Parametric Decay Instability in Magnetized Plasma

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1 Wave analog of the Mathieu equation

It is natural to start the discussion of the theory of parametric decay instabilities by considering parametric resonance (PR). In this case, it is convenient to choose a model that is close to reality and has sufficient simplicity to trace on this model the main qualitative patterns that govern decay parametric instabilities. It is not difficult to choose such a model. In oscillatory systems the discussion of parametric instabilities begins with the analysis of solutions of the Mathieu equation. In our case for a wave medium, such as plasma, it is convenient to refer to the following equation:

$$\partial^2 u/\partial t^2 - v^2 \left[1 + \varepsilon \cos\left(\omega_0 t - k_0 x\right)\right] \partial^2 u/\partial x^2 = 0, \tag{1}$$

where we assume the linear dispersion relation $\omega = kv$.

Obviously, Eq.1 can be considered as one of the simplest generalizations of the Mathieu equation to wave media. In the absence of a term proportional to ε, such equations in linear approximation describe many well-known waves: acoustic, magnetosonic, Alfvénic, etc. Thus, for linear acoustic waves in a homogeneous plasma from the equations:

$$\left.\begin{array}{l} \partial\rho/\partial t + \rho_0 \partial u/\partial x = 0; \\ \rho_0 \partial u/\partial t = -\partial p/\partial x; \quad p/\rho^\gamma = \text{ const }, \end{array}\right\} \tag{2}$$

where ρ, u, p are respectively density, mass velocity and pressure; γ is the adiabatic exponent; the index zero denotes unperturbed quantities, follows the equation

$$\partial^2 u/\partial t^2 - \upsilon^2 \partial^2 u/\partial x^2 = 0,$$

where $\upsilon^2 \equiv \gamma p_0/\rho_0$ (sound speed).

Suppose we are interested in low-amplitude Alfvén waves. The Alfvén wave equation describes the propagation of Alfvén waves in a magnetized plasma. To derive the Alfvén wave equation, consider the basic MHD equations:

1. Momentum equation:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \mathbf{J} \times \mathbf{B}$$

where ρ is the mass density, $\mathbf{u}$ is the velocity, p is the pressure, $\mathbf{J}$ is the current density, and $\mathbf{B}$ is the magnetic field.

2. Induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\eta \mathbf{J})$$

where η is the magnetic diffusivity.

To derive the Alfvén wave equation, we make the following steps and assumptions:

- Linearize the MHD equations by considering small perturbations about a uniform background magnetic field $\mathbf{B}_0$:

$$\mathbf{B} = \mathbf{B}_0 + \tilde{\mathbf{B}}$$

- and small perturbations about a uniform background density ρ_0

$$\rho = \rho_0 + \tilde{\rho}$$

Linearizing the induction and momentum equations, we get:

- Induction equation (linearized):

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \nabla \times (\tilde{\mathbf{u}} \times \mathbf{B}_0)$$

- Linearized momentum equation:

$$\rho_0 \frac{\partial \tilde{\mathbf{u}}}{\partial t} = (\nabla \times \tilde{\mathbf{B}}) \times \mathbf{B}_0$$

From the equations

$$\begin{aligned}\frac{\partial \tilde{\mathbf{B}}}{\partial t} &= \nabla \times (\tilde{\mathbf{u}} \times \mathbf{B}_0)\\ \rho_0 \frac{\partial \tilde{\mathbf{u}}}{\partial t} &= (\nabla \times \tilde{\mathbf{B}}) \times \mathbf{B}_0\end{aligned} \tag{3}$$

we obtain

$$\frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho_0} \frac{\partial^2 \tilde{\mathbf{u}}}{\partial x^2} \tag{4}$$

where μ_0 is the magnetic permeability of free space.
This can be written as a wave equation:

$$\frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} = v_A^2 \frac{\partial^2 \tilde{\mathbf{u}}}{\partial x^2} \tag{5}$$

where $v_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}$ is the Alfvén speed.
Here the x-axis is directed along $\mathbf{B}$, and (u denotes any of the components of the mass velocity perpendicular to $\mathbf{B}_0$.

