglecting the space charge (cf. p. 4). In the opposite limit to that specified by (2.30), a space charge arises and we obtain a dispersion relation similar to the electron dispersion relation (2.24):

$$\omega^2 = \omega_{0i}^2 + k^2 \frac{\gamma_i T_i}{m_i}. \tag{2.32}$$

If the collision frequency of the charges is small compared with the oscillation frequency, Eqs. (2.31) and (2.32) apply only when $\omega^2 \gg \gamma_i k^2 T_i / m_i$; however, as noted above, strong damping occurs in the opposite limit. The latter condition is realized in both cases only when the electron pressure is significantly higher than the ion pressure, i.e., when

$$zT_e \gg T_i. \tag{2.33}$$

When this condition obtains it is legitimate to neglect the gradient of the ion pressure in the ion equation of motion. Thus the ions are subject to the effect of a force zen_iE; it follows from Eq. (2.28) for the electrons (and from the condition that the plasma must be neutral in the stationary state $zn_i = n_e$) that this force is equal to the gradient of the electron pressure. In the case of long-wave excitations, in which the charge is neutralized, the electrons move together with the ions ($v_e = v_i$). Hence, we obtain "ion sound but with the electron temperature" [cf. Eq. (2.31) with $T_i = 0$]. In the limiting case of short wavelengths ($k \to \infty$) the electron velocity in the wave, as follows from Eq. (2.6), is given by $v_{ze} = -i(e\omega/\gamma_e k^2 T_e)E$, and is found to be negligibly small with the ion velocity $v_{zi} = i(e/m_i\omega)E$. The motion of the ions under the effect of the electric field (with the electrons at rest) is manifest in oscillations at the plasma frequency [Eq. (2.32)].

§ 3. Plasma in a Magnetic Field. Hydrodynamic Approximation

The presence of a magnetic field introduces substantial complications into the nature of the plasma motion. For this reason we shall find it expedient to consider first the limiting case of a cold plasma, assuming that the electron and ion pressures are both zero. As indicated in the preceding section, if one neglects the electron and ion pressures in the absence of a magnetic field, the plasma can only exhibit high-frequency electron oscillations. This is the case because the charges oscillate in antiphase with the electric field, producing a current that is opposite in sign to the displacement current. At frequencies below the electron plasma frequency, the current produced by the charges is greater (in absolute magnitude) than the displacement current and the oscillations cannot be excited (cf. p. 13).

If the plasma is located in a magnetic field $\mathbf{B}_0$ the particles are subject to the effect of the wave field and the constant magnetic field, and will execute a rather complicated motion. In the general case, the current associated with the charges will not balance the displacement current. For example, at oscillation frequencies small compared with the cyclotron frequency it is well known that a charge will move in the direction perpendicular to the electric field with a velocity $\mathbf{v} = c[\mathbf{E}\mathbf{B}_0]/B_0^2$.

As a result, it is found that the plasma can exhibit oscillations at low frequencies, even frequencies down to $\omega = 0$.

Let us write the equation of motion for a charge in the field associated with a monochromatic wave:

$$-i\omega\mathbf{v} = \frac{e}{m}\mathbf{E} + [\mathbf{v}\boldsymbol{\omega}_B]; \qquad \boldsymbol{\omega}_B = \frac{e\mathbf{B}_0}{mc} \tag{3.1}$$

(for the ions we have $\omega_B = \omega_{Bi} > 0$ and for the electrons $\omega_B = -\omega_{Be}$, $\omega_{Be} > 0$). Taking the vector and scalar products of this equation with $\boldsymbol{\omega}_B$, we find

$$-i\omega[\mathbf{v}\boldsymbol{\omega}_B] + \mathbf{v}\omega_B^2 - \boldsymbol{\omega}_B(\mathbf{v}\boldsymbol{\omega}_B) = \frac{e}{m}[\mathbf{E}\boldsymbol{\omega}_B]; \tag{3.2}$$

$$-i\omega(\mathbf{v}\boldsymbol{\omega}_B) = \frac{e}{m}(\mathbf{E}\boldsymbol{\omega}_B). \tag{3.3}$$

Now, eliminating $[\mathbf{v}\boldsymbol{\omega}_B]$ and $(\mathbf{v}\boldsymbol{\omega}_B)$ from Eq. (3.1), we have

$$\mathbf{v} = i\frac{e\omega}{m(\omega^2 - \omega_B^2)}\left\{\mathbf{E} - \frac{\boldsymbol{\omega}_B(\mathbf{E}\boldsymbol{\omega}_B)}{\omega^2}\right\} - \frac{e}{m}\frac{[\mathbf{E}\boldsymbol{\omega}_B]}{\omega^2 - \omega_B^2}. \tag{3.4}$$

Substituting the quantity $\mathbf{v}$ in the expression for the electric induction $\mathbf{D} = \mathbf{E} + i(4\pi/\omega)\mathbf{j} = \mathbf{E} + i(4\pi/\omega)\Sigma en\mathbf{v}$, we have

$$\mathbf{D} = \left(1 - \sum\frac{\omega_0^2}{\omega^2 - \omega_B^2}\right)\mathbf{E} + \sum\frac{\omega_0^2}{\omega^2 - \omega_B^2}\frac{\boldsymbol{\omega}_B(\mathbf{E}\boldsymbol{\omega}_B)}{\omega^2} - i\sum\frac{\omega_0^2}{\omega^2 - \omega_B^2}\frac{[\mathbf{E}\boldsymbol{\omega}_B]}{\omega}. \tag{3.5}$$

This result can now be used to write an expression for the dielectric tensor of the plasma. In a coordinate system in which the axis z_0 is directed along the magnetic field, this tensor assumes the form

$$\varepsilon_{\alpha\beta}^0 = \begin{pmatrix} \varepsilon & ig & 0 \\ -ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{pmatrix}, \tag{3.6}$$

where

$$\varepsilon = 1 - \sum \frac{\omega_0^2}{\omega^2 - \omega_B^2}; \quad g = - \sum \frac{\omega_B \omega_0^2}{\omega\left(\omega^2 - \omega_B^2\right)};$$
$$\eta = 1 - \sum \frac{\omega_0^2}{\omega^2}. \tag{3.6a}$$

Let us now assume that the plasma contains ions of a single species with mass m_i and charge ze. We also introduce the notation*

$$M = \frac{m_i}{zm_e}; \quad x = \frac{\omega}{\omega_{Bi}}; \quad A^2 = \frac{\omega_{0i}^2}{\omega_{Bi}^2} + \frac{\omega_{0e}^2}{\omega_{Be}^2} = \frac{4\pi (m_i + zm_e) n_i c^2}{B_0^2}. \tag{3.7}$$

The expressions for ε, g, and η are

$$\left.\begin{aligned} \varepsilon &= 1 - A^2 \frac{M(x^2 - M)}{(x^2 - 1)(x^2 - M^2)}; \\ g &= A^2 \frac{M(M-1)x}{(x^2 - 1)(x^2 - M^2)}; \\ \eta &= 1 - A^2 \frac{M}{x^2}. \end{aligned}\right\} \tag{3.8}$$

The square of the refractive index is determined from Eqs. (1.25) and (1.24a), taking account of the fact that $\xi = f = 0$

$$N^2 = \frac{(\varepsilon^2 - g^2 - \varepsilon\eta)\sin^2\theta + 2\varepsilon\eta \pm \sqrt{(\varepsilon^2 - g^2 - \varepsilon\eta)^2 \sin^4\theta + 4\eta^2 g^2 \cos^2\theta}}{2(\varepsilon \sin^2\theta + \eta\cos^2\theta)}. \tag{3.9}$$

Substituting the expressions in (3.8) in this equation, we find

$$N^2 = \frac{P}{Q};$$
$$\begin{aligned} P = {} & \{M^3A^4 - MA^2[(M^2 - M + 1)x^2 - M^2]\}\sin^2\theta + \\ & + 2(x^2 - MA^2)[(x^2 - 1)(x^2 - M^2) - MA^2(x^2 - M)] \pm \\ & \pm \sqrt{\{M^3A^4 - MA^2[(M^2 - M - 1)x^2 - M^2]\}^2 \sin^4\theta + 4(x^2 - MA^2)^2 A^4 M^2 (M-1)^2 x^2 \cos^2\theta}; \end{aligned} \tag{3.10}$$

*The parameter A represents the ratio of the velocity of light to the Alfvèn velocity. In a deuterium plasma, $A = 0.2\sqrt{n_i}/B_0$.

$$Q = 2\,[x^2(x^2-1)(x^2-M^2) - MA^2x^2(x^2-M)\sin^2\theta - \\ - MA^2(x^2-1)(x^2-M^2)\cos^2\theta] \tag{3.10}$$

In order to plot the function $N^2(\omega)$ graphically, we first find the values of x at which N^2 vanishes and at which it becomes infinite.

It is evident from Eq. (3.9) that N^2 vanishes, regardless of the angle θ, when

$$1)\ \eta = 0; \qquad 2)\ \varepsilon^2 - g^2 = 0. \tag{3.11}$$

The first condition gives the value $x = x_1^0$, where

$$x_1^0 = A\sqrt{M} \quad \text{or} \quad \omega_1^0 = \sqrt{\omega_{0e}^2 + \omega_{0i}^2} \equiv \omega_0. \tag{3.12}$$

The second condition leads to a quadratic equation in $x = x_{2,3}^0$. Omitting the intermediate calculations we present the solution

$$\left.\begin{aligned} x_{2,3}^0 &= \sqrt{MA^2 + \frac{(M+1)^2}{4}} \pm \frac{M-1}{2}; \\ \omega_{2,3}^0 &= \sqrt{\omega_{0e}^2 + \omega_{0i}^2 + \frac{(\omega_{Be}+\omega_{Bi})^2}{4}} \pm \frac{\omega_{Be}-\omega_{Bi}}{2} \end{aligned}\right\} \tag{3.13}$$

Under actual conditions it is usually found that the parameter A is greater than $1/\sqrt{M}$. In this case,

$$x_2^0 < x_1^0 < x_3^0. \tag{3.14}$$

The condition in (3.11) can be obtained directly on the basis of the following considerations. When $N^2 = 0$, the field is independent of the coordinates (i.e., the wavelength is infinite); consequently, the electric induction vanishes ($\mathbf{D} = 0$) [cf. Eq. (1.14a)]. The nontrivial solution for $\mathbf{E}$ is then obtained by writing the condition Det $\varepsilon_{\alpha\beta} = 0$. Since the determinant is an invariant, we use the tensor $\varepsilon^0_{\alpha\beta}$ (3.6) to compute it:

$$\text{Det}\ \varepsilon_{\alpha\beta} = (\varepsilon^2 - g^2)\,\eta. \tag{3.15}$$

The condition that this determinant must vanish leads to the relation in (3.11). In this case, it is evident that the condition $\eta = 0$ is a consequence of the balance between the longitudinal (with respect to $\mathbf{B}_0$) component of the displacement current and the polarization, while the condition $\varepsilon^2 - g^2 = 0$ is a consequence of the balance between the transverse components. The projection of the electric field in the direction of $\mathbf{B}_0$ is an independent quantity, while the projections in the plane perpendicular to $\mathbf{B}_0$ are related by the expressions

$$E_{x_0} = -i\frac{g}{\varepsilon}E_{y_0};\quad E_{y_0} = i\frac{g}{\varepsilon}E_{x_0}.$$

We note further that the values of the frequency ω^0, at which the refractive index vanishes, remain unchanged when the thermal motion of the charges is taken into account. This is the case because the displacement of the charges due to the thermal motion is inconsequential if the wavelength is infinite. Formally this result stems from the fact that the parameter which characterizes thermal motion is the ratio of the thermal velocity of the charges to the phase velocity of the wave v_T/v_Φ. When $N = 0$, obviously $v_\Phi = \infty$ and this parameter vanishes.

Let us now investigate the points at which $N^2 = \infty$. The quantity N^2 becomes infinite for the three values of $x = x^\infty_{1,2,3}$ specified by the condition $Q = 0$. In longitudinal propagation ($\theta = 0$), as is evident from Eq. (3.10) for Q, $x_1^\infty = 1$, x_2^∞ is equal to the smaller of the numbers M and $AM^{\frac{1}{2}}$ and x_3^∞ is the larger of these numbers. The corresponding frequencies

$$\left.\begin{aligned} \omega_1^\infty &= \omega_{Bi};\\ \omega_2^\infty &= \min\ \{\omega_{Be},\ \omega_0\};\\ \omega_3^\infty &= \max\ \{\omega_{Be},\ \omega_0\}.\end{aligned}\right\} \tag{3.16}$$

In order to examine the functional dependence $\omega^\infty_{1,2,3}(\theta)$, we write the condition $Q = 0$ in the form

$$x^6 - \alpha x^4 + \beta x^2 - \gamma = 0, \tag{3.17}$$

where

$$\left.\begin{gathered} \alpha = M^2 + 1 + MA^2;\\ \beta = M^2(1 + A^2) + M(M^2 - M + 1)A^2\cos^2\theta;\\ \gamma = M^3A^2\cos^2\theta.\end{gathered}\right\} \tag{3.18}$$

We shall assume that $A \gtrless 1$; then, since $M \gg 1$, Eq. (3.17) can be replaced by the approximate equation $(x^4 - \alpha x^2 + \beta)(x^2 - \gamma/\beta)$, whose roots (positive) are

$$x_1^\infty = \sqrt{\frac{\gamma}{\beta}} \approx \frac{\cos\theta}{\sqrt{\cos^2\theta + \dfrac{(1 + A^2)\sin^2\theta}{MA^2}}}; \tag{3.19}$$

$$x^\infty_{2,\,3} = \frac{1}{2}\left\{\sqrt{\alpha + 2\sqrt{\beta}} \pm \sqrt{\alpha - 2\sqrt{\beta}}\right\} =$$

$$= \frac{1}{2}\left\{\sqrt{M^2 + MA^2 + 2\,[M^2 + M^2A^2 + M^3A^2\cos^2\theta]^{1/2}} \pm \right.$$
$$\left. \pm \sqrt{M^2 + MA^2 - 2\,[M^2 + M^2A^2 + M^3A^2\cos^2\theta]^{1/2}}\right\}. \quad (3.20)$$

For all values of θ outside the narrow cone,

$$\left|\frac{\pi}{2} - \theta\right| < \frac{1}{\sqrt{M}} \quad (3.21)$$

the approximate values (with accuracy to order $1/M$) of the frequencies ω^∞ are

$$\left.\begin{aligned} &\omega_1^\infty = \omega_{Bi};\\ &\omega_{2,3}^\infty = \frac{1}{2}\left\{\sqrt{\omega_{0e}^2 + \omega_{Be}^2 + 2\omega_{0e}\omega_{Be}\cos\theta} \pm \right.\\ &\qquad \left.\pm \sqrt{\omega_{0e}^2 + \omega_{Be}^2 - 2\omega_{0e}\omega_{Be}\cos\theta}\right\}. \end{aligned}\right\} \quad (3.22)$$

We now use a mnemonic device for remembering these values of the roots $\omega_{2,3}^\infty$. Let $\boldsymbol{\omega}_{Be}$ be a vector directed along B_0 and $\boldsymbol{\omega}_{0e}$ a vector directed along k. Then

$$\omega_{2,3}^\infty = \frac{1}{2}\{|\boldsymbol{\omega}_{0e} + \boldsymbol{\omega}_{Be}| \pm |\boldsymbol{\omega}_{0e} - \boldsymbol{\omega}_{Be}|\}. \quad (3.23)$$

When $\theta = \pi/2$, using Eq. (3.23) to obtain the root ω_3^∞, we find

$$\omega_3^\infty = \sqrt{\omega_{0e}^2 + \omega_{Be}^2}. \quad (3.24)$$

However, the roots ω_1^∞ and ω_2^∞ cannot be found from Eqs. (3.22) and (3.23) when $\theta = \pi/2$. It is evident from Eq. (3.19) that the value of ω_1^∞ falls from ω_{Bi} to zero near $\theta = \pi/2$. In contrast with Eq. (3.23), Eq. (3.20) gives a finite value for ω_2^∞; this value can be obtained easily by expanding the expressions under the radical in Eq. (3.20) with respect to the inner radical (this cannot be done at angles close to $\pi/2$, since the term $M^3A^2\cos^2\theta$ can be large). The corresponding limiting value of ω_2^∞ is:

$$x_2^\infty = \left(M\,\frac{1 + A^2}{M + A^2}\right)^{1/2};$$
$$\omega_2^\infty = \left(\omega_{Bi}\omega_{Be}\,\frac{\omega_{Bi}^2 + \omega_{0i}^2}{\omega_{Bi}\omega_{Be} + \omega_{0i}^2}\right)^{1/2}. \quad (3.25)$$

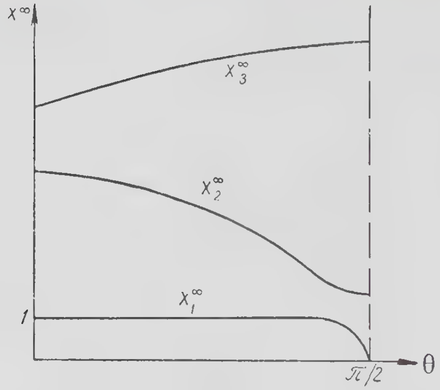


Fig. 5

In Fig. 5 we show schematically the dependence of x^∞ on the angle θ. From a knowledge of the zeros and poles of the function $N^2(x)$, it is easy to determine the general nature of the function. This function is double valued and it is evident from Eq. (3.9) that the two values coincide only when the radical vanishes. The radical vanishes in the following three cases:

$$1)\ \theta = 0,\ g = 0; \quad 2)\ \theta = 0,\ \eta = 0;$$

$$3)\ \theta = \frac{\pi}{2},\ \varepsilon^2 - g^2 - \varepsilon\eta = 0. \tag{3.26}$$

Thus, the curves $N^2(x)$ do not intersect for oblique-wave propagation ($\theta \neq 0$, $\theta \neq \pi/2$) and have points of contact only in the limiting cases ($\theta = 0$ and $\theta = \pi/2$). When $\theta = 0$, the curves have points of contact when $\omega = 0$ and at the plasma frequency $\omega = \omega_0$. When $\theta = \pi/2$, to accuracy of order $1/M$, the frequency ω at which the curves become tangent is given by $x^2 = 1 + A^2$, i.e.,

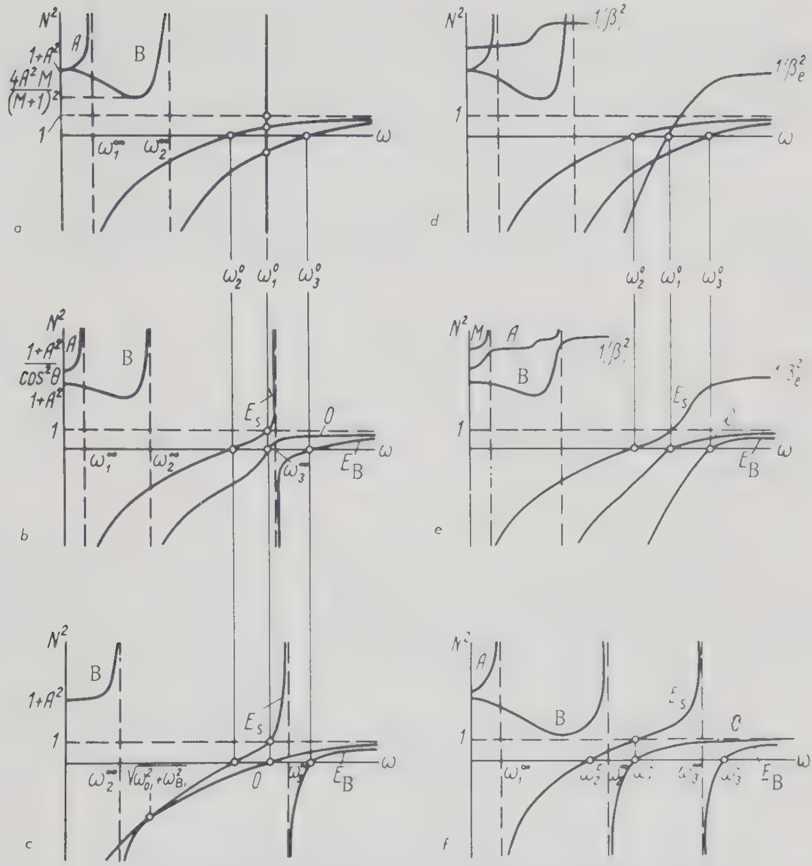
$$\omega = \sqrt{\omega_{0i}^2 + \omega_{Bi}^2}. \tag{3.27}$$

It will be evident that this value of ω is always larger than the value ω_2^∞ (when $\theta = \pi/2$) given by (3.25).

Certain particular values of N^2 are of special interest.

When $\omega \to 0$, the square of the refractive index approaches the values

$$N_1^2 = \frac{1 + A^2}{\cos^2\theta}; \quad N_2^2 = 1 + A^2, \tag{3.28}$$



Frequency dependence of refractive index squared.

a) $\omega_0 > \omega_{Be}$; $p = 0$; $\theta = 0$;

b) $\omega_0 > \omega_{Be}$; $p = 0$; $\theta \neq 0,\ \pi/2$;

c) $\omega_0 > \omega_{Be}$; $p = 0$; $\theta = \pi/2$;

d) $\omega_0 > \omega_{Be}$; $p \neq 0$; $\theta = 0$;

e) $\omega_0 > \omega_{Be}$; $p \neq 0$; $\theta \neq 0,\ \pi/2$;

f) $\omega_0 < \omega_{Be}$; $p = 0$; $\theta \neq 0,\ \pi/2$.

Fig. 6

which correspond to the following dispersion relations:

$$\omega_1^2 = k^2 \frac{c^2 c_A^2}{c^2 + c_A^2} \cos^2\theta; \quad \omega_2^2 = k^2 \frac{c^2 c_A^2}{c^2 + c_A^2}. \tag{3.29}$$

When $c \gg c_A$, we obtain the familiar relations for hydromagnetic waves (in a cold plasma):

$$\omega_1^2 = k^2 c_A^2 \cos^2\theta; \quad \omega_2^2 = k^2 c_A^2. \tag{3.29a}$$

The first relation is to be associated with the Alfvèn wave. It does not change when the plasma pressure is taken into account. The second relation yields one of the magnetoacoustic waves (the "fast" wave) when the Alfvén velocity is appreciably greater than the sound velocity: $c_A \gg c_s$.

We note that the refractive index $N_1^2 \to \infty$ when $\theta \to \pi/2$. This implies zero frequency $\omega_1 = 0$ for any wavelength. The degeneracy that appears is lifted in an inhomogeneous magnetic field, in which the characteristic frequencies can be imaginary, corresponding to an instability in the original state.

When $\omega \to \infty$ both values of N^2 approach unity because the polarization is unimportant at frequencies appreciably greater than the characteristic frequency of the medium. The plasma then has very little effect on the propagation of electromagnetic waves. Under these conditions we find for both polarizations

$$N_1 = N_2 = 1, \quad \omega_1 = \omega_2 = kc. \tag{3.30}$$

As we have noted above, one of the refractive indices vanishes at the plasma frequency. The other, as follows from Eq. (3.9), is given by the expression $N^2 = (\varepsilon^2 - g^2)/\varepsilon$. Substituting $\omega = \omega_{0e}$ we can then find the value of N^2. To an accuracy of order $1/M$ this value is found to be unity.

The results given so far are sufficient to find the behavior of the dispersion curves. In Fig. 6 we show the way in which these curves are deformed as the angle θ varies from 0 to $\pi/2$.

The values of N^2 in the lower halfplane determine the penetration depth of the electromagnetic field in the plasma at a given frequency. The characteristic oscillations of the plasma are described by curves located in the upper halfplane where $N^2 = k^2c^2/\omega^2 > 0$, so that k is a real number. Considering the functional relation $\omega(k)$, i.e., taking account of the curves in the upper halfplane only, we see that there are five wave branches in a cold plasma.

When $\theta \neq 0$, the wave branches are separate. When $\theta \to 0$, three of the branches come together near the plasma frequency $\omega = \omega_0$. The branch associated with the longitudinal electron oscillations $\omega = \omega_0$ at $\theta = 0$ can be regarded as a component part of the three different branches [similarly, when $\theta = \pi/2$, the branch $N^2 = 1 - (\omega_0^2/\omega^2)$ is obtained by fusion of two branches as the angle θ approaches $\pi/2$].

In order to classify the wave branches, it is necessary to consider plasma oscillations in a magnetic field, taking account of the gas-kinetic pressure. It has been shown by Braginskii [9] that when the pressure is introduced the total number of branches of the function $\omega(\mathbf{k})$ is increased to six. The analysis of the dispersion equation is extremely complicated in this case. We shall not attempt such an analysis here, but shall only discuss the general pattern on the basis of certain limiting cases.

When the pressure is taken into account, the equation of motion of a charge in a magnetic field is of the form

$$-i\omega\mathbf{v} = e\mathbf{E} - \frac{1}{mn}\nabla p^{(1)} + [\mathbf{v}\boldsymbol{\omega}_B], \tag{3.31}$$

where

$$p^{(1)} = \frac{\gamma p_0}{i\omega}\operatorname{div}\mathbf{v} = \gamma p_0 \frac{(\mathbf{k}\mathbf{v})}{\omega}. \tag{3.32}$$

Let us consider longitudinal propagation $\mathbf{k} \parallel \mathbf{B}_0$. It is evident that the term $[\mathbf{v}\mathbf{B}_0]$ does not appear in the z component of the equation and that the pressure term does not appear in the x and y components, because it is assumed that $\partial/\partial x = \partial/\partial y = 0$. It is then obvious that the transverse and longitudinal oscillations are completely separate and that the refractive index for the transverse waves is independent of the plasma pressure, while the refractive index for the longitudinal waves is independent of the magnetic field. The general pattern for $\theta = 0$ is obtained by superimposing the curves which describe the longitudinal oscillations for $B_0 = 0$ on the curves which describe the transverse oscillations for $p = 0$. The longitudinal electron oscillations are no longer described by a vertical line $\omega = \omega_0$ but are now described by a diagonal line which "saturates" when $\omega \gg \omega_0$. Comparison with the case $p = 0$ shows that in going from $\theta = 0$ to oblique directions the curves are deformed near the plasma frequency, as shown in Fig.7. As we have already noted, the points at which $N^2 = 0$ are not affected by the pressure.

Let us now consider the low-frequency region, where the ion oscillations are important. It is well known from magnetohydrodynamics that the low-frequency longwave oscillations ($\omega \to 0$, $k \to 0$) comprise three branches. The Alfvèn branch (we assume $c_A \ll c$)

$$\omega = kc_A\cos\theta, \quad N^2 = \frac{c^2}{c_A^2\cos^2\theta} \tag{3.33}$$

and two magnetoacoustic branches (the fast and slow branches):

$$\omega = \frac{k}{2}\left\{\sqrt{c_A^2 + c_s^2 + 2c_Ac_s\cos\theta} \pm \sqrt{c_A^2 + c_s^2 - 2c_Ac_s\cos\theta}\right\} \tag{3.34}$$

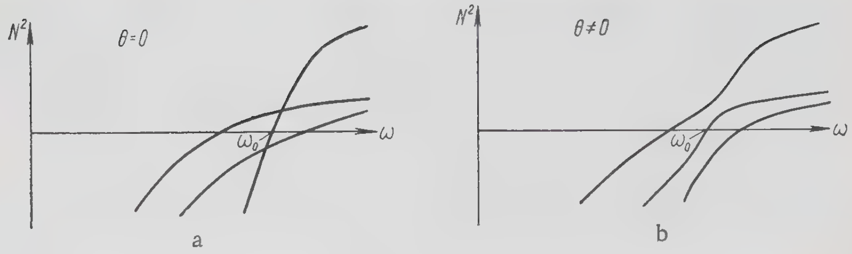


Fig. 7

or, by analogy with Eq. (3.23a),

$$\omega = \frac{1}{2}\{|\mathbf{c}_A + \mathbf{c}_s| \pm |\mathbf{c}_A - \mathbf{c}_s|\}. \tag{3.34a}$$

When $c_A >> c_s$, the fast wave is described by

$$\omega = kc_A; \quad N^2 = \frac{c^2}{c_A^2}, \tag{3.35}$$

and the slow wave by

$$\omega = kc_s \cos\theta; \qquad N^2 = \frac{c^2}{c_s^2 \cos^2\theta}. \tag{3.35a}$$

When $\theta = 0$, the slow magnetoacoustic wave becomes the acoustic wave. When $\theta \neq 0$, the group velocity of this wave is directed along the magnetic field, as is the group velocity of the Alfvèn wave.

In order to determine the behavior of these three branches when the plasma pressure is introduced, we consider the limiting case of short wavelengths $k \to \infty$. Let us assume that the vector $\mathbf{k}$ is directed along the z axis. By comparing the last two terms in the z component of the equation of motion we can make an estimate of $p^{(1)}$:

$$p^{(1)} \simeq \gamma p_0 \frac{kv_z}{\omega} \sim \frac{mnv_x\omega_B}{k}. \tag{3.36}$$

Thus, when $k \to \infty$, the quantity $p^{(1)}$ becomes vanishingly small. We can also estimate v_z when $\omega \sim \omega_B$

$$v_z \sim v_x \frac{mn\omega_B^2}{\gamma p_0 k^2} \sim v_x \frac{\omega_B^2}{k^2 v_T^2}; \quad v_T \sim \sqrt{\frac{p_0}{mn}}. \tag{3.37}$$

When $kv_T \gg \omega_B$, the longitudinal velocity is negligibly small compared with the transverse velocity. The charges then move in a plane perpendicular to the vector **k**. But under these conditions the x and y components of the equation of motion have the same form as in longitudinal propagation, the sole difference being that the motion of the charges is now not affected by the total magnetic field, but only by the projection of the field in the direction of **k**; this projection is given by $B_0 \cos\theta$. Hence, the refractive index is the same for both the transverse waves and the longitudinal waves, except that the quantity ω_B is replaced by $\omega_B \cos\theta$.

It is then evident that the refractive index vanishes when $\omega = \omega_{Bi} \cos\theta$ and when $\omega = \omega_{Be} \cos\theta$ [9]. It should be noted that the wave is actually highly damped when $kv_T \gg \omega_B$; this damping, however, comes from a kinetic analysis. Hence, the investigation of the shortwave region in the hydrodynamic approximation is really useful for a complete classification of the wave branches. In going to an oblique direction $\theta \neq 0$, the curves do not intersect, but separate, as in the case of zero plasma pressure. Taking account of all of these considerations we can then determine the general form of the dispersion curves (cf. Fig. 6e).

It is desirable to set up a classification system for the various wave branches for a number of reasons. The longwave regions show the smallest variation in characteristic properties and are only weakly damped; on the other hand, the shortwave regions are subject to strong damping, so much so that they can only really be discussed on a provisional basis. Furthermore, as we have already seen, the points at which $N^2 = 0$ are actually invariants: these are independent of the angle θ and the plasma pressure. Hence, the plasma wave branches can be classified conveniently on the basis of the properties of the longwave oscillations. The three branches which describe the magnetohydrodynamic waves when $k \to 0$ will be called, respectively, the Alfvèn branch (A), the fast branch (F), and the slow branch (S). The branch which goes through the plasma frequency when $k = 0$ will be called the ordinary wave (O). The other two electron branches will be called the extraordinary waves; the one which is characterized by a phase velocity greater than the velocity of light ($N^2 > 0$, $N < 1$), will be called the fast wave E_F; the other, which exhibits a resonance at a frequency close to ω_0 or ω_{Be} (depending on which of these frequencies is higher), will be called the slow extraordinary wave E_S.*

*We note that when $\omega_0 \gg \omega_{Be}$, the branches A, F, and S are separated by a large frequency interval from the branches O, E_F, E_S. In this case they can be called appropriately the low-frequency and high-frequency branches [9].

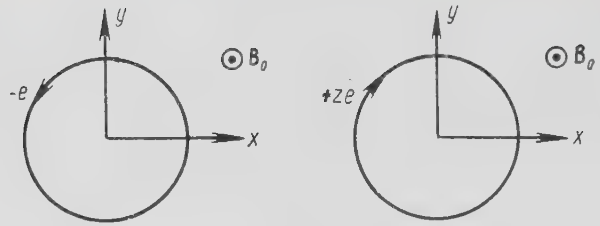


Fig. 8

We now consider in greater detail pure longitudinal propagation and pure transverse propagation in a cold plasma.

In longitudinal propagation $\theta = 0$, $\eta_{xx} = \eta_{yy} = \varepsilon$, $\eta_{xy} = ig$, $N^2 = \varepsilon \pm g$. According to Eq. (1.31), $\alpha_x = \pm 1$, i.e., both branches (the ordinary and the extraordinary) are circularly polarized. In order to determine the direction of rotation of the electric vector, we write the equations for **E**:

$$\left.\begin{aligned} (N^2 - \varepsilon) E_x - igE_y = 0; \\ igE_x + (N^2 - \varepsilon) E_y = 0; \\ -\eta E_z = 0. \end{aligned}\right\} \tag{3.38}$$

For the wave in which $E_x = -iE_y$ (left-handed rotation, in the direction of gyration of an electron in a magnetic field, Fig. 8)*

$$N^2 = \varepsilon - g = 1 - \frac{A^2 M}{(x - M)(x + 1)}; \tag{3.39}$$

while for the wave in which $E_x = iE_y$ (right-handed rotation),

$$N^2 = \varepsilon + g = 1 - \frac{A^2 M}{(x + M)(x - 1)}. \tag{3.39a}$$

*In order to determine the direction of rotation, we write the real components of the field:

$$E_y \sim \operatorname{Re} e^{-i\omega t} = \cos \omega t; \; E_x \sim \operatorname{Re} (-ie^{-i\omega t}) = -\sin \omega t,$$

whence it follows that $\tan \varphi \equiv E_x / E_y = -\tan \omega t$, i.e., $\varphi = -\omega t$.

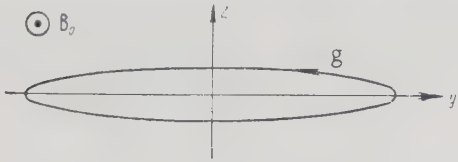


Fig. 9

In the first case there is a resonance at the electron cyclotron frequency; in the second case the resonance occurs at the ion cyclotron frequency. The motion of the charges in the field associated with the wave can easily be determined from Eq. (3.4). Thus, for the wave in which $E_x = iE_y$,

$$x_i = -\frac{ze}{m_i\omega(\omega - \omega_{Bi})} E_x; \quad y_i = i\frac{ze}{m_i\omega(\omega - \omega_{Bi})} E_y. \qquad (3.40)$$

Let $E_x = E_0 e^{i\psi}$, so that $\mathrm{Re}\, E_x = E_0 \cos\psi$. In this case, the real coordinates of the displacement of the charges satisfy the relation

$$x_i^2 + y_i^2 = \frac{z^2e^2}{m_i^2\omega^2(\omega - \omega_{Bi})^2} E_0^2. \qquad (3.40a)$$

In addition to the two waves with circular polarization, in the case being considered ($\theta = 0$) there is a solution $\eta E_z = 0$, i.e., $\eta = 0$ when $E_z \neq 0$. As in the absence of a magnetic field, this solution represents the longitudinal plasma oscillations.

We now consider the case of pure transverse propagation $\theta = \pi/2$. In this case, $N_1^2 = (\varepsilon^2 - g^2)/\varepsilon$, $N_2^2 = \eta$, and α_x assumes the values 0 or ∞, corresponding to linear polarization of the waves (in the plane perpendicular to the direction of propagation). According to Eq. (1.22), in the coordinate system in which the vector **k** is along the z axis we find $\varepsilon_{xy} = -\varepsilon_{yx} = 0$, $\varepsilon_{xz} = \varepsilon_{zx} = 0$ (the magnetic field $\mathbf{B}_0$ is in the x direction).

The system in (1.15) now becomes

$$\left.\begin{aligned} (N^2 - \eta)\, E_x &= 0; \\ (N^2 - \varepsilon_2)\, E_y + igE_z &= 0; \\ -igE_y - \varepsilon_1 E_z &= 0. \end{aligned}\right\} \qquad (3.41)$$

Waves characterized by an electric vector along the magnetic field ($E_y = E_z = 0$, $E_x \neq 0$) see a refractive index which is the same as with no field: $N^2 = \eta$. This is a purely transverse wave with linear polarization.

For a wave with the electric vector in the plane perpendicular to the magnetic field ($E_x = 0$) the square of the refractive index is

$$N^2 = \frac{\varepsilon^2 - g^2}{\varepsilon} = \frac{(x^2-1)(x^2-M^2) - 2MA^2(x^2-M) + M^2A^4}{(x^2-1)(x^2-M^2) - MA^2(x^2-M)}. \tag{3.42}$$

It is important to note that in this wave the longitudinal component of the electric field is nonvanishing ($E_z \neq 0$). The origin of this phenomenon is the fact that the charges drift in the z direction in the crossed fields E_y and B_0. The electron and ion drift velocities are the same only if $\omega \ll \omega_{Bi}$. In the general case, however, the drift velocities will be different and a charge separation arises; this charge separation produces the longitudinal field $E_z = -i(g/\varepsilon)E_y$. At low frequencies we find $g/\varepsilon \to 0$ and the ratio E_z/E_y becomes vanishingly small. However, the situation is completely different at points for which $N^2 \to \infty$. The transverse field vanishes, as is evident from the second equation in (3.41).

The equation that describes the oscillations assumes the form $\varepsilon E_z = 0$, i.e., $\varepsilon = 0$ if $E_z \neq 0$.

Let us consider these longitudinal oscillations in somewhat greater detail. The motion of an electron in the crossed fields E_z and B_x^0 is described by the equations

$$\left.\begin{aligned} -i\omega v_y &= -v_z\omega_{Be}; \\ -i\omega v_z &= -\frac{e}{m}E_z + v_y\omega_{Be}. \end{aligned}\right\} \tag{3.43}$$

Let $E_z = E_0 e^{i\psi}$, so that $\operatorname{Re} E_z = E_0 \cos\psi$, in which case the real displacements are given by

$$y = -\frac{e}{m\left(\omega_{Be}^2 - \omega^2\right)}\frac{\omega_{Be}}{\omega}E_0 \sin\psi; \quad z = \frac{-e}{m\left(\omega_{Be}^2 - \omega^2\right)}E_0 \cos\psi. \tag{3.44}$$

Thus, an electron in the field associated with this wave will describe an ellipse

$$\frac{y^2}{\omega_{Be}^2} + \frac{z^2}{\omega^2} = \frac{e^2E_0^2}{m_e^2\left(\omega_{Be}^2 - \omega^2\right)^2\omega^2}, \tag{3.45}$$

when $\omega \ll \omega_{Be}$, the ellipse is highly elongated in the direction perpendicular to the vectors **k** and **B** (Fig. 9). The electron velocity in the direction of **k** can be appreciably smaller than the velocity along the y axis; hence, the electron velocity can be comparable with the ion velocity in spite of the large difference between the electron and ion masses. As an example, let us consider the case $\omega_{Bi} \ll \omega \ll \omega_{Be}$. To a first approximation the electron executes a drift in the y direction:

$$v_{ye} = \frac{e}{m_e \omega_{Be}} E_z. \tag{3.46}$$

The electron velocity in the direction of propagation of the wave is

$$v_{ze} = i\frac{\omega}{\omega_{Be}} v_y = i\frac{e\omega}{m_e \omega_{Be}^2} E_z. \tag{3.47}$$

If $\omega \gg \omega_{Be}$, we can neglect the effect of the magnetic field in considering the ion motion, so that

$$v_{zi} = i\frac{ze}{m_i \omega} E_z. \tag{3.48}$$

Thus, the current density is given by $j_z = zen_i(v_{zi} - v_{ze}) = i(z^2e^2n_i/m_i\omega) \cdot [1-(m_i/zm_e)(\omega^2/\omega_{Be}^2)]E_z$. The condition $\operatorname{div}\mathbf{D} = 0$, or the equivalent relation $D_z = -i(\omega/c)E_z + (4\pi/c)j_z = 0$, then yields the dispersion relation

$$\omega - \frac{\omega_{0i}^2}{\omega}\left(1 - \frac{m_i}{m_e z}\frac{\omega^2}{\omega_{Be}^2}\right) = 0. \tag{3.49}$$

From this relation we can determine the value of ω_2^∞ which coincides with Eq. (3.25) when $A^2 \gg 1$ (this corresponds to the approximation $\omega \gg \omega_{Bi}$). In particular, when $A^2 \gg M$ the displacement current $-i(\omega/c)E_z$ can be neglected. The value of ω_2^∞ is then determined from the condition $j_z = 0$, i.e., $v_{ze} = v_{zi}$. This quantity is independent of the plasma density and is given by [66, 67]

$$\omega_2^\infty = \sqrt{\omega_{Be}\omega_{Bi}}. \tag{3.50}$$

Under these conditions the electron velocity in the y direction is $\sqrt{M}$ times greater than the velocity in the z direction.

In the transition from pure transverse propagation to oblique propagation ($\theta \neq \pi/2$) the nature of the various longitudinal oscillations changes substantially. If $\theta \neq \pi/2$, there is a component of electric field along $\mathbf{B}_0$ in the longitudinal wave. The electrons can then move along the lines of force of the magnetic field:

$$\left.\begin{aligned} v_{xe} &= i\frac{\omega_{Be}}{\omega}\cos\theta\, v_y; \\ v_{ze} &= v_{xe}\operatorname{ctg}\theta - i\frac{\omega}{\omega_{Be}\sin\theta} v_y. \end{aligned}\right\} \tag{3.51}$$

Under these conditions the electron contribution to the current density j_z will be considerably greater than the ion contribution when $|(\pi/2) - \theta| >> 1/\sqrt{M}$. Using the condition $v_{ze} = 0$ ($j_z = 0$) we can write the frequency of the longitudinal oscillations in the form

$$\omega_2^\infty = \omega_{Be} \cos \theta. \tag{3.52}$$

This same value of ω_2^∞ is obtained from the condition $A^2 >> M$ (i.e., $\omega_{0e}^2 >> \omega_{Be}^2$) if Eq. (3.22) is used.

When $\omega >> \omega_2^\infty$, the ion displacement can be neglected for all values of θ. Let us again consider the longitudinal oscillations for $\theta = \pi/2$. The electron velocity is determined from Eq. (3.43): $v_z = i[e\omega/m_e(\omega_{Be}^2 - \omega^2)]E_z$. Thus, $D_z = 1 - [\omega_{0e}^2/(\omega^2 - \omega_{Be}^2)]$. The condition $D_z = 0$ then determines the oscillation frequency ω_3^∞ [this is to be compared with Eq. (3.24)]:

$$\omega_3^\infty = \sqrt{\omega_{0e}^2 + \omega_{Be}^2}. \tag{3.53}$$

Simplified analytic expressions for N^2 can be obtained in three frequency regions.

The first region is the low-frequency region: $\omega << \sqrt{\omega_{Bi}\omega_{Be}}$. Here,

$$\varepsilon = 1 - \frac{\omega_{0i}^2}{\omega^2 - \omega_{Bi}^2}; \quad g = \frac{-\omega_{0i}^2\omega}{\omega_{Bi}\left(\omega^2 - \omega_{Bi}^2\right)}; \quad \eta = 1 - M\frac{\omega_{0i}^2}{\omega^2}. \tag{3.54}$$

Assuming that η is very large, using Eq. (3.9) we find (first term in an expansion in $1/\eta$)

$$N^2 = \frac{\varepsilon(1 + \cos^2\theta) \pm \sqrt{\varepsilon^2 \sin^4\theta + 4g^2\cos^2\theta}}{2\cos^2\theta}. \tag{3.55}$$

If the unity term can be neglected in ε, the explicit expression for this quantity becomes:

$$N^2 = A^2 \frac{2}{1 + \cos^2\theta \pm \sqrt{1 + 2\left(2\frac{\omega^2}{\omega_{Bi}^2} - 1\right)\cos^2\theta + \cos^4\theta}}. \tag{3.56}$$

This expression holds when $|(\pi/2) - \theta| > 1/\sqrt{M}$, in which case the term $\eta\cos^2\theta$ is still greater than $\varepsilon\sin^2\theta$ in the denominator of Eq. (3.9). Neglecting the term η in N^2 corresponds to neglecting the component of the electric field along the magnetic field $E_{z_0} = (1/\eta)D_{z_0}$. This electric field suppresses motion of the electrons along the lines of force of the magnetic field, the motion being inertialess at low frequencies.

The second frequency range is the high-frequency range: $\omega \gg \sqrt{\omega_{0i}\omega_{Be}}$, where the ion displacement can be neglected completely. In this region,

$$\varepsilon = 1 - \frac{\omega_{0e}^2}{\omega^2 - \omega_{Be}^2}; \; g = \frac{\omega_{0e}^2 \omega_{Be}}{\left(\omega^2 - \omega_{Be}^2\right)\omega}; \; \eta = 1 - \frac{\omega_{0e}^2}{\omega^2}; \tag{3.57}$$

$$N^2 = 1 - \frac{\omega_{0e}^2}{\omega^2} \times$$

$$\times \frac{2\left(\omega_{0e}^2 - \omega^2\right) + \omega_{Be}^2 \sin^2\theta \pm \sqrt{\omega_{Be}^4 \sin^4\theta + 4\frac{\omega_{Be}^2}{\omega^2}\left(\omega_{0e}^2 - \omega^2\right)\cos^2\theta}}{2\left\{\omega_{0e}^2 + \omega_{Be}^2 - \omega^2 - \frac{\omega_{0e}^2\omega_{Be}^2}{\omega^2}\cos^2\theta\right\}}. \tag{3.58}$$

This expression is well known in the theory of propagation of radio waves in the ionosphere. It can be simplified considerably in a dense plasma ($\omega_{0e}^2/\omega_{Be}^2 \gg 1$) for angles that satisfy the condition

$$\frac{\omega^2\omega_{Be}^2}{\left(\omega_{0e}^2 - \omega^2\right)^2} \frac{\sin^4\theta}{4\cos^2\theta} \ll 1, \tag{3.59}$$

Whence we find

$$N^2 = 1 - \frac{\omega_{0e}^2}{\omega\left(\omega \pm \omega_{Be}\cos\theta\right)}. \tag{3.60}$$

This formula is highly reminiscent of the expression for the refractive index for longitudinal wave propagation; the difference is that the quantity ω_{Be} is replaced by $\omega_{Be}\cos\theta$ in the denominator. For this reason we shall call this "quasi-longitudinal" propagation.

The third frequency region is the intermediate range $\omega_{Bi} \ll \omega \ll \omega_{Be}$. In this region the ion displacement must be taken into account, but the effect of the magnetic field on the ion motion can be neglected, so that $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}^{el} - (\omega_{0i}^2/\omega^2)\delta_{\alpha\beta}$, where $\delta\varepsilon_{\alpha\beta}^{el}$ is given by Eq. (3.57) for $\omega \ll \omega_{Be}$. The elements of the tensor $\varepsilon_{\alpha\beta}$ now become

$$\varepsilon = 1 + \frac{\omega_{0e}^2}{\omega_{Be}^2} - \frac{\omega_{0i}^2}{\omega^2}; \; g = -\frac{\omega_{0e}^2}{\omega\omega_{Be}}; \; \eta = 1 - \frac{\omega_{0i}^2}{\omega^2}. \tag{3.61}$$

It is evident that $|\eta| \gg |g|, |\varepsilon|$, so that we can use Eq. (3.55) at angles that satisfy the condition $|(\pi/2) - \theta| > 1/\sqrt{M}$. Furthermore, since $|g| \gg \varepsilon$, this expression can be simplified:

$$N^2 = \pm \frac{g}{\cos\theta} = \pm \frac{\omega_{0e}^2}{\omega\omega_{Be}\cos\theta}. \qquad (3.62)$$

The formula does not apply for angles such that $|(\pi/2) - \theta| < 1/\sqrt{M}$; in this case it is necessary to use the general formula (3.9).

In conclusion, we consider an important feature of the behavior of electromagnetic waves at frequencies corresponding to the cyclotron frequencies of the plasma particles $\omega = \omega_{Bi}$ and $\omega = \omega_{Be}$. We have seen that in the case of longitudinal propagation there are two waves with circular polarization; one of these can exhibit a resonance with the rotation of the charge. The case of pure longitudinal propagation $\theta = 0$ is, however, a special case.

In going to oblique propagation ($\theta \neq 0$) the nature of the polarization changes in a fundamental way. It is found that the component of the electric vector perpendicular to $\mathbf{B}_0$ (when $\omega^2 = \omega_B^2$) also exhibits circular polarization; however, in both waves the rotation is now in the opposite direction to that of the charge. If the polarization vectors are normalized in the form $E_{y1} = E_{y2} = \text{const}$, then when $\omega^2 = \omega_B^2$ the polarization ellipse becomes the cross section of a cylinder with axis along $\mathbf{B}_0$. In Fig. 10 the arrows show the directions of rotation of $\mathbf{E}$ for the waves (denoted by 1 and 2) and for the charge (base of the cylinder).

This feature of the polarization follows directly from the formula that relates $\mathbf{E}$ and $\mathbf{D}$. Using Eq. (3.5) we find that the induction vector $\mathbf{D}$ obeys the relation

$$D_x \pm iD_y = \left(1 - \sum \frac{\omega_0^2}{\omega(\omega \mp \omega_B)}\right)(E_x \pm iE_y). \qquad (3.63)$$

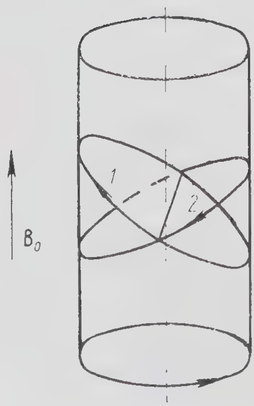


Fig. 10

For finite $\mathbf{E}$ the vector $\mathbf{D} = N^2\mathbf{E}_\perp$ is finite if $\omega^2 = \omega_B^2$, except for the case $\theta = 0$, where $N^2 = \infty$. Hence, using Eq. (3.63) and taking the upper and lower signs in turn, we find

$$E_x = -iE_y; \quad \omega = \omega_{Bi}; \qquad (3.64)$$

$$E_x = iE_y; \quad \omega = \omega_{Be}. \qquad (3.65)$$

Comparison with the case of longitudinal propagation considered above shows that the direction of rotation of $\mathbf{E}$ is opposite to the direction of rotation of the charges.

Problem. Derive an expression for the tensor $\varepsilon_{\alpha\beta}$ in the hydrodynamic approximation taking account of the gas–kinetic pressure of the charged particles.

Solution. We start with the equation of motion

$$mn\frac{\partial \mathbf{v}}{\partial t} = -imn\omega\mathbf{v} = -\nabla p^{(1)} + en\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{B}_0]\right), \tag{1}$$

where $p^{(1)} = \gamma[(\mathbf{kv})/\omega]\,p = \gamma[(\mathbf{kv})/\omega]\,nT$, in accordance with Eq. (2.4). Solving this equation for $\mathbf{v}$, we find the current density $\mathbf{j} = \Sigma en\mathbf{v}$ and the electric induction $\mathbf{D} = \mathbf{E} + i(4\pi/\omega)\mathbf{j}$. By comparing the expression obtained with $D_\alpha = \varepsilon_{\alpha\beta}E_\beta$, we then find $\varepsilon_{\alpha\beta}$. In a coordinate system with z^0 axis along $\mathbf{B}_0$, this tensor is given by (the vector $\mathbf{k}$ lies in the plane of x^0z^0)

$$\varepsilon^0_{\alpha\beta} = \delta_{\alpha\beta} - \sum \frac{\omega_0^2}{\omega^2 - \omega_B^2 - k^2v_{\mathrm{T}}^2 + k^2v_{\mathrm{T}}\frac{\omega_B^2}{\omega^2}\cos^2\theta} \times$$

$$\times\begin{pmatrix} 1-\frac{k_z^2v_{\mathrm{T}}^2}{\omega^2}, & i\frac{\omega_B}{\omega}\left(1-\frac{k_z^2v_{\mathrm{T}}^2}{\omega_B^2}\right), & \frac{k_xk_zv_{\mathrm{T}}^2}{\omega^2} \\ -i\frac{\omega_B}{\omega}\left(1-\frac{k_z^2v_{\mathrm{T}}^2}{\omega^2}\right), & 1-\frac{k^2v_{\mathrm{T}}^2}{\omega^2}, & -i\frac{\omega_B}{\omega}\frac{k_xk_zv_{\mathrm{T}}^2}{\omega^2} \\ \frac{k_xk_zv_{\mathrm{T}}^2}{\omega^2}, & i\frac{\omega_B}{\omega}\frac{k_xk_zv_{\mathrm{T}}^2}{\omega^2}, & 1-\frac{\omega_B^2+k_x^2v_{\mathrm{T}}^2}{\omega^2} \end{pmatrix}. \tag{2}$$

Here, $v_{\mathrm{T}}^2 = \gamma T/m$ and the summation is taken over all species of charged particles.

The tensor obtained is useful for estimating the importance of various components of $\varepsilon_{\alpha\beta}$ in the dispersion equation when the thermal motion of the charged particles is taken into account.

Using the transformation formula (1.22) in the coordinate system with z axis along $\mathbf{k}$, we find

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} - \sum \frac{\omega_0^2}{\omega^2 - \omega_B^2 - k^2v_{\mathrm{T}}^2 + k^2v_{\mathrm{T}}^2\frac{\omega_B^2}{\omega^2}\cos^2\theta} \times$$

$$\times\; 1-\frac{\omega_B^2\sin^2\theta + k^2v_{\mathrm{T}}^2}{\omega^2},\; i\frac{\omega_B}{\omega}\left(1-\frac{k^2v_{\mathrm{T}}^2}{\omega^2}\right)\cos\theta,\; \frac{\omega_B^2}{\omega^2}\sin\theta\cdot\cos\theta \times$$

$$\times \begin{pmatrix} -i\frac{\omega_B}{\omega}\left(1-\frac{k^2v_T^2}{\omega^2}\right)\cos\theta, & 1-\frac{k^2v_T^2}{\omega^2}, & -i\frac{\omega_B}{\omega}\sin\theta \\ \frac{\omega_B^2}{\omega^2}\sin\theta\cdot\cos\theta, & i\frac{\omega_B}{\omega}\sin\theta, & 1-\frac{\omega_B^2}{\omega^2}\cos^2\theta \end{pmatrix}. \quad (3)$$

§ 4. Resonances Due to Thermal Motion

In considering oscillations in the earlier sections, we have assumed that to a zeroth approximation the charge is located at one point and that it is displaced only by the effect of the wave field. Actually, however, in one oscillation period the charge can be displaced by virtue of its thermal motion and in some cases this displacement can be comparable with the wavelength. This displacement is important for slow waves, i.e., waves whose phase velocity is smaller than the velocity of light and comparable with the charge velocity. Under these conditions a number of new resonance effects arise, these effects leading to exchange of energy between the wave and the charges. In order to examine the resonance condition, let us determine the force exerted on a moving charge by a wave described by a factor of the form $e^{i(\mathbf{kr}-\omega t)}$. In the linear approximation this force is obviously

$$\mathbf{F} = e\left\{\mathbf{E} + \frac{1}{c}[\mathbf{v}_0\mathbf{B}]\right\} = e\left\{\mathbf{E}\left(1-\frac{\mathbf{kv}_0}{\omega}\right) + \frac{\mathbf{k}(\mathbf{v}_0\mathbf{E})}{\omega}\right\}. \quad (4.1)$$

Here, $\mathbf{v}_0$ is the velocity associated with the unperturbed motion of the charge, and $\mathbf{B} = (c/\omega)[\mathbf{kE}]$ is the magnetic field of the wave. The electric field $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{kr}-\omega t)}$ must be taken at the point at which the charge is located at time t. Obviously, in the linear approximation by r(t) we mean the radius vector associated with the unperturbed motion of the charge.

Let the initial radius vector of the charge be $\mathbf{r}_0 = \{x_0, y_0, z_0\}$. In the absence of a constant magnetic field, neglecting the interaction between charges we have $\mathbf{v}_0$ = const and the charge moves along a straight line

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t. \quad (4.2)$$

The phase of the wave is

$$\mathbf{kr} - \omega t = \mathbf{kr}_0 + (\mathbf{kv}_0 - \omega)t. \quad (4.3)$$

If

$$\omega - \mathbf{kv}_0 = 0 \quad (4.4)$$

the force acting on the charge remains constant in time. This resonance condition expresses the fact that the projection of the charge velocity on the wave vector coincides with the phase velocity of the wave

$$\frac{\omega}{k} = v_0 \cos(\widehat{\mathbf{k}\mathbf{v}_0}). \tag{4.5}$$

In a plasma with no magnetic field the phase velocity of the transverse waves is larger than the velocity of light, so that the resonance condition cannot be satisfied for transverse waves. On the other hand, a resonance can occur for longitudinal waves since, as we have seen in § 2, the phase velocity of these waves is comparable with the velocity associated with the thermal motion of the particles. The interaction of longitudinal plasma waves with charges leads to a specific kind of plasma wave damping which will be considered below in § § 5 and 10.

Let us now assume that the plasma is located in a uniform magnetic field $\mathbf{B}_0$ directed along the z axis. The particle motion under these conditions is given by

$$\left.\begin{aligned} v_{x0}(t) &= v_{\perp 0}\cos(\omega_B t + \varphi_0) = v_{x0}(0)\cos\omega_B t + v_{y0}(0)\sin\omega_B t;\\ v_{y0}(t) &= -v_{\perp 0}\sin(\omega_B t + \varphi_0) = v_{y0}(0)\cos\omega_B t - v_{x0}(0)\sin\omega_B t;\\ v_{z0}(t) &= v_{z0}. \end{aligned}\right\} \tag{4.6}$$

Here, ω_B is the cyclotron frequency and φ_0 is the initial phase:

$$\omega_B = \frac{eB_0}{mc} = \frac{eB_0}{m_0 c}\sqrt{1-\beta^2}; \quad \operatorname{tg}\varphi = \frac{v_{y0}}{v_{x0}};$$
$$v_{\perp 0} = \sqrt{v_{x0}^2 + v_{y0}^2}. \tag{4.6a}$$

Assume that the wave vector lies in the xz plane:

$$\mathbf{k} = \{k_x,\ 0,\ k_z\}. \tag{4.7}$$

Then the phase at the point at which the charge is located at time t is given by

$$\mathbf{k}\mathbf{r} - \omega t = \mathbf{k}\mathbf{r}_0 - (\omega - k_z v_{z_0})\,t + \frac{k_x v_\perp}{\omega_B}[\sin(\omega_B t + \varphi_0) - \sin\varphi_0]. \tag{4.8}$$

Using the relation (Appendix I)

$$e^{ia\sin x} = \sum_{n=-\infty}^{+\infty} J_n(\alpha)\, e^{inx}, \tag{4.9}$$

we find

$$e^{i(\mathbf{kr}-\omega t)} = e^{i\mathbf{kr}_0} \sum_{n=-\infty}^{+\infty} J_n\left(\frac{k_\perp v_\perp}{\omega_B}\right) e^{in\varphi_0 - i\frac{k_\perp v_\perp}{\omega_B}\sin\varphi_0 - i(\omega - n\omega_B - k_z v_{z_0})t}. \tag{4.10}$$

Thus a charge moving in a magnetic field is subject to a force consisting of terms whose time dependence is given by $e^{-i\omega' t}$, where $\omega' = \omega - n\omega_B - k_z v_{z_0}$. As we have seen in § 3, a resonance occurs when $\pm\omega'$ coincides with the cyclotron frequency: $\omega - (n \pm 1)\omega_B - k_z v_{z_0} = 0$. Introducing the transformation $n \pm 1 \rightarrow n$, we can write the resonance condition in the form

$$\omega - n\omega_B - k_z v_{z0} = 0, \quad n = 0, \ \pm 1, \ \pm 2, \ \ldots \tag{4.11}$$

In accordance with the Doppler relation, the frequency $\tilde{\omega}$ of a radiator that moves along the z axis with velocity v_{z0} is related to the transformed frequency in the laboratory coordinate system by

$$\tilde{\omega} = \frac{\omega - k_z v_{z0}}{\sqrt{1-\beta^2}}. \tag{4.12}$$

The motion of a charge in a magnetic field can be regarded as the motion of an oscillator of finite dimensions (Larmor "circlet") along the lines of force. Assuming that $\omega_B = \omega_B^0\sqrt{1-\beta^2}$, where $\omega_B^0 = eB_0/m_0c$ is the natural frequency of this oscillator in the coordinate system in which it is at rest, we see that the condition in (4.11) implies frequency coincidence between the electromagnetic wave and one of the harmonics of the characteristic oscillator frequency. In particular, when $n = 0$, this condition [like (4.4)] means that the wave frequency is zero (constant field) in the coordinate system in which the charge is at rest.

As we have already noted, charges that move near resonance can effectively exchange energy with the wave. Depending on the form of the velocity distribution of the charges, this interaction can lead to wave damping or wave amplification.

In the linear approximation the equation of motion of a charge for which $\mathbf{v}_0 \neq 0$ can be written in the form

$$m\left(\frac{\partial \mathbf{v}}{\partial t} - [\mathbf{v}\boldsymbol{\omega}_B]\right) = F\left(\mathbf{r}(t), t\right) = F_0 e^{i\left(\mathbf{kr}_0 + \mathbf{k}\int_0^t \mathbf{v}_0(t')\,dt' - \omega t\right)},$$

where $\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{v}_0(t')\,dt'$ is the radius vector of the charge in the absence of the wave.

It is possible to proceed in a somewhat different manner, writing the motion of the particles in terms of a hydrodynamic description (in the Euler variables):

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}_0\nabla)\,\mathbf{v} - [\mathbf{v}\mathbf{w}_B]\right) = \mathbf{F}(\mathbf{r},\ t) = \mathbf{F}_0 e^{i(\mathbf{kr} - \omega t)},$$

where $\mathbf{r}$ is now an independent variable.

In calculations involving the tensor $\varepsilon_{\alpha\beta}$, it is frequently more convenient to use the more formal methods of the kinetic equation.

§ 5. Damping of Plasma Waves

The most characteristic plasma wave is the longitudinal wave, whose frequency is given by (§ 2):

$$\omega^2 = \omega_{0e}^2 + \gamma\,\frac{k^2T_e}{m_e}. \tag{5.1}$$

Plasma oscillations characterized by a frequency $\omega = \omega_{0e}$ were first investigated both theoretically and experimentally by Langmuir and Tonks (1929). Vlasov (1938) was the first to apply the kinetic equation with a self-consistent field in order to determine the dispersion relation for these waves. Somewhat later it was shown by Landau [2] that these plasma waves are damped even in the absence of collisions.

The damping mechanism derived by Landau can be explained on the basis of the following qualitative considerations [68]. Charges that move with a velocity somewhat greater than the phase velocity of the wave ($v_0 > \omega/k$) can be retarded by the electric field of the wave and change the direction of their motion with respect to the wave. However, in the reflection from the potential barrier moving with velocity ω/k the particle energy is changed by an amount *

* The relative velocity of the particle $u = v_0 - (\omega/k)$ changes sign in reflection and becomes $u_1 = (\omega/k) - v_0$; consequently, the absolute velocity of the particle is $v_1 = u_1 + (\omega/k) = 2(\omega/k) - v_0$ after reflection. Subtracting $mv_1^2/2$ from $mv^2/2$, we find $\Delta\varepsilon$ as given by Eq. (5.2).

$$\Delta\varepsilon = 2m\frac{\omega}{k}\left(\frac{\omega}{k} - v_0\right). \tag{5.2}$$

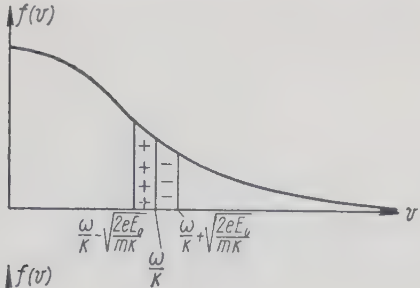
When $v_0 > \omega/k$ the energy change $\Delta\varepsilon < 0$, i.e., the particle loses energy, giving it to the wave. The same formula expresses the change in energy of a particle moving with a velocity somewhat smaller than the phase velocity ($v_0 < \omega/k$). These particles are effectively entrained by the wave, which overtakes them, and acquire an energy $\Delta\varepsilon > 0$. It is evident that the reflection will occur for those particles whose energy in the coordinate system fixed in the wave is smaller than the potential energy of a charge in a field $E = E_0 \sin kx$, i.e.,

$$-\sqrt{\frac{2eE_0}{mk}} < u < \sqrt{\frac{2eE_0}{mk}}. \tag{5.3}$$

The widths of the velocity intervals in which $\Delta\varepsilon > 0$ and $\Delta\varepsilon < 0$ are the same. However, even though the widths of these velocity intervals are the same, in a Maxwellian distribution the number of fast particles (which lose energy) is smaller than the number of slow particles (which acquire energy from the wave). Hence the total energy acquired by the charged particles is positive and the wave is damped. In Fig. 11, the plus signs denote those particles which absorb energy from the wave and the minus signs denote those particles which give up energy. If the distribution function is an increasing function of v, at $v = \omega/k$ (as shown in Fig. 11b) the wave will be amplified. Particles which occupy statistically less probable states of high energy will be transferred into more probable states with low energy by the interaction with the wave, and give up excess energy to the wave. If the oscillation frequency is divided into real and imaginary parts ($\omega = \omega_1 - i\omega_2$) so that positive ω_2 corresponds to wave damping, it is evident from the foregoing that the sign of ω_2 is opposite to the sign of the derivative of the distribution function (with respect to v) at the point $v = \omega/k$:

$$\omega_2 \sim -\left.\frac{\partial f}{\partial v}\right|_{v=\frac{\omega}{k}}. \tag{5.4}$$

The damping (amplification) of the waves is obtained if one takes account of only a single reflection of the charge from the potential barrier. If the wave is not damped (or amplified) in the time required for the charge to traverse the distance between two neighboring barriers, those charges which lose energy in the first reflection will acquire it in the reflection from the opposite barrier, and those which have acquired energy will lose it. On the avera



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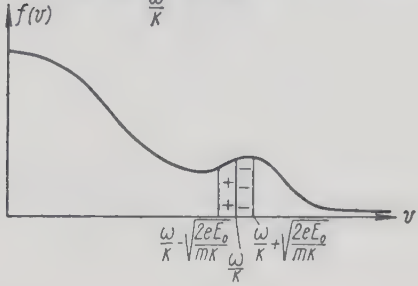

Fig. 11

age there will be no change in the total energy of the charges and the damping (amplification) effect vanishes. The oscillation frequency of a charge in a potential well is approximately ku ~ $\sqrt{(2eE_0/m)}\,k$, so that the condition for strong damping (amplification) is

$$\gamma \gg \sqrt{\frac{2eE_0}{m}}\,k. \tag{5.5}$$

It will be shown below that damping is important when $kv_T \sim \omega_0$, in which case $\gamma \sim \omega_0$. Using these values for k and γ we can now write (5.5) in the form

$$\frac{mv_T^2}{2} \gg \frac{eE_0}{k}. \tag{5.6}$$

Thus the damping we have been considering (Landau damping) operates only in a weak field, in which case the energy acquired by a charge from the electric field in a distance of the order of a wavelength is appreciably smaller than the initial energy of the charge. This is the usual condition that must be satisfied for the linear approximation in electrodynamics. The absence of collisionless damping in the nonlinear approximation can be understood if one adopts a picture in which the interaction between the charges and the wave distorts the distribution function in such a way that a "plateau" appears at v = ω/k:

$$\left.\frac{\partial f}{\partial v}\right|_{v=\frac{\omega}{k}} = 0.$$

Taking account of only a single reflection of the charge from the potential hill is equivalent to introducing a weak dissipative mechanism, for example, collisions of a charge with other particles at a rate ν; this "collision frequency" can be much larger than the frequency of reflection but appreciably smaller than γ [$\gamma >> \nu >> \sqrt{(2eE_0/m)k}$] so that the total effect is independent of ν. The introduction of collisions characterized by a collision frequency which can then be allowed to go to zero is a convenient formal technique for computing collisionless damping.

The dispersion equation for plasma waves with collisionless damping taken into account can be obtained directly from the relation giving the motion of the charges. We shall first obtain this equation in a nonrigorous manner.

Let us consider a group of particles described by density $\Delta n_0 = n_0 f \Delta v_0$, whose velocity is close to v_0. We now write the equation of motion and the equation of continuity for this group of particles in the presence of a wave $E_z = E_0 e^{i(kz - \omega t)}$, taking account of the loss of momentum and the number of particles due to collisions; the collisions are characterized by a collision frequency ν which will later be allowed to go to zero:

$$m\left(\frac{\partial v^{(1)}}{\partial t} + v_0 \frac{\partial v^{(1)}}{\partial t} + \nu v^{(1)}\right) = eE_z; \tag{5.7}$$

$$\frac{\partial (\Delta n^{(1)})}{\partial t} + v_0 \frac{\partial (\Delta n^{(1)})}{\partial z} + \nu \Delta n^{(1)} = -\Delta n^0 \operatorname{div} \mathbf{v}^{(1)}. \tag{5.8}$$

Taking account of the dependence $e^{i(kz - \omega t)}$ we find

$$v^{(1)} = i\frac{e}{m(\omega - kv_0 + i\nu)} E_z; \quad \Delta n^{(1)} = \frac{kv^{(1)}}{\omega - kv_0 + i\nu} \Delta n_0. \tag{5.9}$$

The current density produced by the electric field of the wave is

$$\Delta j_z = e(\Delta n_0 v^{(1)} + \Delta n^{(1)} v_0) = \Delta\sigma_{\parallel} E_z, \tag{5.10}$$

where

$$\Delta\sigma_{\parallel} = i\frac{e^2 \Delta n_0}{m} \frac{\omega + i\nu}{(\omega - kv_0 + i\nu)^2}. \tag{5.11}$$

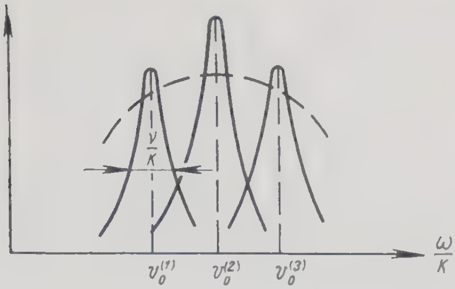

Fig. 12

As we have noted in § 1, the energy absorbed by the medium is determined by the Hermitian (in the present case, real) component of the complex conductivity

$$\Delta Q = \frac{1}{2} \Delta\sigma'_{\parallel} E_0^2. \tag{5.12}$$

From Eq. (5.11) we find

$$\Delta\sigma'_{\parallel} = \mathrm{Re}\,\Delta\sigma_{\parallel} = \nu \frac{e^2 \Delta n_0}{m} \frac{\omega^2 - k^2 v_0^2 + \nu^2}{[(\omega - k v_0)^2 + \nu^2]^2}. \tag{5.13}$$

When $\nu \to 0$ we can neglect ν^2 in the numerator. It is evident from this expression that charges absorb energy ($\Delta\sigma'_{\parallel} > 0$) if their velocity is smaller than the phase velocity of the wave (in accordance with the considerations given above). On the other hand, when $v_0 > \omega/k$, the charges lose energy to the wave. The total effect must depend on the value of the derivative $\partial f/\partial v_0$ at $v_0 = \omega/k$. The expression for the total complex conductivity can be transformed (using an integration by parts) to an integral which contains $\partial f/\partial v_0$:

$$\sigma_{\parallel} = i \frac{e^2 n_0}{m} \int_{-\infty}^{\infty} \frac{\omega + i\nu}{(\omega - k v_0 + i\nu)^2} f(v_0)\, dv_0 =$$

$$= -i \frac{e^2 n_0}{m} \int_{-\infty}^{\infty} \frac{v_0 \frac{\partial f_0}{\partial v_0}}{\omega - kv + i\nu} dv_0. \tag{5.14}$$

The real part of $\sigma_{\|}$ is given by

$$\sigma'_{\|} = -\frac{e^2 n_0}{m} \int\limits_{-\infty}^{\infty} \frac{\nu}{(\omega - kv_0)^2 + \nu^2} v_0 \frac{\partial f_0}{\partial v_0} dv_0. \tag{5.15}$$

For individual groups of particles with velocities close to $v_0^{(1)}$, $v_0^{(2)}$, $v_0^{(3)}$, etc., the integrand has the form of peaks with halfwidths given by ν (Fig. 12). In going over to a continuous distribution of velocities, the spacing between peaks is reduced. As a result of superposition of individual curves we obtain an overall curve (shown by the dashed line) which, when $\nu \to 0$, depends only on the form of the distribution function. The integral in (5.15) is computed very easily. The expression $[\nu/(\omega - kv_0)^2 + \nu^2]$ has the properties of a δ-function. When $\nu \to 0$, it is vanishingly small everywhere except for the narrow region around the point $v_0 = \omega/k$. In this region the function $v_0(\partial f_0/\partial v_0)$ can be regarded as a constant which can be taken outside the integral sign. The remaining integral is then given by

$$\int\limits_{-\infty}^{\infty} \frac{\nu}{(\omega - kv_0)^2 + \nu^2} dv_0 = \frac{\pi}{|k|}. \tag{5.16}$$

Thus,

$$\sigma'_{\|} = -\frac{e^2 n_0}{m} \frac{\pi}{|k|} \left[v_0 \frac{\partial f}{\partial v_0}\right]_{v_0 = \frac{\omega}{k}}. \tag{5.17}$$

The dispersion equation for these oscillations is

$$\varepsilon_{\|} = 1 + i\frac{4\pi\sigma_{\|}}{\omega} = 0. \tag{5.18}$$

In the absence of thermal motion, $\varepsilon_{\|} = 1 - (\omega_{0e}^2/\omega^2)$ and the characteristic frequency $\omega = \omega_{0e}$. Assuming that the thermal correction for the characteristic frequency is small ($kv_0 \ll \omega_{0e}$), we find $\operatorname{Im}\sigma_{\|}$ by expanding the denominator of the integrand (5.14) in terms of kv_0/ω with subsequent integration by parts:

$$\int\limits_{-\infty}^{\infty} \frac{v_0 \frac{\partial f_0}{\partial v_0}}{\omega - kv_0} dv_0 = \frac{1}{\omega} \int\limits_{-\infty}^{\infty} \left(1 + \frac{kv_0}{\omega} + \frac{k^2 v_0^2}{\omega^2} + \dots\right) v_0 \frac{\partial f_0}{\partial v_0} dv_0 =$$

$$= -\frac{1}{\omega}\left(1 + 3\frac{k^2 \overline{v_0^2}}{\omega^2} + \dots\right); \tag{5.19}$$

$$\overline{v_0^2} = \int_{-\infty}^{\infty} v_0^2 f(v_0)\, dv_0. \tag{5.19}$$

Thus,

$$\varepsilon_{\parallel} = 1 - \frac{\omega_0^2}{\omega^2} \times$$

$$\times \left(1 + 3\frac{k^2\overline{v_0^2}}{\omega^2} + \ldots\right) - i\frac{\omega_0^2}{\omega}\frac{\pi}{|k|}\left[v_0\frac{\partial f_0}{\partial v_0}\right]_{v_0 = \frac{\omega}{k}}. \tag{5.20}$$

The condition $\varepsilon_{\parallel} = 0$ then yields

$$\omega^2 = \omega_{0e}^2\left(1 + 3\frac{k^2 v_0^2}{\omega^2} + \ldots\right) + i\omega_{0e}^2\frac{\pi\omega}{|k|}\left[v_0\frac{\partial f}{\partial v_0}\right]_{v_0 = \frac{\omega}{k}}. \tag{5.21}$$

Since the correction associated with the thermal motion is assumed to be small, ω can be replaced by ω_{0e} on the right side of this relation. Also ($\omega_2 \ll \omega_1$)

$$\omega = \omega_1 - i\omega_2, \tag{5.22}$$

so that

$$\omega_1^2 = \omega_{0e}^2 + 3k^2\overline{v_0^2}; \quad \omega_2 = -\pi\frac{\omega_0^2}{2}\left[\frac{v_0}{|k|}\frac{\partial f_0}{\partial v_0}\right]_{v_0 = \frac{\omega}{k}}. \tag{5.23}$$

For the particular case of a Maxwellian distribution $f = (m_e/2\pi T)^{\frac{1}{2}} \cdot \exp - (m_e v_0^2/2T)$ we have [2]

$$\omega_1^2 = \omega_{0e}^2 + 3k^2\frac{T_e}{m_e} = \omega_0^2(1 + 3k^2a^2); \tag{5.24}$$

$$\omega_2 = \omega_0\frac{\sqrt{\pi}}{(2k^2a^2)^{3/2}}\, e^{-\frac{1}{2k^2a^2}}, \tag{5.25}$$

where $a = \sqrt{T/4\pi e^2 n_0}$ is the Debye radius.

The long wave perturbations ($ka \ll 1$) are weakly damped; however, perturbations with wavelength approaching the Debye radius ($ka \sim 1$) are damped in one oscillation period [strictly speaking, when $\omega_2 \sim \omega_0$, Eq. (5.25)

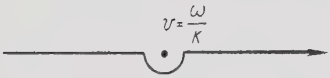

Fig. 13

no longer applies, but the qualitative conclusion as to strong damping still holds]. Thus, as was pointed out in § 1, electron oscillations of a plasma occur in a very narrow range of values near $\omega = \omega_{0e}$.

The nonrigorous aspect of the derivative of the dispersion equation as given above lies in the following. As is evident from Eqs. (5.14) and (5.15), the sign of $\mathrm{Re}\,\sigma_{\parallel}$ is related to the sign of the imaginary part of the denominator of the integrand in Eq. (5.14). For a Maxwellian distribution $\nu > 0$ means $\mathrm{Re}\,\sigma_{\parallel} > 0$ and, correspondingly, $\mathrm{Im}\,\omega = -\omega_2 < 0$, which implies damping. But in computing the integral in (5.14), we have not taken account of the fact that ω is a complex quantity. If we had taken $\omega = \omega_1 - i\omega_2$ for $\nu \to 0$, where $\omega_2 > 0$, the denominator in Eq. (5.14) would have assumed the form $\omega_1 - kv_0 + i(\nu - \omega_2)$, and when $\nu < \omega_2$ we would have found that $\mathrm{Re}\,\sigma_{\parallel} < 0$, corresponding to amplification of the waves (!). But this result is completely incorrect. This follows immediately from the fact that with $\mathrm{Re}\,\sigma_{\parallel} < 0$ the dispersion equation would give $\mathrm{Im}\,\omega > 0$, in contradiction to the assumption made in the calculation of $\mathrm{Re}\,\sigma_{\parallel}$, i.e., that $\omega = \omega_1 - i\omega_2$, $\omega_2 > 0$. This same contradiction would occur if we had written $\omega = \omega_1 + i\omega_2$ $(\omega_2 > 0)$ in computing the integral, corresponding to the excitation of oscillations. In this case, as in the case in which the denominator is of the form $\omega - kv_0 + i\nu$, we find from the dispersion equation that $\mathrm{Im}\,\omega < 0$, in contradiction with the assumed condition $\mathrm{Im}\,\omega = \omega_2 > 0$. These paradoxes derive from the incorrect method that was used for obtaining the dispersion equation with the thermal motion taken into account. The point is that a solution of the form $e^{i(kz - \omega t)}$ that gives a unique dependence of the characteristic frequency on the wave number $\omega = \omega(k)$ cannot hold when the particles exhibit a thermal spread in velocities. The situation is exactly the same and we would not obtain a solution in the form $e^{-i\omega t}$ if the velocity of the charge were definitely specified but the perturbation had the form of a wave packet $f(z) = \int f_k e^{ikz} dk$ $[f_k \neq \delta(k - k')]$. This is formally clear from the dispersion equation itself. This equation is transcendental and has an infinite number of roots. The root that has been found (5.23) is distinguished only by the fact that it has the smallest imaginary part.

A rigorous analysis of the characteristic oscillations of a plasma, such as the one given below (§ 7), shows that a solution of the form $e^{i(kz - \omega t)}$ is an asymptotic one (for large t). In this case it is found that the integral in the dispersion equation must be computed as it has been computed above, i.e., it must be assumed that ω in the denominator of the integrand is real, with a small positive imaginary part $\nu > 0$, which can then be allowed to vanish. If we assume that $\nu = 0$ at the outset, then the rule for computing the integral means that in the integration over v the pole $v = \omega/k$ must be traversed from below when $k > 0$ (Fig. 13). This method of computing the integral is called the "Landau contour." The basis for this rule is given in § 7. First, as a preliminary step, we consider certain general relations for electrodynamic media which exhibit spatial dispersion, i.e., media in which the tensor $\varepsilon_{\alpha\beta}$ depends on **k**.

§ 6. Maxwell's Equations in an Anisotropic Medium with Spatial Dispersion

In an unmagnetized medium ($\mu = 1$) Maxwell's equations can be written in the form

$$\left.\begin{aligned} &\operatorname{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \frac{4\pi}{c}\mathbf{j}(\mathbf{E}) = \frac{4\pi}{c}\mathbf{j}_{\text{spec}};\\ &\operatorname{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0;\\ &\operatorname{div}\mathbf{E} - 4\pi\varrho(\mathbf{E}) = 4\pi\varrho_{\text{spec}};\\ &\operatorname{div}\mathbf{B} = 0. \end{aligned}\right\} \tag{6.1}$$

In the first equation we have isolated that part of the current density $\mathbf{j}(\mathbf{E})$ which is due to the electric field of the wave. By the specified current density $\mathbf{j}_{\text{spec}}$ we are to understand that part of the current density which is independent of **E** – for example, current in conductors located in the plasma, or current produced by separate charges (in considering the field produced by these charges), etc. A corresponding distinction is made in the charge density.

A general relation for the current density $\mathbf{j}(\mathbf{E})$ produced by an electromagnetic field in a uniform medium under stationary conditions can be written in the form of a linear functional (within the framework of the linear Maxwell equations)

$$j_\alpha(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int_{v} \sigma_{\alpha\beta}(\mathbf{r}-\mathbf{r}', t-t') E_\beta(\mathbf{r}', t')\, d\mathbf{r}'. \tag{6.2}$$

Here, and below, it is assumed that there is no electromagnetic field in the medium at $t = -\infty$. The relation in (6.2) expresses the fact that the current at point **r** at time t depends on the electric field at point **r**' at an earlier time $t' < t$. This relation between **j** and **E** stems from the fact that a contribution to the current density $\mathbf{j}(\mathbf{r}, t)$ comes from charges which transfer the effect of the electromagnetic field from point **r**' to **r** in the time interval $t - t'$. In the simplest case, in which we neglect the interaction between these charges, this transfer is realized by the displacement of the charge from $\mathbf{r}'$ to $\mathbf{r}$ as a consequence of its thermal motion. Thus, for a plasma with no magnetic field, a contribution to the current is given by charges which are displaced with velocity $\mathbf{v}_0 = (\mathbf{r} - \mathbf{r}')/(t - t')$ from $\mathbf{r}'$ to $\mathbf{r}$. The kernal $\sigma_{\alpha\beta} \cdot (\mathbf{r} - \mathbf{r}', t - t')$ then depends on the law of motion of the charges in the medium being considered. This dependence is simplest for a medium in statistical equilibrium, in which case the distribution function for the charges is given by

$$D_0(\mathbf{q}_0, \mathbf{p}_0)\, d\mathbf{q}_0\, d\mathbf{p}_0 = \mathrm{const} \cdot e^{-\varepsilon(\mathbf{p}_0, \mathbf{q}_0)/T}\, d\mathbf{q}_0\, d\mathbf{p}_0 \tag{6.3}$$

($\mathbf{q}_0$ and $\mathbf{p}_0$ represent the ensemble of coordinates and momenta of all N charges that appear in the volume being considered). This distribution is distorted by the electromagnetic field:

$$D(t) = D_0 + D_1(t), \tag{6.4}$$

where $D_1(t)$ is governed by the equation

$$\frac{\partial D_1}{\partial t} + [H_0 D_1] - \sum_{i=1}^{N} \frac{e_i}{m_i} \left\{ \mathbf{E}(\mathbf{r}_i, t) + \frac{1}{c} [\mathbf{v}_i \mathbf{B}(\mathbf{r}_i, t)] \right\} \frac{\partial D_0}{\partial \mathbf{p}_i} =$$

$$= \sum_{i=1}^{N} \frac{e_i \mathbf{v}_i \mathbf{E}(\mathbf{r}_i, t)}{T} D_0, \tag{6.5}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the mean electric and magnetic fields as given by Eq. (6.1) and H_0 is the Hamiltonian of the system of charges in the absence of the electromagnetic field.* We now transform from the variables $\mathbf{r}_i$, $\mathbf{p}_i$, t to the variables $\mathbf{r}_{0i}$, $\mathbf{p}_{0i}$, t, the relation between these two sets of variables being given by the motion of the charges in the absence of the wave:

$$\mathbf{r}_i = \mathbf{r}_i(\mathbf{r}_{0i}, t); \quad \mathbf{v}_i = \mathbf{v}_i(\mathbf{v}_{0i}, t). \tag{6.6}$$

Equation (6.5) now assumes the form

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*At this point we are interested in obtaining relations between the basic electrodynamic characteristics of a medium with the greatest physical clarity and will not go into detail concerning the meaning of H_0. In the case of a plasma all the relations obtained in the present section can be obtained by standard methods (cf. Appendix III).

$$\frac{\partial D_1}{\partial t} = \sum_{i=1}^{N} \frac{e_i \mathbf{v}_i(t) \mathbf{E}(\mathbf{r}_i(t), t)}{T} D_0 = \frac{D_0}{T} \int\limits_{v} \mathbf{j}^{\mathrm{M}}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}', t)\, d\mathbf{r}', \tag{6.7}$$

where $\mathbf{j}^{\mathrm{M}}$ is the microscopic current density

$$\mathbf{j}^{\mathrm{M}}(\mathbf{r}, t) = \sum_{i=1}^{N} e_i \mathbf{v}_i(t)\, \delta(\mathbf{r} - \mathbf{r}_i(t)). \tag{6.8}$$

Thus,

$$D_1 = \frac{D_0}{T} \int\limits_{-\infty}^{t} dt' \int\limits_{v} j_\beta^{\mathrm{M}}(\mathbf{r}', t') E_\beta(\mathbf{r}', t')\, d\mathbf{r}'. \tag{6.9}$$

The mean current density is obtained by multiplying $\mathbf{j}^{\mathrm{M}}(\mathbf{r}, t)$ by D_1 and integrating over the coordinates and velocities of all particles:

$$j_\alpha(\mathbf{r}, t) = \frac{1}{T} \int\limits_{-\infty}^{t} dt' \int\limits_{V} \overline{j_\alpha^{\mathrm{M}}(\mathbf{r}, t)\, j_\beta^{\mathrm{M}}(\mathbf{r}', t')} \cdot E_\beta(\mathbf{r}', t')\, d\mathbf{r}'. \tag{6.10}$$

Here, the bar denotes an average over the equilibrium distribution D_0.

Comparison of Eqs. (6.2) and (6.10) for the current density shows that the kernel $\sigma_{\alpha\beta}(\mathbf{r} - \mathbf{r}', t - t')$ is related to the correlation function for the currents

$$G_{\alpha\beta}(\mathbf{R}, \tau) = \overline{j_\alpha^{\mathrm{M}}(\mathbf{r}, t)\, j_\beta^{\mathrm{M}}(\mathbf{r}', t')}; \quad \mathbf{R} = \mathbf{r} - \mathbf{r}', \quad \tau = t - t' \tag{6.11}$$

by the simple expression

$$\sigma_{\alpha\beta}(\mathbf{R}, \tau) = \frac{1}{T} G_{\alpha\beta}(\mathbf{R}, \tau). \tag{6.12}$$

The function $\sigma_{\alpha\beta}(\mathbf{R}, \tau)$, by its meaning, is determined only for $\tau > 0$. But it can be extended (in an arbitrary way) into the region $\tau < 0$. We define this function for $\tau < 0$ in the following way:

$$\sigma_{\alpha\beta}(\mathbf{R}, -\tau) = \sigma_{\beta\alpha}(-\mathbf{R}, \tau), \quad \text{or} \quad \sigma_{\alpha\beta}(-\mathbf{R}, -\tau) = \sigma_{\beta\alpha}(\mathbf{R}, \tau). \tag{6.13}$$

This extension can be easily obtained if Eq. (6.12) is extended to the region $\tau < 0$ on the basis of the obvious property $G_{\alpha\beta}(\mathbf{R}, \tau) = G_{\alpha\beta}(-\mathbf{R}, -\tau)$. The relation in (6.13) can, however, also be used in cases in which (6.12) does not hold (in a nonequilibrium medium).

In the general case, Maxwell's equations are integro-differential equations and the solution of these equations is a task of great difficulty. If the medium is uniform, one possible approach to solving these equations is to use Fourier's method, in which all quantities are expanded in terms of plane waves. Let us now consider the relations that obtain between the Fourier components of **j** and **E**, and $\sigma_{\alpha\beta}$ and $G_{\alpha\beta}$

In carrying out the expansions in plane waves, the Fourier components will be denoted by the same symbol, but the arguments will be changed:

$$\mathbf{j}(\mathbf{r}, t) = \int \mathbf{j}(\mathbf{k}, \omega)\, e^{i(\mathbf{kr}-\omega t)}\, d\mathbf{k}\, d\omega; \tag{6.14}$$

$$\mathbf{j}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^4} \int \mathbf{j}(\mathbf{r}, t)\, e^{-i(\mathbf{kr}-\omega t)}\, d\mathbf{r}\, dt; \tag{6.14a}$$

$$\mathbf{E}(\mathbf{r}, t) = \int \mathbf{E}(\mathbf{k}, \omega)\, e^{i(\mathbf{kr}-\omega t)}\, d\mathbf{k}\, d\omega \text{ etc.} \tag{6.15}$$

Unless otherwise specified, the integration over k_x, k_y, k_z, ω, x, y, z, and t is carried out from $-\infty$ to $+\infty$. Substituting the electric field (6.15) in Eq. (6.2), which relates **j** and **E**, and replacing the variables **r**' and t' by the new variables $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\tau = t - t'$, we obtain an expression for $\mathbf{j}(\mathbf{r}, t)$, in the form of an expansion (6.14) with

$$j_\alpha(\mathbf{k}, \omega) = \sigma_{\alpha\beta}(\mathbf{k}, \omega)\, E_\beta(\mathbf{k}, \omega); \tag{6.16}$$

$$\sigma_{\alpha\beta}(\mathbf{k}, \omega) = \int_0^\infty d\tau \int \sigma_{\alpha\beta}(\mathbf{R}, \tau)\, e^{-i(\mathbf{kR}-\omega\tau)}\, d\mathbf{R}. \tag{6.17}$$

Attention is directed to the fact that the integral over τ is taken between the limits 0 and ∞ (rather than $-\infty$ and $+\infty$). This restriction is directly connected with the causal dependence of **j** on **E**, which is expressed by the fact that the integration in Eq. (6.2) is only carried out over earlier times.

We now write the tensor $\sigma_{\alpha\beta}(\mathbf{k}, \omega)$ in the form of a sum of a Hermitian part and an anti-Hermitian part:

$$\left.\begin{gathered} \sigma_{\alpha\beta} = \sigma'_{\alpha\beta} + i\sigma''_{\alpha\beta}; \\ \sigma'_{\alpha\beta} = \frac{\sigma_{\alpha\beta} + \overset{*}{\sigma}_{\beta\alpha}}{2}; \quad i\sigma''_{\alpha\beta} = \frac{\sigma_{\alpha\beta} - \overset{*}{\sigma}_{\beta\alpha}}{2}. \end{gathered}\right\} \tag{6.18}$$

Carrying out some simple transformations of the variables of integration and making use of Eq. (6.13), from Eq. (6.17) we have

$$\sigma'_{\alpha\beta}(\mathbf{k},\ \omega)=\frac{1}{2}\int\sigma_{\alpha\beta}(\mathbf{R},\ \tau)\,e^{-i(\mathbf{kR}-\omega\tau)}d\mathbf{R}\,d\tau; \tag{6.19}$$

$$\sigma''_{\alpha\beta}(\mathbf{k},\ \omega)=\frac{1}{2i}\int\sigma_{\alpha\beta}(\mathbf{R},\ \tau)\,\mathrm{Sgn}\,\tau\cdot e^{-i(\mathbf{kR}-\omega t)}\,d\mathbf{R}\,d\tau. \tag{6.20}$$

In the integrals (6.19) and (6.20) the integration over τ is carried out from $-\infty$ to $+\infty$. The second integral is actually the difference of two integrals with limits between 0 and $+\infty$ and between $-\infty$ and 0, written in the same form as the first through the use of the discontinuous function

$$\mathrm{Sgn}\,\tau=\begin{cases}\ \ 1, & \tau>0;\\ -1, & \tau<0.\end{cases} \tag{6.21}$$

We shall require the Fourier components of this function below.

We first find the Fourier components of the function $\mathrm{Sgn}\,\tau\cdot e^{-\nu|\tau|}$ and then let ν go to zero. We have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Sgn}\,\tau\cdot e^{-\nu|\tau|}e^{ix\tau}d\tau=\frac{i}{\pi}\,\frac{x}{x^2+\nu^2}. \tag{6.22}$$

The function

$$\frac{P}{x}=\lim_{\nu\to 0}\frac{x}{x^2+\nu^2}, \tag{6.23}$$

is called the principal value of $1/x$ and behaves like $1/x$ everywhere except for the point $x = 0$, where it (the principal value) vanishes. If this function appears under an integral sign, the integral is taken in the sense of the "principal value":

$$\int_a^b f(x)\,\frac{P}{x}\,dx\equiv\int_a^b\frac{f(x)}{x}\,dx. \tag{6.24}$$

From Eq. (6.22) we have

$$\frac{i}{\pi}\,\frac{P}{x}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Sgn}\,\tau\cdot e^{ix\tau}\,d\tau, \tag{6.25}$$

so that

$$\operatorname{Sgn}\tau = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-ix\tau}}{x}\,dx. \tag{6.26}$$

Equations (6.19) and (6.20) yield expressions for the Hermitian and anti-Hermitian parts of the tensor $\sigma_{\alpha\beta}(\mathbf{k}, \omega)$ in terms of the same function $\sigma_{\alpha\beta}(\mathbf{R}, \tau)$. These formulas then allow us to find directly the relation between $\sigma'_{\alpha\beta}(\mathbf{k}, \omega)$ and $\sigma''_{\alpha\beta}(\mathbf{k}, \omega)$ (Kramers–Kronig relation). Carrying out the inverse Fourier transformation we find directly from Eqs. (6.19) and (6.20):

$$\sigma_{\alpha\beta}(\mathrm{R},\ \tau) = \frac{2}{2\pi}\int \sigma'_{\alpha\beta}(\omega', \mathrm{k}')\, e^{i(\mathrm{k}'\mathrm{R}-\omega'\tau)}\,d\mathrm{k}'\,d\omega'; \tag{6.19a}$$

$$\sigma_{\alpha\beta}(\mathrm{R},\ \tau) = \frac{2}{2\pi}\int i\sigma''_{\alpha\beta}(\omega', \mathrm{k}')\operatorname{Sgn}\tau\cdot e^{i(\mathrm{k}'\mathrm{R}-\omega'\tau)}\,d\mathbf{k}'\,d\omega'. \tag{6.20a}$$

Substituting Eq.-(6.19a) in (6.20), Eq. (6.20a) in (6.19), and making use of Eqs. (6.24) and (6.25), we obtain the Kramers–Kronig* relations for the tensor $\sigma_{\alpha\beta}(\mathbf{k}, \omega)$: *

$$\sigma''_{\alpha\beta}(\mathrm{k},\ \omega) = \frac{1}{\pi}\int\frac{\sigma'_{\alpha\beta}(\mathrm{k},\ \omega')}{\omega-\omega'}\,d\omega'; \tag{6.27}$$

$$\sigma'_{\alpha\beta}(\mathrm{k},\ \omega) = -\frac{1}{\pi}\int\frac{\sigma''_{\alpha\beta}(\mathrm{k},\ \omega')}{\omega-\omega'}\,d\omega'. \tag{6.28}$$

As we have noted in §1, it is convenient to eliminate the current density $\mathbf{j}(\mathbf{E})$ in the electric induction in solving Maxwell's equations and to use the dielectric tensor:

$$\varepsilon_{\alpha\beta}(\mathrm{k},\ \omega) = \delta_{\alpha\beta} + i\,\frac{4\pi\sigma_{\alpha\beta}(\mathrm{k},\ \omega)}{\omega}. \tag{6.29}$$

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*We note that in the original formula (6.2) the integration is actually carried out over the "world cone" $R \le c\tau$. Taking account of this limitation on the region of integration leads to an additional relation in the Kramers–Kronig relations which was first pointed out by Leontovich [39]. If the actual value $\sigma_{\alpha\beta}(R,\tau)$ is used, this relativistic effect is introduced automatically since $\sigma_{\alpha\beta}(R,\tau) = 0$ when $R > c\tau$ [cf. Eqs. (6.12) and (8.12)].

This tensor relates the vectors $\mathbf{D}(\mathbf{k}, \omega)$ and $\mathbf{E}(\mathbf{k}, \omega)$:

$$D_\alpha(\mathbf{k}, \omega) = \varepsilon_{\alpha\beta}(\mathbf{k}, \omega) E_\beta(\mathbf{k}, \omega). \tag{6.30}$$

The Hermitian and anti-Hermitian parts of $\varepsilon_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ are related by

$$\varepsilon'_{\alpha\beta} = \delta_{\alpha\beta} - \frac{4\pi\sigma''_{\alpha\beta}}{\omega}; \quad i\varepsilon''_{\alpha\beta} = i\frac{4\pi\sigma'_{\alpha\beta}}{\omega}. \tag{6.31}$$

Since $\sigma_{\alpha\beta}(\mathbf{R}, \tau)$ is a real function, we note, as is evident from Eq. (6.17), that $\sigma_{\alpha\beta}(\mathbf{k}, \omega) = \sigma_{\alpha\beta}(-\mathbf{k}, -\omega)$; consequently,

$$\varepsilon^*_{\alpha\beta}(\mathbf{k}, \omega) = \varepsilon_{\alpha\beta}(-\mathbf{k}, -\omega). \tag{6.32}$$

We shall now use the Fourier method to find the electromagnetic field produced by a specified current in a plasma. Maxwell's equations for the Fourier components (6.1) become an algebraic set of equations in which the differential operations are replaced by the following algebraic operations:

$$\operatorname{div}\mathbf{A} \to i(\mathbf{kA}); \ \operatorname{rot}\mathbf{A} \to i[\mathbf{kA}]; \ \frac{\partial\mathbf{A}}{\partial t} \to -i\omega\mathbf{A}. \tag{6.33}$$

The first two Maxwell equations then assume the form

$$i[\mathbf{kB}] + i\frac{\omega}{c}\mathbf{D} = \frac{4\pi}{c}\mathbf{j}_{\text{spec}}; \tag{6.34}$$

$$i[\mathbf{kE}] - i\frac{\omega}{c}\mathbf{B} = 0.$$

The second two equations follow as a consequence if account is taken of the fact that $\operatorname{div}\mathbf{j} = i\omega\rho$.

