Why does the 2D unforced vorticity equation, despite being a 2D system, exhibit chaotic behavior, which is typically thought to require a 3D phase space?

The 2D unforced vorticity equation, despite being a 2D system, can exhibit chaotic behavior because of the **infinite-dimensional nature of its phase space**.

The Paradox of 2D Chaos

The common understanding that chaos requires a 3D phase space (or higher) applies to **continuous-time systems with a finite number of degrees of freedom**, such as the Lorenz and Rössler attractors. These are systems described by a small number of ordinary differential equations (ODEs). The Poincaré–Bendixson theorem, for example, proves that chaos is impossible in continuous 2D systems with a finite phase space because trajectories can't cross and are thus constrained to fixed points or limit cycles.
The 2D unforced vorticity equation, however, is not a finite-dimensional system. It's a **partial differential equation (PDE)** describing the behavior of a continuous fluid.

How the Vorticity Equation Escapes the Rule

The key to its chaotic behavior lies in the difference between its physical dimension and its mathematical phase space dimension.

- **Physical Space (2D):** The fluid motion is confined to a two-dimensional plane.
- **Phase Space (Infinite-Dimensional):** The "state" of the system at any given time is the **entire vorticity field**, which is a function of two spatial variables. To specify this field, you need an infinite number of values (e.g., the value of vorticity at every point in the plane). This is what makes the phase space of the system infinite-dimensional.

Within this infinite-dimensional phase space, the system can exhibit the hallmarks of chaos, such as:

- **Sensitive dependence on initial conditions:** An infinitesimal change in the initial vorticity field can lead to exponentially diverging trajectories in the phase space. This is often visualized as two nearby fluid particles (or "tracers") starting close together but rapidly separating over time, an effect known as **chaotic mixing**.
- **A strange attractor:** While the physical space is 2D, the system's evolution in its infinite-dimensional phase space can be attracted to a complex, fractal-like structure, or **strange attractor**, which is a geometric signature of chaos.[1]

In essence, the 2D unforced vorticity equation circumvents the 3D phase space rule because it's not a low-dimensional system. It has an infinite number of degrees of freedom, which provides the necessary complexity for chaotic dynamics. This complexity is often simplified in models of **point vortices**, where the continuous vorticity field is approximated by a finite number of singular points.[2] For three or more point vortices, the system's ODEs have a high enough dimension to exhibit chaos, demonstrating how the complexity of the full system can be captured even in a simplified model.

Quellen

Fiveable
fiveable.me

Dynamical Systems: Phase Space & Attractors | Chaos Theory Class Notes | Fiveable

Examples: Oscillating pendulum, predator-prey population cycles. Torus attractors. Represent quasi-periodic behavior with multiple incommensurate frequencies. ...

ResearchGate
www.researchgate.net

Integrable, Chaotic, and Turbulent Vortex Motion in Two-Dimensional Flows -...

Abstract. Consideration is given to a system of point vortices and the effects of finite-area vortices are examined in terms of vortex patches. Vortex points, ...

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The **2D incompressible vorticity equation** can describe chaotic behavior, not because the system itself is 2D in a simple sense, but because the **phase space is effectively infinite-dimensional**. 🤯
Here's the breakdown:

1. The Dimensionality of the System

A 1D differential equation like $\frac{dx}{dt}=f(x)$ has a 1D phase space (a line). A 2D system of ODEs, like a pendulum, has a 2D phase space (a plane). For chaos, you need a phase space of at least three dimensions for the trajectories to stretch, fold, and not cross, as you correctly noted.
The 2D vorticity equation, however, is not a simple system of ordinary differential equations (ODEs). It is a **partial differential equation (PDE)**:
$\frac{\partial \omega}{\partial t}+(v\cdot\nabla)\omega=v\nabla^2\omega$

- ω is the vorticity, a scalar field in 2D.[1]
- v is the velocity vector field.[2]
- v is the kinematic viscosity.

This equation describes the evolution of a **field**, not just a few variables.[3] At every point in space, there's a variable (ω) evolving over time. The state of the system is the entire function $\omega(x,y,t)$, which exists in an **infinite-dimensional space**.

2. The Nonlinearity and Feedback Loop

The key to the chaotic behavior lies in the nonlinear term $(v\cdot\nabla)\omega$, often called the **advection term**. This term describes how vorticity is transported along with the fluid flow. The velocity field v is not an independent variable; it is derived from the vorticity field ω through a nonlocal relationship (the Biot-Savart law in fluid dynamics).
This creates a powerful feedback loop:

1. The vorticity field ω at any given moment determines the velocity field v everywhere in the fluid.
2. The velocity field then advects (transports) the vorticity around.
3. This changes the vorticity field, which in turn changes the velocity field, and the process repeats.

This complex, nonlinear feedback between the vorticity field and the velocity field, which is defined over a continuous space, is what generates the rich, unpredictable dynamics we call turbulence or chaos.

3. Comparison to 3D Fluid Dynamics

In 3D, there's an additional term in the vorticity equation called the **vortex stretching** term, [4]$(\omega \cdot \nabla)v$.[5] This term allows for the direct growth and amplification of vorticity, which is a major contributor to the turbulent energy cascade.

While 2D fluid dynamics lacks this vortex stretching mechanism, it still generates complex, chaotic behavior through the advection term and the infinite-dimensional nature of its phase space. In 2D turbulence, instead of an energy cascade to smaller scales, there's an **inverse energy cascade** where energy flows to larger scales, causing small vortices to merge into bigger ones, which is a key characteristic of 2D turbulent flows.

Quellen

www.ph.unito.it

Vorticity equation in two dimensions

Thus instead of giving a description of the flow in term of the two components of velocity, which are not independent because of the incompressibility ...

Wikipedia, the free encyclopedia
en.wikipedia.org

Navier–Stokes equations - Wikipedia

Flow velocity. The solution of the equations is a flow velocity. It is a vector field—to every point in a fluid, at any moment in a time interval, it gives a ...

Wikipedia
en.wikipedia.org

Vorticity equation - Wikipedia

Physical interpretation - The term DωDt on the left-hand side is the material derivative of the vorticity vector ω. - The term (ω • ∇) u on the right-hand ...