Let the density ρ be modulated by a sinusoidal pump wave of small amplitude, then

$$v_A^2 = v_{A_0}^2 \left[1 + \varepsilon \cos\left(\omega_0 t - k_0 x\right)\right]. \tag{6}$$

Formal substitution of these values into (2) and (5) leads to equations of type (1). A rigorous derivation of the equations describing the propagation of sound or Alfvén waves in a medium with wave density modulation will give rise to additional harmonic terms associated with other nonlinear terms (such as, for example, the convective term). This circumstance significantly complicates the equations, but does not change the nature of the parametric relationship, so for a qualitative analysis it is sufficient to limit ourselves to the choice of the model equation (1). We can say that Eq.(1) describes a wave medium in which the influence of the pump wave is reduced to the modulation of the wave phase velocities.

2 Parametric decay instability (PDI). Regions of instability.

Let's consider how the parametric coupling arises for a pair of waves

$$\omega_1, \mathbf{k}1; \omega 2, \mathbf{k}2$$

described by Eq.(1). It is easy to see that in the absence of a pump wave ($\varepsilon = 0$) Eq.(1) describes independent spatiotemporal harmonics with the **linear dispersion relation** $\omega(k) = kv_A$.

To investigate the coupling of waves, it is convenient to switch to Fourier components in (1) for the spatial variables $V_k = \int u(x)\exp(ikx)dx$ and transfer the term that takes into account the influence of the pump waves to the right-hand side of the equation.

To write the given equation in wavenumber space, we will use the wavenumber representation of $u(x,t)$, which is $V_k(t) = \int_{-\infty}^{\infty} u(x,t)e^{ikx}\, dx$.

Given the equation:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = v_A^2 \left[1 + \varepsilon \cdot \cos(\omega_0 t - k_0 x)\right] \frac{\partial^2 \mathbf{u}}{\partial x^2}$$

First, we take the Fourier transform of both sides of the equation with respect to x.

The Fourier transform of $\frac{\partial^2 \mathbf{u}}{\partial t^2}$ is:

$$\mathcal{F}\left\{\frac{\partial^2 \mathbf{u}}{\partial t^2}\right\} = \frac{\partial^2 V_k(t)}{\partial t^2}$$

The Fourier transform of $\frac{\partial^2 \mathbf{u}}{\partial x^2}$ is:

$$\mathcal{F}\left\{\frac{\partial^2 \mathbf{u}}{\partial x^2}\right\} = -k^2 V_k(t)$$

Now, consider the term with the cosine function:

$$\mathcal{F}\left\{v_A^2 \left[1 + \varepsilon \cos(\omega_0 t - k_0 x)\right] \frac{\partial^2 \mathbf{u}}{\partial x^2}\right\}$$

The cosine term can be written using Euler's formula:

$$\cos(\omega_0 t - k_0 x) = \frac{1}{2}\left(e^{i(\omega_0 t - k_0 x)} + e^{-i(\omega_0 t - k_0 x)}\right)$$

The Fourier transform of $\cos(\omega_0 t - k_0 x)$ will result in two delta functions in the frequency domain:

$$\mathcal{F}\left\{\cos(\omega_0 t - k_0 x)\right\} = \frac{1}{2}\left(e^{i\omega_0 t}\delta(k - k_0) + e^{-i\omega_0 t}\delta(k + k_0)\right)$$

Combining these results, the Fourier transform of the product

$$\left[v_A^2 \varepsilon \cos(\omega_0 t - k_0 x)\right] \frac{\partial^2 u}{\partial x^2}$$

involves the convolution of the individual transforms. Let:

$$f(x) = \cos(\omega_0 t - k_0 x) \;\; and \;\; g(x) = \frac{\partial^2 u}{\partial x^2}$$

The convolution in the Fourier domain is:

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{2} v_A^2 \varepsilon \left[e^{i\omega_0 t}\delta(k - k_0) + e^{-i\omega_0 t}\delta(k + k_0)\right] * (-k^2 V_k(t))$$