Eliminating $\mathbf{B}$, we obtain the system

$$\left(k^2\delta_{\alpha\beta} - \frac{\omega^2}{c^2}\varepsilon_{\alpha\beta} - k_\alpha k_\beta\right) E_\beta = \frac{4\pi i\omega}{c^2} j_\alpha^{\text{spec}} \tag{6.35}$$

(hereinafter the superscript "spec" will be deleted).

If the medium is isotropic, so that

$$\varepsilon_{\alpha\beta} = \varepsilon_\perp\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) + \varepsilon_\parallel\frac{k_\alpha k_\beta}{k^2}, \tag{6.36}$$

this system can be written in vector form:

$$\left(k^2 - \frac{\omega^2}{c^2}\varepsilon_\perp\right)\mathbf{E}_\perp - \frac{\omega^2}{c^2}\varepsilon_\parallel\mathbf{E}_\parallel = \frac{4\pi i\omega}{c^2}\mathbf{j}, \tag{6.37}$$

where $\mathbf{E}_{\|} = [\mathbf{k}(\mathbf{kE})]/k^2$ is the component of $\mathbf{E}$ in the direction of $\mathbf{k}$ (longitudinal field), while $\mathbf{E}_{\perp}[\mathbf{k}(\mathbf{Ek})]/k^2$ is the traverse (with respect to $\mathbf{k}$) field. It follows from this equation that

$$\mathbf{E}_{\|} = -\frac{4\pi i}{\omega\varepsilon_{\|}} \mathbf{j}_{\|}(\mathbf{k}, \omega); \tag{6.38}$$

$$\mathbf{E}_{\perp} = \frac{4\pi i\omega}{c^2} \frac{\mathbf{j}_{\perp}(\mathbf{k}, \omega)}{k^2 - \frac{\omega^2}{c^2}\varepsilon_{\perp}}. \tag{6.39}$$

The total field $\mathbf{E}(\mathbf{r}, t)$ is then given by

$$\mathbf{E}(\mathbf{r}, t) = \int \frac{4\pi i}{\omega} \left\{ \frac{\frac{\omega^2}{c^2}\mathbf{j}_{\perp}(\mathbf{k}, \omega)}{k^2 - \frac{\omega^2}{c^2}\varepsilon_{\perp}(\mathbf{k}, \omega)} - \frac{\mathbf{j}_{\|}(\mathbf{k}, \omega)}{\varepsilon_{\|}(\mathbf{k}, \omega)} \right\} e^{i(\mathbf{kR}-\omega t)}\, d\mathbf{k}\, d\omega. \tag{6.40}$$

Evidently the electric field is made up of two independent parts. The first part satisfies the condition $\operatorname{div}\mathbf{E} = 0$ and represents the transverse field. The second part satisfies the condition $\operatorname{rot}\mathbf{E} = 0$ and represents the longitudinal field.

Let us now consider an anisotropic medium. In order to solve the system of equations (6.35) we make use of the method of normal modes [40,41] in which the field $\mathbf{E}$ is expanded in characteristic vector fields or in characteristic polarization vectors $\mathbf{a}$. These characteristic polarization vectors are defined as a solution of the following system of homogeneous equations:

$$\{\varepsilon_l(\delta_{\alpha\beta} - n_\alpha n_\beta) - \varepsilon_{\alpha\beta}\}\, a_{\beta l} = 0;\ \ \mathbf{n} = \frac{\mathbf{k}}{k}, \tag{6.41}$$

where the characteristic value ε_l ($l = 1,2$) which corresponds to the vector $\mathbf{a}_l$ is given by (1.25a). In addition, we introduce the following auxiliary system of equations for the conjugate vectors $\mathbf{b}^*$:

$$\{\varepsilon_m(\delta_{\alpha\beta} - n_\alpha n_\beta) - \varepsilon_{\beta\alpha}\}\, b^*_{\beta m} = 0. \tag{6.42}$$

[The vectors $\mathbf{a}$ and $\mathbf{b}$ are characteristic functions of complex conjugate operators. If $\varepsilon_{\alpha\beta}$ is a Hermitian matrix ($\varepsilon^*_{\alpha\beta} = \varepsilon_{\beta\alpha}$) the vectors $\mathbf{a}$ and $\mathbf{b}$ are the same.] Multiplying the first equation by $b^*_{\alpha m}$ and the second by $a_{\alpha l}$ and neglecting dummy indices in the second equation, we have

$$\varepsilon_l(\delta_{\alpha\beta} - n_\alpha n_\beta)\, b^*_{\alpha m} a_{\beta l} - \varepsilon_{\alpha\beta} a_{\beta l} b^*_{\alpha m} = 0; \tag{6.43}$$

$$\varepsilon_m (\delta_{\alpha\beta} - n_\alpha n_\beta)\, b^*_{\alpha m} a_{\beta l} - \varepsilon_{\alpha\beta} a_{\beta l} b^*_{\alpha m} = 0. \tag{6.43}$$

We now subtract the second equation from the first

$$(\varepsilon_l - \varepsilon_m)(\delta_{\alpha\beta} - n_\alpha n_\beta)\, b^*_{\alpha m} a_{\beta l} = 0. \tag{6.44}$$

It then follows that the factor that multiplies $(\varepsilon_l - \varepsilon_m)$ vanishes when $l \neq m$. However, it does not vanish when $l = m$ in an anisotropic medium, and can be set equal to unity. This condition then determines the normalization of the vectors **a** and **b**.

Thus, the vectors **a** and **b** satisfy the following conditions:

1) the orthonormality condition*:

$$(\delta_{\alpha\beta} - n_\alpha n_\beta)\, b^*_{\alpha m} a_{\beta l} = \delta_{lm}; \tag{6.45}$$

2) from Eq. (6.43), taken in conjunction with (6.45), the equivalent relation follows:

$$\varepsilon_{\alpha\beta} a_{\beta l} b^*_{\alpha m} = \varepsilon_l \delta_{lm}. \tag{6.46}$$

The solution of the inhomogeneous system of equations (6.35) is now written in the form

$$\mathbf{E} = \sum_{l=1}^{2} E_l \mathbf{a}_l, \tag{6.47}$$

where the E_l are the amplitudes, which are to be determined. We now substitute Eq. (6.47) in Eq. (6.35):

$$\left[\frac{k^2c^2}{\omega^2}(\delta_{\alpha\beta} - n_\alpha n_\beta) - \varepsilon_{\alpha\beta}\right] \sum_{l=1}^{2} E_l a_{\beta l} = \frac{4\pi i}{\omega} j_\alpha(\mathrm{k}, \omega). \tag{6.48}$$

Multiplying this equation by $b^*_{\alpha m}$ and taking account of the orthonormal properties, we have

* The transverse component of the vector field is normalized by (6.45). If the medium is isotropic so that there is an independent longitudinal field, the expression $(\delta_{\alpha\beta} - n_\alpha n_\beta) b^*_{\alpha n} a_{\beta l} = \mathbf{a}_l \mathbf{b}^*_m - (\mathbf{n}\mathbf{a}_l)(\mathbf{n}\mathbf{b}^*_m)$ vanishes identically for the longitudinal field and the method of normalization indicated here is not satisfactory. Instead of normalizing the transverse components, one can normalize in other ways, for example, by normalizing the length of the field vector (cf. Problem 1 of § 6).

$$\left(\frac{k^2c^2}{\omega^2}-\varepsilon_l\right)E_l=\frac{4\pi i}{\omega}\,(\mathbf{j}\mathbf{b}_l^*). \tag{6.49}$$

Using Eqs. (6.47) and (6.49) we obtain the following expression for the Fourier component of the electric field:

$$\mathbf{E}(\mathbf{k};\omega)=\sum_{l=1}^{2}\frac{4\pi i}{\omega}\cdot\frac{\mathbf{a}_l\,(\mathbf{j}\mathbf{b}_l^*)}{\dfrac{k^2c^2}{\omega^2}-\varepsilon_l(\mathbf{k},\omega)}. \tag{6.50}$$

Finally,

$$\mathbf{E}(\mathbf{r},t)=\sum_{l=1}^{2}\int\frac{4\pi i}{\omega}\cdot\frac{\mathbf{a}_l\,(\mathbf{j}\mathbf{b}_l^*)}{\dfrac{k^2c^2}{\omega^2}-\varepsilon_l(\mathbf{k},\omega)}\,e^{i(\mathbf{kr}-\omega t)}d\mathbf{k}\,d\omega. \tag{6.51}$$

The polarization vector in the coordinate system with z axis along $\mathbf{k}$ has been found in § 1 [Eq. (1.27)]:

$$\mathbf{a}=a_y\{i\alpha_x,\ 1,\ i\alpha_z\}, \tag{6.52}$$

where α_x and α_z satisfy Eqs. (1.28) and (1.31). The vector $\mathbf{b}^*$ can be found in similar fashion:

$$\mathbf{b}^*=a_y\{-i\alpha_x,\ 1,\ -i\alpha_z\}. \tag{6.53}$$

(Here we have taken $a_y = a_y^* = b_y$, so that in the case of a Hermitian tensor $\varepsilon_{\alpha\beta}$ the vectors $\mathbf{a}$ and $\mathbf{b}$ not only coincide in direction but in absolute magnitude as well.) The orthonormal property now assumes the form

$$a_{yl}a_{ym}^*(1+\alpha_{xl}\alpha_{xm})=\delta_{lm}. \tag{6.54}$$

According to Eq. (1.32) we have $\alpha_{x1}\alpha_{x2} = -1$, so that this condition is actually satisfied when $l \neq m$. When $l = m$, we find

$$a_y=\frac{1}{\sqrt{1+\alpha_x^2}}. \tag{6.55}$$

In the coordinate system in which the tensor $\varepsilon_{\alpha\beta}$ is usually computed (magnetic field along the z axis, the vector $\mathbf{k}$ in the xz plane forming an angle θ with the vector $\mathbf{B}_0$) the components of the polarization vector are determined by the coordinate transformation formulas and are given by

$$a_{x0} = \frac{i}{\sqrt{1+\alpha_x^2}} \alpha_{x0}; \quad a_{y0} = \frac{1}{\sqrt{1+\alpha_x^2}};$$

$$a_{z0} = \frac{i}{\sqrt{1+\alpha_x^2}} \alpha_{z0}, \tag{6.56}$$

where

$$\alpha_{x0} = \alpha_x \cos\theta + \alpha_z \sin\theta; \quad \alpha_{z0} = \alpha_z \cos\theta - \alpha_x \sin\theta. \tag{6.57}$$

Expressions for α_X, α_Z, α_{X0}, and α_{Z0} in terms of the components of the tensor $\varepsilon^0_{\alpha\beta}$ have been given in § 1 [Eqs. (1.35)-(1.38)].

Since an integration is carried out over **k** in the integral in (6.51), in computing **E** we must have the polarization vector in a fixed coordinate system with arbitrary direction for the vector **k**. Let $\mathbf{B}_0$ be directed along z, and let **k** have the components

$$\mathbf{k} = \{k_\perp \cos\varkappa,\ \mathbf{k}_\perp \sin\varkappa,\ k_z\}. \tag{6.58}$$

The components of the polarization vector in this coordinate system are found from the rules for vector transformation:

$$a_x(\varkappa) = a_{x0}\cos\varkappa - a_{y0}\sin\varkappa, \quad a_y(\varkappa) = a_{x0}\sin\varkappa + a_{y0}\cos\varkappa. \tag{6.59}$$

Problem 1. Derive an expression for determining the field **E** for a specified current density **j**, taking for the normal field vectors the vectors $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ for which the total lengths ($\tilde{\mathbf{a}}_l \tilde{\mathbf{b}}^*_m = \delta_{lm}$) rather than the projected lengths have been normalized.

Solution. It is easy to show that the vectors $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}^*$ must now satisfy the following equations in place of Eqs. (6.41) and (6.42):

$$\left(\tilde{\varepsilon}_l \delta_{\alpha\beta} - \frac{k^2c^2}{\omega^2} n_\alpha n_\beta - \varepsilon_{\alpha\beta}\right) \tilde{a}_{\beta l} = 0; \tag{1}$$

$$\left(\tilde{\varepsilon}_m \delta_{\alpha\beta} - \frac{k^2c^2}{\omega^2} n_\alpha n_\beta - \varepsilon_{\beta\alpha}\right) \tilde{b}^*_{\beta m} = 0. \tag{2}$$

Multiplying (1) by $b^*_{\alpha m}$ and (2) by $\tilde{a}_{\alpha l}$, and subtracting (2) from (1), we find $\delta_{\alpha\beta}(\tilde{\varepsilon}_l - \tilde{\varepsilon}_m)\tilde{a}_{\beta l}\tilde{b}^*_{m\alpha} = 0$, i.e.,

$$\delta_{\alpha\beta}\tilde{a}_{\beta l}\tilde{b}^*_{m\alpha} = \delta_{lm}. \tag{3}$$

When $l = m$, we have

$$\tilde{\mathbf{a}}\,\tilde{\mathbf{b}}^* = 1. \tag{3a}$$

Multiplying by $b^*_{\alpha m}$, and taking account of Eq. (3), from Eq. (1) we find

$$\tilde{\varepsilon}_l = \frac{k^2c^2}{\omega^2}\left(\mathbf{n}\tilde{\mathbf{a}}_l\right)\left(\mathbf{n}\tilde{\mathbf{b}}^*_l\right) + \varepsilon_{\alpha\beta}a_{\beta l}b^*_{l\alpha}\,. \tag{4}$$

For fixed $\mathbf{k}$ and ω the system in (1) has as a solution three characteristic vectors (l = 1,2,3) which correspond to two "quasi-transverse" and one "quasi-longitudinal" polarizations. Writing $\mathbf{E}$ in the form of a sum $\mathbf{E} = \sum_{l=1}^{3} E_l\mathbf{a}_l$ and using Eqs. (3) and (4), by analogy with Eq. (6.51) we find

$$\mathbf{E}(\mathbf{r}, t) = \sum_{l=1}^{3}\int \frac{4\pi i}{\omega}\cdot\frac{\tilde{\mathbf{a}}_l\left(\mathbf{j}\tilde{\mathbf{b}}^*_l\right)}{\dfrac{k^2c^2}{\omega^2} - \tilde{\varepsilon}_l(\mathbf{k}, \omega)}\, e^{i(\mathbf{kr}-\omega t)}d\mathbf{k}\, d\omega. \tag{5}$$

Using this expression, it is easy to carry out the transformation to an isotropic medium, in which case Eq. (1) assumes the form

$$\left(\tilde{\varepsilon}_l - \varepsilon_\perp\right)\tilde{\mathbf{a}}_\perp + \left(\tilde{\varepsilon}_l - \frac{k^2c^2}{\omega^2} - \varepsilon_\parallel\right)\tilde{\mathbf{a}}_\parallel = 0. \tag{6}$$

For the longitudinal field, in particular, assuming that $\tilde{\varepsilon}_\parallel = (k^2c^2/\omega^2) + \varepsilon_\parallel$, as follows from Eq. (6) and that $\tilde{\mathbf{a}}_\parallel\tilde{\mathbf{b}}^*_\parallel = 1$, as follows from Eq. (3a), we have

$$\mathbf{E}_\parallel(\mathbf{r}, t) = -\int \frac{4\pi i}{\omega}\cdot\frac{\mathbf{j}_\parallel}{\varepsilon_\parallel}\, e^{i(\mathbf{kr}-\omega t)}d\mathbf{k}\, d\omega,$$

which coincides with Eq. (6.40).

In the normalization of (3) the quantities $\tilde{\varepsilon}_l$, $\tilde{a}_\alpha$, and $\tilde{b}^*_\beta$ still depend on k even if spatial dispersion is neglected because the factor k^2c^2/ω^2 appears explicitly in Eq. (1). For this reason, Eq. (5) is found to be less convenient than Eq. (6.51) in calculations.

Problem 2. Find $\tilde{\varepsilon}_l$ (see preceding problem) for longitudinal wave propagation (θ = 0) and transverse propagation ($\theta = \pi/2$).

Solution. In longitudinal propagation ($n_x = n_y = 0$, $n_z = 1$), Eq. (1) becomes

$$\left(\tilde{\varepsilon} - \varepsilon_1\right)\tilde{a}_x - ig\tilde{a}_y = 0; \tag{1}$$

$$ig\tilde{a}_x + (\tilde{\varepsilon} - \varepsilon_1)\,\tilde{a}_y = 0;$$
$$(\tilde{\varepsilon} - N^2 - \eta)\,\tilde{a}_z = 0. \tag{1}$$

Setting the determinant equal to zero yields

$$\tilde{\varepsilon}_{1,2} = \varepsilon_1 \pm g;\ \tilde{\varepsilon}_3 = \frac{k^2c^2}{\omega^2} + \eta. \tag{2}$$

For transverse propagation ($n_x = 1$; $n_y = n_z = 0$; $\mathbf{B} = B_0\{0, 0, 1\}$), we have

$$\left.\begin{aligned} \left(\tilde{\varepsilon} - \frac{k^2c^2}{\omega^2} - \varepsilon_1\right) a_x - iga_y = 0;\\ iga_x + (\tilde{\varepsilon} - \varepsilon_2)\, a_y = 0;\\ (\tilde{\varepsilon} - \eta)\, a_z = 0. \end{aligned}\right\} \tag{3}$$

Hence

$$\tilde{\varepsilon}_{1,2} = \frac{\frac{k^2c^2}{\omega^2} + \varepsilon_1 + \varepsilon_2}{2} \pm \sqrt{\frac{\left(\frac{k^2c^2}{\omega^2} + \varepsilon_1 - \varepsilon_2\right)^2}{4} + g^2}\,; \tag{4}$$

$$\tilde{\varepsilon}_3 = \eta. \tag{5}$$

From the equation $k^2c^2/\omega^2 = \tilde{\varepsilon}_3 = \eta$ we determine the refractive index for the ordinary (transverse) wave. The equation $k^2c^2/\omega^2 = \tilde{\varepsilon}_{1,2}$ determines the refractive index for the extraordinary and "plasma waves." It is evident that this equation reduces to the following:

$$1)\ \frac{k^2c^2}{\omega^2} = \varepsilon_2 - \frac{g^2}{\varepsilon_1};$$

$$2)\ \frac{k^2c^2}{\omega^2} \to \infty;\ \varepsilon_{11} = 0.$$

The first of these equations governs the extraordinary wave and the second governs the plasma wave.

§ 7. Characteristic Oscillations of a Plasma and Propagation of Electromagnetic Waves in a Plasma

Let us assume that in a plasma at time $t = 0$ there is specified an electric field $\mathbf{E}(\mathbf{r})$ for which $\mathbf{E}(\mathbf{r}, t) = 0$ when $t < 0$. We wish to determine the

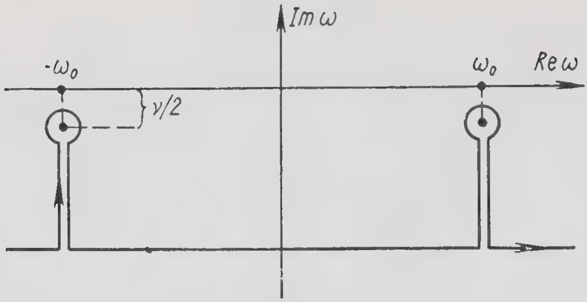

Fig. 14

ultimate fate of the electric field. In order to answer this question we must solve Maxwell's equations subject to the specified initial conditions. It is convenient to solve the problem by using the Laplace transform, which automatically takes account of the condition $\mathbf{E}(t) = 0$ when $t < 0$. When the Laplace transform technique is used, however, the frequency ω is replaced by the variable $p = -i\omega$. Since the tensor $\varepsilon_{\alpha\beta}$ is usually regarded as a function of ω, it is desirable to solve the problem with the given initial conditions using the Fourier method. In the Fourier method the initial condition can be taken into account by including the appropriate specified current in Maxwell's equations. It is evident that the appropriate current density is

$$\mathbf{j}_{\text{spec}} = -\frac{1}{4\pi}\mathbf{E}(\mathbf{r})\,\delta(t). \tag{7.1}$$

We actually assume that there are no other field sources so that $\mathbf{E} = 0$ when $t < 0$. Now, integrating the equation

$$\operatorname{rot}\mathbf{B} = \frac{4\pi}{c}\mathbf{j}(\mathbf{E}) + \frac{1}{c}\cdot\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_{\text{spec}} \tag{7.2}$$

with respect to t from $t = -\varepsilon$ to $t = +\varepsilon$ as $\varepsilon \to 0$, we find

$$\mathbf{E}(\mathbf{r}, t)_{t=+0} = -4\pi\int_{-\varepsilon}^{\varepsilon}\mathbf{j}_{\text{spec}}\,dt = \mathbf{E}(\mathbf{r}). \tag{7.3}$$

Thus, the solution of the inhomogeneous Maxwell's equation with the source given by (7.1) satisfies the specified initial condition.

The Fourier component of $\mathbf{j}_{\text{spec}}(\mathbf{r}, t)$ is

$$\mathbf{j}(\mathbf{k}', \omega) = -\frac{1}{4\pi(2\pi)^4}\int \mathbf{E}(\mathbf{r})\,e^{-i\mathbf{k}'\mathbf{r}}\,d\mathbf{r}. \tag{7.4}$$

In investigating the oscillations it is sufficient to consider a field of the form

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{ikz}. \tag{7.5}$$

The corresponding Fourier component of the specified current density is

$$\mathbf{j}(\mathbf{k}', \omega) = -\frac{1}{8\pi^2}\mathbf{E}_0 \delta(k - k')\,\delta(\mathbf{k}'_\perp). \tag{7.6}$$

Substituting this value of $\mathbf{j}(\mathbf{k}', \omega)$ in the solution of Maxwell's equations (6.40) and (6.51), we obtain an expression for the field in the form of an integral over $d\omega$. For an isotropic plasma

$$E_\parallel(z, t) = i\frac{E^0_\parallel}{2\pi} e^{ikz} \int\limits_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega\varepsilon_\parallel(k, \omega)}\, d\omega; \tag{7.7}$$

$$\mathbf{E}_\perp(z, t) = -i\frac{\mathbf{E}^0_\perp}{2\pi} e^{ikz} \int\limits_{-\infty}^{\infty} \frac{\omega e^{-i\omega t}}{k^2c^2 - \omega^2\varepsilon_\perp(k, \omega)}\, d\omega. \tag{7.8}$$

In an anisotropic medium, the electromagnetic field is determined by the initial conditions in accordance with Eqs. (6.51) and (7.6) through the following expression:

$$\mathbf{E}(z, t) = -\frac{i}{2\pi} e^{ikz} \sum_{l=1}^{2} \int\limits_{-\infty}^{\infty} \frac{\omega \mathbf{a}_l(\mathbf{E}_0 \mathbf{b}^*_l)}{k^2c^2 - \omega^2\varepsilon_\perp(k, \omega)}\, d\omega. \tag{7.9}$$

As an example of calculation of integrals of this kind we shall obtain $\mathbf{E}_\parallel(t)$ in the absence of thermal motion, in which case $\varepsilon = 1 - [\omega_{0e}^2/\omega(\omega + i\nu)]$ (we assume $\nu = 0$ in what follows). The integral in (7.7) can be computed easily by introducing the complex variable ω (Fig. 14). In the plane of this variable the integrand exhibits two poles; when $\nu \ll \omega_{0e}$, these poles are given by

$$\omega = \pm\omega_{0e} - i\frac{\nu}{2}. \tag{7.10}$$

For $t > 0$ we displace the path of integration far into the lower region. In this case the integration reduces to a circuit of the poles, since the integral along the horizontal portion $\operatorname{Im}\omega = -\infty$ vanishes. Thus, when $\nu \to 0$, taking account of the fact that the poles are traversed in the clockwise direction,

$$\mathbf{E}_{\parallel}(t) = i\,\frac{\mathbf{E}_0}{2\pi}\,e^{ikz}\int_{-\infty}^{\infty}\frac{1}{2}\left\{\frac{1}{\omega-\omega_{0e}+i\,\frac{\nu}{2}}+\frac{1}{\omega+\omega_{0e}+i\,\frac{\nu}{2}}\right\}e^{-i\omega t}d\omega =$$

$$= \mathbf{E}_0 e^{ikz}\cos\omega_{0e}t\,e^{-\frac{\nu}{2}t}. \tag{7.11}$$

For $t < 0$ the path of integration is moved into the upper region. Since there are no poles in this region, the integral reduces to an integral over the horizontal path $\operatorname{Im}\omega = +\infty$. Because of the factor $e^{-i\omega t}$ which appears in the integrand the integral vanishes when $t < 0$, i.e., the solution actually satisfies the initial condition.

When the thermal motion is taken into account, $\varepsilon(\mathbf{k}, \omega)$ becomes a transcendental function. The integrand can now exhibit an unlimited number of poles, including essential singularities and branch points. However, when long intervals of time are considered, the integral reduces to the residue for the pole whose imaginary part is the minimum in absolute value.

If the medium is not in statistical equilibrium the integrand can have a pole in the upper halfplane of the complex variable ω. In this case, the electromagnetic field increases exponentially in time and the Fourier expansion no longer applies. But in this case we can use analytic continuation of the integral into the region in which the Fourier method does not apply. The rule for computing the integral in the presence of poles in the upper halfplane follows from the condition that the electric field must be zero when $t < 0$. It follows from this condition that the integration contour must always go over the pole (Fig. 15). If the contour is displaced into the region $\operatorname{Im}\omega = +\infty$, we find, as in the chosen example, that $\mathbf{E} = 0$ when $t < 0$. We note that if the problem is solved by the Laplace transform technique, the field $\mathbf{E}(t)$ is expressed by an integral along the imaginary axis in the plane of the complex variable $p = -i\omega$. In this case the contour must go to the right of all poles; this rule corresponds to the rule indicated above, that the contour must go above all the poles in the plane of the complex variable ω.

It is clear that the problem of finding the characteristic oscillations reduces to that of finding the poles of the integrands, i.e., the solution of the appropriate dispersion equations. In the derivation of the dispersion equation, the frequency ω is regarded as a real quantity, regardless of the way in which the solution is obtained. This was done in § 5 in the investigation of the damping of plasma oscillations.

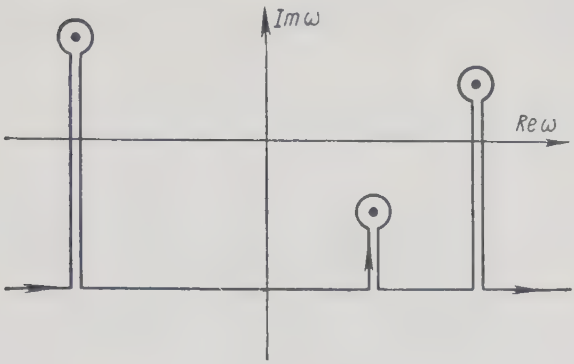

Fig. 15

We now turn to the problem of plasma oscillations in external fields. Let us consider the following idealized problem. Assume that we are given a uniform medium and that in the plane z = 0 we are given surface currents and charges that vary in time in accordance with $e^{-i\omega t}$. It is required to find the field at $z > 0$. As is well known, the surface currents and charges imply discontinuities in the tangential component of the magnetic field and the normal component of the electric field. We denote the magnetic field by $\mathbf{B}_1$ and the electric field by $\mathbf{E}_0$, these fields being taken at z = 0 immediately to the right of the plane of discontinuity; the quantity $\mathbf{n} = \{0, 0, 1\}$ denotes the normal to the surface. The tangential component of the magnetic field at z = +0 is $[\mathbf{n}\mathbf{B}_1]$ and the component at z = −0 is $-[\mathbf{n}\mathbf{B}_1]$. The boundary condition requires that $2[\mathbf{n}\,\mathbf{B}_1] = (4\pi/c)\mathbf{j}_{sur}$. The volume current density (concentrated at z = 0) which corresponds to this value of the surface current density is obviously

$$\mathbf{j}(z) = \frac{c}{2\pi}[\mathbf{n}\mathbf{B}_1]\,\delta(z). \tag{7.12}$$

Similarly, the volume space charge is expressed in terms of $\mathbf{E}_0$ by means of the expression

$$\varrho(z) = \frac{1}{2\pi}(\mathbf{n}\mathbf{E}_0)\,\delta(z). \tag{7.13}$$

This quantity can be included in the current density by invoking the relation $\operatorname{div}\mathbf{j} = \partial j_z/\partial z = i\omega\rho$.

Assuming that*

$$\int_0^z \delta(z')\,dz' = \frac{1}{2}\,\mathrm{Sgn}\,z, \tag{7.14}$$

we find the current density corresponding to the charge in Eq. (7.13):

$$j_z(z) = i\frac{\omega}{4\pi}(\mathbf{nE}_0)\,\mathrm{Sgn}\,z. \tag{7.15}$$

Thus the total specified current density can be expressed in terms of the boundary values of the fields as follows:

$$\mathbf{j}_{\text{spec}}(\mathbf{r},\ t) = \left\{\frac{c}{2\pi}\,[\mathbf{nB}_1]\,\delta(z) + i\frac{\omega}{4\pi}\,(\mathbf{nE}_0)\,\mathbf{n}\cdot\mathrm{Sgn}\,z\right\}e^{-i\omega t}. \tag{7.16}$$

The Fourier component $\mathbf{j}(\mathbf{k},\omega)$ is

$$\mathbf{j}(\mathbf{k},\ \omega') = \frac{c}{4\pi^2}\,[\mathbf{nB}_1]\,\delta(\omega-\omega')\,\delta(\mathbf{k}_\perp) +$$

$$+\frac{\omega}{4\pi^2}\,\mathbf{n}\,(\mathbf{nE}_0)\,\frac{P}{k_z}\,\delta(\omega-\omega')\,\delta(\mathbf{k}_\perp). \tag{7.17}$$

This form of the Fourier component $\mathbf{j}(\mathbf{k},\omega)$ is to be substituted in Eqs. (6.40) and (6.51), which express the field in terms of the specified current in isotropic and anisotropic media. For example, the longitudinal oscillations of an isotropic plasma are described by Eqs. (6.40) and (7.17), and we have

$$E_z(z,\ t) = -\frac{i}{\pi}E_0 e^{-i\omega t}\int_{-\infty}^{\infty}\frac{e^{ikz}}{k\varepsilon_\parallel(k,\ \omega)}\,dk. \tag{7.18}$$

In order to avoid dealing with the principal value of the integral we subtract and add a term $1/\varepsilon_0$ to $1/\varepsilon_\parallel$, where

$$\varepsilon_0 = \varepsilon_\parallel(k=0) = 1 - \frac{\omega_{0e}^2}{\omega^2}. \tag{7.19}$$

*This formula follows from the fact that the δ-function is even and from the condition $\int_{-z}^{z}\delta(z')\,dz' = 1$.

Using the formula for the Fourier integral expansion of the function Sgn z (6.26),

$$\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikz}}{k}\,dk = -\mathrm{Sgn}\, z, \tag{7.20}$$

from Eq. (7.18) we find for $z > 0$

$$\left.\begin{aligned} E(z,\ t) &= \frac{1}{\varepsilon_0}E_0 e^{-i\omega t} + E_1(z,\ t);\\ E_1(z,\ t) &= -\frac{i}{\pi}\frac{E_0 e^{-i\omega t}}{\varepsilon_0}\int_{-\infty}^{\infty}\frac{\varepsilon_0-\varepsilon_{\parallel}(k)}{\varepsilon_{\parallel}(k)}e^{ikz}\frac{dk}{k}. \end{aligned}\right\} \tag{7.21}$$

The principal-value symbol is removed from the integral because the point $k = 0$ is now automatically removed from the region of integration by virtue of the difference $\varepsilon_0 - \varepsilon_{\parallel}(k)$. The singularity of the integrals that appear in the expression for the electric field results because the integrand is not an analytic function of k, since $\varepsilon_{\parallel}(k)$ contains the modulus of k in its imaginary part (cf. §§ 5 and 10). Hence, the integral cannot be evaluated by residues, and the field is generally not of the form $e^{i(\omega/c)Nz}$, as would be the case in the absence of thermal motion. The field expression in (7.21) has been investigated by Landau [2]. When $z \to \infty$, the quantity E_1 vanishes and the field becomes the same as that in which thermal motion is neglected:

$$\underset{(z\to\infty)}{E(z,\ t)} \to \frac{E_0}{\varepsilon_0}e^{-i\omega t}. \tag{7.22}$$

The supplementary field $E_1(z, t)$ at $z \gg a = \sqrt{T_e/4\pi e^2 n_e}$, is given by

$$E_1(z) = \frac{2E_0}{\sqrt{3\varepsilon_0^2}}\left(\frac{\omega}{\omega_{0e}}\right)^{2/3}\left(\frac{z}{a}\right)^{3/2} e^{-\frac{3}{4}\left(\frac{\omega z}{\omega_0 a}\right)^{2/3}} e^{i\left[\frac{3\sqrt{2}}{4}\left(\frac{\omega z}{\omega_{0e}a}\right)^{2/3}+\frac{2}{3}\pi\right]}, \tag{7.23}$$

if the field frequency differs from the plasma frequency $\omega \neq \omega_{0e}$. Near resonance ($\omega \approx \omega_{0e}$) the result depends on the sign of the difference $\omega - \omega_{0e}$. When $\omega < \omega_{0e}$ and $\varepsilon_0 \ll 1$, the total field is given by

$$E(z) = \frac{E_0}{\varepsilon_0}\left(1 - e^{-\frac{z}{a}\sqrt{\frac{|\varepsilon_0|}{3}}}\right). \tag{7.24}$$

When $\omega > \omega_{0e}$ and $|\varepsilon_0| << 1$, we have

$$E(z) = \frac{E_0}{\varepsilon_0}\left[1 - \exp\left\{\frac{z}{a}\left(i\sqrt{\frac{\varepsilon_0}{3}} - \frac{3}{2}\sqrt{\frac{\pi}{2}}\right)\frac{e^{-\frac{3}{2\varepsilon_0}}}{\varepsilon_0^2}\right\}\right]. \quad (7.25)$$

Problem. At time t = 0 a charge e_1 is produced in the plasma. Find the electric field associated with this charge.

Solution. The production of the charge e_1 corresponds to a specified charge density described by the equation

$$\frac{\partial \varrho}{\partial t} = e_1 \delta(\mathbf{r})\, \delta(t),$$

or, using Fourier components, $-i\omega\rho(\mathbf{k}, \omega) = e_1/(2\pi)^4$. Thus, $\mathbf{j}_{\parallel} = (\mathbf{k}/k^2)(\mathbf{k}\mathbf{j}) = (\mathbf{k}/k^2)\,\omega\rho = i(\mathbf{k}/k^2)[e_1/(2\pi)^4]$. Substituting this expression for the longitudinal current in Eq. (6.40), we find the general expression for the electric field:

$$E(\mathbf{r}, t) = -\nabla\varphi;$$

$$\varphi(\mathbf{r}, t) = i\int \frac{4\pi e_1}{k^2\omega\varepsilon_{\parallel}(\mathbf{k}, \omega)}\, e^{i(\mathbf{kr}-\omega t)}\, \frac{d\omega\, d\mathbf{k}}{(2\pi)^4}.$$

The integral over ω reduces to a contour around the poles $\omega = 0$ and $\varepsilon_{\parallel}(\mathbf{k}, \omega_s) = 0$ in the counterclockwise direction:

$$\varphi(\mathbf{r}, t) = \int \frac{4\pi e_1}{k^2\varepsilon_{\parallel}(\mathbf{k}, 0)}\, e^{i\mathbf{kr}}\, \frac{d\mathbf{k}}{(2\pi)^3} + \sum_s \frac{4\pi e_1 \cdot e^{i(kr-\omega_s(k)t)}}{\omega_s k^2 \left[\frac{\partial \varepsilon_{\parallel}}{\partial \omega}\right]_{\omega=\omega_s}}\, \frac{d\mathbf{k}}{(2\pi)^3}.$$

The second term describes the establishment of the field. This term vanishes when $t \to \infty$. The first term, which results from the contour around the pole at $\omega = 0$, describes the static field of the charge in the plasma. Assuming that $\varepsilon_{\parallel}(k, 0) = 1 + (1/k^2d^2)$, where d is the Debye radius $[1/d^2 = \Sigma(4\pi e^2 n/T)$; cf. Eqs. (2.21) and (9.27)], and carrying out the integration over $d\mathbf{k}$, we find

$$\varphi(r, \infty) = e_1 \frac{e^{-\frac{r}{d}}}{r}.$$

§ 8. The Correlation Function for Microcurrents

The correlation function $G_{\alpha\beta}(\mathbf{R}, \tau)$ is important for a number of reasons: in a medium in equilibrium the tensor $\sigma_{\alpha\beta}(\mathbf{R}, \tau)$ is expressed in terms of this correlation function and, in the final analysis, the dielectric tensor derives from this function. If $\mathbf{j}_{spec}$ in Eq. (6.1) is taken to mean the microscopic current density (6.8) the solution of these equations yields a fluctuating field. The mean value of the field is evidently zero; however, there is a great deal of interest in the mean values of quadratic quantities which are associated with the energy, radiation intensity, mean-square fluctuation of the potential, etc.; these quantities do not vanish and are expressed in terms of the correlation function for the microcurrents. In a high-temperature plasma we will be interested in the correlation function for the microcurrents under the assumption that the interaction between charges can be neglected. This function can be computed easily if one knows the usual single-particle distribution function $F(\mathbf{p})$.

Let $\mathbf{r}_i^0$ denote the position of the i-th particle at time t = 0; the quantity $\mathbf{v}_i(t)$ denotes the velocity of this particle at time t. In a uniform medium, which we shall be concerned with here, the quantity $\mathbf{v}_i(t)$ depends only on the initial velocity $\mathbf{v}_i(0)$, but is independent of the initial position. The microscopic current density and charge density are then written in the form

$$\mathbf{j}^M(\mathbf{r}, t) = \sum_{i=1}^{N} e_i \mathbf{v}_i(t)\, \delta\left(\mathbf{r} - \mathbf{r}_i^0 - \int_0^t \mathbf{v}_i(t')\, dt'\right); \tag{8.1}$$

$$\varrho^M(\mathbf{r}, t) = \sum_{i=1}^{N} e_i \delta\left(\mathbf{r} - \mathbf{r}_i^0 - \int_0^t \mathbf{v}_i(t')\, dt'\right). \tag{8.2}$$

Here, the quantity N represents the total number of particles in the volume under consideration. From the definition of the current correlation function $G_{\alpha\beta}(\mathbf{R}, \tau) = \overline{j_\alpha^M(\mathbf{r}, t)\, j_\beta^M(\mathbf{r}', t')}$, we have

$$G_{\alpha\beta}(\mathbf{R}, \tau) = \overbrace{\int_V \cdots \int}^{N} \overbrace{\int_p \cdots \int}^{N} \sum_{i=1}^{N} \sum_{k=1}^{N} e_i e_k v_{i\alpha}(t)\, v_{k\beta}(t')\, \delta(\mathbf{r} - \mathbf{r}_i(t)) \times$$
$$\times \delta(\mathbf{r}' - \mathbf{r}_k(t'))\, D_0\, d\mathbf{r}_1^0 \ldots d\mathbf{r}_N^0\, d\mathbf{p}_1^0 \ldots d\mathbf{p}_N^0. \tag{8.3}$$

If the interaction between charges is neglected, the quantity D_0 is a product of Maxwellian distributions

$$D_0 = \frac{1}{V^N} \prod_{i=1}^{N} A_i e^{-\frac{\varepsilon_i}{T}}; \quad \varepsilon_i = \sqrt{m_{0i}^2 c^2 + c^2 p_{0i}^2};$$

$$A_i = \frac{m_{0i} c^2}{4\pi T \cdot K_2 \left(m_{0i} c^2 / T\right)}$$

and the only nonvanishing terms in the double summation will be those which contain products referring to the same charge (i = k), the coordinates and velocity of the given charge being independent of the coordinates and velocities of the other charges. Carrying out the integration by permutation of the particles (the number of permutations being N − 1), we obtain expressions of the form

$$\overbrace{\int_V \cdots \int}^{N-1} \overbrace{\int_{\mathbf{p}} \cdots \int}^{N-1} D_0 \, d\mathbf{r}_2^0 \ldots d\mathbf{r}_N^0 \, d\mathbf{p}_2^0 \ldots d\mathbf{p}_N^0 = \frac{1}{V} F(\mathbf{p}_0), \qquad (8.4)$$

where $F(\mathbf{p}_0)$ is the single-particle distribution function. We then carry out the integration over $d\mathbf{r}_1^0$ in the remaining expression and obtain the same kind of term for each particle species N_α (N_α is the total number of particles of species α). Denoting the density of particles of a given species by

$$n_\alpha = \frac{N_\alpha}{V}, \qquad (8.5)$$

we then obtain the correlation functions for the current density and charge density:

$$G_{00} = \overline{\varrho^{\text{M}}(\mathbf{r}, t)\, \varrho^{\text{M}}(\mathbf{r}', t')} - \sum e_\alpha^2 n_\alpha^2 = \\ = \sum_\alpha e_\alpha^2 n_\alpha \int \delta\left(\mathbf{R} - \mathbf{R}_\alpha(t, t')\right) F(\mathbf{p}_0)\, d\mathbf{p}_0; \qquad (8.6)$$

$$G_{\alpha\beta} = \sum_\alpha e_\alpha^2 n_\alpha \int v_\alpha(t)\, v_\beta(t')\, \delta\left(\mathbf{R} - \mathbf{R}_\alpha(t, t')\right) F(\mathbf{p}_0)\, d\mathbf{p}_0; \qquad (8.7)$$

$$\mathbf{R}_\alpha(t, t') = \int_{t'}^{t} \mathbf{v}_\alpha(t'')\, dt''. \qquad (8.8)$$

As an example we shall consider the simplest possible case — that of an idealized electron gas in the absence of a magnetic field. In this case,

$$\left.\begin{aligned} \mathbf{v}(t) &= \text{const} = \mathbf{v}_0; \\ \mathbf{R}(t, t') &= \mathbf{v}_0 (t - t'). \end{aligned}\right\} \tag{8.9}$$

The following transformation of variables is carried out in the integrands in Eqs. (8.6) and (8.7):

$$\delta(\mathbf{R} - \mathbf{v}_0(t - t'))\, d\mathbf{p}_0 = \frac{\delta\left(\frac{\mathbf{R}}{\tau} - \mathbf{v}_0\right)}{|\tau|^3} \left(\frac{\varepsilon}{c^2}\right)^2 d\mathbf{v}_0, \tag{8.10}$$

where

$$\varepsilon = \frac{m_0 c^2}{\sqrt{1 - \frac{v_0^2}{c^2}}}, \quad \tau = t - t'.$$

The integration over $\mathbf{v}_0$ yields null values for the distances and time intervals that satisfy the condition $\mathbf{R} > c|\tau|$ and reduces to the replacement of $\mathbf{v}_0$ by $\mathbf{R}/|\tau|$ in the factor that multiplies the δ-function when $R < c\tau$. Thus, for a Maxwellian distribution

$$F(\mathbf{p}_0)\, dp_0 = A e^{-\frac{\varepsilon}{T}} d\mathbf{p}_0; \quad A = \frac{m_0 c^2}{4\pi T \cdot K_2\left(\frac{m_0 c^2}{T}\right)} \tag{8.11}$$

we find $G_{00} = G_{\alpha\beta} = 0$ when $R > c|\tau|$, and

$$\left.\begin{aligned} G_{\alpha\beta} &= \frac{R_\alpha R_\beta}{\tau^2} G_{00}; \\ G_{00} &= \frac{e^2 n A}{(c^2\tau^2 - R^2)^{3/2}} e^{-\frac{m_0 c^2}{T} \frac{c|\tau|}{(c^2\tau^2 - R^2)^{1/2}}}; \end{aligned}\right\} R \leqslant c|\tau|. \tag{8.12}$$

In the nonrelativistic case these expressions become

$$G_{\alpha\beta} = \frac{R_\alpha R_\beta}{\tau^2} G_{00} = e^2 n \frac{R_\alpha R_\beta}{|\tau|^5} \left(\frac{m}{2\pi T}\right)^{3/2} e^{-\frac{mR^2}{2T\tau^2}}. \tag{8.13}$$

When $T \to 0$, the limiting value of the function $G_{\alpha\beta}(\mathbf{R}, \tau)$ is

$$G_{\alpha\beta}(\mathbf{R}, \tau) = T \cdot \frac{e^2 n}{m} \delta(\mathbf{R})\, \delta_{\alpha\beta}. \tag{8.14}$$

These formulas can be more easily obtained directly from Eq. (8.7) by writing $\mathbf{R}(t) = 0$.

In calculations the space–time correlation function for the currents is not as important as the correlation function for the Fourier components of the microcurrents; the latter quantity [in accordance with the definition of the Fourier components for the current density (6.14a)] is given by

$$\overline{j_\alpha^M(\mathbf{k},\omega)\, j_\beta^{*M}(\mathbf{k}',\omega')} = \frac{1}{(2\pi)^8}\int G_{\alpha\beta}(\mathbf{R},\tau)\times$$

$$\times e^{-i(\mathbf{kR}-\omega\tau)+i[(\mathbf{k}'-\mathbf{k})\mathbf{r}'-(\omega'-\omega)t']}\,d\mathbf{r}\,d\mathbf{r}'\,dt\,dt'. \qquad (8.15)$$

Converting from the variables $\mathbf{r}$, $\mathbf{r}'$, t, t' to the variables $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $\mathbf{r}'$, $\tau = t - t'$, t', we find

$$\overline{j_\alpha^M(\mathbf{k},\omega)\, j_\beta^{*M}(\mathbf{k}',\omega')} = G_{\alpha\beta}(\mathbf{k},\omega)\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega'), \quad (8.16)$$

where $G_{\alpha\beta}(\mathbf{k},\omega)$ is found to be nothing more than the Fourier component of $G_{\alpha\beta}(\mathbf{R},\tau)$:

$$G_{\alpha\beta}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int G_{\alpha\beta}(\mathbf{R},\tau)\, e^{-i(\mathbf{kR}-\omega\tau)}\,d\mathbf{R}\,d\tau. \qquad (8.17)$$

In a uniform medium under stationary conditions, in addition to defining a correlation function for the microcurrents it is also possible to introduce correlation functions for other quantities of interest – for example, $\overline{E_\alpha(\mathbf{r},t)E_\beta(\mathbf{r}',t')}$, $\overline{E_\alpha(\mathbf{r},t)H_\beta(\mathbf{r}',t')}$, etc.; by virtue of the uniformity and stationarity, are functions of $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\tau = t - t'$. For reasons of brevity we use the notation

$$(E_\alpha H_\beta)_{\mathbf{R},\tau} \equiv \overline{E_\alpha(\mathbf{r},t)\,H_\beta(\mathbf{r}',t')}. \qquad (8.18)$$

Proceeding by analogy with Eq. (8.16), we write

$$\overline{E_\alpha(\mathbf{k},\omega)\,H_\beta^*(\mathbf{k}',\omega')} = (E_\alpha H_\beta^*)_{\mathbf{k},\omega}\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega'), \quad (8.19)$$

where

$$(E_\alpha H_\beta^*)_{\mathbf{k},\omega} = \frac{1}{(2\pi)^4}\int (E_\alpha H_\beta)_{\mathbf{R},\tau}\, e^{-i(\mathbf{kR}-\omega\tau)}\,d\mathbf{R}\,d\tau; \qquad (8.20)$$

$$(E_\alpha H_\beta)_{\mathbf{R},\tau} = \int (E_\alpha H_\beta^*)_{\mathbf{k},\omega}\, e^{i(\mathbf{kR}-\omega\tau)}\,d\mathbf{k}\,d\omega. \qquad (8.21)$$

In accordance with the definition in Eq. (8.18), the quantity $(E_\alpha H_\beta)_{00}$ is simply the mean value of the product $E_\alpha H_\beta$. On the basis of Eq. (8.21), when $\mathbf{R} = \tau = 0$ we have

$$(E_\alpha H_\beta)_{00} = \int (E_\alpha H_\beta^*)_{\mathbf{k},\,\omega}\, d\mathbf{k}\, d\omega \tag{8.22}$$

and the quantity $(E_\alpha H_\beta^*)_{\mathbf{k},\,\omega}$ is called the spectral density of the mean product $E_\alpha H_\beta$. The latter expression can also be written in the form

$$(E_\alpha H_\beta)_{00} = \frac{1}{2} \int \left\{ (E_\alpha H_\beta^*)_{\mathbf{k},\,\omega} + (E_\alpha^* H_\beta)_{\mathbf{k},\,\omega} \right\} d\mathbf{k}\, d\omega. \tag{8.23}$$

For example, the mean energy absorbed by a medium per unit of time is given by

$$\overline{\mathbf{j}(\mathbf{r},\, t)\, \mathbf{E}(\mathbf{r},\, t)} = \int \sigma'_{\alpha\beta}(\mathbf{k},\, \omega)\, (E_\alpha^* E_\beta)_{\mathbf{k},\,\omega}\, d\mathbf{k}\, d\omega. \tag{8.24}$$

This expression for the absorbed energy, which makes use of $\sigma'_{\alpha\beta}(\mathbf{k}, \omega)$, is valid for strong absorption, in contrast with Eq. (1.10) which is not.

Let us now again consider the correlation function for the microcurrents.

In a medium in static equilibrium, it is evident that the functions $G_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ are related by

$$G_{\alpha\beta}(\mathbf{R},\, \tau) = T\sigma_{\alpha\beta}(\mathbf{R},\, \tau). \tag{8.25}$$

Comparison of $G_{\alpha\beta}(\mathbf{k}, \omega)$ and $\sigma'_{\alpha\beta}(\mathbf{k}, \omega)$ [Eq. (6.19)] taking account of this relation allows us to obtain a relation between the Fourier component of $G_{\alpha\beta}$ and the anti-Hermitian part of the dielectric tensor $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega)$:

$$G_{\alpha\beta}(\mathbf{k},\, \omega) = \frac{2T}{(2\pi)^4}\, \sigma'_{\alpha\beta}(\mathbf{k},\, \omega) = \frac{T}{(2\pi)^5}\, \omega\varepsilon''_{\alpha\beta}(\mathbf{k},\, \omega). \tag{8.26}$$

The function $G_{\alpha\beta}(\mathbf{k}, \omega)$ can be computed either as the Fourier component of $G_{\alpha\beta}(\mathbf{R}, \tau)$ or directly, starting from the expression for the Fourier component of the microscopic current density:

$$j_\alpha^{\mathrm{M}}(\mathbf{k},\, \omega) = \frac{1}{(2\pi)^4} \int \sum_{i=1}^{N} e_i v_{i\alpha}(t)\, \delta\left(\mathbf{r} - \mathbf{r}_{i0} - \int_0^t \mathbf{v}(t')\, dt'\right) e^{-i(\mathbf{kr}-\omega t)}\, d\mathbf{r}\, dt. \tag{8.27}$$

By virtue of the δ-function, the integration over $\mathbf{r}$ can be carried out easily and the statistical mean of the product of two spectral components of the current density (under the assumption of uncorrelated charges) can be obtained using Eqs. (8.4) and (8.5):

$$\overline{j_\alpha^{\mathrm{M}}(\mathbf{k},\ \omega)\, j_\beta^{\mathrm{M}}(\mathbf{k}',\ \omega')} = \sum \frac{e^2 n}{(2\pi)^8} \int v_\alpha(t)\, v_\beta(t') \times$$

$$\times e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{r}_0 - i\left[\mathbf{k}\int_0^t \mathbf{v}dt - \mathbf{k}'\int_0^{t'} \mathbf{v}\,dt\right] - i\omega t + i\omega' t'} F(\mathbf{p}_0)\, dt\, dt'\, dp_0\, d\mathbf{r}_0 \quad (8.28)$$

(the summation is carried out over all particle species). The integration over $\mathbf{r}_0$ gives the δ-function $(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$, so that we can write $\mathbf{k}' = \mathbf{k}$ in the integrand. The factor $v_\alpha(t)\, v_\beta(t')\, e^{-i\mathbf{k}\int_{t'}^{t} \mathbf{v}(t'')dt''}$ which is obtained in this way is found to depend only on the difference $\tau = t - t'$ after averages are taken over the initial velocities, since the time t' does not appear. Introducing the variables τ and t' in place of t and t', and carrying out the integration over t', we obtain another δ-function: $2\pi\delta(\omega - \omega')$. In the remaining integral over τ the integrand depends only on the difference $\tau = t - t'$, so that t' = 0. Thus we obtain an expression of the form in Eq. (8.16) where $G_{\alpha\beta}(\mathbf{k},\ \omega)$ is given by the expression

$$G_{\alpha\beta}(\mathbf{k},\ \omega) = \sum \frac{e^2 n}{(2\pi)^4} \int v_\alpha(t)\, v_\beta(0)\, e^{i\left(\omega t - \mathbf{k}\int_0^t \mathbf{v}(t')dt'\right)} F(\mathbf{p}_0)\, d\mathbf{p}_0\, dt. \quad (8.29)$$

Here, the variable of integration τ has been replaced by t. This expression for the spectral correlation function for the microcurrents holds for any arbitrary particle distribution function that satisfies the conditions of uniformity and stationarity for the unperturbed medium (subject to our initial assumption that the interaction between charges can be neglected).

We now transform this expression for the case of motion in a uniform magnetic field.

Let $k_x = k_\perp \cos\varkappa$ and $k_y = k_\perp \sin\varkappa$. Using the law of motion (4.6), we find

$$\mathbf{k}\int_0^t \mathbf{v}(t')dt' = \frac{k_\perp v_\perp}{\omega_B}\left[\sin(\omega_B t + \varphi_0 + \varkappa) - \sin(\varphi_0 + \varkappa)\right] + k_\parallel v_\parallel t. \quad (8.30)$$

The function $G_{\alpha\beta}(\mathbf{k},\ \omega)$ can be written in the form of an expansion, the individual terms of which correspond to the resonances considered in § 4. We use the expansion (cf. Appendix I)

$$e^{-i\frac{k_\perp v_\perp}{\omega_B}\sin(\omega_B t+\varphi_0+\varkappa)} = \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_\perp v_\perp}{\omega_B}\right) e^{-in(\omega_B t+\varphi_0+\varkappa)} \qquad (8.31)$$

We now introduce a cylindrical coordinate system in momentum space in which $d\mathbf{p} = p_\perp dp_\perp dp_\| d\varphi_0$. Expressing the velocity components in the integral in (8.29), in the form

$$v_x(t) = v_\perp \cos(\omega_B t + \varphi_0) = \frac{v_\perp}{2}\left\{e^{i(\omega_B t+\varphi_0)} + e^{-i(\omega_B t+\varphi_0)}\right\} \qquad (8.32)$$

etc., we obtain (taking account of the axial symmetry of the distribution function) integrals over $d\varphi_0$ and dt in the form

$$\int_0^{2\pi} e^{-i(n\mp m)(\varphi_0+\varkappa)+i\frac{k_\perp v_\perp}{\omega_B}\sin(\varphi_0+\varkappa)} d\varphi_0 = 2\pi J_{n\mp m}\left(\frac{k_\perp v_\perp}{\omega_B}\right), \qquad (8.33)$$

$$\int_{-\infty}^{\infty} e^{i\{\omega-k_\| v_\|-(n\mp p)\omega_B\}t}\, dt = 2\pi\delta(\omega - k_\| v_\| - (n\mp p)\omega_B), \qquad (8.34)$$

where m and p are integers. We now replace the summation index n by $n \pm p$ so that the arguments of the δ-functions have the form $\omega - k_\| v_\| - n\omega_B$ and, after some simple but tedious transformations,

$$G_{\alpha\beta}(\mathrm{k}, \omega) = \sum \frac{e^2 n}{4\pi^2} \int \sum_{n=-\infty}^{\infty} \delta(\omega - k_\| v_\| - n\omega_B) \times$$
$$\times \Pi_{\alpha\beta}^{(n)} F(\mathbf{p})\, p_\perp\, dp_\perp\, dp_\| . \qquad (8.35)$$

Here, $\omega_B = (eB/m_0c^2)\sqrt{1-\beta^2}$, while the quantity $\Pi_{\alpha\beta}^{(n)}$ denotes the following combinations:

$$\left.\begin{aligned}
\Pi_{xx}^{(n)} &= \frac{v_\perp^2}{4}\left\{J_{n+1}^2 + J_{n-1}^2 + 2J_{n+1}J_{n-1}\cos 2\varkappa\right\};\\
\Pi_{yy}^{(n)} &= \frac{v_\perp^2}{4}\left\{J_{n+1}^2 + J_{n-1}^2 - 2J_{n+1}J_{n-1}\cos 2\varkappa\right\};\\
\Pi_{xy}^{(n)} &= \Pi_{yx}^{(n)*} = -i\,\frac{v_\perp^2}{4}\left\{J_{n+1}^2 - J_{n-1}^2 + 2iJ_{n+1}J_{n-1}\sin 2\varkappa\right\};
\end{aligned}\right\} \qquad (8.36)$$

$$\left.\begin{aligned}\Pi_{xz}^{(n)} &= \Pi_{zx}^{(n)*} = \frac{v_\perp v_\parallel}{2}\{J_n J_{n+1} e^{-i\varkappa} + J_n J_{n-1} e^{i\varkappa}\};\\ \Pi_{yz}^{(n)} &= \Pi_{zy}^{(n)*} = i\frac{v_\perp v_\parallel}{2}\{J_n J_{n+1} e^{-i\varkappa} - J_n J_{n-1} e^{i\varkappa}\};\\ \Pi_{zz}^{(n)} &= v_\parallel^2 J_n^2.\end{aligned}\right| \tag{8.36}$$

In all Bessel functions the argument is $k_\perp v_\perp/\omega_B$.

Problem. Find the correlation function $G_{\alpha\beta}(\mathbf{R}, \tau)$ for particles of a given species in a plasma in a uniform magnetic field.

Solution. Using the law of motion in a uniform magnetic field $\mathbf{B}_0 = \{0, 0, B_0\}$, we obtain the following projections of the vector $\mathbf{R}(\tau)$ in Eq. (8.7):

$$X_1(\tau) = \frac{p_\perp c}{eB_0}\left[\sin(\omega_B t + \varphi_0) - \sin(\omega_B t' + \varphi_0)\right] = R_1^\perp \cos\varphi_1;$$

$$Y_1(\tau) = \frac{p_\perp c}{eB_0}\left[\cos(\omega_B t + \varphi_0) - \cos(\omega_B t' + \varphi_0)\right] = -R_1^\perp \sin\varphi_1;$$

$$Z_1(\tau) = \frac{p_\parallel c^2}{\varepsilon}(t - t').$$

Here,

$$\omega_B = \frac{eB_0 c}{\varepsilon};\quad R_1^\perp = \frac{cp_\perp}{eB_0} 2\sin\frac{\omega_B(t-t')}{2};$$

$$\varphi_1 = \varphi_0 + \frac{\omega_B}{2}(t + t');\quad \mathrm{tg}\,\varphi_0 = \frac{p_y}{p_x}.$$

We now convert from the variables p_x, p_y, and p_z to the variables $R_1^\perp$, φ_1, and Z_1:

$$\delta(X - X_1)\,\delta(Y - Y_1)\,\delta(Z - Z_1)\,dp_x dp_y dp_z =$$

$$= \delta\left(R_\perp - R_1^\perp\right)\frac{\delta(\varphi - \varphi_1)}{R_1^\perp}\,\delta(Z - Z_1)\frac{R_1^\perp dR_1^\perp dZ_1 d\varphi_1}{\left[\dfrac{2c}{eB_0}\sin\dfrac{\omega_B(t-t')}{2}\right]^2 \dfrac{c^2(t-t')^2}{\varepsilon}}.$$

Carrying out the simple integration for the condition $R_\perp^2 + Z^2 < c^2\tau^2$ we have

$$G_{\alpha\beta} = v_\alpha(t)\,v_\beta(t')\cdot G_{00} = v_\alpha(t)\,v_\beta(t')\,e^2 n\,\frac{e^2B^2}{4c^2}\,\frac{\varepsilon(\mathrm{p})\,F(\mathrm{p})}{c^2|\tau|\sin^2\dfrac{\omega_B\tau}{2}}.$$

The condition $R_\perp^2 + Z^2 < c^2\tau^2$ follows from the fact that the particle velocity is bounded ($v \le c$). If this condition is not satisfied then $G_{\alpha\beta} = 0$. In the equation for $G_{\alpha\beta}$, the momentum must be expressed in terms of $\mathbf{R}_\perp$ and τ from the conditions $Z = Z_1$ and $R_\perp = R_1^\perp$; this procedure leads to the equations

$$p_\parallel^2 = \left(m_0c^2 + p_\perp^2\right)\frac{Z^2}{c^2\tau^2 - Z^2};$$

$$\sin\frac{eB_0}{2c}\sqrt{\frac{c^2\tau^2 - Z^2}{m_0^2c^2 + p_\perp^2}} = \frac{eB_0}{2c}\frac{R_\perp}{p_\perp}.$$

The components $v_\alpha(t)$ and $v_\beta(t')$ are expressed in terms of $\mathbf{R}$ and τ by means of the equations of motion:

$$v_x(t) = \frac{\omega_B Y}{2} + \frac{\omega_B X}{2}\operatorname{ctg}\frac{\omega_B}{2}\tau; \quad v_x(t') = -\frac{\omega_B Y}{2} + \frac{\omega_B X}{2}\operatorname{ctg}\frac{\omega_B}{2}\tau;$$

$$v_y(t) = -\frac{\omega_B X}{2} + \frac{\omega_B Y}{2}\operatorname{ctg}\frac{\omega_B}{2}\tau; \quad v_y(t') = \frac{\omega_B X}{2} + \frac{\omega_B Y}{2}\operatorname{ctg}\frac{\omega_B}{2}\tau;$$

$$v_z(t) = \frac{Z}{\tau}; \quad v_z(t') = -\frac{Z}{\tau}.$$

In the nonrelativistic approximation, $G_{\alpha\beta}$ becomes

$$G_{\alpha\beta} = v_\alpha(t)\,v_\beta(t')\,e^2n\left(\frac{m}{2\pi T}\right)^{3/2}\frac{\omega_B^2}{4}\,\frac{e^{-\frac{m}{2T}\left(\frac{\omega_B^2R_\perp^2}{4\sin^2\frac{\omega_B}{2}\tau} + \frac{Z^2}{\tau^2}\right)}}{|\tau|\sin^2\frac{\omega_B}{2}\tau}.$$

§ 9. Electrical Permittivity

The electrical permittivity of a medium in statistical equilibrium can be expressed in terms of the correlation function for the microcurrents $G_{\alpha\beta}(\mathbf{k}, \omega)$. For this purpose we express $\varepsilon''_{\alpha\beta}$ in terms of $G_{\alpha\beta}$ [Eq. (8.26)] and use the Kramers–Kronig relation (6.27), employing Eq. (1.8) to express $\varepsilon'_{\alpha\beta}$ in terms of $\varepsilon''_{\alpha\beta}$. However, the desired expression can be obtained in much simpler fashion by comparing Eqs. (6.17), (8.17), and (8.29) for $\sigma_{\alpha\beta}(\mathbf{k}, \omega)$ and $G_{\alpha\beta}(\mathbf{k}, \omega)$. In view of the fact that $\sigma_{\alpha\beta}(\mathbf{R}, \tau) = (1/T) \cdot G_{\alpha\beta}(\mathbf{R}, \tau)$ we find that $\sigma_{\alpha\beta}(\mathbf{k}, \omega)$ is expressed by an integral of the same form as $G_{\alpha\beta}(\mathbf{k}, \omega)$, the only difference being the factors and the limits of integration. In order to obtain $\sigma_{\alpha\beta}(\mathbf{k}, \omega)$ in Eq. (8.17) for $G_{\alpha\beta}(\mathbf{k}, \omega)$, it is necessary to integrate with respect to τ [and in Eq. (8.29) correspondingly with respect to t] and to replace the limits $-\infty$, $+\infty$ by the limits 0 and $+\infty$. It is also necessary to introduce the factor $(2\pi)^4/T$. In this way we find that the tensor $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega) = \delta_{\alpha\beta} + i[4\pi\sigma_{\alpha\beta}(\mathbf{k}, \omega)/\omega]$ is given by [17]:

$$\varepsilon_{\alpha\beta}(\mathbf{k}, \omega) = \delta_{\alpha\beta} + i\sum \frac{4\pi e^2 n}{\omega T} \times$$
$$\times \left\langle \int_0^\infty v_\alpha(t)\, v_\beta(0)\, e^{\left(i\left(\omega t - \mathbf{k}\int_0^t \mathbf{v}(t')\,dt'\right)\right)} dt \right\rangle, \tag{9.1}$$

where the angle brackets denote an average over an equilibrium (Maxwellian) distribution function $F(\mathbf{p}_0)$.

The assumption that the distribution is an equilibrium distribution is expressed in the fact that we have used the condition $\partial D_0/\partial p_i = (\mathbf{v}_i/T)D_0$ in finding the distribution function D_1 (§ 6); in particular, this means that the magnetic field of the wave is neglected. To derive a formula for $\varepsilon_{\alpha\beta}$ without this limitation it is convenient to start directly from the kinetic equation for the single-particle distribution function

$$\frac{\partial f^{(1)}}{\partial t} + [H^0 f^{(1)}] = -e\left\{\mathbf{E} + \frac{1}{c}[\mathbf{vB}]\right\}\frac{\partial F_0}{\partial \mathbf{p}}, \tag{9.2}$$

where H^0 is the Hamiltonian for the particle in the absence of the electromagnetic wave.

The solution of this equation can be written in general form if we convert from the variables $\mathbf{r}$, $\mathbf{v}$, and t to the variables $\mathbf{r}_0$, $\mathbf{v}_0$, and t; the relation between these variables is given by the unperturbed motion of the particle:

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{v}(t')\,dt'. \tag{9.3}$$

In terms of these variables the kinetic equation becomes

$$\frac{\partial f^{(1)}}{\partial t} = -e\left\{\mathbf{E}(\mathbf{r}(t), t) + \frac{1}{c}[\mathbf{v}(t)\,\mathbf{B}(\mathbf{r}(t), t)]\right\}\frac{\partial F_0}{\partial \mathbf{p}(t)}. \tag{9.4}$$

As before, we assume that the electromagnetic wave vanishes at $t = -\infty$ and, correspondingly, $f^{(1)} = 0$ at $t = -\infty$. The solution of this equation is then written in the form

$$f^{(1)} = -e\int_{-\infty}^{t}\left\{\mathbf{E}(\mathbf{r}(t'), t') + \frac{1}{c}[\mathbf{v}(t')\,\mathbf{B}(\mathbf{r}(t'), t')]\right\}\frac{\partial F_0}{\partial \mathbf{p}(t')}\,dt'. \tag{9.5}$$

The current density, averaged over the initial coordinates $\mathbf{r}_0$ and momenta $\mathbf{p}(0)$, can be written in the form

$$j_\alpha(\mathbf{r},\ t) = \sum en \int v_\alpha(t)\, \delta(\mathbf{r} - \mathbf{r}(t))\, f^{(1)}(\mathbf{r}_0,\ \mathbf{p}_0)\, d\mathbf{r}_0\, d\mathbf{p}_0. \tag{9.6}$$

Let us now consider this expression with regard to the individual spectral harmonics. In this case, $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{kr}-\omega t)}$, and $\mathbf{B} = (c/\omega)[\mathbf{kE}]$ and the expression in the curly brackets in Eq. (9.5) becomes

$$\left\{\mathbf{E}_0\left(1 - \frac{\mathbf{kv}(t')}{\omega}\right) + \frac{\mathbf{k}(\mathbf{E}_0\mathbf{v}(t'))}{\omega}\right\} e^{i\left(\mathbf{kr}_0 + \mathbf{k}\int\limits_0^{t'} \mathbf{v}(t'')\,dt'' - \omega t'\right)}. \tag{9.7}$$

Now, substituting the expression for $f^{(1)}$ in Eq. (9.6) and carrying out the integration over $\mathbf{r}_0$ [$\mathbf{r}_0$ appears in the argument of the δ-function through $\mathbf{r}(t)$], we find

$$j_\alpha(\mathbf{r},\ t) =$$

$$= -\sum e^2 n E_{0\beta} e^{i(\mathbf{kr}-\omega t)} \int\limits_{-\infty}^{t} v_\alpha(t) \frac{\partial F_0}{\partial p_\gamma(t')} \times$$

$$\times \left\{\left(1 - \frac{\mathbf{kv}(t')}{\omega}\right)\delta_{\gamma\beta} + \frac{k_\gamma v_\beta(t')}{\omega}\right\} e^{i\left(\omega\tau - \mathbf{k}\int\limits_{t'}^{t} \mathbf{v}(t'')\,dt''\right)} dt'\, d\mathbf{p}_0. \tag{9.8}$$

After the integration over velocity is carried out, the expression in the integral with respect to t' can depend on t and t' only through the difference $\tau = t - t'$ (since t' does not appear explicitly). Thus, we have an integral of the form

$$\int\limits_{-\infty}^{t} f(t - t')\,dt' = -\int\limits_{\infty}^{0} f(\tau)\,d\tau = \int\limits_{0}^{\infty} f(t)\,dt. \tag{9.9}$$

It then follows that we can write t' = 0 in the integral over j_α and the integration over t is carried out from 0 to ∞. Comparing Eq. (9.8) with the relation $j_\alpha = \sigma_{\alpha\beta} E_\beta$, we find $\sigma_{\alpha\beta}(\mathbf{k}, \omega)$ and the tensor $\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + i(4\pi\sigma_{\alpha\beta}/\omega)$:

$$\varepsilon_{\alpha\beta}(\mathbf{k},\ \omega) = \delta_{\alpha\beta} - i\sum \frac{4\pi e^2 n}{\omega} \int d\mathbf{p}_0 \int v_\alpha(t)\left[\left(1 - \frac{\mathbf{kv}_0}{\omega}\right)\frac{\partial F_0}{\partial p_\beta^0} + \right.$$

$$\left. + \frac{v_\beta^0}{\omega}\mathbf{k}\frac{\partial F_0}{\partial \mathbf{p}_0}\right] e^{i\left(\omega t - \mathbf{k}\int\limits_0^t \mathbf{v}(t')\,dt'\right)} dt. \tag{9.10}$$

If $-\partial F_0/\partial \mathbf{p} = \mathbf{v} F_0/T$, Eq. (9.1) is obtained immediately.

Let us now compute $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega)$ for a uniform isotropic plasma in the absence of a magnetic field. In this case, $\mathbf{v}(t) = \text{const} = \mathbf{v}_0$. We choose a coordinate system in which the wave vector is directed along the z axis: $\mathbf{k} = \{0, 0, k\}$. The tensor $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega)$ is then given by

$$\varepsilon_{\alpha\beta}(\mathbf{k}, \omega) = \delta_{\alpha\beta} + i\left\langle \sum \frac{4\pi e^2 n}{\omega T} \int_0^\infty v_\alpha^0 v_\beta^0 e^{i\left(\omega - kv_z^0\right)t}\, dt \right\rangle. \quad (9.11)$$

Hereinafter, in computing the tensor $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega)$, we shall frequently encounter an integral of the form $\zeta(x) = -i\int_0^\infty e^{ixt}\,dt$. In computing this integral we introduce a factor $e^{-\nu t}$ in the integrand, later letting ν go to zero. We have

$$-i\int_0^\infty e^{ixt-\nu t}\,dt = \frac{1}{x+i\nu} = \frac{x}{x^2+\nu^2} - i\frac{\nu}{x^2+\nu^2}. \quad (9.12)$$

We have already encountered the first term [cf. Eq. (6.23)]. When $\nu \to 0$ this term becomes the principal value $1/x$. When $\nu \to 0$ the second term becomes a δ-function multiplied by π:

$$\lim_{\nu\to 0}\frac{x}{x^2+\nu^2} = \frac{P}{x}; \quad \lim_{\nu\to 0}\frac{\nu}{x^2+\nu^2} = \pi\delta(x). \quad (9.13)$$

Thus

$$\zeta(x) = -i\int_0^\infty e^{ixt}\,dt = \frac{P}{x} - i\pi\delta(x). \quad (9.14)$$

The sign of the imaginary part of the function $\zeta(x)$ is uniquely related to the fact that the integration over t is carried out from 0 to ∞; in the final analysis, this is related to the principle of causality, which is expressed by the fact that the current $\mathbf{j}(t)$ is determined by the values of $\mathbf{E}(t')$ at earlier times only. Taking account of (9.13), we now find from Eq. (9.11):

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_\perp = 1 - \sum \frac{4\pi e^2 n}{m\omega}\langle \zeta(\omega - kv_z)\rangle; \quad (9.15)$$

$$\varepsilon_{zz} = \varepsilon_{\parallel} = 1 - \sum \frac{4\pi e^2 n}{m\omega} \left\langle \frac{mv_z^2}{T} \zeta(\omega - kv_z) \right\rangle. \tag{9.16}$$

The remaining elements of $\varepsilon_{\alpha\beta}$ vanish. We note here that in the analysis of plasma oscillations in the presence of beams the distribution function for the particles in the beam is usually assumed to be a Maxwellian distribution, shifted to some beam velocity v_0 [3]:

$$f(v_z)\,dv_z = \left(\frac{m}{2\pi T}\right)^{1/2} e^{-\frac{m(v_z - v_0)^2}{2T}}\,dv_z. \tag{9.17}$$

If the effect of the magnetic field produced by the beam charges is neglected, the medium consisting of the plasma plus the beam can be regarded as uniform. The component $\varepsilon_{\perp}$ in the dielectric tensor is not changed while the component $\varepsilon_{\parallel}$ is evidently given by

$$\varepsilon_{\parallel} = 1 - \sum \frac{\omega_0^2}{\omega} \left\langle \frac{m(v_z - v_0)\,v_z}{T} \zeta(\omega - kv_z) \right\rangle. \tag{9.18}$$

The imaginary parts of $\varepsilon_{\perp}$ and $\varepsilon_{\parallel}$ can be computed easily, since

$$\int_{-\infty}^{\infty} F(v_z)\,\delta(\omega - kv_z)\,dv_z = \frac{1}{k} F\left(\frac{\omega}{k}\right). \tag{9.19}$$

The real parts are usually computed approximately.

We show in Appendix II that for a Maxwellian velocity distribution (in the nonrelativistic case)

$$\langle \zeta(\omega - kv) \rangle = \frac{1}{\omega} Z\left(\frac{\omega}{kv_T}\right); \quad v_T = \sqrt{\frac{2T}{m}}, \tag{9.20}$$

where *

$$Z(x) = X(x) - iY(x); \quad X(x) = 2xe^{-x^2} \int_0^x e^{t^2}\,dt; \tag{9.21}$$

$$Y(x) = \sqrt{\pi}\, xe^{-x^2}.$$

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* The function $Z(x)$ is simply related to the well-known tabulated function [42] $W(x)$: $Z(x) = -i\sqrt{\pi}\,xW(x)$.

The function X(x) has the following asymptotic expansion:

$$X(x) = 1 + \frac{1}{2x^2} + \frac{3}{4x^4} + \ldots \tag{9.22}$$

Using the property of the function $\zeta(x)$ that

$$x\zeta(x) = 1, \tag{9.23}$$

we find

$$kv_z\zeta(\omega - kv_z) = -1 + \omega\zeta(\omega - kv_z); \tag{9.24}$$

$$k^2v_z^2\,\zeta(\omega - kv_z) = \omega^2\zeta(\omega - kv_z) - \omega - kv_z. \tag{9.25}$$

Taking account of (9.22) and the fact that $\langle v_z \rangle = 0$ for a Maxwellian distribution, while $\langle v_z \rangle = v_0$ in the beam, for an isotropic plasma we have, finally,

$$\varepsilon_{\perp} = 1 - \sum \frac{\omega_0^2}{\omega^2} Z\left(\frac{\omega}{kv_T}\right); \tag{9.26}$$

$$\varepsilon_{\parallel} = 1 - \sum \frac{\omega_0^2}{\omega^2}\frac{2\omega^2}{k^2v_T^2}\left[Z\left(\frac{\omega}{kv_T}\right) - 1\right]. \tag{9.27}$$

In accordance with (9.22), when $\omega \gg kv_T$,

$$\varepsilon_{\perp} = 1 - \sum \frac{\omega_0^2}{\omega^2}\left(1 + \frac{k^2v_T^2}{2\omega^2} + \frac{3}{4}\frac{k^4v_T^4}{\omega^4} + \right.$$

$$\left. + \ldots - i\sqrt{\pi}\frac{\omega}{kv_T}e^{-\frac{\omega^2}{k^2v_T^2}}\right); \tag{9.26a}$$

$$\varepsilon_{\parallel} = 1 - \sum \frac{\omega_0^2}{\omega^2}\left(1 + \frac{3}{2}\frac{k^2v_T^2}{\omega^2} + \frac{15}{4}\frac{k^4v_T^4}{\omega^4} + \ldots - \right.$$

$$\left. - i2\sqrt{\pi}\frac{\omega^3}{k^3v_T^3}e^{-\frac{\omega^2}{k^2v_T^2}}\right). \tag{9.27a}$$

If $\omega \ll kv_T$,

$$\varepsilon_{\perp} = i\sqrt{\pi} \sum \frac{\omega_0^2}{\omega k v_T}; \tag{9.26b}$$

$$\varepsilon_{\parallel} = 1 + \sum \frac{2\omega_0^2}{k^2 v_T^2} \left(1 + i\sqrt{\pi} \frac{\omega}{k v_T}\right). \tag{9.27b}$$

We note that the imaginary parts of $\varepsilon_{\parallel}$ and $\varepsilon_{\perp}$ contain odd powers of the modulus of the wave number. Hence, the electrical permittivity is not an analytic function of k.

For the beam,

$$\varepsilon_{\parallel} = 1 - \sum \frac{\omega_0}{\omega^2} \frac{2\omega^2}{k^2 v_T^2} \left[Z\left(\frac{\omega - k v_0}{k v_T}\right) - 1\right]. \tag{9.28}$$

Let us now consider the tensor $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega)$ for a plasma in a magnetic field in the coordinate system in which $\mathbf{B}_0 = \{0, 0, B_0\}$ and $\mathbf{k} = \{k_x, 0, k_z\}$. In the absence of interactions between the charges the distribution function in the unperturbed homogeneous plasma is a function of the transverse energy $\varepsilon_{\perp}$ and the longitudinal energy $\varepsilon_{\parallel}$ of the particles:

$$F_0 = F_0(\varepsilon_{\perp}, \varepsilon_{\parallel}). \tag{9.29}$$

In this case the x and y components of the expression in the rectangular brackets in Eq. (9.10) exhibit the form (we use the x component)

$$v_x \left(1 - \frac{k_z v_z}{\omega}\right)\left(\frac{\partial F_0}{\partial \varepsilon_{\perp}} - \frac{\partial F_0}{\partial \varepsilon_{\parallel}}\right) + v_x \frac{\partial F_0}{\partial \varepsilon_{\parallel}}. \tag{9.30}$$

Adding and subtracting a term $v_x(n\omega_B/\omega)[(\partial F_0/\partial \varepsilon_{\perp}) - (\partial F_0/\partial \varepsilon_{\parallel})]$, we write this expression in the form

$$v_x \left\{F_1 + \frac{\omega - k_z v_z - n\omega_B}{\omega} F_2\right\}, \tag{9.31}$$

where

$$F_1 = \frac{\partial F_0}{\partial \varepsilon_{\parallel}} + \frac{n\omega_B}{\omega}\left(\frac{\partial F_0}{\partial \varepsilon_{\perp}} - \frac{\partial F_0}{\partial \varepsilon_{\parallel}}\right); \tag{9.32}$$

$$F_2 = \frac{\partial F_0}{\partial \varepsilon_{\perp}} - \frac{\partial F_0}{\partial \varepsilon_{\parallel}}. \tag{9.33}$$

The further calculations are similar to those for the correlation function (§ 8) except for the fact that the integration over t is now carried out from 0 to ∞; correspondingly, in place of the δ-function (8.34) we now obtain [in accordance with (9.14)]: $(i/2\pi)\,\zeta(\omega - k_z v_z - n\omega_B)$. The components of the tensor $\varepsilon_{\alpha\beta}$ are transformed similarly for $\beta = z$. Using (8.33), after some straightforward calculations which are not reproduced here, we obtain the tensor elements $\varepsilon_{\alpha\beta}$ as follows *:

$$\left.\begin{aligned}
\varepsilon^0_{xx} &= 1 + \sum \frac{4\pi e^2 n}{\omega}\int d\mathrm{p} \sum_{n=-\infty}^{\infty}\left(\zeta_n F_1 + \frac{1}{\omega}F_2\right)\frac{n^2\omega_B^2}{k_\perp^2} J_n^2;\\
\varepsilon^0_{xy} &= -\varepsilon^0_{yx} = -i\sum \frac{4\pi e^2 n}{\omega}\int d\mathrm{p} \sum_{n=-\infty}^{\infty}\left(\zeta_n F_1 + \frac{1}{\omega}F_2\right)\times\\
&\qquad\times \frac{n\omega_B v_\perp}{k_\perp} J_n J_n';\\
\varepsilon^0_{yy} &= 1 + \sum \frac{4\pi e^2 n}{\omega}\int d\mathrm{p} \sum_{n=-\infty}^{\infty}\left(\zeta_n F_1 + \frac{1}{\omega}F_2\right) v_\perp^2 J_n'^2;\\
\varepsilon^0_{xz} &= \varepsilon^0_{zx} = \sum \frac{4\pi e^2 n}{\omega}\int d\mathrm{p} \sum_{n=-\infty}^{\infty} v_z \zeta_n F_1 \frac{n\omega_B}{k_\perp} J_n^2;\\
\varepsilon^0_{yz} &= -\varepsilon^0_{zy} = i\sum \frac{4\pi e^2 n}{\omega}\int d\mathrm{p} \sum_{n=-\infty}^{\infty} v_z \zeta_n F_1 v_\perp J_n J_n';\\
\varepsilon^0_{zz} &= 1 + \sum \frac{4\pi e^2 n}{\omega}\int d\mathrm{p} \sum_{n=-\infty}^{\infty} v_z^2 \zeta_n F_1 J_n^2.
\end{aligned}\right\} \tag{9.34}$$

*If $F_0 = \mathrm{const}\exp[-\varepsilon_\perp/T_\perp - \varepsilon_\parallel/T_\parallel]$, the integration in Eq. (9.34) can be carried out. The corresponding formulas for the case $T_\perp = T_\parallel$ are given in the review "Oscillations of an Inhomogeneous Plasma," by A. B. Mikhailovskii; in the present volume see page 159. Expressions for $\varepsilon_{\alpha\beta}$ in integral form suitable for the analysis of high-frequency oscillations ($\omega \gg \omega_B$) are given in [21, 43].

Here, $J_n \equiv J_n(k_\perp v_\perp/\omega_B)$ is the Bessel function, $\zeta_n \equiv \zeta(\omega - n\omega_B - k_z v_z)$. For a Maxwellian distribution, F_2 vanishes and $F_1 = -F_0/T$.

The δ-functions that appear in the integrand in the anti-Hermitian components of the tensor $\varepsilon_{\alpha\beta}$ indicate which particles participate in the absorption (or excitation) of the wave. It is evident that the vanishing of the argument of the δ-functions coincides with the resonance conditions considered earlier (§ 4). The relative contribution of a given resonance is determined by the factor that multiplies the δ-function. This factor contains the Bessel function with argument $k_\perp v_\perp/\omega_B$, which is the ratio of the circumference of the Larmor circle $2\pi v_\perp/\omega_B$ to the transverse (with respect to the magnetic field) wavelength $\lambda = 2\pi/k_\perp$. At large wavelengths, the arguments of the Bessel function are small, so that we can write $J_n(x) = 1/n\,(x/2)^n$. When $x = k_\perp v_\perp/\omega_B < 1$, the importance of harmonic resonances ($n = \pm 2, \pm 3 \ldots$) is reduced with increasing n. At short wavelengths ($k_\perp v_\perp/\omega_B > 1$), however, harmonics characterized by $n \gg 1$ can be important.

In longitudinal propagation ($k_\perp = 0$) the argument of the Bessel function vanishes and the components $\varepsilon_{xz} = \varepsilon_{zx}$ and $\varepsilon_{yz} = -\varepsilon_{zy}$ also vanish. The tensor $\varepsilon_{\alpha\beta}$ then assumes the form

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} \varepsilon & ig & 0 \\ -ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{pmatrix}, \tag{9.35}$$

where the only nonvanishing terms in ε and g are the $n = \pm 1$ terms, while the only nonvanishing term in η is the $n = 0$ term. For a Maxwellian distribution with different temperatures parallel and transverse to the magnetic field ($T_\parallel$ and $T_\perp$) the square of the refractive index for circularly polarized transverse waves is ($v_\parallel = \sqrt{2T_\parallel/m}$):

$$N^2 = \varepsilon \pm g = 1 - \sum \frac{\omega_0^2}{\omega}\left\{\frac{1}{\omega \mp \omega_B} Z\left(\frac{\omega \mp \omega_B}{k_z v_\parallel}\right)\left[\frac{T_\perp}{T_\parallel} \pm \frac{\omega_B}{\omega} \times \right.\right.$$
$$\left.\left. \times\left(1 - \frac{T_\perp}{T_\parallel}\right)\right] + \frac{1}{\omega}\left(1 - \frac{T_\perp}{T_\parallel}\right)\right\}. \tag{9.36}$$

The longitudinal dispersion equation

$$\eta = 1 - \sum \frac{\omega_0^2}{\omega^2}\,\frac{2\omega^2}{k_z^2 v_\parallel^2}\left\{Z\left(\frac{\omega}{k_z v_\parallel}\right) - 1\right\} = 0 \tag{9.37}$$

for $k_\perp = 0$ coincides with the dispersion equation $\varepsilon_{\|} = 0$ in the absence of the magnetic field.

In oblique propagation the resonance terms $|n| > 1$ are retained. At wavelengths much greater than the mean radius of the Larmor circle, the ($k_\perp v_\perp / \omega_B \ll 1$) terms and the $n = 0$, $n = \pm 1$ terms, are again important. We write the components of $\varepsilon_{\alpha\beta}$ on the assumption that $k_\perp v_\perp / \omega_B \ll 1$ retaining the $n = 0$ and $n = \pm 1$ terms:

$$\left.\begin{aligned}
\varepsilon^0_{xx} &= 1 - \sum \frac{\omega_0^2}{\omega} \frac{1}{2} \Big\langle \zeta(\omega - \omega_B - k_z v_z) + \\
&\qquad + \zeta(\omega + \omega_B - k_z v_z) \Big\rangle; \\
\varepsilon^0_{xy} &= -\varepsilon^0_{yx} = i \sum \frac{\omega_0^2}{\omega} \frac{1}{2} \Big\langle \zeta(\omega - \omega_B - k_z v_z) - \\
&\qquad - \zeta(\omega + \omega_B - k_z v_z) \Big\rangle; \\
\varepsilon^0_{yy} &= \varepsilon^0_{xx} - \sum \frac{\omega_0^2}{\omega} \frac{2k_\perp^2 T}{m\omega_B^2} \Big\langle \zeta(\omega - k_z v_z) \Big\rangle; \\
\varepsilon^0_{xz} &= \varepsilon^0_{zx} = -\sum \frac{\omega_0^2}{\omega} \frac{1}{2} \Big\langle \frac{k_\perp v_z}{\omega_B} \{\zeta(\omega - \omega_B - k_z v_z) - \\
&\qquad - \zeta(\omega + \omega_B - k_z v_z)\} \Big\rangle; \\
\varepsilon^0_{yz} &= -\varepsilon^0_{zy} = \sum \frac{\omega_0^2}{\omega} \Big\langle \frac{k_\perp v_z}{\omega_B} \Big\{ \zeta(\omega - k_z v_z) - \\
&\qquad - \frac{1}{2} \left[\zeta(\omega - \omega_B - k_z v_z) + \zeta(\omega + \omega_B - k_z v_z) \right] \Big\} \Big\rangle; \\
\varepsilon^0_{zz} &= 1 - \sum \frac{\omega_0^2}{\omega} \Big\langle \frac{m v_z^2}{T} \zeta(\omega - k_z v_z) \Big\rangle.
\end{aligned}\right\} \quad (9.38)$$

Problem. Find the relation between the Fourier components of the correction to the distribution function $f^{(1)}$ and the electric field $\mathbf{E}(\mathbf{k}, \omega)$ in a plasma with no magnetic field.

Solution. Substituting $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{k}, \omega)\, e^{i(\mathbf{kr} - \omega t)}$ in Eq. (9.5) and extracting the factor $e^{i(\mathbf{kr} - \omega t)}$ from the integrand, we find, taking account of the law of motion (9.3), $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v} t$:

$$f^{(1)} = -e \frac{\partial F_0}{\partial \mathbf{p}} \mathbf{E}(\mathbf{k}, \omega)\, e^{i(\mathbf{kr} - \omega t)} \int\limits_{-\infty}^{t} e^{i(\omega - \mathbf{kv})(t - t')}\, dt' \equiv f^{(1)}(\mathbf{k}, \omega)\, e^{i(\mathbf{kr} - \omega t)}.$$

Using Eqs. (9.9) and (9.14), we have

$$f^{(1)}(\mathbf{k},\ \omega) = g(\mathbf{k},\ \omega)\,\mathbf{E}(\mathbf{k},\ \omega),$$

where

$$g(\mathbf{k},\ \omega) = -ie\frac{\partial F_0}{\partial \mathbf{p}}\,\zeta(\omega - \mathbf{kv}) = -e\pi\frac{\partial F_0}{\partial \mathbf{p}}\left\{\delta(\omega - \mathbf{kv}) + \frac{i}{\pi}\,\frac{P}{\omega - \mathbf{kv}}\right\}.$$

§ 10. Oscillations of a Plasma in a Magnetic Field Taking Account of the Thermal Motion with $\mathbf{B}_0 = 0$

In the present section we shall consider briefly the effects of the thermal motion of the particles on plasma oscillations when the interaction between the charges can be neglected. We have already considered the effect of thermal motion on longitudinal electron oscillations. The oscillation frequency and the damping due to the presence of particles in resonance with the waves has been indicated. We have also noted that in contrast with the longitudinal waves, transverse waves in a plasma with no magnetic field are not damped, since the phase velocity of these waves is always greater than the particle velocities; therefore, resonances can never occur.

We now wish to consider the other oscillation branches. First we consider the case of a plasma with no magnetic field. In addition to the electron oscillation branch, which has been already treated, there is a low-frequency ion branch which is the analog of the acoustic waves in an ordinary fluid. In obtaining this branch in § 2 we neglected the electron inertia $-imn\omega\mathbf{v}$ compared with the gradient of the electron pressure $-\nabla p^{(1)}$. Using this condition we now write the requirement

$$\omega \ll kv_{Te}. \tag{10.1}$$

The electron term in Eq. (9.27) for $\varepsilon_{\parallel}$ is simplified considerably under these conditions, since $Z(\omega/kv_{Te}) \approx -i\sqrt{\pi}(\omega/kv_{Te})$. The dispersion equation now assumes the form

$$\varepsilon_{\parallel} = 1 + \frac{4\pi z^2e^2n_i}{k^2zT_e}\left(1 + i\sqrt{\pi}\,\frac{\omega}{kv_{Te}}\right) + \frac{4\pi z^2en_i}{k^2T_i}\times$$
$$\times\left\{1 - Z\left(\frac{\omega}{kv_{Ti}}\right)\right\} = 0. \tag{10.2}$$

When $\omega/kv_{Ti} \sim 1$, the imaginary part of the dispersion equation is comparable in magnitude with the real part. This means that these oscillations are highly damped; the reason is obvious. The condition $\omega \sim kv_{Ti}$ means that most of the ions move with velocities comparable with the phase velocity of the waves, so that they can participate in the absorption. Oscillations are possible only when $\omega/kv_{Ti} \gg 1$. In this case, the only contribution to absorption comes from ions moving with the velocities much greater than the thermal velocity ($v = \omega/k \gg v_{Ti}$), i.e., from ions in the tail of the Maxwellian distribution. Expanding the function $Z(\omega/kv_{Ti})$ for large argument, we find

$$\omega^2 = \frac{k^2 z T_e}{m_i} \frac{1 + \frac{3k^2 T_i}{m_i \omega^2}}{1 + \frac{k^2 z T_e}{4\pi z^2 e^2 n_i}} - i\sqrt{\pi} \frac{\omega^3}{kv_{Ti}} \frac{\sqrt{\frac{T_i}{zT_e}}}{1 + \frac{k^2 z T_e}{4\pi z^2 e^2 n_i}} \times$$
$$\times \left\{ \sqrt{\frac{m_e z}{m_i}} + \left(\frac{zT_e}{T_i}\right)^{3/2} e^{-\frac{\omega^2}{k^2 v_{Ti}^2}} \right\}. \qquad (10.3)$$

We first consider oscillations characterized by wavelengths much greater than the "electron Debye radius"

$$\frac{k^2 z T_e}{4\pi z^2 e^2 n_i} \ll 1. \qquad (10.4)$$

Writing $\omega = \omega_1 - i\omega_2$, we find ($\omega_2 \ll \omega_1$):

$$\omega_1^2 = \frac{k^2 z T_e}{m_i}\left(1 + \frac{3T_i}{zT_e}\right);$$

$$\omega_2 = \sqrt{\frac{\pi}{8}}\, k \sqrt{\frac{zT_e}{m_i}} \left\{ \sqrt{\frac{m_e z}{m_i}} + \left(\frac{zT_e}{T_i}\right)^{3/2} e^{-\frac{zT_e}{2T_i}} \right\}. \qquad (10.5)$$

The criterion for applicability of these formulas $\omega \gg kv_{Ti}$ means that the electron pressure must be much greater than the ion pressure, i.e.,

$$zT_e \gg T_i. \qquad (10.6)$$

By virtue of this condition the phase velocity of the oscillations is much greater than the mean ion velocity. In addition to the ions in the tail of the Maxwellian distribution, the number of which is proportional to $f_i \sim (m_i/T_i)^{1/2} \cdot e^{-zT_e/2T_i}$ a contribution to the absorption comes from the slow electrons (from the maximum in the Maxwellian distribution); the number of these electrons is proportional to $f_e \sim (m_e/T_e)^{\frac{1}{2}}$. The absorption depends on the current density. Assuming that the mean velocity of the particles in the wave is inversely proportional to the pressure $\bar{\nu}_e \sim 1/zT_e$, $\bar{\nu}_i \sim 1/T_i$, we can find the ratio of the energy adsorbed by the ions to the energy absorbed by the electrons:

$$\left(\frac{m_i T_e}{m_e T_i}\right)^{1/2} \frac{zT_e}{T_i} e^{-\frac{zT_e}{2T_i}}. \tag{10.7}$$

Since the absorbed energy is proportional to the imaginary part of the frequency ω_2, it is evident that this is also the ratio of the second term in ω_2 (due to the ions) to the first (due to the electrons).

At short wavelengths,

$$\frac{k^2 zT_e}{4\pi z^2 e^2 n_i} \gg 1, \tag{10.8}$$

in which case the space-charge oscillations are important; under these conditions the solution of the dispersion equation is

$$\left.\begin{aligned} \omega_1^2 &= \frac{4\pi z^2 e^2 n_i}{m_i}\left(1 + \frac{3k^2 T_i}{4\pi z^2 e^2 n_i}\right); \\ \omega_2 &= \sqrt{\frac{\pi}{8}}\left(\frac{4\pi z^2 e^2 n_i}{m_i}\right)^{1/2}\left(\frac{4\pi z^2 e^2 n_i}{k^2 zT_e}\right)^{3/2} \times \\ &\times \left\{\sqrt{\frac{m_e z}{m_i}} + \left(\frac{zT_e}{T_i}\right)^{3/2} e^{-\frac{2\pi z^2 e^2 n_i}{k^2 T_i}}\right\}. \end{aligned}\right\} \tag{10.9}$$

The condition for applicability of (10.8) and the requirement $\omega \gg kv_{Ti}$ (weak damping yield)

$$k^2 T_i \ll 4\pi z^2 e^2 n_i \ll k^2 zT_e. \tag{10.10}$$

Thus, ion plasma oscillations are possible (as are the acoustic oscillations) only if there is an appreciable difference between the electron and ion temperatures.

Let us now summarize briefly the results of our analysis of the oscillations of an isotropic plasma in which collisions between particles can be neglected. The following modes of oscillation are possible in such a plasma.

a) Undamped transverse oscillations at frequencies higher than the plasma frequency

$$\omega^2 = \omega_{0e}^2 + k^2 c^2; \tag{10.11}$$

b) Longitudinal electron oscillations at approximately the plasma frequency

$$\omega^2 = \omega_{0e}^2 + \frac{3k^2T_e}{m_e};\quad \frac{3k^2T_e}{m_e} \ll \omega_{0e}^2; \tag{10.12}$$

c) Longitudinal ion oscillations; at long wavelengths these are essentially "acoustic" oscillations, and at short wavelengths these are charge-density oscillations at the ion-plasma frequency:

$$\left.\begin{aligned} \omega^2 &= \frac{k^2zT_e}{m_i};\quad \frac{k^2zT_e}{4\pi z^2e^2n_i} \ll 1; \\ \omega^2 &= \omega_{0i}^2 + \frac{3k^2T_i}{m_i};\quad \frac{k^2zT_e}{4\pi z^2e^2n_i} \gg 1. \end{aligned}\right\} \tag{10.13}$$

The following condition must be satisfied in order for these oscillations to be excited:

$$zT_e \gg T_i. \tag{10.14}$$

§ 11. Plasma Oscillations in a Magnetic Field Taking Account of Thermal Motion with $\mathbf{B}_0 \neq 0$

In the presence of a magnetic field in a plasma the particles will contribute to absorption when

$$\left.\begin{aligned} \omega - n\omega_B - k_zv_z = 0;\ n = 0,\ \pm 1,\ \pm 2,\dots \\ \omega_B = \omega_B^0\sqrt{1-\beta^2};\ \beta = v/c. \end{aligned}\right\} \tag{11.1}$$

The relation in (11.1) is nothing more than the Doppler formula, which relates the frequency of a radiator $n\omega_B^0$ (ω_B^0 being measured in the rest system) to the frequency ω which is observed in the laboratory coordinate system. If $n < 0$ the effect is called the "anomalous" Doppler effect. Assuming that $\omega > 0$, we find the frequency ω at which a particle moving with velocity v_z along the lines of force is in resonance with the wave as a consequence of the Doppler effect. For the normal Doppler effect ($n > 0$)

$$\omega = \frac{n\omega_B}{1-\beta_zN\cos\theta}. \tag{11.2}$$

The normal Doppler effect obtains when

$$\beta_zN\cos\theta < 1. \tag{11.3}$$

The anomalous Doppler effect ($n < 0$) yields

$$\omega = \frac{|n|\,\omega_B}{\beta_z N\cos\theta - 1}. \tag{11.4}$$

The anomalous Doppler effect can arise only if the charge moves with a "superlight" velocity

$$\beta_z N\cos\theta > 1, \tag{11.5}$$

i.e., in the region in which $N \gg 1$. When $n \neq 0$ the absorption is called cyclotron absorption, and when $n = 0$ it is called Cerenkov absorption because, in this case, the absorption condition $\omega - k_z v_z = 0$ coincides with the Cerenkov radiation condition for a charge moving with a fixed velocity along the z axis.

If the velocity distribution is Maxwellian, the number of charges that contribute to absorption is proportional to the exponential factor $e^{-mv_z^2/2T}$. Taking account of (11.1), we have

$$e^{-\frac{mv_z^2}{2T}} = e^{-\frac{m(\omega - n\omega_B)^2}{2k_z^2 T}} = e^{-\frac{(\omega - n\omega_B)^2}{\omega^2\beta^2 N^2\cos^2\theta}};\quad \beta^2 = \frac{2T}{mc^2}. \tag{11.6}$$

This expression holds if it is legitimate to neglect the dependence of ω_B on β (the relativistic Doppler effect). The relativistic Doppler effect is comparable with the ordinary Doppler effect when $k_z v_z \sim \omega\beta^2$, i.e., $\cos\theta \approx \beta/N$. If $\beta/N \ll 1$, the region in which the relativistic Doppler effect must be considered is very narrow:

$$\left|\frac{\pi}{2} - \theta\right| < \left|\frac{\beta}{N}\right|. \tag{11.7}$$

We shall not consider this region in detail in the present section. It should be noted that since $v_z < c$, absorption occurs only when

$$\left|\frac{\omega - n\omega_B}{k_z}\right| < c. \tag{11.8}$$

It is evident from Eq. (11.6) that when $\beta N\cos\theta > 1$, there is a very wide region of cyclotron absorption ($n \neq 0$). The Cerenkov absorption is also important under these conditions.

In examining regions of absorption for purposes of making estimates, we can take the value of N^2 for a cold plasma. From an analysis of the curves (cf., for example, Fig. 6) of the square of the refractive index (3.10) it will be evident that the Cerenkov absorption occurs on the electron branch $B(\omega_{Bi} \ll \omega < \omega_{Be})$ when $\omega_{0e}^2/\omega_{Be}^2 \gg 1$, if $N^2 \gg 1$. Cerenkov absorption also occurs in the low-frequency ($\omega \ll \omega_{Bi}$) parts of the branches B and M. This absorption leads to damping of the magnetoacoustic waves even in the absence of particle collisions. As we shall see below, there is no Cerenkov absorption on branch A. Finally, Cerenkov absorption is also important near frequencies ω_2^∞ on branch E_M. In this case, however, the region of frequencies for which $N^2 \gg 1$ is relatively small.

One further remark may be appropriate with respect to the regions contiguous to the frequencies $\omega_{1,2,3}^\infty$. When $\omega = \omega^\infty$, we find $N^2 = \infty$ in a cold plasma, which corresponds to zero phase velocity of the wave. When the thermal motion is taken into account the phase velocity of the wave becomes finite when $\omega = \omega^\infty$, but is still appreciably smaller than the velocity of light c. Since $\omega \gtrless \omega^\infty$ in this region the oscillations are highly damped.

Let us now consider in greater detail the region of Cerenkov absorption. We shall first examine the damping of magnetohydrodynamic waves. As in the case of the low-frequency oscillations with $B_0 = 0$, we shall assume that the condition $\omega \ll kv_{Te}$ is satisfied. In addition, the magnetohydrodynamic waves are characterized by the conditions

$$\omega \ll \omega_{Bi}; \quad kv_{Ti} \ll \omega_{Bi}. \tag{11.9}$$

In evaluating $\varepsilon_{\alpha\beta}$, it is easy to show from Eq. (9.38) that when these conditions are satisfied the component $\varepsilon_{xz}^0 = \varepsilon_{zx}^0$ is small compared with the other components and can be neglected. The component ε_{xx}^0 does not contain the $n = 0$ term; hence, when $\omega \ll \omega_{Bi}$ it does not contain a correction for the thermal motion. On the assumption that $\omega_{0i}^2/\omega_{Bi}^2 \gg 1$, we have

$$\varepsilon_{xx}^0 = \varepsilon_1 = \frac{\omega_{0i}^2}{\omega_{Bi}^2} = \frac{c^2}{c_A^2}. \tag{11.10}$$

Similarly, the component $\varepsilon_{xy}^0 = -\varepsilon_{yx}^0$ remains virtually unchanged when the thermal motion is introduced: $\varepsilon_{xy}^0 \approx i\varepsilon_{xx}^0(\omega/\omega_{Bi})$ [cf. Eq. (9.38)]. Since we have assumed that $\omega \ll \omega_{Bi}$, this component of the tensor can also be neglected.

In computing the other components, which contain $\zeta(\omega - k_z v_z)$, it should be kept in mind that the imaginary parts of these components become comparable with the real parts when $\omega \lessapprox kv_{Ti}$. This means that oscillations characterized by $\omega \lessapprox kv_{Ti}$ are not possible (they are highly damped). For this reason we consider the case in which

$$\omega \gg kv_{Ti}. \tag{11.11}$$

Under these conditions the function $Z(\omega/kv_{Ti})$, which appears when $\zeta(\omega - k_z v_z)$ is averaged, is taken in the form ($v_{Ti}^2 = 2T_i/m_i$):

$$Z\left(\frac{\omega}{kv_{Ti}}\right) = \left(1 + \frac{k_z^2 v_{Ti}^2}{2\omega^2} + \cdots\right) - i\sqrt{\pi}\,\frac{\omega}{k_z v_{Ti}} e^{-\frac{\omega^2}{k_z^2 v_{Ti}^2}}. \tag{11.12}$$

In this case,

$$\left.\begin{aligned}
\varepsilon_{yy}^0 &= \varepsilon_2 = \frac{\omega_{0i}^2}{\omega_{Bi}^2}\left\{1 + \frac{2k_z^2}{m_i\omega^2}\left[i\sqrt{\pi}\,\frac{\omega T_i}{k_z v_{Ti}} e^{-\frac{\omega^2}{k_z^2 v_{Ti}^2}} - \right.\right. \\
&\qquad \left.\left. - T_e Z\left(\frac{\omega}{k_z v_{Te}}\right)\right]\right\} \\
\varepsilon_{yz}^0 &= if = i\,\frac{\omega_{0i}^2}{\omega\omega_{Bi}}\left[1 - i\sqrt{\pi}\,\frac{\omega}{k_z v_{Ti}} e^{-\frac{\omega^2}{k_z^2 v_{Ti}^2}} - Z\left(\frac{\omega}{k_z v_{Te}}\right)\right]; \\
\varepsilon_{zz}^0 &= \eta = -\frac{\omega_{0i}^2}{\omega^2} + i\sqrt{\pi}\,\frac{4\pi z^2 e^2 n}{k_z^2 T_i}\,\frac{\omega}{k_z v_{Ti}} e^{-\frac{\omega^2}{k^2 v_{Ti}^2}} + \\
&\qquad + \frac{4\pi z^2 e^2 n}{k_z^2 T_e}\left[1 - Z\left(\frac{\omega}{k_z v_{Te}}\right)\right].
\end{aligned}\right\} \tag{11.13}$$

It is evident that the component η is approximately ω_{Bi}/ω or ω_{Bi}/kv_{Ti} times larger than the other components. This means that the component of the electric field along $\mathbf{B}_0$ is small. By virtue of this condition, we find the dispersion relation can be simplified considerably. From Eq. (1.26) we have

$$\eta_{xx} = \frac{\varepsilon_1}{\cos^2\theta};\ \eta_{yy} = \varepsilon_2 - \frac{f^2}{\eta};\ \eta_{xy} = -\eta_{yx} = 0. \tag{11.14}$$

According to Eq. (1.25a) the dispersion relation is given by*

$$N^2 = \eta_{xx} = \frac{\varepsilon_1}{\cos^2\theta};\ N^2 = \eta_{yy} = \varepsilon_2 - \frac{f^2}{\eta}. \tag{11.15}$$

The first of these relations is to be associated with the Alfvèn wave:

* More precisely, $N^2 = \eta_{xx} = \varepsilon_1/[\cos^2\theta + (\varepsilon_1/\eta)\sin^2\theta]$. This equation is of great importance in investigations of the oscillations of an inhomogeneous plasma (cf. the review by Mikhailovskii in the present volume).

$$\omega^2 = k_z^2 c_A^2. \tag{11.16}$$

There is no Cerenkov absorption here. However, the cyclotron absorption, which is characterized by the exponential $e^{-\frac{\omega_{Bi}^2}{k_z^2 v_{Ti}^2}} = e^{-\frac{\omega_{Bi}^2}{\omega^2}\frac{c_A^2}{v_{Ti}^2}}$, (when $\omega \ll \omega_{Bi}$) is very small; when $\omega < \omega_{Bi}(c_A/c)$ it disappears altogether, in accordance with Eqs. (11.8) and (11.16). Thus, in the absence of collisions, the Alfvèn wave is very weakly damped.

The second dispersion relation (11.15) is related to the fast and slow magnetoacoustic waves. We consider the case $v_{Ti}^2 \ll c_A^2$ or

$$n_i T_i \ll \frac{B_0^2}{8\pi}. \tag{11.17}$$

We will also assume that $v_{Te}^2 \gg c_A^2$ ($kv_{Te} \gg \omega$). If (11.17) is satisfied, the hydrodynamic analysis yields

$$\omega = k_z c_s; \ \ \omega = kc_A.$$

As we have already indicated, oscillations are possible only if $\omega \gg kv_{Ti}$. Hence the slow wave ($\omega = k_z c_s$) will be weakly damped if

$$zTe \gg T_i. \tag{11.18}$$

We now write the dispersion equation in the form

$$\eta = \frac{f^2}{\varepsilon_2 - N^2}. \tag{11.19}$$

It is easy to show that if $\omega \sim kc_s$ the right side is small compared with the terms that appear in η. Thus, the approximate dispersion equation for the slow magnetoacoustic wave is $\eta = 0$. This coincides with the equation for the longitudinal ion oscillations with $B_0 = 0$, with the sole difference that by k we are now to understand k_z:

$$\omega^2 = \frac{T_e}{m_i} k_z^2 \left\{1 - i\frac{\sqrt{\pi}}{2}\left[\sqrt{\frac{zm_e}{m_i}} + \left(\frac{zT_e}{T_i}\right)^{3/2} e^{-\frac{zT_e}{T_i}}\right]\right\}. \tag{11.20}$$

If the condition in (11.17) is satisfied, the fast wave is weakly damped. The damping can be caused either by ions or by electrons. When $\omega/k_z v_{Te} \ll 1$, using Eqs. (11.3) and (11.15) we find [20]:

$$\omega^2 = k^2c_A^2 + k_\perp^2 v_{Ti}^2\left(1+\frac{T_e}{2T_i}\right) - i\,\frac{c_A v_{Ti} k_\perp^2 k}{k_z}\times$$

$$\times\left[\left(1+\frac{T_e}{T_i}+\frac{T_e^2}{2T_i^2}\right)\sqrt{\pi}\,e^{-\frac{c_A^2k^2}{v_{Ti}^2k_z^2}} + \frac{\sqrt{\pi}}{2}\left(\frac{m_eT_e}{m_iT_i}\right)\right], \quad (11.21)$$

and when $\omega/k_z v_{Te} \gg 1$ [i.e., for angles close to $\pi/2$ ($\cos\theta \ll c_A/v_{Te}$)],

$$\omega^2 = k^2c_A^2 + k^2v_{Ti}^2\left(1+\frac{T_e}{T_i}\right) - ik^2c_Av_{Ti}\left(\frac{m_eT_e}{m_iT_i}\right)^{1/2}\frac{2\sqrt{\pi}k}{k_z}\,e^{-\frac{c_A^2k^2}{v_{Te}^2k_z^2}}. \quad (11.22)$$

We note that the thermal corrections to the real part of ω are somewhat different from the corresponding corrections that are obtained in the hydrodynamic approximation. From the formula for the hydrodynamic approximation it follows that when $c_A \gg v_{Ti}$,

$$\omega^2 = (c_A^2 + c_s^2\sin^2\theta)\,k^2. \quad (11.23)$$

This expression coincides with the real part of ω^2 as given by Eq. (11.21) if we take $\gamma_i = 2$ and $\gamma_e = 1$ in the expression $c_s^2 = (\gamma_e T_e + \gamma_i T_i)/m_i$. This means that the ions behave like a gas with two effective degrees of freedom. As far as the electrons are concerned, however, the oscillations are slow, so that at any given instant the electrons are able to set up an equilibrium (Boltzmann) distribution in moving along the lines of force, and this isothermal motion corresponds to $\gamma_e = 1$. For the oscillations in (11.22) we have $\gamma_e = \gamma_i = 2$, i.e., for the waves that propagate across the magnetic field, the electron gas (like the ion gas) has two effective degrees of freedom.

We now consider the Cerenkov absorption of oscillations at frequencies appreciably greater than the ion cyclotron frequency. In this case we neglect the ion contribution to the current density completely. Consider the branch B (cf. Fig. 6) with the condition that $\omega_{0e}^2/\omega_{Be}^2 \gg 1$, in which case the Cerenkov absorption is especially important. If this condition is satisfied, the branch B at all angles θ (with the exception of those close to $\pi/2$) is described by Eq. (3.60) for "quasi-longitudinal propagation." In a cold plasma,

$$N^2 = \frac{\omega_{0e}^2}{\omega(\omega_{Be}\cos\theta - \omega)}. \tag{11.24}$$

Taking account of the thermal motion leads to the appearance of an imaginary part in N^2 and a small correction to the real part of N^2. We shall neglect these corrections, which make only an unimportant change in the phase velocity of the waves, and will be interested only in zero-order effects caused by the thermal motion, specifically the damping. This means that the thermal corrections are introduced only in the anti-Hermitian part of the tensor $\varepsilon_{\alpha\beta}$ which is assumed to be small; thus the Cerenkov absorption mechanism implies $\omega \gg kv_{Te}$.

If the oscillation frequency ω is close to the cyclotron frequency ω_{Be}, in addition to the Cerenkov absorption there is also a cyclotron absorption. We shall assume, however, that the frequency ω is small compared with ω_{Be} and shall take account of the Cerenkov absorption only. In practice, the cyclotron absorption is appreciably smaller than the Cerenkov absorption when the exponential factor $e^{-\frac{(\omega-\omega_B)^2}{k_z^2 v_{Te}^2}}$ becomes comparable with $e^{-\frac{\omega^2}{k_z^2 v_{Te}^2}}$, i.e., when

$$\omega \lesssim \frac{1}{2}\omega_{Be}, \tag{11.25}$$

because of the fact that the cyclotron exponential is multiplied by a smaller factor than the Cerenkov exponential. If the condition in (11.25) is satisfied, as follows from Eq. (9.38) for $\varepsilon_{\alpha\beta}$ and Eqs. (9.22)-(9.25), it is only necessary to consider the imaginary part of the component ε_{zz}^0 because the anti-Hermitian parts of the components ε_{yy}^0 and ε_{yz}^0 are small fractions $k^2v_{Te}^2/\omega\omega_{Be}$ and $k^4v_{Te}^4/\omega^2\omega_{Be}^2$, of Im ε_{zz}^0. Thus, in computing N^2, we start from the tensor in the form in (9.35), where

$$\varepsilon = \frac{\omega_{0e}^2}{\omega_{Be}^2 - \omega^2}; \quad g = \varepsilon\frac{\omega_{Be}}{\omega};$$

$$\eta = \eta_0 + i\eta_1 = -\frac{\omega_{0e}^2}{\omega^2} + i\sqrt{\pi}\frac{2\omega_{0e}^2}{\omega^2(\beta N\cos\theta)^3}e^{-\frac{1}{\beta^2N^2\cos^2\theta}}. \tag{11.26}$$

Assuming that $\eta_1 \ll \eta_0$, ε, g in the linear approximation for N^2 obtained from Eq. (3.9), we have

$$N^2 = \frac{\omega_{0e}^2}{\omega\left(\omega_{Be}\cos\theta - \omega\right)}\left\{1 + i\sqrt{\pi}\,\frac{\omega^{5/2}\left(\omega_{Be}\cos\theta - \omega\right)^{1/2}}{\omega_{0e}^3}\times\right.$$

$$\left.\times\frac{\sin^2\theta}{(\beta\cos\theta)^3}\,e^{-\frac{\omega\left(\omega_{Be}\cos\theta - \omega\right)}{\omega_{0e}^2\beta^2\cos^2\theta}}\right\} \tag{11.27}$$

We now consider the region, $N^2 \gg 1$, in which Cerenkov absorption is also possible. This region is close to the frequency ω_3^∞ where the oscillations become approximately longitudinal. The dispersion equation for the longitudinal oscillations in the presence of a magnetic field is obtained from the condition (cf. § 1): $A = \varepsilon_{zz} = 0$, i.e.,

$$\varepsilon_1\sin^2\theta + \eta\cos^2\theta + \xi\sin 2\theta = 0. \tag{11.28}$$

We assume that ω_3^∞ is not close to ω_{Be}, in which case the cyclotron absorption is unimportant. As in the above, we introduce the thermal corrections only in the anti-Hermitian parts of $\varepsilon_{\alpha\beta}$. In this case, $\mathrm{Im}\,\varepsilon_1 \ll \mathrm{Im}\,\eta$, so that

$$\left.\begin{aligned}\varepsilon_1 &= 1 - \frac{\omega_{0e}^2}{\omega^2 - \omega_{Be}^2};\quad \xi = 0;\\ \eta &= 1 - \frac{\omega_{0e}^2}{\omega^2} + 2i\omega\gamma;\ \gamma = \frac{\sqrt{\pi}\,\omega_{0e}^2}{k^3 v_{Te}^3\cos\theta}\,e^{-\frac{\omega^2}{k^2 v_{Te}^2\cos^2\theta}}\end{aligned}\right\} \tag{11.29}$$

Writing $\omega = \omega_3^\infty - i\omega_2$, with $\omega_3^\infty \gg \omega_2$ we find

$$\omega_2 = \frac{\omega_3^{\infty 2}\left(\omega_3^{\infty 2} - \omega_{Be}^2\right)}{2\omega_3^{\infty 2} - \left(\omega_{0e}^2 + \omega_{Be}^2\right)}\,\gamma. \tag{11.30}$$

When $\theta = 0$, $\omega_2^\infty = \omega_{0e}$ and $\omega_2 = \gamma(\theta = 0)$ coincides with the Landau damping factor.

Let us now consider the region in which cyclotron absorption is important. It is evident from the expression for N^2 for a cold plasma that for almost all angles θ [except for the narrow cone $|(\pi/2) - \theta| \lesssim \sqrt{zm_e/m_i}$], the refractive index becomes infinite at $\omega \approx \omega_{Bi}$. Close to this frequency there is evidently a strong absorption. If $k_\perp v_{Ti} \ll \omega_{Bi}$, the tensor $\varepsilon_{\alpha\beta}$, as is evident from Eq. (9.38), has the same form as for the cold plasma with $\varepsilon \sim g \sim \omega_{0i}^2/\omega_{Bi}^2$, so that $\eta \sim \omega_{0i}^2/k^2 v_{Ti}^2$. Since $\eta \gg \varepsilon$, g, as in the case of a cold plasma, the expression for N^2 is obtained from Eq. (3.55):

$$N^2 = \frac{\varepsilon(1+\cos^2\theta) \pm \sqrt{\varepsilon^2 \sin^4\theta + 4g^2\cos^2\theta}}{2\cos^2\theta}. \tag{11.31}$$

Writing

$$\left.\begin{aligned} &\varepsilon = \varepsilon_0 + i\varepsilon_1; \quad g = g_0 + ig_1; \\ &\varepsilon_0 = \frac{\omega_{0i}^2}{\omega_{Bi}^2 - \omega^2}; \qquad g_0 = \frac{\omega_{0i}^2}{\omega_{Bi}^2 - \omega^2}\frac{\omega}{\omega_{Bi}}; \\ &\varepsilon_1 = g_1 = \frac{\sqrt{\pi}}{2}\frac{\omega_{0i}^2}{\omega^2\beta N\cos\theta}\exp\left\{-\left(\frac{\omega_{Bi}-\omega}{\omega\beta N\cos\theta}\right)^2\right\}, \end{aligned}\right\} \tag{11.32}$$

we obtain the refractive index in the form $N = p(1 + i\varkappa)$, ($\varkappa \ll 1$) with

$$p^2 = \frac{\omega_{0i}^2}{\omega_{Bi}^2}\frac{2}{1+\cos^2\theta \mp \sqrt{1 + 2\left(2\frac{\omega^2}{\omega_{Bi}^2} - 1\right)\cos^2\theta + \cos^4\theta}}; \tag{11.33}$$

$$\varkappa = \frac{\sqrt{\pi}}{4\beta}\frac{\omega_{0i}^2}{\omega^2 p^3\cos\theta}e^{-\left(\frac{\omega_{Bi}-\omega}{\omega\beta p\cos\theta}\right)^2} \times$$

$$\times \frac{(1+\cos^2\theta)\sqrt{\omega_{Bi}^2\sin^4\theta + 4\omega^2\cos^2\theta} \pm (\omega_{Bi}\sin^4\theta + 4\omega\cos^2\theta)}{2\cos^2\theta\sqrt{\omega_{Bi}^2\sin^4\theta + 4\omega^2\cos^2\theta}}. \tag{11.34}$$

Let us now consider absorption at electron cyclotron resonance for the simple case of longitudinal propagation. According to Eq. (9.36),

$$N^2 = \varepsilon \pm g = 1 - \frac{\omega_{0e}^2}{\omega(\omega \mp \omega_{Be})} Z\left(\frac{\omega \mp \omega_{Be}}{k_z v_{Te}}\right). \tag{11.35}$$

When $\omega \sim \omega_{Be}$, the ion term makes a very small contribution and can be neglected. Taking the plus sign in front of ω_{Be} we find $Z \approx 1$, and

$$N^2 = 1 - \frac{\omega_{0e}^2}{\omega(\omega_{Be} + \omega)}.$$

The imaginary part of N^2 is important for the other wave (minus sign in front of ω_{Be}). Writing $N = p(1 + i\varkappa)$ we obtain the following equations for p and $\varkappa$:

$$\left. \begin{aligned} p^2 &= 1 + \frac{2c}{v_{Te}} \frac{\omega_{0e}^2}{\omega^2 p} e^{-z^2} \int_0^z e^{t^2} dt; \\ \varkappa &= \frac{\sqrt{\pi}}{2} \frac{c}{v_{Te}} \frac{\omega_{0e}^2}{\omega^2 p^3} e^{-z^2}; \quad z = \frac{(\omega_{Be} - \omega) c}{\omega v_{Te} p}. \end{aligned} \right\} \tag{11.36}$$

When $\omega_{0e}^2/\omega_{Be}^2 >> 1$, the width of the absorption band can be very large. For example, with $\omega = \frac{1}{2}\omega_{Be}$, we can write

$$p^2 = 1 + \frac{\omega_{0e}^2}{\omega(\omega_{Be} - \omega)} \approx 4 \frac{\omega_{0e}^2}{\omega_{Be}^2} = \frac{16\pi c^2 m_e n}{B_0^2}.$$

In this case,

$$\varkappa = \frac{\sqrt{\pi}}{2} \times \left(\frac{B^2}{32\pi n T} \right)^{1/2} e^{-\frac{B^2}{32\pi n T}}.$$

When $8\pi nT / B_0^2 \sim 0.1$, we find $\varkappa \approx 0.1$. The absorption length $c/\omega\varkappa p$ at $B_0 = 10^4$ is a fraction of a centimeter.

The cyclotron absorption at $\theta \neq 0$ is given by a rather complicated formula which is not reproduced here. This expression can be found in a paper by Gershman [12].

In pure transverse propagation ($\theta = \pi/2$) the resonance condition becomes

$$\omega - n\omega_{Be} = 0, \tag{11.37}$$

and the spreading of the absorption line due to motion of the charges along the lines of force vanishes. The only mechanism that spreads the lines is the relativistic Doppler effect. Assuming that $\omega_{Be} = \omega_{Be}^0 \sqrt{1 - \beta^2}$, we now write (11.37) in the form

$$\omega = n\omega_{Be}^0 \sqrt{1 - \beta^2}. \tag{11.38}$$

A particle with velocity $v = 0$ absorbs at a frequency $\omega = n\omega_{Be}^0$. As the velocity of the particle increases, the frequency at which absorption occurs is reduced. The magnitude of the absorption coefficient is proportional to the number of particles with a given velocity v, this number being related to the frequency ω by Eq. (11.38); thus, the absorption is proportional to the exponential factor

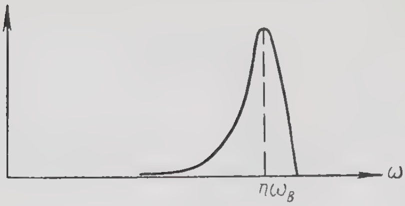

Fig. 16

$$e^{-\frac{m_0c^2}{\sqrt{1-\beta^2}T}} = e^{-\frac{m_0c^2}{T}\frac{n\omega_{Be}}{\omega}}. \tag{11.39}$$

As is evident from Eq. (9.38) for $\varepsilon_{\alpha\beta}$, at small velocities the absorption coefficient is proportional to some power of the particle velocity, so that when $v = 0$, in which case $\omega = n\omega^0_{Be}$, the absorption vanishes; then, as v increases (reduction in the frequency ω), the absorption increases. When $\Delta\omega/\omega = n\omega_{Be} - \omega/\omega > T/m_0c^2$, the further increase of the absorption coefficient is cut off by the exponential factor in Eq. (11.39). The shape of the absorption line is approximately that shown in Fig. 16.

From an analysis of curves of N^2 for $\theta = \pi/2$ in a cold plasma it is evident that characteristic oscillations and wave propagation are possible in the regions ($N^2 > 0$):

$$\left.\begin{array}{ll}
\text{a)} & 0 < \omega < \omega_2^\infty = \left(\omega_{Bi}\omega_{Be}\dfrac{\omega_{Bi}^2+\omega_{0i}^2}{\omega_{Bi}\omega_{Be}+\omega_{0i}^2}\right)^{1/2}; \\
\text{b)} & \sqrt{\omega_0^2+\dfrac{(\omega_{Be}-\omega_{Bi})^2}{4}} - \dfrac{\omega_{Be}-\omega_{Bi}}{2} < \omega < \sqrt{\omega_0^2+\omega_{Be}^2}; \\
\text{c)} & \omega > \omega_0; \\
\text{d)} & \omega > \sqrt{\omega_0^2+\dfrac{(\omega_{Be}+\omega_{Bi})^2}{4}} + \dfrac{\omega_{Be}-\omega_{Bi}}{2}.
\end{array}\right\} \tag{11.40}$$

In the first region absorption occurs at the ion cyclotron frequency and its harmonics $\omega = n\omega_{Bi}$ ($n = 1, 2, \ldots$). The total number of resonances in a dense plasma ($\omega_{0i}^2 \gg \omega_{Bi}\omega_{Be}$) is approximately equal to the square root of the ratio of the ion mass to the electron mass:

$$n_{\max} \sim \sqrt{\frac{m_i}{zm_e}}.$$

When $N\beta << 1$, the absorption at the harmonics is obviously unimportant (just as the radiation is unimportant when the particle velocity is small compared with the phase velocity of the wave). In a plasma, however, it is possible that a large number of particles can have velocities comparable with the phase velocity of the wave ($2T/m_i \sim c^2/N^2$). In the region being considered here $N^2 \sim c^2/c_A^2$ so that the latter condition assumes the form $8\pi nT_i \sim B^2$. Under these conditions the peak absorption (or radiation) can occur at a frequency $\omega = n\omega_{Be}$ with $n >> 1$. This displacement of the peak radiation intensity of a particle moving in a vacuum with a velocity close to the velocity of light is well known from the theory of electron radiation in the synchrotron.

However, the width of the absorption line depends only on the ratio of the mean particle velocity to the absolute velocity of light in vacuum (and not to the phase velocity of the wave, as is the case in the normal Doppler effect when $\theta \neq \pi/2$) and is extremely small for ions. Hence, when $\omega >> \omega_{Bi}$, one finds a large number of narrow absorption lines spaced relatively closely to each other, so that one can speak of a "quasi-continuous" absorption spectrum. In practice, collisions and inhomogeneities in the magnetic field can broaden the absorption line to a width $\Delta\omega \sim \omega_{Bi}$, so that a continuous absorption band is actually observed.

Under these conditions the absorption in the region $\theta = \pi/2$, $\omega >> \omega_{Bi}$ can be treated as follows. As we have already noted at the end of § 3, when $\omega >> \omega_{Bi}$ it is permissible to neglect the effect of the magnetic field on the ion motion so that the ion gas can be treated as an isotropic medium. The tensor $\varepsilon_{\alpha\beta}$ is then made up of the electron dielectric tensor and terms associated with the free motion of the ions. In the coordinate system in which the z axis is directed along **k** we find

$$\left.\begin{aligned} \varepsilon_{xx} &= \varepsilon_{xx}^{\mathrm{el}} - \frac{4\pi z^2e^2n_i}{m_i\omega}\langle\zeta(\omega - k_zv_z)\rangle; \\ \varepsilon_{yy} &= \varepsilon_{yy}^{\mathrm{el}} - \frac{4\pi z^2e^2n_i}{m_i\omega}\langle\zeta(\omega - k_zv_z)\rangle; \\ \varepsilon_{zz} &= \varepsilon_{zz}^{\mathrm{el}} - \frac{4\pi z^2e^2n_i}{m_i\omega}\left\langle\frac{m_iv_z^2}{T}\zeta(\omega - k_zv_z)\right\rangle. \end{aligned}\right\} \qquad (11.41)$$

The ion terms do not appear in the other elements of $\varepsilon_{\alpha\beta}$. It is evident that by means of this tensor $\varepsilon_{\alpha\beta}$ we can now treat the propagation of waves at an arbitrary angle θ. When $\theta = \pi/2$, the equations for the extraordinary wave exhibit the form in (3.41) (magnetic field along the x axis):

$$\left.\begin{aligned} (N^2 - \varepsilon_{yy}) E_y + igE_z - 0; \\ -igE_y - \varepsilon_{zz}E_z = 0. \end{aligned}\right\} \tag{11.42}$$

Since the frequency being considered $\omega \lessgtr \sqrt{\omega_{Be}\omega_{Bi}}$ is appreciably smaller than the electron cyclotron frequency, the corrections in $\varepsilon^{el}_{\alpha\beta}$ due to the thermal motion are unimportant:

$$\varepsilon^{el}_{yy} = \varepsilon^{el}_{zz} \approx \frac{\omega_{0e}^2}{\omega_{Be}^2}; \quad g = \frac{\omega_{0e}^2}{\omega\omega_{Be}}. \tag{11.43}$$

Thus,

$$\left.\begin{aligned} \varepsilon_{yy} &= \frac{\omega_{0e}^2}{\omega_{Be}^2} - \frac{\omega_{0i}^2}{\omega^2} Z\left(\frac{\omega}{kv_{Ti}}\right); \\ g &= \frac{\omega_{0e}^2}{\omega\omega_{Be}}; \\ \varepsilon_{zz} &= \frac{\omega_{0e}^2}{\omega_{Be}^2} - \frac{\omega_{0i}^2}{\omega^2}\frac{2\omega^2}{k^2v_{Ti}^2}\left[Z\left(\frac{\omega}{kv_{Ti}}\right) - 1\right]. \end{aligned}\right\} \tag{11.44}$$

We shall assume that $\omega \gg kv_{Ti}$ (in order-of-magnitude terms $\omega/k \sim c_A$, so that this condition implies, $c_A^2/v_{Ti}^2 = B_0^2/8\pi nT_i \gg 1$). Assuming that the imaginary parts are small, and using the expansion in (9.22), we have

$$\left.\begin{aligned} N^2 &= \varepsilon^0_{yy} - \frac{g^2}{\varepsilon^0_{zz}} + i\sqrt{\pi}\frac{\omega_{0i}^2}{\omega kv_{Ti}}\left[1 + \frac{2\omega^2}{k^2v_{Ti}^2}\left(\frac{g}{\varepsilon^0_{zz}}\right)^2\right] e^{-\frac{\omega^2}{k^2v_{Ti}^2}}; \\ \varepsilon^0_{yy} &= \frac{\omega_{0e}^2}{\omega_{Be}^2} - \frac{\omega_{0i}^2}{\omega^2}\left(1 + \frac{k^2v_{Ti}^2}{2\omega^2} + \ldots\right); \\ \varepsilon^0_{zz} &= \frac{\omega_{0e}^2}{\omega_{Be}^2} - \frac{\omega_{0i}^2}{\omega^2}\left(1 + \frac{3}{2}\frac{k^2v_{Ti}^2}{2\omega^2} + \ldots\right). \end{aligned}\right\} \tag{11.45}$$

When $v_{Ti} = 0$ the quantity ε^0_{zz} vanishes for $\omega = \sqrt{\omega_{Bi}\omega_{Be}}$ and, correspondingly, $N = \infty$, so that the phase velocity of the wave also vanishes. When $\omega > \sqrt{\omega_{Be}\omega_{Bi}}$, we find $N^2 < 0$, and oscillations are impossible when $T_i = 0$. If the thermal motion is introduced, as we have already noted in § 3, the phase velocity does not vanish but approaches the velocity of the particles when $\omega \to \infty$; in this case there is a strong absorption. When $\omega = \sqrt{\omega_{Be}\omega_{Bi}}$, it follows from Eq. (11.45) that the phase velocity is

$$v_p = c\left(\frac{3T_i}{m_i c^2}\right)^{1/4}\left(\frac{m_e}{m_i}\right)^{1/8}. \tag{11.46}$$

At a frequency ω several times smaller than $\sqrt{\omega_{Be}\omega_{Bi}}$, the thermal corrections in $\mathrm{Re}\,N^2$ can be neglected; furthermore, we can write $\varepsilon^0_{yy} = \varepsilon^0_{zz} = -\omega_{0i}^2/\omega^2$. Since $\omega \gg \omega_{Bi}$, then $\varepsilon^0_{yy} \ll g^2/\varepsilon^0_{zz}$ and, consequently, $\mathrm{Re}\,N^2 = -g^2/\varepsilon^0_{zz} = c^2/c_A^2$. Hence, under these conditions,

$$N^2 = \frac{c^2}{c_A^2} + i\sqrt{\pi}\left\{\frac{\omega_{0i}^2}{\omega^2}\frac{c_A}{v_{Ti}} + 2\frac{c_A c^2}{v_{Ti}^3}\right\} e^{-\frac{c_A^2}{v_{Ti}^2}}. \tag{11.47}$$

The ion absorption mechanism here is the Cerenkov mechanism. In the short-wave region (compared with the Larmor radius) the ion motion can be regarded as rectilinear, and when $N \gg 1$, the cyclotron absorption (and radiation) reduces to Cerenkov radiation and absorption. The first term in $\mathrm{Im}\,N^2$ associated with $\mathrm{Im}\,\varepsilon_{yy}$ characterizes the absorption of the energy of the transverse field of the wave, while the second term, which derives from $\mathrm{Im}\,\varepsilon_{zz}$, describes the absorption of energy of the longitudinal component of the electric field in the wave.

In the second region considered above in (11.40): the oscillations at $\omega_{0e} \gg \omega_{Be}$ can be treated in the same way as in the first region. However, the second region is rather narrow $\Delta\omega \approx \omega_{Be}^2/\omega_{0e}$. Here, Eq. (11.45) applies if the quantities ω_{0i} and v_{Ti} are replaced by ω_{0e} and v_{Te}, respectively, while ε^0_{yy}, ε^0_{zz}, and g are given by the following expressions:

$$\varepsilon^0_{yy} = 1 - \frac{\omega^2{}_{0e}}{\omega^2}\left(1 + \frac{k^2 v_{Te}^2}{\omega^2}\right);$$

$$\varepsilon^0_{zz} = 1 - \frac{\omega_{0e}}{\omega^2}\left(1 + \frac{3}{2}\frac{k^2 v_{Te}^2}{\omega^2}\right);$$

$$g = \frac{\omega_{0e}^2\omega_{Be}}{\omega^3}.$$

The phase velocity in a cold plasma vanishes when $\omega = \sqrt{\omega_{0e}^2 + \omega_{Be}^2} \approx \omega_{0e}$. If the thermal motion is introduced, we find

$$v_\Phi = c\left(\frac{12\pi n T_e}{B_0^2}\right)^{1/4}.$$

In order for the Cerenkov absorption to operate we must have $v_\Phi < c$. Hence, absorption occurs only when $12\pi n T_e / B_0^2 \ll 1$.

In the third and fourth regions in which oscillations are possible, the refractive index approaches unity and the phase velocity can be taken everywhere as equal to c. This region is important at relatively high frequencies ($\omega > \omega_B, \omega_0$) in problems concerning the loss of energy in a high-temperature plasma due to radiation by electrons at harmonics of the cyclotron frequency. This question will be discussed in detail in later volumes in this series.

§ 12. Energy Loss of a Charged Particle Moving in a Plasma

A charged particle moving in a medium produces an electromagnetic field. If the motion of the particle is specified, the field can be determined by Eqs. (6.40) and (6.51). It is evident that in turn the field will react back on the particle, thereby affecting its motion. The force exerted by the electromagnetic field (produced by the moving particle) on the charge is called the radiation friction force. The work performed by this force is negative, i.e., the charge loses energy and is decelerated.

In the presence of an ensemble of particles the energy loss per unit volume and unit time can be computed by taking (with reversed sign) the average value of the scalar product of the current density associated with the moving particles and the electric field produced by the particles. This quantity can be expressed in terms of its own spectral density, by analogy with Eq. (8.22), in the following way:

$$\overline{j_\alpha^{\text{spec}}(\mathbf{r}, t)\, E_\alpha(\mathbf{r}, t)} = \int (j_\alpha^{*\text{spec}} E_\alpha)_{\mathbf{k},\,\omega}\, d\mathbf{k}\, d\omega. \tag{12.1}$$

The Fourier component of the electric field $\mathbf{E}(\mathbf{k}, \omega)$ and the Fourier component of the current $\mathbf{j}^{\text{spec}}(\mathbf{k}, \omega)$ are related by an expression of the form

$$E_\alpha(\mathbf{k}, \omega) = L_{\alpha\beta} j_\beta^{\text{spec}}(\mathbf{k}, \omega), \tag{12.2}$$

where the tensor $L_{\alpha\beta}$ in an isotropic or anisotropic plasma is given by the following expressions, which follow from Eqs. (6.38), (6.39), and (6.50):

$$L_{\alpha\beta} = -\frac{4\pi i}{\omega\varepsilon_{\parallel}}\frac{k_\alpha k_\beta}{k^2} + \frac{4\pi i}{\omega}\frac{1}{\frac{k^2c^2}{\omega^2} - \varepsilon_{\perp}}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right); \qquad (12.3)$$

$$L_{\alpha\beta} = \frac{4\pi i}{\omega}\sum_{l=1}^{2}\frac{a_{l\alpha}b^*_{l\beta}}{\frac{k^2c^2}{\omega^2} - \varepsilon_l(\mathbf{k}\ \ \omega)}. \qquad (12.4)$$

Substituting Eq. (12.2) in Eq. (12.1), and assuming that $(j_\alpha^{*\,\mathrm{spec}} j_\beta^{\mathrm{spec}})_{\mathbf{k},\omega} = G_{\beta\alpha}(\mathbf{k}, \omega)$, we have

$$Q = -\overline{j_\alpha^{\mathrm{spec}}(\mathbf{r}, t)\, E_\alpha(\mathbf{r}, t)} = -\int G_{\beta\alpha}(\mathbf{k}, \omega)\, L_{\alpha\beta}(\mathbf{k}, \omega)\, d\mathbf{k}\, d\omega. \qquad (12.5)$$

Making use of the properties $G_{\alpha\beta}(-\mathbf{k}, -\omega) = G^*_{\alpha\beta}(\mathbf{k}, \omega)$ and $L_{\alpha\beta}(-\mathbf{k}, -\omega) = L^*_{\alpha\beta}(\mathbf{k}, \omega)$, which derive from the analogous properties of the tensor $\varepsilon_{\alpha\beta}$ (6.32), we can write an expression for the loss in the form of an integral over positive frequencies:

$$Q = -2\mathrm{Re}\int_0^\infty d\omega \int G_{\beta\alpha}(\mathbf{k}, \omega)\, L_{\alpha\beta}(\mathbf{k}, \omega)\, d\mathbf{k}. \qquad (12.6)$$

Since the tensor $G_{\alpha\beta}$ is Hermitian, the expression for Q can also be written in the form

$$Q = -\int G_{\beta\alpha}L'_{\alpha\beta}\, d\mathbf{k}\, d\omega = -2\int_0^\infty d\omega \int G_{\beta\alpha}L'_{\alpha\beta}\, d\mathbf{k}, \qquad (12.7)$$

where $L'_{\alpha\beta} = (L_{\alpha\beta} + L^*_{\beta\alpha})/2$ is the Hermitian part of the tensor $L_{\alpha\beta}$. It is evident from Eqs. (12.3) and (12.4) for $L_{\alpha\beta}$ that the quantity $L'_{\alpha\beta} = 0$ if the tensor $\varepsilon_{\alpha\beta}$ is Hermitian, so that the energy loss is due only to the absorption properties of the medium.

In order to determine the energy loss experienced by an individual particle we must use the distribution function that appears in Eq. (8.29) for $G_{\alpha\beta}$ in the form $F(\mathbf{v})d\mathbf{v} = \delta(\mathbf{v} - \mathbf{v}_0)d\mathbf{v}$. In this case,

$$G_{\alpha\beta} = nG^1_{\alpha\beta}; \qquad (12.8)$$

$$G^1_{\alpha\beta} = \frac{e_1^2}{(2\pi)^4}\int_{-\infty}^{\infty} v_\alpha(t)\, v^0_\beta\, e^{i\left(\omega t - \mathbf{k}\int_0^t \mathbf{v}(t')dt'\right)}\, dt. \qquad (12.9)$$

The energy loss for an individual particle is given by the expression

$$-\frac{dE}{dt} = -2\mathrm{Re}\int_0^\infty d\omega \int G^1_{\beta\alpha}(\mathbf{k}, \omega)\, L_{\alpha\beta}(\mathbf{k}, \omega)\, d\mathbf{k}. \qquad (12.10)$$

In the absence of a magnetic field [$v_\alpha(t)$ = const = v_α] we have

$$G^1_{\alpha\beta}(\mathbf{k}, \omega) = \frac{e_1^2 v_\alpha v_\beta}{(2\pi)^3}\, \delta(\omega - \mathbf{k}\mathbf{v}). \qquad (12.11)$$

Substituting Eqs. (12.11) and (12.3) in Eq. (12.10), we have

$$-\frac{dE}{dt} = -\frac{e_1^2}{\pi^2}\mathrm{Re}\int_0^\infty i\,\frac{d\omega}{\omega}\int\left\{-\frac{(\mathbf{k}\mathbf{v})^2}{k^2\varepsilon_{\parallel}} + \frac{v^2 - \dfrac{(\mathbf{k}\mathbf{v})^2}{k^2}}{\dfrac{k^2c^2}{\omega^2} - \varepsilon_{\perp}}\right\} \times$$
$$\times\, \delta(\omega - \mathbf{k}\mathbf{v})\, d\mathbf{k}. \qquad (12.12)$$

We now divide $\varepsilon_{\parallel}$ and $\varepsilon_{\perp}$ into real and imaginary parts:

$$\varepsilon_{\parallel} = \varepsilon_{\parallel}^{(1)} + i\varepsilon_{\parallel}^{(2)}; \qquad \varepsilon_{\perp} = \varepsilon_{\perp}^{(1)} + i\varepsilon_{\perp}^{(2)},$$

so that the factors appearing in $\varepsilon_{\parallel}$ and $\varepsilon_{\perp}$ in the integrand can be written

$$\mathrm{Re}\left(\frac{i}{\varepsilon_{\parallel}}\right) = \frac{\varepsilon_{\parallel}^{(2)}}{|\varepsilon_{\parallel}|^2}; \quad \mathrm{Re}\left(\frac{-i}{\dfrac{k^2c^2}{\omega^2} - \varepsilon_{\perp}}\right) = \frac{\varepsilon_{\perp}^{(2)}}{\left|\dfrac{k^2c^2}{\omega^2} - \varepsilon_{\perp}\right|^2}. \qquad (12.13)$$

If the imaginary parts of the electric permittivity are vanishingly small, by analogy with Eq. (9.13) these expressions can be written in the form

$$\mathrm{Re}\left(\frac{i}{\varepsilon_{\parallel}}\right) = \pi\delta(\varepsilon_{\parallel}); \quad \mathrm{Re}\left(\frac{-i}{\dfrac{k^2c^2}{\omega^2} - \varepsilon_{\perp}}\right) = \pi\delta\left(\frac{k^2c^2}{\omega^2} - \varepsilon_{\perp}\right). \qquad (12.14)$$

It is evident from Eqs. (9.26) and (9.27) for $\varepsilon_{\perp}$ and $\varepsilon_{\parallel}$ that these imaginary parts will in fact be small when $\omega \gg kv_T$. According to Eq. (12.12), the contribution to the energy loss comes from frequencies

$$\omega = \mathbf{k}\mathbf{v} = kv\cos\theta. \qquad (12.15)$$

Consequently, Eq. (12.14) can be used in the case in which the velocity of the particle is appreciably greater than the mean thermal velocity of the plasma particles ($v \gg v_T$).

We shall confine our attention to this case. Let us first investigate the second part of the integral in (12.12), which corresponds to the transverse wave. According to the second relation in (12.14),

$$k = \frac{\omega}{c}\sqrt{\varepsilon_\perp}. \tag{12.16}$$

Substituting this value of k in Eq. (12.15) we find the condition for which the energy is lost in the form of transverse waves:

$$1 = \frac{v}{c}\sqrt{\varepsilon_\perp}\cos\theta. \tag{12.17}$$

This condition, which is called the Cerenkov condition, is obviously not satisfied in the plasma since $\varepsilon_\perp < 1$; consequently, transverse waves are not emitted in a cold plasma ($v \gg v_T$).

In a medium for which $\varepsilon_\perp > 1$ the second part of the integral in Eq. (12.12) is of the form ($\mu \equiv \cos\theta$)

$$-\left(\frac{dE}{dt}\right)_\perp = \frac{e_1^2 v}{c^2}\int_0^\infty \omega\, d\omega \int_{-1}^{1}(1-\mu^2)\, d\mu \int_0^\infty \delta\left(k^2 - \frac{\omega^2}{c^2}\varepsilon_\perp\right) \times$$

$$\times\, \delta\left(\mu - \frac{\omega}{kv}\right) dk^2. \tag{12.18}$$

Carrying out the simple integration over k^2 and μ, we find

$$-\left(\frac{dE}{dt}\right)_\perp = \frac{e_1^2 v}{c^2}\int_0^\infty \left(1 - \frac{c^2}{\varepsilon_\perp v^2}\right)\omega\, d\omega. \tag{12.19}$$

This is the well-known expression for Cerenkov radiation.

We now investigate the energy loss in a plasma associated with the longitudinal field (the so-called polarization losses). Assuming that $\delta(\varepsilon_{\|}) = \delta(1 - \omega_0^2/\omega^2) = (\omega^3/2\omega_0^2)\,\delta(\omega - \omega_0)$, we can write the first part of the integral in (12.12) in the form

$$-\left(\frac{dE}{dt}\right)_{\|} = \int e_1^2 v k \mu^2 \delta\left(\mu - \frac{\omega}{kv}\right)\delta(\omega - \omega_0)\, dk\, d\mu\, d\omega. \tag{12.20}$$

We first carry out the integration over ω and μ. Since the maximum value of μ is 1, it follows from the first δ-function that the minimum value of k for which the integrand is nonvanishing is given by

$$k_{\min} = \frac{\omega_0}{v}. \tag{12.21}$$

Thus

$$-\left(\frac{dE}{dt}\right)_{\parallel} = \frac{e_1^2\omega_0^2}{v}\int\limits_{\omega_0/v}^{k_0}\frac{dk}{k} = \frac{e_1^2\omega_0^2}{v}\ln\frac{k_0 v}{\omega_0}. \tag{12.22}$$

The divergence of the integral at the upper limit is associated with the fact that large values of k in the expression for the electric field (6.40) correspond to small distances from the moving particle, in which case the medium can no longer be regarded as continuous. In order to avoid this divergence the integration is usually carried out to some specific limit. Akhiezer and Sitenko [44] take $k_0 = 2/\gamma\rho_0$; $\gamma = 1.78\ldots$, where ρ_0 is an impact parameter which divides the interactions between the particle and the electrons in the medium into binary interactions ($\rho < \rho_0$) and collective interactions ($\rho > \rho_0$). This parameter satisfies the condition

$$a \ll \varrho_0 \ll \frac{v}{\omega_0}; \qquad a = \frac{e_1 e\,(m_1 + m_e)}{m_1 m_e v^2}. \tag{12.23}$$

Under these conditions the theory of binary collisions leads to the following expression for the loss:

$$-\left(\frac{dE}{dt}\right)_{\varrho<\varrho_0} = \frac{e_1^2\omega_0^2}{v}\ln\frac{\varrho_0}{a}, \tag{12.24}$$

so that the total loss is given by

$$-\frac{dE}{dt} = \frac{e_1^2\omega_0^2}{v}\ln\frac{2v}{\gamma\omega_0 a}. \tag{12.25}$$

Landau and Lifshits [25] use a different method, relating the upper limit of the integration k_0 to the transfer of momentum ($q = \hbar k_0$) to the electron from the medium if this momentum transfer is sufficiently large, so that the electron can be regarded as free.* According to classical mechanics, the momentum transfer is related to the impact parameter by the following relation ($\rho \gg a$):

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*It is known from quantum mechanics that in the interaction of a field of the form $e^{i\mathbf{kr}}$ with a free electron the electron receives a momentum $\hbar\mathbf{k}$.

$$q = \frac{2e_1 e}{v}\frac{1}{\varrho}. \tag{12.26}$$

Consequently, the formula for binary collisions can be rewritten in the form (to be definite we take $m_1 >> m_e$):

$$-\left(\frac{dE}{dt}\right)_{q>q_0} = \frac{e_1^2\omega_0^2}{v}\ln\frac{2m_1 v}{q_0}. \tag{12.27}$$

Comparing this with (12.22) we write the energy losses in the form

$$-\frac{dE}{dt} = \frac{e_1^2\omega_0^2}{v}\ln\frac{2m_1 v^2}{\hbar\omega_0}. \tag{12.28}$$

Equations (12.25) and (12.28) correspond to the Bohr energy loss formula, derived by classical methods, and the Bethe formula, obtained by a quantum-mechanical method (in the Born approximation). To obtain the relation between these formulas it is more accurate to consider the collective interaction. We regard the region $\rho < \rho_0$ as a vacuum which contains charges, while the region $\rho > \rho_0$ is regarded as a medium with an electrical permittivity $\varepsilon = 1 - (\omega_0^2/\omega^2)$. The energy loss in this medium is given by the expression (cf. Problem 1 of §12):

$$-\left(\frac{dE}{dt}\right)_{\varrho>\varrho_0} = \frac{e_1^2\omega_0^2}{v}\int_0^{k_0}\frac{J_0(k_\perp\varrho_0)}{k_\perp^2 + \frac{\omega_0^2}{v^2}}k_\perp\, dk_\perp. \tag{12.29}$$

When $k_0\rho_0 >> 1$, the integration can be taken to infinity, in which case the integral is equal to $K_0(\rho_0\omega_0/v) \approx \ln[(2/\gamma)(v/\rho_0\omega_0)]$. Combining this result with Eq. (12.24), we obtain the classical formula, Eq. (12.25). When $k_0\rho_0 << 1$, we can write $J_0(k_\perp\rho_0) = 1$. In this case the integral is given by $\ln(k_0 v/\omega_0)$. Combining this result with Eq. (12.27) we obtain the quantum-mechanical formula, Eq. (12.28). According to Eq. (12.26), $k_0\rho_0 = q_0\rho_0/\hbar = 2ee_1/\hbar v$. Consequently, Eq. (12.25) is valid when $2ee_1/\hbar v >> 1$, whereas Eq. (12.28) is valid when $2ee_1/\hbar v << 1$, this being the well-known criterion for the applicability of the Born approximation.

We now consider the energy losses in a plasma in a magnetic field [45-57]. The expression $G_{\alpha\beta}L_{\alpha\beta}$ is an invariant with respect to rotation of the coordinate axes, so that it can be computed in a coordinate system x, y, z in which B_0 is along the z axis, while **k** lies in the xz plane. According to Eq.

(8.35), in this coordinate system the tensor $G'_{\alpha\beta}(\mathbf{k}, \omega)$ is given by

$$G^{1}_{\alpha\beta}(\mathbf{k}, \omega) = \frac{e_1^2}{8\pi^3} \sum_{n=-\infty}^{\infty} \pi^{(n)}_{\alpha\beta} \delta(\omega - k_z v_z - n\omega_B), \tag{12.30}$$

where $\pi^{(n)}_{\alpha\beta}$ is the value of $\Pi^{(n)}_{\alpha\beta}$ when $\varkappa = 0$

$$\left.\begin{aligned}
&\pi^{(n)}_{xx} = v_\perp^2 \frac{n^2}{x^2} J_n^2(x); \quad \pi^{(n)}_{xy} = -\pi^{(n)}_{yx} = i v_\perp^2 \frac{n}{x} J_n(x) J_n'(x);\\
&\pi^{(n)}_{yy} = v_\perp^2 J_n'^2(x); \quad \pi^{(n)}_{yz} = -\pi^{(n)}_{zy} = -i v_\perp v_z \frac{n}{x} J_n(x) J_n'(x);\\
&\pi^{(n)}_{zz} = v_z^2 J_n^2(x); \quad \pi^{(n)}_{xz} = \pi^{(n)}_{zx} = v_\perp v_z \frac{n}{x} J_n^2(x);\\
&x \equiv \frac{k_\perp v_\perp}{\omega_B}; \quad \omega_B = \omega_B^0 \sqrt{1-\beta^2}.
\end{aligned}\right\} \tag{12.31}$$

In this coordinate system the polarization vectors **a** and **b** are

$$\mathbf{a} = a_y \{i\alpha_x^0, 1, i\alpha_z^0\}; \quad \mathbf{b}^* = a_y \{-i\alpha_x^0, 1, -i\alpha_z^0\}. \tag{12.32}$$

Taking account of the fact that $a_y^2 = 1/(1 + \alpha_x^2)$ [cf. Eq. (6.55)], it is easy to show that

$$\pi^{(n)}_{\beta\alpha} a_\alpha b_\beta^* = \frac{1}{1+\alpha_x^2} \left\{\left(\alpha_x^0 \frac{v_\perp n}{x} + \alpha_z^0 v_z\right) J_n(x) + v_\perp J_n'(x)\right\}^2. \tag{12.33}$$

Now, substituting the values of $L_{\alpha\beta}$ (12.4) and $G_{\alpha\beta}$ (12.30) [taking account of Eq. (12.33)] in Eq. (12.10), we find an expression for the energy loss of a charge moving along a helical trajectory:

$$-\frac{dE}{dt} = -\operatorname{Re} \sum_{n=-\infty}^{\infty} \sum_{l=1}^{2} \frac{e_1^2}{\pi^2} \int_0^\infty i \frac{d\omega}{\omega} \int \frac{d\mathbf{k}}{1+\alpha_{xl}^2} \times$$
$$\times \frac{\left\{\left(\alpha_{xl}^0 \frac{v_\perp n}{x} + \alpha_{zl}^0 v_z\right) J_n + v_\perp J_n'\right\}^2}{\frac{k^2c^2}{\omega^2} - \varepsilon_l(\omega, \mathbf{k})} \delta(\omega - k v_z \cos\theta - n\omega_B). \tag{12.34}$$

In deriving this formula we have not assumed that the absorption is weak or that the plasma is cold. Hence, in principle, Eq. (12.34) can be used to determine the deceleration of a charge in the general case.

However, we will consider Eq. (12.34) for the simple case of a cold plasma. This formula contains the losses due to radiation and the losses due to polarization which appear in the form of longitudinal waves. In the absence of a magnetic field, the radiation of the longitudinal waves occurs at the Langmuir frequency ω_0, this being indicated, for example, by the presence of the δ-function $\delta(\omega - \omega_0)$ in Eq. (12.20) for the polarization losses. As we have seen in § 2, in the presence of a magnetic field the frequency of the longitudinal waves in a cold plasma is determined by the condition $\varepsilon_l = \infty$, which depends on the angle θ. In this case the calculations become extremely complicated and will not be given here. We note, however, that like Eq. (12.22), which gives the polarization losses for $B_0 = 0$, Eq. (12.34) also contains a divergent integral; this divergent integral arises because of the improper use of the equations at small distances from the charged particle. The divergence can be avoided by the method used above in the $B_0 = 0$ case.

Departing from the polarization losses, for the time being we shall not consider the region of frequencies and angles in Eq. (12.34) where $\varepsilon_l = \infty$, and where a divergence appears. In a cold plasma, the expression $\varepsilon_l = N_l^2$ is independent of k and is given by Eq. (3.10).

In a transparent plasma, proceeding by analogy with Eq. (12.14), we can write

$$\operatorname{Re} \frac{-i}{\dfrac{k^2c^2}{\omega^2} - N_l^2} = \pi\delta\left(\frac{k^2c^2}{\omega^2} - N_l^2\right). \tag{12.35}$$

Since $k^2c^2/\omega^2 > 0$, the argument of the δ-function can vanish only when $N_l^2 > 0$. Thus the charged particle radiates only at frequencies and angles for which propagation is possible. Carrying out the integration over k and replacing n by $-n$ in the summation over n from $-\infty$ to 0, we have

$$-\frac{dE}{dt} = \sum_{l=1}^{2} \int_{-1}^{1} d\mu \int_{0}^{\infty} d\omega \, \frac{e_1^2\omega^2}{c^3} \frac{N_l}{1+\alpha_x^2} \left\{ I_0 + \sum_{n=1}^{\infty} (I_n + I_{-n}) \right\}; \tag{12.36}$$

$$I_0 = (\alpha_z^0 v_z J_0 + v_\perp J_0')^2 \, \delta(\omega - \omega\beta_\parallel N_l \mu); \tag{12.37}$$

$$I_n = \left[\left(\alpha_x^0 \frac{v_\perp n}{x_l} + \alpha_z^0 v_z\right) J_n + v_\perp J_n'\right]^2 \delta(\omega - n\omega_B - \omega\beta_\parallel N_l \mu); \quad (12.38)$$

$$I_{-n} = \left[\left(\alpha_x^0 \frac{v_\perp n}{x_l} - \alpha_z^0 v_z\right) J_n - v_\perp J_n'\right]^2 \delta(\omega + n\omega_B - \omega\beta_\parallel N_l \mu). \quad (12.39)$$

Here, $\beta_\parallel = v_z/c$; the argument of the Bessel function is

$$x_l = \frac{\omega}{\omega_B} \frac{v_\perp}{c} N_l \sqrt{1-\mu^2}. \quad (12.40)$$

The arguments of the δ-function indicate that I_0 governs the Cerenkov radiation, while I_n is responsible for the cyclotron radiation [for the normal Doppler effect (with $\omega_B > 0$)] and I_{-n} corresponds to the cyclotron radiation for the anomalous Doppler effect (this term is nonzero only when $N_l \gg 1$). If the radiator is an electron, then $\omega_B < 0$. In this case it is more convenient to write Eqs. (12.37)-(12.39) with a positive cyclotron frequency $\omega_{Be} = -\omega_B > 0$:

$$I_0 = (\alpha_z^0 v_z J_0 - v_\perp J_0')^2 \delta(\omega - \omega\beta_\parallel N_l \mu); \quad (12.37a)$$

$$I_n = \left[\left(\alpha_x^0 \frac{v_\perp n}{x_l} + \alpha_z^0 v_z\right) J_n - v_\perp J_n'\right]^2 \delta(\omega - n\omega_{Be} - \omega\beta_\parallel N_l \mu); \quad (12.38a)$$

$$I_{-n} = \left[\left(\alpha_x^0 \frac{v_\perp n}{x_l} - \alpha_z^0 v_z\right) J_n + v_\perp J_n'\right]^2 \delta(\omega + n\omega_{Be} - \omega\beta_\parallel N_l \mu). \quad (12.39a)$$

Here, we now take $x_l = (\omega/\omega_{Be})(v_\perp/c) N_l \sqrt{1-\mu^2}$. Equation (12.38a) is obtained from Eq. (12.39) while Eq. (12.39a) is obtained from Eq. (12.38) if ω_B is replaced by ω_{Be}. Using Eq. (12.36), it is an easy matter to obtain a formula for the radiation of the charged particle in vacuum. In doing this one must take account of the fact that $\alpha_z = 0$, because the electromagnetic waves in the vacuum are transverse waves. Since $\alpha_{x1}\alpha_{x2} = -1$ [cf. Eq. (1.32)], using the transformation formula (6.57) and writing $\alpha_{x1} = \alpha_x$, we have

$$\left.\begin{aligned} \alpha_{x1}^0 &= \alpha_x \cos\theta; & \alpha_{z1}^0 &= -\alpha_x \sin\theta; \\ \alpha_{x2}^0 &= -\frac{1}{\alpha_x}\cos\theta; & \alpha_{z2}^0 &= \frac{1}{\alpha_x}\sin\theta. \end{aligned}\right\} \quad (12.41)$$

The terms I_0 and I_{-n} make contributions only when the velocity of the charged particle is greater than the velocity of light; these terms vanish in vacuum. In what follows, we assume that $N_1 = N_2 = 1$, and combine both polarizations so that

$$-\frac{dE}{dt}=\sum_{n=1}^{\infty}\int_{-1}^{1}d\mu\int_{0}^{\infty}d\omega\,\frac{e_1^2\omega^2}{c^3}\left\{\left(\frac{v_\perp n}{x}\cos\theta-v_z\sin\theta\right)^2 J_n^2(x)+v_\perp^2 J'^2(x)\right\}\times$$
$$\times\,\delta(\omega-n|\omega_B|-\omega\beta_\parallel\mu). \qquad (12.42)$$

(The factor $1+\alpha_x^2$ in the denominator is canceled by the same factor in the numerator.) The same expression can be obtained from the general formula (12.7) by assuming, in accordance with Eqs. (12.3) and (12.14):

$$L'_{\alpha\beta}=-\frac{4\pi^2}{\omega}\,\delta\left(\frac{k^2c^2}{\omega^2}-1\right)\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right). \qquad (12.43)$$

Let us examine the qualitative dependence of the radiation intensity of the n-th harmonic on harmonic number n. We first consider the case $v_z = 0$. In this case, according to Eq. (12.38), the radiation occurs at frequencies $\omega = n\omega_B$ and, according to Eq. (12.36), the radiation intensity is proportional to the square of the frequency (consequently, to the square of the harmonic number n^2). Thus, approximately, the intensity is proportional to the square of the Bessel function $J_n^2[n(v_\perp/v_p)\sqrt{1-\mu^2}]$, where $v_p = c/N$ is the phase velocity of a wave emitted in the direction θ with respect to the magnetic field. If the velocity of the particle is smaller than the phase velocity, the function $J_n[n(v_\perp/v_p)\sqrt{1-\mu^2}]$ diminishes monotonically as the number n increases, and this reduction is faster, the smaller the value of the ratio v/v_p. If $v_\perp/v_p \ll 1$, in particular, the product nJ_n is a maximum at $n = 1$. When $v_\perp \sim v_p$, the Bessel function falls off with increasing n at a slower rate and because of the factor n, the product nJ_n can reach a maximum at relatively large values of n. Thus, in particular, when $1-(v/v_p) \ll 1$ in an isotropic medium the maximum radiation intensity is shifted in the direction of higher frequencies:

$$n\sim\frac{1}{\left(1-\frac{v}{v_\Phi}\right)^{3/2}}.$$

In the betatron, for example, the radiation from an ultrarelativistic electron can be appreciable at $n \sim 10^5$. In a plasma, even a small particle velocity ($v \ll c$) can be comparable with the velocity of propagation of the radiated waves $v_p = c/N$, so that the refractive index of plasma can be much greater than unity in some region of frequencies. Obviously, the peak radiation of a nonrelativistic particle cannot be shifted into a region of extremely high frequencies because the refractive index approaches unity when $\omega >> \omega_0$. However, when $\omega \lesssim \omega_0$, the distribution of radiation intensity from a charged particle in a plasma can be appreciably different from that in a vacuum.

The displacement of the peak radiation intensity toward higher values of n is found in purest form for nonrelativistic motion of ions [52]. As we have seen in § 2, the refractive index is approximately $N \sim \omega_{0i}/\omega_{Bi}$ over a very wide frequency range (extending from $\omega = 0$ to $\omega \sim \omega_{Bi}\sqrt{m_i/m_e}$. Thus, when $v \sim c(\omega_{Bi}/\omega_{0i}) = c_A$, the region in which the phase velocity is comparable with the velocity of the particle extends to values of n in the tens ($n \lesssim \sqrt{m_i/m_e}$). A calculation [52] shows, for example, that when $\omega_{0i} > \omega_{Bi}$, the maximum radiation intensity (in the B branch) occurs at n = 3 when $(v/c)(\omega_{0i}^2/\omega_{Bi}^2) = 0.5$, and at n = 25 when $(v/c)(\omega_{0i}^2/\omega_{Bi}^2) = 0.9$.

Aside from effects associated with the fact that wave propagation velocities can be slow ($v_p < c$), there is an additional effect in a plasma which leads to a marked difference between the radiation intensity in a plasma and in vacuum, even when the particle velocity is much smaller than the phase velocity ($v << v_p$) [49]. This effect stems from a feature of the polarization of waves at $\omega = \omega_{Bi}$ and $\omega = \omega_{Be}$ that has been noted in § 3. As we have indicated several times, the effective radiation (absorption) of waves by charged particles occurs when there is a resonance between the particle and the wave. If $v_z = 0$, resonance requires that the vector **E** rotate in the direction of rotation of the particle. But at $\omega = \omega_{Bi}$ and $\omega = \omega_{Be}$, there is no wave with this polarization (except for $\theta = 0$); hence, when $v << v_p$ one expects an absence of radiation in a plasma at these frequencies. When $x << 1$, the radiation intensity at frequency $\omega = n\omega_B$ in a plasma is

$$J_n = \frac{e_1^2\omega_B^2 v_\perp^2}{c^3} \sum_{l=1}^{2} \int_{-1}^{1} \frac{N_l}{1+\alpha_{xl}^2} \left(\frac{x^{n-1}}{n!\,2^n}\right)^2 (1 \pm \alpha_x^0)^2\, d\mu. \qquad (12.44)$$

The upper sign here refers to ion radiation and the lower to radiation from an electron.

Correspondingly, in vacuum we find

$$J_n = \frac{e_1^2\omega_B^2 v_\perp^2}{c^3} \int_{-1}^{1} \left(\frac{x^{n-1}}{n!\,2^n}\right)^2 (1+\mu^2)\, d\mu.$$

With n = 1 in a plasma, we find $\alpha_X^0 = -1$ for the ions and $\alpha_X^0 = +1$ for the electrons [cf. Eqs. (3.64) and (3.65)] so that $J_1 = 0$. However, in a vacuum in the nonrelativistic case ($\lambda \ll 1$), all of the radiation is found at the fundamental (n = 1).

If the velocity of the particle is nonzero along the lines of force ($v_z \neq 0$), as a consequence of the Doppler effect the frequency of radiation will differ from a cyclotron frequency and can fall into a region in which $N^2 \gg 1$ [for the ions this is branch A (cf. Fig. 6); for the electrons it is branch B with $\omega_0 \gg \omega_{Be}$ and branch E_S when $\omega_0 \ll \omega_{Bg}$]. Under these conditions there can be a strong radiation at frequencies close to the cyclotron frequency. The results of numerical calculations of the intensity of this radiation are given in [56].

In concluding this section we consider the thermal radiation of a self-absorbing plasma. In exactly the same way as we have done for a single charged particle, we can determine the energy loss for an ensemble of particles by taking the appropriate tensor $G_{\alpha\beta}$. If the radiators are the same plasma particles that are responsible for the electrical permittivity, the radiation can be called fluctuation radiation or thermal radiation. Let us find the intensity of the thermal radiation for a Maxwellian distribution of plasma particles for the case in which the absorption in the plasma is negligibly small (radiation transparent). If the absorption is weak, the polarization vectors **a** and **b** are the same, so that the Hermitian part of the tensor $L_{\alpha\beta}$ is

$$L'_{\alpha\beta} = -\frac{4\pi^2\omega}{c^2}\sum_l a_{l\alpha}\overset{*}{a}_{l\beta}\delta\left(k^2 - \frac{\omega^2}{c^2}N_l^2\right). \tag{12.46}$$

For a Maxwellian velocity distribution, the tensor $G_{\alpha\beta}$ can be expressed in terms of the dielectric tensor (8.26):

$$G_{\alpha\beta}(\mathbf{k}, \omega) = \frac{T}{(2\pi)^5}\,\omega\,\frac{\varepsilon_{\alpha\beta} - \overset{*}{\varepsilon}_{\alpha\beta}}{2i}. \tag{12.47}$$

Invoking the orthonormal properties of the vectors **a** (6.46) we have

$$\frac{(\varepsilon_{\alpha\beta} - \overset{*}{\varepsilon}_{\beta\alpha})\,a_{\beta l}\overset{*}{a}_{\alpha l}}{2i} = \operatorname{Im} N_e^2. \tag{12.48}$$

Substituting $G_{\alpha\beta}$ and $L_{\alpha\beta}$ in Eq. (12.7), taking account of the last relation, and introducing the absorption coefficient $\tilde{\alpha} = (\operatorname{Im} N^2/N)(\omega/c)$, we have

$$Q = \sum_{l=1}^{2} \int d\Omega \int_0^\infty \frac{T\omega^2}{8\pi^3 c^2} N^2 \tilde{\alpha}\, d\omega. \quad (12.49)$$

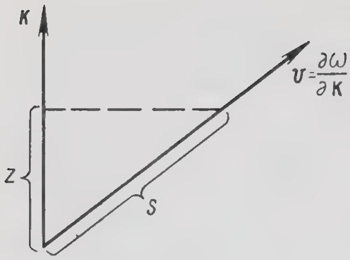

Fig. 17

Here, $d\Omega$ is an element of solid angle in the direction of the wave vector. In order to obtain an expression for the emissivity of the medium, we must write Eq. (12.49) in the form

$$Q = \int d\Omega_0 \sum_{l=1}^{2} \int_0^\infty \eta_{\omega l}\, d\omega, \quad (12.50)$$

where $d\Omega_0$ is an element of solid angle in the direction of energy propagation, i.e., in the direction of the group velocity $\mathbf{v}_{gr} = \nabla_k \omega$. In addition to transforming the solid angles, we must replace the absorption coefficient $\tilde{\alpha}$ in the direction of the wave vector α by an absorption coefficient ω along the ray, having defined the relation between these quantities by $e^{-\tilde{\alpha}z} = e^{-\tilde{\alpha}s}$. Evidently $s = z/\cos(\widehat{\mathbf{k}\mathbf{v}}_{gr})$ (Fig. 17); consequently,

$$\alpha = \tilde{\alpha} \cos(\widehat{\mathbf{k}\mathbf{v}_{gr}}). \quad (12.51)$$

Comparing Eqs. (12.49) and (12.50) we can find an expression for the emissivity:

$$\eta_{\omega l} = \frac{1}{2} I_{0\omega} N^2 \frac{d\Omega}{d\Omega_0} \frac{\alpha}{\cos(\widehat{kv_{gr}})}, \quad (12.52)$$

where $I_{0\omega}$ is the equilibrium intensity of the radiation in vacuum

$$I_{0\omega} = \frac{T\omega^2}{4\pi^3 c^2}. \quad (12.53)$$

The values of $d\Omega/d\Omega_0$ and $\cos(\widehat{kv}_{gr})$ are computed in the monographs [26, 36]. We present the results here without derivation:

$$\left.\begin{aligned} \frac{d\Omega}{d\Omega_0} &= -\frac{\sin\theta}{\frac{\partial}{\partial\theta}\left\{\frac{\omega N\cos\theta + \sin\theta\,\partial(\omega N)/(\partial\theta)}{\sqrt{\omega^2 N^2 + [\partial(\omega N)/\partial\theta]^2}}\right\}}; \\ \frac{1}{\cos(\widehat{kv_{gr}})} &= \sqrt{1 + \frac{1}{\omega^2 N^2}\left[\frac{\partial(\omega N)}{\partial\theta}\right]^2}. \end{aligned}\right\} \quad (12.54)$$

In an isotropic medium we have $d\Omega/d\Omega_0 = 1$ and the total emissivity is

$$\eta_\omega = a_\omega I_{0\omega} N^2. \tag{12.55}$$

The group velocity and the wave velocity are in the same direction in an anisotropic medium in the frequency region in which the refractive index is close to unity ($N^2 \approx 1$). In this case,

$$\eta_{\omega l} = \frac{1}{2} I_{0\omega} a_l. \tag{12.56}$$

Equations (12.52), (12.55), and (12.56), which relate the emissivity of the medium and the absorption coefficient, represent Kirchhoff's law for weakly absorbing media; this law states that the ratio of the emissivity to the absorption coefficient is equal to the intensity of the equilibrium radiation of the medium:

$$\frac{\eta_\omega}{a_\omega} = I_\omega. \tag{12.57}$$

Problem 1. A charged particle e_1 moves with velocity v along the axis of a channel of radius ρ_0 surrounded by a plasma ($\varepsilon = 1 - \omega_0^2/\omega^2$). Determine the polarization loss when $\rho_0 \ll v/\omega_0$.

Solution. We start with the expression

$$-\frac{dE}{dt} = -2\,\mathrm{Re}\int_0^\infty d\omega \int e_1 v E_z(\mathbf{k}, \omega)\, d\mathbf{k} =$$

$$= 2e_1 v \int_0^\infty d\omega k_z \,\mathrm{Im}\, \varphi(\mathbf{k}, \omega)\, d\mathbf{k}, \tag{1}$$

where $\varphi(\mathbf{k}, \omega)$ is the Fourier component of the scalar potential. This can be expressed as follows:

$$\varphi(\mathbf{k}, \omega) = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\chi \int_0^\infty f(k_z, \varrho)\, e^{-ik_\perp \varrho \cos\chi} \varrho\, d\varrho, \tag{2}$$

where $f(k_z, \rho)$, in turn, is the Fourier component of the scalar potential expanded in time and in the longitudinal coordinate z. The function f satisfies the equation ($\rho \neq 0$)

$$\frac{\partial^2 f}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial f}{\partial \varrho} - k_z^2 f = 0. \tag{3}$$

The solutions for $\rho < \rho_0$ and $\rho > \rho_0$ are:

$$f_i = \frac{e_1}{\pi} K_0(k_z\varrho)\,\delta(\omega - k_z v) + BI_0(k_z\varrho); \tag{4}$$

$$f_e = CK_0(k_z\varrho). \tag{5}$$

The constants B and C are found from the continuity conditions on the potential and the radial component of the electric induction. Using the relation

$$\mathrm{Im}\,\frac{1}{\varepsilon} = \pi\delta(\varepsilon) = \frac{\pi}{2}\,\omega_0\delta(\omega - \omega_0) \tag{6}$$

and the condition $k_z\rho_0 = (\omega_0/v)\rho_0 \ll 1$, we have

$$\mathrm{Im}\,C = \mathrm{Im}\,B = \frac{e_1\omega_0}{2}\,\delta(\omega - k_z v)\,\delta(\omega - \omega_0); \tag{7}$$

$$\mathrm{Im}\,\varphi(\mathbf{k},\omega) = \frac{e_1\omega_0}{2v}\,\delta\left(\frac{\omega}{v} - k_z\right)\delta(\omega - \omega_0)\,\frac{1}{2\pi}\,\frac{J_0(k_\perp\varrho_0)}{k_\perp^2 + k_z^2}. \tag{8}$$

Carrying out the integration in Eq. (1) with respect to ω and k_z, and assuming that fundamental quantum-mechanical considerations set an upper limit on $k_\perp$, the critical value being designated by k_0 (cf. footnote on page 108), we find

$$-\frac{dE}{dt} = \frac{e_1^2\omega_0^2}{v}\int_0^{k_0} \frac{J_0(k_\perp\varrho_0)}{k_\perp^2 + \frac{\omega_0^2}{v^2}}\,k_\perp\,dk_\perp. \tag{9}$$

Problem 2. Derive a formula for the polarization loss in a plasma in the presence of a magnetic field.

Solution. The polarization loss can be computed by taking the negative of the work done on a charged particle by the longitudinal field produced by this particle in a plasma. From the equation $\mathrm{div}\,\mathbf{D} = 4\pi\rho^{\mathrm{spec}}$ we have

$$ik\varepsilon_{\alpha\beta}E_\beta(k,\omega) = 4\pi\varrho^{\mathrm{spec}}(\mathbf{k},\omega). \tag{1}$$

Taking $\mathrm{rot}\,\mathbf{E} = 0$ or $E_\beta = (k_\beta/k)E$, and expressing $\rho(\mathbf{k},\omega)$ in terms of $\mathbf{j}(\mathbf{k},\omega)$ through the equation of continuity $[\rho = (\mathbf{kj})/\omega]$, we have

$$\mathbf{E}(\mathbf{k}, \omega) = i\frac{4\pi}{\omega}\frac{\mathbf{k}(\mathbf{kj})}{k^2\varepsilon_{\parallel}}, \tag{2}$$

where $\varepsilon_{\parallel} = (k_\alpha k_\beta/k^2)\,\varepsilon_{\alpha\beta}$ is the longitudinal electrical permittivity, an expression for which is obtained from Eq. (9.10):

$$\varepsilon_{\parallel} = 1 - i\sum\frac{4\pi e^2 n}{\omega k^2}\int d\mathbf{p}_0\int_0^\infty (\mathbf{k}\mathbf{v}(t))\left(\mathbf{k}\frac{\partial F_0}{\partial \mathbf{p}_0}\right)\times$$

$$\times\, e^{i\left(\omega t - \mathbf{k}\int_0^t \mathbf{v}(t')\,dt'\right)}\,dt. \tag{3}$$

If $F_0 = F_0(\varepsilon_\perp, \varepsilon_\parallel)$, in accordance with Eq. (9.34) with $\varepsilon_\parallel = \varepsilon^0_{xx}\sin^2\theta + \varepsilon^0_{zz}\cos^2\theta + \varepsilon^0_{xz}\sin 2\theta$, we have

$$\varepsilon_{\parallel} = 1 + \sum\frac{4\pi e^2 n}{\omega}\int J_n^2\left(\frac{k_\perp v_\perp}{\omega_B}\right)\left\{\left(\frac{n\omega_B}{k_\perp}\sin\theta + v_z\cos\theta\right)^2 F_1\zeta_n + \right.$$

$$\left. + \frac{n^2\omega_B^2}{k_\perp^2\omega}F_2\sin^2\theta\right\}d\mathbf{p}. \tag{4}$$

In a cold plasma,

$$\varepsilon_{\parallel} = 1 - \sum\frac{\omega_0^2}{\omega^2}\,\frac{\omega^2 - \omega_B^2\cos^2\theta}{\omega^2 - \omega_B^2}. \tag{5}$$

Using Eq. (2), we now obtain the following formula for the energy loss:

$$-\frac{dE}{dt} = -\frac{(\overline{\mathbf{jE}})}{n} = 8\pi\int_0^\infty d\omega\int\frac{G^1_\parallel(\mathbf{k}, \omega)}{\omega\,|\varepsilon_\parallel|^2}\,\mathrm{Im}\,\varepsilon_\parallel\,d\mathbf{k}, \tag{6}$$

where $G^1_\parallel = (k_\alpha k_\beta/k^2)\,G^1_{\alpha\beta}$. In accordance with Eq. (12.9),

$$G^1_\parallel = \frac{e_1^2}{(2\pi)^4 k^2}\int_{-\infty}^{+\infty}(\mathbf{k}\mathbf{v}(t))\,(\mathbf{k}\mathbf{v}_0)\,e^{i\left(\omega t - \mathbf{k}\int_0^t \mathbf{v}(t')\,dt'\right)}\,dt. \tag{7}$$

Using the expansion in (12.30), we then find

$$G_{\parallel}^{1} = \frac{e_1^2 \omega^2}{8\pi^3 k^2} \sum_{n=-\infty}^{+\infty} \left(\frac{n\omega_B}{k_\perp} \sin\theta + v_t \cos\theta\right)^2 \times$$

$$\times J_n^2\left(\frac{k_\perp v_\perp}{\omega_B}\right) \delta(\omega - k_z v_z - n\omega_B). \tag{8}$$

§ 13. Effect of Radiation Friction and Fluctuating Fields on Motion of a Charged Particle in a Plasma

Deceleration of a Charged Particle in a Plasma. In § 12 the energy loss of a charged particle was calculated by computing A,the negative work of the frictional force exerted on the moving charge. In certain cases it is of interest to determine the mean value of this frictional force. The frictional force density is given by

$$\mathbf{F}(\mathbf{r}, t) = \varrho(\mathbf{r}, t)\mathbf{E}(\mathbf{r}, t) + \frac{1}{c}[\mathbf{j}(\mathbf{r}, t)\mathbf{B}(\mathbf{r}, t)]. \tag{13.1}$$

The mean value of the frictional force can be obtained by the general rule (cf. § 8) for expressing the spectral density of this force:

$$\overline{\mathbf{F}} = \int \left\{(\varrho^*\mathbf{E})_{\mathbf{k},\omega} + \frac{1}{c}([\mathbf{j}^*\mathbf{B}])_{\mathbf{k},\omega}\right\} d\mathbf{k}\, d\omega = \int \mathbf{F}_{\mathbf{k},\omega}\, d\mathbf{k}\, d\omega. \tag{13.2}$$

Since $\mathbf{B}(\mathbf{k},\omega) = (c/\omega)[\mathbf{k}\mathbf{E}(\mathbf{k},\omega)]$ and, from the equation of continuity (for the Fourier component)

$$\varrho(\mathbf{k}, \omega) = \frac{\mathbf{k}\mathbf{j}(\mathbf{k}, \omega)}{\omega}, \tag{13.3}$$

we find (expanding the double vector product and collecting terms that contain ρ and $\mathbf{kj}$) the following expression for the spectral density of the frictional force:

$$\mathbf{F}_{\mathbf{k},\omega} = \frac{\mathbf{k}}{\omega} A_{\mathbf{k},\omega}, \tag{13.4}$$

where $A_{\mathbf{k},\omega} = (\mathbf{j}^*\mathbf{E})_{\mathbf{k},\omega}$ is the spectral density of the work of the frictional force (per unit volume).

By multiplying and dividing the expression in (13.4) by Planck's constant $\hbar$ we obtain a quantum-mechanical interpretation of the expression for the frictional force. The quantity $A_{\mathbf{k},\omega}\cdot\hbar\omega$ evidently represents the spectral density of the number of photons emitted per unit time by the moving charge. The momentum carried by each photon is $\hbar\mathbf{k}$. The product of these quantities, $\mathbf{F}_{\mathbf{k},\omega}$, is consequently the spectral density of the momentum lost by the charge per unit time.

Thus, the procedure for obtaining the average frictional force is very simple: in the integrand of the expression that gives the energy change of the charge $dE/dt = \int A_{\mathbf{k},\omega} d\mathbf{k} d\omega$ we simply introduce the factor $\mathbf{k}/\omega$. The general expression for the frictional force is obtained from Eq. (12.10):

$$\overline{\mathbf{F}} = 2\int_0^\infty \frac{d\omega}{\omega}\int \mathbf{k} G^1_{\beta\alpha} L'_{\alpha\beta}\, d\mathbf{k}, \tag{13.5}$$

where $G^1_{\beta\alpha}$ is given by Eq. (12.9), while $L'_{\alpha\beta}$ is the Hermitian part of $L_{\alpha\beta}$, which is given by Eqs. (12.3) and (12.4).

For the case of a charged particle moving in a plasma with no magnetic field, using Eq. (12.12) we find

$$\overline{\mathbf{F}} = -\frac{e_1^2}{\pi^2}\int_0^\infty \frac{d\omega}{\omega^2}\int \mathbf{k}\left\{\frac{(\mathbf{kv})^2}{k^2}\frac{\operatorname{Im}\varepsilon_\parallel}{|\varepsilon_\parallel|^2} + \frac{v^2 - \frac{(\mathbf{kv})^2}{k^2}}{\left|\varepsilon_\perp - \frac{k^2c^2}{\omega^2}\right|^2}\operatorname{Im}\varepsilon_\perp\right\}\times$$
$$\times\,\delta(\omega - \mathbf{kv})\, d\mathbf{k}. \tag{13.6}$$

Since the particle velocity $\mathbf{v}$ = const (in the zeroth approximation), the work of the frictional force $\overline{A} = \overline{\mathbf{vF}} = \mathbf{v}\overline{\mathbf{F}}$. Multiplying the frictional force (13.6) by v, we obtain (taking account of the fact that the δ function implies $\mathbf{kv}/\omega = 1$) an expression for dE/dt that coincides with Eq. (12.12).

The mean frictional force on a charge moving in a plasma in the presence of a magnetic field can be obtained from Eq. (12.34). We write this formula in the form

$$-\frac{dE}{dt} = -\int_0^\infty d\omega \int A_{\mathbf{k},\omega}\, d\mathbf{k} = \int_0^\infty d\omega \sum_{n=-\infty}^{\infty} P_n(\omega); \tag{13.7}$$

where $P_n(\omega)$ is the radiation intensity (Cerenkov radiation for n = 0, cyclotron radiation due to the normal Doppler effect when n = 1,2,3,... and cyclotron radiation due to the anomalous Doppler effect when n = −1, −2, −3, ...):

$$P_n(\omega) = -\operatorname{Re}\sum_{l=1}^{2} i\,\frac{e_1^2}{\pi^2}\int \frac{d\mathbf{k}}{\omega\left(1+\alpha_{xl}^2\right)}\times$$

$$\times\frac{\left\{\left(\alpha_{xl}^0\frac{v_\perp n}{x_l}+\alpha_{zl}^0 v_z\right)J_n\left(\frac{k_\perp v_\perp}{\omega_B}\right)+v_\perp J_n'\left(\frac{k_\perp v_\perp}{\omega_B}\right)\right\}^2}{\frac{k^2c^2}{\omega^2}-\varepsilon_l(\omega,\,\mathbf{k})}\times$$

$$\times\,\delta\left(\omega-k_z v_z-n\omega_B\right). \tag{13.8}$$

In accordance with Eqs. (13.2) and (13.4), the average frictional force is then

$$\overline{\mathbf{F}} = \int_0^\infty d\omega\int\frac{\mathbf{k}}{\omega}A_{\mathbf{k},\,\omega}\,d\mathbf{k}. \tag{13.9}$$

Using this frictional force, we can find how the radiation causes a redistribution of the energy of the translational and rotational motion of the charge [70-73]. Since the velocity along the magnetic field is constant (v_z = const), the work of the frictional force on the translational degree of freedom $A_{\|} = \overline{v_z F_z} = v_z\overline{F_z}$, i.e., in accordance with Eq. (13.4),

$$A_{\|} = \int\frac{k_z v_z}{\omega}A_{\mathbf{k},\,\omega}\,d\mathbf{k}\,d\omega. \tag{13.10}$$

By virtue of the δ function in the expression for $A_{\mathbf{k},\omega}$, the factor $k_z v_z/\omega$ in each term of the summation over n is replaced by $(\omega - n\omega_B)/\omega$. In going to a summation with respect to positive n, as in the derivation of Eq. (12.36), the energy loss for the translational motion is found to be

$$-\frac{dE_{\|}}{dt} = \int P_0\,d\omega + \sum_{n=1}^{\infty}\left\{\int\frac{\omega-n\omega_B}{\omega}P_n\,d\omega + \right.$$

$$\left. + \int\frac{\omega+n\omega_B}{\omega}P_{-n}\,d\omega\right\}. \tag{13.11}$$

The energy loss for the rotational motion of the charge is found by subtracting Eq. (13.11) from Eq. (13.7):

$$-\frac{dE_{\perp}}{dt} = -\left(\frac{dE}{dt}-\frac{dE_{\|}}{dt}\right) = \sum_{n=1}^{\infty}\left\{\int\frac{n\omega_B}{\omega}P_n\,d\omega - \right.$$

$$\left. - \int\frac{n\omega_B}{\omega}P_{-n}\,d\omega\right\}. \tag{13.12}$$

The meaning of this expression is clear from a quantum-mechanical point of view. We recall that P_n is the radiation intensity at the normal Doppler frequencies

$$\omega = \frac{n\omega_B}{1 - \frac{k_z v_z}{\omega}}; \qquad \left(\frac{k_z v_z}{\omega} = N\beta_{\parallel}\cos\theta < 1\right), \tag{13.13}$$

while P_{-n} is the radiation intensity at the anomalous Doppler frequencies

$$\omega = -\frac{n\omega_B}{1 - \frac{k_z v_z}{\omega}}; \qquad \left(\frac{k_z v_z}{\omega} = N\beta_{\parallel}\cos\theta > 1\right). \tag{13.14}$$

From the quantum-mechanical point of view, the emission of a normal Doppler photon changes the oscillator energy (rotating charge) by an amount $\hbar n\omega_B$, i.e., the transverse energy of the charge is reduced. However, the emission of an anomalous Doppler photon changes the oscillator energy by an amount $-\hbar n\omega_B$, i.e., the transverse energy is increased; this increase is a consequence of the reduction in translational energy. The number of photons of the first kind per unit frequency interval is $P_n/\hbar\omega$ and the number of photons of the second kind is $P_{-n}/\hbar\omega$. Hence, the energy associated with the photons is given by (13.12).

The energy expressions (13.11) and (13.12), or (12.34) and (12.36), are not convenient for separate analysis of the losses due to plasma polarization and the excitation of longitudinal (plasma) oscillations. The polarization losses can be found more conveniently by assuming that the waves are longitudinal (rot $\mathbf{E}$ = 0) at the very outset. The total polarization losses obtained in this way are given by the following (cf. Problem 2 of § 12):

$$-\frac{dE}{dt} = \int_0^\infty \frac{d\omega}{\omega} \int \frac{8\pi G^1_{\parallel}(\mathbf{k}, \omega)}{|\varepsilon_{\parallel}|^2} \operatorname{Im} \varepsilon_{\parallel}\, d\mathbf{k}, \tag{13.15}$$

so that the mean frictional force is

$$\overline{\mathbf{F}} = 8\pi \int_0^\infty \frac{d\omega}{\omega^2} \int \mathbf{k}\, \frac{G^1_{\parallel}(\mathbf{k}, \omega)}{|\varepsilon_{\parallel}|^2} \operatorname{Im} \varepsilon_{\parallel}\, d\mathbf{k}. \tag{13.16}$$

Expressions for $\varepsilon_{\parallel}$ and $G_{\parallel}$ are given in Problem 2 of § 12. We introduce the notation

$$P''_n = \frac{e_1^2\omega}{\pi^2} \int J_n^2\left(\frac{k_\perp v_\perp}{\omega_B}\right) \frac{\operatorname{Im} \epsilon_{\parallel}}{k^2|\epsilon_{\parallel}^2|^2}\, \delta(\omega - n\omega_B - k_z v_z) d\mathbf{k} \tag{13.17}$$

The formulas for the energy losses of the translational and rotational motion are found to agree with Eqs. (13.11) and (13.12) if the substitution $P_n \to P_n''$ ($n = 0, \pm 1, \pm 2, \ldots$) is made in the latter.

Absorption of a Fluctuating Field by a Moving Charge. In addition to the self-field, which produces a frictional force, a charge moving in a plasma is subject to the fluctuating field produced by all of the other charges. The mean value of this fluctuating field is zero; however, the work done by the fluctuations on the charge velocity is quadratic in the field, and is nonvanishing, so that the charge absorbs energy. The energy absorbed per unit time and unit volume by an ensemble of charges described by a distribution function $F_0(\mathbf{v})$ can be computed from Eq. (8.24):

$$Q = \mathbf{jE} = \int \sigma'_{\alpha\beta}(\mathbf{k}, \omega) \left(E^*_\alpha E_\beta\right)_{\mathbf{k}, \omega} d\mathbf{k}\, d\omega, \tag{13.18}$$

where $\sigma'_{\alpha\beta}(\mathbf{k},\omega)$ is the Hermitian part of the conductivity tensor as computed from the known distribution function $F_0(\mathbf{v})$ using Eq. (9.10). Taking $F_0(\mathbf{v})$ to be the distribution function which corresponds to a single moving charge, we can obtain the energy absorbed per unit time by this charge. For a charge moving with a rectilinear velocity $\mathbf{v}_0$, this function is

$$F_0(\mathbf{v}) = \delta(\mathbf{v} - \mathbf{v}_0). \tag{13.19}$$

Consider the case of a charge moving in an isotropic medium. For this case, the absorbed energy can be written as a sum $Q = Q_\perp + Q_\parallel$, where $Q_\perp$ and $Q_\parallel$ are the parts associated with absorption of the transverse and longitudinal fields respectively:

$$Q_\perp = \int \sigma'_\perp(\mathbf{k}, \omega) \left(E^2_\perp\right)_{\mathbf{k}, \omega} d\mathbf{k}\, d\omega; \tag{13.20}$$

$$Q_\parallel = \int \sigma'_\parallel(\mathbf{k}, \omega) \left(E^2_\parallel\right)_{\mathbf{k}, \omega} d\mathbf{k}\, d\omega. \tag{13.21}$$

Using Eq. (9.10) and the relation $\sigma_{\alpha\beta} = i(4\pi/\omega)(\varepsilon_{\alpha\beta} - \delta_{\alpha\beta})$, we find

$$\sigma'_\parallel(\mathbf{k}, \omega) = -\frac{\pi e_1^2}{m_1} \int \frac{(\mathbf{kv})\,\mathbf{k}}{\mathbf{k}^2} \frac{\partial F_0}{\partial \mathbf{v}} \delta(\omega - \mathbf{kv})\, d\mathbf{v}; \tag{13.22}$$

$$\sigma'_\perp(\mathbf{k}, \omega) = -\frac{\pi e_1^2}{m_1} \int \left[v^2 - \frac{(\mathbf{kv})^2}{k^2}\right] \frac{\mathbf{k}}{\omega} \frac{\partial F_0}{\partial \mathbf{v}} \delta(\omega - \mathbf{kv})\, d\mathbf{v}. \tag{13.23}$$

Substituting Eqs. (13.22) and (13.23) in Eqs. (13.20) and (13.21), and carrying out the simple integration over ω and $\mathbf{v}$, we have

$$Q_{\parallel} = \frac{\pi e_1^2}{m_1} \int \left[\frac{\partial}{\partial \omega} \left(\omega |E_{\parallel}^2|_{\mathbf{k},\,\omega} \right) \right]_{\omega = \mathbf{k v}_0} d\mathbf{k}; \tag{13.24}$$

$$Q_{\perp} = \frac{\pi e_1^2}{m_1} \int \left[k^2 v_0^2 - (\mathbf{k v}_0)^2 \right] \left[\frac{\partial}{\partial \omega} \left(\frac{(E_{\perp}^2)_{\mathbf{k},\,\omega}}{\omega} \right) \right]_{\omega = \mathbf{k v}_0} d\mathbf{k}. \tag{13.25}$$

Under stationary conditions, and in the absence of external sources, the spectral densities $(E_{\parallel}^2)_{\mathbf{k},\omega}$ and $(E_{\perp}^2)_{\mathbf{k},\omega}$ that appear in these formulas can be expressed in the following way, in accordance with Eq. (6.40):

$$(E_{\parallel}^2)_{\mathbf{k}\omega} = \frac{16\pi^2}{\omega^2} \frac{G_{\parallel}}{|\varepsilon_{\parallel}|^2}; \quad (E_{\perp}^2)_{\mathbf{k},\,\omega} = \frac{16\pi^2}{\omega^2} \frac{2G_{\perp}}{\left| \frac{k^2 c^2}{\omega^2} - \varepsilon_{\perp} \right|^2}, \tag{13.26}$$

where $G_{\parallel} = (k_\alpha k_\beta / k^2) G_{\alpha\beta}$ and $2G_{\perp} = [\delta_{\alpha\beta} - (k_\alpha k_\beta / k^2)] G_{\alpha\beta}$ are determined by Eqs. (12.8) and (12.11):

$$\left. \begin{aligned} G_{\parallel} &= \sum \frac{e^2 n}{(2\pi)^3} \left\langle \frac{(\mathbf{k v})^2}{k^2} \delta(\omega - \mathbf{k v}) \right\rangle; \\ 2G_{\perp} &= \sum \frac{e^2 n}{(2\pi^3)} \left\langle \left[v^2 - \frac{(\mathbf{k v})^2}{k^2} \right] \delta(\omega - \mathbf{k v}) \right\rangle. \end{aligned} \right\} \tag{13.27}$$

Here, the summation is carried out over all particle species in the plasma and the angle brackets denote averages over velocity. If the plasma is in thermodynamic equilibrium (Maxwellian velocity distribution with the same temperature T for all charges), then $G_{\parallel}$ and $G_{\perp}$ are related to Im $\varepsilon_{\parallel}$ and Im $\varepsilon_{\perp}$ by Eq. (8.26) (with $\hbar\omega \ll T$):

$$G_{\parallel} = \frac{T}{(2\pi)^5} \omega \operatorname{Im} \varepsilon_{\parallel}; \; G_{\perp} = \frac{T}{(2\pi)^5} \omega \operatorname{Im} \varepsilon_{\perp}. \tag{13.28}$$

In order to examine the role of absorption of the fluctuating field, we consider the case in which the charge velocity is small compared with the thermal velocity of the plasma electrons ($\omega \ll kv_{Te}$) but large compared with the thermal velocity of the ions (fixed ions); the basic contribution to the change of the energy of the charge comes from the longitudinal field, so that the transverse field is neglected. The total change in the energy of the charge, which includes the absorption of energy (13.24) and the polarization losses (12.12), is then found to be

$$\frac{dE}{dt} = \frac{e_1^2 T}{2\pi^2 m_1} \int \left[\frac{\partial}{\partial \omega} \left(\frac{\operatorname{Im} \varepsilon_{\parallel}}{|\varepsilon_{\parallel}|^2} \right) \right]_{\omega = \mathbf{k}\mathbf{v}^0} d\mathbf{k} - \frac{e_1^2}{\pi^2} \int \frac{\mathbf{k}\mathbf{v}_0}{k^2} \times \left(\frac{\operatorname{Im} \varepsilon_{\parallel}}{|\varepsilon_{\parallel}{}^2|} \right)_{\omega = \mathbf{k}\mathbf{v}_0} d\mathbf{k}. \tag{13.29}$$

For the approximation used here, $kv_{Ti} \ll \omega \ll kv_{Te}$, Eq. (10.2), the expression for $\varepsilon_{\parallel}$, assumes the form

$$\varepsilon_{\parallel} = 1 + \frac{1}{k^2 d_e^2} \left(1 + i \sqrt{\pi} \frac{\omega}{k v_{Te}} \right); \; d_e^2 = \frac{T_e}{4\pi z e^2 n_e}, \tag{13.30}$$

so that

$$\operatorname{Im} \varepsilon_{\parallel} = i \frac{\sqrt{\pi}\, \omega}{k^3 d_e^2 v_{Te}}; \quad \frac{1}{|\varepsilon_{\parallel}{}^2|} \approx \frac{k^4 d_e^4}{\left(k^2 d_e^2 + 1\right)^2}. \tag{13.31}$$

Substituting Eq. (13.31) in Eq. (13.29), we find ($k_d \equiv 1/d_e$)

$$\frac{dE}{dt} = \frac{4 e_1^2 \omega_0^2}{\sqrt{\pi}\, v_{Te}^3} \left(\frac{T}{m_1} - \frac{2}{3} v_0^2 \right) \int_0^{k_{\max}} \frac{k^3 dk}{\left(k^2 + k_d^2\right)^2} \approx \frac{4 e_1^2 \omega_0^2}{\sqrt{\pi}\, v_{Te}^3} \times \left(\frac{T}{m_1} - \frac{2}{3} v_0^2 \right) \ln \frac{k_{\max}}{k_d} \tag{13.32}$$

(where the meaning of the upper limit k_{max} is discussed in § 12 and in Problem 1 of § 12).

The second term here is responsible for the retardation of the charge; in contrast with the case of a fast charge, [Eq. (12.22)], the retardation increases with increasing velocity. The first term, which is independent of v_0 determines the energy acquired by the charge.

Equation (12.32) does not apply when $v_0^2 \leq T/m_1$; when the velocity of the charge becomes comparable with the thermal velocity, its trajectory can no longer be regarded as rectilinear. However, this expression does give a qualitative indication of the necessity for taking account of the interaction of a charge with velocity $v \sim \sqrt{T/m_1}$ with the fluctuating field produced by the other charges in the plasma.

In regions in which the plasma is transparent, where the characteristic oscillation frequencies [solutions of the dispersion equations $\varepsilon_{\parallel} = 0$, $(k^2c^2/\omega^2) - \varepsilon_{\perp} = 0$] do not exhibit imaginary parts, the spectral densities

$(E^2_{\parallel})_{\mathbf{k},\omega}$ and $(E^2_{\perp})_{\mathbf{k},\omega}$ can, in principle, be much greater than the thermal level given by Eq. (13.26). If the spectra $(E^2_{\parallel})_{\mathbf{k},\omega}$ and $(E^2_{\perp})_{\mathbf{k},\omega}$ are known, then Eqs. (13.24) and (13.25) can be used to compute the energy acquired by the charge in the "superthermal" fluctuating electromagnetic fields.

Expression for the Collision Term in Terms of the Electric Permittivity of a Plasma. In the preceding presentation the basic electromagnetic properties of a plasma, the dielectric tensor $\varepsilon_{\alpha\beta}$ and the correlation function for the currents $G_{\alpha\beta}$, have been obtained as approximations, since they were computed for noninteracting charges. The corresponding kinetic equation (in the absence of perturbations) for a particle of species a is

$$\frac{\partial f_a}{\partial t} + \mathbf{v}_a \frac{\partial f_a}{\partial \mathbf{r}} + \mathbf{F}_0 \frac{\partial f_a}{\partial \mathbf{p}_a} = 0, \tag{13.33}$$

where

$$\mathbf{F}_0 = \frac{e_a}{c}[\mathbf{v}_a \mathbf{B}].$$

Actually, however, as we have seen above, the charge is subject to a radiative friction force and to a fluctuating field.

Since it is small, the radiative friction force can be computed using the unperturbed motion of the charge (as is done in electrodynamics) by suitably modifying the original equation of motion of the charge. The fluctuation field $\mathbf{E}$ can also be computed starting with the unperturbed motion of the charge. We treat this field $\mathbf{E}$ as a perturbation so that $f_a^{(1)}$, the fluctuating part of the distribution function f_a, can be found; the effect of the fluctuating field is then introduced in the "quasilinear approximation." The corresponding total correction to the kinetic equation (right-hand side), which is called the collision term, is then given by

$$S(f_a) = -\frac{\partial}{\partial \mathbf{p}_a}\left\{\overline{F_{\mathrm{gr}} \cdot f_a} + \overline{F^{(1)} \cdot f_a^{(1)}}\right\}, \tag{13.34}$$

where

$$\mathbf{F}^{(1)} = e_a\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_a \mathbf{B}]\right).$$

Let us consider the function $S(f_a)$ for a plasma in the absence of a magnetic field. The largest contribution to $S(f_a)$ is due to the longitudinal field (the transverse field is important only in a relativistic plasma) which is the one we consider. The correction to the distribution function $f_a^{(1)}\,\mathbf{k},\omega)$ is related to the longitudinal field $\mathbf{E}(\mathbf{k},\omega)$ by the expression (cf. the problem in § 9):

$$f_a^{(1)}(\mathbf{k}, \omega) = \mathbf{g}(\mathbf{k}, \omega)\, \mathbf{E}(\mathbf{k}, \omega), \tag{13.35}$$

where

$$\mathbf{g}(\mathbf{k}, \omega) = -ie\frac{\partial f_a}{\partial \mathbf{p}_a}\left\{\frac{P}{\omega - \mathbf{k}\mathbf{v}_a} - i\pi\delta(\omega - \mathbf{k}\mathbf{v}_a)\right\}. \tag{13.36}$$

The corresponding required electrical permittivity is obtained from Eq. (9.10):

$$\varepsilon_{\parallel}(\mathbf{k}, \omega) = \frac{k_\alpha k_\beta}{k^2}\varepsilon_{\alpha\beta} =$$

$$= 1 + \sum_b \left\langle \frac{4\pi e_b^2}{\omega}(\mathbf{k}\mathbf{v}_b)\left(\mathbf{k}\frac{\partial f_b}{\partial \mathbf{p}_b}\right)\left\{\frac{P}{\omega - \mathbf{k}\mathbf{v}_b} - i\pi\delta(\omega - \mathbf{k}\mathbf{v}_b)\right\}\right\rangle, \tag{13.37}$$

where the summation is carried out over all particle species in the plasma.

The first member of the collision term is the friction force of the charge and is due to the longitudinal field produced by the charge (the so-called dynamical friction term), being given by the first term in Eq. (13.6). Substituting the explicit expression for Im $\varepsilon_{\parallel}$ from Eq. (13.37), we have

$$\mathbf{F}_{gr} = \sum_b 2e_a^2 e_b^2 \frac{\partial f_b}{\partial p_{b\beta}} \int \frac{\mathbf{k}k_\beta}{k^4 |\varepsilon_{\parallel}|^2}\delta(\omega - \mathbf{k}\mathbf{v}_a)\,\delta(\omega - \mathbf{k}\mathbf{v}_b)\, d\omega\, d\mathbf{k}. \tag{13.38}$$

The mean value of the product $\mathbf{F}^{(1)} f_a^{(1)} = e_a f_a^{(1)}\mathbf{E}$, which appears in the second term of the expression for $S(f_a)$, can, in accordance with the general rule (§ 8), be written in the form of an integral over ω and $\mathbf{k}$ of the corresponding spectral density $e_a(\mathbf{E}f_a^{(1)})_{\mathbf{k},\omega}$. Taking account of (13.35), by analogy with Eq. (8.24) we have

$$\overline{f_a^{(1)} E_\alpha} = e_a \int g'_\beta(\mathbf{k}, \omega)\,(E_\alpha^* E_\beta)_{\mathbf{k},\omega}\, d\mathbf{k}\, d\omega, \tag{13.39}$$

where g'_β is the Hermitian part of the operator $\mathbf{g}(\mathbf{k},\omega)$:

$$g'_\beta(\mathbf{k}, \omega) = -\pi e_a \frac{\partial f}{\partial p_\beta}\delta(\omega - \mathbf{k}\mathbf{v}_a). \tag{13.40}$$

In accordance with Eqs. (6.38), (12.2), and (12.3), the longitudinal field $\mathbf{E}(\mathbf{k},\omega)$ produced by the charges is of the form

$$\mathbf{E} = -\frac{4\pi i}{\omega\varepsilon_{\parallel}}\frac{\mathbf{k}(\mathbf{k}\mathbf{j})}{k^2}. \tag{13.41}$$

Using Eqs. (8.29), (12.8), and (12.11), we obtain an expression for the correlation function for the currents

$$(j_\alpha^* j_\beta)_{\mathbf{k},\,\omega} \equiv G_{\alpha\beta} = \sum_b \int \frac{e_b^2 v_{b\alpha} v_{b\beta}}{(2\pi)^3}\, \delta(\omega - \mathbf{k}\mathbf{v}_b)\, f(\mathbf{p}_b)\, d\mathbf{p}_b, \quad (13.42)$$

so that the correlation function for the microfields that appears in (13.39) becomes

$$(E_\alpha^* E_\beta)_{\mathbf{k},\,\omega} = \frac{2k_\alpha k_\beta}{\pi\omega^2 \,|\, \varepsilon_\parallel \,|^2} \sum_b \frac{e_b^2\, (\mathbf{k}\mathbf{v}_b)^2}{k^4} \times$$
$$\times\, \delta(\omega - \mathbf{k}\mathbf{v}_b)\, f(\mathbf{p}_b)\, d\mathbf{p}_b. \quad (13.43)$$

Substituting Eqs. (13.40) and (13.43) in Eq. (13.39), we have

$$f_a^{(1)} E_\alpha = -\sum_b 2e_a^2 e_b^2 \int \frac{\partial f_a}{\partial p_{\alpha\beta}}\, f_b\, d\mathbf{p}_b \int \frac{k_\alpha k_\beta}{k^4 \,|\, \varepsilon_\parallel \,|^2} \times$$
$$\times\, \delta(\omega - \mathbf{k}\mathbf{v}_a)\, \delta(\omega - \mathbf{k}\mathbf{v}_b)\, d\mathbf{k}\, d\omega. \quad (13.44)$$

Now, substituting the expressions for the frictional force (13.38) and the product $\overline{f_a^{(1)} \mathbf{E}^{(1)}}$ [Eq. (13.44)] in Eq. (13.34), we obtain the final expression for the collision term (for greater detail see [69])

$$S(f_a) = -\sum_b 2e_a^2 e_b^2 \frac{\partial}{\partial p_{a\alpha}} \int I_{\alpha\beta} \left(f_a \frac{\partial f_b}{\partial p_{b\beta}} - f_b \frac{\partial f_a}{\partial p_{a\beta}} \right) d\mathbf{p}_b, \quad (13.45)$$

where

$$I_{\alpha\beta} = \int \frac{k_\alpha k_\beta}{k^4 \,|\, \varepsilon_\parallel(\omega, k) \,|^2}\, \delta(\mathbf{k}\mathbf{v}_a - \mathbf{k}\mathbf{v}_b) \times$$
$$\times\, \delta(\omega - \mathbf{k}\mathbf{v}_a)\, \delta(\omega - \mathbf{k}\mathbf{v}_b)\, d\omega\, d\mathbf{k}. \quad (13.46)$$

§ 14. Fluctuations in a Plasma. Scattering of Waves [58-60]

As we have already seen in § 8, the solution of Maxwell's equations (6.1) in which the quantity $\mathbf{j}_{\text{spec}}$ is understood to mean the microscopic current density $\mathbf{j}^M$ [(6.8)], allows us to find various fluctuating quantities: for instance, the energy of the fluctuating magnetic field, the fluctuations in the current density and charge density, etc.

In the final analysis, all of these quantities are expressed in terms of two fundamental characteristics of the medium – the dielectric tensor $\varepsilon_{\alpha\beta}$ and the correlation function for the microcurrents $G_{\alpha\beta}$ (or specific parts of these which refer to charges of a given species).

In the present section we shall consider in greater detail some other important fluctuation quantities. These are: 1) the energy of the Coulomb interaction of the charges, which is directly related to the energy of the longitudinal electric field $E_{\parallel}^2/8\pi$; and, 2) the fluctuation in the total charge of the plasma and the electron density, in terms of which the cross section for scattering of electromagnetic waves can be expressed.

Energy of the Coulomb Interaction. In accordance with the general rule [cf. Eq. (8.22)], the energy of the longitudinal electric field can be written in the form of an integral over a spectrum

$$\frac{\overline{E_{\parallel}^2}}{8\pi} = \int \frac{(E_{\parallel}^2)_{\mathbf{k}}}{8\pi}\, d\mathbf{k}, \tag{14.1}$$

where, from Eqs. (13.26) and (13.28), for a plasma in thermodynamic equilibrium,

$$(E_{\parallel}^2)_{\mathbf{k}} = \int_{-\infty}^{\infty} (E_{\parallel}^2)_{\mathbf{k},\,\omega}\, d\omega = -\frac{T}{2\pi^3} \int_{-\infty}^{\infty} \frac{1}{\omega}\, \mathrm{Im}\, \frac{1}{\varepsilon_{\parallel}}\, d\omega. \tag{14.2}$$

In order to compute this integral, we shall consider an auxiliary integral $\oint (d\omega/\omega\,\varepsilon_{\parallel})$ for which the contour of integration goes around the upper half-plane of the complex variable ω, enclosing the point $\omega = 0$. Since the equation $\varepsilon_{\parallel} = 0$ has roots $\omega = \mathrm{Re}\,\omega + i\,\mathrm{Im}\,\omega$, with $\mathrm{Im}\,\omega < 0$ (corresponding to stable equilibrium) the function $1/\varepsilon_{\parallel}$ has no poles in the upper half-plane of the complex variable ω. Consequently, the integral over the closed contour is equal to zero; this integral can be divided into three parts: (1) the principle value along the real axis, (2) the integral over the semicircle $|\omega| \to 0$, which goes around the point $\omega = 0$ in the clockwise direction, and, (3) the integral over the arc $|\omega| \to \infty$ (counterclockwise direction). Thus,

$$\int \frac{d\omega}{\omega\varepsilon_{\parallel}} - \frac{i\pi}{\varepsilon_{\parallel}(0)} + i\pi = 0. \tag{14.3}$$

Separating real and imaginary parts, we find

$$\int \operatorname{Im} \frac{1}{\varepsilon_{\parallel}} \frac{d\omega}{\omega} = -\pi \left(1 - \frac{1}{\varepsilon_{\parallel}(0, \mathbf{k})}\right). \tag{14.4}$$

The principal value symbol is not used here, since $\operatorname{Im}(1/\varepsilon_{\parallel}) \sim \omega$.

The static electric permittivity which appears in Eq. (14.4) can be obtained from Eq. (9.27)

$$\varepsilon_{\parallel}(0, \mathbf{k}) = 1 + \frac{1}{k^2 d^2}, \tag{14.5}$$

where d is the Debye radius:

$$\frac{1}{d^2} = \sum_a \frac{4\pi e_a^2 n_a}{T}. \tag{14.6}$$

From Eqs. (14.2), (14.4), and (14.5), we then have

$$\frac{\overline{E_{\parallel}^2}}{8\pi} = \int \frac{T}{2} \frac{1}{1 + k^2 d^2} \frac{d\mathbf{k}}{(2\pi)^3}. \tag{14.7}$$

This integral diverges as $k \to \infty$ because the energy in the electromagnetic field (14.7) includes the "unobservable" energy of the Coulomb field of the noninteracting point charges which can be determined from Eqs. (13.26) and (13.27):

$$\frac{\overline{E_{\parallel}^2}}{8\pi} = \sum_a \frac{e_a^2 n_a}{4\pi^2 k^2} \delta(\omega - \mathbf{k}\mathbf{v}_a)\, d\omega\, d\mathbf{k} = \int \frac{T}{2} \frac{1}{k^2 d^2} \frac{d\mathbf{k}}{(2\pi)^3}. \tag{14.8}$$

Subtracting Eq. (14.8) from Eq. (14.7), we obtain the energy of the Coulomb interaction of the plasma charges [61]:

$$U = -\frac{T}{2} \int \frac{1}{k^2 d^2 (1 + k^2 d^2)} \frac{d\mathbf{k}}{(2\pi)^3} = -\frac{T}{8\pi d^3}. \tag{14.9}$$

Charge Fluctuations. Knowing the fluctuations in the longitudinal electric field, we can now use the equation $\operatorname{div} \mathbf{E} = 4\pi\rho$ to obtain the charge fluctuation in the plasma. The Fourier space component of the charge density can be obtained from this equation: $\rho_{\mathbf{k}} = -i(kE_{\parallel}/4\pi)$, so that

$$(\varrho^2)_{\mathbf{k}} = \frac{k^2 (E_{\parallel}^2)_{\mathbf{k}}}{16\pi^2}. \tag{14.10}$$

For a plasma in thermodynamic equilibrium, using Eqs. (14.1) and (14.7), we find

$$(E_{\parallel}^2)_{\mathbf{k}} = \frac{T}{2\pi^2}\frac{1}{1+k^2d^2},$$

whence

$$(\varrho^2)_{\mathbf{k}} = \frac{T}{32\pi^4}\frac{k^2}{1+k^2d^2} = \frac{\sum e_a^2 n_a}{(2\pi)^3}\frac{k^2d^2}{1+k^2d^2}. \tag{14.11}$$

Applying the general rule [cf. Eq. (8.21)] and Eq. (14.11) it is not difficult to obtain the space–time correlation function for the charge density:

$$\overline{\varrho(\mathbf{r}, t)\varrho(\mathbf{r}', t')} = \int (\varrho^2)_k e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} d\mathbf{k} =$$
$$= \sum e_a^2 n_a \{\delta(\mathbf{r}-\mathbf{r}') + \nu(\mathbf{r}-\mathbf{r}')\}, \tag{14.12}$$

where the correlation function $\nu(\mathbf{r} - \mathbf{r}')$ is given by the formula

$$\nu(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi d^2}\frac{e^{-\frac{|\mathbf{r}-\mathbf{r}'|}{d}}}{|\mathbf{r}-\mathbf{r}'|}. \tag{14.13}$$

The relation in (14.11) can also be obtained from the total current density (microscopic current plus that induced by the field fluctuation):

$$j_{\parallel} = j_{\parallel}^{\mathrm{M}} + \sigma_{\parallel} E_{\parallel}, \tag{14.14}$$

where $\mathbf{E}_{\parallel}$ is determined by Eq. (6.38): $\mathbf{E}_{\parallel} = -(4\pi i/\omega\varepsilon_{\parallel})\mathbf{j}_{\parallel}^{\mathrm{M}}$. Thus

$$\mathbf{j}_{\parallel} = \mathbf{j}_{\parallel}^{\mathrm{M}}\left(1 - i\frac{4\pi\sigma_{\parallel}}{\omega\varepsilon_{\parallel}}\right) = \frac{\mathbf{j}_{\parallel}^{\mathrm{M}}}{\varepsilon_{\parallel}}. \tag{14.15}$$

Using the relation between ρ and $\mathbf{j}_{\parallel}$ [cf. Eq. (13.3)], we have

$$(\varrho^2)_{\mathbf{k},\omega} = \frac{k^2 G_{\parallel}}{\omega^2\varepsilon_{\parallel}}. \tag{14.16}$$

If the value of $G_{\parallel}$ for thermodynamic equilibrium is used and if (13.28) and Eq. (14.16) are integrated over ω, the result in (14.11) is obtained.

Fluctuations in Electron Density. If it is necessary to compute the charge fluctuations for one species only (for example the electrons or the ions), we take $\mathbf{j}_{\parallel}^{\mathrm{M}}$ and $\sigma_{\parallel}$ in Eq. (14.14) to refer to that species only. It is recalled that the quantities $\mathbf{j}^{\mathrm{M}}$, σ, and G are additive:

$$\mathbf{j}^{\mathrm{M}} = \sum_a \mathbf{j}_a^{\mathrm{M}}; \quad \sigma = \sum_a \sigma_a; \quad G = \sum_a G_a. \tag{14.17}$$

Let us consider the fluctuations in electron charge in a two-component plasma. In this case,

$$\varepsilon_{\parallel} = 1 + i\frac{4\pi}{\omega}(\sigma^e_{\parallel} + \sigma^i_{\parallel}) \tag{14.18}$$

(the letters e and i refer to the electrons and ions, respectively);

$$\sigma^e_{\parallel} = -ie^2 n_e \left\langle \frac{(\mathbf{kv})}{k^2}\left(\mathbf{k}\frac{\partial F_e}{\partial \mathbf{p}}\right)\zeta(\omega - \mathbf{kv}) \right\rangle. \tag{14.19}$$

If the electrons exhibit a Maxwellian velocity distribution (9.27):

$$\frac{4\pi\sigma^e_{\parallel}}{\omega} = \frac{2\omega^e_{0e}}{k^2 v^2_{Te}}\left\{Y\left(\frac{\omega}{kv_{Te}}\right) + i\left[X\left(\frac{\omega}{kv_{Te}}\right) - 1\right]\right\}. \tag{14.20}$$

Similar relations hold for $\sigma^i_{\parallel}$.

The total density of longitudinal current associated with the electrons is given by

$$j^e_{\parallel} = j^{eM}_{\parallel} + \sigma^e_{\parallel}E_{\parallel} = j^{eM}_{\parallel}\left(1 - i\frac{4\pi\sigma^e_{\parallel}}{\omega\varepsilon_{\parallel}}\right) - i\frac{4\pi\sigma^e_{\parallel}}{\omega\varepsilon_{\parallel}}j^{iM}_{\parallel}. \tag{14.21}$$

The expression in the brackets can be written in somewhat different form through the use of Eq. (14.18):

$$1 - i\frac{4\pi\sigma^e_{\parallel}}{\omega\varepsilon_{\parallel}} = \frac{1 + i\dfrac{4\pi\sigma^i_{\parallel}}{\omega}}{\varepsilon_{\parallel}}. \tag{14.22}$$

Since $\overline{\mathbf{j}^{eM}_{\parallel}\mathbf{j}^{iM}_{\parallel}} = 0$, the spectral density of the fluctuations in the longitudinal current becomes

$$(j^{e^2}_{\parallel})_{\mathbf{k},\omega} = \frac{G^e_{\parallel}}{|\varepsilon_{\parallel}|^2}\left|1 + i\frac{4\pi\sigma^i_{\parallel}}{\omega}\right|^2 + \frac{16\pi^2|\sigma^e_{\parallel}|^2}{\omega^2|\varepsilon_{\parallel}|^2}G^i_{\parallel}. \tag{14.23}$$

The fluctuations in electron charge are related to the fluctuations in longitudinal current by the simple expression $(\rho^2_e)_{\mathbf{k},\omega} = (k^2/\omega^2)(j^e_{\parallel})^2_{\mathbf{k},\omega}$, which follows from the equation of continuity. If the electrons and ions exhibit Maxwellian velocity distributions at temperatures T_e and T_i respectively:

$$(\varrho^2_e)_{\mathbf{k},\omega} = \frac{e^2\bar{n}_e}{8\pi^4\omega|\varepsilon|^2}\left[Y(x_e)\left\{\left[1 + \frac{1}{k^2d_i^2}(1 - X(x_i))\right]^2 + Y^2(x_i)\right\} + \right.$$

$$+ Y(x_i)\left\{\frac{1}{k^4 d_e^4}(1 - X(x_e))^2 + Y^2(x_e)\right\}\Big]. \tag{14.24}$$

Here,

$$\left.\begin{aligned} |\varepsilon|^2 = \left\{1 + \frac{1}{k^2 d_e^2}[1 - X(x_e)] + \frac{1}{k^2 d_i^2}[1 - X(x_i)]\right\}^2 + \\ + \left[\frac{1}{k^2 d_e^2} Y(x_e) + \frac{1}{k^2 d_i^2} Y(x_i)\right]^2; \\ x_e = \frac{\omega}{|k|\, v_{Te}}; \quad x_i = \frac{\omega}{|k|\, v_{Ti}}; \\ v_{Te} = \sqrt{\frac{2T_e}{m_e}}; \; v_{Ti} = \sqrt{\frac{2T_i}{m_i}}; \\ X(x) = 2x\mathrm{e}^{-x^2}\int_0^x \mathrm{e}^{t^2}dt; \; Y(x) = \sqrt{\pi}x\mathrm{e}^{-x^2}. \end{aligned}\right\} \tag{14.25}$$

It is evident from Eq. (14.21) that the density of electron current (as well as the electron density itself) consists of three terms. The first of these, $\mathbf{j}_{\parallel}^{eM}$, is associated with the "bare" electron, while the second and third are associated with the polarization clouds around the electron and ion, respectively. The existence of a polarization cloud around an ion explains why the fluctuations in electron density are affected by the ions.

If $T_e = T_i = T$, the fluctuations in electron density can be written in the form*

$$(\varrho_e^2)_{\mathbf{k},\,\omega} = \frac{k^2 T}{(2\pi)^5 \omega}\,\mathrm{Im}\frac{(\varepsilon_e - 1)\,\varepsilon_i}{\varepsilon},$$

where

$$\varepsilon_e = 1 + i\frac{4\pi\sigma^e}{\omega}; \quad \varepsilon_i = 1 + i\frac{4\pi\sigma^i}{\omega}.$$

This method of writing the expression allows us to use the same integration over ω as is used in computing $(E^2)_{\mathbf{k}}$ (with the difference that here the integral over the arc $|\omega| \to \infty$ vanishes). Carrying out this integration, we find

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*This method of writing the formula has been indicated to the author by O.P. Pogutse.

$$(\varrho_e^2)_{\mathbf{k}} = \frac{e^2 n_e}{(2\pi)^3} \frac{k^2 d_i^2 + 1}{k^2 d_i^2 + 1 + d_e^2/d_i^2}.$$

The time–space correlation of the electron charge density is then found by analogy with Eq. (14.12):

$$\overline{\varrho_e(\mathbf{r}, t)\, \varrho_e(\mathbf{r}', t)} = e^2 n_e \left\{ \delta(\mathbf{r} - \mathbf{r}') - \frac{1}{4\pi d_i^4/d_e^2} \frac{e^{-\frac{r}{d_i}\sqrt{1+\frac{d_e^2}{d_i^2}}}}{r}. \right\}$$

In these expressions,

$$d_e^2 = \frac{T}{4\pi e^2 n_e}; \quad d_i^2 = \frac{T}{4\pi e_i^2 n_i}.$$

The spectral density of the electron-charge fluctuations (14.24) is of great importance because it can be used to express the cross section for scattering of electromagnetic waves on density fluctuations.

Scattering of Electromagnetic Waves in a Plasma. Let us consider a transverse wave propagating in a plasma (without magnetic field); the electric field of this wave is described by

$$\mathbf{E} = \mathbf{E}_0 \cos(\varkappa \mathbf{r} - \Omega t). \tag{14.26}$$

The equation of motion of the i-th electron of the plasma in the field produced by the wave is

$$\frac{\partial \mathbf{v}_i}{\partial t} = -\frac{e\mathbf{E}_0}{m} \cos(\varkappa \mathbf{r}_i - \Omega t), \tag{14.27}$$

where $\mathbf{r}_i(t)$ is the radius vector of the electron being considered. Using Eq. (14.27) we can find the change in the microscopic electron current density $\mathbf{j} = -\Sigma e \mathbf{v}_i\, \delta[\mathbf{r} - \mathbf{r}_i(t)]$ due to the wave:

$$\frac{\partial \delta \mathbf{j}}{\partial t} = -\sum e \frac{\partial \mathbf{v}_i}{\partial t} \delta(\mathbf{r} - \mathbf{r}_i(t)) = -\frac{e\mathbf{E}_0}{m} \varrho_e \cos(\varkappa \mathbf{r} - \Omega t). \tag{14.28}$$

Here, $\rho_e = -\Sigma\, e\, \delta[\mathbf{r} - \mathbf{r}_i(t)]$ is the microscopic electron charge density.

In the usual fashion, we write ρ_e and $\delta \mathbf{j}$ in terms of Fourier integrals:

$$\varrho_e = \int \varrho_e(\mathbf{k}', \omega')\, e^{i(\mathbf{k}'r - \omega' t)}\, d\mathbf{k}'\, d\omega'; \tag{14.29}$$

$$\delta \mathbf{j} = \int \delta \mathbf{j}(\mathbf{k}, \omega)\, e^{i(\mathbf{k}r - \omega t)}\, d\mathbf{k}'\, d\omega. \tag{14.30}$$

The relation between the Fourier components of these quantities is then obtained from Eq. (14.28):

$$\delta \mathbf{j}(\mathbf{k}, \omega) = i\,\frac{eE_0}{m\omega}\,\frac{\varrho_e(\mathbf{k}-\varkappa, \omega-\Omega)+\varrho_e(\mathbf{k}+\varkappa, \omega+\Omega)}{2}. \quad (14.31)$$

We then obtain the correlation function for the currents induced by the wave:

$$\overline{\delta j_\alpha^*(k, \omega)\,\delta j_\beta(k', \omega')} = \delta G_{\alpha\beta}\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega'); \quad (14.32)$$

$$\delta G_{\alpha\beta} = \frac{e^2 E_\alpha^0 E_\beta^0}{m^2\omega^2}\,\frac{(\varrho_e^2)_{\mathbf{k}-\varkappa,\,\omega-\Omega}+(\varrho_e^2)_{\mathbf{k}+\varkappa,\,\omega+\Omega}}{4}. \quad (14.33)$$

The radiation induced by the wave (14.26) is now determined from Eq. (12.7):

$$\delta Q = -\int \delta G_{\beta\alpha} L'_{\alpha\beta}\,d\mathbf{k}\,d\omega, \quad (14.34)$$

where, in accordance with Eq. (12.42),

$$L'_{\alpha\beta} = -\frac{4\pi^2}{\omega}\,\delta\left(\frac{k^2c^2}{\omega^2}-\varepsilon(\omega)\right)\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right). \quad (14.35)$$

We note that $(\rho_e^2)_{\mathbf{k}+\varkappa,\omega+\Omega} = (\rho_e^2)_{-\mathbf{k}-\varkappa,-\omega-\Omega}$. Using this relation, and replacing the variables of integration $\omega \to -\omega$ and $\mathbf{k} \to -\mathbf{k}$ in that part of the integral (14.24) which contains $(\rho_e^2)_{\mathbf{k}+\varkappa,\omega+\Omega}$, we can obtain the energy associated with the scattered radiation (per unit volume)

$$\delta Q = \int_{-\infty}^{\infty} d\omega \int \frac{2\pi^2 e^2 E_\alpha^0 E_\beta^0}{m^2\omega^3}\,(\varrho_e^2)_{\mathbf{k}-\varkappa,\,\omega-\Omega}\times$$
$$\times\,\delta\left(\frac{k^2c^2}{\omega^2}-\varepsilon(\omega)\right)\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)d\mathbf{k}. \quad (14.36)$$

Assume that the electric vector of the incident wave (14.26) is directed along the x axis while the wave vector $\varkappa$ is directed along the z axis. Then,

$$E_\alpha^0 E_\beta^0\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right) = E_0^2\left(1-\frac{k_x^2}{k^2}\right) = E_0^2(1-\sin^2\theta\cos^2\varphi), \quad (14.37)$$

where θ is the angle between the wave vectors of the incident and scattered waves, while φ is the angle between the projection of the wave vector of the scattered wave on the xy plane and the electric vector of the incident wave. The energy flux in the incident wave is

$$\overline{S} = \frac{c}{8\pi}E_0^2\sqrt{\varepsilon(\Omega)}. \quad (14.38)$$

Writing $d\mathbf{k} = k^2 dk dO$, and carrying out the integration over k in Eq. (14.36) using the δ function, we obtain an expression for the scattering cross section

$$\left.\begin{aligned} \sigma &= \frac{\delta Q/\bar{n}_e}{\bar{S}} = \int \sigma_\omega d\omega\, dO; \\ \sigma_\omega &= \left(\frac{e^2}{mc^2}\right)^2 \sqrt{\frac{\varepsilon(\omega)}{\varepsilon(\Omega)}}(1-\sin^2\theta\cos^2\varphi)\frac{(2\pi)^3\left(\varrho_e^2\right)_{\mathbf{k}-\boldsymbol{\varkappa},\,\omega-\Omega}}{e^2\bar{n}_e}. \end{aligned}\right\} \quad (14.39)$$

The relative frequency shift in scattering is usually small: $(\omega - \Omega)/\omega \ll 1$; hence we can write $|\mathbf{k}| = |\boldsymbol{\varkappa}| = (\Omega/c)\sqrt{\varepsilon(\omega)}$. Under these conditions,

$$(\mathbf{k} - \boldsymbol{\varkappa})^2 = 4\frac{\Omega^2}{c^2}\varepsilon(\Omega)\sin^2\frac{\theta}{2}. \quad (14.40)$$

In the limiting case of short wavelengths $kd \gg 1$ [i.e., $\omega \gg \omega_0(c/v_T)$] we have

$$\varepsilon = 1; \ \left(\varrho_e^2\right)_{\mathbf{k}-\boldsymbol{\varkappa},\,\omega-\Omega} = \frac{e^2\bar{n}_e}{8\pi^4(\omega-\Omega)}Y\left(\frac{\omega-\Omega}{|\mathbf{k}-\boldsymbol{\varkappa}|\,v_{Te}}\right),$$

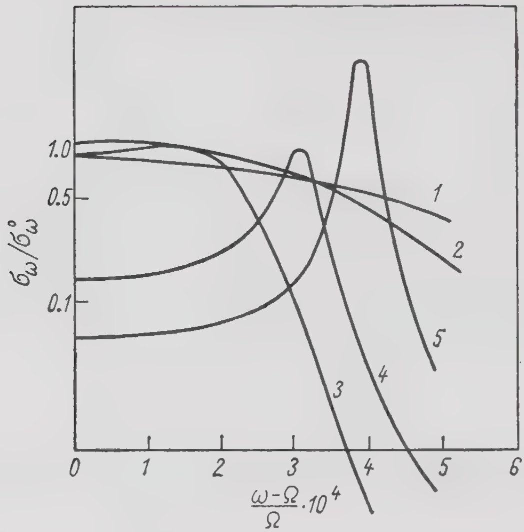
so that

$$\sigma_\omega = \left(\frac{e^2}{mc^2}\right)^2(1-\sin^2\theta\cos^2\varphi)\frac{c}{2\sqrt{\pi}\Omega v_{Te}\sin\frac{\theta}{2}}e^{-\frac{(\omega-\Omega)^2}{4\Omega^2\sin^2\frac{\theta}{2}}\frac{c^2}{v_{Te}^2}}. \quad (14.41)$$

This is the scattering cross section due to individual electrons that exhibit a thermal velocity spread. When $v_{Te} \to 0$, the factor that takes account of the Doppler broadening of the scattered line becomes $\delta(\omega - \Omega)$:

$$\sigma_\omega \to \left(\frac{e^2}{mc^2}\right)^2(1-\sin^2\theta\cos^2\varphi)\,\delta(\omega-\Omega). \quad (14.42)$$

We now consider the case in which $kd \ll 1$, i.e., $\omega \ll \omega_0(c/v_T)$ (but, obviously, $\omega > \omega_0$ because, otherwise, the wave cannot propagate through the plasma). In this case, the polarization clouds around the electrons and ions make contributions to the scattering. Part of the scattering intensity at the unshifted frequency $\omega = \Omega$ comes from charges with velocities close to zero. For comparable electron and ion temperatures, the number of ions (per unit velocity interval) with velocities $v = 0$ is $\sqrt{m_i/m_e}$ times greater than the number of electrons; thus, it is clear that the shape of the scattering line at $\omega - \Omega = 0$ is determined by the polarization clouds associated with the ions. For $\omega - \Omega = 0$, the cross section in (14.39) yields the following value:



Differential scattering cross section for a plasma with different electron and ion temperatures

Curve	T_e, eV	T_i, eV	σ_{tot}/r_0^2
1	5	50	0.812
2	5	25	0.808
3	5	5	0.500
4	25	5	0.236
5	50	5	0.360

Fig. 18

$$\sigma_\omega^0 = \left(\frac{e^2}{mc^2}\right)^2 \frac{1-\sin^2\theta\cos^2\varphi}{2\sqrt{\varepsilon(\Omega)}\,\Omega\sin\frac{\theta}{2}} \times$$

$$\times \sqrt{\frac{m_i c^2}{2\pi T_i}}\, \frac{1+\left(\frac{m_e T_e^3}{m_i T_i^3}\right)^{1/2}\left[1+(\mathbf{k}-\varkappa)^2 d_i^2\right]^2}{\left[1+(\mathbf{k}-\varkappa)^2 d_e^2 + d_e^2/d_i^2\right]^2}. \qquad (14.43)$$

It is evident that when $kd \ll 1$, the quantity $\sigma_\omega^0 \sim (e^2/mc^2)^2(1/\varkappa v_{Ti})$. When $\omega - \Omega > kv_{Ti}$, the number of resonance ions becomes exponentially small and we have $\sigma_\omega \ll \sigma_\omega^0$.

The scattering cross section will exhibit maxima at those values of the frequency difference $\omega - \Omega$ for which the electric permittivity that appears in the denominator of the density fluctuation (14.24) approaches zero, i.e., when the frequency of the scattered waves is shifted by an amount equal to the frequency of the characteristic plasma oscillations. A sharp peak in the scattering cross section obtains at the plasma frequency $\omega - \Omega = \omega_0$ (when $kd \ll 1$). However, the integrated cross section in this frequency region only yields a small fraction (about $k^2d^2 \ll 1$) of the total cross section. In a highly nonisothermal plasma with $T_e \gg T_i$, the scattering cross section exhibits a maximum at a frequency shifted by an amount equal to the ion-acoustic frequency (10.5): $\omega - \Omega = |\mathbf{k} - \boldsymbol{\varkappa}|\sqrt{zT_e/m_i}$. The dependence of the differential scattering cross section σ_ω on frequency for scattering at right angles ($\theta = \pi/2$) for various values of the plasma parameters is shown in Fig. 18, which is taken from a paper by Rosenbluth and Rostoker [60]. The frequency of the incident wave here is taken to be equal to twice the plasma frequency $\Omega = 2\omega_0$, $\Omega/c = 3.76\ \text{cm}^{-1}$. These curves show the maxima that appear at the ion-acoustic frequency as the ratio T_e/T_i is increased.

The total (integrated over frequency) differential cross section is of the order $\sigma_{tot} \sim r_0^2 = (e^2/mc^2)^2$. In an isothermal plasma ($T_e = T_i$), for example, in which the integration over frequency can be carried out analytically, the ratio of the scattering cross section for the plasma to the cross section for scattering on an individual electron varies from 1 (for $kd \gg 1$) to $\frac{1}{2}$ (for $kd \ll 1$).

Problem. Find the fluctuations in the magnetic field and the transverse electric field in a plasma when $\mathbf{B}_0 = 0$.

Solution. Using Eq. (13.26),

$$\left(E_\perp^2\right)_{\mathbf{k},\,\omega} = \frac{16\pi^2}{\omega^2} \frac{2G_\perp}{\left|\dfrac{k^2c^2}{\omega^2} - \varepsilon_\perp\right|^2}. \tag{1}$$

Since $\mathbf{B} = (c/\omega)[\mathbf{kE}]$

$$(B^2)_{\mathbf{k},\,\omega} - \frac{c^2k^2}{\omega^2}\left(E_\perp^2\right)_{\mathbf{k},\,\omega} = \frac{16\pi^2}{\omega^2} \frac{2G_\perp}{\left|\dfrac{k^2c^2}{\omega^2} - \varepsilon_\perp\right|^2} \frac{c^2k^2}{\omega^2}. \tag{2}$$

For a plasma in thermodynamic equilibrium, using Eq. (8.26) we have

$$\left(E_\perp^2\right)_{\mathbf{k},\,\omega} = -\frac{T}{\pi^3}\frac{1}{\omega}\,\mathrm{Im}\frac{1}{\varepsilon_\perp - \dfrac{k^2c^2}{\omega^2}}; \tag{3}$$

$$(B^2)_{\mathbf{k},\,\omega} = \frac{k^2c^2}{\omega^2}\left(E_\perp^2\right)_{\mathbf{k},\,\omega}. \tag{3}$$

As in §14, we can carry out the integration over ω. The integrals

$$\oint \frac{d\omega}{\omega\left(\varepsilon_\perp - \frac{k^2c^2}{\omega^2}\right)}$$

and

$$\oint \frac{k^2c^2 d\omega}{\omega^3\left(\varepsilon_\perp - \frac{k^2c^2}{\omega^2}\right)}$$

taken around a closed contour which encompasses the upper half-plane reduce to integrals along the real axis and over a semicircle: $|\omega| \to \infty$ for the first integral and $|\omega| \to 0$ for the second. Both integrals yield the same result. Finally, we have

$$\left(E_\perp^2\right)_{\mathbf{k}} = \int_{-\infty}^{\infty} \left(E_\perp^2\right)_{\mathbf{k},\,\omega} d\omega = \frac{T}{\pi^2}; \quad (B^2)_{\mathbf{k}} = \frac{T}{\pi^2}. \tag{4}$$

The time–space correlation functions are

$$\overline{\mathbf{E}_\perp(\mathbf{r}, t)\,\mathbf{E}_\perp(\mathbf{r}', t')} = \int \left(E_\perp^2\right)_k e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\, dk = 8\pi T\delta(\mathbf{r}-\mathbf{r}'); \tag{5}$$

$$\overline{\mathbf{B}(\mathbf{r}, t)\,\mathbf{B}(\mathbf{r}', t')} = 8\pi T\delta(\mathbf{r}-\mathbf{r}'). \tag{6}$$

The total energy densities of the electric and magnetic fields can be obtained from (4) and Eq. (8.22):

$$\frac{\overline{E_\perp^2}}{8\pi} = \frac{\overline{B^2}}{8\pi} = \int T\frac{d\mathbf{k}}{(2\pi)^3}. \tag{7}$$

§ 15. Equation for Energy Transfer of Electromagnetic Waves [62-64]

Using Maxwell's equations (6.1), we can write the Poynting theorem in the following form:

$$\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}\right) + \operatorname{div}\frac{c}{4\pi}[\mathbf{EB}] + \mathbf{j}(\mathbf{E})\,\mathbf{E} = -\mathbf{j}^M\mathbf{E}. \tag{15.1}$$

We now apply this theorem to a quasi-monochromatic wave (wave packet)

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}, t)\, e^{i(\mathbf{kr}-\omega t)}, \tag{15.2}$$

where $\mathbf{E}_0(\mathbf{r}, t)$ is a complex amplitude that varies slowly in time and space while ω and $\mathbf{k}$ are the real frequency and the wave vector. This wave exists in a frequency region in which $\varepsilon''_{\alpha\beta}$, the anti-Hermitian part of the dielectric tensor $\varepsilon_{\alpha\beta}$, is small. Hence, in what follows, we will take $\varepsilon''_{\alpha\beta}$ and the rate of change of the wave amplitude to be first-order quantities:

$$\omega\frac{\partial}{\partial t} \sim \mathbf{k}\frac{\partial}{\partial \mathbf{r}} \sim \frac{\varepsilon''_{\alpha\beta}}{\varepsilon_{\alpha\beta}} \ll 1. \tag{15.3}$$

To obtain the relation between real values of ω and $\mathbf{k}$, we neglect the anti-Hermitian part of the dielectric tensor $\varepsilon_{\alpha\beta}$ and the dependence of the amplitude of $\mathbf{E}_0$ on $\mathbf{r}$ and t. Then the homogeneous Maxwell equations for the amplitude are given by Eq. (1.14)

$$D_{\alpha\beta}(\mathbf{k}, \omega) E_\beta = 0, \tag{15.4}$$

where

$$D_{\alpha\beta}(\mathbf{k}, \omega) = k^2\delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2}\varepsilon_{\alpha\beta}. \tag{15.5}$$

We now find the relation between the current density $\mathbf{j}(\mathbf{E})$ and the quasi-monochromatic wave (15.2). For this purpose, we substitute Eq. (15.2) in Eq. (6.2), which relates $\mathbf{j}(\mathbf{E})$ and $\mathbf{E}(\mathbf{r}, t)$ and isolate the factor $e^{i(\mathbf{kr}-\omega t)}$:

$$j_\alpha(\mathbf{r}, t) = e^{i(\mathbf{kr}-\omega t)} \int_{-\infty}^{t} dt' \int \sigma_{\alpha\beta}(\mathbf{r}-\mathbf{r}', t-t') \times \\ \times E_0(\mathbf{r}', t')\, e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')+i\omega(t-t')}\, d\mathbf{r}'. \tag{15.6}$$

Since the dependence of the amplitude $E_0(\mathbf{r}, t)$ on coordinates and time is slow, we can expand in terms of $\mathbf{r}-\mathbf{r}'$ and $t-t'$:

$$\mathbf{E}_0(\mathbf{r}', t') = \mathbf{E}_0(\mathbf{r}, t) + \frac{\partial \mathbf{E}_0}{\partial \mathbf{r}}(\mathbf{r}'-\mathbf{r}) + \frac{\partial \mathbf{E}_0}{\partial t}(t'-t) + \cdots \tag{15.7}$$

From the definition of the complex conductivity tensor (6.17), we have:

$$\left.\begin{aligned} &\int_{-\infty}^{t} dt' \int \sigma_{\alpha\beta}(\mathbf{r}-\mathbf{r}', t-t')\, e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')+i\omega(t-t')}\, d\mathbf{r}' = \sigma_{\alpha\beta}(\mathbf{k}, \omega); \\ &\int_{-\infty}^{t} dt' \int \sigma_{\alpha\beta}(\mathbf{r}-\mathbf{r}', t-t')(\mathbf{r}'-\mathbf{r})\, e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')+i\omega(t-t')}\, d\mathbf{r}' = \\ &\qquad = -i\frac{\partial}{\partial \mathbf{k}}\sigma_{\alpha\beta}(\mathbf{k}, \omega); \end{aligned}\right\} \tag{15.8}$$

$$\left.\begin{aligned}\int\limits_{-\infty}^{t} dt' \int \sigma_{\alpha\beta}(\mathbf{r}-\mathbf{r}', t-t')(t'-t)\,e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')+i\omega(t-t')}\,d\mathbf{r}' = \\ = -i\frac{\partial}{\partial\omega}\sigma_{\alpha\beta}(\mathbf{k},\omega).\end{aligned}\right\} \quad (15.8)$$

Substitution of (15.7) in Eq. (15.6) then yields

$$j_\alpha(\mathbf{r},t) = e^{i(\mathbf{kr}-\omega t)}\left\{\sigma_{\alpha\beta}(\mathbf{k},\omega)\,E_0(\mathbf{r},t) - i\frac{\partial\sigma_{\alpha\beta}}{\partial\mathbf{k}} \times \right.$$
$$\left. \times \frac{\partial E_{0\beta}(\mathbf{r},t)}{\partial\mathbf{r}} + i\frac{\partial\sigma_{\alpha\beta}}{\partial\omega}\frac{\partial E_{0\beta}(\mathbf{r},t)}{\partial t}\right\}. \quad (15.9)$$

We make the substitution $i\sigma_{\alpha\beta} = (\varepsilon_{\alpha\beta} - \delta_{\alpha\beta})(\omega/4\pi)$ in the second and third terms. The third term on the left side of Eq. (15.1) then becomes:*

$$\overline{\mathbf{jE}} = \left[\frac{\partial(\omega\varepsilon'_{\alpha\beta})}{\partial\omega} - \delta_{\alpha\beta}\right]\frac{\partial}{\partial t}\frac{E_\alpha^* E_\beta}{16\pi} - \omega\frac{\partial\varepsilon_{\alpha\beta}}{\partial\mathbf{k}}\frac{\partial}{\partial r}\frac{E_\alpha^* E_\beta}{16\pi} +$$
$$+\frac{1}{2}\sigma'_{\alpha\beta}E_\alpha^* E_\beta. \quad (15.10)$$

In the last term in Eq. (15.1) which gives the magnitude of the fluctuation radiation (cf. § 12), we isolate that part of the radiation which appears in a wave with the polarization and frequency being considered:

$$\overline{-\mathbf{j}^{\text{M}}\mathbf{E}} \to \int \eta_\omega\, d\Omega_0. \quad (15.11)$$

Substituting Eqs. (15.10) and (15.11) in Eq. (15.1) and collecting terms that contain derivatives with respect to time and coordinates, we now write the Poynting theorem in the form

$$\frac{\partial W}{\partial t} + \mathbf{div S} + \frac{1}{2}\sigma'_{\alpha\beta}E_\beta^* E_\alpha = \int \eta_\omega\, d\Omega_0, \quad (15.12)$$

where W is the energy density and **S** is the energy flux associated with the electromagnetic oscillations; these are given by the expressions

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*We recall that the mean value (over an oscillation period) of a quantity that is bilinear in the field (15.2) is given by an expression of the form

$$\overline{\mathrm{Re}E_\alpha \mathrm{Re}E_\beta} = \frac{1}{2}\mathrm{Re}E_\alpha^* E_\beta.$$

$$W = \frac{\partial(\omega\varepsilon'_{\alpha\beta})}{\partial\omega}\frac{E^*_\alpha E_\beta}{16\pi} + \frac{B^2}{16\pi}; \tag{15.13}$$

$$\mathbf{S} = \frac{c}{8\pi}[\mathbf{EB}^*] - \omega\frac{\partial\varepsilon'_{\alpha\beta}}{\partial\mathbf{k}}\frac{E^*_\alpha E_\beta}{16\pi}. \tag{15.14}$$

If the electromagnetic field satisfies the homogeneous Maxwell equation (15.4), or (1.12) and (1.13), then

$$\overline{B^2} = \overline{\mathbf{DE}} = \frac{1}{2}\varepsilon'_{\alpha\beta}E^*_\alpha E_\beta, \tag{15.15}$$

by means of which the energy expression (15.13) can also be written in the form

$$W = \frac{\partial(\omega^2\varepsilon'_{\alpha\beta})}{\omega\partial\omega}\frac{E^*_\alpha E_\beta}{16\pi}. \tag{15.16}$$

Making use of the notation $\mathbf{B} = (c/\omega)[\mathbf{kE}]$, we can write the energy flux $\mathbf{S}$ in the form

$$\mathbf{S} = \frac{c^2}{8\pi\omega}\left\{\mathbf{k}E^2 - \frac{(\mathbf{kE})\,\mathbf{E}^* + (\mathbf{kE}^*)\,\mathbf{E}}{2}\right\} - \omega\frac{\partial\varepsilon'_{\alpha\beta}}{\partial\mathbf{k}}\frac{E^*_\alpha E_\beta}{16\pi}. \tag{15.17}$$

The energy flux $\mathbf{S}$ that appears in Eq. (15.12), and the dissipative energy $\sigma'_{\alpha\beta}E^*_\alpha E_\beta$ can be expressed in terms of W if we introduce the group velocity $\mathbf{v} = d\omega/d\mathbf{k}$ and the damping factor (growth factor) for the waves. For this purpose we consider Eq. (15.4) (taking account of the anti-Hermitian part $\varepsilon''_{\alpha\beta}$) for a wave $\mathbf{E} = \text{const}\, e^{i(\mathbf{kr} - \omega t)}$ with real $\mathbf{k}$. In this case, the frequency ω is complex: $\omega = \omega_1 - i\omega_2$, where $\omega_2 \ll \omega_1$, in view of the fact that $\varepsilon''_{\alpha\beta}$ is assumed to be small. In the tensor $D_{\alpha\beta}$, which appears in Eq. (15.4), we now introduce an anti-Hermitian part which is due to the presence of ω_2 and $\varepsilon''_{\alpha\beta}$:

$$\left.\begin{aligned} D_{\alpha\beta} &= D'_{\alpha\beta} - i\frac{\omega^2}{c^2}\varepsilon''_{\alpha\beta} + i\frac{\partial(\omega^2\varepsilon'_{\alpha\beta})}{c^2\partial\omega}\omega_2; \\ D'_{\alpha\beta} &= k^2\delta_{\alpha\beta} - k_\alpha k_\beta - \frac{\omega^2}{c^2}\varepsilon'_{\alpha\beta}. \end{aligned}\right\} \tag{15.18}$$

Multiplying Eq. (15.4) by E^*_α, separating real and imaginary parts and setting the imaginary part equal to zero, we then have

$$\frac{1}{2}\sigma'_{\alpha\beta}E^*_\alpha E_\beta = 2\omega_2 W \equiv \gamma W. \tag{15.19}$$

In obtaining the relation between $\mathbf{S}$ and W, we limit ourselves to the zeroth approximation [since taking account of corrections due to absorption would give terms of second order in Eq. (15.1)]. Hence, we can start from the equation

$$D'_{\alpha\beta} E_\alpha = 0. \tag{15.20}$$

Differentiation of this equation with respect to $\mathbf{k}$ yields

$$\frac{\partial D_{\alpha\beta}}{\partial \mathbf{k}} E_\beta + D'_{\alpha\beta} \frac{\partial E_\beta}{\partial \mathbf{k}} = 0. \tag{15.21}$$

Multiplying the expression that has been obtained by E^*_α, and assuming $D'_{\alpha\beta} E^*_\alpha = D^{*\prime}_{\beta\alpha} E^*_\alpha$ [by virtue of Eq. (15.20)], we have

$$\frac{\partial D'_{\alpha\beta}}{\partial \mathbf{k}} E_\beta E^*_\alpha = 0. \tag{15.22}$$

Carrying out the differentiation with respect to $\mathbf{k}$, taking account of the function $\omega(\mathbf{k})$, we obtain the familiar relation between the energy flux and the energy:

$$S = \frac{d\omega}{d\mathbf{k}} W, \tag{15.23}$$

where $\mathbf{S}$ and W are given by Eqs. (15.17) and (15.16).

Taking account of the relations that have been obtained [(15.19) and (15.23)], we can now write the energy equation (15.12) in the form

$$\left.\begin{aligned} &\frac{\partial W}{\partial t} + \operatorname{div}(\mathbf{v}W) + \gamma W = \int \eta_\omega \, d\Omega_0; \\ &\mathbf{v} = \frac{d\omega}{d\mathbf{k}}, \end{aligned}\right\} \tag{15.24}$$

where $\gamma = 2\omega_2$, as follows from Eq. (15.19).

The corrections in the expressions for W and $\mathbf{S}$ that stem from the $\mathbf{jE}$ term have the meaning of the work done by the field on the charges and obviously represent the energy and energy flux associated with the oscillations of the charges. For example, let us consider a plasma in the absence of a magnetic field. For the transverse waves in the hydrodynamic approximation we have $\varepsilon_{\alpha\beta} = \varepsilon_\perp[\delta_{\alpha\beta} - (k_\alpha k_\beta / k^2)]$, $\varepsilon_\perp = 1 - (\omega_0^2/\omega^2)$, so that the energy density is

$$W = \frac{\overline{B^2}}{8\pi} + \frac{\overline{E^2_\perp}}{8\pi} \frac{\partial(\varepsilon_\perp \omega)}{\partial \omega} = \frac{\overline{B^2} + \overline{E^2_\perp}}{8\pi} + \frac{\omega^2_{0e}}{\omega^2} \frac{\overline{E^2_\perp}}{8\pi}. \tag{15.25}$$

The last term here is the energy associated with the electron oscillations. The electron velocity in the wave is given by $v = i(e/m\omega)E$, so that

$$W_{el} = \frac{nm\overline{v^2}}{2} = \frac{\omega_0^2}{\omega^2}\frac{\overline{E^2}}{8\pi}. \tag{15.26}$$

For longitudinal waves (considering high frequencies only), in the weak damping region ($k^2T_e \ll m_e\omega^2$), we have

$$\varepsilon = 1 - \frac{\omega_0^2}{\omega^2}\left(1 + \gamma\frac{k^2T_e}{m_e\omega^2}\right).$$

The energy flux as a whole is due to the correction to S associated with $d\varepsilon_{\alpha\beta}/dk$ (since $B = 0$):

$$S = -\omega\frac{\partial\varepsilon_{\parallel}}{\partial \mathbf{k}}\frac{\overline{E^2}}{8\pi} = \frac{2\gamma \mathbf{k}T_e}{m_e\omega}\frac{\omega_{0e}^2}{\omega^2}\frac{\overline{E^2}}{8\pi} = \frac{\gamma p_0 v^2}{\omega}\mathbf{k}. \tag{15.27}$$

As is well known from hydrodynamics [65], the mean energy flux associated with the small oscillations is $\mathbf{q} = p^{(1)}\mathbf{v}$. In view of the fact that a change in pressure is to be associated with the velocity by the relation $p^{(1)} = \gamma(kv/\omega)p_0$, we have $\mathbf{q} = (\gamma p_0 v^2/\omega)\mathbf{k}$, i.e., we obtain a formula that coincides with the expression for the energy flux in the electrodynamic formulation.

Appendix I

Derivation of Eqs. (4.9). The Bessel function can be given by the following integral representation:

$$J_n(x) = \frac{1}{2\pi}\int e^{-ix\sin\varphi+in\varphi}\,d\varphi = \frac{1}{2\pi}\int e^{ix\sin\varphi-in\varphi}\,d\varphi =$$
$$= \frac{i^n}{2\pi}\int e^{-ix\sin\varphi+in\varphi}\,d\varphi = \frac{(-i)^n}{2\pi}\int e^{ix\cos\varphi+in\varphi}\,d\varphi. \tag{I.1}$$

The integration is carried out over any interval of length 2π. We expand $e^{ix\sin\varphi}$ in a Fourier series

$$e^{ix\sin\varphi} = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}. \tag{I.2}$$

Multiplying Eq. (I.2) by $e^{-in\varphi}$ and integrating with respect to φ from 0 to 2π, taking account of Eq. (I.1), we have

$$a_m = \frac{1}{2\pi}\int e^{ix\sin\varphi-in\varphi}\,d\varphi = J_m(x), \tag{I.3}$$

so that

$$e^{ix\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\varphi}. \tag{I.4}$$

Appendix II

Derivation of Eq. (9.20). We consider the following integral ($k > 0$):

$$I(\nu) = \left\langle \frac{1}{\omega - n\omega_B - kv_z + i\nu} \right\rangle =$$

$$= \left(\frac{m}{2\pi T}\right)^{1/2} \int_{-\infty}^{\infty} \frac{e^{-\frac{mv_z^2}{2T}}\, dv_z}{\omega - n\omega_B - kv_z + i\nu} = \frac{1}{\sqrt{\pi}\, kv_T} \int_{-\infty}^{\infty} \frac{e^{-t^2}\, dt}{z - t}, \tag{II.1}$$

where $v_T = \sqrt{2T/m}$ and $z = (\omega - n\omega_B + i\nu)/kv_T$. Since $\operatorname{Im} z = (\nu/kv_T) > 0$, we write

$$\frac{1}{z - t} = -i \int_0^{\infty} e^{i(z-t)\xi}\, d\xi. \tag{II.2}$$

Furthermore,

$$\int_{-\infty}^{\infty} \frac{e^{-t^2}\, dt}{z - t} = -i \int_0^{\infty} d\xi \cdot e^{iz\xi - \frac{\xi^2}{4}} \int_{-\infty}^{\infty} e^{-\left(t + i\frac{\xi}{2}\right)^2} dt =$$

$$= -i\sqrt{\pi}\, e^{-z^2} \int_0^{\infty} e^{-\left(\frac{\xi}{2} - iz\right)^2} d\xi =$$

$$= -i\sqrt{\pi}\, e^{-z^2} \left(2\int_0^{\infty} e^{-y^2}\, dy - 2\int_0^{-iz} e^{-y^2}\, dy \right) =$$

$$= -i\pi e^{-z^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^{z} e^{t^2}\, dt \right). \tag{II.3}$$

When $\nu \to 0$, $I(\nu) \to \langle \zeta(\omega - n\omega_B - kv_z) \rangle$. Thus ($v_T \equiv \sqrt{2T/m}$)

$$\langle \zeta (\omega - n\omega_B - kv_z) \rangle = \frac{1}{\omega - n\omega_B} Z\left(\frac{\omega - n\omega_B}{kv_T}\right), \tag{II.4}$$

where

$$\left.\begin{aligned} Z(z) &= X(z) - iY(z); \\ X(z) &= 2ze^{-z^2} \int_0^z e^{t^2}\, dt; \\ Y(z) &= \sqrt{\pi} z e^{-z^2}. \end{aligned}\right\} \tag{II.5}$$

Consider the integral $\int_0^z e^{t^2}\, dt$ for $|z| >> 1$. We write $t = z - \alpha$. Then,

$$\int_0^z e^{t^2}\, dt = e^{z^2} \int_0^z e^{-2\alpha z} e^{\alpha^2}\, d\alpha = e^{z^2} \int_0^z e^{-2\alpha z} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!}\, d\alpha.$$

To exponential accuracy the upper limit for $|z| >> 1$ can now be set equal to infinity. Making use of the familiar relation

$$\int_0^\infty e^{-t} t^x\, dt = x!,$$

we have

$$\int_0^z e^{t^2}\, dt = e^{z^2} \sum_{n=0}^{\infty} \frac{(2n)!}{n!\,(2z)^{2n+1}} = \frac{e^{z^2}}{2z}\left(1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \frac{15}{8z^6} + \cdots\right).$$

Thus, when $|z| >> 1$,

$$X(z) = 1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \frac{15}{8z^6} + \cdots$$

Appendix III

Derivation of the Maxwell Equations Used to Write the Fluctuation Fields. Using the exact equations of motion for all the charges in the plasma and the exact Maxwell equations for vacuum (in other words the Lorentz equations), we obtain the system of Maxwell's equations (6.1) which can be used to describe the mean and fluctuation fields in a medium. The derivation of these equations can be carried out conveniently using the "microscopic distribution function" for the phase density of particle species a:*

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*This method has been developed by Yu. L. Klimontovich et al. Below in our presentation we shall also make use of works of O.P. Pogutse (in press).

$$N_a(\mathbf{r}, \mathbf{p}, t) = \sum \delta(\mathbf{r} - \mathbf{r}_{ai}(t))\, \delta(\mathbf{p} - \mathbf{p}_{ai}(t)). \tag{1}$$

The mean value of the function N_a is proportional to the usual distribution function f_a:

$$\overline{N_a}(\mathbf{r}, \mathbf{p}, t) = f_a = n_a F_a(\mathbf{r}, \mathbf{p}), \tag{2}$$

where n_a is the density of particles of species a.

Let the equation of motion for the i-th charge be of the form

$$\frac{d\mathbf{p}_{ai}}{dt} = \mathbf{F}_\Sigma(\mathbf{r}_{ai}, t); \tag{3}$$

$$\frac{d\mathbf{r}_{ai}}{dt} = \mathbf{v}_{ai} = \frac{\mathbf{p}_{ai}}{m_a}, \tag{4}$$

where $\mathbf{F}_\Sigma$ represents the total force acting on the i-th charge. This force is made up of the external force $\mathbf{F}_e$, which is independent of the distribution of charges in the plasma (for example, gravitational forces, the Lorentz force due to electric and magnetic fields), and the electromagnetic force $\mathbf{F}$, which arises because of the interaction of the entire ensemble of charges in the plasma with the charge being considered:

$$\mathbf{F}_\Sigma(\mathbf{r}_{ai}, t) = \mathbf{F}_e(\mathbf{r}_{ai}, t) + \mathbf{F}(\mathbf{r}_{ai}, t). \tag{5}$$

The equation for N_a can now be written in the form

$$\hat{K}_e N_a + \frac{\partial}{\partial \mathbf{p}} \{ \mathbf{F}(\mathbf{r}, t) N_a \} = 0, \tag{I}$$

where $\hat{K}_e$ is the kinetic operator

$$\hat{K}_e = \frac{\partial}{\partial t} + (\mathbf{v}\nabla) + \frac{\partial}{\partial \mathbf{p}} \{ \mathbf{F}_e(\mathbf{r}, t) \}. \tag{6}$$

Here, the subscript e indicates the fact that $\hat{K}_e$ contains the external force $\mathbf{F}_e$. The validity of Eq. (I) can be established by direct differentiation, taking account of (3) and (4).

In addition to (I), we must make use of the microscopic Maxwell's equations, which we write in symbolic form for the quantity $\mathbf{F}$:

$$\hat{M}_0 \mathbf{F}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t). \tag{II}$$

($\mathbf{F}_e$ satisfies the homogeneous equation $\hat{M}_0 \mathbf{F}_e = 0$ in the volume occupied by the plasma.)

The total microscopic current density $\mathbf{j}(\mathbf{r},t)$ can be written making use of N_a:

$$\mathbf{j}(\mathbf{r},\ t) = \sum_a \int e_a \mathbf{v}_a N_a \, d\mathbf{p}_a. \tag{III}$$

The subscript 0 on the Maxwell operator $\hat{M}$ means that the Maxwell equation is written in vacuo.

The subsequent procedure in the derivation consists of separating the system of equations (I) and (II) into equations for the mean quantities and for fluctuations about the mean quantities.

The departure of N_a from the Mean value $\overline{N_a} = f_a$ is denoted by ΔN_a:

$$N_a = \overline{N_a} - \Delta N_a. \tag{7}$$

In view of the fact that we wish to obtain equations which will be useful for describing propagation of electromagnetic waves in a plasma, in addition to the description of the inherent properties, we introduce in $\overline{N}_a$ a small correction $N_a^{(1)} \ll N_a^{(0)}$ which is to be associated with the field of the wave $\overline{N_a} = N_a^{(0)} + N_a^{(1)}$. Similarly, we write two terms in the fluctuation parts: $\Delta N_a = \Delta N_a^{(0)} + \Delta N_a^{(1)}$. Thus, the function N_a is finally written in the form

$$N_a = N_a^{(0)} + \Delta N_a^{(0)} + N_a^{(1)} + \Delta N_a^{(1)}. \tag{8}$$

Now, substituting N_a in (III) for the current, we obtain a corresponding separation of the total current density

$$\mathbf{j} = \mathbf{j}^{(0)} + \Delta \mathbf{j}^{(0)} + \mathbf{j}^{(1)} + \Delta \mathbf{j}^{(1)}. \tag{9}$$

By virtue of the linearity of Maxwell's equations (II), the "field" $\mathbf{F}$ can also be divided into four parts:

$$\mathbf{F} = \mathbf{F}^{(0)} + \Delta \mathbf{F}^{(0)} + \mathbf{F}^{(1)} + \Delta \mathbf{F}^{(1)}, \tag{10}$$

which satisfy the following equations:

$$\hat{M}_0 \mathbf{F}^{(0)} = \mathbf{j}^{(0)}; \tag{IIa}$$

$$\hat{M}_0 \Delta \mathbf{F}^{(0)} = \Delta \mathbf{j}^{(0)}; \tag{IIb}$$

$$\hat{M}_0 \mathbf{F}^{(1)} = \mathbf{j}^{(1)}; \tag{IIc}$$

$$\hat{M}_0 \Delta \mathbf{F}^{(1)} = \Delta \mathbf{j}^{(1)}; \tag{IId}$$

where

$$\mathbf{j}^{(0)} = \int \sum_a e_a \mathbf{v} N_a^{(0)}\, d\mathbf{p}; \tag{IIIa}$$

$$\Delta \mathbf{j}^{(0)} = \int \sum_a e_a \mathbf{v}\, \Delta N_a^{(0)}\, d\mathbf{p}; \tag{IIIb}$$

$$\mathbf{j}^{(1)} = \int \sum_a e_a \mathbf{v} N_a^{(1)}\, d\mathbf{p}; \tag{IIIc}$$

$$\Delta \mathbf{j}^{(1)} = \int \sum_a e_a \mathbf{v}\, \Delta N_a^{(1)} d\mathbf{p}. \tag{IIId}$$

We recall that $N^{(0)}$, $N^{(1)}$, $\mathbf{F}^{(0)}$, $\mathbf{F}^{(1)}$, $\mathbf{j}^{(0)}$, and $\mathbf{j}^{(1)}$ are "mean quantities"; the symbol Δ denotes a fluctuation quantity whose mean is zero; the "zero" subscripts denote quantities in the absence of the electromagnetic wave and the subscript "one" denotes quantities associated with the electromagnetic wave. All quantities with the subscript "one" are first-order quantities and linearization is carried out to this order (i.e., we neglect products of the form $\mathbf{F}^{(1)}N^{(1)}$, $\Delta\mathbf{F}^{(1)}N^{(1)}$, etc.).

Carrying out the averaging procedure in Eq. (I) and separating zero-order and first-order quantities, we obtain two equations for the mean quantities:

$$\widehat{K} N_a^{(0)} + \frac{\partial}{\partial \mathbf{p}} \left\langle \Delta \mathbf{F}^{(0)}\, \Delta N_a^{(0)} \right\rangle = 0; \tag{11}$$

$$\widehat{K} N_a^{(1)} + \frac{\partial}{\partial \mathbf{p}} \left\{ \mathbf{F}_a^{(1)} N^{(0)} + \left\langle \Delta \mathbf{F}^{(0)}\, \Delta N_a^{(1)} \right\rangle + \left\langle \Delta \mathbf{F}^{(1)}\, \Delta N_a^{(0)} \right\rangle \right\} = 0. \tag{12}$$

where the operator $\hat{K}$ is given by

$$\widehat{K} = \widehat{K}_e + \frac{\partial}{\partial \mathbf{p}} \{ \mathbf{F}^{(0)}(\mathbf{r},\ t) \cdot \} = \frac{\partial}{\partial t} + (\mathbf{v}_a \nabla) + \frac{\partial}{\partial \mathbf{p}} \{ (\mathbf{F}_e + \mathbf{F}^{(0)}) \cdot \}. \tag{13}$$

We recall that $\mathbf{F}_e$ is the external force, while $\mathbf{F}^{(0)}$ is the internal force [$\mathbf{F}^{(0)}$ depends on the form of the distribution function].

Now, subtracting Eqs. (11) and (12) from Eq. (I), and considering first-order quantities, we obtain two equations for the fluctuation quantities:

$$\widehat{K} \Delta N_a^{(0)} + \frac{\partial}{\partial \mathbf{p}} \left\{ \Delta \mathbf{F}^{(0)} N_a^{(0)} + \left(\Delta N_a^{(0)} \Delta \mathbf{F}^{(0)} - \langle \Delta N_a^{(0)} \Delta \mathbf{F} \rangle \right) \right\} = 0; \tag{14}$$

$$\begin{aligned} &\widehat{K} \Delta N_a^{(1)} + \frac{\partial}{\partial \mathbf{p}} \left\{ \mathbf{F}^{(1)}\, \Delta N_a^{(0)} + N_a^{(1)}\, \Delta \mathbf{F}^{(0)} + N_a^{(0)} \Delta \mathbf{F}^{(1)} + \right. \\ &\left. + \left(\Delta \mathbf{F}^{(0)} \Delta N_a^{(1)} - \langle \Delta \mathbf{F}^{(0)} \Delta N_a^{(1)} \rangle \right) + \left(\Delta \mathbf{F}^{(1)} \Delta N_a^{(0)} - \langle \Delta \mathbf{F}^{(1)} \Delta N_a^{(0)} \rangle \right) \right\} = 0. \end{aligned} \tag{15}$$

The system of equations in (11)-(15) actually represent another way of writing Eq. (I) [to accuracy of second order in the electromagnetic wave $\mathbf{F}^{(1)}$; when $\mathbf{F}^{(1)} = 0$, Eqs. (11) and (14) are identical with Eq. (I)].

The usual simplification of equations that have been obtained (or their equivalent) makes use of the smallness of the interaction energy compared with the kinetic energy of the charges ($e^2 n^{1/3} \ll m\overline{v^2}$). When this condition is invoked, the terms in the curly brackets in Eqs. (14) and (15) can be omitted. We also assume that the frequency of the electromagnetic wave $\mathbf{F}^{(1)}$ is considerably higher than the collision frequency. Then, we can also omit products of the fluctuation quantities (taking account of "collisions") in Eq. (12). The final system of equations for the distribution function then assumes the form:

$$\hat{K} N_a^{(0)} + \frac{\partial}{\partial \mathbf{p}} \left(\langle \Delta \mathbf{F}^{(0)} \Delta N_a^{(0)} \rangle \right) = 0; \tag{Ia}$$

$$\hat{K} \Delta N_a^{(0)} + \frac{\partial}{\partial \mathbf{p}} \left(\Delta \mathbf{F}^{(0)} N_a^{(0)} \right) = 0; \tag{Ib}$$

$$\hat{K} N_a^{(1)} + \frac{\partial}{\partial \mathbf{p}} \left(\mathbf{F}^{(1)} N_a^{(0)} \right) = 0; \tag{Ic}$$

$$\hat{K} \Delta N_a^{(1)} + \frac{\partial}{\partial \mathbf{p}} \left(\Delta \mathbf{F}^{(1)} N_a^{(0)} + \mathbf{F}^{(1)} \Delta N_a^{(0)} + \Delta \mathbf{F}^{(0)} N_a^{(1)} \right) = 0. \tag{Id}$$

The equations for $\Delta N_a^{(0)}$ and $\Delta N_a^{(1)}$ can be conveniently divided into two parts. In $\Delta N_a^{(0)}$ we isolate the part associated with the interaction $\Delta \mathbf{F}^{(0)}$

$$\Delta N_a^{(0)} = \Delta N_a^{00} + \Delta N_a \left(\Delta \mathbf{F}^{(0)} \right); \tag{16}$$

$$\hat{K} \Delta N_a^{00} = 0; \tag{17}$$

$$\hat{K} \Delta N_a \left(\Delta \mathbf{F}^{(0)} \right) + \frac{\partial}{\partial \mathbf{p}} \left(\Delta \mathbf{F}^{(0)} N_a^{(0)} \right) = 0. \tag{18}$$

Equation (17) describes the fluctuations in a system of charges moving in the external field $\mathbf{F}_e$ and in the averaged (self-consistent) field $\mathbf{F}^{(0)}$, but which do not interact microscopically. The solution of this equation is

$$\Delta N_a^{00} = \sum_i \delta \left(\mathbf{r} - \mathbf{r}_{ai}^{(0)}(t) \right) \delta \left(\mathbf{v} - \mathbf{v}_{ai}^{(0)}(t) \right) - N_a^{(0)}. \tag{19}$$

From the definition of the operator $\hat{K}$ (10) it follows that $\mathbf{r}_{ai}^{(0)}(t)$ and $\mathbf{v}_{ai}^{(0)}(t)$ are defined by the equations

$$\frac{d\mathbf{p}_{ai}^{0}(t)}{dt}=\mathbf{F}_{e}\left(\mathbf{r}_{ai}^{0}(t),\ t\right)+\mathbf{F}^{0}\left(r_{ai}^{0}(t),\ t\right); \tag{20}$$

$$\frac{d\mathbf{r}_{ai}^{0}(t)}{dt}=\mathbf{v}_{ai}^{0}(t). \tag{21}$$

Equation (Id) divides into two parts in the following way:

$$\Delta N_{a}^{(1)}=\delta N_{a}+\Delta N_{a}\left(\Delta\mathbf{F}^{(1)}\right); \tag{22}$$

$$\widehat{K}\delta N_{a}+\frac{\partial}{\partial\mathbf{p}}\left(\mathbf{F}^{(1)}\,\Delta N_{a}^{(0)}+\Delta\mathbf{F}^{(0)}N_{a}^{(1)}\right)=0; \tag{23}$$

$$\widehat{K}\,\Delta N_{a}\left(\Delta\mathbf{F}^{(1)}\right)+\frac{\partial}{\partial\mathbf{p}}\left(\Delta\mathbf{F}^{(1)}N_{a}^{(0)}\right)=0. \tag{24}$$

This division allows us to introduce the "specified" currents: these are the microscopic current

$$\mathbf{j}^{\mathrm{M}}=\int\sum_{a}e_{a}\mathbf{v}_{a}\,\Delta N_{a}^{00}\,d\mathbf{p}_{a}=\sum_{i,\,a}e_{a}\mathbf{v}_{ai}(t)\,\delta\left(\mathbf{r}_{a}-\mathbf{r}_{ai}^{0}(t)\right)-\mathbf{j}^{0}, \tag{25}$$

which is the source of the fluctuation fields $\Delta\mathrm{F}^{(0)}$, and the current induced by the electromagnetic wave $\mathbf{F}^{(1)}$

$$\delta\mathbf{j}=\int\sum_{a}e_{a}\mathbf{v}_{a}\delta N_{a}\,d\mathbf{p}_{a}, \tag{26}$$

which is the source of the scattered wave $\mathbf{F}^{(1)}$.

It is convenient to combine the Maxwell equations for $\mathbf{F}^{(1)}$, $\Delta\mathbf{F}^{(0)}$, and $\Delta\mathbf{F}^{(1)}$. We now introduce a notation for the sum of these fields:

$$\mathbf{E}=\mathbf{F}^{(1)}+\Delta\mathbf{F}^{(0)}+\Delta\mathbf{F}^{(1)} \tag{27}$$

and for the sum of the corresponding distribution functions:

$$f_{a}^{(1)}(\mathbf{E})=N_{a}^{(1)}+\Delta N_{a}\left(\Delta\mathbf{F}^{(1)}\right)+\Delta N_{a}^{0}\left(\Delta\mathbf{F}^{0}\right). \tag{28}$$

Adding Eqs. (IIb)-(IId) and Eqs. (Ib)-(Id), we have

$$\widehat{M}\mathbf{E}-\mathbf{j}(\mathbf{E})=\mathbf{j}^{\mathrm{M}}+\delta\mathbf{j}; \tag{29}$$

$$\mathbf{j}(\mathbf{E})=\int\sum_{a}e_{a}\mathbf{v}_{a}f_{a}^{(1)}\,d\mathbf{v}_{a}; \tag{30}$$

$$\hat{K} f_a^{(1)} + \frac{\partial}{\partial \mathbf{p}} \left(\mathbf{E} f_a^0 \right) = 0. \tag{31}$$

When $\mathbf{j}^0 = 0$ (uniform plasma), and $\delta \mathbf{j} = 0$ (neglecting scattering), we obtain a system which coincides with the equations used in the text (§§ 6, 9, and 13).

The fluctuation field $\Delta \mathbf{F}^0$ that appears in Eqs. (23) and (Ia) is determined by Eq. (29) with $\delta j = 0$ [which corresponds to Eq. (6.1) used in the text]. The fluctuation part of the distribution function $\Delta N_a(\Delta \mathbf{F}^0)$, which is due to the interaction of the charges, as is evident from a comparison of Eqs. (18) and (31), satisfies the same equation as the corrections to the distribution function caused by the electromagnetic wave.

The system of equations in (29)-(31) is written in symbolic form. In explicit form these equations become

$$\operatorname{rot} \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \frac{4\pi}{c} \mathbf{j}(\mathbf{E}) = \frac{4\pi}{c} (\mathbf{j}^M + \delta \mathbf{j}); \tag{32}$$

$$\operatorname{rot} \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0; \tag{33}$$

$$\operatorname{div} \mathbf{E} - 4\pi \varrho (\mathbf{E}) = 4\pi (\varrho^M + \delta \varrho); \tag{34}$$

$$\operatorname{div} \mathbf{B} = 0; \tag{35}$$

$$\mathbf{j}(\mathbf{E}) = \sum_a \int e_a \mathbf{v} f_a^{(1)} \, d\mathbf{p}_a; \quad \varrho(\mathbf{E}) = \sum_a \int e_a f_a^{(1)} \, d\mathbf{p}_a; \tag{36}$$

$$\frac{\partial f_a^{(1)}}{\partial t} + \mathbf{v} \frac{\partial f_a^{(1)}}{\partial r} + e \left(\mathbf{E}^0 + \frac{1}{c} [\mathbf{v} \mathbf{B}^0] \right) \frac{\partial f_a^{(1)}}{\partial \mathbf{p}} = - e_a \left(\mathbf{E} + \frac{1}{c} [\mathbf{v} \mathbf{B}] \right) \frac{\partial f_a^0}{\partial \mathbf{p}}. \tag{37}$$

The specified current density and charge density are obtained from Eqs. (20), (21), (23), (25), and (26):

$$\mathbf{j}^M = \sum_a e_a \mathbf{v}_{ai}(t) \, \delta (\mathbf{r} - \mathbf{r}_{ai}(t)) - \mathbf{j}^0; \quad \varrho^M = \sum_a e_a \delta \left(\mathbf{r} - \mathbf{r}_{ai}^0 (t) \right) - \varrho^0; \tag{38}$$

$$\frac{d\mathbf{r}_{ai}^0(t)}{dt} = \mathbf{v}_{ai}^0(t) = \frac{\mathbf{p}_{ai}^0(t)}{m_a}; \quad \frac{d\mathbf{p}_{ai}^0}{dt} = e_a \left(\mathbf{E}^0 + \frac{1}{c} [\mathbf{v} \mathbf{B}^0] \right); \tag{39}$$

$$\delta \mathbf{j} = \int \sum_a e_a \mathbf{v}_a \delta N_a \, d\mathbf{v}; \quad \delta \varrho = \int \sum e_a \delta N_a \, d\mathbf{v}; \tag{40}$$

$$\frac{\partial \delta N}{\partial t} + \mathbf{v} \frac{\partial \delta N}{\partial \mathbf{r}} + e \left(\mathbf{E}^0 + \frac{1}{c} [\mathbf{v} \mathbf{B}^0] \right) \frac{\partial \delta N}{\partial \mathbf{p}} =$$

$$= - \frac{\partial}{\partial p} \left\{ e_a \left(\mathbf{E}^{(1)} + \frac{1}{c} \left[\mathbf{v}_a \mathbf{B}^{(1)} \right] \right) \Delta N_a^{(0)} + e_a \left(\mathbf{E}^M + \frac{1}{c} [\mathbf{v} \mathbf{B}^M] \right) f_a^{(1)} \left(\mathbf{E}^{(1)} \right) \right\}. \tag{41}$$

Here, $\mathbf{E}^{(1)}$ and $\mathbf{B}^{(1)}$ are the solutions for the homogeneous Maxwell equations (32)-(35); $\mathbf{E}^{M}$ and $\mathbf{B}^{M}$ are the microscopic fields, the solutions of Eqs. (32)-(35) for $\delta j = 0$; $f_a^{(1)}[\mathbf{E}^{(1)}]$ is the solution of Eq. (37) in the right side of which the quantities $\mathbf{E}$ and $\mathbf{B}$ are replaced by $\mathbf{E}^{(1)}$ and $\mathbf{B}^{(1)}$. The quantities f_a^0, $\mathbf{E}^0$, $\mathbf{B}^0$, and $\mathbf{j}^0$, which appear in Eqs. (37), (38), (39), and (41), are the mean quantities which are the solutions of Eqs. (Ia) and (IIa). These equations represent a system of self-consistent Maxwell equations and the kinetic equations with a collision term:

$$\operatorname{rot} \mathbf{B}^0 - \frac{1}{c}\frac{\partial \mathbf{E}^0}{\partial t} = \frac{4\pi}{c}\mathbf{j}^0;$$

$$\operatorname{rot} \mathbf{E}^0 + \frac{1}{c}\frac{\partial \mathbf{B}^0}{\partial t} = 0;$$

$$\operatorname{div} \mathbf{E}^0 = 4\pi\varrho^0;$$
$$\operatorname{div} \mathbf{B}^0 = 0;$$

$$j^0 = \sum_a e_a \mathbf{v} f_a^0\, dv; \quad \varrho^0 = \sum_a e_a f_a^0\, d\mathbf{v};$$

$$\frac{\partial f_a^0}{\partial t} + \mathbf{v}\frac{\partial f_a^0}{\partial \mathbf{r}} + e\left(\mathbf{E}^0 + \frac{1}{c}[\mathbf{v}\mathbf{B}^0]\right)\frac{\partial f_a^0}{\partial \mathbf{p}} = S\left(f_a^0\right);$$

$$S\left(f_a^0\right) \equiv -\frac{\partial}{\partial \mathbf{p}}\left\langle e_a\left(\mathbf{E}^M + \frac{1}{c}[\mathbf{v}\mathbf{B}^M]\right)\Delta N_a^0\right\rangle;$$

$$\Delta N_a^0 = \sum_i \delta\left(\mathbf{r} - \mathbf{r}_{ai}^0(t)\right)\delta\left(\mathbf{v} - \mathbf{v}_{ai}^0(t)\right) - f_a^0 + f_a^{(1)}(\mathbf{E}^M).$$

Here, $\mathbf{E}^M$, $\mathbf{B}^M$, and $b^{(1)}(\mathbf{E}^M)$ have the same meaning as in the preceding system of equations.

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OSCILLATIONS OF AN INHOMOGENEOUS PLASMA

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Introduction

In this review we shall be concerned with the oscillations of an inhomogeneous plasma.

The review by V. D. Shafranov in the present volume [1] has considered certain characteristic plasma oscillations under the assumption that effects due to inhomogeneities in the equilibrium plasma parameters are small. It will be shown below that this approach is valid so long as the phase velocity of a wave ω/k is appreciably greater than the drift velocity of the particles $v_{dr} \sim (\rho/a)v_T$ (ρ is the particle Larmor radius, v_T is the thermal velocity, and a is the characteristic scale size of the plasma inhomogeneity). It has been found in recent years, however, that a great deal of importance should be assigned to slow wave $\omega/k \lesssim v_{dr}$, which are usually called drift waves; this importance derives from the fact that certain kinds of plasma instabilities are associated with these waves. These slow waves represent the principal subject of interest of the present review.

It is assumed that the reader is already familiar with the theory of oscillations of a uniform plasma as given in the present volume in the review by Shafranov. Using this work as a starting point, our analysis will make use of the same methods and notation as [1].

§ 1. Dielectric Constant of an Inhomogeneous Plasma

A general expression for the current density induced by a low-amplitude electromagnetic field in a plasma is given, for example, in § 9 of [1]. For the case of a nonrelativistic plasma contained by a constant magnetic field along the z axis (the plasma is inhomogeneous in the y direction), the current density can be written in the form

$$j_\alpha(r,t) = \int d\mathbf{k}\, d\omega \left\{ \frac{\omega}{4\pi i} (\varepsilon_{\alpha\beta} - \delta_{\alpha\beta}) e^{i(\mathbf{kr}-\omega t)} E_\beta(\mathbf{k},\omega) \right\}, \tag{1.1}$$

$$\varepsilon_{\alpha\beta}(\mathbf{k},\omega,y) =$$

$$= \delta_{\alpha\beta} - i \sum_{i,e} \frac{4\pi e^2}{m\omega} \int d\mathbf{v}_0 v_\alpha(t) \int_{-\infty}^{t} \frac{\partial F_0}{\partial v_\gamma(t')} \left\{ \left(1 - \frac{\mathbf{kv}(t')}{\omega}\right) \delta_{\gamma\beta} + \right.$$

$$\left. + \frac{k_\gamma v_\beta(t')}{\omega} \right\} \times e^{i\left[\omega(t-t') - \mathbf{k}\int_{t'}^{t} \mathbf{v}(t'')\,dt''\right]} dt'. \tag{1.2}$$

The notation here is the same as in [1].* Thus, the current density in an inhomogeneous plasma is simply related to the quantity $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega, y)$, which is analogous to the dielectric tensor for a uniform plasma in the $\mathbf{k}$, ω representation.

It is clear that the form of $\varepsilon_{\alpha\beta}$ will depend on the motion of the individual plasma particles in the equilibrium fields. Consider a portion of the plasma in which the equilibrium electric field is zero ($\mathbf{E}^0 = 0$), and in which the plasma pressure is small compared with the magnetic pressure ($\beta = 8\pi p/B_0^2 \ll 1$), so that quantities such as ∇B_0 can be neglected. Under these conditions the motion of the particles (electrons and ions) will be the same as in a homogeneous plasma: each particle gyrates around a Larmor circle in the planes perpendicular to the lines of force and simultaneously moves with constant velocity along the lines of force. It then follows from Eq. (1.2) that the dielectric properties of a low-pressure inhomogeneous plasma in which $\mathbf{E}^0 = 0$ are determined only by the form of the equilibrium distribution function F_0.

It is also clear that F_0 cannot be a completely arbitrary function of coordinates and velocity; this function must satisfy the equilibrium kinetic equation

$$\mathbf{v}\nabla F_0 + [\mathbf{v}_0 \boldsymbol{\omega}_B] \frac{\partial F_0}{\partial \mathbf{v}} = 0. \tag{1.3}$$

In other words, F_0 must be a function only of the integrals of the motion, i.e.,

*The sole exception is the function F_0 which is the distribution function normalized to unit volume; in [1] it represents the distribution function per particle.

quantities that are conserved in the particle motion. The equations of motion

$$\frac{d\mathbf{v}}{dt} = [\mathbf{v}, \boldsymbol{\omega}_B], \quad \frac{d\mathbf{r}}{dt} = \mathbf{v} \tag{1.4}$$

show that the integrals of the motion, which cannot contain the time explicitly, are the following: the total energy of a particle $\varepsilon = mv^2/2$, the longitudinal velocity of a particle $V_z = v_z$, and the quantities $Y = y - v_x/\omega_B$ and $X = x + v_y/\omega_B$. The most general form for an equilibrium distribution function that is a function of the y coordinate only is

$$F^0(\mathbf{v}, y) = f_0\left(\varepsilon, V_z, y - \frac{v_x}{\omega_B}\right), \tag{1.5}$$

where f_0 is an arbitrary function.

The meaning of f_0 is a simple one: this quantity represents the distribution function for the Larmor centers of the gyrating particles. If the Larmor radius of the particles is negligibly small ($\rho = v_\perp/\omega_B \to 0$) the number of particles at point y coincides with the number of centers at the same point [$F_0(y) = f_0(y)$]; if ρ is finite, the number of particles and circles is different. This same situation is responsible for the fact that current can flow across a magnetic field in an inhomogeneous plasma even though the individual particles are not displaced on the average (i.e., the centers of the circles remain fixed). If the quantity v_x/ω_B is small, the right side of Eq. (1.5) can be expanded; multiplying Eq. (1.5) by v_x and integrating, we find

$$n_0 v_{0x} = \int F_0 v_x \, d\mathbf{v} = -\frac{1}{\omega_B}\frac{\partial}{\partial y}\int f_0 v_x^2 \, d\mathbf{v} = -\frac{1}{m\omega_B}\frac{\partial p}{\partial y} \neq 0. \tag{1.6}$$

These points have been considered in detail in the review by Braginskii in this series [2]. All effects that will be considered below derive from these same features (which are sometimes called paradoxes) of an inhomogeneous plasma, i.e., the equilibrium "flows" in the direction perpendicular to the magnetic field and the inhomogeneity, even though all the particles remain fixed on the average.

Having established the relation between the function F_0 and f_0 [Eq. (1.5)], we now return to the principal task of the present section, that of computing the tensor (1.2), which will then be used in the analysis of oscillation problems. We shall limit ourselves to a simpler form of the function f_0, assuming it to be independent of the integral of the motion V_z, i.e., we shall not treat an anisotropic plasma or a plasma that contains beams (the tensor $\varepsilon_{\alpha\beta}$ is computed for the case $\partial F_0/\partial V_z \neq 0$ in Appendix I). Then,

$$\frac{\partial F_0}{\partial v_\gamma} = \frac{\partial F_0}{\partial \varepsilon} m v_\gamma - \frac{\partial F_0}{\partial Y}\frac{1}{\omega_B}\delta_{\gamma x}. \tag{1.7}$$

The quantities $\partial F_0/\partial \varepsilon$ and $\partial F_0/\partial Y$ are integrals of the motion and can be taken out from under the integral with respect to t' in Eq. (1.2), so that

$$\int_{-\infty}^{t} dt' \frac{\partial F_0}{\partial v_\gamma(t')} \left\{\left(1-\frac{\mathbf{kv}(t')}{\omega}\right)\delta_{\gamma\beta}+\frac{k_\gamma v_\beta(t')}{\omega}\right\} A(t,t')\,dt' =$$
$$= m\frac{\partial F_0}{\partial \varepsilon}\int_{-\infty}^{t} v_\beta(t')\,A(t,t')\,dt' -$$
$$-\frac{1}{\omega_B}\frac{\partial F_0}{\partial Y}\int_{\infty}^{t}\left\{\left(1-\frac{\mathbf{kv}(t')}{\omega}\right)\delta_{\beta x}+\frac{k_x v_\beta(t')}{\omega}\right\} A(t,t')\,dt';$$
$$A(t,t') = \mathbf{e}^{i\left[\omega(t-t')-\mathbf{v}\int_{t'}^{t}\mathbf{k}(t'')\,dt''\right]}. \tag{1.8}$$

The last term on the right side can be reduced to a simpler form by means of the identity

$$\frac{d}{dt'}A(t,\ t') = -i\,[\omega-\mathbf{kv}(t')]\,A(t,\ t').$$

Hence, the first term in the curly brackets on the right side of Eq. (1.8) represents an integral of the total time derivative and

$$\varepsilon_{\alpha\beta}(\mathbf{k},\ \omega,\ y) = \delta_{\alpha\beta} - i\sum_{i,\,e}\frac{4\pi e^2}{m\omega}\int d\mathbf{v}_0 v_\alpha(t)\times$$
$$\times\left[\left(m\frac{\partial F_0}{\partial\varepsilon}-\frac{k_x}{\omega\omega_B}\frac{\partial F_0}{\partial Y}\right)\int_{-\infty}^{t} v_\beta(t')\,A(t,\ t')\,dt' - \frac{i\delta_{\beta x}}{\omega\omega_B}\frac{\partial F_0}{\partial Y}\right]. \tag{1.9}$$

It is interesting to compare this expression with the corresponding result for a homogeneous plasma ($\partial F_0/\partial Y = 0$). The difference lies in the fact that the term $m\partial F_0/\partial\varepsilon$, which appears in $\varepsilon_{\alpha\beta}$ for a homogeneous plasma (cf. [1]), is replaced by the following combination for an inhomogeneous plasma:

$$m\frac{\partial F_0}{\partial\varepsilon}-\frac{k_x}{\omega\omega_B}\frac{\partial F_0}{\partial Y}. \tag{1.10}$$

The second term is comparable with the first if

$$\frac{k_x\,|v\mathrm{dr}|}{\omega} = \left|\frac{k_x}{\omega\omega_B}\frac{1}{n_0}\frac{\partial p}{\partial y}\right| \gtrsim 1. \tag{1.11}$$

This relation has a very simple meaning if one keeps Eq. (1.6) in mind: terms associated with the inhomogeneity are important if the velocity associated with the Larmor drift is not negligible compared with the phase velocity along the x axis, i.e., if $v_{dr} \gtrsim \omega / k_x$. It is precisely in this region of phase velocities that one expects to find the characteristic features of the oscillations of an inhomogeneous plasma.

If the plasma is weakly inhomogeneous over a Larmor radius the function f_0 in Eq. (1.5) can be expanded in powers of v_x/ω_B, in which case Eq. (1.10) assumes the simpler form

$$m\frac{\partial F_0}{\partial \varepsilon} - \frac{k_x}{\omega\omega_B}\frac{\partial F_0}{\partial Y} = \Phi - \frac{v_x(t)}{\omega_B}\frac{\partial \Phi}{\partial y}. \tag{1.12}$$

Here,

$$\Phi = \Phi(v^2, y) = m\frac{\partial f_0(\varepsilon, y)}{\partial \varepsilon} - \frac{k_x}{\omega\omega_B}\frac{\partial f_0(\varepsilon, y)}{\partial y}. \tag{1.13}$$

Substituting these expressions in Eq. (1.9) and writing $d\mathbf{v}_0$ in the form $d\mathbf{v}_0 = dv_{\perp 0}^2 dv_{z0} d\alpha_0$, we have

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{i,\,e}\frac{4\pi e^2}{m\omega}\left[2\pi\int dv_{\perp 0}^2\, dv_{z0}\left(\Phi Q_{\alpha\beta} - \frac{1}{\omega_B}\frac{\partial \Phi}{\partial y}P_{\alpha\beta}\right) + \right.$$

$$\left. + \frac{\delta_{\alpha x}\delta_{\beta x}}{\omega\omega_B^2}\int \frac{v_{\perp 0}^2}{2}\frac{\partial^2 f_0}{\partial y^2}\, d\mathbf{v}_0\right], \tag{1.14}$$

where

$$Q_{\alpha\beta} = -\frac{i}{2\pi}\int_0^{2\pi} d\alpha_0 v_\alpha(t)\int_{-\infty}^{t} v_\beta(t')\,A(t, t')\,dt'; \tag{1.15}$$

$$P_{\alpha\beta} = -\frac{i}{2\pi}\int_0^{2\pi} d\alpha_0 v_\alpha(t)\, v_x(t)\int_{-\infty}^{t} v_\beta(t')\,A(t, t')\,dt'. \tag{1.16}$$

The tensors $Q_{\alpha\beta}$ and $P_{\alpha\beta}$ are computed in the following way. Making use of the equations of motion (1.4), we find

$$\left.\begin{aligned} v_x(t) &= v_{\perp 0}\cos(\alpha_0 - \omega_B t);\\ v_y(t) &= v_{\perp 0}\sin(\alpha_0 - \omega_B t);\\ v_z(t) &= v_{z0}. \end{aligned}\right\} \tag{1.17}$$

Then

$$\mathbf{k}\int_{t'}^{t}\mathbf{v}(t'')\,dt'' = \xi\,[\sin(\alpha_0-\omega_B t'-\psi)-$$

$$-\sin(\alpha_0-\omega_B t-\psi)] + k_z v_{z0}(t-t'). \qquad (1.18)$$

Here, $\xi = k_\perp v_{\perp 0}/\omega_B$, and $k_\perp = \sqrt{k_x^2+k_y^2}$, $\psi = \arctan(k_y/k_x)$. An exponential of the form $\exp[-i\xi\sin(\alpha_0-\omega_B t'-\psi)]$ can be expanded in terms of Bessel functions [3]:

$$\exp[-i\xi\sin(\alpha_0-\omega_B t'-\psi)] = \sum_{n=-\infty}^{+\infty} J_n(\xi)\,e^{-in(\alpha_0-\omega_B t'-\psi)}.$$

Integrating with respect to t', we then have

$$Q_{\alpha\beta} = \frac{1}{2\pi}\int_0^{2\pi} d\alpha_0 v_\alpha(t)\,e^{i\xi\sin(\alpha_0-\omega_B t-\psi)}\sum_{n=-\infty}^{+\infty} q_\beta\zeta_n e^{-in(\alpha_0-\omega_B t'-\psi)} \qquad (1.19)$$

with a similar expression for $P_{\alpha\beta}$. Here, the function $\zeta_n \equiv \zeta_n(\omega - n\omega_B - k_z v_{z0})$ has the same meaning as in § 9 of [1]:

$$\zeta(x) = \frac{P}{x} - i\pi\delta(x). \qquad (1.20)$$

The vector q_β is made up of the components

$$\left.\begin{aligned} q_x &= v_{\perp 0}\left(\frac{nJ_n}{\xi}\cos\psi - iJ_n'\sin\psi\right),\\ q_y &= v_{\perp 0}\left(iJ_n'\cos\psi + \frac{nJ_n}{\xi}\sin\psi\right),\\ q_z &= v_{z0}J_n. \end{aligned}\right\} \qquad (1.21)$$

We now carry out an averaging with respect to α_0, making use of the following relation [3]:

$$\frac{1}{2\pi}\int_0^{2\pi} dx\,e^{i\xi\sin x - inx} = J_n(\xi). \qquad (1.22)$$

As a result we find

$$Q_{\alpha\beta} = \sum_{n=-\infty}^{+\infty} \zeta_n q_\alpha^* q_\beta; \tag{1.23}$$

$$P_{\alpha\beta} = \sum_{n=-\infty}^{+\infty} \zeta_n p_\alpha q_\beta, \tag{1.24}$$

where q_α^* is the vector which is the complex conjugate of q_α while the vector p_α is made up of the components

$$\left.\begin{aligned} p_x &= \frac{v_{\perp 0}^2}{2}\left[J_n + (J_n + 2J_n'')\cos 2\psi + 2in\left(\frac{J_n}{\xi}\right)'\sin 2\psi\right];\\ p_y &= \frac{v_{\perp 0}^2}{2}\left[-2in\left(\frac{J_n}{\xi}\right)'\cos 2\psi + (J_n + 2J_n'')\sin 2\psi\right];\\ p_z &= v_{\perp 0} v_{z0}\left(\frac{nJ_n}{\xi}\cos\psi + iJ_n'\sin\psi\right). \end{aligned}\right\} \tag{1.25}$$

Thus, we have derived an expression for the currents induced in a weakly inhomogeneous low-pressure plasma characterized by an isotropic velocity distribution for the Larmor centers. This expression follows from Eqs. (1.1), (1.13), (1.14), (1.20), (1.21), (1.23), (1.24), and (1.25).

Basic interest attaches to the case in which the function f_0 is a Maxwellian $f_0 = n_0(m/2\pi T)^{3/2} e^{-\varepsilon/T}$ because it is closest to being an "equilibrium" function in velocity space. Under these conditions the integration over velocity in Eq. (1.14) is carried out in the following way. The integrals over dv_{z0} are computed by means of the formula (cf. Appendix II of [1]):

$$\left(\frac{m}{2\pi T}\right)^{1/2} \int_{-\infty}^{+\infty} \frac{dv_{z0} e^{-mv_{z0}^2/2T}}{\omega - n\omega_B - k_z v_{z0}} = -i\frac{\sqrt{\pi}}{k_z v_T} W\left(\frac{\omega - n\omega_B}{k_z v_T}\right), \tag{1.26}$$

where

$$W(x) = e^{-x^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt\right); \quad v_T = \sqrt{\frac{2T}{m}}; \tag{1.27}$$

where W is the Cramp function of complex argument, which is tabulated in [4].

In the integration over transverse velocity, we make use of the formula [3]:

$$\int\limits_0^\infty e^{-\sigma^2 x^2} J_n(\alpha x) J_n(\beta x) x\,dx =$$
$$= \frac{1}{2\sigma^2} \exp\left(-\frac{\alpha^2+\beta^2}{4\sigma^2}\right) I_n\left(\frac{\alpha\beta}{2\sigma^2}\right), \tag{1.28}$$

where I_n is the Bessel function of imaginary argument. As a result, we find

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{i,\,e}\left\{ \varepsilon^{\mathrm{I}}_{\alpha\beta} + \frac{k_x}{m\omega\omega_B}\cdot\frac{\partial}{\partial y} T\varepsilon^{\mathrm{I}}_{\alpha\beta} + \right.$$
$$\left. + \frac{1}{k_\perp}\cdot\frac{\partial}{\partial y}\left(\varepsilon^{\mathrm{II}}_{\alpha\beta} + \frac{k_x}{m\omega\omega_B}\cdot\frac{\partial}{\partial y} T\varepsilon^{\mathrm{II}}_{\alpha\beta}\right)\right\} +$$
$$+ \delta_{\alpha x}\delta_{\beta x}\sum_{i,\,e}\frac{4\pi e^2}{m^2\omega^2\omega_B^2}\cdot\frac{\partial^2}{\partial y^2} n_0 T. \tag{1.29}$$

The components of $\varepsilon^{\mathrm{I}}_{\alpha\beta}$ are then

$$\left.\begin{aligned}
\varepsilon^{\mathrm{I}}_{xx} &= \varepsilon^0_{xx}\cos^2\psi + \varepsilon^0_{yy}\sin^2\psi;\\
\varepsilon^{\mathrm{I}}_{xy} &= \varepsilon^0_{xy} + (\varepsilon^0_{xx} - \varepsilon^0_{yy})\sin\psi\cos\psi;\\
\varepsilon^{\mathrm{I}}_{yx} &= \varepsilon^0_{yx} + (\varepsilon^0_{xx} - \varepsilon^0_{yy})\sin\psi\cos\psi;\\
\varepsilon^{\mathrm{I}}_{yy} &= \varepsilon^0_{xx}\sin^2\psi + \varepsilon^0_{yy}\cos^2\psi;\\
\varepsilon^{\mathrm{I}}_{xz} &= \varepsilon^0_{xz}\cos\psi - \varepsilon^0_{yz}\sin\psi;\\
\varepsilon^{\mathrm{I}}_{yz} &= \varepsilon^0_{yz}\cos\psi + \varepsilon^0_{xz}\sin\psi;\\
\varepsilon^{\mathrm{I}}_{zx} &= \varepsilon^0_{zx}\cos\psi - \varepsilon^0_{zy}\sin\psi;\\
\varepsilon^{\mathrm{I}}_{zy} &= \varepsilon^0_{zy}\cos\psi + \varepsilon^0_{zx}\sin\psi;\\
\varepsilon^{\mathrm{I}}_{zz} &= \varepsilon^0_{zz},
\end{aligned}\right\} \tag{1.30}$$

where $\varepsilon^0_{\alpha\beta}$ is of the same form as the dielectric tensor for a homogeneous plasma as computed in a coordinate system in which $k_\perp = k_x$, i.e., $\psi = 0$. This tensor is the conventional one, and is given by

$$\left.\begin{aligned}
\varepsilon^0_{xx} &= -\sum_n \frac{\omega_0^2}{\omega k_z v_T}(-i\sqrt{\pi}W_n)\, e^{-z}\frac{n^2}{z} I_n;\\
\varepsilon^0_{xy} &= -\varepsilon^0_{yx} = i\sum_n \frac{\omega_0^2}{\omega k_z v_T}(-i\sqrt{\pi}W_n)\, e^{-z} n\,(I_n - I'_n);
\end{aligned}\right|$$

$$\left.\begin{aligned}
\varepsilon^0_{yy} &= \varepsilon^0_{xx} - \sum_n \frac{\omega_0^2}{\omega k_z v_T} (-i\sqrt{\pi} W_n)\, e^{-z} \cdot 2z\,(I_n - I'_n);\\
\varepsilon^0_{xz} &= \varepsilon^0_{zx} = \sum_n \frac{\omega_0^2}{\omega\omega_B} \cdot \frac{k_x}{k_z} (-i\sqrt{\pi} W_n)\, x_n \frac{n}{z}\, e^{-z} I_n;\\
\varepsilon^0_{yz} &= -\varepsilon^0_{zy} = i\sum_n \frac{\omega_0^2}{\omega\omega_B} \frac{k_x}{k_z} (-i\sqrt{\pi} W_n)\, x_n e^{-z}\,(I_n - I'_n);\\
\varepsilon^0_{zz} &= \sum_n \frac{2\omega_0^2}{\omega k_z v_T}\, e^{-z} I_n x_n (1 + i\sqrt{\pi} x_n W_n).
\end{aligned}\right\} \quad (1.31)$$

Here

$$z = \frac{k_\perp^2 T}{m\omega_B^2}; \quad x_n = \frac{\omega - n\omega_B}{k_z v_T}; \quad \omega_0^2 = \frac{4\pi e^2 n_0}{m};$$
$$W_n = W(x_n); \quad I_n = I_n(z).$$

The tensor $\varepsilon^{II}_{\alpha\beta}$ cannot be expressed in terms of $\varepsilon^0_{\alpha\beta}$. We shall write it here in the form in which it is not integrated over velocity, since the integration does not result in any simplification. Thus,

$$\varepsilon^{II}_{\alpha\beta} = \frac{4\pi e^2 n_0}{\omega} \sum_n S_{\alpha\beta}; \quad (1.32)$$

$$S_{\alpha\beta} = \left(\frac{m}{2\pi T}\right)^{3/2} \frac{2\pi}{T} \int dv^2_{\perp 0}\, dv_{z0}\, e^{-\varepsilon/T} \zeta_n \frac{\xi}{v_{\perp 0}}\, p_\alpha q_\beta. \quad (1.33)$$

In problems on plasma oscillations, one of Maxwell's equations

$$(\mathrm{rot}\,\mathrm{rot}\,\mathbf{E})_\alpha = \frac{\omega^2}{c^2} \hat{\varepsilon}_{\alpha\beta} E_\beta \quad (1.34)$$

is frequently replaced by Poisson's equation

$$\mathrm{div}\,\mathbf{E} = 4\pi\varrho. \quad (1.35)$$

By means of the equation of continuity

$$\frac{\partial\varrho}{\partial t} + \mathrm{div}\,\mathbf{j} = 0 \quad (1.36)$$

the charge density ρ can be expressed in terms of current density so that

$$\varrho = \int d\mathbf{k}\, d\omega e^{i(\mathbf{kr}-\omega t)} \chi_\beta E_\beta(\mathbf{k}), \tag{1.37}$$

where

$$\chi_\beta(\mathbf{k},\ \omega,\ y) = \frac{1}{4\pi i}\left[k_\alpha(\varepsilon_{\alpha\beta} - \delta_{\alpha\beta}) - i\frac{\partial \varepsilon_{y\beta}}{\partial y}\right]. \tag{1.38}$$

If the function f_0 is a Maxwellian, the vector χ_β is given by

$$\chi_\beta = \sum_{i,\ e}\left(\chi_\beta^{\mathrm{I}} + \frac{k_x}{m\omega\omega_B}\frac{\partial}{\partial y}T\chi_\beta^{\mathrm{I}}\right) + \\ + \sum_{i,\ e}\frac{1}{k_\perp}\frac{\partial}{\partial y}\left(\chi_\beta^{\mathrm{II}} + \frac{k_x}{m\omega\omega_B}\frac{\partial}{\partial y}T\chi_\beta^{\mathrm{II}}\right). \tag{1.39}$$

Here,

$$\left.\begin{aligned} \chi_x^{\mathrm{I}} &= \chi_x^0 \cos\psi - \chi_y^0 \sin\psi; \\ \chi_y^{\mathrm{I}} &= \chi_y^0 \cos\psi + \chi_x^0 \sin\psi; \\ \chi_z^{\mathrm{I}} &= \chi_z^0. \end{aligned}\right\} \tag{1.40}$$

The vector χ_β^0 is made up of the components

$$\left.\begin{aligned} \chi_x^0 &= \frac{ie^2 n_0}{T}\sum_n \frac{n\omega_B}{k_\perp k_z v_T}\left(-i\sqrt{\pi}W_n\right) I_n e^{-z}; \\ \chi_y^0 &= \frac{e^2 n_0}{T}\sum_n \frac{\omega_B}{k_\perp k_z v_T}\left(-i\sqrt{\pi}W_n\right) z\left(I_n' - I_n\right) e^{-z}; \\ \chi_z^0 &= -i\frac{e^2 n_0}{Tk_z}\sum_n \left(1 + i\sqrt{\pi}x_n W_n\right) I_n e^{-z}. \end{aligned}\right\} \tag{1.41}$$

The vector χ_β^{II} is given by

$$\chi_\beta^{\mathrm{II}} = -\frac{ie^2 n_0}{T}\sum_n 2\pi\int f_0 \frac{k_\perp}{\omega_B} q_x^* q_\beta \zeta_n\, dv_{\perp 0}^2\, dx_{z0}. \tag{1.42}$$

We have seen above at what phase velocities the dielectric constant of an inhomogeneous plasma differs from that of a homogeneous plasma [cf. Eq. (1.11)]. Now we wish to examine qualitatively the frequencies ω at which this difference becomes important. For this purpose we form the ratio of the "characteristic drift frequency" $\omega^* = k_x v_{dr}$ to the ion–cyclotron frequency ω_{Bi}, and compute the order of magnitude of this ratio. It is found that

$$\frac{\omega^*}{\omega_{Bi}} \approx \frac{\varrho_i}{a} k_x \varrho_i. \tag{1.43}$$

Here we have made use of the fact that in order-of-magnitude terms $v_{dr} \approx \rho_i v_{Ti}/a$ [cf. Eq. (1.11)], where a is the characteristic scale size of the inhomogeneity. It is evident from Eq. (1.43) that $\omega^*/\omega_{Bi} \ll 1$ if

$$k_x \varrho_i \ll \frac{a}{\varrho_i}. \tag{1.44}$$

The quantity on the right is large. Hence (in any case for wavelengths larger than, or of the order of, the ion Larmor radius), the characteristic drift frequency is appreciably smaller than the ion–cyclotron frequency. As the wave number increases still further, the ratio ω^*/ω_{Bi} remains small until the following value is reached:

$$k_x^* \sim \frac{a}{\varrho_i^2}, \tag{1.45}$$

which depends on the degree of inhomogeneity of the plasma.

Conventionally the plasma is called "weakly inhomogeneous" if $k_x^* < 1/\rho_e$, in which case

$$\frac{\varrho_i}{a} < \left(\frac{m_e}{m_i}\right)^{1/2}, \tag{1.46}$$

and "highly inhomogeneous" if $k_x^* > 1/\rho_e$, in which case

$$\frac{\varrho_i}{a} > \left(\frac{m_e}{m_i}\right)^{1/2}, \tag{1.47}$$

where ρ_e is the electron Larmor radius. In a weakly inhomogeneous plasma the characteristic drift frequency ω^* will be smaller than ω_{Bi} until wavelengths of the order of the electron Larmor radius are reached. The analysis can be limited to low-frequency oscillations in such a plasma without risking the possibility that any effect associated with the characteristic dielectric properties will be lost. In the case of a highly homogeneous plasma (1.47), it is possible that there are effects which are important at frequencies of the order of the ion–cyclotron frequency and wavelengths smaller than the ion Larmor radius.

The analysis of low-frequency oscillations ($\omega \ll \omega_{Bi}$) is especially simple. This feature is due to the fact that it is not necessary to carry out complicated summations over n (the harmonic number of the cyclotron frequency) in the expressions for $\varepsilon_{\alpha\beta}$ and χ_β. An example of these infinite summations is given by the right side of Eq. (1.31). When $\omega \ll \omega_{Bi}$, the contribution of terms with different indices to the dielectric constant of the plasma is found to be different: the largest contribution comes from the n = 0 term, while the smallest contributions come from terms characterized by $n \neq 0$. This situation arises because

$$\frac{W_{n\neq 0}}{W_{n=0}} \sim \frac{\omega}{\omega_B} \ll 1.$$

In certain cases, however, it is necessary to take account of the higher-order terms in the summation because certain elements in the tensors $\varepsilon_{\alpha\beta}$ and γ_β vanish for the zeroth harmonic, for example, ε^0_{xx}, ε^0_{xy}, and ε^0_{xz}.

In order to simplify expressions of the form (1.31) for $\omega \ll \omega_{Bi}$, we make one further assumption:

$$k_z v_T \ll \omega_B, \tag{1.48}$$

which implies that effects due to cyclotron absorption are small (cf. § 11 in [1]). If this is not the case, the oscillations are highly damped. In the case of wavelengths short in the x and y directions ($k_\perp \rho_i \gtrsim 1$), it follows from (1.48) that $k_z \ll k_\perp$, i.e., weakly damped waves can only propagate across the magnetic field. We will always assume the case $k_z \ll k_\perp$ below, because, at frequencies $\omega \sim k_\perp v_{dr}$ and for $k_z \sim k_\perp$, the waves are highly damped by Cerenkov absorption (cf. §11 in [1]).

Introducing these simplifications (ω and $k_z v_T \ll \omega_B$ and $k_z \ll k_\perp$), we can write $\varepsilon^0_{\alpha\beta}$ in the form

$$\left.\begin{aligned}
\varepsilon^0_{xx} &= \frac{\omega_0^2}{\omega_B^2}\,\frac{1 - I_0 e^{-z}}{z};\\
\varepsilon^0_{yx} &= -\,\varepsilon^0_{xy} = -i\,\frac{\omega_0^2}{\omega\omega_B}\,e^{-z}(I_0 - I_1);\\
\varepsilon^0_{yy} &= \varepsilon^0_{xx} - \frac{\omega_0^2}{\omega k_z v_T}\left(-i\sqrt{}W_0\right)\cdot 2z\,(I_0 - I_1)\,e^{-z};\\
\varepsilon^0_{zx} &= \varepsilon^0_{xz} \approx 0;\\
\varepsilon^0_{yz} &= -\,\varepsilon^0_{zy} = \frac{i\omega_0^2}{\omega\omega_B}\,\frac{k_x}{k_z}\,(I_0 - I_1)\,e^{-z}\left(1 + i\sqrt{}\,x_0 W_0\right);\\
\varepsilon^0_{zz} &= \frac{2\omega_0^2}{k_z^2 v_T^2}\,I_0 e^{-z}\left(1 + i\sqrt{}\,x_0 W_0\right).
\end{aligned}\right\} \tag{1.49}$$

As we have noted above, certain elements of $\varepsilon^0_{\alpha\beta}$ are small as a result of the "magnetization" of the transverse motion of the particles when $\omega \ll \omega_{Bi}$; for this reason, it is necessary to take account of the contributions in the appropriate elements of $\varepsilon_{\alpha\beta}$ due to terms in $\varepsilon^{II}_{\alpha\beta}$ [cf. Eq. (1.29)]. In a number of cases the following elements of $\varepsilon^{II}_{\alpha\beta}$ are found to be important: ε^{II}_{xx}, ε^{II}_{xz}, and ε^{II}_{zx}. For example,

$$\varepsilon^{II}_{xx} = -\frac{\omega_0^2}{\omega\omega_B}\left[1 - e^{-z}(I_0 - I_1)\right]\cos\psi. \tag{1.50}$$

It will be evident that $(1/k_\perp)(\partial\varepsilon^{II}_{xx}/\partial y)$ is of the same order as ε^0_{xx}.

With the same assumptions ($\omega \ll \omega_B$, $k_z v_T \ll \omega_B$, $k_z \ll k_\perp$), the expressions for χ^0_β and χ^{II}_β become:

$$\left.\begin{aligned} \chi^0_x &= -\frac{ie^2 n_0}{Tk_\perp}\left(1 - I_0 e^{-z}\right); \\ \chi^0_y &= \frac{e^2 n_0}{Tk_\perp}\,\frac{i\sqrt{\pi}W_0}{k_z v_T}\,z\left(I_0 - I_0'\right); \\ \chi^0_z &= -\frac{ie^2 n_0}{Tk_z}\left(1 + i\sqrt{\pi}x_0 W_0\right) I_0 e^{-z}; \end{aligned}\right\} \tag{1.51}$$

$$\chi^{II}_y = -\frac{e^2 n_0}{Tk_\perp}\,z e^{-z}\left(I_0 - I_0'\right). \tag{1.52}$$

These simplified expressions for $\varepsilon_{\alpha\beta}$ and χ_α will be used below in all problems concerning low-frequency oscillations of a plasma.

Above, the lines of force of the fixed magnetic field are assumed parallel. If this is not so the above expressions for $\varepsilon_{\alpha\beta}$ and χ_β must be modified. The curvature of the lines of force can usually be introduced qualitatively by using the following approach. The lines of force are assumed to be straight, as before, but a fictitious gravitational field **g** is introduced; this field acts in the directions of the plasma inhomogeneity. The quantity g is chosen in such a way that the gravitational drift of the ions is the same as the centrifugal drift (due to curvature). The appropriate relation is

$$g \simeq \frac{T_i}{m_i R}, \tag{1.53}$$

where R is the radius of curvature. The dielectric constant for a plasma in gravitational field is computed in Appendix II.

Another convenient model which takes account of both curvature and shear in the lines of force is a helical magnetic field of cylindrical symmetry.

This model is treated in Appendix III. The modifications of $\varepsilon_{\alpha\beta}$ and χ_β required by the introduction of the inhomogeneous magnetic field are treated there (i.e., the magnetic drift of the particles).

§ 2. Drift Instability of a Plasma

Oscillations of an inhomogeneous plasma characterized by phase velocities $\omega/k_\perp$ which are of the order of, or smaller than, the velocities associated with the Larmor drift of the particles, are of interest in that they can be excited spontaneously (i.e., the plasma can be "unstable"). Making use of the expressions derived above for $\varepsilon_{\alpha\beta}$, one can determine the conditions under which oscillations are to be expected, and it is found that spontaneous excitation occurs when $\omega/k_\perp \lessapprox v_{dr}$. We shall illustrate this feature making use of waves whose longitudinal phase velocity ω/k_z satisfies the relation

$$v_{Ti} \ll \frac{\omega}{k_z} \ll v_{Te}. \tag{2.1}$$

Energy transferred from the wave to the plasma particles is characterized by the imaginary part of the electron term in ε_{zz} (the imaginary part of the ion term in ε_{zz} is exponentially small, and the other terms in $\varepsilon_{\alpha\beta}$ are negligible):

$$W_{\text{wave}\to\text{particle}} \sim \frac{\omega}{4\pi}\operatorname{Im}\varepsilon_{zz} =$$

$$= \frac{\sqrt{\pi}e^2\omega^2}{k_z^3 T_e}\left(1+\frac{k_x T_e}{m_e\omega_{Be}\omega}\frac{\partial}{\partial y}\right)\frac{n_0}{v_{Te}} I_0(z_e)\,e^{-z_e}. \tag{2.2}$$

In a homogeneous plasma ($\partial/\partial y = 0$) this quantity is always positive, so that the wave must necessarily be damped. On the other hand, in an inhomogeneous plasma, the sign of the energy term can be reversed if

$$\left(1+\frac{k_x T_e}{m_e\omega_{Be}\omega}\frac{\partial}{\partial y}\right)\frac{n_0}{v_{Te}} I_0 e^{-z} < 0. \tag{2.3}$$

This condition then represents the criterion for instability for a plasma with respect to perturbations characterized by frequency ω. If $\nabla T_e = 0$, the instability criterion assumes the simpler form

$$1-\frac{k_x v_0^e}{\omega} < 0, \tag{2.4}$$

where $v_0^e = -(T_e/m_e\omega_{Be})(\partial \ln n_0/\partial y)$ is the velocity of the electron Larmor drift. Thus, if an inhomogeneous plasma can support waves with low phase velocity, these waves can be unstable. In this section we show that these very slow waves exist if the lines of force of the magnetic field do not deviate greatly from being parallel. The problem of waves in a curved magnetic field is treated in § 5.

The equations for the slow waves can be obtained in the following way. The charge density and current density found in § 1 for the case $\omega \ll \omega_{Bi}$ are used in Maxwell's equations and the Poisson equation (1.34) and (1.35). The projection of Eq. (1.31) along the y axis is then

$$k_x^2 E_y + ik_x \frac{\partial E_x}{\partial y} = \hat{\varepsilon}_{y\beta} E_\beta. \tag{2.5}$$

In the case of interest ($\omega/k_x \lesssim v_{dr} \ll v_{Ti}$), the right side ($\hat{\varepsilon}_{y\beta} E_\beta$) is appreciably smaller than the terms on the left side. Hence, as an approximation we have

$$E_y = -\frac{i}{k_x}\frac{\partial E_x}{\partial y}, \tag{2.5'}$$

so that the transverse components of the electric field are irrotational, or potential. This relation is valid to terms of order

$$\beta = \frac{8\pi p}{B_0^2} \quad \text{and} \quad \left(\frac{k_x v_{dr}}{\omega}\right)\beta.$$

Equation (2.5) and the expression for $\varepsilon_{\alpha\beta}$ show that terms of order β

$$\hat{\varepsilon}_{zx} E_x + \hat{\varepsilon}_{zy} E_y \approx 0. \tag{2.6}$$

We now make use of Eqs. (2.5) and (2.6), the projection of Eq. (1.34) along the z axis, and Poisson's equation, to obtain the following system:

$$\left.\begin{aligned} &\int e^{ik_y y} dk_y \left\{\frac{1}{k_x}\left(k_\perp^2 \varepsilon_\perp - ik_y \tilde{\varepsilon}'_\perp\right) E_x(k_y) + k_z \varepsilon_\parallel E_z(k_y)\right\} = 0; \\ &\int e^{ik_y y} dk_y \left\{\left(k_\perp^2 - \frac{\omega^2}{c}\varepsilon_\parallel\right) E_z(k_y) - \frac{k_z}{k_x} k_\perp^2 E_x(k_y)\right\} = 0. \end{aligned}\right\} \tag{2.7}$$

Here we have introduced the notation

$$\left.\begin{aligned} \varepsilon_\perp &= 1 + \sum_{i,e} \frac{4\pi e^2}{m\omega_B^2}\left(1 + \frac{k_x}{m\omega\omega_B}\frac{\partial}{\partial y} T\right) n_0 \frac{1 - I_0 e^{-z}}{z}; \\ \tilde{\varepsilon}_\perp &= 1 + \sum_{i,e} \frac{4\pi e^2}{m\omega_B^2}\left(1 + \frac{k_x}{m\omega\omega_B}\frac{\partial}{\partial y} T\right) n_0 e^{-z}(I_0 - I_1); \\ \varepsilon_\parallel \equiv \varepsilon_{zz} &= 1 + \sum_{i,e} \frac{4\pi e^2}{k_z^2 T}\left(1 + \frac{k_x T}{m\omega\omega_B}\frac{\partial}{\partial y}\right) n_0 e^{-z} I_0 \left(1 + i\sqrt{\pi} x W\right). \end{aligned}\right\} \tag{2.8}$$

In § 3 we shall consider various particular cases of waves and oscillations described by (2.7) and (2.8). Here, we shall be concerned with one particular case which, in our opinion, is of the greatest interest and allows us to illustrate the characteristic features of drift waves and drift instabilities.

We shall limit ourselves to $\lambda_y/a \ll 1$, considering waves that vary rapidly in amplitude in the direction of the plasma inhomogeneity. We introduce the wave number $K_y(y)$, which varies slowly as a function of distance $\lambda_y \sim 1/K_y$; the field is thus written in the form

$$E(y) \sim e^{i \int K_y(y)\, dy}. \tag{2.9}$$

Now, neglecting terms such as dK_y/dy and $(K_y a)^{-1}$, and using Eqs. (2.7) and (2.8), we obtain the following dispersion relation:

$$\varepsilon_{\parallel}\left(1 - \frac{\omega^2}{c^2 k_z^2}\varepsilon_{\perp}\right) + \frac{K_{\perp}^2}{k_z^2}\varepsilon_{\perp} = 0, \tag{2.10}$$

where $K_{\perp}^2(y) = k_x^2 + K_y(y)$, while $\varepsilon_{\perp}$ and $\varepsilon_{\parallel}$ are defined by Eq. (2.8) in which k_y is to be replaced by $K_y(y)$.

It is immediately evident from the form of the dispersion equation that the waves of interest here are related in some way to the ion-acoustic waves and the Alfvén waves which propagate in a uniform plasma. If effects due to the plasma inhomogeneity are neglected in Eq. (2.10) and if the wavelengths are assumed to be large ($K_{\perp}^2 \to 0$), this equation divides up into two equations [1]: one of these describes the ion-acoustic wave

$$\varepsilon_{\parallel} = 0, \tag{2.11}$$

and the other describes the Alfvén wave

$$1 - \frac{\omega^2}{c^2 k_z^2}\varepsilon_{\perp} = 0. \tag{2.12}$$

When the drift terms are taken into account in the expressions for $\varepsilon_{\parallel}$ and $\varepsilon_{\perp}$ in this same approximation, i.e., when $K_{\perp}^2 \to 0$, using Eq. (2.10) we can obtain equations that are of the same form as Eqs. (2.11) and (2.12), although the quantities $\varepsilon_{\parallel}$ and $\varepsilon_{\perp}$ are somewhat different. By analogy with the case of the uniform plasma, it is convenient to call the wave described by Eq. (2.11) the ion-acoustic wave and the wave described by Eq. (2.12) the Alfvén wave. If terms containing $K_{\perp}^2$ are retained in Eq. (2.10), and if $K_{\perp}^2$ is not neglected, this equation cannot be written as a system of two separate equations (so long as $k_z \neq 0$). However, under these conditions, it is assumed that we are still dealing with the ion-acoustic and Alfvén waves, albeit in highly modified versions.

We assume below that the plasma is dense, in which case $c_A^2 \equiv B_0^2 4\pi n_0 m_i \ll c^2$ (c_A is the Alfvén velocity) and that it is also hot, i.e., $\beta = 8\pi p/B_0^2 \gg m_e/m_i$ (but obviously subject to the restriction $\beta \ll 1$!). The wavelength is assumed to be larger than the electron Larmor radius $z_e \ll 1$, and the longitudinal phase velocity is assumed to satisfy the condition $v_{Ti} \ll \omega/k_z \ll v_{Te}$; the ion and electron temperatures are assumed to be independent of coordinates, so that $\nabla T_i = \nabla T_e = 0$. Under these assumptions, Eq. (2.10) becomes

$$\left.\begin{aligned} &\left[\left(1-\frac{k_x v_0^e}{\omega}\right)\left(1+i\sqrt{\pi}\frac{\omega}{k_z v_{Te}}\right)-\frac{k_z^2 T_e}{m_i\omega^2} I_0 e^{-Z}\left(1-\frac{k_x v_0^i}{\omega}\right)\right]\times \\ &\qquad\times\left(\omega^2-k_x v_0^i\omega-\frac{Z}{1-I_0e^{-Z}}k_z^2c_A^2\right)= \\ &\qquad\qquad= Z\frac{T_e}{T_i}k_z^2c_A^2\left(1-\frac{k_x v_0^i}{\omega}\right); \\ &\qquad Z=\frac{K_\perp^2 T_i}{m_i\omega_{Bi}^2}; \quad v_0^i=-\frac{T_i}{T_e}v_0^e. \end{aligned}\right\} \tag{2.13}$$

It is now assumed that the perturbation (a wave packet), having been produced in the vicinity of some fixed point $y = y^*$, grows significantly in time and becomes nonlinear before propagating a distance of the order of the characteristic scale size of the inhomogeneity a. This assumption is valid if there are solutions of Eq. (2.13) $\omega = \omega(\mathbf{K})$ for which

$$\gamma \equiv \operatorname{Im}\omega \gg \frac{1}{a}\frac{\partial \operatorname{Re}\omega}{\partial K_y}. \tag{2.14}$$

If (2.14) is satisfied, the solutions of Eq. (2.13) in the vicinity of $y = y^*$ can be found without considering the fate of the perturbation at points remote from the initial point; in particular, it is not necessary to consider the solution in terms of a characteristic function for some boundary value problem. Solutions of this kind will be called localized solutions. [In § 4 we consider the opposite limit to (2.14), in which case localized solutions are no longer meaningful.]

We now assume that $T_e = T_i$. It then follows from Eq. (2.13) that

$$\left(1+i\sqrt{\pi}\frac{\omega}{k_z v_{Te}}\right)\left(1-\frac{k_x v_0^e}{\omega}\right)\left(\omega^2+k_x v_0^e\omega-\frac{Zk_z^2c_A^2}{1-I_0e^{-Z}}\right)= \\ = Zk_z^2c_A^2\left(1+\frac{k_x v_0^e}{\omega}\right). \tag{2.15}$$

When $Z \ll 1$, the three roots of $\omega = \omega(\mathbf{K})$ are

$$\omega_1 = k_x v_0^e - k_z c_A Z \frac{k_z c_A}{k_x v_0^e}\left[\frac{1}{2}\left(\frac{k_z c_A}{k_x v_0^e}\right)^2 - 1\right]^{-1} +$$

$$+ i\sqrt{\pi} k_z c_A \sqrt{\frac{m_e}{m_i \beta}} \frac{Z}{\frac{1}{2}\left(\frac{k_z c_A}{k_x v_0^e}\right)^2 - 1}; \qquad (2.16)$$

$$\omega_2 = -\frac{1}{2} k_x v_0^e + \sqrt{\frac{1}{4}(k_x v_0^e)^2 + c_A^2 k_z^2} + (\text{terms of order } Z) -$$

$$- i\sqrt{\pi} k_z c_A \sqrt{\frac{m_e}{m_i \beta}} \frac{1}{4}\left(\frac{k_z c_A}{k_x v_0^e}\right)^2 \left(1 + \frac{3}{\sqrt{1 + 4\left(\frac{c_A k_z}{k_x v_0^e}\right)^2}}\right) \times$$

$$\times \frac{Z}{\frac{1}{2}\left(\frac{k_z c_A}{k_x v_0^e}\right)^2 - 1}; \qquad (2.17)$$

$$\omega_3 = -\frac{1}{2} k_x v_0^e - \sqrt{\frac{1}{4}(k_x v_0^e)^2 + c_A^2 k_z^2} + (\text{terms of order } Z) -$$

$$- i\sqrt{\pi} k_z c_A \sqrt{\frac{m_e}{m_i \beta}} \frac{2Z\left(\frac{k_z c_A}{k_x v_0^e}\right)^2}{1 + \frac{4 c_A^2 k_z^2}{k_x^2 v_0^{e2}} + 3\sqrt{1 + 4\left(\frac{c_A k_z}{k_x v_0^e}\right)^2}}. \qquad (2.18)$$

It is evident that when $k_z c_A \sim k_x v_0^e$ the condition in (2.14) means

$$\frac{K_y}{K_\perp^2 a} \ll \left(\frac{m_e}{m_i \beta}\right)^{1/2}. \qquad (2.19)$$

Thus, a necessary condition for the existence of a localized solution (perturbation for which the boundary conditions are not important) is that the transverse wavelength be small compared with the characteristic scale size of the plasma inhomogeneity.

It follows from Eqs. (2.16)-(2.18) that the wave associated with the root ω_3 is damped for all values of k_z, whereas the waves associated with ω_1 and ω_2 can either be damped waves or growing waves: when $k_z c_A > \sqrt{2}|k_x v_0^e|$, the wave characterized by $\omega = \omega_1$ is excited, and when $k_z c_A < \sqrt{2}|k_x v_0^e|$, the wave characterized by $\omega = \omega_2$ is excited. The relations in (2.16)-(2.18) furnish a good example of the general excitation criteria formulated at the beginning of the present section [cf. (2.4)]: those waves are excited for which $\omega / k_x v_0^e < 1$.

Thus, it is evident that very slow waves can exist in the plasma [the existence condition is (2.19)] and that these are unstable when $\omega / k_x v_0^e < 1$. An instability of this kind will be called the drift instability (it can be shown that the drift instability is not limited to $\nabla T = 0$, but that it appears for any relation between ∇T and ∇n_0; cf. § 3).

We note that if terms of order Z are neglected in Eqs. (2.16)-(2.18), the resulting expression for $\omega = \omega(\mathbf{K})$ is a solution of Eqs. (2.11) and (2.12) in which $\omega = \omega_1(k)$ corresponds to the ion-acoustic wave and $\omega = \omega_{2,3}(K)$ corresponds to the Alfvén waves. The real parts of $\omega_{1,2,3}$ are shown in Fig. 1 as functions of k_z. In accordance with Eq. (2.13), at large k_z we have taken account of the term that corresponds to the longitudinal motion of the ions (about k_z^2), and have shown schematically the transition from the ordinary ion-acoustic branches in a uniform plasma to the branch $\omega = \omega_1(k_z)$. It is also evident that the second ion-acoustic branch (lower) is highly retarded; if k_z is small, the condition $\omega / k_z \gg v_{Ti}$ is not satisfied, even in a highly nonisothermal plasma ($T_e \gg T_i$).

If $k_z c_A \approx \sqrt{2}\, |k_x v_0^e|$, the frequency of the fast ion-acoustic wave is almost the same as the frequency of the slow Alfvén wave ($\omega_1 \approx \omega_2$). In this case, Eqs. (2.16) and (2.17) are no longer meaningful. Using Eq. (2.15), we can show that the growth rate γ (as a function of k_z) reaches a maximum when $k_z \approx \sqrt{2}\, |k_x v_0^e| / c_A$, being given by

$$\gamma = \sqrt{\frac{\pi}{6}}\, k_x v_0^e \sqrt{\frac{m_e}{m_i \beta} Z}\,. \tag{2.20}$$

It is evident that γ increases with Z. Hence, the most interesting region of wave numbers lies in the range $Z \gtrsim 1$. Using Eq. (2.15), we can show that if Z is fixed and smaller than $\beta m_i / m_e$, the growth rate is a maximum when $k_z = k_z^* \sim \beta^{1/2} \varkappa$; for larger Z, $k_z^* \sim [(m_e/m_i)Z]^{1/2} \varkappa (\varkappa = \partial \ln n_0 / \partial y)$. If $k_z = k_z^*$, and the wavelength is reduced (with increasing Z), the growth rate continues to increase up to $Z \lesssim \beta(m_i/m_e)$, and then becomes constant:

$$\gamma_{\max} = \frac{\varkappa v_{Ti}}{4\sqrt{\pi}}\,. \tag{2.21}$$

The real part of ω [when $k_z \sim k_z^*$ and $Z > \beta(m_i/m_e)$] is independent of Z and is of the order of the growth rate $\gamma_{\max}$. The functions $k_z^* = k_z^*(Z)$ and $\gamma_{\max}(Z)$ are shown in Figs. 2 and 3.

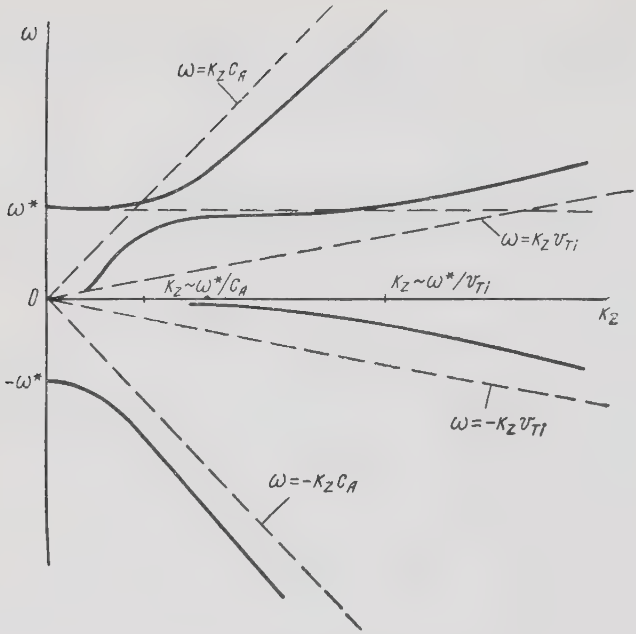

Fig. 1. The function $\omega = \omega(k_z)$ for the Alfvén wave and the ion-acoustic wave in an inhomogeneous plasma with $\nabla T = 0$. The quantity $\omega^* = k_x(cT/eB_0)(\partial \ln n_0/\partial y)$ is the characteristic drift frequency.

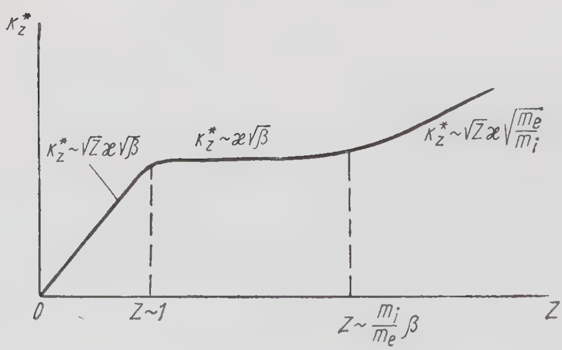

Fig. 2. The wave number k_z^*, which corresponds to the maximum growth rate, as a function of $Z = K_\perp^2 \rho_i^2$.

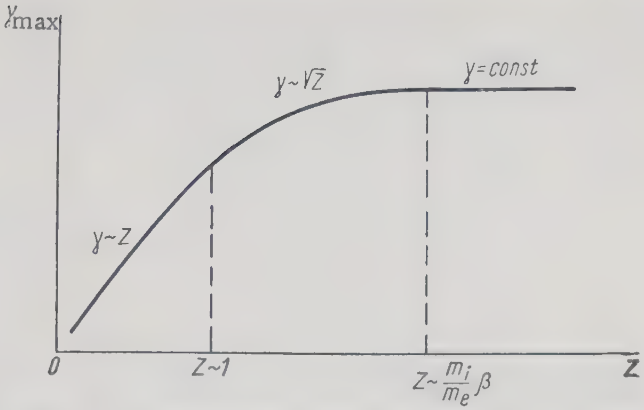

Fig. 3. The dependence of the growth rate γ_{max} on Z.

Thus, a plasma confined by a magnetic field with parallel lines of force is unstable against the excitation of drift waves; this is the so-called drift instability. One expects that this instability will be dangerous for plasma confinement experiments, inasmuch as the plasma particles can be scattered by high-amplitude waves and thus escape the confinement volume.

§ 3. Review of Work on Drift Instabilities

The history of kinetic instabilities in an inhomogeneous plasma starts with the work of Tserkovnikov [5]. The kind of instability considered by Tserkovnikov differs from that analyzed in § 2 because the instability in [5] is caused by magnetic drift. Nonetheless, this work served as the point of departure for later authors who investigated instabilities associated with the Larmor drift. In the present section we shall review the work of these authors.

Initial Efforts. The initial investigation of kinetic instabilities associated with the Larmor drift was that of Rudakov and Sagdeev, who reported on this topic at the beginning of 1961 [6,7]. At the present time, some two years later, the results obtained in this early work can be formulated in the following way.

One considers the curl-free oscillations of a plasma (rot $\mathbf{E}$ = 0) at wavelengths appreciably greater than the ion Larmor radius $\rho_i^2 K_\perp^2 \to 0$ with $k_z \neq 0$ (i.e., longwave ion-acoustic waves with $\omega / k_z \ll c_A$). The dispersion equation for these waves is [this follows, for example, from Eq. (2.10) above]:

$$\varepsilon_{\parallel}\big|_{Z\to 0} = 0\,. \tag{3.1}$$

The authors considered the frequency range $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$, in which case Eq. (3.1) is very similar to the dispersion equation for ion-acoustic waves in a uniform plasma

$$\left.\begin{array}{c} 1-\dfrac{k_x v_0^e}{\omega}-\dfrac{k_z^2 T_e}{m_i \omega^2}\left(1-\dfrac{k_x v_0^i}{\omega}\right)+\dfrac{i\sqrt{\pi}\omega}{k_z v_{Te}}\left[1-\dfrac{k_x v_0^e}{\omega}\times\right. \\ \left.\times\left(1-\dfrac{\eta_e}{2}\right)\right]=0; \\ v_0^e=-\dfrac{T_e}{m_e \omega_{Be}}\dfrac{\partial \ln n_0}{\partial y}; \\ v_0^i=-\dfrac{1}{m_i \omega_{Bi}}\dfrac{1}{n_0}\dfrac{\partial p_i}{\partial y}; \qquad \eta_e=\dfrac{\partial \ln T_e}{\partial \ln n_0}. \end{array}\right\} \tag{3.2}$$

When $T_e \simeq T_i$, terms corresponding to the longitudinal motion of the ions can be neglected in this equation, so that the waves are described by the simpler relation

$$1-\frac{k_x v_0^e}{\omega}+\frac{i\sqrt{\pi}\omega}{k_z v_{Te}}\left[1-\frac{k_x v_0^e}{\omega}\left(1-\frac{\eta_e}{2}\right)\right]=0. \tag{3.3}$$

It then follows that

$$\omega = k_x v_0^e-\frac{i\sqrt{\pi}}{2}\frac{(k_x v_0^e)^2}{k_z v_{Te}}\frac{\partial \ln T_e}{\partial \ln n_0}. \tag{3.4}$$

It is evident that $\gamma \equiv \operatorname{Im}\omega > 0$ if

$$\frac{\partial \ln T_e}{\partial \ln n_0}<0. \tag{3.5}$$

Thus, an isothermal* plasma is unstable if there is a temperature gradient in addition to the density gradient, and if these gradients are in opposite directions [returning to Fig. 1, we note that here we are considering the excitation of the fast (cf. p. 177) ion-acoustic wave].

In a highly nonisothermal plasma ($T_e \gg T_i$), the real part of the frequency (as a function of the wave number k_z) can assume the following two values [cf. Eq. (3.2)]:

$$\operatorname{Re}\omega_{1,2}=\frac{1}{2}k_x v_0^e \pm \sqrt{\left(\frac{1}{2}k_x v_0^e\right)^2+\frac{k_z^2 T_e}{m_i}} \tag{3.6}$$

*In the Russian literature isothermal means $T_e = T_i$ [translator's note].

(i.e., there are fast and slow ion-acoustic waves). When the small imaginary parts in Eq. (3.2) are considered, it is found that the criterion for the excitation of waves with the plus sign (fast branch) is the same as before [cf. (3.5)]. The second wave (the slow wave in Fig. 1) can also be excited if

$$1-\frac{k_x v_0^e}{\mathrm{Re}\,\omega_2}<0. \tag{3.7}$$

Thus, when k_z is small, we obtain the instability criterion

$$\frac{\partial \ln T_e}{\partial \ln n_0}>2. \tag{3.8}$$

The work of Rudakov and Sagdeev indicated that ion-acoustic waves can be unstable when certain relations obtain between the temperature and density gradients in an inhomogeneous plasma confined by a straight magnetic field. This instability is due to the interaction of resonance electrons with the wave; in contrast with the usual situation in a uniform plasma, this interaction does not necessarily result in damping (cf. also the beginning of § 2).

It was also shown in this work that the instability is not necessarily to be associated with imaginary terms in the dispersion equation. Thus, when $\partial \ln T/\partial \ln n_0 \to \infty$, even if one neglects the imaginary terms in Eq. (3.2), it follows that (if $k_z^2 T_e/m_i\omega^2 \ll 1$),

$$\omega^3=-\frac{k_z^2 T_e}{m_i}k_x v_0^i, \tag{3.9}$$

and one of the three roots corresponds to an instability.

These are the most important results of the initial efforts on plasma drift instabilities [6,7].

Refinement of the Rudakov – Sagdeev Theory by Introduction of $(T_e/T_i)K_\perp^2\rho_i^2)$ Terms. The Rudakov–Sagdeev analysis was based on the use of the so-called "drift kinetic equation" [8] and could not be extended into the shortwave region $K_\perp\rho_i \gtrsim 1$ and, similarly, could not take account of $K_\perp^2\rho_i^2$ terms. At the time this shortcoming was not considered important, since it was generally believed that any wave would be highly damped at wavelengths of the order of the ion-Larmor radius and, hence, not of any great interest. For this reason, following the appearance of [6,7] there was no immediate attempt to investigate the question of whether finite values of $K_\perp^2\rho_i^2$ have any effect on the drift instability. Another question appeared to be more important: this was the question of whether an inhomogeneous plasma with zero temperature gradient could be unstable; according to the Rudakov–Sagdeev analysis, an instability could arise only if $\partial \ln T/\partial \ln n_0 \neq 0$. The answer to this question was furnished by Timofeev, who showed that a nonisothermal plasma ($T_e >> T_i$) can be unstable even when $\nabla T = 0$ if the "transverse ion inertia" is taken into account. More precisely, this means when the $(T_e/T_i)K_\perp^2 \rho_i^2$ terms are not neglected in the dispersion equation for the curl-free oscillations

$$\varepsilon_\perp K_\perp^2 + \varepsilon_\parallel k_z^2 = 0, \tag{3.10}$$

which is a particular case of Eq. (2.10). If it is assumed that the frequency satisfies the condition $k_z v_{Ti} << \omega << k_z v_{Te}$, terms of the type $\varepsilon_\perp K_\perp^2$ are then terms which appear in the expression for $\varepsilon_\parallel k_z^2$ in the form $K_\perp^2 \rho_i^2 (T_e/T_i)$.

In this case, the following dispersion relation is obtained:

$$1 + \frac{T_e K_\perp^2}{m_i \omega_{Bi}^2} - \frac{k_x v_0^2}{\omega} - \frac{T_e k_z^2}{m_i \omega^2} + i\sqrt{\pi}\frac{\omega}{k_z v_{Te}}\left(1 - \frac{k_x v_0^e}{\omega}\right) = 0. \tag{3.11}$$

It then follows that for small k_z,

$$\omega = \frac{k_x v_0^e}{1 + \dfrac{K_\perp^2 T_e}{m_i \omega_{Bi}^2}} + i\sqrt{\pi}\frac{(Re\omega)^2}{k_z v_{Te}}\frac{K_\perp^2 T_e}{m_i \omega_{Bi}^2}, \tag{3.12}$$

and the instability can arise when $\nabla T_e = 0$.

It then became clear that taking account of $(T_e/T_i)K_\perp^2 \rho_i^2$ terms does not lead to stabilization of the instabilities that were already known; on the contrary, these terms are the origin of new instabilities. No stabilizing effects have been observed at shorter wavelengths and there is no reason to believe that they will be.

Development of the Mathematical Apparatus for Investigation Oscillations with Arbitrary $K_\perp^2 \rho_i^2$. Further progress in the theory of drift instabilities was, to a large extent, due to the success of Rosenbluth, Krall, and Rostoker [9] and later workers [10, 11], who were concerned with another kind of plasma instability – the flute instability. (The theory of the flute instability in a plasma with finite ion-Larmor radius is treated in the present review in § 6.) The work of Rosenbluth et al. [9] is especially important because a method was developed for solving the kinetic equation in an inhomogeneous plasma with an arbitrary value of the ratio $\rho/\lambda_\perp$. Rudakov has also shown [10] that if one uses hydrodynamic equations including the magnetic viscosity [2] it is possible to take account of terms of the form $k_x v_0^i/\omega$, which are necessary in the theory of drift waves with $K_\perp^2 \rho_i^2 << 1$. The present author has considered the case in which $K_\perp^2 \rho_i^2 \gtrsim 1$

[11], and has shown that in certain cases these waves are also unstable (cf. § 6). Thus, progress in the theory of the flute instability with finite ion-Larmor radius has led to a better understanding of the importance of instabilities with $K_\perp \rho_i \sim 1$, and has provided the mathematical apparatus required for studying these instabilities.

Subsequent progress in the theory of oscillations of a nonuniform plasma has made it possible to treat the following features: wavelengths comparable with the ion-Larmor radius; arbitrary ratio of oscillation frequency to cyclotron frequency; arbitrary direction of the wave vector; nonpotential electric fields.

An important step in this direction was taken in [12]: using the broad assumptions listed above, it was possible to find an expression for the currents induced in a plasma by a wave field. The results of this work and certain refinements made later by the author comprise § 1 of this review. This work furnished a broad base for the investigation of various kinds of instabilities in an inhomogeneous plasma. However, the physical significance of many aspects of the problem is still not clear.

Investigation of the Drift Instability for Arbitrary $K_\perp^2 \rho_i^2$. Progress in the shortwave region was made by three groups of workers almost simultaneously [13, 14, 15].

A paper by Kadomtsev and Timofeev [13] was a development of earlier results obtained by Timofeev, being devoted to the question of stability in a plasma with zero temperature gradient ($\nabla T = 0$). The authors treated the case of curl-free oscillations so that their dispersion equation could be written in the form of (3.10) although it was necessary to retain all powers of Z in the expressions for $\varepsilon_\perp$ and $\varepsilon_\parallel$ [cf. Eq. (2.10)]. It was shown in [13] that the drift instability can occur for an arbitrary ratio of ion-Larmor radius to wavelength. Since the authors assumed the oscillations to be curl-free, the results they obtained pertaining to the region $\omega \gtrsim k_z v_{Te}$ are valid only when $\beta \ll m_e/m_i$ and $K_\perp^2 \gg \omega_{0e}^2/c^2$ (when $K_\perp \rho_i \lesssim 1$). If these conditions are satisfied, it turns out that the growth rate is a maximum when $k_z \simeq \omega/v_{Te}$, and that it is of the same order as the real part of the frequency: $\gamma \simeq \mathrm{Re}\,\omega \approx k_x v_0^e$ (for $k_x^2 \rho_i^2 \lesssim 1$). When $k_x^2 \rho_i^2 \gg 1$ (the authors assume $k_x^2 \gg K_y^2$), the growth rate, treated as a function of k_x, tends to a finite limit (2.21). It is shown in this work that the plasma is stable when $k_z/\varkappa > 0.16$ ($\varkappa \sim 1/a$ is the reciprocal scale length of the inhomogeneity). This result obtains because, at these large values of k_z, the interaction between the wave and resonance ions becomes strong and these ions absorb energy.

A paper by Galeev et al. [15] was devoted to the question of stability in a plasma with arbitrary ratio between the temperature and density gradients.

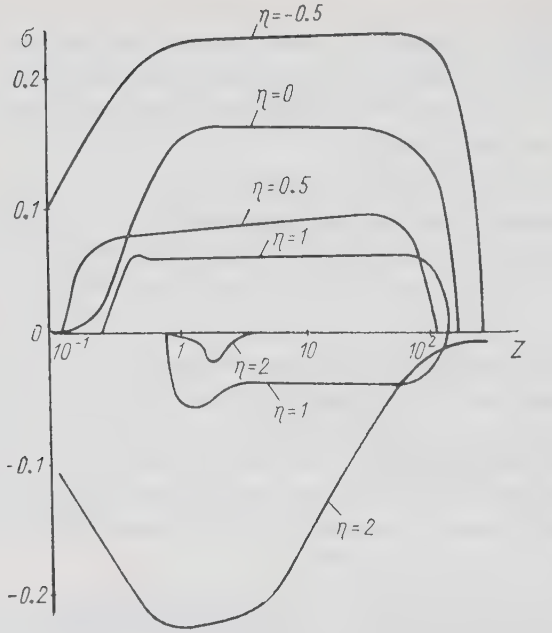

Fig. 4. Stability boundary of a plasma for various values of $\eta = \partial \ln T / \partial \ln n_0$. The unstable regions lie between a given curve $\sigma(Z)$ and the abscissa axis. The quantity $\sigma = k_z / \varkappa$, the ratio of the characteristic scale size for the density variation to the longitudinal wavelength $\varkappa = \partial \ln n_0 / \partial y$, is the ordinate; the abscissa is the quantity $Z = K_{\perp}^2 \rho_i^2$.

The starting assumptions in this work are the same as those of Kadomtsev and Timofeev, with the exception of the condition $\nabla T \neq 0$. It is found that the difference from the $\nabla T = 0$ case is important only in the longwave region ($K_{\perp} \rho_i \lesssim 1$). In the shortwave region, the plasma is unstable for any ratio between the temperature and density gradients. In view of this property of the instability, the authors [15] called it a "universal" instability. In the present work we use the term "drift instability," following Kadomtsev and Timofeev.

A paper by Mikhailovskii and Rudakov [14] is more general than the work presented in [13, 15] in that it does not assume rot $\mathbf{E} = 0$, i.e., nonpotential oscillations are treated. In particular, this makes it possible to investigate the oscillations of a plasma in which the pressure is not necessarily low ($\beta \gtrsim m_e/m_i$). Certain of the results of this investigation form the content of § 2 of the present review. Other results obtained in [14], and certain others

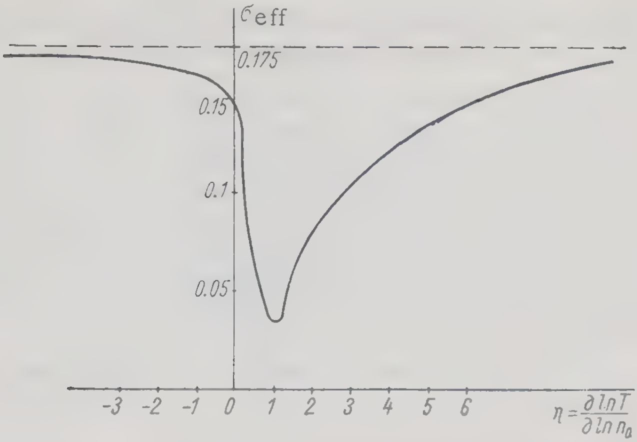

Fig. 5. The ratio of the effective transverse dimension of the plasma inhomogeneity to the shortest wavelength of the instability [$\sigma_{eff} \equiv |\sigma_{max}|/(1+|\eta|)$] as a function of $\eta = \partial \ln T/\partial \ln n_0$.

of [13, 15] will be presented later; in particular, we wish to call attention to a later work [16], in which the question of instability limits for a plasma with arbitrary ratios of the temperature and density gradients was considered. Sections of a paper that has already been mentioned [12] are also of interest; general equations for drift waves were derived in this work.

The results that follow from the last group of papers can be summarized as follows:

1. If $K_\perp^2 \rho_i^2 \gtrsim 1$, and if k_z is small, a plasma is subject to perturbations which are unstable for any ratio between the temperature and density gradients (even when $T_e = T_i$). One is easily convinced of this from an examination of Eqs. (2.8) and (2.10) with $1 \ll K_\perp^2 \rho_i^2 \ll m_i/m_e$, $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$. Under these conditions,

$$\omega = \frac{\frac{k_x}{K_\perp}\left(\frac{T_e}{T_i}\right)^{1/2}\sqrt{\frac{T_e}{2\pi m_i}}\,\varkappa\left(1-\frac{\eta_i}{2}\right)}{1+\frac{T_e}{T_i}+\frac{\beta_e}{2}\frac{k_x^2}{K_\perp^2}\frac{\varkappa^2}{k_z^2}-2_i\sqrt{\pi}\,\frac{m_e}{m_i}\frac{v_{Te}}{|\omega_{Bi}|}\frac{k_x}{k_z}\varkappa\left(1-\frac{\eta_e}{2}\right)};$$

$$\beta_e = \frac{8\pi n_0 T_e}{B_0^2}; \quad \varkappa = \frac{\partial \ln n_0}{\partial y}; \quad \eta = \frac{\partial \ln T}{\partial \ln n_0}. \qquad (3.13)$$

The sign of $\gamma \equiv \mathrm{Im}\,\omega$ is determined by the sign of the product $[1-(\eta_i/2)] \cdot [1-(\eta_e/2)]$, and the latter is always positive when $T_i = T_e$.

If it is assumed that $k_z \gtrsim \pi/L$, where L is the longitudinal dimension (length) of the device, then for any value of $\partial \ln T/\partial \ln n_0$ the instability is suppressed if L is not too large as compared with transverse dimension a, so that $a/L \gtrsim {}^1\!/_{10} - {}^1\!/_{30}$ (this statement holds, at least for systems in which the lines of force are frozen in conductors at the ends). This criterion follows qualitatively from the requirement for small ion damping ($\omega \gg k_z v_{Ti}$). If the quantity ω is replaced by $k_x v_0$, and if k_x is taken to be $\sim 1/\rho_i$, this condition means $k_z \ll \varkappa$, i.e., $L/a \gg 1$ (cf. also Figs. 4 and 5).

2. The excitation of longwave oscillations is determined by different criteria, depending on the value of the quantity $\beta_e \equiv 8\pi p_e/B_0^2$. Specifically:

2a. In a highly rarefied plasma ($\beta \ll m_e/m_i$), in which the Alfvén speed is greater than the electron thermal velocity ($c_A \gg v_{Te}$), the frequency region $\omega \gg k_z v_{Te}$ is important in addition to the region $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$. If $\omega \gg k_z v_{Te}$, and $K_\perp^2 \rho_i^2 \ll 1$, the dispersion equation (2.10) assumes the form [14]:

$$(\omega^2 + \omega\omega^* - k_z^2 c_A^2)(\omega - \omega^*) = -\frac{c^2 K_\perp^2}{\omega_{0e}^2}\,\omega^2(\omega + \omega^*);$$

$$\omega^* = k_x v_0^e (1+\eta) \equiv -\frac{k_x(\partial p_e/\partial y)}{m_e \omega_{Be} n_0}; \quad T_i \equiv T_e \equiv T. \tag{3.14}$$

It is evident that the temperature gradient appears in a sum with the density gradient so that all the instability criteria obtained for $\nabla T = 0$ remain valid for any ∇T (except $\partial \ln T/\partial \ln n_0 \approx -1$, in which case $\omega^* \to 0$).

If the wavelength is very large ($k_\perp^2 \ll \omega_{0e}^2/c^2$), the right side of Eq. (3.14) is small, so that a method of successive approximations can be used to solve this equation. We then find an intersection of the branches (the fast ion-acoustic wave and the slow Alfvén wave) near $k_z^{*2} = 2\omega^{*2}/c_A^2$; this effect is similar to that considered in § 2 (cf. also Fig. 1). Near this value of k_z the frequencies of both branches are complex, the growth rate being given by

$$\gamma = \pm\sqrt{\frac{2}{3}}\,\frac{cK_\perp}{\omega_{0e}}\,\omega^*. \tag{3.15}$$

When $\nabla T = 0$, this expression is very similar to the growth rate in (2.20) obtained for the case $\beta \gg m_e/m_i$; the only difference is a numerical factor of order unity. In contrast with the growth rate for the present case, it should be noted that (2.20) derives from the electron pole, i.e., it is due to the interaction between the wave and electrons with velocities close to ω/k_z.

The range Δk_z in which the oscillations are excited is of the order

$$\frac{\Delta k_z}{k_z^*} \lesssim \frac{cK_\perp}{\omega_{0e}}. \tag{3.16}$$

Outside of this range, in the direction of higher k_z, the instability is quenched up to $k_z \lessgtr k_x v_0^2/v_{Te}$; beyond this point terms associated with the electron pole become important and can lead to the kinetic instability. Now, however, the question of whether the kinetic instability can grow depends on the relation between the temperature gradient and the density gradient, since the latter do not appear additively everywhere as is the case when $\omega/k_z \gg v_{Te}$. It can be shown from Eq. (2.10) that for all k_z down to $k_z \lessgtr \omega/v_{Ti}$, the stability limit is given by the approximate relation

$$\frac{\partial \ln T}{\partial \ln n_0} < 4K_\perp^2 \varrho_i^2. \tag{3.17}$$

At still larger k_z, i.e., $k_z \gtrless \omega/v_{Ti}$, it is found that an isothermal plasma ($T_e \approx T_i$) is unstable if $\partial \ln T/\partial \ln n > 2$ [16], as in the instability found for $T_e \gg T_i$ by Rudakov and Sagdeev [7].

This, then, is the instability pattern for a highly rarefied plasma ($\beta \ll m_e/m_i$) for longwave perturbations ($K_\perp^2 c^2/\omega_{0e}^2 \ll 1$).

It is now of interest to ask how this pattern is changed as the wavelength is reduced. The region of hydrodynamic instability expands with k_z and when $(\omega_{0e}^2/c^2) \ll K_\perp^2 \ll 1/\rho_i^2$, all three waves (the two Alfvén waves and the fast ion-acoustic wave) tend to fuse; Eq. (3.14) then assumes the form [13,15]:

$$\frac{\omega - \omega^*}{\omega + \omega^*} = \frac{m_e}{m_i} \frac{K_\perp^2}{k_z^2} \frac{\omega^2}{\omega_{Bi}^2}. \tag{3.18}$$

The instability criterion is now [13]:

$$k_z v_{Te} < 3.3\omega^* K_\perp \varrho_i. \tag{3.19}$$

It is evident that as $K_\perp \rho_i$ increases, the region of hydrodynamic instability starts to overlap the region of kinetic instability, while the range of stable $\partial \ln T/\partial \ln n_0$ is reduced [cf. Eq. (3.17)]. At still shorter wavelengths ($K_\perp^2 \rho_i^2 \gg 1$), the instability pattern is described by Eq. (3.13).

2b. In a plasma of higher density ($\beta \gg m_e/m_i$), the region corresponding to the hydrodynamic drift instability vanishes. The instability of a plasma characterized by $\nabla T \neq 0$ and $\beta \gg m_e/m_i$ has already been considered in §2. When $\nabla T \neq 0$ and $k_z \gtrless \omega/v_{Ti}$, the situation is very much the same as for the case $\beta \ll m_e/m_i$. For curl-free modes, it appears that there are no longwave instabilities $K_\perp^2 \rho_i^2 \ll 1$ if $0 < \partial \ln T / \partial \ln n_0 < 2$ in this kind of a plasma $\beta \gg m_e/m_i$. Actually, however, an Alfvén instability is excited in this case [14]. This result can be demonstrated in the following way. According to Eq. (2.10), longwave Alfvén oscillations in a plasma for which $\eta \equiv \partial \ln T/\partial \ln n_0 \neq 0$ are described by the equation

$$\omega^2 + \omega k_x v_0^e (1+\eta) - c_A^2 k_z^2 = -i\sqrt{\pi}\frac{\omega}{k_z v_{Te}} c_A^2 k_z^2 \varrho_i^2 K_\perp^2 \times$$
$$\times \frac{\left[1 + \frac{k_x v_e^0}{\omega}(1+\eta)\right]\left[1 - \frac{k_x v_0^e}{\omega}\left(1 - \frac{\eta}{2}\right)\right]}{\left(1 - \frac{k_x v_0^e}{\omega}\right)^2}. \tag{3.20}$$

Assuming for definiteness that $k_x v_0^e > 0$, we find that the wave given in Eq. (2.18) is damped for any η, whereas the growth rate of the wave in (2.17) is given by

$$\operatorname{Sgn}\gamma = \operatorname{Sgn}\left\{\frac{3}{2} k_x v_0^e - \sqrt{\left[\frac{k_x v_0^e}{2}(1+\eta)\right]^2 + c_A^2 k_z^2}\right\}. \tag{3.21}$$

It then follows that this wave is unstable when $(-4) < (\partial \ln T/ \partial \ln n_0) < 2$.

2c. A plasma of intermediate pressure $\beta \approx m_e/m_i$ is distinguished by the fact that the electron thermal velocity is approximately equal to the Alfvén velocity ($v_{Te} \approx c_A$). Since the maximum longwave growth rate usually lies in the region $k_z \sim \omega / c_A$, and since the argument of the electronic W function is of the order of unity for these values of k_z, it turns out that this case is not convenient for analysis. However, when $\nabla T = 0$, it is possible to obtain simple results which lead to an understanding of the relation between the limiting cases that have already been considered, $\beta \ll m_e/m_i$ and $\beta \gg m_e/m_i$. When $\nabla T = 0$ and $K_\perp^2 \rho_i^2 \ll 1$, we find from Eq. (2.10)

$$(\omega - k_x v_0^e)(\omega^2 + k_x v_0^e - c_A^2 k_z^2) = \frac{\varrho_i^2 K_\perp^2 c_A^2 k_z^2 (\omega + k_x v_0^e)}{1 + i\sqrt{\pi}\frac{\omega}{k_z v_{Te}} W\left(\frac{\omega}{k_z v_{Te}}\right)}. \tag{3.22}$$

The contribution of the small right-hand side is a maximum when $k_z^2 = 2(k_x v_0^e / c_A)^2$. For this value of k_z the frequencies of two of the three oscillation branches are close to $k_x v_0^e$ and the corrections are given by

$$\delta\omega_{1,2} = \pm \frac{2}{3} \frac{\varrho_i K_\perp k_x v_0^e}{\left\{1 + i\sqrt{\pi}\left(\frac{m_e}{m_i\beta}\right)^{1/2} W\left[\left(\frac{m_e}{m_i\beta}\right)^{1/2}\right]\right\}^{1/2}}; \tag{3.23}$$

$$\beta = 8\pi \cdot 2 n_0 T / B_0^2.$$

It then follows that when $\beta \sim m_e/m_i$, the growth rate is approximately $\gamma \sim \rho_i K_\perp k_x v_0^e$. Using the formulas obtained here, we can also obtain the growth rate for $\beta \gg m_e/m_i$ (2.20) and the growth rate for $\beta \ll m_e/m_i$ (3.15).

These are the principal results that have been obtained in the work on drift instabilities of a plasma with finite ion-Larmor radius.

Investigation of Plasma Drift Instabilities at Frequencies of the Order of the Ion – Cyclotron Frequency. The investigation of drift instabilities at low frequencies ($\omega \ll \omega_{Bi}$) was supplemented by simultaneous work on drift instabilities at higher frequencies. This topic has been treated in a paper by Mikhailovskii and Timofeev [17], and to some extent in [18].

The analysis in [17] is devoted to the case of curl-free oscillations (the equations for electrostatic oscillations at an arbitrary frequency are given in Appendix IV). If the plasma inhomogeneity is strong, and if the wavelength is short, the drift frequency can be of the same order as the ion-cyclotron frequency (cf. §1 for a discussion of this case); it is then possible that oscillations at frequencies close to harmonics of the ion-cyclotron frequency will be unstable (cyclotron instability). The results of this work can be easily understood by reference to Eq. (IV.1) of Appendix IV. There, for example, we find that for $k_z = 0$ and certain reasonable simplifying assumptions $\nabla T_i = \nabla T_e = 0$, $T_e = 0$ and $Z_i \gg 1$, the dispersion equation for electrostatic oscillations assumes the form

$$1 + K_\perp^2\left(d_i^2 + \frac{m_e}{m_i}\varrho_i^2\right) - \frac{k_x v_0^i}{\omega} = \frac{1}{\sqrt{2\pi Z}}\,\frac{\omega - k_x v_0^i}{\omega - n\omega_{Bi}}; \qquad (3.24)$$

$$v_0^i = -\frac{\varkappa T_i}{m_i\omega_{Bi}}; \quad d_i^2 = \frac{T_i}{4\pi e^2 n_0}; \quad \varkappa = \frac{\partial \ln n_0}{\partial y}.$$

Here, again, we encounter the intersection of branches that has been noted above if

$$\varkappa\varrho_i \gtrsim 2n\left[\frac{m_e}{m_i}\left(1 + \frac{\omega_{Be}^2}{\omega_{0e}^2}\right)\right]^{1/2}; \quad n = 1, 2, \ldots \qquad (3.25)$$

Intersections occur between the cyclotron oscillations $\omega \approx n\omega_{Bi}$ (cf., for example, [19]) and the fast Alfvén wave $\omega = k_x v_0^i$ [cf. Eq. (2.18)]. Solving the quadratic equation (3.24), we find that if the branches intersect, i.e., if the condition (3.25) is satisfied, the oscillation frequency becomes complex, so that one of the waves grows while the other is damped. The characteristic growth rates $\gamma \sim (m_e/m_i)^{1/4}\omega_{Bi}$ and the characteristic wavelengths are of the order of the electron Larmor radius.

The relations between the oscillations are somewhat different when $\omega/k_z \ll v_{Te}$. In this case, the dispersion equation is

$$1 + \frac{4\pi e^2 n_0}{K_\perp^2}\left\{\frac{1}{T_e} + \frac{1}{T_i} + \frac{i\sqrt{\pi}}{T_e k_z v_{Te}}(\omega - k_x v_0^2)\, I_0(Z_e)\, e^{-Z_e} - \right.$$

$$\left. - \frac{1}{T_i\sqrt{2\pi Z_i}} \frac{\omega - k_x v_0^i}{\omega - n\omega_{Bi}}\right\} = 0. \quad (3.26)$$

whence we find that when $\mathrm{Re}\,\omega \approx n\omega_{Bi}$, the growth rate is

$$\gamma = \mathrm{Im}\,\omega = -\frac{I_0(Z_e)\, e^{-Z_e}}{\sqrt{2\pi Z_i} k_z v_{Te}} \frac{(n\omega_{Bi} - k_x v_0^i)(n\omega_{Bi} - k_x v_0^e)}{\left(1 + \frac{T_i}{T_e} + K_\perp^2 d_i^2\right)^2}, \quad (3.27)$$

while the excitation condition becomes

$$\left(1 - \frac{k_x v_0^e}{\omega}\right)\left(1 - \frac{k_x v_0^i}{\omega}\right) < 0. \quad (3.28)$$

The drift excitation of the ordinary wave that propagates across the magnetic field ($k_z = 0$) is treated in [18]. It is evident from Fig. 1 that when $k_z \to 0$ the frequency of the fast ion-acoustic wave is not zero: $\omega = k_x v_0^e$. However, when k_z is reduced, the polarization of this wave is highly modified, so that when $k_z = 0$ the ion-acoustic wave becomes purely transverse ($\mathrm{div}\,\mathbf{E} = 0$) with a polarization characteristic of the ordinary wave ($\mathbf{E} \parallel \mathbf{B}_0$) [1].

The equations that describe this kind of oscillation are given in Appendix IV. If $k_x v_0^e$ is comparable with the ion-cyclotron frequency, the drift branch intersects the cyclotron branch characteristic of a uniform plasma [20] and both branches can exhibit complex frequencies. The dispersion equation which describes the excitation of the ordinary wave near cyclotron harmonics is of the form (cf. Appendix IV):

$$\frac{1}{\omega} - \zeta\frac{k_x v_0^e}{\omega^2} + \zeta\frac{m_e}{m_i}\frac{I_n\, e^{-Z_i}}{\omega - n\omega_{Bi}}\left(1 - \frac{k_x v_0^i}{\omega}\right) = 0;$$

$$\zeta = \left(1 + \frac{c^2 K_\perp^2}{\omega_{0e}^2}\right)^{-1}. \quad (3.29)$$

It then follows that the instability criterion is

$$\rho_e \varkappa \gtrsim \frac{m_e}{m_i}\frac{1}{\sqrt{\beta_e}}, \quad (3.30)$$

and that the maximum growth rate is of the order $\gamma_{max} \approx (m_e/m_i)^{1/2}\omega_{Bi}$.

This kind of instability can appear in highly inhomogeneous transition layers that contain hot electrons.

§ 4. Progress in Methods of Investigating the Oscillations of an Inhomogeneous Plasma

Recent work in the analysis of drift waves has led to further understanding of the following two important topics:

a) methods of describing waves in an inhomogeneous plasma in terms of a medium in which the parameters vary in space;
b) the effect of curvature of magnetic lines of force on the excitation of oscillations.

In the present section, which is an extension of the material in § 3, we shall touch on a number of papers which pertain to the first of these topics; in § 5 we indicate papers devoted to the question of curvature of the lines of force.

We have already seen that homogeneous and inhomogeneous plasma [cf., for example, Eqs. (1.31) and (2.8)] can be differentiated by the fact that the quantities $\varepsilon_{\perp}$ and $\varepsilon_{\parallel}$ in the inhomogeneous plasma contain drift terms that are not present in the homogeneous plasma case. Throughout our presentation we have at all times emphasized this characteristic feature of the inhomogeneous plasma which, in the final analysis, is the origin for a number of interesting effects, in particular, the drift instability.

However, there is still another important difference between a homogeneous plasma and an inhomogeneous plasma: this is the fact that the quantities $\varepsilon_{\perp}$ and $\varepsilon_{\parallel}$ are functions of the coordinates for the inhomogeneous plasma, but are obviously independent of the coordinates for the homogeneous plasma. This feature means that the properties of a wave in an inhomogeneous plasma must be different at different points in space. Mathematically this implies that Eq. (2.7) cannot admit of plane-wave solutions and we are now confronted with the problem of studying the spatial structure of the waves.

The methods brought to bear in solving the problem depend on the ratio between the characteristic scale size of the plasma inhomogeneity a and the assumed scale size for the spatial variation of the wave in the direction of the inhomogeneity λ_y. Evidently the simplest possibilities obtain for two limiting cases: the case in which $\lambda_y/a \ll 1$, i.e., when the "wavelength" is small compared with the characteristic scale size of the plasma inhomogeneity, and the case $\lambda_y/a \gg 1$, i.e., when the wavelength is very long (or the plasma is highly inhomogeneous). In the first case the properties of the wave are analyzed by analogy with the waves in a uniform plasma, i.e., the spatial variation of the field is characterized by a wave number $K_y = K_y(y)$, which is a weak function of the coordinates:

$$E(y) \approx E_0 \exp\left\{ i \int\limits_{y_0}^{y} K_y(y)\, dy \right\}. \tag{4.1}$$

Here, E_0 and y_0 are constants. This approach is called the method of geometric optics (semiclassical or WKB approximation).

In the second case, we are actually concerned with the problem of wave propagation in two uniform media separated by an infinitely thin surface layer. The appropriate approximation will be called the surface-layer approximation.

It is then hoped that once the properties of the waves are known for $\lambda_y/a \ll 1$ and for $\lambda_y/a \gg 1$, one can obtain an approximate picture for the oscillations at an arbitrary wavelength.

In §§ 2 and 3, we have considered cases in which $\lambda_y/a \ll 1$. Hence, we have used the semiclassical representation of the field (4.1) [cf. Eq. (2.9)] and reduced the problem to the solution of the dispersion equation (2.10). We investigated the problem in the spirit of a paper by Tserkovnikov [5], assuming a wave packet localized in the vicinity of some fixed point and studying its behavior (growth of the perturbation in time) for a time much smaller than that required to propagate a distance of the order of a. It is assumed that the wave packet is able to grow significantly in this time interval, i.e., the condition in (2.14) must be satisfied.

Thus, we have actually required that there be a perturbation whose dispersion equation, written for some fixed point $y = y^*$, can be satisfied for some positive K_y^2, which is assigned beforehand. For perturbations whose initial amplitude is not infinitesimally small (actually we always deal with perturbations of this kind), and which grow rapidly in time, this assumption can only be verified or justified by a nonlinear theory. However, one can use a more formal approach in which the initial amplitude of the growing perturbations is assumed to be infinitesimally small so that the wave amplitude remains small enough for the analysis to be valid for an arbitrary time interval. In this approach we remain within the framework of the linear approximation and can trace the behavior of even a rapidly growing perturbation over the course of a time interval during which it propagates a distance of the order of the characteristic dimension of the plasma inhomogeneity a. The formal scheme for the analysis of rapidly growing perturbations is then obviously no different from the method used for studying slowly growing perturbations [for which Eq. (2.14) is not satisfied] and time damped ($\gamma < 0$) or neutral ($\gamma = 0$) perturbations.

Starting from these considerations and remaining within the framework of the linear theory, we now evaluate the rationale behind the "localization method" used in §§ 2 and 3, i.e., we shall investigate the validity of the assumptions that have been made above. Furthermore, since we do not require that Eq. (2.14) be satisfied, we can include in our analysis perturbations whose growth rates (as determined by the localization method) are small or negative.

The features of interest here can be obtained in the following way. We still assume that Eq. (2.10) can be satisfied at the point $y = y^*$ for some $K_y^2 > 0$. We can then find the oscillation frequency ω. Our objective is to follow a perturbation that propagates from the point $y = y^*$ into some other region of space. The dispersion equation (2.10) allows us to establish a relation between the frequency ω and the wave number $K_y = K_y(y^*)$, and this yields the possibility of following the behavior of the wave number at other points in space $K_y = K_y(y,\omega)$, where $y \neq y^*$. We shall illustrate these considerations using the example of longwave oscillations $K_\perp^2 \rho_i^2 \ll 1$, in which case the quantities $\varepsilon_\parallel$ and $\varepsilon_\perp$ in Eq. (2.10) are independent of K_y^2. Under these conditions, the square of the wave number at the point y is given by

$$K_{\prime\prime}^2(y) = -\left[k_x^2 + \frac{k_z^2 \varepsilon_\parallel}{\varepsilon_\perp}\left(1 - \frac{\omega^2}{c^2 k_z^2}\varepsilon_\perp\right)\right], \tag{4.2}$$

while the frequency, which appears on the right side of this relation, is determined from the condition that $K_y^2 = K_y^{*2} \equiv K_y^2(y^*)$ is a fixed positive number assigned beforehand. Using Eqs. (4.1) and (4.2), we can determine the amplitude of the wave at any point in space once it is known for $y = y^*$.

However, we cannot be sure that the resulting expression for the field E(y) will satisfy the proper boundary conditions; for example, these might require that the field E vanish in a region in which there is no plasma. The need for introducing the boundary conditions leads to certain restrictions on the choice of $K_y^2(y^*)$ or, what is the same thing, restrictions on the allowed values of the frequency. Here, it is useful to recall the analogous situation in the semiclassical problem of motion of a particle in a potential well in which taking account of the boundary conditions forces a quantization of the particle energy. However, in our case, the situation is complicated by the fact that the right side of Eq. (4.2) will generally contain complex quantities, whereas the corresponding expression in the analogous semiclassical equation (the potential energy of the particle) is always real. Silin [21] has shown that the "quantization rule," i.e., the condition that the wave number must satisfy (and consequently the frequency) in problems on drift instabilities, is formally the same as in the semiclassical approximation in quantum theory.

The use of the quantization rule for problems involving drift instabilities can be illustrated as follows. Having solved Eq. (2.10) for $K_y^2(y)$, we obtain a relation of the form

$$K_y^2(y) = Q(\omega, y), \tag{4.3}$$

where Q is some function of the coordinates and the frequency of oscillation [for longwave oscillations the function Q is of the form of the right side in Eq. (4.2)]. Assume that the imaginary terms in Q are much smaller than the real terms, as they are in most cases of interest $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$; we then separate the real and imaginary parts in the complex equation (4.3) and obtain the following relations for the real and imaginary parts of the wave number:

$$K_1^2(y) = Q_1(\omega_1, y); \tag{4.4}$$

$$2K_1(y)K_2(y) = Q_2(\omega_1, y) + \gamma \frac{\partial Q_1}{\partial \omega_1}. \tag{4.5}$$

Here, $K_1 = \mathrm{Re}\, K_y$, $K_2 = \mathrm{Im}\, K_y \ll K_1$, $\omega_1 = \mathrm{Re}\,\omega$, $\gamma \equiv \omega_2 = \mathrm{Im}\,\omega \ll \omega_1$, $Q_1 = \mathrm{Re}\, Q(\omega = \omega_1)$, $Q_2 = \mathrm{Im}\, Q(\omega = \omega_1) \ll Q_1$.

Returning to Eq. (4.1) we note that the quantity $K_1(y)$ characterizes the phase variation of the wave field as the perturbation propagates in space while the quantity $K_2(y)$ characterizes the amplitude variation (here we are considering the spatial part of the wave field). Assume now that the perturbation has traversed the entire region of localization, which extends from the turning point y_1 to the turning point y_2, where y_1 and y_2 satisfy the relation

$$Q_1(\omega_1, y_1) = Q_1(\omega_1, y_2) = 0. \tag{4.6}$$

In being reflected from the two boundaries and returning to the original point, the perturbation must acquire its initial phase and initial amplitude [since E(y) is a unique function]. Consequently, the phase variation must satisfy the condition

$$\oint K_1(y)\, dy = 2\pi n, \tag{4.7}$$

where n is an integer appreciably greater than unity; similarly, the total amplitude variation must be zero, i.e.,

$$\oint K_2(y)\, dy = 0. \tag{4.8}$$

Here, the integration is taken from y_1 to y_2 and back again, while K_1 is the positive root of Eq. (4.4).

The relations in (4.7) and (4.8) represent the required additional restrictions on the possible values of the frequency and are analogous to the Bohr quantization rule. We note that both of these relations can be written in the form of a single formula which has exactly the same form as the Bohr quantization rule:

$$\oint K_y(y)\,dy = 2\pi n. \tag{4.9}$$

Now we analyze the properties that follow from (4.7) and (4.8), especially with regard to the localization method used in investigating the instability in §§ 2 and 3. The relation in (4.7) indicates that when $y = y^*$ the real part of the square of the wave number $K_1^2(y^*)$ cannot be a completely arbitrary positive number; rather, it must be a number such that the associated frequency ω_1 satisfies the relation

$$\oint \sqrt{Q_1(\omega_1, y)}\,dy = 2\pi n. \tag{4.10}$$

Hence, the real parts of the frequency and $K_1^2(y^*)$ are quantized. As an approximation, however, we can assume that the spectrum is continuous because the distance between neighboring levels is small for large values of n. We then conclude that taking account of the additional condition (4.7) only leads to unimportant corrections to the real part of the frequency as determined by the localization method. If we are not interested in the growth rates, the validity of the assumptions made earlier (cf. page 192) can be assumed to be completely demonstrated.

Since effects associated with the imaginary terms in the dispersion equation are of great importance (these are obviously responsible for any instability), we must remember that there is a second condition to be satisfied, that in (4.8). The consequences that follow from this relation, as far as the localization method is concerned, can be obtained in the following way.

We have assumed that it is proper (in the vicinity of an arbitrary fixed point $y = y^*$) to write the dispersion equation (2.10) assuming that K_y^2 is equal to some positive number. Returning to Eq. (4.5), we note this equation can be valid only if $K_2(y^*) = 0$, i.e., only if

$$Q_2(\omega_1, y^*) + \gamma\,\frac{\partial Q(\omega_1, y^*)}{\partial \omega_1} = 0. \tag{4.11}$$

This requirement supplies a relation between γ and y^* since the frequency ω_1 can be expressed in terms of the plasma parameters at the point $y = y^*$ by writing Eq. (4.4) at that point. On the other hand, Eq. (4.8) still relates γ and y^*. These considerations imply the following necessary refinement of the basic premise of the localization method. The point $y = y^*$, at which K_y^2 is real, must be determined in some way.

Of all the refinements of the localization method which stem from a more detailed analysis than that given in the present section, thus last is the most important. The validity of the localization method this depends on how important it is to choose the point $y = y^*$ out of all points in the region of localization of the perturbation.

In order to examine this question, we must ascertain the significance of the point y^*. For this purpose, we introduce the notion of a "local growth rate" $\gamma_0 = \gamma_0(y)$, i.e., the notion of some function of coordinates which satisfies the equation [cf. Eq. (4.11)]:

$$Q_2(\omega_1, y) + \gamma_0(y)\frac{\partial Q_1(y, \omega_1)}{\partial \omega_1} = 0. \tag{4.12}$$

Comparing this equation with Eq. (4.5), we note that the local growth rate can also be written in the form

$$\gamma_0(y) = \gamma - \frac{2K_1K_2}{\partial Q_1/\partial\omega_1} \equiv \gamma - \frac{K_2(y)}{\partial K_1/\partial\omega_1}. \tag{4.13}$$

The quantity $1/(\partial K_1/\partial\omega_1) \equiv \partial\omega_1/\partial K_1$ obviously has the meaning of the group velocity of the perturbation v_{gr}, while K_2 characterizes the growth of the perturbation as it propagates in space. Thus, the product $K_2 v_{gr} dy$ is equal to the growth exponential of the wave amplitude in the time that the wave packet propagates a distance dy. The function γ_0, which takes account of the growth in both time and space, is thus properly described as a local growth rate.

Using the notion of the local growth rate, we can now understand the meaning of the point $y = y^*$ quite simply: this is the point at which the local growth rate (γ_0) coincides with the true growth rate (γ), i.e.,

$$\gamma_0(y^*) = \gamma. \tag{4.14}$$

Equation (4.8) indicates that there are regions of spatial growth as well as regions of spatial damping. This result means that somewhere within the interval y_1, y_2 there is at least one point at which $K_2 = 0$, i.e., there is a point that has the meaning assigned to y^*. Thus, the point $y = y^*$, which is so important for the localization method, lies somewhere within the region of localization of the wave. In §§ 2 and 3, all points lying within the region of the inhomogeneity of the plasma were assumed to be equivalent. More precisely, in obtaining the instability criterion we did not consider situations in which the instability criterion is satisfied at one point and not at another; hence, there is no reason to doubt the validity of the results in §§ 2 and 3 (cf. also [22]).

We note that Eq. (4.8) can also be written in the form

$$\gamma = \frac{\oint \frac{\gamma_0 \, dy}{v_{gr}}}{\oint \frac{dy}{v_{gr}}}. \tag{4.15}$$

It is evident that we can obtain the true growth rate γ by averaging the local growth rate in space with an appropriate weighting factor. In particular, if the region of localization of the perturbation is small compared with the dimensions of the plasma inhomogeneity, the true growth rate and the local growth rate are the same. This statement holds if the "quantum number" n is reasonably small, i.e., if the effective "well" is deep. If the region of localization is small* because of nonlinear effects, it again follows that $\gamma = \gamma_0$. In both of these cases the two methods yield results that agree quantitatively as well as qualitatively.

Let us now summarize the principal results of the analysis.

1. The investigation of fine-scale (microscopic) oscillations in an inhomogeneous plasma can be carried out by two related methods: for large growth rates, which satisfy the condition $\gamma \gg a^{-1}\partial \operatorname{Re} \omega / \partial K_y$, it is physically reasonable to use the localization method; for small growth rates ($\gamma < a^{-1}\partial \operatorname{Re} \omega / \partial K_y$) it is convenient to use the method based on the quantization rule.

2. A formal linear analysis of rapidly growing waves $[a^{-1}(\partial \operatorname{Re} \omega / \partial K_y) \ll \gamma \ll \omega]$ by the quantization technique verifies qualitatively the instability criterion obtained by the localization method.

We have considered methods of description of oscillations at wavelengths appreciably smaller than the scale size of the plasma inhomogeneity ($\lambda_y / a \ll 1$), in which case the semiclassical representation (4.1) can be used. The case of a highly inhomogeneous plasma ($\lambda_y / a \gg 1$), as we have noted earlier in this section, is more appropriately treated in terms of surface waves. We illustrate the application of this method in the following example.

Consider longwave ($\rho_i K_\perp \ll 1$) irrotational oscillations of a nonisothermal plasma ($T_e \gg T_i$) at frequencies $k_z v_{Ti} \ll \omega \ll k_z v_{Te}$. Using Eqs.

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*Kadomtsev has investigated nonlinear oscillations and has concluded that it is completely possible to have a situation in which wave—wave scattering does not "allow" emission of another wave.

(2.7) and (2.8), we obtain the following differential equation for the potential φ:

$$(\varepsilon_\perp \varphi')' - (\varepsilon_\perp k_x^2 + \varepsilon_\parallel k_z^2)\,\varphi = 0. \qquad (4.16)$$

Here,

$$\varepsilon_\perp = \frac{4\pi e^2 n_0}{m_i \omega_{Bi}^2}; \qquad \varepsilon_\parallel = \frac{4\pi e^2}{T_e k_z^2}\left(1 + \frac{k_x T_e}{m_e \omega_{Be}\omega}\frac{\partial}{\partial y}\right) n_0 \left(1 + i\sqrt{\pi}\frac{\omega}{k_z v_{Te}}\right).$$

Assuming that the potential does not change greatly over the width of the inhomogeneous layer, and integrating over this layer, we find

$$n_2 k_2 - n_1 k_1 - \frac{k_x \omega_{Bi}}{\omega}\left(1 + i\sqrt{\pi}\frac{\omega}{k_z v_{Te}}\right) n_0 \Big|_1^2 = 0, \qquad (4.17)$$

where k_2 and k_1 describe the decay of the electric field in the uniform regions 1 and 2, i.e., $E_2 = E_0 e^{-k_2 y}$, $E_1 = E_0 e^{k_1 y}$ (here, y is the distance from the layer). Thus, where the plasma is uniform we write the usual dispersion equations, thereby obtaining expressions for k_2 and k_1:

$$k_2^2 = \frac{1}{\varrho_2^2}\left(1 + i\sqrt{\pi}\frac{\omega}{k_z v_{Te2}}\right);$$

$$k_1^2 = \frac{1}{\varrho_1^2}\left(1 + i\sqrt{\pi}\frac{\omega}{k_z v_{Te1}}\right);$$

$$\varrho_{1,2}^2 = T_{1,2}^e / m_i \omega_{Bi}^2. \qquad (4.18)$$

Substituting the values of k_1 and k_2 in Eq. (4.17), we find the frequency and growth rate of the surface wave:

$$\mathrm{Re}\,\omega = -k_x \sqrt{\frac{T_e}{m_i}}\,\frac{n_2 - n_1}{n_2 + n_1\sqrt{\frac{T_2}{T_1}}}; \qquad (4.19)$$

$$\gamma = \mathrm{Im}\,\omega = -\frac{\sqrt{\pi}}{2}\frac{\omega^2}{k_z v_{Te}}\sqrt{\frac{T_2}{T_1}}\frac{n_2 + n_1}{n_2 - n_1}\left(1 - \frac{n_2}{n_1}\frac{T_1}{T_2}\right). \qquad (4.20)$$

It is evident that this wave can be excited only if

$$\frac{T_2}{T_1} < \frac{n_2}{n_1}. \qquad (4.21)$$

In order to compare this result with the result of §3, we assume that $|T_2 - T_1| \ll T_1$; $n_2 - n_1 \ll n_1$, in which case (4.21) can be written in the form

$$\frac{\Delta \ln T}{\Delta \ln n_0} < 1, \qquad (4.22)$$

i.e., a highly inhomogeneous plasma is unstable if $\partial \ln T/\partial \ln n_0$ is negative (or if $\partial \ln T/\partial \ln n_0$ is small).

§ 5. Drift Instabilities in a Plasma in a Sheared Magnetic Field

A characteristic feature of the slow waves considered in §§ 2 and 3 is the fact that the oscillation frequency ω is very sensitive to the value of the wave number k_z; small changes in k_z lead to important changes in the wave properties. The ion-acoustic drift wave requires $k_z/k_x \leq \rho_i/a$; the Alfvén drift wave requires values of k_z that are still smaller: $k_z/k_x \lessapprox (\rho_i/a)\sqrt{\beta}$ (for large k_z both the ion-acoustic and the Alfvén waves exhibit frequencies ω that are appreciably higher than the characteristic drift frequency $k_x v_{dr}$; hence, these branches are stable at high k_z).

In view of these considerations, one might hope that the drift instability could be inhibited in an experimental device which is short in some appropriate sense, and in which the lines of force are frozen in conductors at the ends of the device. Under these conditions, k_z is bounded from below by a value $\sim 1/L$ (L is the length of the device); then, if $k_x \sim 1/\rho_i$, the instability is known to be suppressed if L is small enough (cf. § 3).

Another possibility exists for stabilization of the drift instability: in this case, the projection of the wave vector in the direction of the lines of force is a function of coordinates but is not necessarily small at all points occupied by the plasma. This situation can be realized, for example, in a helical cylindrically symmetric field in which the pitch of the field is a function of radius. In this case, the projection of $\mathbf{k}$ in the direction of B_0 is given by the quantity $k_{\|} = (1/B)[B_\varphi(m/r) + B_z k_z]$ (cf. Appendix III). If $k_{\|}(r_0) = 0$ at some point, $r = r_0$, at neighboring points where $r \neq r_0$,

$$k_{\|}(r) = (r - r_0)\, k'_{\|}(r_0), \tag{5.1}$$

where $k'_{\|}$ is the radial derivative. Introducing the projection of the wave vector along the binormal to the line of force $k_b = [B_\varphi k_z - (m/r)B_z]/B_0$ (this quantity plays the role of k_x) it is easy to show that

$$k'_{\|} = -r_0 \mu' k_b, \tag{5.2}$$

where $\mu \equiv 2\pi/h = B_\varphi/rB_z$ is the reciprocal pitch of the lines of force. Thus it is evident that if the pitch of the lines of force is a function of radius ($\mu' \neq 0$), the quantity $k_{\|}$ must vary from point to point.

It is clear that in §§2 and 3 we are correct in taking $k_{\parallel}$ to be a constant only if the dependence of the pitch on r is not too strong, in which case the change in $k_{\parallel}$ over the region of localization of the perturbation can be neglected:

$$\delta k_{\parallel} \ll k_{\parallel}, \tag{5.3}$$

where $k_{\parallel}$ is the characteristic longitudinal wave number of interest. It is evident from §§2 and 3 that the maximum value of $k_{\parallel}$ for values of interest for the drift wave is of the order

$$k_{\parallel}^{\max} \sim \frac{\varrho_i}{a} k_b; \tag{5.4}$$

at higher values the wave is highly damped because of the resonance interaction of the ions with the wave. Hence, a necessary condition for the excitation of drift waves is [taking account of Eqs. (5.1)-(5.4)]:

$$\theta \ll \varrho/\delta \tag{5.5}$$

where $\theta = a r_0 \mu'$; this quantity denotes the angle between lines of force which are separated from each other in the radial direction by a distance of the order of a (we recall that a is the characteristic scale size of the inhomogeneity in density or temperature); δ is the characteristic dimension of the localization of the perturbation.

As θ increases, the condition in (5.5) is first violated for perturbations which are highly extended in the radial direction. For example, if a perturbation is localized over a region which is comparable with the scale of the plasma inhomogeneity a, this perturbation will be highly damped when [23]

$$\theta \gtrsim \varrho_i/a. \tag{5.6}$$

We have seen in §2 that when $k_{\perp}\rho_i \sim 1$, perturbations can be localized in a region δ, whose dimensions are not much larger than $(\partial \operatorname{Re}\omega/\partial k_y)/\gamma$. However, these perturbations will not be excited if $\theta \neq 0$ and if

$$\theta \gtrsim \left(\frac{m_e}{m_i}\right)^{1/2}. \tag{5.7}$$

Thus, even a modest shear in the lines of force can have an important effect on perturbations such as those considered in §2, possibly leading to the stabilization of the drift instability.

On the other hand, the criteria in (5.6) and (5.7) do not always give the complete picture if one is interested in the values of θ for which the drift instability is suppressed in the general case. The relation in (5.5) shows that the most dangerous instabilities are due to highly localized perturbations; it is then of great interest to see whether these are still the most dangerous when $\theta \neq 0$.

The equations for slow waves in a helical field, which are required to answer this question, can be obtained approximately by the same method as (2.7)-(2.8) (cf. Appendix III). Assuming, as in § 2, that $E(r) \sim \exp[i\int K_r(r)dr]$, we again obtain a dispersion equation in the form of (2.10) in which the quantities k_x, k_y, and k_z are now replaced by k_b, K_r, and $k_{\parallel}$. In spite of the outward similarity of the dispersion equations for the straight field and the helical field, the form of the solution is very different when $\theta \neq 0$. This difference arises from the fact that there is now another parameter to characterize the inhomogeneity of the plasma: the rate of change of the longitudinal wave number $k_{\parallel}$ in the radial direction. When $\theta = 0$, the plasma inhomogeneity is related to $a \approx (\partial \ln n_0/\partial r)^{-1}$; now, however, there are two inhomogeneity parameters, a and $a^* = (\partial \ln k_{\parallel}/\partial r)^{-1}$. When $k_{\parallel} \approx (\rho/a)k_b$ [cf. Eq. (5.4)], $a^* \approx \rho/\theta$; hence, when θ increases, the dimension a^* becomes comparable with a when $\theta \sim \rho/a$ and for still larger values of θ we find $a^* < a$, a feature which is completely verified by the relation described earlier (5.6). Hence, we shall assume below that $a^* \ll a$ and try to ascertain whether drift waves can, in fact, propagate over the entire region of localization in which this inequality is satisfied.

Assuming that (5.7) is also satisfied, we find that we can not use the local solution, and that it is now necessary to re-examine the behavior of perturbations at points far from the point r^*, assuming that these perturbations are oscillatory near $r = r^*$ [i.e., we assume $\mathrm{Re}\, K_r(r^*) \gg \mathrm{Im}\, K_r(r^*)$]. If it turns out that the wave is spatially damped on both sides of r^*, for example, for $r < r_1$, $r > r_2$ $(r_1 < r^* < r_2)$, and that the conditions $\omega/k_{\parallel} \gg v_{Ti}$, $\omega < k_b v_0^e$ are satisfied everywhere between r_1 and r_2, this will mean that the drift waves can propagate and are unstable.

Close to the points r_1, r_2, where $K_r \approx 0$, we can write $K_r\rho_i \ll 1$; it then follows from Eq. (2.10) that

$$K_r^2(r) = \frac{1}{\varrho_i^2} \frac{\varepsilon_{\parallel}\left(1 - \frac{\omega^2}{c^2k_{\parallel}^2} \varepsilon_{\perp}\right) + \frac{k_b^2}{k_{\parallel}^2}\varepsilon}{\varepsilon_{\parallel}\frac{\omega^2}{c^2k_{\parallel}^2}\frac{\partial\varepsilon_{\perp}}{\partial z} - \frac{1}{k_{\parallel}^2\varrho_i^2}\frac{\partial}{\partial z}(z\varepsilon_{\perp})}, \tag{5.8}$$

where all functions of z on the right side are taken at $z = k_b^2 T/m_i\omega_{Bi}^2$ $(z_e \approx 0)$.

Assuming, as in § 2, that $\beta \gg m_e/m_i$; $\omega/k_{\parallel} \ll v_{Te}$; $T_e = T_i = T$; $\nabla T = 0$; $z_e \ll 1$, and, using Eq. (5.8), we have

$$K_r^2 = \frac{1}{\varrho^2} \times$$

$$\times \frac{I_0 e^{-z}\left(1 + \frac{k_b v_0}{\omega}\right) - 2 + \frac{\omega^2}{c_A^2 k_{\parallel}^2}\left[\left(1 - \frac{k_b^2 v_0^2}{\omega^2}\right)\frac{1 - I_0 e^{-z}}{z} - \varepsilon_{\parallel}^i\left(1 - \frac{\omega^2}{c^2 k_{\parallel}^2}\varepsilon_{\perp}\right)\right]}{\left(1 + \frac{k_b v_0}{\omega}\right)\left\{(I_0 - I_1)e^{-z} + \frac{\omega^2}{c_A^2 k_{\parallel}^2}\left(1 - \frac{k_b v_0}{\omega}\right)\left[\frac{1}{z^2}(1 - I_0 e^{-z}) - \frac{e^{-z}}{z}(I_0 - I_1)\right]\right\}}. \tag{5.9}$$

Here, $\rho \equiv \rho_i$, $v_0 \equiv v_0^e$, and $\varepsilon_{\parallel}^i$ is the ion contribution to $\varepsilon_{\parallel}$. The small imaginary terms due to the electron pole can be neglected.

It then follows that even when $\theta > \rho/a$ there are actually only two points r_1 and r_2 at which $k_r^2 = 0$; and between these points $K_r^2 > 0$ (so long as $\sqrt{\beta} < \rho/a$). The case $z \ll 1$ is a good example because the expression on the right side can be simplified considerably:

$$K_r^2 = \frac{1}{\varrho^2} \times$$

$$\times \frac{\left[\frac{k_b v_0}{\omega} - 1 - \frac{k_{\parallel}^2 v_{Ti}^2}{2\omega^2}\left(1 + \frac{k_b v_0}{\omega}\right)\right]\left[1 - \frac{\omega^2}{c_A^2 K_{\parallel}^2}\left(1 + \frac{k_b v_0}{\omega}\right)\right]}{\left(1 + \frac{k_b v_0}{\omega}\right)\left[1 + \frac{3}{4}\frac{\omega^2}{c_A^2 k_{\parallel}^2}\frac{k_b v_0}{\omega}\left(\frac{k_b v_0}{\omega} - 1\right)\right]}. \tag{5.10}$$

Evidently the region of localization of the perturbation is bounded by certain points; at one of these the oscillation frequency coincides with the local frequency of the ion acoustic wave and at the other, with the local frequency of the slow Alfvén wave [cf. Eqs. (2.16) and (2.17)]. It is also obvious that $k_b v_0 \gtrsim \omega$ in the entire region of localization, i.e., the instability can be excited when the electron pole is taken into consideration.

The condition that the well be deep enough leads to the following requirement on the value of the shear parameter θ:

$$\theta \ll \left(\frac{\varrho}{a}\right)^{2/3}. \tag{5.11}$$

This expression is the criterion for the existence of highly localized longwave ($k_b \rho_i \ll 1$) perturbations which lead to the instability. It is clear qualitatively that this result holds up to $k_b \rho_i \simeq 1$.

On the other hand, the criterion in (5.11) is not meaningful for the shorter wavelengths ($k_b \rho_i \gg 1$); it is found that the shortwave perturbations can be stabilized when $\theta < \rho_i/a$.

Thus, shear of the lines of force leads to stabilization of the drift instability. Important stabilization effects are to be expected when $\theta \sim \rho_i/a$, certain perturbations are stabilized at smaller values of θ, and others are stabilized at somewhat higher values.

§ 6. Flute Instability of a Plasma with Finite Ion Larmor Radius

A plasma confined by a magnetic field with curved lines of force can be unstable against flute instabilities. The hydrodynamic theory of the flute instability has been given in a review by Kadomtsev in the present series [24]. The results of the hydrodynamic analysis can be formulated as follows. A plasma will be stable in devices in which the "effective" radius of curvature of the lines of force is in the same direction as the plasma pressure gradient (lines of force convex into the plasma) and unstable if the effective radius of curvature is in the opposite direction.

In recent years, the theory of the flute instability has been improved considerably, chiefly by virtue of the work of Rosenbluth, Krall, and Rostoker [9], who showed that the flute instability can be stabilized if the ion Larmor radius is not negligible compared with the dimensions of the system, so that

$$\varrho_i^2/a^2 > a/R, \tag{6.1}$$

where R is the effective radius of curvature.

In the present section we present certain results of the kinetic theory of the flute instability which have been developed in the work of Rosenbluth et al., cited above [9], and in later papers [10-11]. The kinetic theory allows us to take account of the finite ion Larmor radius in a simple way, although it is desirable to consider the simplest possible magnetic-field configuration; this limitation arises because the curvature of the lines of force is simulated by the introduction of a fictitious gravitational field, the procedure being based on the well-known hydrodynamic analogy between the flute instability in a curved magnetic field and in a gravitational field g of order T/m_iR, where T is the plasma temperature and m_i is the ion mass.

Flute Instability in a Gravitational Field. We assume that the plasma is confined by a uniform magnetic field with straight lines of force, but that there is also present a gravitational field directed along the plasma inhomogeneity: $\mathbf{g} = (0, -g, 0)$. In order to investigate the characteristic oscillations of a plasma configuration of this kind, we require a knowledge of the dielectric tensor. Since the motion of the particles in the presence of $\mathbf{g}$ is different from the case $\mathbf{g} = 0$, we cannot make direct use of the expression for $\varepsilon_{\alpha\beta}$ that has been computed in §1. The modification of $\varepsilon_{\alpha\beta}$ that is required to take account of $\mathbf{g}$ is derived in Appendix II. Using Maxwell's equations and Poisson's equations and the calculations in Appendix II, we can obtain a system of equations which describe slow waves in a plasma in a gravitational field. This system is analogous to Eqs. (2.6) and (2.7) and can be written as follows:

$$\left.\begin{aligned} \int e^{ik_y y}\, dk_y \left\{\frac{1}{k_x}\left[k_\perp^2 \varepsilon_\perp^g - ik_y \tilde{\varepsilon}_g'\right] E_x(k_y) + k_z \varepsilon_\parallel^g E_z(k_y)\right\} &= 0; \\ \int e^{ik_y y} dk_y \left\{\left(k_\perp^2 - \frac{\omega^2}{c^2}\varepsilon_\parallel^g\right) E_z\left(k_y\right) - \frac{k_z}{k_x} k_\perp^2 E_x(k_y)\right\} &= 0; \\ E_y(k_y) = \frac{k_y}{k_x} E_x(k_y). & \end{aligned}\right\} \quad (6.2)$$

Here, $\varepsilon_\perp^g = \varepsilon_\perp + \delta\varepsilon_\perp$, where $\varepsilon_\perp$ is of the same form as in Eq. (2.8), while the quantities $\delta\varepsilon_\perp$ and $\tilde{\varepsilon}_g$, $\varepsilon_\parallel^g$ are given by

$$\delta\varepsilon_\perp = -\sum_{i,\,e} \frac{4\pi e^2}{Tk_\perp^2} \frac{gk_x}{\omega\omega_B}\left\{1 + \left[1 + \frac{gk_x}{\omega\omega_B} + \frac{Tk_x}{m\omega\omega_B}\frac{\partial}{\partial y}\right] i\sqrt{\pi}xW(x)\right\}\times$$

$$\times\, n_0 I_0 e^{-z};$$

$$\tilde{\varepsilon}_g = 1 + \sum_{i,\,e} \frac{4\pi e^2}{m\omega_B^2}\left(1 + \frac{k_x}{m\omega\omega_B}\frac{\partial}{\partial y}T\right) n_0 e^{-z}(I_0 - I_1)\times$$

$$\times\left[1 + \frac{gk_x}{\omega\omega_B} i\sqrt{\pi}xW(x)\right];$$

$$\varepsilon_\parallel^g = 1 + \sum_{i,\,e} \frac{4\pi e^2}{k_z^2 T}\left(1 + \frac{gk_x}{\omega\omega_B}\right)\left(1 + \frac{k_x T}{m\omega\omega_B}\frac{\partial}{\partial y}\right)\times$$

$$\times\left(1 + i\sqrt{\pi}xW(x)\right) n_0 e^{-z} I_0(z);$$

$$x = \left(\omega + \frac{gk_x}{\omega_B}\right)\Big/ k_z v_T. \quad (6.3)$$

Representing the field in the form of Eq. (2.9), and taking $K_y a \gg 1$, we obtain a dispersion equation which is analogous to Eq. (2.10):

$$\varepsilon_\parallel^g\left(1 - \frac{\omega^2}{ck_z^2}\varepsilon_\perp^g\right) + \frac{K_\perp^2}{k_z^2}\varepsilon_\perp^g = 0. \quad (6.4)$$

We note that the structure of the dispersion equation for the slow waves in the presence of a gravitational field is exactly the same as for $g = 0$, as is evident from a comparison of Eqs. (2.10) and (6.4).

Hence the restriction that the slow waves must be the Alfvén wave or the ion-acoustic wave, which has been given in § 2, remains in force.

Rosenbluth et al. [9] have considered the case of electrostatic waves propagating across a magnetic field. Using Eq. (6.4) we see that this is the limiting case of the Alfvén wave which, when $k_z = 0$, is described by the equation

$$\varepsilon_{\perp}^{g} = 0. \tag{6.5}$$

It has been assumed that the plasma density is large ($c_A^2 \ll c^2$) and that the ion Larmor radius is small compared with the wavelength ($K_{\perp}^2 \rho_i^2 \ll 1$). It is evident that under these conditions Eq. (6.5) becomes

$$\omega^2 - k_x v_0^i \omega + g\varkappa = 0; \tag{6.6}$$

$$\left(v_0^i = -\frac{T_i \varkappa}{m_i \omega_{Bi}}; \quad \nabla T_i = 0; \quad \varkappa = \partial \ln n_0 / \partial y; \quad k_x \gg k_y \right).$$

It follows that if $k_x v_0^i \ll \omega$, as is usually the case in hydrodynamics, there will be an instability (flute instability) with growth rate

$$\gamma \equiv \operatorname{Im} \omega = \sqrt{g\varkappa}. \tag{6.7}$$

Rosenbluth, Krall, and Rostoker [9] have called attention to the fact that taking $k_x v_0^i/\omega$ terms into account is very important and can lead to qualitatively new results. For instance, if $k_x v_0^i$ is considered, we find from (6.6) that the flute instability is damped if

$$(k_x v_0^i)^2 > 4g\varkappa. \tag{6.8}$$

Thus, it has been shown [9] that a stabilizing effect arises and this effect is called "finite Larmor radius stabilization." The meaning of this terminology is clear if g in (6.8) is replaced by the equivalent expression $T/m_i R$, and if v_0^i is replaced by the equivalent expression $v_0^i = -T_i \varkappa / m_i \omega_{Bi}$. In this case the criterion in (6.8) assumes the form

$$\varrho_i^2 k_x^2 > 4a/R. \tag{6.9}$$

It is evident that the stabilization disappears if the Larmor radius is formally taken to be infinitesimally small compared with the wavelength.

It follows from (6.9) that the longwave perturbations are the most dangerous. If it is assumed that a given experimental device is circular, writing $k_x^{min} \simeq 1/a$ it is possible to obtain a qualitative criterion for stabilization of the flute instability; this criterion is in the form of (6.1).

A stabilization effect also arises if there is a sharp boundary between the plasma and the vacuum, so that the transition layer a can be regarded as small compared with the transverse dimensions of the device (in the plane-wave approximation this implies the possibility of very small wave numbers k_x, such that $k_x a \ll 1$). Kruskal and Schwarzchild [25] have considered the stability of a sharp boundary using ordinary hydrodynamics and have shown that the plasma is unstable for any value of g directed in the vacuum direction. However, using our equations (6.2) and (6.3), which take account of the finite ion Larmor radius, we now find that the plasma can be stable for a sharp boundary (but not so sharp that $\rho_i/a \ll 1$) and for very small values of g. Assuming that $k_\perp \rho_i \ll 1$ (for $k_z = 0$, $c_A^2 \ll c^2$), using these equations we can derive the following differential equation:

$$\left[n_0\left(1-\frac{k_x v_0^i}{\omega}\right)E_x'\right]-k_x^2 n_0\left(1-\frac{k_x v_0^i}{\omega}+\frac{g\varkappa}{\omega^2}\right)E_x=0. \tag{6.10}$$

This equation is integrated along the transition layer, assuming the layer to be thin ($k_x a \ll 1$) and neglecting the variation in E_x. If $E \sim e^{-k_x|y|}$ outside the layer we have

$$\omega^2+\frac{k_x^2 T_i n_0}{m_i \omega_{Bi}}\omega+gk_x=0. \tag{6.11}$$

In the hydrodynamic approximation ($\omega \gg \omega_{Bi} Z_i$) one then obtains the result of [25], which indicates that the plasma is unstable with growth rate

$$\gamma=\sqrt{gk_x}. \tag{6.12}$$

However, for very small values of g, in which case

$$k_x^2 \varrho_i^2 > 2\sqrt{gk_x}, \tag{6.13}$$

the instability is suppressed. At the limit of applicability ($k_x \sim 1/a$) this criterion agrees qualitatively with the stability criterion (6.1) obtained by Rosenbluth et al. [9] for a smooth transition layer.

Following the publication of the paper by Rosenbluth et al. [9], several authors [10, 26] were able to show that the stabilization effect can be described by a hydrodynamic analysis in which the usual equations (used, for example in the energy-principle analysis of stability [27]), must be supplemented by magnetic–viscosity terms [2].

The case $k_z = 0$, which has been considered above, is the most dangerous. If we do not assume $k_z = 0$, Eq. (6.5) becomes

$$1 - \frac{\omega^2}{c^2 k_z^2} \varepsilon_\perp^g = 0, \tag{6.14}$$

whence, when $c_A^2 \ll c^2$, $\omega \gg k_z v_{Ti}$, it follows that

$$\omega^2 - k_x v_0^i \omega + g\varkappa - c_A^2 k_z^2 = 0. \tag{6.15}$$

In place of the inequality in (6.8), the stabilization criterion is now given by

$$(k_x v_0^i)^2 > 4\,(g\varkappa - c_A^2 k_z^2). \tag{6.16}$$

It is then evident that the instability is suppressed for an arbitrarily small ion Larmor radius so long as

$$k_z > \sqrt{g\varkappa}/c_A. \tag{6.17}$$

If the lines of force are anchored at the ends of the apparatus, in which case $k_z \gtrsim \pi/L$, the criterion for the suppression of the flute instability, equation (6.17), becomes [28]:

$$Ra > \frac{\beta}{\pi^2} L^2. \tag{6.18}$$

After the discovery of the finite ion Larmor radius stabilization of the flute instability, it was hoped that taking the finite ion Larmor radius into account in the proper way would lead to stabilization of other instabilities that had been predicted earlier. However, these hopes were not justified. As we have seen in §§ 2-5, when $\omega \lesssim k_x v_0$, i.e., in the stabilization region for the flute instability, there is an instability of another kind, the drift instability; this instability does not vanish even at wavelengths of order ρ_i (indeed, this instability is strongest when $K_\perp \rho_i \sim 1$).

Furthermore, it turns out that even the flute instability is not always stabilized when the wavelength of the perturbation is comparable with the ion Larmor radius. Here we are considering a plasma in which the density is low, $c_A \gtrsim c$. As shown in [11], when $c_A \gtrsim c$, the dispersion equation for the

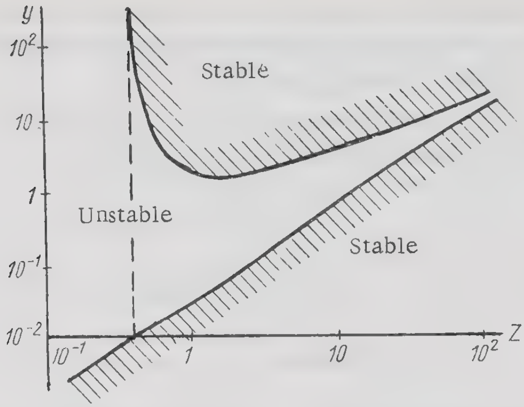

Fig. 6. The flute instability in a highly rarefied plasma for $g = \frac{1}{12}(\varkappa T/m_i)$. Along the abscissa, the quantity $Z = K_\perp^2 \rho_i^2$ is plotted; the density ($y = 4\pi n_0 m_i c^2/B_0^2$) is the ordinate.

flute instability with arbitrary $k_x\rho_i$ assumes the form [this follows from Eq.(6.4)]:

$$1 + \frac{4\pi e^2 n_0}{k_x^2 T_i}\left(1 - I_0 e^{-z}\right)\left(1 - \frac{k_x v_0^i}{\omega}\right) +$$

$$+ \frac{4\pi e^2 n_0}{m_i \omega_{Bi}^2} \frac{g\varkappa I_0 e^{-z}}{\omega(\omega + gk_x/\omega_B)} = 0, \tag{6.19}$$

$$\left(z = k_x^2 T_i/m_i\omega_{Bi}^2; \quad K_y \ll k_x; \quad \nabla T = 0\right).$$

The stability limits found from this equation exhibit a pattern such as that shown in Fig. 6. It is evident from this figure that even perturbations with $K_\perp \rho_i \gtrless 1$ will be unstable in a plasma which is highly rarefied. The stability of a plasma with an extremely low density (lower part of the figure) is explained by the fact that the motion of a particle in a plasma of this kind is independent of the motion of the other particles [29]. When $K_\perp \rho_i \gg 1$, there is a peculiar kind of shortwave instability which is somewhat reminiscent of the two-stream instability because here we find $\omega/K \approx -g/\omega_B$, i.e., the phase velocity is equal to the gravitational drift velocity of the ions.

Flute Instability in a Helical Magnetic Field with Cylindrical Symmetry [30]. We now consider the flute instability in a more pertinent example (although we must still make a number of idealizing assumptions): the plasma is confined in a helical field with cylindrical symmetry. This kind of magnetic configuration has been used in § 5, in which the effect of shear on the drift instability was illustrated.

It follows from (III.22) in Appendix III that spatially smooth perturbations corresponding to longwave ($K_\perp^2 \rho_i^2 \ll 1$) Alfvén waves, are described by the equation

$$\left\{\left[1 - \frac{\omega^2}{c_A^2 k_\parallel^2}\left(1 - \frac{k_b v_0^i}{\omega}\right)\right]\left(\frac{\partial^2}{\partial r^2} - k_b^2\right) + \frac{2k_\parallel'}{k_\parallel}\frac{\partial}{\partial r} - \frac{8\pi p'}{B_0^2 R}\frac{k_b^2}{k_\parallel^2}\right\} E_b = 0. \tag{6.20}$$

In order to study this equation we convert to the new variables

$$F = E_b \sqrt{-\lambda}\,(s^2 + 1)^{1/2}; \quad s^2 = -(r - r_0)^2/\lambda,$$

where

$$\lambda = \omega^2\left(1 - \frac{k_b v_0^i}{\omega}\right) \Big/ k_\parallel'^2 c_A^2.$$

We assume that $\lambda < 0$. In this case, Eq. (6.20) can be written in the form of a Schrödinger equation

$$\frac{d^2F}{ds^2} + \lambda k_b^2 F + \left[\frac{\chi^2}{s^2 + 1} - \frac{1}{(s^2 + 1)^2}\right] F = 0 \tag{6.21}$$

for a particle with energy

$$\varepsilon = \lambda k_b^2 \tag{6.22}$$

in a potential

$$U = -\frac{\chi^2}{s^2 + 1} + \frac{1}{(s^2 + 1)^2}. \tag{6.23}$$

Here,

$$\chi^2 = -\frac{8\pi p'}{r B_z^2}\left(\frac{\mu}{\mu'}\right)^2. \tag{6.24}$$

In the magnetohydrodynamic approximation ($\omega \gg k_b v_0^i$) the oscillation frequency appears in quadratic form in Eq. (6.21): $\lambda = \omega^2 / c_A^2 k_\parallel'^2$. Hence the stability condition in this approximation is an existence condition for solutions of Eq. (6.21) with $\lambda < 0$ (i.e., the existence of bound states). The existence condition for the solution (the limiting value for the appearance of first level) can be found by writing $\lambda \to 0$, $s \to \infty$ in Eq. (6.21); this equation then assumes the form of a Bessel equation for a function of imaginary argument. The solution that converges at infinity is of the form

$$F \sim K_\nu(s); \quad \nu = \sqrt{\varkappa^2 - 1/4}, \tag{6.25}$$

where K_s is the MacDonald function. Since $\varepsilon \equiv \lambda k_b^2 < 0$, this function must not oscillate at infinity; thus, we require

$$\chi^2 > 1/4. \tag{6.26}$$

If this relation is satisfied, the plasma is unstable within the framework of the hydrodynamic approximation. This criterion was first given by Suydam [31]. Since χ^2 is inversely proportional to the square of the shear of the lines of force ($\chi^2 \sim 1/\theta^2$) the inequality in (6.26) indicates that shear in the lines of force plays a stabilizing role, as in the drift instability (cf. § 5).

Taking account of terms of order $k_b v_0^i/\omega$ leads to a more favorable stability condition. When $k_b v_0^i \neq 0$, the existence condition for the solution (6.26) does not necessarily mean an instability because

$$\omega = \frac{k_b v_0^i}{2} \pm \sqrt{\left(\frac{k_b v_0^i}{2}\right)^2 + \varepsilon_n \left(\frac{k'_\parallel c_A}{k_b}\right)^2} \tag{6.27}$$

in this case. This relation follows from $\lambda k_b^2 = \varepsilon_n$, where ε_n is the characteristic value of the energy ε, and the plasma is stable if

$$\left(\frac{k_b v_0^i}{2}\right)^2 > |\varepsilon_n| \left(\frac{k'_\parallel c_A}{k_b}\right)^2. \tag{6.28}$$

Since $|\varepsilon_n|$ does not, in any case, exceed the modulus of the potential minimum U, it follows from Eqs. (6.23) and (6.27) that the plasma is stable even for large χ^2 if the ion Larmor radius is large enough to satisfy the condition

$$\left(\frac{m}{r} \varrho_i\right)^2 > \frac{a}{R}\left(1 + \frac{T_e}{T_i}\right). \tag{6.29}$$

Here, m is the azimuthal wave number (cf. Appendix III).

This stability criterion is in agreement with the analogous results (6.13) in the problem involving the gravitational force.

§ 7. Drift Excitation of Flute Instabilities

As we have noted in § 6, a large value of the ion Larmor radius tends to stabilize the flute instability. However, taking account of interactions between the wave and resonance particles can mean that the flute perturbations will be unstable again, although with smaller growth rates. The importance of this essentially nonhydrodynamic effect was first considered in the work of Rosenbluth et al. [9], in which the stabilizing effect of the finite ion Larmor radius was first considered. It was found that the interaction between the wave and resonance electrons drifting under the effect of the inhomogeneity in the magnetic field can destabilize the flute instability. However, this effect is exponentially small and falls beyond the limits of accuracy of the approximations being used.

In the present section we consider flute perturbations characterized by $k_{||} \neq 0$. Under these conditions one finds that the resonance excitation mechanisms are important; these mechanisms are connected with the motion of particles along the lines of force and appear when $\omega < k_x v_0$, where v_0 is the Larmor drift velocity. If the plasma pressure is small ($\beta \ll 1$), the Larmor drift velocity is appreciably greater than the velocity associated with the magnetic drift (a factor of β^{-1}), so that one expects the excitation of oblique flute instabilities to be much stronger than that considered in [9].

<u>Drift Excitation of Flute Instabilities in a Gravitational Field.</u> Longwave plasma oscillations ($K_\perp^2 \rho_i^2 \ll 1$) in a gravitational field are described, as follows from Eqs. (6.2) and (6.3), by the following fourth-order differential equation:

$$\Big\{\left(k_x^2 - \frac{\partial^2}{\partial y^2}\right)\frac{1}{\varepsilon_{||} k_z^2}\left[(\varepsilon_\perp + \delta\varepsilon_\perp)\ k_x^2 - \frac{\partial}{\partial y}\left(\varepsilon_\perp \frac{\partial}{\partial y}\right)\right] + \\ + \left[1 - \frac{\omega^2}{c^2 k_z^2}(\varepsilon_\perp + \delta\varepsilon_\perp)\right]k_x^2 - \\ - \frac{\partial}{\partial y}\left[\left(1 - \frac{\omega^2}{c^2 k_z^2}\varepsilon_\perp\right)\frac{\partial}{\partial y}\right]\Big\} E_x = 0, \tag{7.1}$$

where

$$\varepsilon_\perp = \frac{c^2}{c_A^2}\left(1 - \frac{k_x v_0^i}{\omega}\right), \quad \delta\varepsilon_\perp = \frac{c^2}{c_A^2}\frac{g\varkappa}{\omega^2};$$

$$\varepsilon_{||} = \frac{4\pi e^2 n_0}{k_z^2 T_e}\left(1 - \frac{k_x v_0^e}{\omega}\right)\left(1 + i\sqrt{\pi}\,\frac{\omega}{k_z v_{Te}} W_0\right). \tag{7.2}$$

For simplicity we take $\nabla T_i = \nabla T_e = 0$; $c^2 \gg c_A^2$.

Comparing the relative magnitudes of the terms in Eq. (7.1), we conclude that the higher derivatives are multiplied by a small parameter. When

$$k_z \gtrsim \omega / v_{Te} \tag{7.3}$$

this parameter is a quantity of order ρ_i^2, and when

$$k_z \lesssim \omega / v_{Te} \tag{7.4}$$

this parameter is of order c^2/ω_{0e}^2.

Assume that ρ_i and c/ω_{0e} are small compared with the other characteristic transverse dimensions (there are two such dimensions: the characteristic scale size of the inhomogeneity a and the wavelength in the x direction, given by $1/k_x$). Under these conditions, two of the four solutions of Eq. (7.1) correspond to spatially smooth perturbations with wavelengths (along y) determined by the characteristic dimensions a and $1/k_x$; the other two solutions correspond to perturbations with shorter wavelengths, associated with ρ_i or c/ω_{0e}. Hence, we can obtain two approximate equations from Eq. (7.1) one for the large-scale perturbations

$$\left\{\left(1 - \frac{\omega^2}{c^2 k_z^2}\varepsilon_\perp\right)\left(\frac{\partial^2}{\partial y^2} - k_x^2\right) - \frac{\omega^2}{c^2 k_z^2}\varepsilon_\perp' \frac{\partial}{\partial y} + \frac{\omega^2}{c^2 k_z^2}\delta\varepsilon_\perp \cdot k_x^2\right\} E_x = 0, \tag{7.5}$$

and the other for the fine-scale perturbations

$$\varepsilon_\perp E_x^{\mathrm{IV}} - k_z^2 \varepsilon_\parallel \left(1 - \frac{\omega^2}{c^2 k_z^2}\varepsilon_\perp\right) E_x'' = 0. \tag{7.6}$$

The latter equation is equivalent to a second-order equation for the field E_z

$$\varepsilon_\perp E_z'' - k_z^2 \varepsilon_\parallel \left(1 - \frac{\omega^2}{c^2 k_z^2}\varepsilon_\perp\right) E_z = 0. \tag{7.7}$$

This equation is typical for problems that arise in the analysis of drift instabilities; it has already been considered in detail in § 4. In what follows principal attention will be directed to the investigation of the other class of perturbations – the flute perturbations described by the approximate equation (7.5) or by the more rigorous equation (7.1). Let us consider Eq. (7.5). We have already considered this equation in § 6. If the wavelength of the perturbation is small compared with the scale size of the plasma inhomogeneity, we can write the field in the form $E_x \sim \exp[i\int K_y(y)dy]$ and obtain the dispersion equation

$$k_x^2\left(\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2 + g\varkappa\right) + \\ + K_y^2\left(\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2\right) = 0; \tag{7.8}$$

a particular case of this equation was treated in § 6 [Eq. (6.15)], where it was assumed that $K_y^2 \ll k_x^2$.

Writing the quantity $K_y^2(y,\omega)$ in the form

$$K_y^2(\omega,\ y) = -k_x^2\left(1 + \frac{g\varkappa}{\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2}\right), \tag{7.9}$$

we see that it is possible to have a wave whose wave number K_y is real over some portion of the space only when $g \neq 0$. Furthermore, it is necessary that the following condition be satisfied in the region of localization of the wave, where $K_y^2 > 0$:

$$1 + \frac{g\varkappa}{\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2} < 0, \tag{7.10}$$

it being assumed that the region of localization is much larger than $1/k_x$. The boundaries of the localization region are defined by points at which either

$$\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2 + g\varkappa = 0, \tag{7.11}$$

in which case $K_y = 0$ at these points, or

$$\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2 = 0, \tag{7.12}$$

in which case $K_y^2 \to \pm\infty$. The first case is a typical one in the theory of oscillations of an inhomogeneous plasma and has been discussed in § 4. However, if (7.12) holds at some point $y = y_0$, our simplified equation (7.5) is no longer valid, since we have violated the condition of large-scale perturbations which was assumed in the derivation. A point of this kind ($y = y_0$) will be called a critical point.

The significance of the critical point is the following: at this point the local Alfvén frequency corresponds to the oscillation frequency ω. Thus, we must solve the complete equation (7.1) in the vicinity of $y = y_0$ and then join this solution to the solutions of the simplified equations (7.5) and (7.6). It is found that solutions that satisfy the matching condition do not pertain to large-scale or small-scale perturbations individually but are a mixture of both.

Let us assume that there is no critical point and that the region of localization is bounded by two ordinary turning points, which satisfy Eq. (7.11)

The existence of two points, y_1 and y_2, at which Eq. (7.11) is satisfied is typical for density distributions in which $\varkappa \equiv (1/n_0)\partial n_0/\partial y$ has a maximum, for example, when $\varkappa = a/(y^2 + a^2)$.

If we work with Eq. (7.7) and retain only the zero-order terms in $K_{\perp}^2\rho^2$ we obtain a spectrum of real values of ω under the condition that the ion Larmor radius is large enough and that the stabilization effect of Rosenbluth et al. holds [9]. This question has been treated in § 6. However, if we take account of imaginary quantities in terms of order $K_{\perp}^2\rho_i^2$, Eq. (7.9) becomes

$$K_y^2 = -k_x^2\left\{1 + \frac{g\varkappa}{\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2} - \frac{i(g\varkappa)^2 (k_x c/\omega)^2}{\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2}\,\mathrm{Im}\,\frac{1}{\varepsilon_{\parallel}}\right\}. \tag{7.13}$$

Here, the oscillation frequency is complex and its imaginary part γ is of the form [we make use of Eq. (4.8)]

$$\gamma = -\int_{y_1}^{y_2} \frac{1}{K_y}\,\frac{(g\varkappa)^2\left(\frac{k_x c}{\omega}\right)^2 \mathrm{Im}\,\frac{1}{\varepsilon_{\parallel}}}{\left(\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2\right)^2}\,dy \times \left\{\int_{y_1}^{y_2} \frac{dy}{K_y}\,\frac{g\varkappa\left(2\omega - k_x v_0^i\right)}{\left(\omega^2 - \omega k_x v_0^i - c_A^2 k_z^2\right)^2}\right\}^{-1}. \tag{7.14}$$

In particular, for the lowest levels $(y_2 - y_1) \ll a$, $\omega \ll k_z v_{Te}$, $T_e = T_i$, whence it follows that

$$\gamma = -\frac{k_z c^2{}_A}{v_{Te}}\,k_x^2 \varrho_i^2\,\frac{g\varkappa}{\left(\omega_{1,\,2} + k_x v_0^i\right)\left(2\omega_{1,\,2} - k_x v_0^i\right)}, \tag{7.15}$$

where

$$\omega_{1,\,2} = \frac{k_x v_0^i}{2} \pm \sqrt{\left(\frac{k_x v_0^i}{2}\right)^2 - g\varkappa}.$$

The growth rate for the slow wave (minus sign) is positive and of the form

$$\gamma = \frac{k_z c_A^2}{v_{Te}}\,k_x^2 \varrho^2\,\frac{g\varkappa}{\sqrt{\left(\frac{k_x v_0^i}{2}\right)^2 - g\varkappa}\left(\sqrt{\left(\frac{k_x v_0^i}{2}\right)^2 - g\varkappa} + \left(\frac{k_x v_0^i}{2}\right)\right)}. \tag{7.16}$$

Thus, taking account of resonance electrons means that the flute perturbations are unstable even in the presence of the stabilizing effect described by Rosenbluth et al. [9]; however, the growth rate is smaller.

Appendix I

The Tensor $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega, y)$ for a Plasma with an Anisotropic Distribution Function of Guiding Centers. If f_0 depends on V_z, Eq. (1.9) in the text becomes

$$\varepsilon_{\alpha\beta}(\mathbf{k}, \omega, y) = \delta_{\alpha\beta} - i \sum_{i, e} \frac{4\pi e^2}{m\omega} \int d\mathbf{v}_0 \cdot v_\alpha(t) \times$$

$$\times \left[\left(m \frac{\partial F_0}{\partial \varepsilon} - \frac{k_x}{\omega\omega_B} \frac{\partial F_0}{\partial Y} + \frac{k_z}{\omega} \frac{\partial F_0}{\partial V_z} \right) \int_{-\infty}^{t} v_\beta(t') A(t, t') dt' - \right.$$

$$\left. - \frac{i\delta_{\beta x}}{\omega\omega_B} \frac{\partial F_0}{\partial Y} + \frac{i\delta_{\beta z}}{\omega} \frac{\partial F_0}{\partial V_z} \right]. \qquad (I.1)$$

After integration over t' and α_0 we have

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{i, e} \frac{4\pi e^2}{m\omega} \left[2\pi \int dv_{\perp 0}^2 dv_{z0} \left\{ \tilde{\Phi} Q_{\alpha\beta} - \frac{1}{\omega_B} \frac{\partial \tilde{\Phi}}{\partial y} P_{\alpha\beta} \right\} - \right.$$

$$- \frac{\delta_{\beta x}}{\omega\omega_B} \left\{ \delta_{\alpha z} \int v_{z0} \frac{\partial f_0}{\partial y} d\mathbf{v}_0 - \delta_{\alpha x} \int \frac{v_{\perp 0}^2}{2\omega_B} \frac{\partial^2 f_0}{\partial y^2} d\mathbf{v}_0 \right\} +$$

$$+ \frac{\delta_{\beta z}}{\omega} \left\{ \int v_{z0} \left(\frac{\partial f_0}{\partial v_z} - m v_z \frac{\partial f_0}{\partial \varepsilon} \right) d\mathbf{v}_0 \cdot \delta_{\alpha z} - \delta_{\alpha x} \frac{\partial}{\partial y} \int \frac{v_\perp^2}{2\omega_B} \times \right.$$

$$\left. \left. \times \left(\frac{\partial f_0}{\partial v_z} - m v_z \frac{\partial f_0}{\partial \varepsilon} \right) d\mathbf{v}_0 \right\} \right], \qquad (I.2)$$

where

$$\tilde{\Phi} = m \frac{\partial f_0}{\partial \varepsilon} \left(1 - \frac{k_z v_z}{\omega} \right) - \frac{k_x}{\omega\omega_B} \frac{\partial f_0}{\partial y} + \frac{k_z}{\omega} \frac{\partial f_0}{\partial v_z}, \qquad (I.3)$$

while $Q_{\alpha\beta}$ and $P_{\alpha\beta}$ are determined by Eqs. (1.21), (1.23), (1.24), and (1.25), as before.

Appendix II

The Tensor $\varepsilon_{\alpha\beta}(\mathbf{k}, \omega, y)$ for a Plasma in a Gravitational Field. The motion of the particles in the gravitational field is given by

$$v_x = v_{\perp 0} \cos(\alpha_0 - \omega_B t) - \frac{g}{\omega_B};$$

$$v_y = v_{\perp 0} \sin(\alpha_0 - \omega_B t);$$

$$v_z = v_{z0}; \qquad (II.1)$$

$$
\begin{aligned}
x &= x_0 - (v_{\perp 0}/\omega_B)\,[\sin(\alpha_0 - \omega_B t) - \sin\alpha_0] - \frac{gt}{\omega_B};\\
y &= y_0 + (v_{\perp 0}/\omega_B)\,[\cos(\alpha_0 - \omega_B t) - \cos\alpha_0];\\
z &= z_0 + v_{z0} t;
\end{aligned}
\tag{II.1}
$$

where

$$
v_{\perp 0} = \left[\left(v_{x0} + \frac{g}{\omega_B}\right)^2 + v_{y0}\right]^{1/2}; \quad \alpha_0 = \operatorname{arc\,tg}\,[v_{y0}/(v_{x0} + g/\omega_B)];
$$

$$
v_{x0} = v_x(0); \quad v_{y0} = v_y(0); \quad v_{z0} = v_z(0); \quad \mathbf{g} = (0,\ -g,\ 0).
$$

The constants of the motion are $\varepsilon = mv^2/2 + mgy$; $Y = y - v_x/\omega_B$; $V_z = v_z$. If f_0 is isotropic, we find that Eq. (1.9) still holds in the presence of g. The following relations obtain:

$$
\left.
\begin{aligned}
\frac{\partial F_0}{\partial Y} &= \frac{\partial f_0(\varepsilon_0 y)}{\partial y} - mg\frac{\partial f_0(\varepsilon_0,\ y)}{\partial \varepsilon_0} - \frac{v_x(t)}{\omega_B}\frac{\partial^2 f_0(\varepsilon_0,\ y)}{\partial y^2};\\
\frac{\partial F_0}{\partial \varepsilon_0} &= \frac{\partial f_0(\varepsilon_0,\ y)}{\partial \varepsilon_0} - mg\frac{v_x(t)}{\omega_B}\frac{\partial^2 f_0}{\partial \varepsilon_0^2} - \frac{v_x(t)}{\omega_B}\frac{\partial^2 f_0(\varepsilon_0,\ y)}{\partial \varepsilon_0 \partial y},
\end{aligned}
\right\}
\tag{II.2}
$$

where

$$
\varepsilon_0 = mv_0^2/2 + mgy; \quad (g/\omega_B \ll v_\perp).
$$

The final expression for $\varepsilon_{\alpha\beta}$ is

$$
\begin{aligned}
\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{i,\,e}\frac{4\pi e^2}{m\omega}\Bigg\{2\pi\int dv_{\perp 0}^2\, dv_{z0}\Bigg[\Phi_1\left(Q_{\alpha\beta} - \frac{g}{\omega_B}G_{\alpha\beta}\right) - \\
-\frac{1}{\omega_B}\left(\frac{\partial \Phi_2}{\partial y} + m^2 g\frac{\partial^2 f_0}{\partial \varepsilon_0^2}\right)\left(P_{\alpha\beta} - \frac{g}{\omega_B}H_{\alpha\beta}\right)\Bigg] + \frac{\delta_{\alpha x}\delta_{\beta x}}{\omega\omega_B}\int\frac{v_{\perp 0}^2}{2\omega_B}\frac{\partial f_0}{\partial y}dv_0\Bigg\},
\end{aligned}
\tag{II.3}
$$

where

$$
\left.
\begin{aligned}
\Phi_1 &= m\frac{\partial f_0}{\partial \varepsilon_0}\left(1 + \frac{gk_x}{\omega\omega_B}\right) - \frac{k_x}{\omega\omega_B}\frac{\partial f_0}{\partial y};\\
\Phi_2 &= m\frac{\partial f_0}{\partial \varepsilon_0} - \frac{k_x}{\omega\omega_B}\frac{\partial f_0}{\partial y};\\
G_{\alpha\beta} &= \sum_n \zeta_n J_n\left(\delta_{\alpha x} q_\beta + \delta_{\beta x} q_\alpha^*\right);\\
H_{\alpha\beta} &= \sum_n \zeta_n\left[J_n \delta_{x\beta} p_\alpha + q_\alpha^* q_\beta + q_x^* q_\beta \delta_{\alpha x}\right].
\end{aligned}
\right|
\tag{II.4}
$$

The polarizability $\chi_\beta(y, k, \omega)$ is then

$$\chi_\beta = \sum_{i,e}\left(\frac{-ie^2}{m}\right)\left\{2\pi\int dv_{\perp 0}^2 dv_{z0}\left[\Phi_1 S_\beta - \frac{1}{\omega_B}\left(\frac{\partial\Phi_2}{\partial y} + m^2 g\frac{\partial^2 f_0}{\partial\varepsilon_0^2}\right)T_\beta\right] + \right.$$

$$\left. + \frac{\delta_{\beta x}}{\omega\omega_B} mg\int\frac{\partial f_0}{\partial\varepsilon}\,d\mathbf{v}_0\right\}, \qquad \text{(II.5)}$$

where

$$S_\beta = \sum_{n=-\infty}^{+\infty}\zeta_n J_n\left(q_\beta - \delta_{\beta x}\frac{g}{\omega_B}J_n\right);$$

$$T_\beta = \sum_{n=-\infty}^{+\infty}\zeta_n\left(q_\beta - \delta_{\beta x}\frac{g}{\omega_B}J_n\right)\left(q_x^* - \frac{g}{\omega_B}J_n\right). \qquad \text{(II.6)}$$

Appendix III

Dielectric Constant of a Plasma in a Helical Magnetic Field [32]. We assume that the plasma occupies an infinitely long cylinder which is located in a magnetic field with components $B_\varphi = B_\varphi(r)$, $B_z = B_z(r)$, and $B_r = 0$. In place of the Cartesian coordinate system which has been used up to this point we now work in a cylindrical coordinate system r, φ, and z. The wave numbers k_x, k_y, and k_z are now replaced by m, k_r, and k_z. The dependence of the wave field on coordinates is now given by

$$E = \sum_m\int dk_r dk_z E(m, k_z, k_r)\,e^{im\varphi + ik_z z + ik_r r}. \qquad \text{(III.1)}$$

It is found convenient to construct certain combinations of the cylindrical wave numbers m, k_r, and k_z; these combinations have the meaning of projections of the wave vector in directions associated with the line of force, i.e., along the line of force, normal to the line of force (i.e., along the radius) and along the binormal. These combinations, which are analogous to the projections k_x, k_y, and k_z for a plane layer, will be the quantities k_b, k_r, and k_τ, where k_r has already been defined above while k_b and k_τ are given by

$$k_b = \frac{B_\varphi}{B} k_z - \frac{B_z}{B} \frac{m}{r};$$

$$k_\tau \equiv k_\parallel = \frac{B_z}{B} k_z + \frac{B_\varphi}{B} \frac{m}{r}. \quad \text{(III.2)}$$

In the same way, it is convenient to replace the cylindrical field components E_r, E_φ, and E_z by the components E_b, E_r, and E_τ, where

$$E_b = \frac{B_\varphi}{B} E_z - \frac{B_z}{B} E_\varphi;$$

$$E_\tau \equiv E_\parallel = \frac{B_z}{B} E_z + \frac{B_\varphi}{B} E_\varphi, \quad \text{(III.3)}$$

similarly, the projections of the current j_r, j_φ, and j_z are replaced by j_b, j_r, and j_τ, where

$$\left.\begin{aligned} j_b &= \frac{B_\varphi}{B} j_z - \frac{B_z}{B} j_\varphi; \\ j_\tau \equiv j_\parallel &= \frac{B_z}{B} j_z + \frac{B_\varphi}{B} j_\varphi. \end{aligned}\right\} \quad \text{(III.4)}$$

Then, it is easy to show that the relation between j and E can be expressed by formulas analogous to Eqs. (1.1) and (1.2):

$$j_\alpha(r,\ t) = \sum_m \int d\omega\, dk_r\, dk_z \left\{\frac{\omega}{4\pi i}(\varepsilon_{\alpha\beta} - \delta_{\alpha\beta}) e^{i(\mathbf{kr} - \omega t)} E_\beta(\mathbf{k},\ \omega)\right\}; \quad \text{(III.5)}$$

$$\varepsilon_{\alpha\beta}(\mathbf{k},\ \omega,\ r) = \delta_{\alpha\beta} - i \sum_{i,\ e} \frac{4\pi e^2}{M\omega} \int d\mathbf{v}_0 v_\alpha(t) \int_{-\infty}^{t} \frac{\partial F_0}{\partial v_\gamma(t')} \times$$

$$\times \left\{\left(1 - \frac{\mathbf{kv}(t')}{\omega}\right)\delta_{\gamma\beta} + \frac{k_\gamma v_\beta(t')}{\omega}\right\} e^{i\left[\omega(t - t') - \int_{t'}^{t} \mathbf{kv}(t'')\, dt''\right]} dt'. \quad \text{(III.6)}$$

Here, the subscripts α and β assume the values b, r, and τ.

In order to obtain a complete description of the dielectric constant, we must integrate the particle equations of motion and take averages in Eq. (III.6) for some equilibrium distribution function F_0.

First we express $\partial F_0/\partial v_\gamma$ in terms of the integrals of the motion. The integrals of the motion in a helical magnetic field are the following:

$$\left.\begin{aligned} \varepsilon &= \frac{Mv^2}{2}; \\ P_z &= v_z - \int\limits_{r_0}^{r} \frac{eB_\varphi}{Mc}\, dr; \\ P_\varphi &= rv_\varphi + \int\limits_{r_0}^{r} \frac{eB_z}{Mc}\, r\, dr. \end{aligned}\right\} \tag{III.7}$$

Then,

$$\frac{\partial F_0}{\partial v_\gamma} = Mv_\gamma \frac{\partial F_0}{\partial \varepsilon} + \delta_{b\gamma}\left(-r\frac{B_z}{B}\frac{\partial F_0}{\partial P_\varphi} + \frac{B_\varphi}{B}\frac{\partial F_0}{\partial P_z}\right) + $$
$$+ \delta_{\tau\gamma}\left(r\frac{B_\varphi}{B}\frac{\partial F_0}{\partial P_\varphi} + \frac{B_z}{B}\frac{\partial F_0}{\partial P_z}\right). \tag{III.8}$$

Here, we have used the following relations between the unit vectors $\mathbf{b}$, $\boldsymbol{\tau}$, and $\mathbf{e}_z$, $\mathbf{e}_\varphi$:

$$\boldsymbol{\tau} = \frac{B_z}{B}\mathbf{e}_z + \frac{B_\varphi}{B}\mathbf{e}_\varphi;$$
$$\mathbf{b} = \frac{B_\varphi}{B}\mathbf{e}_z - \frac{B_z}{B}\mathbf{e}_\varphi. \tag{III.9}$$

Using a transformation such as that in (1.8) we now write Eq. (III.6) in the following form [cf. Eq. (1.1)]:

$$\varepsilon_{\alpha\beta}(\mathbf{k}, \omega, r) = \delta_{\alpha\beta} - i\sum_{i,\,e}\frac{4\pi e^2}{M\omega}\int d\mathbf{v}_0 v_\alpha(t) \times$$
$$\times \left\{\left(M\frac{\partial F_0}{\partial \varepsilon} - \frac{k_b}{\omega}\Phi_\perp - \frac{k_\tau}{\omega}\Phi_\parallel\right)\int\limits_{-\infty}^{t} v_\beta(t')\, A(t, t')\, dt' - \right.$$
$$\left. - \frac{i\delta_{b\beta}}{\omega}\Phi_\perp + \frac{i\delta_{b\tau}}{\omega}\Phi_\parallel\right\}, \tag{III.10}$$

where

$$\Phi_\perp = \frac{B_z}{B} r \frac{\partial F_0}{\partial P_\varphi} - \frac{B_\varphi}{B}\frac{\partial F_0}{\partial P_z}, \qquad \Phi_\parallel = \frac{B_z}{B}\frac{\partial F_0}{\partial P_z} + \frac{B_\varphi}{B} r \frac{\partial F_0}{\partial P_\varphi}.$$

The motion of the particles in the helical magnetic field with cylindrical symmetry can be described as follows:

$$\left.\begin{aligned}
&\frac{dr}{dt}=w\sin\alpha;\quad \frac{d\varphi}{dt}=\frac{1}{r}\left(u\frac{B_\varphi}{B}-w\cos\alpha\frac{B_z}{B}\right);\\
&\frac{dz}{dt}=u\frac{B_z}{B}+w\cos\alpha\frac{B_\varphi}{B};\\
&\frac{dw}{dt}=\omega_B(u_2\sin\alpha+2u_4\sin 2\alpha);\\
&\frac{du}{dt}=-\frac{w}{u}\frac{dw}{dt};\\
&\frac{d\alpha}{dt}=-\omega_B\left\{1+\frac{1}{w}[-\cos\alpha\,(u_2+2u_3)+2u_4(1-\cos 2\alpha)+u_5]\right\},
\end{aligned}\right\}\qquad (\text{III}.11)$$

where

$$u_2=\frac{u^2}{r\omega_B}\left(\frac{B_\varphi}{B}\right)^2;\quad u_3=\frac{w^2}{2r\omega_B}\left(\frac{B_z}{B}\right)^2;$$

$$u_4=\frac{uw}{4\omega_B}r\mu'\left(\frac{B_z}{B}\right)^2;\ u_5=\frac{2uw}{\omega_B}\mu\left(\frac{B_z}{B}\right)^2;\quad \mu=\frac{B_\varphi}{rB_z}.$$

Here, the primes denote differentiation with respect to r; w, α, and u are the velocity components in the cylindrical coordinate system.

Assuming that the magnetic field varies slightly in distances of the order of the Larmor radius, we can now find the solution of this system of equations. For the present purposes it is convenient to write the solution in the form

$$\left.\begin{aligned}
&w=w_0+u_2\cos\alpha\Big|_1^2+u_4\cos 2\alpha\Big|_1^2;\\
&u-u_0=-\frac{w_0}{u_0}(w-w_0);\\
&\alpha=\alpha_0-\omega_Bt-\frac{1}{w_0}(2u_1+u_2+2u_3)\sin\alpha\Big|_1^2-\frac{u_4}{w_0}\sin 2\alpha\Big|_1^2;\\
&r(t)=r+\frac{w_0}{\omega_B}\cos\alpha\Big|_1^2;\\
&m\varphi+k_zz=m\varphi_0+k_zz_0+tk_\tau u_0-\frac{k_\perp w_0}{\omega_B}\times\\
&\times\sin(\alpha-\psi)\Big|_1^2+t(u_1+u_2)k_b.
\end{aligned}\right\}\qquad (\text{III}.12)$$

Here we have introduced the notation

$$f(\alpha)\Big|_1^2 = f(\alpha_0 - \omega_B t') - f(\alpha_0 - \omega_B t);$$

$$u_1 = -\frac{w_0^2}{2\omega_B}\frac{B'}{B}; \quad k_\perp = \left(k_b^2 + k_r^2\right)^{1/2}; \quad \psi = \operatorname{arc\,tg}\frac{k_r}{k_b}.$$

In the exponential we neglect small terms of the form $k_\perp^2 \rho^2/L_B$, where L_B is the characteristic scale size for the variation of the magnetic field; as a result it is transformed into the same form as in § 1,except that the quantities k_x and k_z are replaced by k_b and k_τ. (In the following we will sometimes omit terms of the form $k_b u_1/\omega$ which are of the order of the ratio of the microscopic drift velocity to the phase velocity of the wave.) Terms containing the derivative of the magnetic field appear in the factor that multiplies the exponential (III.10). These are small compared with the others (of order ρ/L_B). The order of higher-order terms can be reduced still further when averages are taken over angles; the resultant contribution of the higher-order terms can then be of order $(\rho/L_B)(\omega_B/\omega) \sim v_{Ti}/L_B\omega$.

It is evident that these terms also must be taken into account at long wavelengths where $v_{Ti}/L_B\omega \sim (a/L_B)(\lambda_\perp/\rho_i) \gtrsim 1$, i.e., when $k_\perp \rho_i \lesssim a/L_B$, these terms can be neglected at shorter wavelengths. The factors that multiply the exponentials also contain terms such as ρ/r, which must be considered.

Introducing these considerations and choosing F_0 to be approximately a Maxwellian (so that the guiding-center function is a Maxwellian) we can characterize the dielectric constant of a plasma in a helical magnetic field for $\omega/k_b \gg u_1, u_2$ by expressions of the form

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}^{\mathrm{pl}} + \delta\varepsilon_{\alpha\beta},$$

$$\chi_\beta = \chi_\beta^{\mathrm{pl}} + \delta\chi_\beta, \tag{III.13}$$

where $\varepsilon_{\alpha\beta}^{\mathrm{pl}}$ and χ_β^{pl} are of the same form as $\varepsilon_{\alpha\beta}$ and χ_β in § 1 with the substitutions indicated by k_x, $k_z \to k_b$, k_τ, and $\partial/\partial y \to \partial^0/\partial r \equiv (dn_0/dr) \cdot (\partial/\partial n_0) + (dT/dr)(\partial/\partial T)$, while $\delta\varepsilon_{\alpha\beta}$ and $\delta\chi_\beta$ are important only for the longwave, low-frequency oscillations, in which case the nonvanishing elements are of the form

$$\delta\varepsilon_{\tau b} = -\sum_{i,\,e}\frac{4\pi e^2}{T\omega}\int dw_0^2 \cdot 2\pi du_0 \cdot f_0 \cdot u_0 \zeta_0 \cdot \left(u_1 + u_2 + u_3\right); \tag{III.14}$$

$$\delta\chi_b = \sum_{i,\,e} \frac{ie^2}{T} \int dw_0^2 \cdot 2\pi du_0 f_0 \zeta_0 \left(u_1 + u_2 + u_3\right) \tag{III.14}$$

(the other components of $\delta\,\varepsilon_{\alpha\beta}$ and χ_β are unimportant).

In obtaining (III.13) and (III.14), we have not taken account of the dependence of F_0 on the combinations P_φ and P_z which are analogous to V_z in § 1; thus we have assumed that the equilibrium current, which flows along the lines of force (since $B_\varphi \neq 0$), is not important for the oscillations with small k_τ considered in the present work.

We now use the expressions for $\varepsilon_{\alpha\beta}$ and χ_β obtained above to investigate slow waves in a helical magnetic field.

Using Maxwell's equation for the current in the r direction, we obtain the following relation, which is analogous to Eq. (2.5'), but which takes account of terms of order β:

$$E_r = -\frac{i}{k_b} E_b' - \frac{i}{rk_b}\left(\frac{B_z}{B}\right)^2 E_b +$$
$$+ \frac{\omega^2}{k_b^2 c^2} \sum_{i,\,e} \left[\hat{\varepsilon}_{rb} E_b + \hat{\varepsilon}_{rr} \left(-\frac{i}{k_b} E_b' \right) + \varepsilon_{r\tau} E_\tau \right] \equiv$$
$$\equiv \frac{-i}{k_b} E_b' + \hat{\Lambda}_b E_b + \hat{\Lambda}_\tau E_\tau . \tag{III.15}$$

Using the expression for $\varepsilon_{\alpha\beta}$ from § 1, we now find that the combination $\varepsilon_{rb}E_b + \varepsilon_{rr}\,[-(i/k_b)E_b']$ can be written in the form

$$\hat{\varepsilon}_{rb} E_b + \hat{\varepsilon}_{rr} \left(-\frac{i}{k_b} E_b' \right) =$$
$$= \sum_{i,\,e} \int e^{ik_r r}\, dk_r \frac{k_\perp}{k_b} \left(1 + \frac{k_b}{M\omega\omega_B} \frac{\partial^0}{\partial r} T \right) \times$$
$$\times \left(\varepsilon_{rb}^0 \cos\psi + \varepsilon_{bb}^0 \sin\psi \right) E_b\left(k_r\right). \tag{III.16}$$

Substituting E_r expressed in terms of E_b and E_τ in the other two electrodynamic equations used in § 1, we have

$$4\pi \sum_{i,\,e} \left\{ \left(\hat{\chi}_b E_b - \frac{i}{k_b} \hat{\chi}_r E_b' + \hat{\chi}_\tau E_\tau \right) + \hat{\chi}\left(\hat{\Lambda}_b E_b + \hat{\Lambda}_\tau E_\tau \right) \right\} = \operatorname{div} E; \tag{III.17}$$

$$\left(k_b^2 - \frac{\partial^2}{\partial r^2}\right) E_\tau - k_\tau k_b E_b + 2\frac{k_\tau'}{k_b} E_b' + \frac{k_\tau}{k_b} E_b'' = \frac{\omega^2}{c^2} \sum_{i,\,e} \times$$

$$\times \left\{ \widehat{\varepsilon}_{\tau b} E_b + \widehat{\varepsilon}_{\tau r} \left(-\frac{iE_b'}{k_b}\right) + \varepsilon_{\tau\tau} E_\tau + \widehat{\varepsilon}_{\tau r} \left(\widehat{\Lambda}_b E_b + \widehat{\Lambda}_\tau E_\tau\right) \right\}. \qquad \text{(III.18)}$$

It is evident that terms containing $\hat{\Lambda}$ must be retained only for long wavelengths ($k_\perp^2 \rho_i^2 << 1$), in which case the corrections $\delta\varepsilon_{\tau b}$ and $\delta_{\chi b}$ are important. In those cases in which these expressions are not important, (III.17) and (III.18) assume a simpler form, analogous to the system in (2.7):

$$\left.\begin{aligned} &\int e^{ik_r r} dk_r \left\{ \frac{1}{k_b} \left(k_\perp^2 \varepsilon_\perp - ik_r \tilde{\varepsilon}_\perp'\right) E_b(k_r) + \right.\\ &\left. + k_\tau \varepsilon_\parallel E_\tau(k_r) \right\} = 0;\\ &\int e^{ik_r r} dk_r \left\{ \left(k_\perp^2 - \frac{\omega^2}{c^2}\varepsilon_\parallel\right) E_\tau(k_r) - \right.\\ &\left. - \frac{k_\tau}{k_b}\left(k_\perp^2 - 2ik_r \frac{k_\tau'}{k_\tau}\right) E_b(k_r) \right\} = 0, \end{aligned}\right\} \qquad \text{(III.19)}$$

where the quantities $\varepsilon_\perp$, $\tilde{\varepsilon}_\perp$, and $\varepsilon_\parallel$ are of the same form as in § 2.

We now examine the range of validity of the simplified equations. For this purpose, it is assumed that $k_\perp^2 \rho_i^2 << 1$, and the Bessel functions in $\varepsilon_{\alpha\beta}$ are expanded. Equations (III.17) and (III.18) can then be written in the form

$$\left.\begin{aligned} &E_b \left\{ \varepsilon_\perp^* + \sum_{i,\,e} \frac{4\pi e^2}{M\omega\omega_B} \frac{1}{k_b R} \left(1 + \frac{k_b}{M\omega\omega_B} \frac{\partial^0}{\partial r} T\right) \times \right.\\ &\left. \times n_0 \left[ix\sqrt{\pi} W_0 + 2x^2 \left(1 + i\sqrt{\pi} x W_0\right)\right] \right\} -\\ &- \frac{1}{k_b^2} \frac{\partial}{\partial r} \left(E_b' \varepsilon_\perp\right) + E_\tau \frac{k_\tau}{k_b} \left\{\varepsilon_\parallel^* + (\text{terms of order } \beta)\right\} = 0; \end{aligned}\right. \qquad \text{(III.17')}$$

$$\begin{aligned} &\left(k_b^2 - \frac{\partial^2}{\partial r^2}\right) E_\tau - \frac{k_\tau}{k_b}\left(k_b^2 - \frac{\partial^2}{\partial r^2}\right) E_b + \frac{2k_\tau'}{k_\tau} E_b' =\\ &= \frac{\omega^2}{c^2} \left\{ \left(\varepsilon_\parallel^* + \frac{\omega^2}{c^2 k_b^2} \varepsilon_{\tau b} \varepsilon_{b\tau}\right) E_\tau + E_b \sum_{i,\,e} \frac{4\pi e^2}{M\omega\omega_B} \times \right.\\ &\left. \times \frac{1}{k_\tau R} \left(1 + \frac{k_b}{M\omega\omega_B} \frac{\partial^0}{\partial r} T\right) n_0 \left[2 + i\sqrt{\pi} x W_0 + 2x^2 \left(1 + i\sqrt{\pi} x W_0\right)\right] \right\}. \end{aligned} \qquad \text{(III.18')}$$

Here, $\varepsilon_{\perp}^{*}$ and $\varepsilon_{\parallel}^{*}$ are determined by equations of the form in (2.8) for $z \to 0$, i.e.,

$$\left.\begin{aligned} \varepsilon_{\perp}^{*} &= 1 + \frac{4\pi e^2 n_0}{M_i \omega_{Bi}^2}\left(1 - \frac{k_b v_0^i}{\omega}\right) \equiv 1 + \frac{c^2}{c_A^2}\left(1 - \frac{k_b v_0^i}{\omega}\right); \\ \varepsilon_{\parallel}^{*} &= 1 + \sum_{i,\, e} \frac{4\pi e^2}{T k_\tau^2}\left(1 + \frac{k_b T}{M \omega \omega_B}\frac{\partial^0}{\partial r}\right) n_0 \left(1 + i\sqrt{\pi} x W_0\right); \\ v_0^i &= -\frac{1}{M_i n_0 \omega_{Bi}}\frac{\partial p_i}{\partial r}; \quad x = \frac{\omega}{k_\tau v_T}. \end{aligned}\right\} \tag{III.20}$$

It is evident that the corrections considered here will sometimes have only an unimportant effect on the coefficients that multiply E_τ, changing the latter only by a quantity of order $\beta \ll 1$. However, the coefficients of E_b are more sensitive to these corrections: these are important in the longwave region, where

$$\varrho_i^2 k_\perp^2 \lesssim \frac{a}{R}. \tag{III.21}$$

We can now formulate the conditions for applicability of the simplified equations (III.19): the waves must not be so long that $k_\perp^2 \rho_i^2 \gg a/R$. At very long wavelengths ($\rho_i^2 k_\perp^2 \lesssim a/R$), (III.19) is replaced by an equation that follows from (III.17') and (III.18'):

$$\begin{aligned} & k_b^2\left(k_b^2 - \frac{\partial^2}{\partial r^2}\right)\frac{1}{\varepsilon_{\parallel}^{*} k_\tau}\Bigg\{\varepsilon_{\perp}^{*} - \sum_{i,\, e}\frac{4\pi e^2}{M\omega\omega_B}\frac{1}{R k_b} \times \\ & \times \left(1 + \frac{k_b}{M\omega\omega_B}\frac{\partial^0}{\partial r} T\right) n_0 \left[2x^2\left(1 + i\sqrt{\pi} x W_0\right) + i\sqrt{\pi} x W_0\right] - \\ & - \frac{1}{k_b^2}\frac{\partial}{\partial r}\left(\varepsilon_{\perp}^{*}\frac{\partial}{\partial r}\right)\Bigg\} E_b + k_\tau\left(k_b^2 - \frac{\partial^2}{\partial r^2}\right) E_b - 2k_\tau' E_b' = \\ & = \frac{k_b^2}{k_\tau}\left[\left(\frac{\omega^2}{c^2}\varepsilon_{\perp}^{*} + \frac{8\pi p'}{R B_0^2}\right) E_b - \frac{\omega^2}{c^2 k_b^2}\frac{\partial}{\partial r}\left(\varepsilon_{\perp}^{*} E_b'\right)\right]. \end{aligned} \tag{III.22}$$

The system of equations (III.19) and (III.22) has been investigated in §§ 5-7.

This is the situation for the starting equations for waves for which the phase velocity ω/k_b is appreciably greater than the microscopic drift velocity u_1 and u_2. If, however, $\omega/k_b \lesssim u_1, u_2$, it can be shown that $\varepsilon_{\alpha\beta}$ can be found by means of Eq. (1.14) in which $(x,y,z) \to (b,r,\tau)$; $\partial/\partial y \to \partial^0/\partial r$; ζ_n

$\rightarrow [\omega - n\omega_B - k_T u - k_b(u_1 + u_2)]^{-1}$; similarly, in Eqs. (1.23) and (1.24) we take q_α^*, q_r, q_T, and p_α as before, whereas q_b is replaced by $q_b + \delta q_b$, where

$$\delta q_b = (u_1 + u_2 + u_3) J_n. \tag{III.23}$$

Substituting the resulting value of $\varepsilon_{\alpha\beta}$ in Maxwell's equations, we obtain a system of equations for the electric field of the wave.

Appendix IV

Initial Equations for Certain Kinds of Oscillations

1. We require an equation that describes the electrostatic oscillations of a plasma for an arbitrary ratio of oscillation frequency to cyclotron frequency.

Writing $\mathbf{E} = -\nabla\varphi$ and using Eqs. (1.35),(1.37)-(1.42), we find the equation

$$\int e^{ik_y y} \left(k^2\varepsilon_0 - ik_y \frac{\partial^2}{\partial y \partial k_\perp^2} k^2\varepsilon_0 \right) \varphi(k_y)\, dk_y = 0, \tag{IV.1}$$

where ε_0 plays the role of a scalar dielectric constant:

$$\varepsilon_0 = 1 + \sum_{i,\,e} \frac{4\pi e^2}{k^2 T} \left(1 + \frac{k_x T}{m\omega\omega_B} \frac{\partial}{\partial y} \right) n_0 \times$$

$$\times \left(1 + i\sqrt{\pi} \frac{\omega}{k_z v_T} \sum_{n=-\infty}^{+\infty} I_n e^{-z} W_n \right). \tag{IV.2}$$

2. We require an equation that describes the ordinary wave that propagates across the magnetic field ω/ω_B.

When $k_z = 0$, the system of Maxwell's equations breaks up into two subsystems, one of which represents the required equation for the ordinary wave ($E_z \neq 0$, $E_x = E_y = 0$):

$$\int \left[k_\perp^2 - \frac{\omega^2}{c^2} \varepsilon_{zz}(k_y) \right] e^{ik_y y} E_z(k_y)\, dy = 0. \tag{IV.3}$$

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THEORY OF A WEAKLY TURBULENT PLASMA

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§ 1. Plasmon – Particle Interactions

A characteristic feature of a plasma is the existence of a spectrum of collective oscillations or plasma waves (plasmons). The frequency and velocity of propagation of these waves are determined by the wave vector and by the gross parameters of the plasma such as the density, the mean velocity spread, the magnetic field, etc., and this situation is a reflection of the fact that all of the particles in the plasma are involved in the plasma oscillations. The situation is different, however, when one examines the damping (or growth) of the oscillations. Damping (growth) is determined by the "fine details" of the particle distribution in phase space, for example, by the derivative of the velocity distribution function; this situation reflects the specific role played by resonance particles (i.e., particles for which the following condition is satisfied: $\omega_{\mathbf{k}} - \mathbf{kv} = n\omega_H$; $n = 0, 1, 2, \ldots$; here, $\omega_{\mathbf{k}}$ and $\mathbf{k}$ are the frequency and wave vector that characterize the wave, $\mathbf{v}$ is the particle velocity, and $\omega_H = eH/mc$). These particles are capable of exchanging energy with the waves and can thus amplify or damp it.

The important role played by resonance particles in the damping of plasma waves is evident from the fact that the damping of a wave characterized by frequency ω and wave number k in a plasma in thermodynamic equilibrium is found to be proportional to the derivative of the electron distribution function $f'(v)$ taken at the point $v = \omega_k/k$; this result was first established by Landau [1] through the use of the self-consistent field equations [2,3].

Many later authors have verified this result, i.e., that the damping rate (or growth rate) for waves in a low-density plasma is proportional to the derivative of the distribution function for the resonance particles (cf. [4]).

A detailed physical analysis of the interaction of the plasma particles with plasma waves (as well as problems touching on the propagation of plasma waves) has been given by Bohm, Gross, and Pines [5], who have indicated the importance of resonance particles for a given wave mode. The importance of resonance particles in both absorption and growth of plasma waves may be regarded as well established.

We shall also be concerned with the effect of plasma waves on transport processes in a low-density plasma. This question arises in connection with the problem of treating the "Coulomb logarithm" in the collision term that appears in the kinetic equation for a low-density plasma [3, 6]. Davydov [7] has estimated the contribution of plasma waves to the kinetic coefficients for a plasma close to a state of thermodynamic equilibrium and has shown that taking account of the emission and absorption of plasma waves by particles (in addition to the usual binary collisions between particles) can modify the value of the Coulomb logarithm substantially.*

It is clear, however, that treating these two processes separately in a plasma close to thermodynamic equilibrium is not consistent with the available accuracy, since the exact values of the quantities that appear in the Coulomb logarithm remain unknown when this approach is used. In order to make a more consistent formal calculation [12], the kinetic coefficients associated with collisions between particles and with the emission and absorption of waves by particles cannot be treated separately – they must be treated together.

The situation is different, however, if one considers a weakly turbulent plasma in which the energy density contained in the waves (plasma oscillations) is small compared with the thermal energy density, but appreciably greater than the energy density associated with the equilibrium thermodynamic plasma noise (this situation is frequently realized in low-density plasmas, cf. below). Under these conditions, one need not necessarily take account of collisions between particles at the outset; the plasma can be treated by means of the self-consistent field equations. It turns out that these equations can be replaced by the simpler equations of the quasi-linear theory [4, 13-15]: an equation for the rate of energy growth (damping) of the plasma waves, and a diffusion-like equation for the distribution function of the resonance particles in the plasma (the diffusion coefficient being proportional to the energy density of the waves in the turbulent plasma).

It must be emphasized that this derivation of the quasi-linear equations from the self-consistent field equations can only be carried through when the

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*The ideas in [7] have been developed further by Galitskii and also by Romanov and Filippov [9], who have derived a system of kinetic equations for a system of electrons and plasma waves by analogy with the kinetic equations for electrons and phonons in a solid. Similar equations have been studied by Silin and Klimontovich [10] and by Pines and Schriffer [11].

resonance particles comprise a small group which does not have an important effect on the "gross" characteristics of the plasma (density, temperature, etc.).

The quasi-linear theory, which is described briefly below, describes the dynamics of the interaction between the resonance particles and the waves. A consistent derivation of the equations and an analysis of these processes can be carried out only when the energy contained in the collective degrees of freedom, i.e., the plasma oscillations, is much smaller than the energy associated with the random motion of all the particles (but, at the same time, greater than the energy associated with the thermal noise in the collective degrees of freedom).

The essence of the quasi-linear method lies in separating the distribution function for the resonance particles into a rapidly varying part and a slowly varying part, and then taking account of the average quadratic effect of the rapidly varying part on the slowly varying part (the method is similar to the well-known method of Van der Pol used in nonlinear mechanics). When this is done, it turns out that the behavior of the slow part of the distribution function is described by a diffusion equation in phase space and that the rate of growth (or damping) of the fast oscillations (plasma waves) is determined by the formulas of the linear theory, with the nonoscillating part of the distribution function varying slowly in time.

In a homogeneous low-density plasma in which collisions between particles are not important, there is a large degree of arbitrariness in the choice of the stationary velocity distribution function. The quasi-linear theory indicates the existence of well-defined states to which the unstable plasma evolves as a result of the development of perturbations.

These states are characterized by the fact that in certain regions of phase space the distribution function f becomes a constant (i.e., a plateau appears on the function f); in the corresponding regions of wave-number space the plasma oscillations are present in the form of noise at a level appreciably greater than the thermal level.

§ 2. Basic Equations of the Quasi-Linear Theory

We shall first derive the basic equations for the quasi-linear theory assuming a fully ionized, low-density plasma. We assume that the distribution function f_α for particles of species α with charge e_α and mass m_α obeys the self-consistent field equations

$$\frac{\partial f_\alpha}{\partial t} + v\frac{\partial f_\alpha}{\partial x} + \frac{e_\alpha}{m_\alpha}\left(E + \frac{v}{c} \times H\right)\frac{\partial f_\alpha}{\partial v} = 0, \tag{1}$$

where the self-consistent fields E and H are determined by the distribution of plasma particles:

$$\left.\begin{aligned} &\frac{\partial E}{\partial t}=4\pi\sum_{\alpha}e_{\alpha}\int f_{\alpha}\,dv;\ \nabla\times E=-c^{-1}\,\partial H/\partial t;\\ &\nabla\times H=4\pi c^{-1}\sum_{\alpha}e_{\alpha}\int vf_{\alpha}\,dv+c^{-1}\,\partial E/\partial t.\end{aligned}\right\} \tag{2}$$

The system of self-consistent field equations (1)-(2) yields a proper description of the plasma if the plasma is almost ideal,* i.e., if the average amplitude of the Coulomb scattering e^2/T (where T is the mean kinetic energy of a particle) is much smaller than the mean distance between particles $r \approx n^{-1/3}$ (where n is the plasma density). Under these conditions, the number of particles in a Debye sphere $N_D \approx n(T/ne^2)^{3/2}$ is appreciably greater than unity and the quantity N_D^{-1} is a small parameter in terms of which one usually expands the exact equations of motion for the plasma particles; in the first approximation this procedure leads to the system (1)-(2) and in higher approximations to the appearance of a collision term on the right side of Eq. (1) [3]. However, if one is interested in the dynamics of such a plasma,† it turns out that the parameter N_D^{-1} is not the only small parameter in a low-density plasma. A low-density plasma can exhibit plasma oscillations of various kinds. In the absence of a magnetic field, these are the electron plasma oscillations and the ion-acoustic oscillations; the frequencies and propagation velocities of these waves are determined by the gross properties of the plasma (the density, mean velocity spread, etc.). The damping (or growth rate) for these waves depends on the fine details of the distribution function in phase space. The plasma particles experience the random effect of the electric fields associated with many waves and diffuse in phase space, and under these conditions changes occur in precisely those details of the distribution function which are responsible for the wave damping. On the other hand, the gross properties of this system are not changed in these processes; the wave energy

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*In addition we assume that quantum degeneracy effects are not important (this condition imposes a further limitation on the plasma density: degeneracy effects can be neglected if $\lambda \ll r$, where $\lambda \approx \hbar/mv$ is the mean wavelength associated with the particle).

†This is in contrast with the case of an ideal plasma in thermodynamic equilibrium in which the ratio of the mean scattering amplitude for binary collisions to the mean distance between particles is the only small parameter, and in terms of which a consistent expansion procedure can be used to find the equation of state [17].

is so small that there is no appreciable effect on the mean plasma density, on the various moments of the distribution function, etc.

The particle diffusion rate in velocity space as a result of the waves is proportional to the energy density in the waves ε. If the ratio ε/nT is appreciably greater than N_D^{-1}, the effects due to "wave" diffusion are greater than those due to collisions between particles (which also cause diffusion in velocity space); in this case collisions between particles can be neglected in a first approximation.

The ratio of the energy density in the nonequilibrium plasma oscillations to the kinetic energy density ε/nT represents the second small parameter in the dynamics of a low-density plasma. In what follows we assume that

$$1 \gg \frac{\varepsilon}{nT} \gg N_D^{-1},$$

implying that the energy density in the collective degrees of freedom of the plasma-wave exceeds appreciably the density in the Coulomb interactions: $\varepsilon \gg nT/N_D$.* At the same time, ε is much smaller than the thermal energy nT.

The equations of the quasi-linear theory are obtained by expanding the self-consistent field equations (1)-(2) in terms of the small parameter ε/nT, and taking account of terms that are quadratic in the amplitude of the plasma oscillations.

To be definite, let us consider the case of longitudinal plasma oscillations in a plasma in the absence of a magnetic field. The point of departure is the self-consistent field equation for the electron distribution function

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\frac{eE}{m}\frac{\partial f}{\partial v} \tag{3}$$

and the equation

$$4\pi en\int vf\,dv = -\partial E/\partial t\,. \tag{4}$$

(Here, n is the equilibrium plasma density, so that $\int f_0 dv = 1$ at equilibrium.) As is well known from the linear theory, the harmonic (in time) solutions of (3) and (4) describe plasma oscillations; the damping rate for these oscillations in an equilibrium plasma is small compared to the frequency if the

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*To within an accuracy of order unity the quantity nT/N_D is equal to the familiar Debye correction to the free energy of a Coulomb particle system; the same quantity characterizes the energy density of the thermal plasma waves.

wavelength is much greater than the Debye length. We can write the linearized equations (3)-(4) in terms of Fourier space components

$$f = f_0 + \sum_k G_k e^{ikx};$$

$$E = \sum_k E_k e^{ikx},$$

and find [neglecting the $v(\partial f/\partial x)$ term at long wavelengths]

$$\dot{G}_k = -\frac{e}{m} E_k \frac{\partial f_0}{\partial v};$$

$$\dot{E}_k = -4\pi en \int vG_k\, dv,$$

whence it follows that the longwave component of the field satisfies an oscillator equation

$$\ddot{E}_k = -\omega_p^2 E_k \qquad (5)$$

characterized by the plasma frequency $\omega_p = (4\pi ne^2/m)^{\frac{1}{2}}$. All of the plasma particles participate in these oscillations and the kinetic energy associated with the motion of all the particles in this wave is equal to the electrostatic energy (virial theorem):

$$\frac{1}{2} nm \left| \int vG_k\, dv \right|^2 = \frac{E_k^2}{8\pi},$$

so that the total energy associated with the oscillations is $E_k^2/4\pi$. Oscillations for which the wave number k is large are not damped if there are no particles characterized by velocities $v \gtrsim \omega_p/k$ [under these conditions, we can neglect the term $v\partial f/\partial x \sim kvG_k$ as compared with $\partial f/\partial t \sim \omega_p G_k$, as has been done in deriving Eq. (5)].

On the other hand, if the plasma does contain electrons whose velocity coincides with the phase velocity of any of the plasma waves ω_p/k, it then becomes possible to have an energy exchange between the waves and these "resonance" particles. We shall assume that the number of resonance particles is small and neglect any effect they may have on the dispersion properties of the plasma (the oscillation frequency, the phase velocity, and the group velocity – but not the damping!). In the case being considered here, this means that in the presence of the resonance electrons we still take the frequency of the plasma oscillations to be ω_p.

The interaction between the plasma oscillations and the resonance particles leads to two effects: first, there is a change in the mean energy of the plasma oscillators $E_k^2/4\pi$; second, there is a simultaneous change in the distribution of resonance electrons in velocity space. To derive equations that describe these processes we proceed as follows.

We write the distribution function for the resonance particles F in the form of a sum of a rapidly oscillating term $\sum_k F_k e^{ikx}$ and a slowly varying function $\overline{F}$; in this case, the electric field $E = \sum_k E_k e^{ikx}$ is in the form of a product of rapidly oscillating space-time functions multiplied into a slowly varying amplitude (we assume the mean field to be zero). The average taken over a time period much greater than the period of the plasma oscillations leads to zero values for the oscillating parts of the distribution function and the electric field:

$$\langle F_k \rangle = \langle E \rangle = 0,$$

so that $\overline{F}$ represents the mean value of the total distribution function for the resonance electrons F. In the kinetic equation for the resonance electrons,

$$\frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} = -\frac{eE}{m} \frac{\partial F}{\partial v} \tag{6}$$

we now take an average over space:

$$\frac{\partial \overline{F}}{\partial t} = -\overline{\frac{eE}{m} \frac{\partial F}{\partial v}} = -\frac{\partial}{\partial v} \frac{e}{m} \sum_k E_k^+ F_k. \tag{7a}$$

Subtracting Eq. (7a) from Eq. (6), and neglecting the difference* $E\partial F/\partial v - \overline{E\partial F/\partial v}$, we find

$$\dot{F}_k + ikvF_k = -\frac{eE_k}{m} \frac{\partial \overline{F}}{\partial v}. \tag{7b}$$

From the kinetic equations for the nonresonance particles, we have

$$\dot{G}_k = -\frac{eE_k}{m} \frac{\partial f_0}{\partial v},$$

*In Eq. (7b) this difference would yield terms that are nonlinear in E, these terms representing the interaction of plasma waves with themselves. This effect can be neglected for weak excitation of the plasma.

so that the time derivative of the current density produced by all the plasma particles (except for the resonance particles) is

$$ne \int vG_k = \frac{ne^2}{m} E_k.$$

Making use of this relation and using the equation for the total current (4), we have

$$\ddot{E}_k + \omega_p^2 E_k = 4\pi en \int v\dot{F}_k\, dv. \tag{7c}$$

We now integrate Eq. (7b)

$$F_k(t) = F_k(0)\, e^{-ikvt} - \frac{e}{m} \int_0^t dt' E_k(t') \frac{\partial \bar{F}(t')}{\partial v} e^{ikv(t-t')}$$

and substitute this expression in Eq. (7c); multiplying both sides of the resulting relation by $\dot{E}_k^+$ and adding the complex conjugate, we have

$$\dot{E}_k^+ (\ddot{E}_k + \omega_p^2 E_\kappa) + c.\,c. = \frac{d}{dt} (|\dot{E}_k|^2 + \omega_p^2 |E_k^2|) =$$

$$= 4\pi en\dot{E}_k^+ \int v\, dv \left\{ -ikvF_k(0)\, e^{-ikvt} - \frac{eE_k(t)}{m} \frac{\partial \bar{F}(t)}{\partial v} + \right.$$

$$\left. + \frac{e}{m} ikv \int_0^t dt' E_k(t') \frac{\partial \bar{F}(t')}{\partial v} e^{-ikv(t-t')} \right\} + c.\,c.$$

Substituting the relation $E_k(t) = \sqrt{\varepsilon_k(t)}\, e^{-i\omega_p t}$ in this expression, and taking the slowly varying functions ε_k and $\partial \bar{F}/\partial v$ outside the integral over t', we obtain an equation for the square of the wave amplitude:

$$\frac{d}{dt} 2\omega_p^2 \varepsilon_k = -4\pi nei\omega_p \sqrt{\varepsilon_k} \int v\, dv \int \left\{ -ikvF_k(0)\, e^{-ikvt} + \right.$$

$$\left. + ikv \frac{e}{m} \sqrt{\varepsilon_k} \frac{\partial \bar{F}}{\partial v} \int_0^t e^{i(\omega_p - kv)(t-t')} dt' \right\} + c.\,c.$$

Now, going to the limit $t \to \infty$, we find that the first term in the curly brackets vanishes and that the second, in accordance with the formula

$$\lim_{t \to \infty} \left(\int_0^t e^{i\alpha(t-t')} dt' + c.\,c. \right) = 2\pi\delta(\alpha),$$

yields

$$\frac{d\varepsilon_k}{dt} = \varepsilon_k \pi \omega_p^2 \int v \frac{\partial \bar{F}}{\partial v} \delta(\omega_p - kv)\, dv,$$

i.e., the growth rate for the energy in a given harmonic

$$\frac{d\varepsilon_k}{dt} = 2\gamma_k \varepsilon_k, \tag{8}$$

where

$$\gamma_k = \frac{\pi}{2} \frac{\omega_p^3}{k^2} \int k \frac{\partial \bar{F}}{\partial v} \delta(\omega_p - kv)\, dv. \tag{8a}$$

Thus, in the quasi-linear theory the rate of growth (or damping) of the energy of a given Fourier harmonic is determined by the formula of the linear theory except that the unperturbed "linear" distribution function in the expression for the growth rate (damping) is replaced by the averaged function $\bar{F}$.

The second equation in the quasi-linear theory is obtained by substituting the following expression in Eq. (7a):

$$F_k(t) = F_k(0)\, e^{-ikvt} - \frac{e}{m} \int_0^t dt' E_k(t') \frac{\partial \bar{F}(t')}{\partial v} e^{-ikv(t-t')}$$

and adding the resulting expression to its complex conjugate:

$$\frac{\partial \bar{F}}{\partial t} = -\frac{1}{2} \frac{\partial}{\partial v} \frac{e}{m} \sum_k E_k^+ \left\{ F_k(0)\, e^{-ikvt} - \right.$$

$$\left. - \frac{e}{m} \int_0^t dt' E_k(t') \frac{\partial \bar{F}(t')}{\partial v} e^{-ikv(t-t')} \right\} + c.c.$$

Replacing $E_k(t)$ by $\sqrt{\varepsilon_k(t)}\, e^{-i\omega_k t}$ as in the above, we now obtain the following equation for the averaged function describing the distribution of resonance particles $\bar{F}$:

$$\frac{\partial \bar{F}}{\partial t} = \frac{\partial}{\partial v} \frac{e^2}{m^2} \sum_k \varepsilon_k \pi \delta(\omega_p - kv) \frac{\partial \bar{F}}{\partial v}.$$

Thus, the second equation is of the form [4, 14, 15]

$$\frac{\partial \overline{F}}{\partial t} = \frac{\partial}{\partial v_\alpha} D_{\alpha\beta} \frac{\partial \overline{F}}{\partial v_\beta}, \tag{9}$$

where

$$D_{\alpha\beta} = \pi \frac{e^2}{m^2} \sum \varepsilon_k \delta(\omega_k - kv) \frac{k_\alpha k_\beta}{k^2}. \tag{9a}$$

Equations (8) and (9) represent a closed* system of equations of the quasi-linear theory for the spectral density ε_k and the averaged distribution function $\overline{F}(v)$.† Equation (9) is in the form of a diffusion equation in which the diffusion coefficient D is proportional [as follows from Eq. (9a)] to the energy density of the plasma waves which, in turn, depends on the distribution function.

The system of quasi-linear equations (8)-(9) which has been obtained from the self-consistent field equations (3)-(4) obviously contains less information than the original equations (for instance, we can only find the amplitudes $\sqrt{\varepsilon_k}$ and not the phases of the fast oscillations). Furthermore, the region of applicability of the quasi-linear theory is much narrower than that of the original system. However, these shortcomings are balanced by the relative simplicity of the equations of the quasi-linear theory.

The system of quasi-linear equations (8)-(9) which describes the interaction of resonance particles with plasma waves exhibits an energy integral. Let us consider the time derivative of the total energy of the system of resonance particles and waves Q. The quantity Q is made up of the kinetic energy of the resonance electrons, the electrostatic energy in the plasma waves $\sum_k \frac{\varepsilon_k}{8\pi}$ and the kinetic energy of all the plasma electrons that participate in the oscillations; by the virial theorem the latter energy is equal to the electrostatic field energy. Thus,

$$\frac{dQ}{dt} = \frac{d}{dt}\left(n \int \frac{mv^2}{2} F\, dv + \sum_k \frac{\varepsilon_k}{4\pi}\right).$$

*The dependence of ω_k on k is determined by the gross parameters of the plasma and is assumed to be known.

†Hereinafter the averaging symbol will be omitted over the distribution function.

Substituting the value of $\partial F/\partial t$ from Eq. (9) and the value of $d\varepsilon/dt$ from Eq. (8) and integrating by parts, we find

$$\frac{dQ}{dt} = 0,$$

i.e., the total energy of the plasmon–particle system is conserved.

In order to understand the physical meaning of the quasi-linear theory, and in order to generalize the equations that have been obtained [(8)-(9)], we view a plasma with highly excited collective degrees of freedom as a system of two gases: a particle gas (fermions), which we will assume to be nondegenerate, and a plasmon gas (bosons).

We now consider the equation showing the balance in the number of particles and waves in phase space assuming that the system is homogeneous and that the condition $N_D^{-1} \ll \varepsilon/nT \ll 1$ is satisfied (the density of the wave gas is considerably greater than the thermodynamic equilibrium value). Since the particle–particle and wave–wave interactions are unimportant,* to a first approximation we need only consider the interaction between particles and waves.

The basic process which we wish to consider is the first-order radiation (Fig. 1a) or absorption (Fig. 1b) of a plasmon q by a particle k.

The process denoted *a* is the Cerenkov emission of a plasmon by an electron moving in the plasma with a velocity v which exceeds the phase velocity of the plasma wave ω_k/k:

$$v = \frac{\omega_k}{k}\,\frac{1}{\cos\vartheta};$$

the inverse process b is the Cerenkov absorption of a plasmon by a particle.

In the case we are considering, in which the density of waves in phase space N_q is large, the matrix elements for these processes are proportional to

*It should be understood that the "waves" actually represent collective oscillations in which all of the plasma particles participate; however, here, by "particles" we mean only a small group of "resonance" particles which occupy a small volume in phase space, but which exhibit a strong interaction with the "waves."

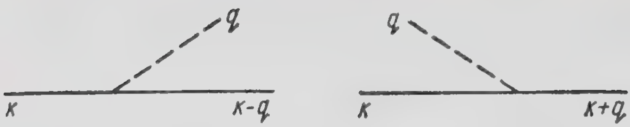

Fig. 1

$\sqrt{N_q}$ so that the probability for both processes W is the same, being given by

$$W(k,\ q) = N_q w_{k,\ k+q} \delta(\varepsilon_i - \varepsilon_f);$$

$$w_{k,\ k+q} = w_{k+q,\ k}.$$

As a result of the emission or absorption of a wave, the particle changes its momentum and is transferred to another point in phase space.

The change in the number of particles at point k in phase space is made up of loss terms due to the absorption of plasmons

$$-\sum_q F_k N_q w_{k,\ k+q} \delta(\varepsilon_k + \hbar\omega_q - \varepsilon_{k+q})$$

and due to the emission of plasmons

$$-\sum_q F_k N_q w_{k,\ k-q} \delta(\varepsilon_k - \hbar\omega_q - \varepsilon_{k-q})$$

and of gain terms due to the absorption of plasmons

$$+\sum_q F_{k-q} N_q w_{k-q,\ k} \delta(\varepsilon_{k-q} + \hbar\omega_q - \varepsilon_k)$$

and the emission of plasmons

$$+\sum_q F_{k+q} N_q w_{k+q,\ k} \delta(\varepsilon_{k+q} - \hbar\omega_q - \varepsilon_k).$$

Here, F_k is the particle distribution function in phase space; ε_k is the kinetic energy of a particle with wave vector k; $\hbar\omega_q$ is the energy of the wave denoted by q.

Summing the contributions of the various processes, we obtain the following equation for the particle distribution function F:

$$\partial F_k / \partial t = \sum_q N_q (\Psi_{k+q,\ q} - \Psi_{k,\ q}), \tag{9b}$$

where

$$\Psi_{k,\,q} = (F_k - F_{k-q})\, w_{k,\,k-q}\, \delta(\varepsilon_k - \varepsilon_{k-q} - \hbar\omega_q).$$

An equation for the wave distribution function N_q can be obtained in similar fashion. The change in N_q occurs as a result of the same processes of emission and absorption of plasmons by particles, so that in the spatially homogeneous case being considered here,

$$\partial N_q/\partial t = N_q \sum_k \Psi^*_{k+q,\,q}. \tag{8b}$$

In order to obtain the equations for a low-density plasma [(8)-(9)] from Eqs. (8b) and (9b),* we take account of the fact that the relative change in the momentum of a particle in the emission (absorption) of a wave in a low-density plasma is always small ($q/k \to 0$) and make use of the following formulas for the probability w and the number of photons associated with the plasma oscillations N_q:

$$w_{k,\,k-q} = 4\pi^2 e^2 \omega_0/q^2; \quad N_q = |E_q^2|/4\pi\hbar\omega_0$$

where ω_0 is the plasma frequency. Under these conditions, Eq. (9b) coincides with Eq. (9) and Eq. (8b) becomes the formula for the growth rate (8).

In practice it is easier to obtain the kinetic equation for the plasmon distribution function (for the spectral density of the noise) by solving the linearized kinetic equation with the self-consistent field and determining the growth rate (damping rate) γ; in this case, the quantity γ is a functional of the averaged resonance particle distribution function F against the background of which the small oscillations occur. Thus, in place of Eq. (8b) we have

$$\frac{1}{|E_k^2|}\,\frac{d|E_k^2|}{dt} = 2\gamma\{F\}. \tag{8c}$$

Equations (8b) and (9b) describe the interaction between the plasmons and particles in a weakly turbulent plasma.

Problem I. Derive Eqs. (8b) and (9b) from the equations for the density matrix of the plasma [20].

Solution. As in the case of the classical plasma, we start from the equation with the self-consistent field φ; in this case we obtain the following

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*Equations (8b) and (9b) can be derived from the equations for the density matrix of the plasma (cf. Problem I).

expression for the density matrix in the Wigner representation:

$$f_{xp} = \sum_{\xi} e^{-i\xi p} \varrho \left(x - \frac{\xi}{2}, x + \frac{\xi}{2} \right),$$

where $\rho(y, z)$ satisfies the equation *

$$i\partial\varrho(y, z)/\partial t = [-\Delta_y/2 + \Delta_z/2 + e\varphi(y) - e\varphi(z)]\,\varrho(y, z) =$$
$$= \left[\nabla_x\nabla_\xi + e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right)\right]\varrho\left(x + \frac{\xi}{2}, x - \frac{\xi}{2}\right),$$

where we have

$$\frac{\partial f_{xp}}{\partial t} = \frac{1}{i}\sum_{\xi} e^{-i\xi p}\left[\nabla_x\nabla_\xi + e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right)\right]\sum_q e^{i\xi q} f_{xq} =$$
$$= -p\frac{\partial f_{xp}}{\partial x} + \frac{1}{i}\sum_{\xi q}\left[e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right)\right] e^{i\xi(q-p)} f_{xq}. \qquad \text{(A)}$$

Equation (A), together with Poisson's† equation

$$\Delta_x\varphi = 4\pi ne\left(\sum_p f_{xp} - 1\right) \qquad \text{(B)}$$

in the quasi-linear plasma theory are replaced by a system of equations for the mean value of the quantum distribution function $f_p^0 = \langle f_{xp} \rangle$ and for the time oscillatory deviation of the distribution function f_{xp} from its mean value (this deviation is assumed to be small).

In Eqs. (A) and (B) we isolate the oscillatory terms and convert to Fourier space components

$$\varphi(x) - \sum_k \varphi_k e^{ikx}; \quad f_{xp} - \langle f_{xp} \rangle = \sum_k f^1_{kp} e^{ikx},$$

* We take $\hbar = m = 1$ and, for simplicity, consider a quadratically isotropic spectrum $\varepsilon_p = p^2/2$.

† For reasons of simplicity we consider the case of longitudinal oscillations of an electron plasma with a positive space charge background.

thereby obtaining, for the spatially homogeneous case ($\nabla_x f^0 = 0$)

$$\dot{f}^1_{kp} + ikpf^1_{kp} = e\varphi_k \frac{\overset{0}{f}_{p+\frac{k}{2}} - \overset{0}{f}_{p-\frac{k}{2}}}{i}; \quad \varphi_k = -4\pi nek^2 \sum_p f^1_{kp}. \tag{C}$$

On the other hand, carrying out an averaging over x in Eq. (A), we find

$$\frac{\partial f^0_p}{\partial t} = i \sum_k e\varphi^+_k \left[\overset{1}{f}_{k,\, p+\frac{k}{2}} - \overset{1}{f}_{k,\, p-\frac{k}{2}}\right]. \tag{D}$$

Substituting the solution of the ordinary differential equation (C) in (D), and introducing the notation

$$w_{p,\,p'} = 4\pi^2 e^2 \frac{\omega_{p-p'}}{|p-p'|^2}, \quad N_k = \frac{k^2 |\varphi^2_k|}{4\pi\omega_k}, \quad F_p = f^0_p,$$

we have

$$\frac{\partial F_p}{\partial t} = \sum_k w_{p,\,p+k} N_k \left\{(F_{p-k} - F_p)\,\delta\left(\omega_k - k\left[p + \frac{k}{2}\right]\right) - \right.$$

$$\left. - (F_p - F_{p-k})\,\delta\left(\omega_k - k\left[p - \frac{k}{2}\right]\right)\right\}.$$

Similarly, we find that N_k obeys the equation

$$\partial N_k/\partial t = N_k \sum_p w_{p+\frac{k}{2},\, p-\frac{k}{2}} \left(F_{p+\frac{k}{2}} - F_{p-\frac{k}{2}}\delta\,(\omega_k - kp\right).$$

§ 3. Relaxation of Plasma Oscillations

We now wish to consider the damping of plasma oscillations within the framework of the quasi-linear theory. The linear theory predicts an exponential damping in a time of order $1/\gamma$. But the damping rate for this case γ is determined in the linear theory by a thermodynamic equilibrium (Maxwellian) distribution function, since it is assumed that the plasma is in thermodynamic equilibrium when the oscillations are excited. Thus, the infinitesimally small perturbation produced in the plasma decays gradually in accordance with the linear theory, and the system returns to the thermodynamic equilibrium state.

However, if the energy of the initial plasma oscillations is appreciably greater than the energy of the equilibrium thermal noise, the process by which the oscillations are damped is somewhat different. As long as the wave energy density ε is much larger than the thermal energy density nT/H_D, particle collisions are unimportant and the wave diffusion process equalizes the distribution function in the region of phase space that corresponds to resonance particle velocities. As a result of this equalization process particles from low-velocity regions are transferred to regions of higher velocities and the damping of the plasma oscillations is accompanied by an increase in the kinetic energy of the particles (the quantity γ is negative); the total energy of the wave-particle system is conserved in the process. This quasi-linear absorption process is terminated when γ becomes zero. Under these conditions, the energy of the plasma oscillations is finite and appreciably greater than the level of the thermal noise. At this point the oscillations are not longer damped, since $\gamma = 0$ and the distribution function remains unchanged. Subsequently, because of particle collisions, there is a slow diffusion in velocity space which eventually leads to the establishment of a thermodynamic equilibrium (Maxwellian) distribution and the reduction of the oscillations to the thermal noise level; this second stage requires a time interval much longer than the first. In the present section we only consider the first stage, the quasi-linear relaxation of the oscillations.

Let us consider the simplest case of electron plasma oscillations. We assume that at an initial time $t = 0$ in a plasma in thermodynamic equilibrium (the electron velocity distribution is Maxwellian) uniformly in all space over some range of wave vectors k there are produced plasma waves with a spectral energy density $\varepsilon_k(0)$ which is appreciably greater than the thermal noise. All the vectors k are assumed to be parallel to each other, i.e., we are considering a one-dimensional problem. In this case the equations are simplified considerably and an analytic solution can be found. For the one-dimensional spectrum and long wavelengths the velocity of the resonance particles is related uniquely to the wave vector by the simple expression

$$v = \omega_0/k, \tag{10}$$

where ω_0 is the plasma frequency. The coefficient for wave diffusion is then

$$D(v) = \frac{e^2}{2m^2}\frac{|E_k^2|}{v}, \tag{11}$$

while the damping (growth) is given by

$$\gamma = \frac{\pi}{2}\frac{\omega_0^3}{k^2}\frac{\partial f}{\partial v}, \tag{12}$$

where f is the normalized ($\int f\,dv = 1$) average electron distribution function for the velocity component in the direction of the wave vector k.

Thus the system of quasi-linear equations assumes the form

$$\partial\varepsilon/\partial t = A\varepsilon\partial f/\partial v; \tag{13}$$

$$\partial f/\partial t = \frac{\partial}{\partial v}\left(B\varepsilon\frac{\partial f}{\partial v}\right), \tag{14}$$

where $\varepsilon = E_k^2/8\pi$ and f is a function of time t and velocity $v = \omega_0/k$, while the coefficients A and B depend on the velocity but not time:

$$A = \pi\omega_0^2 v^2; \quad B = \omega_0^2/nmv. \tag{14a}$$

The initial conditions for Eqs. (13) and (14) are the following: when $t = 0$ the quantity $\varepsilon = \varepsilon_0(0, v)$, $f = f_M(v)$; here the spectral density $\varepsilon_0(0, v)$ is nonzero in a finite range of velocities $v_1 < v < v_2$, while f_M is the Maxwellian distribution function.

Under the effect of wave diffusion in the region $v_1 < v < v_2$ the negative derivative of the distribution function is increased, i.e., the slope of the distribution function becomes steeper. At the same time, the waves are damped and the diffusion coefficient is reduced. If the initial spectral density of the noise $\varepsilon(0, v)$ is sufficiently large, the value of $\partial f/\partial v$ becomes zero and the noise density $\varepsilon(\infty, v)$ remains finite. Under these conditions, the system goes to a state in which $\partial f/\partial v = 0$ in the range $v_1 < v < v_2$, while $f = f_M$ outside this velocity range. The diffusion coefficient D (and the energy density of the plasma waves) will be nonzero in the region $v_1 < v < v_2$ and zero outside of this region. According to the quasi-linear theory, this state with a plateau in the distribution function should be stationary, since Eqs. (13)-(14) are satisfied in velocity space under these conditions. Actually, as we have already indicated, particle collisions, which are not considered in Eqs. (13)-(14), lead to a slow particle diffusion in velocity space and to the gradual establishment of thermodynamic equilibrium. Thus, the "plateau" distribution described here is quasi-stationary. If slow processes are neglected, however, it can be regarded as stationary.

The equations of the quasi-linear theory (13)-(14) can be used to relate the spectral energy density of the plasma waves $\varepsilon(\infty, v)$ in the stationary state to the initial spectral density $\varepsilon(0, v)$. Substituting $\varepsilon\partial f/\partial v = A^{-1}\partial\varepsilon/\partial t$ from Eq. (13) in Eq. (14), we see that the quantity $f - (\partial/\partial v)BA^{-1}\varepsilon$ is conserved in the relaxation process:

$$\frac{\partial}{\partial t}\left\{f - \frac{\partial}{\partial v} BA^{-1}\varepsilon\right\} = 0. \tag{15}$$

Hence, at any instant of time,

$$f - \frac{\partial}{\partial v} BA^{-1}\varepsilon = f_M - \frac{\partial}{\partial v} BA^{-1}\varepsilon_0.$$

In particular, in the final state (when $t \to \infty$),

$$f_\infty - \frac{\partial}{\partial v} BA^{-1}\varepsilon_\infty = f_M - \frac{\partial}{\partial v} BA^{-1}\varepsilon_0,$$

so that

$$\varepsilon(\infty, v) = \varepsilon(0, v) - AB^{-1}\int_{v_1}^{v} (f_M - f_\infty)\, dv. \tag{16}$$

Since the height of the plateau f_∞ is a known constant (it is determined by the conservation of the total number of resonance particles* $\int_{v_1}^{v_2} (f_M - f_\infty)\, dv = 0$, i.e., $f_\infty = (v_2 - v_1)^{-1} \int_{v_1}^{v^2} f_M\, dv$) the relation in (16) determines the spectral density of the wave energy in the final stationary state.

The reduction in the wave energy as a result of the quasi-linear relaxation process is compensated by the growth in kinetic energy of the particles: as a result of diffusion in phase space there is a net particle transfer to regions of higher velocity. It follows from Eq. (16) that

$$2\int_{\omega/v_2}^{\omega/v_1} \{\varepsilon(0, v) - \varepsilon(\infty, v)\}\frac{dk}{2\pi} =$$

$$= \int_{v_1}^{v_2} dv' nmv' \int_{v_1}^{v'} (f_M - f_\infty)\, dv. \tag{17}$$

*This relation follows from the conservation of the total number of particles: since the distribution function $f(v)$ does not change for $v < v_1$ or $v > v_2$, the total number of resonance particles ($v_1 < v < v_2$) must be conserved.

Integrating the right side of this relation by parts we obtain the energy conservation relation:

$$2 \int [\varepsilon(0, v) - \varepsilon(\infty, v)] \frac{dk}{2\pi} = \int_{v_1}^{v_2} n \frac{mv^2}{2} (f_M - f_\infty)\, dv. \tag{18}$$

It can be shown that the initial energy of the waves is not sufficient to establish a plateau on the electron distribution [Eq. (16) leads to a meaningless negative expression for ε_∞]. In this case, the stationary state is not reached and the system goes to thermodynamic equilibrium in a time of the order of the mean time between binary particle collisions.

Up to this point we have been considering a rarefied plasma in which particle collisions can be neglected. In general, the effect of collisions on particle motion will be comparable with the wave effect only when the wave is in equilibrium, i.e., when the wave amplitude is not greater than the amplitude of the corresponding mode in the thermal noise spectrum. For thermal noise, processes such as Cerenkov emission of a wave by the moving particle, emission in collisions, wave Landau damping and collisional absorption, are comparable. Indeed, the level of thermal noise of the plasma waves is determined by the balance between these phenomena.

For high-amplitude waves (superthermal) the particle collisions can still be quite important in certain phenomena, for instance, resonance absorption.

The effect of the wave is to produce a strong distortion of the distribution function in the region of the resonance particles. However, collisions partially restore the Maxwellian distribution function and establish the stationary wave absorption. All other effects due to collisions are negligibly small. Formally, the equation that describes the behavior of the averaged distribution function in time is obtained as the first term in an expansion of the exact kinetic equation in the quantity $1/N_D$ (the ratio of the thermal noise to the thermal energy of the plasma):

$$\frac{df}{dt} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} + Sf, \tag{19}$$

where the last term describes the binary collisions of resonance particles with the other plasma particles.

There are various ways of writing the collision term for a plasma. For example, this term can be written in the form given by Landau [3, 6] and then linearized since the number of resonance particles is small:

$$Sf = L\frac{\partial}{\partial v_i}v^{-3}\left[v_i f + \left(v^2\delta_{ik} - v_i v_k - \frac{T}{m}\,\frac{v^2\delta_{ik} - 3v_i v_k}{v^2}\right)\frac{\partial f}{\partial v_k}\right],$$

where $L = \lambda\omega_0^4/n$ (here λ is the Coulomb logarithm).

However, if we are interested in the particle distribution for only one velocity component $v_{\parallel}$ and integrate the collision term over the other components, we find

$$\int Sf\,dv_{\perp} = \frac{\partial}{\partial v_{\parallel}}\nu\left(v_{\parallel}f + \frac{T}{m}\frac{\partial f}{\partial v_{\parallel}}\right), \tag{19a}$$

where T is the electron temperature while $\nu \approx \lambda\omega_0^4/nv_{\parallel}^3$ is the collision frequency.

Binary collisions between particles lead to the gradual disappearance of the plateau on the distribution function and bring the system to the thermodynamic equilibrium state. The characteristic time for the system to reach the Maxwellian distribution can be estimated as follows. The quasi-linear equation for the distribution function (taking account of diffusion due to emission and absorption of waves and binary collisions) is of the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_{\parallel}}D\frac{\partial f}{\partial v_{\parallel}} + \frac{\partial}{\partial v_{\parallel}}\nu\left(v_{\parallel}f + \frac{T}{m}\frac{\partial f}{\partial v_{\parallel}}\right). \tag{20}$$

We now integrate Eq. (20) in time, assuming that $D(\partial f/\partial v) = A^{-1}(\partial D/\partial t)$, where A is given by Eq. (14a), thus obtaining (the subscript on $v_{\parallel}$ is omitted)

$$\left(f - \frac{\partial}{\partial v}A^{-1}D - \frac{\partial}{\partial v}\nu\frac{T}{m}A^{-1}\ln D\right)\Big|_0^t = \frac{\partial}{\partial v}\int_0^t dt\,\nu v f. \tag{20a}$$

If we consider the case in which the change in the distribution function $f_0 - f_M$ is appreciably smaller than $\partial D_0 A^{-1}/\partial v$, the first term on the left side can be neglected. Then we integrate Eq. (20a) with respect to v from $-\infty$ to v, making use of the fact that the distribution function f is not changed appreciably in the damping process (this is obviously not true for the derivative of this function $\partial f/\partial v$); hence we can replace f under the integral sign on the right side of Eq. (20a) by the thermodynamic equilibrium function $f_M = (2\pi T/m)^{-\frac{1}{2}}\exp-(mv^2/2T)$. Carrying out this procedure, we obtain the following transcendental equation for D(t, v) which gives the time dependence of the diffusion coefficient D (or the wave energy $\varepsilon = B^{-1}D$):

$$\left(-\frac{m}{T\nu}D - \ln D\right)\Big|_0^t = -A\frac{\partial f_M}{\partial v}t.$$

It is evident from this equation that the noise decays in linear fashion in the initial stage $D_0 - D \sim t$, and that the decay only becomes exponential $\ln(D_0/D) \sim t$ in the later stages of the process, when the noise level has become small.

For this reason, the feedback effect of the waves on the particles, which is introduced in the quasi-linear theory, leads to a sharp reduction in absorption; the resonance particles are redistributed and a plateau is formed on the distribution function; however, the collisions gradually smooth the edge of the plateau and a stationary state is established in which

$$\frac{\partial f}{\partial v} = -\frac{\nu v f}{\nu \frac{T}{m} + D}.$$

The main effect due to wave feedback is to change the derivative of the distribution function rather than the distribution function itself, so that

$$\frac{\partial f}{\partial v} = \frac{1}{1 + D\frac{m}{T\nu}} \frac{\partial f_M}{\partial v} = \frac{1}{1 + \lambda \frac{\bar{k}}{\Delta k}\left(\frac{v}{v_T}\right)^3 \frac{\varepsilon}{nT/N_D}} \frac{\partial f_M}{\partial v}, \tag{21}$$

where $\bar{k}$ is the mean wave number of the packet; Δk is the halfwidth; $v_T = \sqrt{T/m}$; $\lambda \approx 1$.

From Eqs. (13) and (21) we have

$$\frac{\partial D}{\partial t} = D \frac{1}{1 + D\frac{m}{T\nu}} \frac{\partial f_M}{\partial v}.$$

Integrating this ordinary differential equation for D(t) (the velocity v appears as a parameter), we obtain Eq. (20b).

Substituting Eq. (21) in Eq. (8), we have (Fig. 2):

$$2\gamma = \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial t} = \frac{1}{1 + \lambda \frac{\bar{k}}{\Delta k}\left(\frac{v}{v_T}\right)^3 \frac{\varepsilon}{nT/N_D}} 2\gamma_0. \tag{22}$$

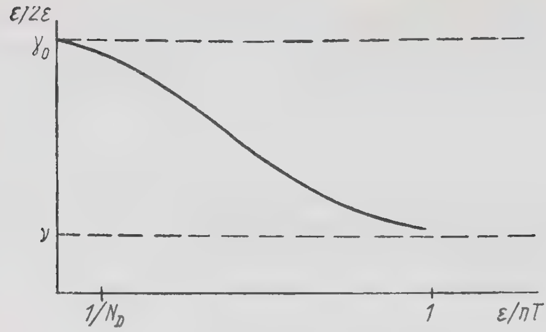

Fig. 2

When $\varepsilon << nT/N'_D$ (here N'_D is the number of particles in a sphere of radius v/ω), the quantity $\gamma \to \gamma_0$ where γ_0 is obtained from the linear theory. When $\varepsilon/nT \approx 1$, the quantity $\gamma = (\gamma_0/N_D) \approx \nu$. Hence, the effective damping time for the packet is changed from a quantity of order $1/\gamma_0$, for waves below the thermal noise level, to a time of the order of the collision time, for high-amplitude waves.

The factor that reduces the derivative $\partial f/\partial v$ is a consequence of the distortion of the distribution function and can be written in the form

$$\frac{1}{1 + D\dfrac{m}{T\nu}} = \frac{1}{1 + \dfrac{e^2}{mT\nu k}\,\dfrac{E_k^2 \Delta k}{\Delta v}}$$

for a "monochromatic" wave (in which $\Delta v \approx \sqrt{e\varphi_0/m}$; $E_k^2 \Delta k = E^2$) this factor is

$$\frac{1}{1 + \dfrac{A' e^2}{mT\nu k}\,\dfrac{E^2}{\sqrt{e\varphi_0/m}}} = \frac{1}{1 + A\dfrac{(e\varphi_0)^{3/2}}{\sqrt{m}\,T\nu\lambda}}, \qquad (23)$$

where A and A' are approximately equal to 1.

Let us now consider the propagation, through a plasma layer, of plasma waves which are generated continuously at the boundary of the layer.*

*In the framework of the quasi-linear theory we can only consider the propagation of a wave packet with some finite minimum width; this requirement arises since it is necessary to satisfy the condition $\Delta(\omega/k) > \sqrt{e\varphi/m}$, where φ is the mean amplitude of the potential associated with the waves.

The linear theory of small oscillations in a low-density plasma predicts a collisionless damping of waves that propagate in the plasma. In particular, a consequence of this collisionless damping is the attenuation of longitudinal plasma waves that are excited at the boundary of the plasma by an external electric field with frequency $\omega > \omega_0$; these waves are assumed to propagate perpendicularly to the boundary. For plasma waves, which are the only ones we consider, the variation in wave amplitude with distance into the plasma is given by the expression [1]*

$$\varepsilon_k^{-1} \frac{\partial \varepsilon_k}{\partial x} = \frac{\pi}{3} \frac{\omega_0^4}{k^3} \frac{m}{T} \frac{\partial f}{\partial v}, \tag{24}$$

where $k = \omega/v$ is the wave vector; $\omega^2 = \omega_0^2 + 3(T/m)k^2$; f is the electron distribution function for the velocity component parallel to the direction of wave propagation (perpendicular to the boundary). Thus, the linear theory, in which the energy of the wave packet is assumed to be infinitesimally small, leads to exponential damping of the wave packet as a function of distance. The damping factor is given by (24) with

$$f = f_M(v) = (2\pi T/m)^{-1/2} \exp - mv^2/2T.$$

Actually, however, the wave energy is finite and the wave diffusion effect causes an equalization of the distribution function for the resonance electrons with a consequent reduction in damping. If we take account of the fact that the parameter $N_D \varepsilon / nT$ (where ε is the energy density in the wave) is appreciably greater than unity, and neglect collisions, the equations of the quasilinear theory then indicate that the waves will produce a plateau on the distribution function at some distance from the boundary:

$$v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} = 0, \quad D \neq 0;$$

beyond this point is characterized by zero damping:

$$f(v, x) = \text{const}; \qquad \frac{\partial \varepsilon_k}{\partial x} = 0.$$

In order to obtain a finite absorption it is then necessary to introduce a collision term in the equation for the particle distribution function

$$v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} + Sf. \tag{25}$$

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*This expression (which only holds for distances of the order of several wavelengths away from the boundary) follows from Eq. (8b) since

$$\partial N_q/\partial t \to [H_q, N_q] = 3 \frac{q}{\omega} \frac{T}{m} \frac{\partial N}{\partial x}.$$

Equations (24) and (25) with the initial conditions $\varepsilon(0, v) = \varepsilon_0(v)$ and $f(0, v) = f_0(v)$ determine the spectral energy density of the wave and the distribution function as functions of distance. In order to simplify the calculation, however, we shall limit ourselves to the case of strong waves:

$$\frac{\varepsilon}{nT} \gg \frac{1}{\sqrt{N_D}}.$$

In this case the $v\partial f/\partial x$ term in Eq. (25) can be neglected. Then

$$\frac{\partial f}{\partial v} = -\nu \frac{vf}{D}, \tag{26}$$

where the quantity $(T/m)(\partial f/\partial v)$ is negligibly small compared with vf in the velocity region of interest. If the plasma wave packet is not very broad, the distribution function does not change appreciably (this remark obviously does not apply to the derivative of the distribution function, which can exhibit a substantial change) and f in the right side of Eq. (26) can be replaced by the Maxwellian function f_M. Substituting the value found for $\partial f/\partial v$ in Eq. (24), we then have

$$\frac{\partial \varepsilon_k}{\partial x} = -\frac{\pi}{3}\frac{\nu}{k}\frac{v^2}{T/m} nmv^2 f_M,$$

so that the energy of the wave packet decreases linearly with increasing distance from the boundary:

$$\frac{\varepsilon_k(x)}{\varepsilon_k(0)} = 1 - \frac{x}{L},$$

while the characteristic damping length L is directly proportional to the wave energy at the boundary, being of order

$$L \approx \frac{1}{k}\frac{\varepsilon N_D}{nT} \text{ (when } v \approx v_T).$$

Thus, the quantity L is appreciably greater than the damping length L_{lin} given by the linear theory. For wavelengths of the order of the Debye radius we find $L/L_{lin} \approx \varepsilon N_D/nT$.

The formula for the quasi-linear damping rate of a wave in an anisotropic plasma is complicated, but its general structure is very much the same as that in Eqs. (22)-(23):

$$\gamma = \frac{\gamma_0}{1 + \frac{\nu'}{\nu_e}};$$

where ν_e is the electron collision frequency while ν' is the reciprocal time for formation of the quasi-linear plateau.

§ 4. Growth of Perturbations in an Unstable Plasma

Using the quasi-linear theory we now wish to consider the development of a perturbation in an unstable low-density plasma. We shall first investigate the dynamics of a system which is unstable against the excitation of electron plasma oscillations. In order to simplify the problem we consider the case in which the wave vectors characteristic of the growing waves are parallel to each other and in which the wave spectrum is one-dimensional.* We assume that the initial electron distribution function $f(0, v)$ exhibits a rising part in some small range of velocities (the mean velocity in this range is appreciably greater than the mean thermal velocity of the plasma electrons), so that df/dv is positive in this region. Under these conditions the plasma is unstable and the spectral energy density ε_k in the corresponding range of wave numbers $k = \omega_0/v$ starts to grow in accordance with the relation

$$\frac{\partial \varepsilon_k}{\partial t} = 2\gamma\varepsilon_k; \quad \gamma = \frac{\pi}{2}\frac{\omega_0^3}{k^2}\frac{\partial f}{\partial v}. \tag{27}$$

The growing oscillations lead to an increased diffusion coefficient for the resonance particles that interact with the waves:

$$D = \frac{e^2}{2m^2}\frac{|E_k^2|}{v}. \tag{28}$$

Simultaneously, the distribution function is smoothed and the region of instability expands:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v}. \tag{29}$$

The wave growth and the diffusion of resonance electrons continue until a plateau is formed on the distribution function, i.e., a region in which $\partial f/\partial v = 0$. After this point the waves no longer grow and a stationary state is established. The electron distribution in the final state $f_\infty(v)$ can be found from

*This situation holds when a plasma exhibits a preferred direction (external magnetic field, axis of the plasma tube, etc.) and the growth rate is a maximum for wave vectors in this direction.

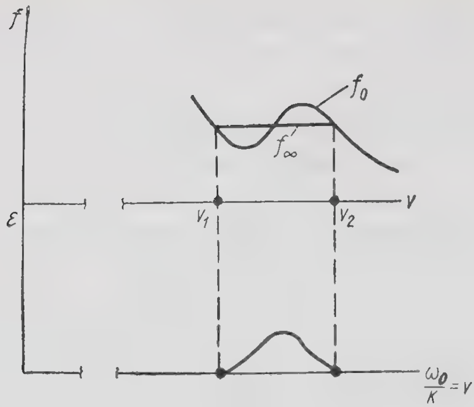

Fig. 3

the conservation of the total number of resonance particles that have diffused to lower velocities in phase space in the establishment of the stationary state:

$$\int_{v_1}^{v_2} f(0, v)\, dv = \int_{v_1}^{v_2} f(\infty, v)\, dv. \tag{30}$$

The velocities $v_{1,2}$ are determined by the boundaries of the plateau region and must be found simultaneously with $f(\infty, v)$ by solving Eq. (30) together with the relation

$$f(0, v_1) = f(0, v_2) = f(\infty); \tag{31}$$

this means that the area under the curves $f(0, v)$ and $f(\infty, v)$ (Fig. 3) must be the same (the points $v_{1,2}$ are shown in the figure).

The system of equations (30)-(31) determines uniquely the value of $f(\infty, v)$ in the plateau region and the boundaries of this region $v_{1,2}$; outside the plateau $f(\infty, v) = f(0, v)$.

As in the quasi-linear relaxation of the plasma oscillations considered in the preceding section, the spectral density of the noise in the final state ε_∞ is related to the initial spectral density ε_0 and the change in the distribution function $f(0, v) - f(\infty, v)$ by the expression

$$\varepsilon(\infty, v) - \varepsilon(0, v) = -AB^{-1} \int_{v_1}^{v} (f_0 - f_\infty)\, dv, \tag{32}$$

where the functions A and B are determined by Eq. (14a). If the initial noise level in the system is thermal, the quantity $\varepsilon(0, v)$ can be neglected, so that the spectrum of plasma waves in the final state is determined only by the initial electron distribution in the vicinity of the growth region f_0 [14, 18]:

$$\varepsilon(v, \infty) = AB^{-1} \int_{v_1}^{v} (f_\infty - f_0)\, dv. \tag{33}$$

The energy density of the waves that are established at the termination of the diffusion process is of order

$$\frac{E^2(\infty)}{8\pi} \approx \delta n \left(mv_2^2 - mv_1^2\right),$$

where $v_{1,2}$ represent the plateau boundaries, while δn is the density of electrons that move to regions of lower energy in velocity space.

The time τ required for excitation of waves and relaxation of the electron distribution (establishment of the plateau) can be estimated from the diffusion time in velocity space by using the expression for the diffusion coefficient D_∞ in the final state:

$$\tau \approx \frac{(v_2 - v_1)^2}{D_\infty} \approx \frac{1}{\omega_0} \frac{(v_2 - v_1)^2}{v^2} \frac{n}{\delta n},$$

where v is the mean value of the velocity in the plateau region.

The development of the instability means that the kinetic energy of the resonance particles is converted into electrostatic energy associated with the plasma waves and into the kinetic energy of all of the plasma electrons, which participate in these collective oscillations; the total energy of the plasma is obviously conserved.

§ 5. Interaction of a Beam with Plasma

It is well known that a system consisting of a plasma and a beam of charged particles that passes through the plasma can be unstable under certain conditions. This so-called electrostatic instability has been the subject of a large number of experimental and theoretical papers. According to the linear theory [4], the electrostatic instability is somewhat different in two limiting cases. When the beam is dense and mono-energetic, and moves with a high velocity with respect to the plasma, the plasma exhibits growing oscillations; the frequency and growth rate are determined by the parameters of the entire system. On the other hand, if the velocity and density of the beam are not very large, and if the velocity spread in the beam is not too small, the frequency of oscillation is equal to the plasma frequency of the plasma and it is only the growth rate that is determined by the properties of the overall system; this growth rate is then proportional to the velocity derivative of the combined distribution function for the plasma electrons and the beam electrons (at the point $v = \omega/k$).

The quasi-linear theory of the earlier sections can be used to investigate the dynamics of a beam–plasma interaction in the second case only.

In analyzing the interaction of a beam with the plasma, as in the preceding sections we shall limit ourselves to one-dimensional electron plasma waves. Assume that the beam moves through the plasma in the positive x direction; at the point $x = 0$ we are given the distribution functions for the electrons in the plasma and in the beam, as well as the spectral density of the noise $\varepsilon_k = |E_k^2|/8\pi$. If particles that are in resonance with the plasma waves ($v = \omega/k$) satisfy the condition $\partial f/\partial v > 0$, the waves will grow. Simultaneously there is a diffusion of the electrons in the beam and plasma in velocity space; this tends to smooth the distribution function in the region in which the wave diffusion coefficient is nonvanishing, thus reducing the growth rate. As the beam continues to move, the velocity derivative of the distribution function diminishes while the wave energy increases. At $x \to \infty$ the system is in the stationary state described above: there is a plateau on the electron distribution function for the beam–plasma system and there is a corresponding region of wave vectors in which there are undamped plasma waves. Since the energy density of these waves is larger than at the input to the system ($x = 0$), it is evident that the kinetic energy of the beam electrons has been reduced. As a result of the formation of a plateau on the electron distribution function, a group of particles has been displaced toward the origin of coordinates in velocity space, indicating a reduction of the kinetic energy of the beam (i.e., the beam is retarded). The quasi-linear theory can be used to find the energy loss of the beam and to determine the shape of the spectrum of plasma waves in the system.

In the case at hand the quasi-linear equations assume the form*

$$\left.\begin{aligned} v_g \frac{\partial \varepsilon}{\partial x} &= A\varepsilon \frac{\partial f}{\partial v}; \\ v \frac{\partial f}{\partial x} &= \frac{\partial}{\partial v} B\varepsilon \frac{\partial f}{\partial v}, \end{aligned}\right\} \tag{34}$$

*These follow from the general equations (8b) and (9b). However, it is simpler to obtain them from Eqs. (8) and (9) by making the obvious substitutions

$$\frac{\partial \varepsilon}{\partial t} \to \frac{\partial \omega_k}{\partial k}\frac{\partial \varepsilon}{\partial x} = v_g \frac{\partial \varepsilon}{\partial x}; \quad \frac{\partial f}{\partial t} \to \frac{\partial (p^2/2m)}{\partial p}\frac{\partial f}{\partial x} = v\frac{\partial f}{\partial x}.$$

where v_g is the group velocity of the plasma waves, while A and B, which are independent of the x coordinate, are given by Eq. (14a). The system of equations in (34) must be solved with the following boundary conditions: $f(0, v) = f_0(v)$, $\varepsilon(0, v) = \varepsilon_0(v)$. It may be assumed that the wave vector and the velocity of the resonance particles are related by the expression $\omega = kv$, where $\omega = \omega_0 + {}^3\!/_2 k^2 (T/m\omega_0)$, while ω_0 is the electron-plasma frequency and T is the electron temperature.

The level of the plateau formed on the distribution function can be determined from the conservation of the total number of resonance electrons

$$\int_{v_1}^{v_2} f(0, v)\, dv = \int_{v_1}^{v_2} f_\infty\, dv,$$

so that

$$f(\infty) = (v_2 - v_1)^{-1} \int_{v_1}^{v_2} f(0, v)\, dv. \tag{35}$$

Here, $v_{1,2}$ are points in velocity space which define the boundaries of the plateau; the values $v_{1,2}$ can be found simultaneously with f_∞ by solving Eq. (35) together with the equation

$$f(0, v_1) = f(0, v_2) = f_\infty.$$

The spectral energy density of the plasma waves at $x \to \infty$ can be determined as follows. The value of $\varepsilon \partial f/\partial v$ from the first equation in (34) is used in the second, yielding

$$\frac{\partial}{\partial x}\left(vf - \frac{\partial}{\partial v} BA^{-1} v_g \varepsilon\right) = 0,$$

i.e.,

$$\varepsilon(v, \infty) = \varepsilon(v, 0) + AB^{-1} v_g^{-1} \int_{v_1}^{v} v\,(f_\infty - f_0)\, dv. \tag{36}$$

Thus, the development of the two-stream instability and the smearing of the peak in the electron velocity distribution cause some of the kinetic energy of the beam electrons to be converted into plasma wave energy. Obviously the total energy flux remains constant; this can be shown as follows. Consider the case in which the noise level at the input to the system is thermal: $\varepsilon(v, 0) = 0$. Multiplying both sides of Eq. (36) by $2v_g$, and integrating

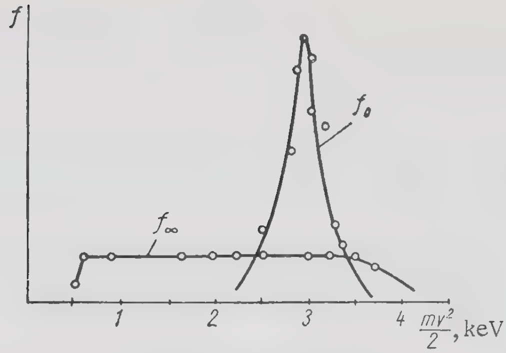

Fig. 4

with respect to wave number $k = \omega/v$, we find that the energy flux through the cross sections $x = 0$ and $x = \infty$ are the same:

$$\sum_k v_g 2\varepsilon_k + \int_{v_1}^{v_2} v \frac{mv^2}{2} (f_\infty - f_0)\, dv = 0. \tag{37}$$

The limits of integration in Eq. (37) can be extended to infinity because the spectral density $\varepsilon(\infty)$ is zero outside the range $v_1 < v < v_2$, and the functions f_0 and f_∞ are identical.

In conclusion, we note that the theoretical conclusions concerning the relaxation of an unstable plasma toward a state with a plateau on the distribution function have been observed experimentally (Fig. 4) [16, 21, 25].

Problem 2. Investigate the development of perturbations when the boundaries of the instability region are fixed (Fig. 5).

Solution. Since $\partial f/\partial v \to -\infty$ at two points $v_0 \pm u$ (Fig. 5) these points represent the boundaries of the instability region and the distribution function can only change for $-u < v - v_0 < u$. If $f(0, v)$, the initial electron distribution, is smooth in this region, it can be expanded in a series in which we limit ourselves to the first two terms:

$$f(0,\ v) = \text{const} + A_0 (v - v_0).$$

It then follows from the conservation of the number of particles that the constant in this expression is $f(\infty)$.

It can be easily shown that Eqs. (13) and (14) have the solution

$$F(v, t) = f(\infty) + A(t)(v - v_0);$$

$$D(v, t) = \frac{u^2 - (v - v_0)^2}{2} B(t),$$

where

$$A(t) = \frac{A_0 - B_0}{1 + \frac{B_0}{A_0} \exp(A_0 + B_0)t}; \quad B(t) = A_0 + B_0 - A(t).$$

Here, $A_0 = \partial f(0, v)/\partial v$, while the quantity B_0 is proportional to the amplitude of the initial noise.

In the solution given here, the distribution functions remains linear throughout the entire process, while the spectrum is parabolic.

Problem 3. Find the spectrum for ion-acoustic waves excited by an electric current in a weakly ionized plasma.

Solution. In the stationary state the equation for the averaged electron distribution function in the resonance region of interest f is given by

$$-\frac{eE_0}{m} \frac{\partial f}{\partial v} + \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} + Sf = 0 \tag{A}$$

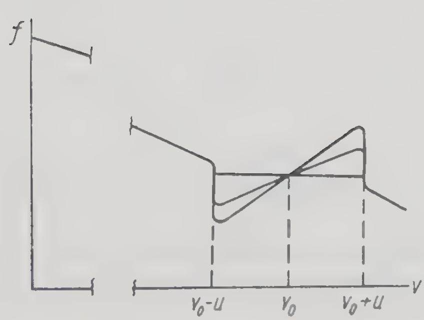

Fig. 5

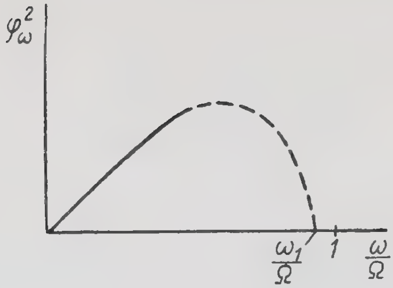

Fig. 6

where E_0 is the external electric field and Sf is the collision term.

The equation for the waves reduces to an equality for the rate of production of waves by electrons and the absorption due to collisions of ions with neutrals [4]

$$\frac{\pi}{m}\frac{\partial f}{\partial v} = \frac{2}{M}\frac{\nu_i}{v^3 k}, \tag{B}$$

where m and M are the masses of the electron and ion, respectively; ν_i is the ion-neutral collision frequency, and k is the wave number. The shape of the spectrum in the low-frequency region can be found as follows. Since

$$D = \frac{\pi e^2}{m^2} \sum_k E_k^2 \delta(\omega_k - kv) \sim \frac{E_k^2}{v - v_g}$$

(where v_g is the group velocity), and since (A) and (B) indicate that for small k

$$D \sim \left(\frac{\partial f}{\partial v}\right)^{-1} \sim k,$$

it then follows that

$$E_k^2 \sim (v - v_g)\, k. \tag{C}$$

Substituting the expressions for $v = \omega_k/k$ and $v_g = d\omega_k/d_k$ in (B), we find the dispersion equation

$$\left(\frac{\omega_k}{\Omega}\right)^2 = \frac{(kR_D)^2}{1 + (kR_D)^2}$$

where Ω is the ion-plasma frequency and R_D is the Debye radius; the noise density in the low-frequency region of the spectrum is then

$$E_k^2 \sim \left(\frac{\omega}{\Omega}\right)^3, \quad (\omega \ll \Omega),$$

and the spectral density of the square of the potential is

$$\varphi_\omega^2 \sim \frac{E_k^2}{k^2 v_g} \sim \frac{\omega}{\Omega}.$$

Thus, in the weakly turbulent state, the quantity φ_ω^2 increases linearly with frequency ω when $\omega \ll \Omega$ (Fig. 6).

At still higher frequencies, the quantity φ_ω^2 reaches a peak and is then reduced, reaching zero when $\omega = \omega_1 \sim \Omega$.

§ 6. Threshold for Wave Absorption in a Plasma and Turbulent Heating

If one considers the propagation of a wave in a plasma at an amplitude exceeding some given threshold value (depending on the type of wave and its period), in certain cases the plasma can be unstable. When this happens, part of the ordered energy of the wave is converted into the energy associated with the spectrum of the nonequilibrium plasma oscillations.

In order to illustrate the effect we consider the excitation of high-amplitude, one-dimensional, ion-acoustic waves in a plasma. It is assumed that the waves cause the electrons to acquire a mean velocity U (with respect to the ions which are at rest), and that this mean velocity is greater than the critical velocity $c_s \approx \sqrt{T_e/M}$. Under these conditions, the ion-acoustic waves grow, causing electron diffusion in velocity space by virtue of wave diffusion. As a result of the equalization of the electron distribution function, the region of instability expands in velocity space and soon encompasses the entire range of allowed values of phase velocities for the ion-acoustic waves $c_i < v < c_s$ ($c_i \approx \sqrt{T_i/M}$).

If a wave of sufficiently high amplitude propagates in the plasma the electron distribution periodically passes through the region $-c_s < v < c_s$ in which the ion-acoustic waves are excited; hence, the quasi-linear diffusion coefficient in velocity space D is different from zero. The electron distribution function will gradually be smoothed and will exhibit a plateau after a period of time (Fig. 7):

$$f(\infty, v) = \begin{cases} f_\infty = \text{const}; & (|v| < U); \\ f_0(v); & (|v| > U), \end{cases}$$

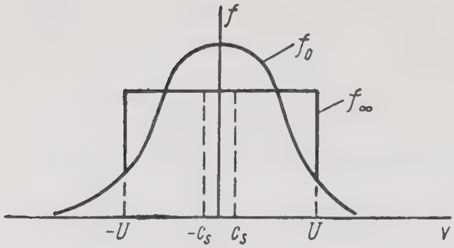

Fig. 7

where U is the maximum displacement of the electron distribution in velocity space caused by the external field (assuming that $U \gg c_s$), while $f_\infty = (1/2U) \times \int_{-U}^{U} f_0(v)\,dv$. The smoothing of the distribution function is described by the diffusion equation

$$\frac{\partial f}{\partial t} = k\,\frac{\partial^2 f}{\partial v^2}$$

[with the boundary condition $(\partial f/\partial v)_{\pm u} = 0$], where $k \approx \omega c_s^2$ for $U > c_s$; $k = 0$ for $U < c_s$; c_s is the width of the region in which the superthermal noise is excited and ω is the frequency of the external field. Thus, the smoothing time is given by $\tau \approx (U^2/k) \approx (1/\omega)(U/c_s)^2$.

Hence, an external field whose intensity is greater than the threshold value will displace electrons in velocity space and do work; by exciting collective oscillations of the plasma this field can provide "collisionless" heating of the electrons.

Turbulent heating of electrons in an unstable plasma, in which the electrons move with respect to the ions, is characterized by intense high-frequency oscillations in the plasma; the electrons then execute random motions in these high-frequency fields [23]. The presence of these oscillations tends to stabilize the system: for example, assume that high-frequency oscillations are excited in a plasma consisting of a cold electron gas (at rest) and a moving cold ion gas; then the plasma will be stable if the random velocity of the electrons $\sqrt{\langle v^2 \rangle}$ (due to the oscillations) is greater than the relative velocity of the ions with respect to the electrons U_0 (cf. Problems 4-5).

Problem 4. Derive equations to describe slow processes in a plasma in which plasma waves are excited.

Solution. In the presence of the high-frequency waves the plasma particles are subject to a force

$$f = -\sum_k \frac{e^2}{4m\omega_k^2} \nabla E_k^2. \tag{A}$$

The slow (compared with the characteristic period of the high-frequency waves $<1/\omega>$) variations of the distribution function F are described by the following equation (in the absence of external forces):

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} + \frac{f}{m}\frac{\partial F}{\partial v} = 0. \tag{B}$$

The next equation we seek is the equation for the spectral energy density in the high-frequency waves; the variation in spectral density is given by the following equation for $N_k = E_k^2/\omega_k$:

$$\frac{\partial N_k}{\partial t} + v_g\frac{\partial N_k}{\partial x} - \frac{\partial \omega_k}{\partial x}\frac{\partial N_k}{\partial k} = 0 \tag{C}$$

(v_g is the group velocity).

Problem 5. Investigate the stability of an ion stream moving through a cold electron gas in which plasma waves are excited.

Solution. We compute the first two moments of (B) in Problem 4, obtaining

$$\frac{\partial n}{\partial t} + \frac{\partial nU}{\partial x} = 0; \tag{A}$$

$$\frac{dU}{dt} = \frac{f}{m}, \tag{B}$$

where f is given by (A) of Problem 4.

Linearizing the first of these two relations and (B) of Eq. (4), we find δN_k, the change in the quantity N_k, in a plane wave in which all quantities are proportional to $e^{-i\Omega t + iqx}$:

$$-i\Omega\delta N_k - kiqU\frac{\partial N_k}{\partial k} - \frac{iq\omega_0}{2}\frac{\delta n}{n}\frac{\partial N_k}{\partial k} = 0;$$

$$-i\Omega\delta n + iqnU = 0.$$

i.e.,

$$\delta N_k = -\frac{\delta n}{n}\left(k + q\,\frac{\omega_0}{2\Omega}\right)\frac{\partial N_k}{\partial k}.$$

The change in the spectral density of the high-frequency waves is

$$\delta E_k^2 = \delta(\omega_k N_k) = \omega_k \delta N_k + N_k\left(\frac{\omega_0}{2}\,\frac{\delta n}{n} + kU\right) =$$

$$= \frac{\delta n}{n}\left(1 + 2\,\frac{k}{q}\,\frac{\Omega}{\omega_0}\right)\left(\frac{\omega_0 N_k}{2} + \frac{\omega_0}{\Omega}\,q\,\frac{\partial N_k}{\partial k}\right).$$

Substituting this value of δE_k^2 in (B) (considering, for simplicity, the case $\bar{k} = \sum_k k N_k / \sum_k N_k = 0$), we have

$$\frac{dU}{dt} = -iq\,\frac{\delta n}{n}\sum_k \frac{e^2}{4m^2\omega_0^2}\,E_k^2. \tag{C}$$

It is evident from (C) that the presence of the waves is equivalent to the presence of a thermal velocity spread for the electrons (i.e., an electron pressure):

$$\langle v^2 \rangle = \sum_k \frac{e^2}{4m^2\omega_0^2}\,E_k^2. \tag{D}$$

Hence, from the linear theory of stability of an ion stream moving with velocity U_0 through an electron gas [4], we find that the plasma is stable when

$$U_0 < \sqrt{\langle v^2 \rangle},$$

where $\langle v^2 \rangle$ is given by (D).

§ 7. Plasmon – Plasmon Interactions

Up to this point we have been considering a weakly turbulent plasma, assuming that the wave energy density is small enough so that interaction between waves could be neglected; in this case the important processes are the emission and absorption of collective plasmon oscillations by resonance particles. When the wave energy increases, the interaction between waves becomes important; since many waves are excited simultaneously in a turbulent plasma, and since their phases are random, the interaction between waves reduces to wave "collisions" and can be described on the basis of a kinetic equation for the wave distribution function (plasmon equation) in phase space.*

It is convenient to derive the wave kinetic equation starting with the Lagrangian expanded in powers of the amplitude of the collective plasma oscillations. The complete Lagrangian L for a plasma can be written in the following form (cf. [8]):

$$L = \sum_{\nu} \iint dx\, dv\, f_{\nu} \left\{ \frac{m_{\nu}(v + D_{\nu} y_{\nu})^2}{2} - e_{\nu} V_0 (x + y_{\nu}) - \right.$$
$$\left. - e_{\nu} \varphi (x + y_{\nu}) + e_{\nu} (v + D_{\nu} y_{\nu}) (A_0 (x + y_{\nu}) + \alpha (x + y_{\nu})) \right\} +$$
$$+ \frac{1}{8\pi} \int dx \{ (E_0 + e)^2 - (B_0 + b)^2 \}. \tag{38}$$

Here, $y_{\nu}(x, v, t)$, φ, and α are variables in terms of which the variation is taken, while $f_{\nu}(x, v)$ is the stationary distribution function for particles of species ν in the stationary fields $E_0 = -\nabla V_0$ and $B_0 = \nabla \times A_0$, which satisfy the Maxwell equations

$$\left.\begin{aligned} \nabla \times E_0 = 0; \quad \nabla \cdot E_0 = \sum_{\nu} \int e_{\nu} f_{\nu}(x, v)\, dv; \\ \nabla \cdot B_0 = 0; \quad \nabla \times B_0 = \frac{4\pi}{c} \sum_{\nu} \int e_{\nu} v f_{\nu}(x, v)\, dv. \end{aligned}\right| \tag{39}$$

The operator D_{ν} in Eq. (38) is the total time derivative along the trajectory of particles of species ν in the fields E_0 and B_0:

$$D_{\nu} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} + a_{0\nu} \frac{\partial}{\partial v}, \tag{40}$$

where

$$a_{0\nu} = \frac{e_{\nu}}{m_{\nu}} \left\{ E_0(x) + \frac{v}{c} \times B_0(x) \right\}. \tag{41}$$

The quantities y, φ, and α represent the displacement of the particles and the deviations of the scalar and vector potentials from equilibrium values in the stationary state. Expanding the Lagrangian L in powers of y, φ, and α we obtain the Lagrangians for the zeroth, first, second, etc., orders L_0, L_1, L_2, ... The zero-order Lagrangian L_0 does not contain y, φ, and α; the

*The interaction between plasmons is analogous to the interaction between phonons in condensed media and the kinetic equations for the two cases are analogous.

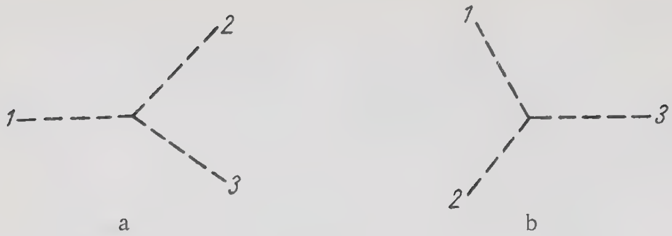

Fig. 8

Lagrangian L_1 vanishes identically; the Lagrangian L_2 describes the harmonic oscillations of the plasma [8]. The third- and fourth-order Lagrangians L_3 and L_4 describe the interaction between these harmonic oscillations (plasmons).

For reasons of simplicity we shall first limit ourselves to the analysis of longitudinal waves in a uniform isotropic plasma ($E_0 = B_0 = 0$); in this case, Eq. (38) yields

$$L_3 = -\sum_\nu \frac{e_\nu}{2} \iint dx\, dv f_\nu (x, v)\, y_\alpha^\nu y_\beta^\nu \nabla_\alpha \nabla_\beta \varphi; \tag{42}$$

$$L_4 = -\sum_\nu \frac{e_\nu}{6} \iint dx\, dv f_\nu (x, v)\, y_\alpha^\nu y_\beta^\nu y_\gamma^\nu \nabla_\alpha \nabla_\beta \nabla_\gamma \varphi. \tag{43}$$

§ 8. Three-Plasmon Processes

We shall first consider interactions between waves in which three plasmons participate. These processes are (a) decay of a single plasmon into two plasmons, and (b) combination of two plasmons into one (Fig. 8).

In these processes we must satisfy frequency conservation ω and wave-vector conservation k (otherwise, the transition probability amplitude vanishes); for cases (a) and (b) these conservation relations are

$$\text{a) } k_1 = k_2 + k_3;\ \omega_1 = \omega_2 + \omega_3; \tag{44}$$

$$\text{b) } k_1 + k_2 = k_3;\ \omega_1 + \omega_2 = \omega_3. \tag{45}$$

In an isotropic plasma in which there are two kinds of plasmons – the ion-acoustic plasmons (s) and the plasma (longitudinal) (l) oscillations (we assume that the necessary condition of weakly damped ion-acoustic waves is satisfied; the electron pressure exceeds the ion pressure and only weakly damped longwave plasma oscillations are considered), the conservation relations [(44)-(45)] allow only those three-plasmon processes in which two plasma plasmons and one acoustic plasmon participate. The longwave l-plasmons have approximately the same frequency $\omega \approx \omega_{0e}$, so that a single l-plasmon cannot split into two; conversely, the two l-plasmons cannot combine into one. The ion-acoustic oscillations cannot interact between themselves in a three-plasmon process because their spectrum is a "nondecay" type – the frequency of these oscillations ω increases with wave number k at a slower rate than linear (cf. Problem 6). Finally, three-plasmon processes in which two s-plasmons and one l-plasmon participate are not possible in a plasma with a small ratio of electron mass m to ion mass M because the maximum possible frequency of the ion-acoustic wave is much smaller than the frequencies of the plasma waves and the frequency conservation relation cannot be satisfied.

We then consider only the allowed three-plasmon processes, in which one s-plasmon and two l-plasmons participate: we express the displacement y^e in the Lagrangian in (42) in terms of the potential φ (the ion term in L_3 can be neglected since its contribution is small if $m/M \ll 1$, in which case the ion velocity and displacement are small compared with the electron velocity and displacement):

$$y^e = \sum_k \frac{e}{m} ik \left\{ \frac{\varphi_k^s}{(kv)^2} + \frac{\varphi_k^l}{-\omega_{0e}^2} \right\} e^{ikx}, \tag{46}$$

where φ_k^s and φ_k^l are the spatial Fourier components of the potential in the ion-acoustic wave and plasma waves, respectively. Substituting this value of y^e in Eq. (42), we obtain the following expression for the Lagrangian for three-plasmon processes in an isotropic plasma:

$$L_3 = \sum_{p+q+r=0} \Lambda_{p;\,qr} \varphi_p^s \varphi_q^l \varphi_r^l, \tag{47}$$

where

$$\Lambda_{p;\,qr} = \frac{e^3}{2m^2} \int \frac{f dv}{(n\cdot v)^2} \frac{q\cdot r}{2\omega_{0e}^2} \approx \frac{e q\cdot r}{T}. \tag{48}$$

Starting with the classical Lagrangian (47) by introducing second quantization we can write a system of kinetic equations for the distribution functions for the l- and s-plasmons. For the distribution function of the l-plasmons n_k we find

$$\dot{n}_1 = \sum w_{12} \{ -n_1(n_3+1)(N_2+1) + (n_1+1) n_3 N_2 \} +$$
$$+ \sum w_{21} \{ -n_1(n_3+1) N_2 + (n_1+1) n_3 (N_2+1) \}, \tag{49}$$

where w_{12} is the probability for decay of an l-plasmon k_1 into an s-plasmon k_2 and an l-plasmon k_3, while w_{21} is the probability for combination of an l-plasmon k_1 and an s-plasmon k_2 into another l-plasmon k_3. The summation in Eq. (49) is taken over wave numbers k_2 and k_3, which satisfy the conservation relation (44) in the first collision integral and the conservation relation (45) in the second collision integral.

The analogous equation for the distribution function for the s-plasmons N_k is

$$\dot{N}_2 = \sum w_{21} \{-N_2 n_1 (n_3 + 1) + (N_2 + 1)(n_1 + 1) n_3\}. \tag{50}$$

Here, w_{21} is the probability for combination of an s-plasmon k_2 and an l-s plasmon k_1 into an l-plasmon k_3. The second collision integral vanished in Eq. (50), since the decay of a low-frequency s-plasmon into two high-frequency l-plasmons is forbidden by the frequency conservation relation (as is the inverse process, combination of two l-plasmons into one s-plasmon).

The probabilities w in (49) and (50) can be expressed in terms of the matrix elements of the Lagrangian (42):

$$w_{12} = \frac{2\pi}{\hbar} |\langle n_1, N_2, n_3 | L_3 | n_1 - 1, N_2 + 1, n_3 + 1 \rangle|^2;$$

$$w_{21} = \frac{2\pi}{\hbar} |\langle n_1, N_2, n_3 | L_3 | n_1 - 1, N_2 - 1, n_3 + 1 \rangle|^2;$$

so that

$$w_{12} = w_{21} = \frac{2\pi}{\hbar} |\Lambda_{p;\,qr}|^2 |\varphi^s_{k_2}|^2 |\varphi^l_{k_1}|^2 |\varphi^l_{k_3}|^2. \tag{51}$$

Here, φ^s and φ^l are the matrix elements for the potential for the s and l waves:

$$\varphi^s_k = \frac{1}{k} \sqrt{2\pi\hbar\omega_s(k)}; \tag{52}$$

$$\varphi^l_k = \frac{1}{k} \sqrt{2\pi\hbar\omega_l(k)}. \tag{53}$$

Substituting the value of w_{21} (51) in the kinetic equation for the s-plasmons, we have

$$\dot{N}_2 = \int \frac{dk_1}{(2\pi)^3} \frac{dk_3}{(2\pi)^3} \delta(k_1 + k_2 - k_3)\, \delta(\omega_1 + \omega_2 - \omega_3) \frac{2\pi}{\hbar^2} |\Lambda_{k_2;\, k_1 k_3}|^2 \times$$

$$\times \frac{2\pi\hbar\omega_s(k_2)}{k_2^2} \frac{2\pi\hbar\omega_l(k_1)}{k_1^2} \frac{2\pi\hbar\omega_l(k_3)}{k_3^2} \{-N_2 n_1 (n_3 + 1) +$$

$$+(N_2+1)(n_1+1)\,n_3\}. \tag{54}$$

Writing $n_k = |E_k^l|^2/\hbar\omega_l$ in this equation, we find the order of the characteristic "collision frequency" of s-plasmons with l-plasmons

$$\gamma_3 = \frac{\dot N_2}{N_2} \approx \omega_3 \frac{|E^l|^2}{nT} \tag{55}$$

[the expression for γ_3 can conveniently be written in the form

$$\gamma_3 \approx \omega_s \left(\frac{v_\sim}{U}\right)^2,$$

where $v_\sim$ is the random velocity of the electrons in the oscillations, while U is the phase (or group) velocity for waves characterized by $k \lessapprox R_D^{-1}$].

The quantity γ_3 determines a number of the characteristic features of a weakly turbulent plasma: for example, the characteristic decay length L of the ion-acoustic waves in a plasma with highly excited plasma oscillations is approximately equal to the mean free path for collisions of an s-plasmon with l-plasmons, and is thus related to the collision frequency γ_3 by the expression

$$L \approx c_s/\gamma_3. \tag{56}$$

After substitution of the value of the transition probability w (51), the kinetic equation (50) for the distribution function of l-plasmons assumes the form

$$\dot n_1 = \int \frac{dk_2}{(2\pi)^3}\frac{dk_3}{(2\pi)^3}\frac{2\pi}{\hbar^2}\,|\Lambda_{k_2;\,k_1 k_3}|^2\,\frac{2\pi\hbar\omega_s(k_2)}{k_2^2}\times$$

$$\times\frac{2\pi\hbar\omega_l(k_1)}{k_1^2}\,\frac{2\pi\hbar\omega_l(k_3)}{k_3^2}\cdot\{(-n_1(N_2+1)(n_3+1)+$$

$$+(n_1+1)N_2n_3)\,\delta(k_1-k_2-k_3)\,\delta(\omega_1-\omega_2-\omega_3)+$$

$$+(-n_1N_2(n_3+1)+(n_1+1)(N_2+1)\,n_3)\,\delta(k_1+k_2-k_3)\times$$

$$\times\,\delta(\omega_1+\omega_2-\omega_3)\}. \tag{57}$$

The system of equations in (54) and (57) determines completely the dynamics of a weakly turbulent isotropic plasma in the absence of resonance particles. When there are resonant particles, it is then necessary to use a sys-

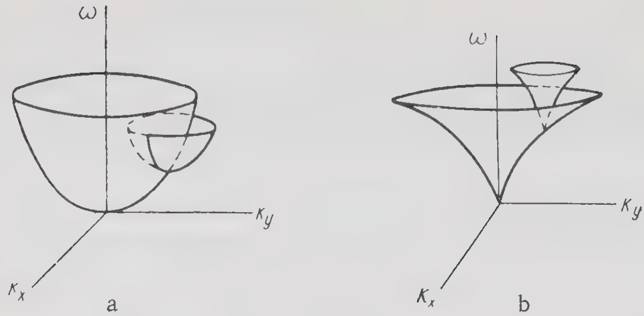

Fig. 9

tem of equations for the waves and resonant particles which takes account of the interaction of the waves (plasmons) in the form of the collision integrals in (54) and (57). It should be noted that the structure of the collision integrals (54) and (57) is simplified to some extent in a classical plasma. Since the mean "population number" N_k is related to the spectral energy density of the oscillations Q_k by $N_k = Q_k/\hbar\omega_k$,* in a classical plasma ($\hbar \to 0$), in treating the collision integral we need only retain the highest-order terms in N or n; in this case, the collision integrals (54) and (57) are quadratic in N and n:

$$-n_1 N_2 (n_3 + 1) + (n_1 + 1)(N_2 + 1) n_3 \underset{\hbar \to 0}{\to} -N_2 n_1 + N_2 n_3 + n_1 n_3, \quad (58)$$

$$-n_1 (N_2 + 1)(n_3 + 1) + (n_1 + 1) N_2 n_3 \underset{\hbar \to 0}{\to} -n_1 N_2 - n_1 N_3 + N_2 n_3 \quad (59)$$

and after the substitution $N_k = Q_k/\hbar\omega_k$, Planck's constant disappears from (54) and (57).†

*For example, for longwave plasma oscillations,

$$Q_k = \frac{nmv_k^2}{2} + \frac{E_k^2}{8\pi} = \frac{E_k^2}{4\pi}; \quad N_k = \frac{E_k^2}{4\pi\hbar\omega_0}.$$

†The direct derivation of the classical ($\hbar = 0$) collision integrals for plasmons using the hydrodynamic equations has been given in [27, 28].

The kinetic equations that describe three-plasmon processes in an anisotropic (and inhomogeneous) plasma can be obtained similarly by isolating third-order terms in the oscillation amplitude in the Lagrangian [Eq. (38)]. These three-plasmon processes are responsible for a number of important features of a turbulent plasma: in particular, the turbulent transport coefficients for matter, momentum, and energy. A knowledge of these coefficients is required to solve a number of problems: for instance, the structure of the turbulent front of a shock wave in a rarefied plasma [27, 28], or the evaluation of the "anomalous" diffusion coefficient (Problem 7), etc.

Problem 6. What functional relation $\omega = \omega(k)$ must be displayed by the dispersion relation to satisfy the frequency and wave-number conservation relations for three-plasmon interactions between plasmons of one kind?

Solution. For clarity we consider the case in which the frequency ω depends only on the modulus of the two-dimensional wave vector $\mathbf{k} = \{k_x, k_y\}$; in this case, the function $\mathbf{k} = k_x k_y$ represents a surface of rotation about the ω axis in $(k_x k_y \omega)$ space (Fig. 9).

The conservation relations allow three-plasmon processes if the equation $\omega(\mathbf{k}) + \omega(\mathbf{q}) = \omega(\mathbf{k} + \mathbf{q})$ has a solution, i.e., if the ω surface intersects a similar surface but drawn in a coordinate system whose origin lies on the ω surface (Fig. 9a); if this condition is not satisfied, the three-plasmon processes are forbidden (Fig. 9b). It is evident from Fig. 9 that the three-plasmon interactions are forbidden for spectra in which ω increases more slowly than k.

Problem 7. Estimate the value of the "anomalous" diffusion coefficient in a weakly inhomogeneous rarefied plasma in a magnetic field.

Solution. The origin of the anomalous diffusion is the excitation of drift waves in the unstable inhomogeneous plasma [24]. These waves are emitted by electrons and absorbed by ions; as a result there is a transfer of momentum, i.e., a frictional force between the electron and ion gases

$$f \approx \gamma' N \hbar k_\perp = \nu_{ef}\, nmU. \tag{A}$$

Here, $N \approx nMv_\sim^2/\hbar\omega$ is the density of the gas of drift waves; ω and $k_\perp$ are the characteristic frequency and wave vector, U is the drift velocity, γ' is the frequency of emission of waves by the electrons (the growth rate in the linear theory).

In the stationary turbulent state, the frequency of emission of waves γ' must be equal to the frequency of collisions between waves in three-plasmon processes; when $k_\perp \rho_i \approx 1$,

$$\gamma' \approx \frac{k_\perp^2 v_\sim^2}{\omega}. \tag{B}$$

Taking the value of $v_\sim^2$ from (B) and substituting in (A), we find the effect of collision frequency ν_{ef} and the coefficient of anomalous diffusion

$$D_a \approx \frac{\gamma'^2}{\omega\omega_{Hi}} \frac{cT}{eH} \approx \frac{m}{M\beta} \frac{\varrho_i}{a} \frac{cT}{eH}.$$

§ 9. Higher-Order Processes

In a number of cases the dispersion relation for the collective plasma oscillations is such that the three-plasmon interactions are forbidden by the frequency and wave-vector conservation relations. In this case it is necessary to consider processes in which four waves participate.

The matrix elements for four-plasmon processes (consequently, the transition probabilities) can be obtained by means of the Lagrangians L_3 and L_4 as in the three-plasmon processes (the contribution to the probability for four-plasmon processes is associated with matrix elements of first order in the perturbation theory in L_4 and second-order in the perturbation theory in L_3).

In the case of an isotropic plasma, which we consider below, the conservation relations forbid three-plasmon processes in which only plasma waves or ion-acoustic waves participate. For this reason the electron plasma waves excited in the plasma are described by an equation which considers the interaction of four plasma waves in addition to the three-plasmon processes considered above. However, if ion-acoustic waves are excited in the plasma, the kinetic situation is described by the interaction of four of these waves.

In general, in the interaction of four plasmons, one can find the following processes: (a) conversion of two plasmons into two other plasmons; (b) decay of one plasmon into three plasmons; (c) combination of three plasmons into one (Fig. 10a, b, c). Hence, the kinetic equation for waves in which four-plasmon processes occur is

$$\begin{aligned}\frac{dN_1}{dt} &= \sum w_{2,2} \{(N_1+1)(N_2+1)N_3N_4 - N_1N_2(N_3+1)(N_4+1)\} + \\ &+ \sum w_{1,3} \{(N_1+1)N_2N_3N_4 - N_1(N_2+1)(N_3+1)(N_4+1)\} + \\ &+ \sum w_{3,1} \{(N_1+1)(N_2+1)(N_3+1)N_4 - N_1N_2N_3(N_4+1)\}.\end{aligned} \tag{60}$$

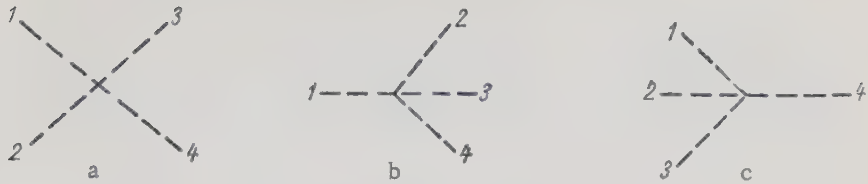

Fig. 10

The summation in Eq. (60) is taken over the wave numbers k_2, k_3, k_4, taking account of the conservation relations which, for the first, second, and third terms on the right side of Eq. (60) can be written as follows:

$$a)\ k_1 + k_2 = k_3 + k_4;\ \omega_1 + \omega_2 = \omega_3 + \omega_4;$$

$$b)\ k_1 = k_2 + k_3 + k_4;\ \omega_1 = \omega_2 + \omega_3 + \omega_4;$$

$$c)\ k_1 + k_2 + k_3 = k_4;\ \omega_1 + \omega_2 + \omega_3 = \omega_4.$$

The kinetic equation for the ion-acoustic waves only contains three terms; as far as the longwave electron plasma oscillations are concerned, we find that processes (b) and (c) are forbidden (since the frequency for these processes is approximately the same, ω_{0e}), so that Eq. (60) assumes the form

$$\dot{N}_1 = \sum w_{2.2}\{(N_1+1)(N_2+1)N_3N_4 - - N_1N_2(N_3+1)(N_4+1)\}. \tag{61}$$

Thus, the collision integral for four-plasmon processes is

$$\dot{N}_1 = \int B\{(N_1+1)(N_2+1)N_3N_4 - N_1N_2(N_3+1)(N_4+1)\} \times \times \delta\ (\omega_1 + \omega_2 - \omega_3 - \omega_4)\ dk_2\,dk_3, \tag{62}$$

where $k_4 = k_1 + k_2 - k_3$, and

$$B \approx \frac{\hbar^2 k^4}{n^2 m^2}. \tag{63}$$

Substituting $N_k = E_k^2/\hbar\,\omega_k$ from Eq. (62), we find the characteristic collision frequency in the gas of plasma waves:

$$\gamma_4 = \frac{\dot{N}}{N} \approx \omega\left(\frac{E^2}{nT}\right)^2, \tag{64}$$

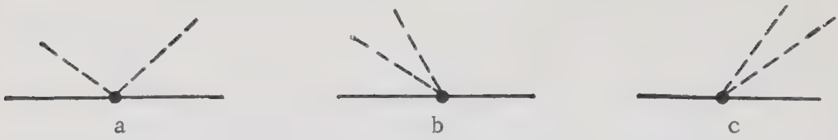

Fig. 11

where $\omega \approx \omega_{0e}$, while $\frac{E^2}{4\pi} = \sum_k E_k^2/4\pi$ is the energy density of the waves.

An estimate of the collision frequency γ_4 in the gas of these waves can be obtained as follows. Since the collision integral is proportional to N^3, the frequency γ_4 must be proportional to N^2, i.e., the fourth power of the ratio of the random velocity of the electrons in the plasma waves to the phase (or group) velocity U for these waves (with wave number $k < 1/R_D$). The coefficient of proportionality must be equal to the oscillation frequency (as follows from dimensional considerations), so that

$$\gamma_4 \approx \omega \left(\frac{v_\sim}{U}\right)^4. \tag{65}$$

Substituting $v_\sim \approx eE/m\omega$, $U \approx \sqrt{T/m}$, we obtain (64).

The frequency γ_4 is responsible for a number of characteristic features of a turbulent plasma. The decay time for the turbulence spectrum of plasma waves is approximately γ_4^{-1}; the frequency γ_4 (or the mean-free-path for plasma wave collisions) also determines the energy flux q in a gas of plasma waves; $q \approx (v_T^2/\gamma_4)\nabla E^2$; the damping for a nonlinear plasma wave of finite amplitude is also of order γ_4.

As we have noted above, the structure of the collision integral that describes ion waves colliding with ion waves is more complicated than for the plasma waves; the frequency of collisions in a gas of ion-acoustic waves can be estimated from Eq. (65) by making the substitutions

$$v_\sim \approx eE/M\omega; \quad U \approx \sqrt{T/M}; \quad \omega \approx \omega_{0i};$$

$$\gamma_4^s \approx \omega_{0i}\left(\frac{E^2}{nT}\right)^2. \tag{66}$$

It is possible to have situations in which the description of the physical effects in a weakly turbulent plasma requires higher-order processes than the emission and absorption of plasmons by particles or the three- and four-plasmon processes we have considered above. Some of the higher-order processes are scattering of a plasmon by a particle (Fig. 11a) and simultaneous absorption (Fig. 11b), or emission (Fig. 11c), of two plasmons by a particle. The need for considering these processes can arise because the frequency and wave-vector conservation rules do not allow absorption or emission of a single plasmon by a particle [26].

The collision term for plasmons described by the diagram in Fig. 11a is of the form

$$\dot{N}_1 = \sum w_{12;\,34} \{-N_1 (N_2 + 1) f_3 (1 - f_4) + \\ + (N_1 + 1) N_2 (1 - f_3) f_4\} \tag{67}$$

(the summation is carried out taking account of the conservation of frequency and wave vector); in a rarefied plasma this yields the following expression for the relative change in the number of plasmons:

$$\frac{\dot{N}_1}{N_1} = -\sum w_{12;\,34} N_2 (f_3 - f_4).$$

Proceeding in similar fashion, we can find the contribution of this process in the particle-collision term.

In treating the dynamics of a turbulent plasma, it may be necessary to sum the series in perturbation theory just as is done in solid-state theory; however, in most investigations of a weakly turbulent plasma, taking account of the emission or absorption of plasmons by electrons and ions and of three- and four-plasmon processes is sufficient.

Thus, the quasi-linear equations (8)-(9) with the plasmon collision integrals (54), (57), (62), and (67), represent a closed system for the investigation of a weakly turbulent plasma.

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SYMMETRIC MAGNETOHYDRODYNAMIC FLOW AND HELICAL WAVES IN A CIRCULAR PLASMA CYLINDER

L. S. Solov'ev

Introduction

The designation symmetric, or two-parameter, stationary flow refers to a flow in which all quantities depend on only two spatial coordinates q_1 and q_2, being independent of the third coordinate q_3. The existence of this kind of symmetry allows the introduction of a number of integrals of the motion and simplifies the analysis considerably. The most general form of spatial symmetry is helical symmetry, in which all quantities are constant along the helices $\varphi - \alpha z = \text{const}$, $r = \text{const}$, characterized by a fixed pitch $L = 2\pi/\alpha$. In order to formulate a general theory of symmetric flow which includes the case of helical symmetry, it is necessary to carry out the calculations in a nonorthogonal curvilinear coordinate system q_i (cf. Appendix).

Symmetric magnetohydrodynamic flow was first considered by Chandrasekhar [1], who considered the case of axially symmetric flow of an incompressible fluid. Tkalich [2,3] obtained equations for symmetric flow in an incompressible fluid in an arbitrary orthogonal coordinate system – but these equations only pertain to translational and axial symmetry. In the present review we consider the general flow of a compressible fluid characterized by helical symmetry; for purposes of simplicity, however, we shall make use of the usual cylindrical coordinate system r, φ, and z inasmuch as the transformation of the equations to new coordinates is not a difficult problem. The equations that apply for axial and translational symmetry are obtained from the case of helical symmetry by taking appropriate limits, allowing the pitch of the helix L to approach zero or infinity, respectively.

Symmetric magnetohydrodynamic flow includes particular cases of symmetric equilibrium plasma configurations [4-7] and symmetric hydrodynamic flow [8]. Helical magnetohydrodynamic flow is of special interest, since it includes the case of helical waves in a plasma cylinder, these waves being the most dangerous from the point of view of stability of a pinch that carries a longitudinal current.

§ 1. Stationary Helical Flow

In the absence of dissipative processes, the stationary flow of an ideally conducting fluid in a magnetic field is described by the following system of magnetohydrodynamic equations:

$$\varrho\,(\mathbf{v}\nabla)\,\mathbf{v} = -\nabla p + \frac{1}{4\pi}\,[\mathrm{rot}\,\mathbf{B}\mathbf{B}] - \varrho\nabla\Phi; \tag{1.1}$$

$$\mathrm{rot}\,[\mathbf{v}\mathbf{B}] = 0; \quad (\mathbf{v}\nabla S) = 0; \tag{1.2}$$

$$\mathrm{div}\,\varrho\mathbf{v} = 0; \quad \mathrm{div}\,\mathbf{B} = 0. \tag{1.3}$$

Here, $\mathbf{v}$ is the fluid velocity, ρ is the density of the fluid, p is the pressure, S is the entropy, **B** is the magnetic field, and Φ is the potential associated with external forces of nonelectromagnetic origin, for example, the force of gravity. The first equation in (1.2) expresses the condition that the lines of force of the magnetic field are frozen in the fluid, while the second expresses the conservation of entropy along the line of flow of the fluid. We introduce the heat function W in accordance with the thermodynamic relation

$$dW = \frac{dp}{\varrho} + T\,dS, \tag{1.4}$$

where T is the temperature.

In order to obtain greater symmetry in our relations, we introduce the notation

$$\mathbf{H} = \frac{\mathbf{B}}{\sqrt{4\pi}}; \qquad w = W + \frac{v^2}{2} + \Phi; \tag{1.5}$$

$$\mathbf{j} = \mathrm{rot}\,\mathbf{H}; \qquad \mathbf{j}_0 = \mathrm{rot}\,\mathbf{v}. \tag{1.6}$$

Equations (1.1)-(1.3) can now be written in the form

$$\mathrm{div}\,\mathbf{H} = 0; \qquad \mathrm{div}\,\varrho\mathbf{v} = 0; \tag{1.7}$$

$$[\mathbf{v}\mathbf{H}] = \nabla\Phi_E; \qquad [\mathbf{j}_0\mathbf{v}] - \frac{1}{\varrho}\,[\mathbf{j}\mathbf{H}] = -\nabla w + T\nabla S, \tag{1.8}$$

where Φ_E is a function which is proportional to the potential of the electric field.

We now assume that the flow is helically symmetric, i.e., that all quantities depend only on the two variables

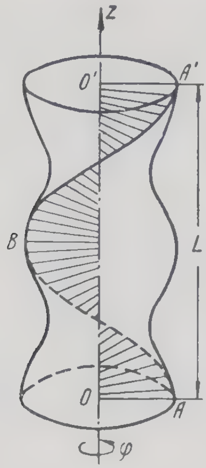

Fig. 1

$$q_1 = r; \qquad q_2 = \theta = \varphi - \alpha z, \tag{1.9}$$

where r, φ, and z are the usual cylindrical coordinates; $\alpha = 2\pi/L$; L = const is the pitch of the helix (Fig. 1).

1. The fact that the divergence of the vectors $\rho\mathbf{v}$, $\mathbf{H}$, $\mathbf{j}_0$, and $\mathbf{j}$ vanishes allows us to introduce the stream functions ψ_0, ψ, I_0, and I, which are defined by the relations

$$rv_r = \frac{1}{\varrho}\frac{\partial\psi_0}{\partial\theta}; \quad v_\varphi - \alpha r v_z = -\frac{1}{\varrho}\frac{\partial\psi_0}{\partial r}; \quad rH_r = \frac{\partial\psi}{\partial\theta}; \tag{1.10}$$

$$H_\varphi - \alpha r H_z = -\frac{\partial\psi}{\partial r};$$

$$rj_{0r} = \frac{\partial I_0}{\partial\theta}; \quad j_{0\varphi} - \alpha r j_{0z} = -\frac{\partial I_0}{\partial r}; \quad rj_r = \frac{\partial I}{\partial\theta}; \tag{1.11}$$

$$j_\varphi - \alpha r j_z = -\frac{\partial I}{\partial r}.$$

It is easy to show that the streamlines of the fluid lie on the surfaces $\psi_0(r, \theta)$ = const, those of the magnetic lines of force on the surfaces $\psi(r, \theta)$ = const, those of the electric current flow on the surfaces $I(r, \theta)$ = const, and those of the vortex lines on the surfaces $I_0(r, \theta) = \text{const}$; it can also be shown that all of these surfaces exhibit helical symmetry.*

The stream function ψ for the magnetic field is expressed in terms of A_z and A_φ, the components of the vector potential ($\mathbf{H} = \operatorname{rot} \mathbf{A}$)

$$\psi = A_z + arA_\varphi.$$

The quantity ψ is proportional to the magnetic flux passing through the helicoidal surface formed by the normals to the z axis drawn to a helical line on the helical magnetic surface being considered. This flux is given by $\int \mathbf{H} d\mathbf{S} = \oint \mathbf{A} d\mathbf{l} = LA_z + 2\pi r A_\varphi = L\psi$, where the integration is carried out over the path OABA'O'O. Similarly, the function ψ_0 is proportional to the fluid flow through the same surface, while $I(r, \theta)$ and $I_0(r, \theta)$ are proportional to the current and vortex flow $\operatorname{rot} \mathbf{v}$ through the surface.

2. Multiplying the first equation in (1.8) successively by $\mathbf{v}$ and $\mathbf{H}$, we obtain the Jacobian relations $\partial(\psi_0, \Phi_E)/\partial(r, \theta) = 0$ and $\partial(\psi, \Phi_E)/\partial(r, \theta) = 0$, from which it follows that $\Phi_E = \Phi_E(\psi)$ and $\psi = \psi(\psi_0)$, i.e., the magnetic surfaces are equipotentials and the fluid flows along these surfaces. Without giving preference to either one of the functions ψ and ψ_0, we introduce the new variable ξ [2], the normalization of which remains arbitrary. Then, we write

$$\psi = \psi(\xi); \qquad \psi_0 = \psi_0(\xi). \tag{1.12}$$

3. It follows from the definitions of the vectors $\mathbf{j}_0$ and $\mathbf{j}$ that I_0 and I are proportional to the helical components of $\mathbf{v}$ and $\mathbf{H}$:

$$I_0 = v_z + arv_\varphi; \qquad I = H_z + arH_\varphi, \tag{1.13}$$

and that the following relations hold for the components of $\mathbf{j}_0$ and $\mathbf{j}$:

$$j_{0z} + arj_{0\varphi} = -\frac{\beta}{\varrho}\Delta^*\psi_0 + \frac{1}{\varrho^2}(\nabla\varrho\nabla\psi_0) + \frac{2\alpha}{\beta}I_0;$$

$$j_z + arj_\varphi = -\beta\Delta^*\psi + \frac{2\alpha}{\beta}I. \tag{1.14}$$

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*That is to say, these surfaces can be formed from helical lines with constant pitch L; in the planes perpendicular to the z axis, however, the cross section of these surfaces can be arbitrary. For example, Fig. 1 shows a helical surface whose transverse cross section is an ellipse.

Here we have introduced the notation

$$\Delta^* \equiv \frac{1}{r}\frac{\partial}{\partial r}\frac{r}{\beta}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}; \qquad \beta = 1 + \alpha^2 r^2.$$

4. The first equation in (1.8) yields the following Jacobian relation:

$$\frac{\partial\left(\psi, \frac{I_0}{\beta}\right)}{\partial (r, \theta)} = \frac{\partial\left(\psi_0, \frac{I}{\beta\varrho}\right)}{\partial (r, \theta)}. \tag{1.15}$$

5. The second equation in (1.8) yields the further Jacobian relation

$$\frac{\partial (\psi, I)}{\partial (r, \theta)} = \frac{\partial (\psi_0, I_0)}{\partial (r, \theta)}. \tag{1.16}$$

We find from Eqs. (1.15) and (1.16) that

$$I_0\psi_0' - I\psi' = A(\xi); \quad I\frac{\psi_0'}{\varrho} - I_0\psi' = \beta B(\xi). \tag{1.17}$$

The primes here and below denote derivatives with respect to ξ; A and B are arbitrary functions that depend only on ξ. Using the terminology of [9], we may say that these are surface quantities.

Solving Eq. (1.17) for I_0 and I, we have

$$I_0 = \frac{1}{s}\left(A\frac{\psi_0'}{\varrho} + \beta B\psi'\right); \qquad I = \frac{1}{s}(A\psi' + \beta B\psi_0'), \tag{1.18}$$

where

$$s \equiv \frac{\psi_0'^2}{\varrho} - \psi'^2. \tag{1.19}$$

6. Transformation of the transverse components of the second equation in (1.18) leads to the system

$$\frac{\partial}{\partial q_i}\left(w + \frac{\beta B^2}{s}\right) = D\frac{\partial \xi}{\partial q_i} + Q\frac{\partial \varrho}{\partial q_i}, \tag{1.20}$$

where

$$D = \frac{1}{\varrho}\left\{\frac{\psi_0'}{\varrho}\Delta^*\psi_0 - \frac{\psi_0'}{\beta\varrho^3}(\nabla\varrho\nabla\psi_0) - \psi'\Delta^*\psi\right\} +$$

$$+ \frac{1}{2\beta\varrho}\frac{\partial}{\partial\xi}\frac{A^2}{s} + \frac{\beta}{2}\frac{\partial}{\partial\xi}\frac{B^2}{s} - \frac{2\alpha A}{\beta^2\varrho} + TS'; \; Q = -\frac{AB\psi'\psi_0'}{\varrho^2 s^2},$$

and the partial derivative with respect to ξ is taken at constant density ρ.

From Eq. (1.20) we find that

$$\frac{\partial (D, \xi)}{\partial (r, \theta)} = - \frac{\partial (Q, \varrho)}{\partial (r, \theta)}; \tag{1.21}$$

whence, since $Q = Q(\rho, \xi)$, it follows that $\partial(D, \xi)/\partial(r, \theta) = (\partial Q/\partial \xi)[\partial(\rho, \theta)/\partial(r, \theta)]$. Now, converting from the variables ρ and ξ to the variables r and θ we obtain the equation $\partial D/\partial \rho = \partial Q/\partial \xi$, integration of which yields

$$D = - \frac{\partial}{\partial \xi} \frac{AB\psi_0'}{\varrho s \psi'} + U'(\xi), \tag{1.22}$$

where $U'(\xi)$ is an arbitrary function of ξ. The function $U(\xi)$ can be determined from Eq. (1.20) since

$$d\left(w + \frac{\beta B^2}{s}\right) = D\, d\xi + Q\, d\varrho = d\left(- \frac{AB\psi_0'}{\varrho s \psi'} + U\right). \tag{1.23}$$

Integrating Eq. (1.23) and transforming the expressions for $\Delta^* \psi_0$ and $\Delta^* \psi$ in Eq. (1.20) to the variable ξ, finally we have

$$\frac{s}{\varrho} \Delta^* \xi + \frac{1}{2\beta\varrho} \frac{\partial s}{\partial \xi} (\nabla \xi)^2 - \frac{\psi_0'^2}{\beta \varrho^3} (\nabla \varrho \nabla \xi) + \frac{1}{2\beta\varrho} \frac{\partial}{\partial \xi} \frac{A^2}{s} +$$

$$+ \frac{\beta}{2} \frac{\partial}{\partial \xi} \frac{B^2}{s} - \frac{2\alpha A}{\beta^2 \varrho} + \frac{\partial}{\partial \xi} \frac{AB\psi_0'}{\varrho s \psi'} + TS' - U' = 0; \tag{1.24}$$

$$w = - \frac{\beta B^2}{s} - \frac{AB\psi_0'}{\varrho s \psi'} + U. \tag{1.25}$$

In principle, Eq. (1.25) allows us to express ρ, which appears only in Eq. (1.24), in terms of ξ and its partial derivatives with respect to r and θ [cf. Eqs. (1.4)-(1.5)]. Thus, the original system of equations for stationary helical flow of an ideally conducting fluid in a magnetic field is reduced to a single equation for ξ [Eq. (1.24)]. This equation is nonlinear and contains six arbitrary functions of ξ: ψ_0, ψ, A, B, S, and U. In the problem of helical wave propagation in an initially circular plasma cylinder, these functions are determined by the unperturbed distribution of density $\rho(r)$, velocity $\mathbf{v}(r)$, and magnetic field $\mathbf{B}(r)$.

The complete system of equations that describes stationary flow of an ideally conducting fluid characterized by helical symmetry is then*

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*Since the components of the vectors $\mathbf{v}$ and $\mathbf{H}$ are expressed in terms of ξ by symmetric formulas, it is convenient to write them in matrix form.

$$\left.\begin{aligned}
&\frac{s}{\varrho}\Delta^*\xi+\frac{1}{2\beta\varrho}\frac{\partial s}{\partial\xi}(\nabla\xi)^2-\frac{\psi_0'^2}{\beta\varrho^3}(\nabla\varrho\nabla\xi)+\frac{1}{2\beta\varrho}\frac{\partial}{\partial\xi}\frac{A^2}{s}+\\
&+\frac{\beta}{2}\frac{\partial}{\partial\xi}\frac{B^2}{s}+\frac{\partial}{\partial\xi}\frac{AB\psi_0'}{\varrho s\psi'}-\frac{2\alpha A}{\beta^2\varrho}+TS'-U'=0;\\
&W+\frac{v^2}{2}+\Phi=-\frac{\beta B^2}{s}-\frac{AB\psi_0'}{\varrho s\psi'}+U;\\
&r\begin{pmatrix}v_r\\ H_r\end{pmatrix}=\begin{pmatrix}\psi_0'/\varrho\\ \psi'\end{pmatrix}\frac{\partial\xi}{\partial\theta},\quad \begin{pmatrix}v_\varphi\\ H_\varphi\end{pmatrix}-\alpha r\begin{pmatrix}v_z\\ H_z\end{pmatrix}=-\begin{pmatrix}\psi_0'/\varrho\\ \psi'\end{pmatrix}\frac{\partial\xi}{\partial r};\\
&\begin{pmatrix}v_z\\ H_z\end{pmatrix}+\alpha r\begin{pmatrix}v_\varphi\\ H_\varphi\end{pmatrix}=\frac{A}{s}\begin{pmatrix}\psi_0'/\varrho\\ \psi'\end{pmatrix}+\frac{\beta B}{s}\begin{pmatrix}\psi'\\ \psi_0'\end{pmatrix}.
\end{aligned}\right\}\quad\text{(I)}$$

Here, $s \equiv (\psi_0'^2/\rho) - \psi'^2$; the primes denote total differentiation with respect to ξ; the partial differentiation with respect to ξ is taken for fixed ρ: $\Delta^* \equiv (1/r)(\partial/\partial r)(r/\beta)(\partial/\partial r) + (1/r^2)(\partial^2/\partial\theta^2)$.

The equations for axially symmetric flow [$\xi = \xi(r, z)$] are then obtained by the limiting transition $\alpha \to \infty$. In this way we obtain the following system:

$$\left.\begin{aligned}
&\frac{s}{\varrho}\Delta^*\xi+\frac{1}{2\varrho r^2}\frac{\partial s}{\partial\xi}(\nabla\xi)^2-\frac{\psi_0'^2}{r^2\varrho^3}(\nabla\varrho\nabla\xi)+\frac{1}{2\varrho r^2}\frac{\partial}{\partial\xi}\frac{A^2}{s}+\\
&+\frac{r^2}{2}\frac{\partial}{\partial\xi}\frac{B^2}{s}+\frac{\partial}{\partial\xi}\frac{AB\psi_0'}{\varrho s\psi'}+TS'-U'=0;\\
&W+\frac{v^2}{2}+\Phi=-\frac{r^2B^2}{s}-\frac{AB\psi_0'}{\varrho s\psi'}+U;\\
&r\begin{pmatrix}v_r\\ H_r\end{pmatrix}=-\begin{pmatrix}\psi_0'/\varrho\\ \psi'\end{pmatrix}\frac{\partial\xi}{\partial z};\ r\begin{pmatrix}v_z\\ H_z\end{pmatrix}=\begin{pmatrix}\psi_0'/\varrho\\ \psi'\end{pmatrix}\frac{\partial\xi}{\partial r};\\
&r\begin{pmatrix}v_\varphi\\ H_\varphi\end{pmatrix}=\frac{A}{s}\begin{pmatrix}\psi_0'/\varrho\\ \psi'\end{pmatrix}+\frac{r^2B}{s}\begin{pmatrix}\psi'\\ \psi_0'\end{pmatrix}.
\end{aligned}\right\}\quad\text{(II)}$$

Here, as in the general case (I), the quantities ψ_0, ψ, A, B, S, and U are arbitrary functions that depend only on ξ; the operator Δ^* is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial z^2}.$$

In the particularly important case in which the magnetic field has only one component B_φ ($\psi' = 0$), the equation (II) yields*

$$\left.\begin{aligned}
&\frac{1}{\varrho^2}\Delta^*\xi - \frac{1}{r^2\varrho^3}(\nabla\varrho\nabla\xi) + \frac{AA'}{r^2} + \varrho r^2 BB' + TS' - U' = 0;\\
&W + \frac{v^2}{2} + \Phi = -\varrho r^2 B^2 + U;\\
&rv_r = -\frac{1}{\varrho}\frac{\partial\xi}{\partial z}, \qquad rv_z = \frac{1}{\varrho}\frac{\partial\xi}{\partial r}; \quad rv_\varphi = A;\\
&H_\varphi = \varrho r B,
\end{aligned}\right\} \quad \text{(II')}$$

where $\xi = \psi_0$.

For the plane case, in which there is no dependence on the longitudinal coordinate z, using the other limiting transition $\alpha \to 0$, we find

$$\left.\begin{aligned}
&\frac{s}{\varrho}\Delta\xi + \frac{1}{2\varrho}\frac{\partial s}{\partial\xi}(\nabla\xi)^2 - \frac{\psi_0'^2}{\varrho^3}(\nabla\varrho\nabla\xi) + \frac{1}{2\varrho}\frac{\partial}{\partial\xi}\frac{A^2}{s} +\\
&+\frac{1}{2}\frac{\partial}{\partial\xi}\frac{B^2}{s} + \frac{\partial}{\partial\xi}\frac{AB\psi_0'}{\varrho s\psi'} + TS' - U' = 0;\\
&W + \frac{v^2}{2} + \Phi = -\frac{B^2}{s} - \frac{AB\psi_0'}{\varrho s\psi'} + U;\\
&r\begin{pmatrix} v_r \\ H_r \end{pmatrix} = \begin{pmatrix} \psi_0'/\varrho \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial\varphi}; \quad \begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = -\begin{pmatrix} \psi_0'/\varrho \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial r};\\
&\begin{pmatrix} v_z \\ H_z \end{pmatrix} = \frac{A}{s}\begin{pmatrix} \psi_0'/\varrho \\ \psi' \end{pmatrix} + \frac{B}{s}\begin{pmatrix} \psi' \\ \psi_0' \end{pmatrix},
\end{aligned}\right\} \quad \text{(III)}$$

where Δ is the Laplacian and $\xi = \xi(r, \varphi)$. It is an easy matter to write the appropriate equations for a Cartesian coordinate system $\xi = \xi(x, y)$; in this case, we need only change the expressions for the transverse components of **v** and **H** in (III):

$$\begin{pmatrix} v_x \\ H_x \end{pmatrix} = \begin{pmatrix} \psi_0'/\varrho \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial y}; \quad \begin{pmatrix} v_y \\ H_y \end{pmatrix} = -\begin{pmatrix} \psi_0'/\varrho \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial x}.$$

*The term containing ψ' in the denominator vanishes in this case, as can be easily seen from the derivation of Eq. (1.24).

With the exception of § 6, we shall limit our further analysis of waves to incompressible fluids, in which case all of the equations obtained above are simplified considerably. As in ordinary hydrodynamics, the criterion for the applicability of the incompressible fluid model is the condition that the velocity must be small compared with the velocity of sound.

We now write the complete system of equations for helical flow for the case in which the fluid can be regarded as incompressible. Under these conditions, ρ = const and Eq. (1.24) contains only the single unknown function ξ. Setting the density ρ equal to unity, W = p, S = const, and using the notation

$$\frac{A}{s} \equiv a; \quad \frac{B}{s} \equiv b; \quad U - \frac{AB\psi_0'}{s\psi'} \equiv -su,$$

in accordance with (I) we can now write the equation for helical flow in the form [10]:

$$s\Delta^*\xi + \frac{s'}{2\beta}(\nabla\xi)^2 + \frac{1}{2\beta}(a^2 s)' + \frac{\beta}{2}(b^2 s)' - \frac{2\alpha a s}{\beta^2} + (us)' = 0; \quad (1.26)$$

$$p + \frac{v^2}{2} + \Phi = -s(u + \beta b^2); \; s \equiv \psi_0'^2 - \psi'^2; \quad (1.27)$$

$$r\begin{pmatrix} v_r \\ H_r \end{pmatrix} = \begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial\theta}; \quad \begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} - \alpha r\begin{pmatrix} v_z \\ H_z \end{pmatrix} = -\begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial r}; \quad (1.28)$$

$$\begin{pmatrix} v_z \\ H_z \end{pmatrix} + \alpha r\begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = a\begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix} + \beta b\begin{pmatrix} \psi' \\ \psi_0' \end{pmatrix}.$$

Here, ψ_0, ψ, a, b, and u are functions of ξ which can be assigned arbitrarily. The normalization of the function ξ can be chosen in various ways, depending on the convenience of the solution for a given problem. For example, following Tkalich we can define ξ from the requirement s = const, i.e., from the equation

$$\psi_0'^2 - \psi'^2 = s = \text{const}. \quad (1.29)$$

In this case, Eq. (1.26) is simplified considerably and we find

$$\Delta^*\xi + \frac{aa'}{\beta} + \beta bb' - \frac{2\alpha a}{\beta^2} + u' = 0. \quad (1.30)$$

However, in the general case, Eq. (1.30) appears in conjunction with a complicated equation (1.29) and in many cases it is more convenient to use Eq. (1.26) directly. In particular, in considering helical waves below we shall use another relation between ξ and the stream functions ψ_0 and ψ. The term $(2\alpha/\beta^2)as$ in Eq. (1.26) is characteristic for general symmetric flow, but vanishes in the particular limiting cases of axial ($\alpha \to \infty$) and translational ($\alpha \to 0$) symmetry.

Representative particular cases of Eqs. (1.29)-(1.30) are the following.

1. An equation for helical equilibrium magnetohydrodynamic configurations [6,7] ($\psi_0 = I_0 = 0$, whence it follows that $s = -1$, $\xi = \psi$, $a = I$, $b = 0$, $u = w$)

$$\Delta^*\psi + \frac{1}{2\beta}\frac{dI^2}{d\psi} - \frac{2\alpha I}{\beta^2} + \frac{dw}{d\psi} = 0; \tag{1.31}$$

$$w = p + \Phi.$$

2. An equation for helical flow in ordinary hydrodynamics for an incompressible fluid ($\psi = I = 0$, i.e., $s = 1$, $\xi = \psi_0$, $a = I_0$, $b = 0$, $u = -w$)

$$\Delta^*\psi_0 + \frac{1}{2\beta}\frac{dI_0^2}{d\psi_0} - \frac{2\alpha I_0}{\beta^2} - \frac{dw}{d\psi_0} = 0;$$

$$w = p + \frac{v^2}{2} + \Phi. \tag{1.32}$$

In these equations, the functions $w(\psi)$ and $I(\psi)$ [or $w(\psi_0)$ and $I_0(\psi_0)$] can be assigned arbitrarily.

It follows from Eq. (1.28) that the square of the velocity and the square of the magnetic field can be written in the form

$$v^2 = \frac{\psi_0'^2}{\beta}(\nabla\xi)^2 + \frac{1}{\beta}(a\psi_0' + \beta b\psi')^2;$$

$$H^2 = \frac{\psi'^2}{\beta}(\nabla\xi)^2 + \frac{1}{\beta}(a\psi' + \beta b\psi_0')^2. \tag{1.33}$$

Correspondingly, the total pressure is given by

$$p + \frac{H^2}{2} + \Phi = -\frac{s}{2\beta}(\nabla\xi)^2 - \frac{a^2 s}{2\beta} - \frac{\beta b^2 s}{2} - us. \tag{1.34}$$

The relations in (1.33) and (1.34) will be used below in deriving boundary conditions for a free plasma surface.

In concluding this section, we note that in the derivation of Eqs. (1.26)-(1.28) it is sufficient to assume that the density ρ is constant on the magnetic surfaces $\psi(r, \theta)$ = const [10]. In this case $\mathbf{v}$ is replaced by $\sqrt{\rho}\mathbf{v}$ where $\mathbf{v}$ is the fluid velocity and ρ is the density of the fluid.

For the sake of completeness we now present equations for axially symmetric and plane symmetric flow of an incompressible ideally conducting fluid [1-3]; these relations follow from Eqs. (1.26)-(1.28) if one makes the limiting transitions $\alpha \to \infty$ and $\alpha \to 0$.

A. For axially symmetric flow, in which case there is no dependence on the azimuthal coordinate φ, from Eqs. (1.26)-(1.28) we find

$$s\Delta^*\xi + \frac{s'}{2r^2}(\nabla\xi)^2 + \frac{1}{2r^2}(a^2s)' + \frac{r^2}{2}(b^2s)' + (us)' = 0; \quad (1.35)$$

$$s = \psi_0'^2 - \psi'^2; \; p + \frac{v^2}{2} + \Phi = -s(u + r^2b^2); \quad (1.36)$$

$$r\begin{pmatrix} v_r \\ H_r \end{pmatrix} = -\begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial z}; \; r\begin{pmatrix} v_z \\ H_z \end{pmatrix} = \begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial r};$$

$$r\begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = a\begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix} + r^2b\begin{pmatrix} \psi' \\ \psi_0' \end{pmatrix}, \quad (1.37)$$

where ψ_0, ψ, a, b, and u are arbitrary functions that depend only on ξ, while the operator Δ^* is given by $\Delta^* \equiv (1/r)(\partial/\partial r)(1/r)(\partial/\partial r) + (1/r^2)(\partial^2/\partial z^2)$.

The equations for axially symmetric magnetohydrodynamic flow (1.35)-(1.37) can be regarded as the generalization of the corresponding equations for equilibrium plasma configurations ($\mathbf{v}$ = 0) or for the flow of an incompressible fluid with no magnetic field ($\mathbf{H}$ = 0). For these cases, Eq. (1.35) becomes

1. If $\mathbf{v} = 0$; $w = p + \Phi$:

$$\Delta^*\psi + \frac{1}{2r^2}\frac{dI^2}{d\psi} + \frac{dw}{d\psi} = 0. \quad (1.38)$$

2. If $\mathbf{H} = 0$; $w = p + (v^2/2) + \Phi$:

$$\Delta^*\psi_0 + \frac{1}{2r^2}\frac{dI_0^2}{d\psi_0} - \frac{dw}{d\psi_0} = 0. \quad (1.39)$$

The quantities $I_0 = rv_\varphi$, $I = rH_\varphi$, and w are arbitrary functions of ψ_0 or, correspondingly, of ψ. For the axially symmetric problem ψ is determined by A_φ (the component of the vector potential $\psi = rA_\varphi$) and is proportional to the magnetic flux through a transverse cross section of the magnetic surface ψ = const. Similarly, ψ_0 is proportional to the liquid flow within the surface ψ_0 = const.

B. For the plane problem, in which case the dependence on the longitudinal coordinate z vanishes, using Eqs. (1.26)-(1.28) with $\alpha \to 0$, we find

$$s\Delta\xi + \frac{s'}{2}(\nabla\xi)^2 + \frac{1}{2}(a^2 s)' + \frac{1}{2}(b^2 s)' + (us)' = 0; \qquad (1.40)$$

$$s = \psi_0'^2 - \psi'^2;\; p + \frac{v^2}{2} + \Phi = -s(u + b^2); \qquad (1.41)$$

$$r\begin{pmatrix} v_r \\ H_r \end{pmatrix} = \begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial\varphi};\; \begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = -\begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix}\frac{\partial\xi}{\partial r};$$
$$\begin{pmatrix} v_z \\ H_z \end{pmatrix} = a\begin{pmatrix} \psi_0' \\ \psi' \end{pmatrix} + b\begin{pmatrix} \psi' \\ \psi_0' \end{pmatrix}. \qquad (1.42)$$

It is evident from Eq. (1.42) that for planar flow of an incompressible fluid the quantities v_z and H_z depend only on ξ.

The relations that have been derived above allow us to analyze various kinds of symmetric magnetohydrodynamic flow. However, we shall limit ourselves to more specialized problems; in particular, we shall consider the helical waves that can propagate in a circular plasma cylinder.

§ 2. Helical Waves

We consider helical waves that propagate in a circular plasma cylinder; these waves deform the magnetic surfaces, which are initially cylindrical. Assume that in the unperturbed state the cylinder exhibits a specified distribution of magnetic field $B_z = B_z(r)$, $B_\varphi = B_\varphi(r)$, and velocity $v_z = v_z(r)$, $v_\varphi = v_\varphi(r)$ (plasma jet), and that it is confined by an external field which also has the components $B_{ze}(r)$ and $B_{\varphi e}(r)$ (Fig. 2).

In general, the analysis of helical waves under steady-state conditions in a plasma cylinder with a free boundary r = R reduces to a single nonlinear equation for the function ξ which must satisfy a nonlinear boundary condition at its perturbed surface $\xi = \xi_\Sigma$ = const; the boundary condition derives from the pressure balance condition. We shall limit ourselves to steady-state waves that represent stationary helical flow in a coordinate system moving with the velocity v_Φ. A characteristic feature of waves that distinguishes them from arbitrary stationary flow, is the fact that the wave amplitude can be reduced

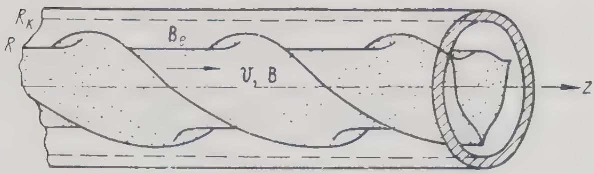

Fig. 2

to zero, i.e., there is an unperturbed state – the cylindrical jet. Mathematically this means that the equation for the waves must also contain solutions that represent cylindrical magnetic surfaces in the absence of waves.

1. In the equilibrium state $\mathbf{H} = \mathbf{H}(r)$ and $\mathbf{v} = \mathbf{v}(r)$, the quantity $\xi = \bar{\xi}$ can only be a function of r. We take this function to be

$$\bar{\xi} = \frac{r^2}{2}. \tag{2.1}$$

With this choice of the function $\bar{\xi}(r)$ the unperturbed stream functions ψ_0 and ψ are given by the following equations [cf. Eq. (1.28)]:

$$d\psi_0 = J_0\,(r)\,r\,dr, \quad d\psi = J\,(r)\,rdr, \tag{2.2}$$

where J_0 and J denote the equilibrium quantities

$$J_0 = \alpha v_z - \frac{v_\varphi}{r}; \quad J = \alpha H_z - \frac{H_\varphi}{r}. \tag{2.3}$$

We now write $\xi = \xi(r)$ in Eq. (1.26) and substitute the value of us obtained from Eq. (1.27); in this way we obtain the following equation for the equilibrium pressure distribution inside the plasma cylinder*

$$\frac{d}{dr}\left(p + \frac{H^2}{2}\right) + \frac{1}{r}\left(H_\varphi^2 - v_\varphi^2\right) = 0. \tag{2.4}$$

The field distribution in the unperturbed state can be specified arbitrarily; under these conditions the expressions for the current density components are

$$j_\varphi = -\frac{dH_z}{dr}; \quad j_z = \frac{1}{r}\frac{d}{dr} rH_\varphi; \tag{2.5}$$

similar relations hold for the component $\mathbf{j}_0 = \mathrm{rot}\,\mathbf{v}$.

*Here and below we take $\Phi = 0$, i.e., forces of nonhydromagnetic origin are neglected.

2. Assume that a helical wave propagates along a plasma cylinder; the magnetic field outside the cylinder is then also subject to a helical perturbation. In the coordinate system in which the fluid motion is represented by a stationary flow the external magnetic field is time-independent and exhibits helical symmetry. It is also convenient to describe this field by a stream function $\psi_e(r, \theta)$, since the boundary condition that H_e must be continuous at the perturbed plasma surface can be written most simply in terms of this function ($\psi_e|_\Sigma$ = const). The equation which is satisfied by the function $\psi_e(r, \theta)$ for the external curl-free magnetic field can be obtained from the general equations (1.26)-(1.28) by writing $\xi = \psi_e$ and setting $\mathbf{v}$ and p equal to zero. In this way we find

$$\Delta^* \psi_e - \frac{2\alpha}{\beta^2} I_e = 0; \tag{2.6}$$

$$rH_{re} = \frac{\partial \psi_e}{\partial \theta}; \quad \alpha r H_{ze} - H_{\varphi e} = \frac{\partial \psi_e}{\partial r}; \quad H_{ze} + \alpha r H_{\varphi e} = I_e. \tag{2.7}$$

Here, the quantity I_e is a function of ψ_e only. The form of this function is determined by the unperturbed state; in the unperturbed state the longitudinal field H_z is uniform, while the azimuthal field $H_{\varphi e} = H_{\varphi R}(R/r)$, so that I_e = const. Writing

$$\psi_e = \overline{\psi}_e + \tilde{\psi}_e = H_{ze} \frac{\alpha r^2}{2} - H_{\varphi e} r \ln r + \tilde{\psi}_e, \tag{2.8}$$

we find that $\tilde{\psi}_e$ satisfies the homogeneous equation $\Delta^* \tilde{\psi}_e = 0$; a solution of this equation, which is periodic in φ, is of the form $\psi_e = f_e(r) \cos m\theta$, where the function f_e for the m-th mode is given by the equation

$$\left(\frac{r}{\beta} f_e'\right)' - \frac{m^2}{r} f_e = 0. \tag{2.9}$$

The solutions of Eq. (2.9) are expressed in terms of derivatives of the Bessel functions of imaginary argument: $f_e(r) = A_e r I_m'(\alpha m r) + B_e r K_m'(\alpha m r)$.

Thus, the external magnetic field exhibits helical magnetic surfaces $\psi_e(r, \theta)$ = const, one of which can be identified with the perturbed surface of the plasma jet. Linear waves can be decomposed into individual modes; each of these modes can be matched to only one mode of the external field ψ_{me}. In the case of nonlinear waves, however, in general satisfaction of the boundary conditions will require an ensemble of modes.

3. In deriving the equation for the function ξ which describes the helical waves inside the plasma cylinder, we make further use of the circumstance

that $\psi_0(\xi)$, $\psi(\xi)$, $a(\xi)$, $b(\xi)$, and $u(\xi)$ are surface quantities and that the form of these functional dependences must be the same for the perturbed and equilibrium states.

Since $\overline{\xi} = r^2/2$, using the invariance of the functions $\psi_0(\xi)$ and $\psi(\xi)$, and introducing Eq. (2.2) for the equilibrium state, we find that $\psi_0(\xi)$ and $\psi(\xi)$ satisfy the equations

$$d\psi_0 = J_{0\xi}\, d\xi; \quad d\psi = J_\xi\, d\xi. \tag{2.10}$$

The subscript ξ means that a given quantity is taken in the equilibrium state, and that the argument r is to be replaced by $\sqrt{2\xi}$. Correspondingly, we obtain the following expression for $s(\xi)$:

$$s = J_{0\xi}^2 - J_\xi^2. \tag{2.11}$$

The wave equation for ξ can now be obtained from Eq. (1.26) where use is made of the fact that the quantity u is a function of ξ only; consequently, it can also be determined from equilibrium state. In Eq. (1.26) we write $\xi = \overline{\xi} = r^2/2$, and determine (us)' (as in the foregoing, we use the subscript ξ for the equilibrium quantities) and subtract it from Eq. (1.26). After some straightforward but lengthy calculations, we obtain the following equation for $\widetilde{\xi} = \xi - (r^2/2)$:

$$\Delta^*\widetilde{\xi} + \frac{rs'}{\beta s}\frac{\partial\widetilde{\xi}}{\partial r} + \left\{-\frac{4\alpha^2 a_1}{\beta^2 s} + \frac{4\alpha^2 a_1^2}{\beta s^2} + \frac{2a_1'}{\beta s} + \right.$$
$$\left. + \frac{1}{s}\left(\frac{a_1^2 - b_1^2}{s}\right)'\right\}\widetilde{\xi} + \frac{s'}{2\beta s}(\nabla\widetilde{\xi})^2 + \frac{2\alpha^2}{\beta s}\left(\frac{a_1^2}{s}\right)'\widetilde{\xi}^2 = 0. \tag{2.12}$$

Here, the primes denote differentiation with respect to ξ, while a_1 and b_1 denote functions of ξ given by

$$a_1 = \left(J_0\frac{v_\varphi}{r} - J\frac{H_\varphi}{r}\right)_\xi; \quad b_1 = \left(J_0\frac{H_\varphi}{r} - J\frac{v_\varphi}{r}\right)_\xi. \tag{2.13}$$

Thus, the form of Eq. (2.12) is determined by the equilibrium distribution of magnetic field **H**(r) and velocity **v**(r) inside the plasma cylinder. The equation that has been obtained is linear only when s and a_1 are constants. The latter situation holds if $\overline{v}_\varphi \sim r$, $\overline{v}_z = \text{const}$, $\overline{H}_\varphi \sim r$, and $\overline{H}_z = \text{const}$, i.e., for a cylinder with a uniform longitudinal current rotating as a whole about its own axis. In this case, the equation for $\widetilde{\xi}$ is linear

$$\Delta^*\widetilde{\xi} + \left(-\frac{4\alpha^2 a_1}{\beta^2 s} + \frac{4\alpha^2 a_1^2}{\beta s^2}\right)\widetilde{\xi} = 0, \tag{2.14}$$

and its solution, which is periodic in φ, is of the form

$$\tilde{\xi} = f(r) \cos m\theta, \tag{2.15}$$

where $f(r)$ satisfies the equation

$$\frac{1}{r}\left(\frac{r}{\beta} f'\right)' + \left(\frac{\varepsilon^2}{\beta} - \frac{2\alpha\varepsilon}{\beta^2} - \frac{m^2}{r^2}\right) f = 0, \tag{2.16}$$

in which $\varepsilon = 2\alpha a_1/s$ = const. The solutions of Eq. (2.16) are expressed in terms of Bessel functions:

$$f(r) = \varepsilon J_m(\varkappa r) - \varkappa \alpha r J'_m(\varkappa r);$$
$$\varkappa^2 = \varepsilon^2 - \alpha^2 m^2. \tag{2.17}$$

In this case, the analysis of waves inside the plasma cylinder can be carried through analytically.

Equation (2.12) will generally be nonlinear for an arbitrary equilibrium distribution of magnetic field and velocity. However, it is easy to write the appropriate linearized equation. The quantity $\tilde{\xi}$ is assumed to be small and quadratic terms in $\tilde{\xi}$ in Eq. (2.12) are neglected. Furthermore, in s, the quantities a_1 and b_1 can be written $\xi \simeq \overline{\xi} = r^2/2$, and the derivative $d/d\xi \simeq (1/r) \cdot (d/dr)$. Under these conditions we obtain a linear equation for ξ and the periodic solution also has the form of Eq. (2.15), in which the radial part $f(r)$ satisfies the equation

$$\left(\frac{rs}{\beta} f'\right) + \left\{- \frac{m^2 s}{r} + \frac{4\alpha^2 r a_1^2}{\beta s} + \left(\frac{2a_1}{\beta} + \frac{v_\varphi^2 - H_\varphi^2}{r^2}\right)\right\} f = 0. \tag{2.18}$$

The coefficients of this equation are known functions of r, which are determined by the equilibrium distribution of magnetic fields and velocity. For the case in which the plasma cylinder is at rest [$v_\varphi = 0$, $v_z = (\omega/k)$], equations equivalent to Eqs. (2.18) have been obtained in [14, 15].

4. In the general case, the function ξ must satisfy a nonlinear equation (2.12), the condition that it be finite inside the plasma cylinder, and the condition ξ = const at the perturbed cylinder boundary. The last condition guarantees continuity of both the velocity as well as the internal magnetic field at the surface of an ideally conducting plasma Σ. The pressure balance relation $p + (H^2/2) = (H_e^2/2)$ must also be satisfied at this surface. Since the external magnetic field H_e is described by the stream function ψ_e, then ψ_e must also be a constant at the boundary between the plasma and the external field $\psi_e/_\Sigma$ = const, and must satisfy appropriate boundary conditions at ex-

ternal conductors or dielectrics if there are such, or must fall off smoothly at $r \to \infty$ if we are considering a plasma in free space.

According to Eqs. (1.27)-(1.28), the pressure inside and outside the plasma is given by

$$p + \frac{H^2}{2} = -\frac{s}{2}\left\{\frac{1}{\beta}(\nabla\xi)^2 + \frac{a^2}{\beta} + \beta b^2 + 2u\right\};$$
$$\frac{H_e^2}{2} = \frac{1}{2\beta}(\nabla\psi_e)^2 + \frac{I_e^2}{2\beta}; \tag{2.19}$$

hence, if s and u are functions of ξ (and are constants at the boundary of the plasma Σ), the pressure-balance boundary condition can be written in the form

$$s\left\{\frac{1}{\beta}(\nabla\xi)^2 + \frac{a^2}{\beta} + \beta b^2\right\}_\Sigma + \left\{\frac{1}{\beta}(\nabla\psi_e)^2 + \frac{I_e^2}{\beta}\right\}_\Sigma = \text{const.}$$

Transforming this relation, we now express ξ and ψ_e in terms of the perturbations $\tilde{\xi}$ and $\tilde{\psi}_e$. As a result, we have

$$s\left\{\frac{1}{\beta}(\nabla\tilde{\xi})^2 + \frac{2r}{\beta}\frac{\partial\tilde{\xi}}{\partial r} + \left(\frac{2a_1}{\beta} + \frac{a_1^2 - b_1^2}{s}\right)\frac{2\tilde{\xi}}{s} + \right.$$
$$\left. + \frac{4\alpha^2 a_1^2}{\beta s^2}\tilde{\xi}^2\right\}_\Sigma + \left\{\frac{1}{\beta}(\nabla\tilde{\psi}_e)^2 + \frac{2J_e r}{\beta}\frac{\partial\tilde{\psi}_e}{\partial r} + H_{\varphi e}^2\right\}_\Sigma = \text{const.} \tag{2.20}$$

Here, a_1, b_1, and s are known functions of ξ given by Eqs. (2.11) and (2.13), $J_e = \alpha H_{ze} - (H_{\varphi e}/r)$, where H_{ze} = const and $H_{\varphi e} = H_{\varphi R}R/r$ represents the perturbed component of the external magnetic field.

Thus, we find that there are three conditions that must be satisfied at the free boundary between the plasma and the magnetic field: ξ_Σ = const, $\psi_{e\Sigma}$ = const, and the condition in (2.20). The boundary condition in (2.20) is always nonlinear, so that the problem of waves in a plasma with free boundaries can only be solved exactly by numerical methods. A characteristic feature of the present problem is the specification of the boundary conditions at the unknown (to be determined) boundary. The analogous problem of stationary nonlinear waves on the plane surface of a heavy liquid in ordinary hydrodynamics has available certain existence theorems [11-13]; however, the effective solution of even these simpler problems can only be effected by means of approximation methods.

The boundary conditions for $\tilde{\xi}$ [which satisfies Eq. (2.12)] can be simplified considerably if the plasma cylinder is surrounded by an ideally conducting hollow metal cylinder at r = R. Under these conditions, it is only required that $\tilde{\xi}$ satisfy a single simple boundary condition $\tilde{\xi}|_R = 0$; the problem of wave propagation in this kind of cylinder can be solved exactly if Eq. (2.12) is linear [cf. Eq. (2.14)].

For small oscillations, the linearized boundary conditions lead to a single condition for the function $f(r)$ given by Eq. (2.18). The linear waves are then represented by a superposition of modes given by

$$\xi = \frac{r^2}{2} + f(r)\cos m\theta;$$

$$\psi_e = H_{ze}\frac{ar^2}{2} - H_{\varphi e} r \ln r + f_e(r)\cos m\theta, \tag{2.21}$$

and can be treated individually in terms of propagation of single modes. In accordance with Eq. (2.20) the boundary conditions can be written in the form

$$\xi_\Sigma = \text{const}; \quad \psi_{e\Sigma} = \text{const}; \quad sP_{i\Sigma} + P_{e\Sigma} = \text{const}, \tag{2.22}$$

where P_i and P_e (to linear terms in $\tilde{\xi}$ and $\tilde{\psi}_e$) are given by

$$P_i = \frac{2r}{\beta}\frac{\partial\tilde{\xi}}{\partial r} + \left(\frac{4a_1}{\beta s} + \frac{2}{s}\frac{v_\varphi^2 - H_\varphi^2}{r^2}\right)\tilde{\xi};$$

$$P_e = \frac{H_{\varphi R}^2 R^2}{r^2} + \frac{2rJ_e}{\beta}\frac{\partial\tilde{\psi}_e}{\partial r}. \tag{2.23}$$

At the boundary between the plasma cylinder and the external magnetic field Σ we have

$$r = R + \varrho_1 \cos m\theta, \tag{2.24}$$

where ρ_1 = const and $\tilde{\xi}$ is a first-order quantity. Taking account of the boundary conditions in (2.22), as a first approximation we have

$$R\varrho_1 + f = 0;$$

$$RJ_e\varrho_1 + f_e = 0;$$

$$s\left\{\frac{2R}{\beta}f' + \left(\frac{4a_1}{\beta s} + \frac{2}{s}\frac{v_\varphi^2 - H_\varphi^2}{R^2}\right)f\right\} - \frac{2H_{\varphi e}^2}{R}\varrho_1 + \frac{2RJ_e}{\beta}f'_e = 0. \tag{2.25}$$

Then, eliminating ρ_1, finally we obtain the boundary condition on the function $f(r)$:

$$(J_0^2 - J^2)\frac{Rf'}{f} + J_e^2 \frac{Rf_e'}{f_e} + \beta\left(\frac{2a_1}{\beta} + \frac{v_\varphi^2 - H_\varphi^2 + H_{\varphi e}^2}{R^2}\right) = 0, \quad (2.26)$$

in which all quantities are taken at the unperturbed boundary $r = R$.

Equation (2.18) and the associated boundary condition (2.26), together with the condition $f(0) = 0$, represent the complete system of equations for determining the phase velocity v_Φ of the linear helical waves $[J_0 = \alpha(v_z^0 - v_\Phi) - (v_\varphi/r)]$. Since the logarithmic derivative of f_e (for the external field) is a known function of r, the condition in (2.26) represents a linear Sturm–Liouville boundary-value problem which relates the function $f(R)$ to its derivative $f'(R)$.

5. All of the relations obtained above can be easily generalized to the case in which the plasma is imbedded in a neutral gas characterized by a pressure p_e. We treat the external medium as an incompressible medium (for velocities much smaller than the velocity of sound), so that it is possible to use the calculational technique given above, expressing the flow function for the external gas in the form $\xi_0 = (r^2/2) + \tilde{\xi}_0$. Omitting the calculations, which are similar to those given above, we write the linearized equation for $\tilde{\xi}_0 = f_0(r)\cdot \cos m\theta$ and the boundary condition equivalent to (2.26) for a plasma taking account of the external pressure:

$$\left(\frac{rJ_{0e}}{\beta} f_0'\right)' + \left\{-\frac{m^2 J_{0e}}{r} + \frac{4\alpha^2 v_{\varphi e}^2}{\beta r} + \left(\frac{2J_{0e}v_{\varphi e}}{\beta r} + \frac{v_{\varphi e}^2}{r^2}\right)'\right\} f_0 = 0; \quad (2.27)$$

$$(J_0^2 - J^2)\frac{Rf'}{f} - J_{0e}^2 \frac{Rf_0'}{f_0} + J_e^2 \frac{Rf_e'}{f_e} + \beta\left(2\frac{a_1 - J_{0e}v_\varphi/R}{\beta} + \frac{v_\varphi^2 - v_{\varphi e}^2 - H_\varphi^2 + H_{\varphi e}^2}{R^2}\right) = 0. \quad (2.28)$$

Here, $J_{0e} = \alpha(v_{ze}^0 + v_\Phi) - (v_{\varphi e}/r)$; $v_{ze}(r)$ and $v_{\varphi e}(r)$ are the longitudinal and azimuthal unperturbed velocities of the external gas and all quantities in Eq. (2.28) are taken at the boundary $r = R$.

§ 3. Stability of a Cylindrical Plasma Jet in a Magnetic Field

We now consider certain applications of the results that have been obtained above; in particular, we wish to investigate the stability of various stationary plasma configurations. In the linear approximation the equations that describe helical waves in a circular plasma cylinder located in an external field are

$$\left(\frac{rs}{\beta} f'\right)' + \left\{-\frac{m^2 s}{r} + \frac{4\alpha^2 r a_1^2}{\beta s} + \left(\frac{2a_1}{\beta} + \frac{v_\varphi^2 - H_\varphi^2}{r^2}\right)'\right\} f = 0; \quad (3.1)$$

$$\left(\frac{r}{\beta} f_e'\right)' - \frac{m^2}{r} f_e = 0. \quad (3.2)$$

The boundary condition at r = R is essentially the dispersion equation for determining the phase velocity $v_\Phi = \omega/k$ of the individual helical wave modes: $\sim \exp i(kz - m\varphi - \omega t)$:

$$s\frac{rf'}{f} + J_e^2 \frac{rf_e'}{f_e} + \beta\left(\frac{2a_1}{\beta} + \frac{v_\varphi^2 - H_\varphi^2 + H_{\varphi e}^2}{r^2}\right) = 0. \quad (3.3)$$

Here, $s = J_0^2 - J^2$; $ra_1 = J_0 v_\varphi - JH_\varphi$;

$$J_0 = \alpha(v_z + v_\Phi) - \frac{v_\varphi}{r}; \quad J = \alpha H_z - \frac{H_\varphi}{r};$$

$$\beta = 1 + \alpha^2 r^2; \quad \alpha = \frac{k}{m}; \quad \mathbf{v}(r) \text{ and } \mathbf{B}(r) = \sqrt{4\pi\varrho}\,\mathbf{H}(r)$$

are the velocity and magnetic field, while the subscript e means that a given quantity refers to the magnetic field external to the plasma.

1. We first consider a plasma carrying a uniform current which rotates as a whole around its own axis (H_φ, $v_\varphi \sim r$; H_z, v_z = const). In this case, Eq. (3.1) has the solution

$$f = \varepsilon J_m(\varkappa r) - \alpha\varkappa r J_m'(\varkappa r),$$

where $\varkappa^2 = \varepsilon^2 - k^2$; $\varepsilon = 2ka_1/ms$ = const.

a. If the pinch is isolated from the wall, the dispersion equation (3.3) is written in the form

$$(J_0^2 - J^2)\frac{\varkappa^2 J_m(\varkappa R)}{\varepsilon J_m(\varkappa R) - \alpha\varkappa R J_m'(\varkappa R)} +$$

$$+ J_e^2 \frac{\alpha m^2 F_e(R)}{RF_e'(R)} + \frac{\alpha}{R^2}\left(v_\varphi^2 - H_\varphi^2 + H_{\varphi e}^2\right) = 0. \tag{3.4}$$

Here $f_e = rF_e'$, where the function $F_e(r)$ satisfies the Bessel equation; in the presence of an ideally conducting wall at $r = R_k$ we have

$$F_e(r) = K_m'(kR_k)\, I_m(kr) - I_m'(kR_k)\, K_m(kr). \tag{3.5}$$

We shall limit ourselves to the case of longwave oscillations ($kR \ll 1$; $\varepsilon R \ll 1$) in which case the phase velocity v_Φ is given by the following expression:

$$kv_\Phi = (m-1)\, v_\varphi \pm$$

$$\pm \sqrt{(m-1)\left(mH_\varphi^2 - 2kH_\varphi H_z - v_\varphi^2\right) + k^2H_z^2 - mH_{\varphi e}^2 - \frac{m^3 J_e^2 F_e}{F_e'}}. \tag{3.6}$$

Here, we have written $R = 1$. It follows from the relation given above that the H_φ and v_φ do not appear in the dispersion equation when $m = 1$. Consequently, neither the uniform current [16] nor the rotation have any effect on the stability criterion with respect to longwave curvature of the pinch $m = 1$. The stability condition is that the radical in Eq. (3.6) must be positive.

For the case of a pinch that is not located in a chamber ($R_k \to \infty$), the stability criterion can be written in the form

$$k\left(1 + \varepsilon_1^2\right)\frac{H_z}{H_{\varphi e}} > (m-1)\,\alpha_1 + m\varepsilon_1 +$$

$$+ \sqrt{\begin{matrix}(m-1)\left(1+\varepsilon_1^2\right)\alpha_2^2 + m\varepsilon_1^2\left[1-(m-1)\,\alpha_1^2\right] - \\ -(m-1)\left(\alpha_1^2 - 2\alpha_1 m\varepsilon_1 + m\right)\end{matrix}}, \tag{3.7}$$

where we have introduced the notation $\alpha_1 = H_\varphi/H_{\varphi e}$, $\varepsilon_1 = H_{ze}/H_z$, and $\alpha_2 = v_\varphi/H_{\varphi e}$. The criterion (3.7) has been obtained by Shafranov for the case of a cylinder at rest ($v_\varphi = 0$) [16]. Equation (3.7) indicates that the rotation of the cylinder raises the magnitude of the longitudinal field H_z required for stabilization and, therefore, favors instability. We shall show below that the situation is reversed when the plasma touches the chamber wall ($r = R$), i.e., rotation of the cylinder is then a necessary condition for stability. We note further that for a cylinder at rest ($v_\varphi = 0$) with no surface currents ($H_z = H_{ze}$, $H_\varphi = H_{\varphi e}$), the following stability condition follows from Eq. (3.7):

$$H_z > \frac{mH_\varphi}{kR}, \tag{3.8}$$

or $J(R) > 0$. It is easy to show that this condition holds regardless of the value of $F_e(r)$, i.e., this relation holds in the presence or absence of a chamber wall. We shall also show below that $J(R) = 0$ also represents the stability limit in the case of an arbitrary distribution of current inside the plasma cylinder.

b. The problem of stationary waves in a plasma cylinder in contact with an ideally conducting wall at $r = R$ can be solved exactly: for a uniform current and uniform rotation the linear equation (3.1) is not an approximation and the boundary condition $f(R) = 0$ can also be satisfied exactly.

Using the expression given for $f(r)$ in the preceding section, we now write the condition $f(R) = 0$ in the form

$$m\sqrt{k^2R^2 + x^2}\, J_m(x) + kRxJ_{m+1}(x) = 0. \tag{3.9}$$

We denote the roots of Eq. (3.9) by μ_n [for linear waves ($kR \ll 1$), for example, the μ_n are the roots of the Bessel function $J_m(x)$]. Writing $x \equiv \varkappa R = \mu_n$, we obtain the following expression for J_0:

$$J_0 = \frac{kv_\varphi}{m\mu_n} \pm \sqrt{J^2 - \frac{2kH_\varphi}{m\mu_n}J + \left(\frac{kv_\varphi}{m\mu_n}\right)^2}. \tag{3.10}$$

It then follows that the stability condition for a nonrotating cylinder ($v_\varphi = 0$) is $J > 2kH_\varphi/\mu_n$. For longwave oscillations ($kR \ll 1$) this condition is approximately $J > 0$, i.e., it is approximately the stability criterion for the case in which the plasma is detached from the walls when the fields H_z and H_φ are continuous at the surface.

If the plasma cylinder rotates ($v_\varphi \neq 0$), we can treat the radical in Eq. (3.10) as a quadratic form in J and the condition that it be positive definite is $v_\varphi^2 > H_\varphi^2$. On the other hand, if the pressure is to be a diminishing function of radius $p'(r) < 0$, we require $v_\varphi^2 < 2H_\varphi^2$, as follows from Eq. (2.4). Thus, there exists a range of rotational velocities

$$B_\varphi^2 < 4\pi\varrho v_\varphi^2 < 2B_\varphi^2, \tag{3.11}$$

for which a pinch with a parabolic pressure distribution that vanishes at the wall $r = R$ is stable against arbitrary helical perturbations.

2. It is of interest to obtain local stability criteria that are independent of boundary conditions. If some distribution of internal magnetic field and velocity yields a local solution of Eq. (3.1) for the phase velocity v_Φ which contains an imaginary part, this distribution is unstable. The phase velocity only appears in Eq. (3.1) in J_0; consequently, a necessary condition for stability is that there be no solutions of Eq. (3.1) with complex J_0.

a. We shall limit ourselves to the case $\mathbf{v} \parallel \mathbf{H}$ and consider the vicinity of the point $r = r_S$ at which $J(r) = 0$, i.e., a perturbation directed along some internal helical line of force of $\mathbf{H}$. We shall assume that at the point r_S the quantity $J_0^2(r) < 0$, and that this quantity is small in absolute magnitude, and that $m \gg 1$. Expanding (3.1) in the vicinity of $r = r_S$, we obtain the equation

$$\frac{d^2F}{d\zeta^2} + \left\{ -E - \frac{N}{(1+\zeta^2)^2} + \frac{M}{1+\zeta^2} \right\} F = 0, \tag{3.12}$$

where

$$F = \sqrt{\frac{rs}{\beta}}\, f; \quad \zeta = \frac{\varrho}{\varrho_0} + i\frac{J_0'}{J'}; \qquad \varrho = r - r_s;$$

$$\varrho_0^2 = \frac{J_0^2 J'^2}{(J'^2 - J_0'^2)^2}; \qquad E = \frac{m^2\beta}{r^2}\varrho_0^2;$$

$$N = 1 + \frac{4\alpha^2 H_\varphi^2}{J'^2 r^2}; \quad M = -\frac{2\alpha^2}{r^2}\,\frac{\left(p + \frac{v^2}{2}\right)'}{J'^2 - J_0'^2}.$$

If the variable $\zeta = x$ is real, the condition for the existence of localized solutions of Eq. (3.12) is $M > 1/4$ [17]. A local solution on the real axis $\zeta = x$ can be analytically continued into the complex plane $\zeta = x + iy$. Consequently, when $\zeta = x + i(J_0'/J')$ there are also localized solutions if $|J_0'| < |J'|$, i.e., if the parallel displacement from the x axis is smaller than the distance to the singular points of Eq. (3.12) located at $y = \pm i$. The latter condition is equivalent to the requirement $\mathbf{v}^2 < \mathbf{H}^2$. In expanded form the necessary condition for stability $M < 1/4$ can be written

$$\frac{8\pi}{B_z^2}\left(p + \frac{\varrho v^2}{2}\right)' + \frac{r}{4}\left(\frac{\mu'}{\mu}\right)^2\left(1 - \frac{4\pi\varrho v^2}{B^2}\right) > 0. \tag{3.13}$$

Here, $\mu = B_\varphi/rB_z$ and we have taken account of the equilibrium condition in (2.4). The criterion in (3.13) is a generalization of the Suydam stability criterion [18] to the case of nonzero velocities along the magnetic field $\mathbf{v}^2 < (\mathbf{B}^2/4\pi\rho)$. When $B_\varphi \sim r$, it becomes the condition obtained above, $v_\varphi^2 < (B_\varphi^2/4\pi\rho)$.

b. In the particular case of axially symmetric oscillations ($m = 0$) of a rotating plasma cylinder with no longitudinal magnetic field ($B_z = 0$), in which the analysis just given does not apply, Eq. (3.1) yields ($v_\Phi = \omega/k$):

$$\left(\frac{1}{r}f'\right)' + \left\{-\frac{k^2}{r} + \frac{k^2}{\omega^2}\left[\frac{(r^2 v_\varphi^2)'}{r^4} - \left(\frac{H_\varphi^2}{r^2}\right)'\right]\right\} f = 0. \tag{3.14}$$

It follows from the general Sturm–Liouville theory [22] that the necessary and sufficient condition for the existence of characteristic values for which $\omega^2 > 0$ for the boundary value $f(0) = f(R) = 0$ is that the expression in the square brackets in Eq. (3.14) be positive. We then obtain the following stability criterion:

$$4\pi\left(\varrho r^2 v_\varphi^2\right)' - r^4\left(B_\varphi^2/r^2\right)' > 0, \tag{3.15}$$

which becomes the well-known Rayleigh stability criterion when $B_\varphi = 0$; when $B_\varphi \sim r$ and $v_\varphi \sim r$, this becomes the condition $v_\varphi^2 > 0$, which can be obtained from the actual solution of the appropriate problem (for $m = 0$, $B_z = 0$). The criterion in (3.15), like that in (3.14), is determined by the internal parameters and is independent of the boundary conditions at the surface of the plasma cylinder. This condition can also be obtained from an analysis of the balance of forces acting on a given volume element in the plasma [19].

3. From the boundary condition (3.3) for the case of continuous fields at the free plasma surface (i.e., the absence of surface currents), it follows that $J(R) \equiv (kH_z/m) - (H_\varphi/R) = 0$ represents the boundary of the stability region. If J and v_φ are small so that quadratic terms in these quantities can be neglected in Eq. (3.3), and if $H_\varphi = H_{\varphi e}$ and $H_z = H_{ze}$, then

$$J_0 \simeq -\frac{v_\varphi}{R^2}\frac{f}{f'} \pm \sqrt{\frac{2H_\varphi f}{R^2 f'}\,J}\,. \tag{3.16}$$

The quantity J_0 and, consequently, the phase velocity v_Φ, acquire imaginary parts when J goes through zero. Thus, when

$$B_z = \frac{mB\varphi}{kR} \tag{3.17}$$

an instability arises which develops at the surface of the cylinder for $\mathbf{k} \parallel \mathbf{B}$. This result, which is independent of the distribution of internal magnetic field, is consequently independent of the Suydam condition. In the case being considered, in which there are no surface currents, the helical lines of force of the internal field and the external field do not intersect at the boundary of the plasma cylinder and the plasma can leak through the lines of force in the outward direction.* In § 5, making use of nonlinear terms, we shall show that the plasma actually forms "narrow" tongues directed along the helical lines of force at the surface of the cylinder.

*It follows from Eq. (2.28) that this "surface" instability also arises in the presence of an external neutral gas.

§ 4. Nonlinear Longwave Axisymmetric Oscillations of a Plasma Cylinder

In the present section we shall be concerned with the axisymmetric waves in a plasma cylinder located inside a coaxial, ideally conducting chamber (Fig. 3). It is assumed that the wavelength is large compared with the radius of the chamber; on the other hand, the amplitude of the oscillations of the surface of the plasma cylinder need not be small compared with the radius.

We assume that in the unperturbed state the plasma cylinder is at rest and that the internal (B_z) and external (B_{ze}) magnetic fields are uniform; we also assume that there is a longitudinal surface current which produces an azimuthal magnetic field $B_{\varphi a}(a/r)$ outside the plasma cylinder. In this case the stream functions $\psi_0(r, z)$ for the velocity fields and $\psi_e(r, z)$ for the external magnetic field (in the coordinate system that moves with the wave) satisfy the linear equation

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=0, \tag{4.1}$$

where $\mathbf{v}$ and $\mathbf{H}_e$ can be expressed in terms of ψ_0 and ψ_e by means of the relations

$$rv_z=\frac{\partial\psi_0}{\partial r};\qquad rv_r=-\frac{\partial\psi_0}{\partial z};$$

$$rH_z=\frac{\partial\psi_e}{\partial r};\qquad rH_r=-\frac{\partial\psi_e}{\partial z}, \tag{4.2}$$

while the internal magnetic field is described by the function ψ_i, which is proportional to ψ_0. It will be evident that for a uniform (unperturbed) magnetic field the appropriate stationary flow is irrotational and current-free ($\operatorname{rot}\mathbf{v} = \operatorname{rot}\mathbf{H} = 0$).

In order to determine ψ approximately, we assume that $\psi = \psi(r, \varepsilon z)$, where ε is a small parameter, and that the solution of Eq. (4.1) can be written

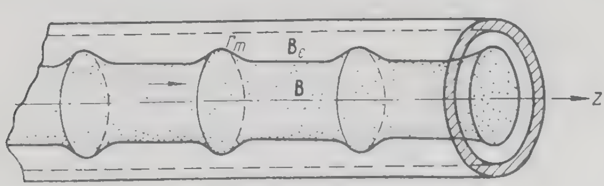

Fig. 3

in the form of a series in ε. It turns out that this series contains only even powers of ε. By successive approximations, retaining terms of order ε^2 inclusively, we find $\psi = \psi^{(0)} + \psi^{(1)} + \ldots$, where

$$\psi^{(0)} = fr^2 + f_1; \qquad \psi^{(1)} = -f''\frac{r^4}{8} - \\ - f_1''\frac{r^2}{4}(\ln r^2 - 1) + f_2\frac{r^2}{2} + f_3, \tag{4.3}$$

where $f_i = f_i(z)$ represents an arbitrary function of z.

If we are to determine the streamline for the fluid (or the lines of force of the internal field), the expansion (4.3) must be valid in the region including the z axis. For this reason $f_1 = f_3 = 0$ from the condition that the quantities remain finite at $r = 0$; furthermore, we write $f_2 = 0$ in order that the velocity at the z axis be described completely by the function f. Thus

$$\psi_0(r, z) = f_0(z)r^2 - f_0''(z)\frac{r^4}{8} + \cdots, \tag{4.4}$$

where $f_0(z) = (v_z/2)|_{r=0}$.

In order to find the magnetic surfaces of the external field we require that the cylinder $r = 1$* also be a magnetic surface, i.e., we require $H_r|_{r=1} = 0$. Then $f_1 = -f$ and $f_2 = {}^1/_2 f''$ from the condition that the field at $r = 1$ be described completely by the function f while $f_3 = {}^1/_8 f''$ from the boundary condition at $r = 1$; consequently,

$$\psi_e(r, z) = f_e(z)(r^2 - 1) + \frac{1}{4}f_e''(z)\left\{r^2\ln r^2 - \frac{1}{2}(r^4 - 1)\right\}, \tag{4.5}$$

where $f_e(z) = (H_z/2)|_{r=1}$.

The equations $\psi(r, z) = \text{const}$ are the equations for axisymmetric magnetic surfaces with an axial line of force $r = 0$ in the first case (4.4) and with a cylindrical magnetic surface $r = 1$ in the second case (4.5) (to accuracy of $\sim\varepsilon^2$, inclusively). These relations contain the arbitrary functions $f_0(z)$ and $f_e(z)$ which are determined by the distribution of longitudinal field at $r = 0$ and $r = 1$, respectively.

Assume that the surface of the plasma cylinder which is to be determined is given by the equation $r = r(z)$ in the coordinate system that moves

*The radius of the chamber is taken to be unity.

with the wave. Then, in accordance with the expansion in (4.4) and (4.5), in the zeroth approximation $f_0(z) = \psi_0/r^2$ and $f_e(z) = \psi_e/(r^2 - 1)$, where ψ_0 = const and ψ_e = const are the values of $\psi_0(r, z)$ and $\psi_e(r, z)$ at the surface $r = r(z)$. Substituting these expressions in Eqs. (4.4) and (4.5), in the next approximation we find

$$f_0 = \psi_0\left\{\frac{1}{r^2} + \frac{r^2}{8}\left(\frac{1}{r^2}\right)''\right\};$$

$$f_e = \psi_e\left\{\frac{1}{r^2 - 1} - \frac{2r^2 \ln r^2 - (r^4 - 1)}{8\,(r^2 - 1)}\left(\frac{1}{r^2 - 1}\right)''\right\}. \tag{4.6}$$

We now determine the square of the velocity and the external magnetic field, also expressing these in terms of $r(z)$. To the accuracy used here (taking account of the external azimuthal field), we find

$$v^2 = 2\psi_0^2\left\{\frac{2}{r^4} + \frac{r''}{r^3} - \frac{r'^2}{r^4}\right\}; \tag{4.7}$$

$$H_e^2 = 2\psi_e^2\left\{\frac{2}{(1 - r^2)^2} + \frac{2 \ln r^2 + (1 - r^2)(3 - r^2)}{(1 - r^2)^4}\, rr'' - \right.$$

$$\left. - \frac{2\,(1 - r^2)^3 + (1 + 3r^2)\,[2 \ln r^2 + (1 - r^2)(3 - r^2)]}{(1 - r^2)^5}\, r'^2\right\} + H_{\varphi a}^2 \frac{a^2}{r^2}. \tag{4.8}$$

The differential equation for the boundary of the plasma cylinder in the external magnetic field $r = r(z)$ is obtained from the pressure-balance condition at this boundary: $v^2 - H_i^2 + H_e^2 = P$ = const. Substituting (4.7) and (4.8) in the left side of this relation, we obtain a second-order differential equation for $r(z)$. This equation has the integrating factor rr'. We denote by a the value of r at the extreme of $r(z)$ and express the constant P in terms of the second derivative r_a'' at the point $r = a$; then, by means of a single integration, we find

$$r'^2\left\{\psi_0^2 - \psi_i^2 + \psi_e^2 r^4\, \frac{2 \ln r^2 + (1 - r^2)(3 - r^2)}{(1 - r^2)^4}\right\} =$$

$$= \frac{2\,(r^2 - a^2)^2}{a^4}\left\{\psi_0^2 - \psi_i^2 - \psi_e^2\, \frac{a^4 r^2}{(1 - a^2)^2\,(1 - r^2)} + \right.$$

$$+ \frac{H_{\varphi a}^2 a^4 r^2}{4}\, \frac{r^2 - a^2 - a^2 \ln r^2/a^2}{(r^2 - a^2)^2} + \frac{r_a'' a r^2}{2\,(r^2 - a^2)}\left[\psi_0^2 - \right.$$

$$\left.\left. - \psi_i^2 + \psi_e^2 a^4\, \frac{2 \ln a^2 + (1 - a^2)(3 - a^2)}{(1 - a^2)^4}\right]\right\}. \tag{4.9}$$

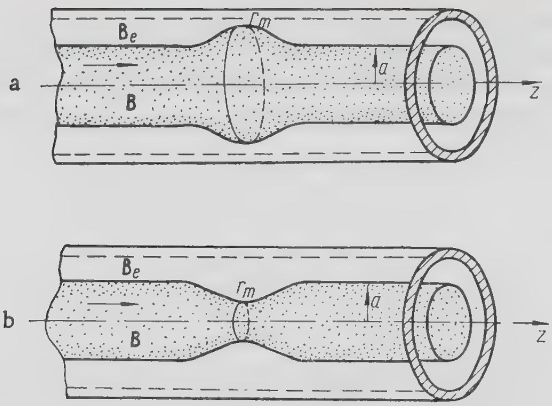

Fig. 4

In Eq. (4.9), which gives the wave profile, the constants ψ_0, ψ_i, and ψ_e determine the fluid flow for the internal and external magnetic fields, while the quantity r''_a characterizes the curvature of the profile at the extremum $r = a$. It is evident that bounded solutions of this equation are either periodic or in the form of solitary waves, i.e., single peaks or valleys.

To obtain an equation that describes a solitary wave (Fig. 4), we require that the radius $r = a$ at $z = \pm\infty$, where $r'_a = r''_a = 0$. In this case, in accordance with the expansions in Eqs. (4.4) and (4.5), the quantities ψ_0, ψ_i, and ψ_e are expressed in terms of the unperturbed velocity and magnetic fields (for $z = \pm\infty$, $r = a$):

$$\psi_0 = v_z \frac{a^2}{2}; \quad \psi_i = H_{zi} \frac{a^2}{2}; \quad \psi_e = H_{ze} \frac{a^2 - 1}{2}. \tag{4.10}$$

The quantity v_z is the phase velocity of the wave. The equation for the profile of the solitary wave is written in the form

$$\begin{aligned} r'^2 \Big\{ v_z^2 - H_{zi}^2 + H_{ze}^2 \frac{(1-a^2)^2}{a^4} \frac{2 \ln r^2 + (1-r^2)(3-r^2)}{(1-r^2)^4} r^4 \Big\} = \\ = \frac{2(r^2-a^2)^2}{a^4} \Big\{ v_z^2 - H_{zi}^2 - H_{ze}^2 \frac{r^2}{1-r^2} + \\ + H_{\varphi a}^2 \frac{r^2 - a^2 - a^2 \ln r^2/a^2}{(r^2-a^2)^2} r^2 \Big\}. \end{aligned} \tag{4.11}$$

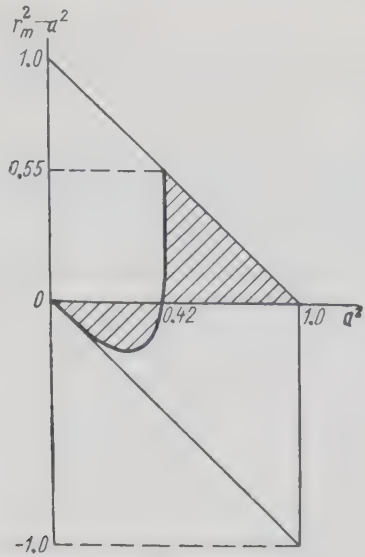

Fig. 5

The phase velocities for the periodic wave and for the solitary wave can be determined by setting the right sides of Eqs. (4.9) and (4.11) equal to zero at the second extremum of the solution $r = r_m$ (different from $r = a$). The velocity depends on the magnitudes of both extrema a and r_m and on the curvature of the wave profile r_a'' at the extremum $r = a$. The quantities ψ_0, ψ_i, and ψ_e for the periodic waves are expressed in terms of the phase velocity v_z and the unperturbed fields H_{zi} and H_{ze} by the same formulas (4.10) as for the solitary wave if the quantity a is replaced by the unperturbed cylinder radius R. However, the quantity R remains undetermined if the values of the extrema a and r_m are assigned; in this sense there are difficulties in the determination of the phase velocity and the unperturbed magnetic fields.* Hence, in treating periodic waves it is found most convenient to operate directly with the flows ψ_0, ψ_i, and ψ_e which are single-valued unique quantities. Below we shall limit ourselves to a consideration of solitary waves only. The phase velocity of these waves is given by

*A similar difficulty arises in ordinary hydrodynamics [8] in the theory of so-called cnoidal waves.

$$v_z^2 = H_{zi}^2 + H_{ze}^2 \frac{r_m^2}{1-r_m^2} - H_{\varphi a}^2 r_m^2 \frac{r_m^2 - a^2 - a^2 \ln r_m^2/a^2}{(r_m^2 - a^2)^2}. \tag{4.12}$$

For a given magnetic field v_z^2 depends on the radius of the plasma cylinder a and on the extremum r_m. The region for the existence of stationary waves is determined by the boundary $v_z^2 = 0$ beyond which there are no real values of v_z. It follows from Eq. (4.12) that if the perturbation amplitudes are large, it is always possible to reach the boundary of the stability region $v_z = 0$.

Analysis of Eq. (4.11) shows that in addition to the boundary determined by the vanishing of the phase velocity v_z there is also a boundary for the region of solutions which is connected with the steepening of the crest or the foot of the wave. In Fig. 5 we show the region in which solitary waves can exist for the case $H_{zi} = 0$ and $H_{\varphi a} = 0$. The diagonal lines represent the axis $r = 0$ and the wall $r = 1$, and in approaching these the crest or the foot of the wave is steepened; this feature can be shown easily by computing the second derivative of $r(z)$ at the point r_m. It is evident that an angle cannot be formed at the vertex of the waves; only the tendency to approach the angle appears, since the approximation that is used assumes the wave to be smooth. Furthermore, in Fig. 5 we show the boundary for the existence of the waves as determined by the equation

$$\frac{a^4}{(1-a^2)^2} = -\frac{r_m^2}{(1-r_m^2)^3}\left[2\ln r_m^2 + (1-r_m^2)(3-r_m^2)\right], \tag{4.13}$$

upon approaching which the wave is also steepened. The curve $r_m^2 - a^2$ intersects the abscissa axis at the point $a^2 \simeq 0.42$. If the radius of the cylinder is less than 0.65 of the radius of the chamber, only a contraction configuration or "sausage" can be propagated; if $a > 0.65$, only a convex "bulge" configuration can be propagated. If $a \simeq 0.65$, a low-amplitude wave is steepened; a peak or valley can be steepened depending on whether $a > 0.65$ or $a < 0.65$. In the presence of a longitudinal current ($H_{\varphi a} \neq 0$) the critical radius of the plasma cylinder is increased [21].

The steepening of a low-amplitude wave for magnetohydrodynamic surface waves is a characteristic feature with no analog in ordinary hydrodynamics. As we have already noted above, wave steepening is also observed in the development of a surface instability in a pinch in the absence of surface currents. In the next section we shall consider nonlinear helical waves, making use of an expansion in the small amplitude of the wave. However, both methods of analysis (that used in the present section and the expansion) can only indicate the tendency toward wave steepening; neither can be used to give a detailed description.

§ 5. Nonlinear Helical Waves

We now consider finite-amplitude effects in helical waves that propagate in a plasma cylinder with a free surface. It will be assumed that the fundamental mode, the m-th mode, is a first-order quantity, that the 2m-th mode is a second-order quantity, etc. In order to find the corrections to the phase velocity, which determine the nonlinear stability characteristics, it is necessary to carry out the calculations to third order. At the outset, we only consider second-order terms; this allows us to analyze the situation as far as wave steepening is concerned. We then find the correction to the phase velocity of the linear approximation for the "skin" in a plasma cylinder.

I. In general, for an equilibrium distribution of velocity $\mathbf{v}(r)$ and magnetic field $\mathbf{B}(r)$ the stream functions ξ and ψ_e must satisfy Eqs. (2.6) and (2.12). We write $\xi(r, \theta)$ and $\psi_e(r, \theta)$ in the form of a sum of harmonics

$$\xi = \frac{r^2}{2} + f_m(r)\cos m\theta + f_{2m}(r)\cos 2m\theta + \ldots;$$

$$\psi_e = f_{0e}(r) + f_{me}(r)\cos m\theta + f_{2me}(r)\cos 2m\theta + \ldots, \tag{5.1}$$

where f_m and f_{me} are $\sim\varepsilon$ while f_{2m} and f_{2me} are $\sim\varepsilon^2$. The boundary conditions on ξ and ψ_e are

$$\xi_\Sigma = \text{const}; \quad \psi_{e\Sigma} = \text{const}; \quad sP_{i\Sigma} + P_{e\Sigma} = \text{const} \tag{5.2}$$

and must be satisfied at the boundary between the plasma cylinder and the external magnetic field Σ. The equations for this boundary are written in the form

$$r = R + \rho_1 \cos m\theta + \rho_2 \cos 2m\theta + \cdots, \tag{5.3}$$

where $\rho_1 \sim \varepsilon$, $\rho_2 \sim \varepsilon^2$, etc., respectively. The quantity $s = s(\xi)$ is constant at the surface Σ, while P_i and P_e are given by

$$P_i = \frac{2r}{\beta}\,\frac{\partial\tilde{\xi}}{\partial r} + N\tilde{\xi} + \cdots;$$

$$P_e = \frac{H^2_{\varphi R} R^2}{r^2} + \frac{2f'_{0e}}{\beta}\cdot\frac{\partial\tilde{\psi}_e}{\partial r} + \cdots \tag{5.4}$$

Here we have omitted terms $\sim\tilde{\xi}^2_e, \tilde{\psi}^2_e$ and higher-order terms, while f'_{0e} and N are given respectively by $f'_{0e} = rJ_e(r)$; $N = (4\alpha_1/\beta s) + (2/s)[(v^2_\varphi - H^2_\varphi)/r^2]$, where α_1 and s are determined in § 2.

To accuracy of order ρ_1^2 inclusively, at the boundary (5.3) we have the following expressions for the quantities that appear in the boundary conditions (5.2):

$$\left.\begin{aligned}
\xi_\Sigma &= (R\varrho_1 + f_m)\cos m\theta + (R\varrho_2 + f_{2m} + \alpha_2)\times \\
&\quad \times \cos 2m\theta + \cdots; \\
\psi_{e\Sigma} &= (f_0'\varrho_1 + f_{me})\cos m\theta + (f_{0e}'\varrho_2 + f_{2me} + \alpha_{2e})\times \\
&\quad \times \cos 2m\theta + \cdots; \\
P_{i\Sigma} &= \left(\frac{2R}{\beta} f_m' + N f_m\right)\cos m\theta + \\
&\quad + \left(\frac{2R}{\beta} f_{2m}' + f_{2m}N + \beta_2\right)\cos 2m\theta + \cdots; \\
P_{e\Sigma} &= \left(-\frac{2H_{\varphi e}^2}{R}\varrho_1 + \frac{2f_{0e}'}{\beta} f_{me}'\right)\cos m\theta + \\
&\quad + \left(-\frac{2H_{\varphi e}^2}{R}\varrho_2 + \frac{2f_{0e}'}{\beta} f_{2me}' + \beta_{2e}\right)\cos 2m\theta + \cdots
\end{aligned}\right\} \tag{5.5}$$

Here, all quantities are taken at $r = R$; α_2, α_{2e}, β_2, and β_{2e} are second-order quantities and do not contain the characteristics of the second harmonic. In evaluating the steepening of the wave we do not need actual expressions for these quantities.

Substituting (5.5) in the boundary conditions (5.2), we have:

1. First approximation

$$\left.\begin{aligned}
&R\varrho_1 + f_m = 0;\ \ f_{0e}'\varrho_1 + f_{me} = 0; \\
&s\left(\frac{2R}{\beta} f_m' + N f_m\right) - \frac{2H_{\varphi e}^2}{R}\varrho_1 + \frac{2f_{0e}'}{\beta} f_{me}' = 0.
\end{aligned}\right\} \tag{5.6}$$

Eliminating ρ_1 we find the dispersion relation for the linear waves

$$sR\frac{f_m'}{f_m} + J_e^2 R\frac{f_{me}'}{f_{me}} + \beta\left(\frac{2a_1}{\beta} + \frac{v_\varphi^2 - H_\varphi^2 + H_{\varphi e}^2}{R^2}\right) = 0, \tag{5.7}$$

which has been obtained in §2.

2. In the second approximation,

$$R\varrho_2 + f_{2m} + \alpha_2 = 0;\ \ f_{0e}'\varrho_2 + f_{2me} + \alpha_{2e} = 0;$$

$$\left.\begin{aligned} & s\left(\frac{2R}{\beta} f'_{2m} + N f_{2m} + \beta_2\right) - \frac{2H^2_{\varphi e}}{R}\varrho_2 + \\ & + \frac{2f'_{0e}}{\beta} f'_{2me} + \beta_{2e} = 0. \end{aligned}\right\} \tag{5.8}$$

whence the amplitude of the second harmonic at the cylindrical surface is given by

$$\varrho_2 = \frac{\Gamma}{D}, \tag{5.9}$$

where

$$\begin{aligned} \Gamma = s & \left\{\beta_2 - \left(\frac{2R}{\beta}\frac{f'_{2m}}{f_{2m}} + N\right)\alpha_2\right\} \\ & + \left\{\beta_{2e} - \frac{2J_e R}{\beta}\frac{f'_{2me}}{f_{2me}}\alpha_{2e}\right\}; \end{aligned} \tag{5.10}$$

$$\begin{aligned} D = \frac{2R}{\beta} & \left\{sR\frac{f'_{2m}}{f_{2m}} + J_e^2 R\frac{f'_{2me}}{f_{2me}} + \beta \times \right. \\ & \left. \times\left(\frac{2a_1}{\beta} + \frac{v^2_\varphi - H^2_\varphi + H^2_{\varphi e}}{R^2}\right)\right\}. \end{aligned} \tag{5.11}$$

The vanishing of D determines the wave steepening at the plasma surface. The quantity in the curly brackets in Eq. (5.11) differs from the left side of the dispersion equation for the linear waves (5.7) only in that the logarithmic derivatives f'_m/f_m and f'_{me}/f_{me} are replaced by f'_{2m}/f_{2m} and f'_{2me}/f_{2me}. Consequently, D vanishes in a dispersionless medium (in which these logarithmic derivatives coincide) or if the dispersion equation is satisfied for arbitrary f_m and f_{me}, as is the case for a "surface" instability [$J = (k/m) \cdot H_z - (H_\varphi/R) = 0$] by continuity of the fields H_z and H_φ at the boundary (cf. § 3). Thus, the development of a surface instability is generally associated with the steepening of the wave at the surface of the cylinder. If Γ and D have the same sign, the crest of the wave is steepened; if Γ and D have opposite signs, the foot of the wave is steepened. In the concrete example of waves in a plasma cylinder with a uniform current it can be shown that the crest is steepened, so that the plasma forms narrow tongues oriented along the lines of force at the surface of the cylinder. If there are no surface currents, the latter also holds for the fundamental mode of the helical wave m = 1. As the boundary of the stability region is approached, this boundary being given by Kruskal–

Shafranov criterion $B_z = B_\varphi/kR$, two tongues appear on the opposite sides of the pinch.

The effect of steepening is to make the method of calculation used here inapplicable so that we cannot investigate the effect of nonlinearity in the development of surface instability. For this reason we shall only consider the "skin" in a plasma cylinder with a longitudinal surface current.

II. We now consider a plasma cylinder which, in the equilibrium state, exhibits a uniform longitudinal field B_z, a longitudinal surface current which produces a field outside the plasma $B_{\varphi R}(R/r)$, and an azimuthal surface current which gives rise to a discontinuity in the longitudinal magnetic field. Under these conditions, as we have shown in § 2, the equations for the stream functions are linear and the difficulties in solving the problem of nonlinear waves only requires satisfaction of the nonlinear boundary condition which derives from the pressure balance at the free surface between the plasma and the external magnetic field.

The equations for the successive approximations are obtained in the same way as in the preceding section if it is assumed that the fundamental m-mode is a first-order quantity, that the 2m-th mode is a second-order quantity, etc. In this case it is easy to show that the expressions for the 3m-th mode can be neglected but that it is necessary to take account of third-order terms inclusively in the expressions for the m-th and 2m-th modes (5.5).

1. To a first approximation we find

$$f_0'\varrho_1 - \frac{RF_m'}{m} = 0;\ f_e'\varrho_1 - \frac{RF_{me}'}{m} = 0; \tag{5.12}$$

$$\frac{RH_{\varphi e}^2}{m^2} + t_{me}f_e'^2 + t_m(f_0'^2 - f_i'^2) = 0. \tag{5.13}$$

Here, $f_i' = \alpha RH_{zi}$; $f_e' = \alpha RH_{ze} - H_{\varphi e}$; $f_0' = \alpha rv_z$; v_z is the phase velocity; $\alpha = k/m$; $k = 2\pi/\lambda$; λ is the wavelength; $t_m = F_m(R)/F_m'(R)$; $t_{me} = F_{me}(R)/F_{me}'(R)$; $F_m(r)$ and $F_{me}(r)$ are Bessel functions in terms of which we express the solutions of the inner and outer problems, given respectively by

$$F_m = A_m I_m(kr);\ F_{me} = A_{me}\{K_m'(kR_k) I_m(kr) - \\ - I_m'(kR_k) K_m(kr)\}, \tag{5.14}$$

where R_k is the radius of the ideally conducting chamber.

The relation in (5.13) is the dispersion equation which, in the linear approximation, determines the phase velocity $v_z = \omega/k$ of the m-th mode of the helical wave.

2. In the second approximation we have

$$f'_0\varrho_2 - \frac{RF'_{2m}}{2m} + \alpha_2 = 0; \quad f'_e\varrho_2 - \frac{RF'_{2me}}{2m} + \alpha_{2e} = 0; \tag{5.15}$$

$$D\varrho_2 = \Gamma; \ D = \frac{RH^2_{\varphi e}}{4m^2} + t_{2me}f'^2_e + t_{2m}(f'^2_0 - f'^2_i); \tag{5.16}$$

$$\Gamma = -t_{2me}\alpha_{2e}f'_e - \frac{t_{2m}\alpha_2}{f'_0}(f'^2_0 - f'^2_i) +$$

$$+ \frac{R^2}{8m^2}\left\{\beta_{2e} + \frac{\beta_2}{f'^2_0}(f'^2_0 - f'^2_i)\right\}. \tag{5.17}$$

Equations (5.16) and (5.17) determine the amplitude ρ_2 of the second mode of the surface of the plasma cylinder $r = R$ (below we shall write $R = 1$). The quantities α_{2e}, β_{2e}, α_2, and β_2 that appear are given by

$$\alpha_{2e} = \left(\frac{H_{\varphi e}}{2f'_e} + \frac{1}{4} - \frac{m^2\beta t_{me}}{2}\right) f'_e\varrho_1^2;$$

$$\beta_{2e} = \left(\frac{3H^2_{\varphi e}}{2f'^2_e m^2} - \frac{2H_{\varphi e}t_{me}}{f'_e} - \frac{3}{2} + \frac{m^2\beta t^2_{me}}{2}\right) m^2 f'^2_e \varrho_1^2; \tag{5.18}$$

$$\alpha_2 = \left(\frac{1}{4} - \frac{m^2\beta t_m}{2}\right) f'_0\varrho_1^2; \quad \beta_2 = \left(-\frac{3}{2} + \frac{m^2\beta t^2_m}{2}\right) m^2 f'^2_0 \varrho_1^2. \tag{5.19}$$

We now eliminate the phase velocity [as given by the linear dispersion equation (5.13)] in Eq. (5.16) to obtain the following expression for D; the vanishing of D corresponds to the steepening of the wave profile at the surface:

$$t_m D = (t_m t_{2me} - t_{me}t_{2m}) f'^2_e - \frac{H^2_{\varphi e}}{4m^2}(4t_{2m} - t_m). \tag{5.20}$$

3. The equations of the third approximation are obtained by setting the coefficients of $\cos m\theta$ in the boundary conditions (5.2) equal to zero; taking account of third-order terms in ρ_1 we have

$$f'_0\varrho_1 - \frac{F'_m}{m} + \alpha_3 = 0; \quad f'_e\varrho_1 - \frac{F'_{me}}{m} + \alpha_{3e} = 0; \tag{5.21}$$

$$\{2H^2_{\varphi e} + 2m^2 t_{me}f'^2_e + 2m^2 t_m(f'^2_0 - f'^2_i)\}\varrho_1 =$$

$$= -2m^2 t_{me}f'_e\alpha_{3e} + \beta_{3e} + \frac{f'^2_0 - f'^2_i}{f'^2_0}(-2m^2 t_m f'_0\alpha_3 + \beta_3). \tag{5.22}$$

The last equation can be used to find the correction to the linear phase velocity. We write $v_z^2 = v_{z1}^2 + \Delta v_z^2$, where v_{z1} is the linear phase velocity as given by Eq. (5.13) and find that the left side of Eq. (5.22) is equal to $\alpha^2 t_m \Delta v_z^2 \rho_1$. By transforming the right side of Eq. (5.22) we obtain the following expression for the correction to the square of the phase velocity Δv_z^2:

$$\frac{t_m}{\varrho_1^2}\Delta v_z^2 = 8D\frac{\varrho_2^2}{\varrho_1^4} + \left\{-8t_{2me}\bar{\alpha}_{2e}^2 + \frac{m^2 t_{me}^2}{2}(\beta-3) - \frac{1}{2}\right\} f_e'^2 +$$
$$+ \left\{-8t_{2m}\bar{\alpha}_2^2 + \frac{m^2 t_m^2}{2}(\beta-3) - \frac{1}{2}\right\}(f_0'^2 - f_i'^2) -$$
$$- \frac{3H_{\varphi e}^2}{8}\left(4\beta - \frac{3}{m^2}\right) - 2H_{\varphi e} f_e'(1 - t_{me}). \qquad (5.23)$$

Here, $\bar{\alpha}_2 = \alpha_2/f_0'\rho_1^2$, $\bar{\alpha}_{2e} = \alpha_{2e}/f_e'\rho_1^2$, where α_2 and α_{2e} are given by Eqs. (5.18) and (5.19); $\rho_2 = \Gamma/D$, where Γ and D are determined by Eqs. (5.16) and (5.17), while the quantities f_0', f_i', and f_e' are related by the first-approximation dispersion relation (5.13).

Equation (5.23) shows that in general Δv_z^2 is proportional to the square of the amplitude of the fundamental (m-th) harmonic ρ_1, but that it becomes infinite when $D \to 0$, i.e., for those values of the parameters at which wave steepening occurs at the surface of the pinch.

We now consider several particular cases:

A. Azimuthal Waves. Azimuthal waves, i.e., waves that propagate in the azimuthal direction φ around the cylinder (Fig. 6), are obtained from general helical waves by taking the limit $\alpha \to 0$, $\alpha v_z \to \omega$, where ω is the angular velocity of the crest of the azimuthal wave. In order to obtain the appropriate expressions for the azimuthal waves in the preceding sections, we must write $f_e' = -H_{\varphi e}$, $f_i' = 0$. In this case,

$$t_m = \frac{1}{m}; \; t_{me} = -\frac{1}{m}\,\frac{x^{2m}+1}{x^{2m}-1}; \; \left(x \equiv \frac{R_k}{R}\right), \qquad (5.24)$$

while the linear dispersion equation becomes

$$\omega^2 = \left(-mt_{me} - \frac{1}{m}\right) H_{\varphi e}^2. \qquad (5.25)$$

The correction to the square of the angular velocity ω is given by

$$\frac{\Delta\omega^2}{m\varrho_1^2} = \frac{8\Gamma^2}{D\varrho_1^4} + \left\{-8t_{2me}\bar{\alpha}_{2e}^2 - m^2 t_{2me} - \frac{1}{2}\right\} H_{\varphi e}^2 +$$

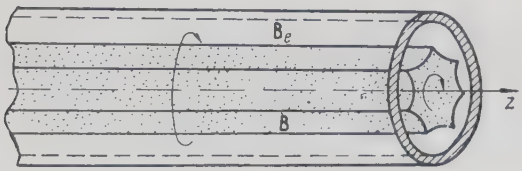

Fig. 6

$$+\left\{-8t_{2m}\bar{\alpha}_2^2-m^2t_{2m}-\frac{1}{2}\right\}\omega^2+$$

$$+\left(\frac{1}{2}-\frac{9}{8m^2}-2t_{me}\right)H^2_{\varphi e}, \tag{5.26}$$

where $\Gamma/D=\rho_2$ is the amplitude of the second (2m-th) mode at the surface

$$D=\left(\frac{1}{4m^2}+t_{2me}\right)H^2_{\varphi e}+t_{2m}\omega^2; \tag{5.27}$$

$$\frac{\Gamma}{\varrho_1^2}=\left\{-t_{2me}\bar{\alpha}_{2e}+\frac{1}{8m^2}\bar{\beta}_{2e}\right\}H^2_{\varphi e}+$$

$$+\left\{-t_{2m}\bar{\alpha}_2+\frac{1}{8m^2}\bar{\beta}_2\right\}\omega^2, \tag{5.28}$$

while

$$\left.\begin{aligned}&\bar{\alpha}_2=\frac{1}{4}-\frac{m}{2};\ \bar{\alpha}_{2e}=-\frac{1}{4}-\frac{m^2t^2_{me}}{2};\\&\bar{\beta}_2=-m^2;\ \bar{\beta}_{2e}=\frac{3}{2}-\frac{3m^2}{2}+2m^2t_{me}+\frac{m^4t^2_{me}}{2}.\end{aligned}\right\} \tag{5.29}$$

a. In the case of a pinch with no chamber $t_{me}=-1/m$, and the linear dispersion equation

$$\omega^2=\left(1-\frac{1}{m}\right)H^2_{\varphi e}, \tag{5.30}$$

i.e., the equilibrium conditions with respect to displacement of the pinch as a whole in the transverse direction (m = 1) and with respect to azimuthal waves with m > 1 are different. In accordance with Eq. (5.26), the correction to ω^2 is found to be

$$\Delta\omega^2=-\frac{H^2_{\varphi e}}{m}\{2m(m-2)^2+3m-4\}(m-1)\varrho_1^2. \tag{5.31}$$

This correction vanishes when m = 1, so that, as expected, in the nonlinear approximation the equilibrium condition with respect to transverse displacements is different. With the exception of the m = 1 mode, the correction to ω^2 is negative for any m; consequently, there is a tendency toward development of an instability in the azimuthal waves.

b. If the pinch is located inside an ideally conducting chamber at r = R_k, the equation for the first approximation for ω^2 is

$$\omega^2 = \frac{H_{\varphi e}^2}{m} \, \frac{(m-1)\,x^{2m} + m + 1}{x^{2m} - 1}. \tag{5.32}$$

In this case, the pinch is stable with respect to azimuthal waves for any m, including m = 1. We find

$$D = -\frac{H_{\varphi e}^2}{4m^2} \, \frac{x^{4m} - 4mx^{2m} - 1}{x^{4m} - 1}. \tag{5.33}$$

This quantity vanishes when R = R_{cr}, where $R_{cr} = (2\,m + \sqrt{2m^2+1})^{-1/2m} R_k$. Consequently, for any radius of the pinch there is a steepening of the azimuthal surface waves. Considering the case $d \equiv R_k - R \ll R$, we find the following expression for $\Delta\omega^2$:

$$\Delta\omega^2 = \frac{13H_{\varphi e}^2}{8md^3}\,\varrho_1^2. \tag{5.34}$$

This quantity is positive, so that an ideally conducting chamber located close to a plasma cylinder tends to stabilize the nonlinear azimuthal oscillations.

B. Axisymmetric Waves. The case of axisymmetric waves is obtained by taking the limit $\alpha \to \infty$, $\alpha m \to k$, where $k = 2\pi/\lambda$ is a finite quantity. We obtain the following expressions for t_n and t_{ne} (n = 1, 2):

$$t_n = \frac{I_0(nk)}{nkI_1(nk)};$$

$$t_{ne} = \frac{1}{2nk} \, \frac{I_0(nk)\,K_1(nkR_k) + I_1(nkR_k)\,K_0(nk)}{I_1(nk)\,K_1(nkR_k) - I_1(nkR_k)\,K_1(nk)}. \tag{5.35}$$

The dispersion equation for linear waves proportional to cos kz is

$$(v_z^2 - H_z^2)\,t_1 + H_{ze}^2 t_{1e} + \frac{H_{\varphi e}^2}{k^2} = 0. \tag{5.36}$$

The amplitude of the second mode (~cos 2kz) is given by $\rho_2 = \Gamma/D$, where

$$D = (v_z^2 - H_z^2)\,t_2 + H_{ze}^2 t_{2e} + \frac{H_{\varphi e}^2}{4k^2}; \tag{5.37}$$

$$\frac{\Gamma}{\varrho_1^2} = (v_z^2 - H_z^2)\left\{-t_2\bar{\alpha}_2 - \frac{\bar{\beta}_2}{8k^2}\right\} - H_{ze}^2 \times$$

$$\times\left\{-t_{2e}\bar{\alpha}_{2e} + \frac{\bar{\beta}_{2e}}{8k^2}\right\}, \tag{5.38}$$

where the quantities $\bar{\alpha}_2$, $\bar{\alpha}_{2e}$, $\bar{\beta}_2$, and $\bar{\beta}_{2e}$ are given, respectively, by

$$\bar{\alpha}_2 = \frac{1}{4} - \frac{k^2 t_1}{2}; \quad \bar{\beta}_2 = \frac{k^4 t_1^2}{2} - \frac{3k^2}{2};$$

$$\alpha_{2e} = \frac{1}{4} - \frac{k^2 t_{1e}}{2}; \quad \bar{\beta}_{2e} = \frac{k^4 t_{1e}^2}{2} - \frac{3k^2}{2} + \frac{3H_{\varphi e}^2}{2H_{ze}^2}. \tag{5.39}$$

Using Eqs. (5.36) and (5.37), we can write the expression for D in the form

$$t_1 D = (t_1 t_{2e} - t_{1e} t_2) H_{ze}^2 - (4t_2 - t_1)\frac{H_{\varphi e}^2}{4k^2}. \tag{5.40}$$

By expanding the Bessel functions in their arguments, we can easily show that setting D equal to zero for long waves ($kR_k \ll 1$) leads to those values of the parameters for wave steepening which were obtained in § 4 by expanding in the inverse wavelength (if the amplitude is assumed to be small). Thus, the periodic waves also exhibit a critical radius R_{cr}; when $R > R_{cr}$, the wave crest is steepened at the surface, and when $R < R_{cr}$, the wave foot is steepened. If there is no longitudinal current, this radius $R_{cr} \simeq 0.65 R_k$.

The formula for the correction to the square of the phase velocity Δv_z^2 for axisymmetric waves is

$$\frac{t_1 \Delta v_z^2}{\varrho_1^2} = \frac{8\Gamma^2}{D\varrho_1^4} + (v_z^2 - H_z^2)\left\{-8t_2\bar{\alpha}_2^2 + \frac{k^2 t_1^2}{2} - \frac{1}{2}\right\} +$$

$$+ H_{ze}^2\left\{-8t_{2e}\bar{\alpha}_{2e}^2 + \frac{k^2 t_{1e}^2}{2} - \frac{1}{2}\right\} - \frac{3H_{\varphi e}^2}{8k^2}(3 + 4k^2). \tag{5.41}$$

We shall only consider long waves ($k \ll 1$); since effects due to chamber walls are considered in § 4, we shall only consider axisymmetric waves in a plasma without surrounding walls. For wavelengths such that $\ln(1/k) \gg 1$, the quantities Γ and D are given approximately by

$$\Gamma = -\frac{9}{16}\ln k \cdot H_{ze}^2 - \frac{H_{\varphi e}^2}{8k^2}; \quad D = \frac{3}{4}\left(\ln k \cdot H_{ze}^2 - \frac{H_{\varphi e}^2}{8}\right). \tag{5.42}$$

The quantity D is negative and does not vanish, i.e., the wave is not steepened in the absence of a chamber. The correction to the square of the phase velocity is

$$\Delta v_z^2 = \frac{2\varrho_1^2}{3k^2} \cdot \frac{H_{\varphi e}^4 + 3H_{ze}^2 k^2 \ln k + 18\left(H_{ze}^2 k^2 \ln k\right)^2}{8H_{ze}^2 \ln k - H_{\varphi e}^2} . \qquad (5.43)$$

We note that the denominator is negative while the numerator contains a positive definite quadratic form. Consequently, taking account of finite-wave amplitude ρ_1 for longwave axisymmetric oscillations of a plasma in the absence of a chamber gives rise to an additional instability. As is shown in § 4, a similar result holds for axisymmetric waves in the presence of a chamber.

C. Helical Waves. In analyzing nonlinear helical waves, we assume that the longitudinal field component B_z is continuous at the boundary of the plasma cylinder. Under these conditions, a pinch inside an ideally conducting chamber is stable against all modes of oscillation except the m = 1 mode, i.e., bending of the pinch; the instability arises when $B_z = B_\varphi / kR$ (Kruskal–Shafranov criterion). We consider the effect of finite-wave amplitude in this interesting case at the boundary of the linear stability region $f_e' = \alpha R H_z - H_\varphi = 0$. We write m = 1; $H_{ze} = H_z$; $f_e' = 0$; $f_0' = 0$, and note that for long wavelengths,

$$t_m \simeq \frac{1}{m}; \; t_{me} \simeq -\frac{1}{m} \cdot \frac{x^{2m}+1}{x^{2m}-1}, \qquad (5.44)$$

so that the amplitude of the second mode $\rho_2 = \Gamma / D$:

$$D = -\frac{H_{\varphi e}^2}{4}; \quad \Gamma = \frac{3H_{\varphi e}^2}{16} \varrho_1^2 . \qquad (5.45)$$

In this case, $D \neq 0$ and the correction to the square of the phase velocity is proportional to the square of the amplitude of the fundamental mode ρ_1:

$$\Delta v_z^2 = \frac{3 - x^4}{x^4 - 1} H_z^2 \varrho_1^2 . \qquad (5.46)$$

It follows from the expression obtained above that the nonlinearity stabilizes the instability when $x^4 \equiv (R_k/R)^4 < 3$ or $R > 0.76R_k$; on the other hand, if $R < 0.76R_k$, the finite amplitude enhances the instability.

In conclusion, we emphasize that the results obtained apply only in the presence of a surface current. If the magnetic field is continuous at the boundary of the plasma cylinder, the linear theory (cf. § 3) indicates the appearance of an instability when $B_z = mB_\varphi / kR$. Taking account of nonlinear terms shows (n1) that in this case the waves at the surface are steepened and the latter circumstance makes it impossible to compute the phase velocity of the waves (within the framework of the methods used here).

§ 6. Linear Waves in a Compressible Plasma

In § 2 we derived an equation for linear waves in a plasma jet, assuming it to be incompressible. In the present section we shall relax this restriction. The initial system of equations is (I) in § 1, which describes the stationary helical flow of an ideally conducting fluid in a magnetic field.

I. We assume that in the equilibrium state all quantities depend only on r, and that the velocity and magnetic field have only azimuthal and longitudinal components $\mathbf{v} = \{0, v_\varphi, v_z\}$, $\mathbf{B} = \{0, B_\varphi, B_z\}$. It is then an easy matter to relate the equilibrium distributions of density, pressure, velocity, and magnetic field $\mathbf{B} \equiv \sqrt{4\pi}\,\mathbf{H}$. Writing $\Phi = 0$, we have

$$\frac{1}{r}\frac{d}{dr}\left(p + \frac{H^2}{2}\right) + h_\varphi^2 - \varrho v_\varphi^2 = 0, \tag{6.1}$$

where $v_\varphi = v_\varphi/r$; $h_\varphi = H_\varphi/r$.

We take the stream function ξ in the equilibrium state to be $\bar{\xi} = r^2/2$. Then, in the presence of waves, we have $\xi = (r^2/2) + \tilde{\xi}(r, \theta)$. The equilibrium density $\bar{\rho}(r)$ also exhibits a perturbation $\tilde{\rho}(r, \theta)$, so that $\rho = \bar{\rho}(r) + \tilde{\rho}(r, \theta)$. The relation between $\tilde{\rho}$ and $\tilde{\xi}$ is obtained from the second equation of (I), making use of the fact that U depends only on ξ. The subsequent calculations are analogous to those in § 2. In the linear approximation in $\tilde{\rho}$ and $\tilde{\xi}$, taking

$$\tilde{\xi} = f(r)\,e^{im\theta}; \qquad \tilde{\varrho} - \frac{\bar{\varrho}'(r)}{r}\tilde{\xi} = g(r)\,e^{im\theta} \tag{6.2}$$

and making use of the fact that $S = S(\xi)$, we have

$$g = -\frac{rs}{\beta G} f' - \frac{Q - b^2}{G} f, \tag{6.3}$$

where we have used the notation $G = \dfrac{c_T^2 s}{\varrho J_0^2} + \dfrac{H^2}{\varrho} - \dfrac{r^2 J_0^2}{\beta}$;

$$Q = \frac{2a}{\beta} + \varrho v_\varphi^2 - h_\varphi^2; \quad s = \varrho J_0^2 - J^2; \quad a = \varrho J_0 v_\varphi - J h_\varphi,$$

$$J_0 b = J_0 h_\varphi - J v_\varphi; \quad J = \alpha H_z - h_\varphi;$$

$$J_0 = \alpha(v_z - v_\Phi) - \nu_\varphi; \qquad \alpha = \frac{k}{m}; \quad \beta = 1 + \alpha^2 r^2;$$

$\theta = \varphi - \alpha z$; $v_\Phi = \omega/k$ is the phase velocity; $c_T = (\partial p/\partial \rho)_S^{\frac{1}{2}} = (\gamma p/\rho)^{\frac{1}{2}}$ is the velocity of sound.

The equation for the perturbation $\tilde{\xi}$ is obtained from the first equation in (I) making use of the relation $U' = U'(\xi)$, the equation of state $p = \rho kT$, and the adiabatic relation $p\rho^{-\gamma} = \exp[(\gamma - 1)/k]S$. The equation for $\tilde{\xi}$ contains $\tilde{\rho}$. If g(r) is eliminated by means of Eq. (6.3), we obtain an equation for $f(r)$

$$\left(\frac{rs}{1-\varkappa}\frac{f'}{\beta}\right)' + \left\{-\frac{m^2 s}{r} + \frac{4\alpha^2 r a^2}{\beta s} + \left(\frac{Q - \varkappa b^2}{1-\varkappa}\right)' - \right.$$

$$\left. - \frac{\varkappa}{1-\varkappa}\frac{\beta}{rs}(Q - b^2)^2\right\} f = 0. \tag{6.4}$$

Here,

$$\varkappa = \frac{r^2 J_0^2}{\beta}\left(\frac{c_T^2 s}{\varrho J_0^2} + \frac{H^2}{\varrho}\right)^{-1}. \tag{6.5}$$

As in the foregoing, if the plasma is bounded by an ideally conducting shield at r = R, the boundary condition is $f(R) = 0$. However, if the plasma cylinder is confined by an external magnetic field $\mathbf{B}_e$, the boundary condition is the pressure-balance condition $p + (H^2/2) = (H_e^2/2)$ at the perturbed surface of the cylinder $r = R + \delta R(\theta)$.

We now write the perturbed external field stream function $\psi_e = (\alpha r^2/2)H_z^e - H_\varphi^e r \ln r + f_e(r)e^{im\theta}$ The components of the field are defined in terms of $\psi_e(r, \theta)$ by the relations $rH_r^e = \partial\psi_e/\partial\theta$, $\alpha r H_z^e - H_\varphi^e = \partial\psi_e/\partial r$ and $H_z^e + \alpha r H_\varphi^e = \text{const}$. The radial part of the perturbation $f_e(r)$ satisfies the linear equation (2.9), the solutions of which are derivatives of the Bessel functions $rI_m'(\alpha m r)$ and $rK_m'(\alpha m r)$.

The following boundary conditions must be satisfied at the unperturbed surface of a plasma cylinder that is bounded by an external magnetic field:

$$\left\{\frac{rs}{1-\varkappa}\frac{f'}{f} + \right.$$

$$\left. + \beta\frac{Q - \varkappa b^2}{1-\varkappa} + rJ_e^2\frac{f_e'}{f_e} + \beta h_{\varphi e}^2\right\}_{r=R} = 0, \tag{6.6}$$

This relation derives from the pressure-balance condition and the requirement

$$\xi\big|_{R+\delta R} = \text{const} \quad \text{and} \quad \psi_e\big|_{R+\delta R} = \text{const}.$$

The quantities $J_e = \alpha H_z^e - h_{\varphi e}$ and $h_{\varphi e} = H_\varphi^e / r$ are analogous to those derived above for the internal field. The function $f_e(r)$ is either chosen from the requirement that it fall off as $r \to \infty$ if one is considering a plasma cylinder in free space, or from the condition $f_e(R_e) = 0$ if there is an ideally conducting shield at $r = R_e$, etc. The ratio f_e'/f_e is a uniquely determined known function; as in the case of the incompressible plasma, Eq. (6.4) and a Sturm–Liouville boundary condition (6.6) are used for $f(r)$.

We note that from the definition of ξ [cf. third relation in (I)], that the radial part $f(r)$ of the perturbation $\tilde{\xi}(r, \theta)$ is proportional to the radial parts v_r and H_r, i.e., $f \sim r v_r / J_0 \sim r H_r / J$ and, correspondingly, $f_e \sim r H_z^e$.

Near the boundary $v_\Phi = 0$ the boundary of the instability region of a plasma cylinder at rest ($v_z = v_\varphi = 0$), the quantity $\varkappa \to 0$; with the exception of the case $J = 0$, in Eq. (6.4) we can neglect all terms $\sim\varkappa$. Furthermore, $\varkappa \to 0$ when $c_T^2 \to \infty$ and when $H^2 \to \infty$. In those cases in which terms of order $\varkappa$ can be neglected in Eqs. (6.4) and (6.6), we obtain the equations [20]

$$\left(\frac{rs}{\beta} f'\right)' + \left(-\frac{m^2 s}{r} + \frac{4\alpha^2 r a^2}{\beta s} + Q'\right) f = 0; \tag{6.7}$$

$$\left\{\frac{rs}{\beta}\frac{f'}{f} + Q + \frac{rJ_e^2}{\beta}\frac{f_e'}{f_e} + h_{\varphi e}^2\right\}_{r=R} = 0, \tag{6.8}$$

which coincide with those obtained in § 2.

Since Eqs. (6.4)-(6.6) describe waves of a general helical type $\sim \exp i(kz - m\varphi - \omega t)$, when $k \to 0$ and $k \to \infty$ we can easily obtain the equations for azimuthal and axisymmetric waves, respectively.

II. We now consider certain problems in the stability of a rotating plasma. We limit ourselves to the case $J = \alpha H_z - h_\varphi = 0$ or $\mathbf{kH} = 0$, in which the perturbations are directed along the lines of force of the magnetic field. If the perturbations are helical, since α = const, this case is realized when $\mu \equiv h_\varphi / H_z$ = const; for axisymmetric waves we have $H = H_\varphi$, and for azimuthal waves, $H = H_z$.

When $J = 0$, Eq. (6.4) becomes

$$\left(\frac{\varrho r J_0^2 c_s^2 f'}{\beta c_s^2 - r^2 J_0^2}\right)' + \left\{-\frac{m^2 \varrho J_0^2}{r} + \frac{4\alpha^2 r \varrho v_\varphi^2}{\beta} + \right.$$

$$+\left(\frac{\beta c_s^2 Q - r^2 J_0^2 h_\varphi^2}{\beta c_s^2 - r^2 J_0^2}\right)' - \frac{r\beta(Q - h_\varphi^2)^2}{\varrho\left(\beta c_s^2 - r^2 J_0^2\right)}\Bigg\} f = 0, \tag{6.9}$$

where $c_S^2 \equiv c_T^2 + (H^2/\rho)$. We now consider certain cases in which Eq. (6.9) can be used to obtain the instability condition rather simply, i.e., the condition under which there are solutions with frequency ω that contains an imaginary part.

1a. Assume that the plasma rotates as a whole* (ν_φ = const) so that $J_0 = -\alpha v_\Phi - \nu_\varphi$ = const. When $J_0 \to 0$, Eq. (6.9) assumes the form

$$\left(\frac{\varrho r}{\beta} f'\right)' + \left\{ -\frac{m^2\varrho}{r} + \frac{1}{J_0^2}\left[\frac{4\alpha^2 r\varrho v_\varphi^2}{\beta} + (\varrho v_\varphi^2 - h_\varphi^2) - \right.\right.$$
$$\left.\left. - \frac{r\left(\varrho v_\varphi^2 - 2h_\varphi^2\right)^2}{\varrho c_s^2}\right]\right\} f = 0. \tag{6.10}$$

It then follows [22] that the necessary condition for stability ($J_0^2 > 0$) for a plasma cylinder in contact with an ideally conducting chamber (with respect to helical perturbations) is that the expression in the square brackets in Eq. (6.10) must be positive:

$$\frac{4\alpha^2 r\varrho v_\varphi^2}{\beta} + (\varrho v_\varphi^2 - h_\varphi^2)' - \frac{r\left(\varrho v_\varphi^2 - 2h_\varphi^2\right)^2}{\varrho c_s^2} > 0. \tag{6.11}$$

b. If a plasma cylinder carrying a uniform longitudinal current (h_φ = const) rotates as a whole (ν_φ = const), for long waves ($\alpha^2 r^2 \ll 1$) in the case $v_\varphi^2 \ll c_s^2$, Eq. (6.9) yields

$$(\varrho r f')' + \left\{ -\frac{m^2\varrho}{r} + \frac{\varrho'}{J_0^2}\left(2J_0 v_\varphi + v_\varphi^2\right)\right\} f = 0. \tag{6.12}$$

We consider a cylinder with a free boundary and no surface currents; then $J_e(R) = 0$ and the boundary condition (6.6) is written in the form

$$\left\{J_0^2 \frac{rf'}{f} + 2J_0 v_\varphi + v_\varphi^2\right\}_{r=R} = 0. \tag{6.13}$$

Equation (6.12) and the boundary condition (6.13) are satisfied by the solution $f \sim r^m$, so that the frequency ω is given by

$$\omega = v_\varphi\left(1 - m \pm \sqrt{1 - m}\right). \tag{6.14}$$

*This limitation is not important.

If there is no longitudinal current the case being considered here, $J = 0$, corresponds to an azimuthal wave. The azimuthal wave instability is described in [23] and for a particular density distribution $\rho \sim \exp(-qr^2)$ in [24]. When ρ = const, Eq. (6.7) can be solved exactly [cf. (3.6)] and we obviously obtain the same formula for ω for long waves, since it does not contain ρ. Thus, for an arbitrary dependence $\rho(r)$, a plasma pinch that rotates as a whole with a free boundary is hydrodynamically unstable against perturbations parallel to **B**. The growth-rate $\nu_\varphi \sqrt{m-1}$ increases with m, indicating the tendency of a rotating plasma to break up into filaments.

2. For the case of axisymmetric waves in a rotating plasma with $H = H_\varphi$, Eq. (6.9) assumes the form

$$\left(\frac{\varrho c_s^2}{c_s^2 - v_\Phi^2}\,\frac{f'}{r}\right)' + \left\{ -\frac{k^2\varrho}{r} + \frac{4\varrho v_\varphi^2}{r v_\Phi^2} + \left[\frac{c_s^2\left(\varrho v_\varphi^2 - h_\varphi^2\right) - v_\Phi^2 h_\varphi^2}{v_\Phi^2\left(c_s^2 - v_\Phi^2\right)}\right]' - \right.$$
$$\left. - \frac{r\left(\varrho v_\varphi^2 - 2h_\varphi^2\right)^2}{\varrho v_\Phi^2\left(c_s^2 - v_\Phi^2\right)}\right\} f = 0. \tag{6.15}$$

When $v_\Phi^2 \to 0$, this yields an equation of the form (6.10), whence the necessary condition for stability becomes

$$\frac{4\varrho v_\varphi^2}{r} + \left(\varrho v_\varphi^2 - h_\varphi^2\right)' - \frac{r\left(\varrho v_\varphi^2 - 2h_\varphi^2\right)^2}{\varrho c_s^2} > 0. \tag{6.16}$$

The fact that this condition is sufficient with respect to axisymmetric perturbations is established in [19]. For a nonrotating plasma ($\nu_\varphi = 0$), the condition in (6.16) coincides with the necessary condition for stability of a plasma with closed lines of force which derives from the Kadomtsev [25] criterion

$$-w\left(\frac{\nabla u}{u}\right)^2 < \nabla u \nabla p < \gamma p\,\frac{(\nabla u)^2}{|u|}, \tag{6.17}$$

where $u = -\oint \frac{dl}{H}$, $w = \oint H\,dl$ when $H = H_\varphi$.

3. For azimuthal waves $[J_0 = -(\omega/m) - \nu_\varphi]$ and $H = H_z$, Eq. (6.9) yields

$$\left(\frac{\varrho r J_0^2 c_s^2 f'}{c_s^2 - r^2 J_0^2}\right)' + \left\{ -\frac{m^2 J_0^2 \varrho}{r} + \left[\frac{c_s^2\left(2\varrho J_0 v_\varphi - \varrho v_\varphi^2\right)}{c_s^2 - r^2 J_0^2}\right]' - \right.$$
$$\left. - \frac{r\left(2\varrho J_0 v_\varphi + \varrho v_\varphi^2\right)^2}{\varrho\left(c_s^2 - r^2 J_0^2\right)}\right\} f = 0. \tag{6.18}$$

Setting J_0 equal to zero, as in the foregoing, we obtain a criterion for stability against azimuthal (flute) perturbations

$$(\varrho v_\varphi^2)' - \frac{\varrho r v_\varphi^4}{c_s^2} > 0. \tag{6.19}$$

Equation (6.19) shows that in the case of uniform rotation (v_φ = const), the condition $\rho' > 0$ is required for stability.

Appendix. Curvature and Torsion of the Coordinate Line x_3 when $\partial g_{ik}/\partial x_3 = 0$

We show that the requirement $\partial g_{ik}/\partial x_3 = 0$ is equivalent to the requirement of helical symmetry in the problem and, consequently, that the general form of two-parameter flow is helical flow. The proof is based on the fact that the curvature $1/R$ and the torsion $1/T$ of the coordinate line x_3 are expressed in terms of the components of the metric tensor g_{ik}. Since we assume $\partial g_{ik}/\partial x_3 = 0$, then R and T are constant along the line x_3, whence it follows that x_3 is a helical line.

We start with the Frenet formulas

$$\frac{d\boldsymbol{\tau}}{ds} = \frac{\mathbf{n}}{R}; \quad \frac{d\mathbf{n}}{ds} = -\frac{\boldsymbol{\tau}}{R} + \frac{\mathbf{b}}{T}; \quad \frac{d\mathbf{b}}{ds} = -\frac{\mathbf{n}}{T}, \tag{1}$$

where $\boldsymbol{\tau} = d\mathbf{r}/ds$ is the unit vector for the tangent to the line x_3, $\mathbf{n}$ is the normal, and $\mathbf{B}$ is the binormal. The quantities R and T are defined by the expressions

$$\frac{1}{R^2} = \left(\frac{d\boldsymbol{\tau}}{ds}\right)^2; \qquad \frac{1}{T} = R^2\left[\boldsymbol{\tau}\,\frac{d\boldsymbol{\tau}}{ds}\right]\frac{d^2\boldsymbol{\tau}}{ds^2}, \tag{2}$$

which are easily obtained from Eq. (1), since $\mathbf{b} = [\boldsymbol{\tau}\mathbf{n}]$.

In the curvilinear coordinates x_i, the differential $d\mathbf{r}$ is written in the form $d\mathbf{r} = \mathbf{l}_i dx_i$ and the unit vector $\boldsymbol{\tau}$ and the element of arc length along x_3 are

$$\boldsymbol{\tau} = \frac{\mathbf{l}_3}{\sqrt{g_{33}}}; \qquad ds = \sqrt{g_{33}}\,dx_3. \tag{3}$$

Substituting these expressions in Eq. (2), we have

$$\frac{1}{R^2} = \frac{1}{g_{33}^2}\left(\frac{\partial \mathbf{l}_3}{\partial x_3}\right)^2; \quad \frac{1}{T} = \frac{R^2}{g_{33}^3}\left[\mathbf{l}_3\,\frac{\partial \mathbf{l}_3}{\partial x_3}\right]\frac{\partial^2 \mathbf{l}_3}{\partial x_3^2}. \tag{4}$$

The derivatives $\mathbf{l}_i$ are expressed in terms of $g_{ik} = (\mathbf{l}_i\mathbf{l}_k)$ by the following equations of vector analysis:

$$\frac{\partial \mathbf{l}_i}{dx_k} = \Gamma^l_{ik}\mathbf{l}_l \tag{5}$$

where

$$\Gamma^l_{ik} = g^{lm}\Gamma_{mik};\ 2\Gamma_{jkl} = \frac{\partial g_{ik}}{\partial x_l} + \frac{\partial g_{il}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_i}. \tag{6}$$

Thus, the statement made above that $\partial R/\partial x_3 = \partial T/\partial x_3 = 0$, $\partial g_{ik}/\partial x_3 = 0$, is proved.

To find explicit relations for T and R in terms of g_{ik}, we substitute Eqs. (5) and (6) in Eq. (4). Making some straightforward but lengthy transformations using the tensor identities

$$g_{il}g^{lk} = \mathbf{l}_i\mathbf{l}^k = \delta_{ik},\quad [\mathbf{l}_i\mathbf{l}_k] = \sqrt{g}\varepsilon^{ijk}\,\mathbf{l}^j,$$

$$g_{ik} = gG_{ik},\ g^{ik} = \frac{G_{ik}}{g},$$

where G_{ik} and G^{ik} are the minors of g^{ik} and g_{ik} while $g = \mathrm{Det}(g_{ik})$, we find

$$\frac{1}{R^2} = \frac{g^{ik}}{4g_{33}^2}\frac{\partial g_{33}}{\partial x_i}\frac{\partial g_{33}}{\partial x_k} = \frac{1}{4g_{33}^2}\left[g^{11}\left(\frac{\partial g_{33}}{\partial x_1}\right)^2 + \right.$$

$$\left. + 2g^{12}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{33}}{\partial x_2} + g^{22}\left(\frac{\partial g_{33}}{\partial x_2}\right)^2\right]; \tag{7}$$

$$\frac{1}{T} = \frac{R^2}{8\sqrt{g}g_{33}}\left[g^{11}\left(\frac{\partial g_{33}}{\partial x_1}\right)^2 - g^{22}\left(\frac{\partial g_{33}}{\partial x_2}\right)^2\right]\left[\frac{\partial}{\partial x_1}\left(\frac{g_{23}}{g_{33}}\right) - \frac{\partial}{\partial x_2}\left(\frac{g_{13}}{g_{33}}\right)\right]. \tag{8}$$

Starting from the Frenet formulas it is easy to show that lines characterized by constant R and T will be helices in the general case.

Equations (7)-(8) also show that in orthogonal coordinates ($g_{12} = g_{13} = g_{23} = 0$) the torsion of the line x_3 is zero; consequently, the requirement that $\partial g_{ik}/\partial x_3 = 0$ is equivalent to the limitation to axial or translational symmetry.

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