The convolution of a delta function with another function shifts the argument of the function:

$$-\frac{1}{2} v_A^2 \varepsilon \left[(k - k_0)^2 V_{k-k_0}(t) e^{i\omega_0 t} + (k + k_0)^2 V_{k+k_0}(t) e^{-i\omega_0 t}\right]$$

Simplifying, we get:

$$\mathcal{F}\left\{\left[v_A^2 \varepsilon \cos(\omega_0 t - k_0 x)\right] \frac{\partial^2 u}{\partial x^2}\right\} = -\frac{1}{2} v_A^2 \varepsilon \left[(k - k_0)^2 V_{k-k_0}(t) e^{i\omega_0 t} + (k + k_0)^2 V_{k+k_0}(t) e^{-i\omega_0 t}\right]$$

Combining everything, the equation in wavenumber space is:

$$\frac{\partial^2 V_k(t)}{\partial t^2} + v_A^2 k^2 V_k(t) + \frac{1}{2} v_A^2 \varepsilon \left[(k - k_0)^2 e^{i\omega_0 t} V_{k-k_0}(t) + (k + k_0)^2 e^{-i\omega_0 t} V_{k+k_0}(t)\right] = 0$$

or, taking into account that $\omega(k) = v_A^2 k^2$, we have:

$$\frac{\partial^2 V_k(t)}{\partial t^2}+\omega(k)V_k(t)=-\frac{1}{2}\varepsilon v_A^2\left[(k-k_0)^2e^{i\omega_0 t}V_{k-k_0}(t)+(k+k_0)^2e^{-i\omega_0 t}V_{k+k_0}(t)\right] \tag{7}$$

The equation we obtained is actually not a single equation. **Considering the continuous spectrum of waves, any waves, it can be said that this is a system of two equations for coupled waves with wave numbers k and $k \pm k_0$.** Note that ε is a small quantity.

First of all, we note that in the zeroth approximation in ε, all V_k oscillate with their own frequencies ω_k. The weak coupling does not significantly change the frequency of the oscillator. However, in the case when the forcing force in the right-hand side of Eq.(7) falls into resonance with the natural frequency, the oscillator can be excited.

The resonance condition for the first term in the right-hand side of Eq.(7) has the form: $\omega(k-k_0)+\omega_0=\omega(k)$, for the second: $\omega(k+k_0)-\omega_0=\omega(k)$.

Let the first condition be fulfilled, then the second term is non-resonant and can be neglected.

In turn, the Fourier component V_{k-k_0} of the resonance part of the equation Eq.(7) is described by the equation

$$\begin{gathered} d^2V_{k-k_0}/dt^2+\omega^2\left(k-k_0\right)V_{k-k_0}=-(\varepsilon/2)(k-2k_0)^2v_A^2\exp\left(\mathrm{i}\omega_0 t\right)V_{k-2k_0}- \\ -(\varepsilon/2)k^2v_A^2\exp\left(-\mathrm{i}\omega_0 t\right)V_k \end{gathered} \tag{8}$$

In this equation the second term is resonant. Therefore, considering only the resonant interaction of two coupled oscillators (**designated oscillator 1 and oscillator 2**), we obtain the following shortened system:

$$\begin{aligned} \frac{\partial^2 V_{k_1}(t)}{\partial t^2}+\omega^2(k_1)V_{k_1} &= -\frac{1}{2}v_A^2\varepsilon\cdot(k_2)^2e^{-i\omega_0 t}V_{k_2}^*(t) \\ \frac{\partial^2 V_{k_2}(t)}{\partial t^2}+\omega^2(k_2)V_{k_2} &= -\frac{1}{2}v_A^2\varepsilon\cdot(k_1)^2e^{i\omega_0 t}V_{k_1}(t) \end{aligned} \tag{9}$$

where the notation $k_2=k_0-k_1$ is introduced.

Taking into account the resonance conditions for frequencies, it can be said that parametrically related waves are those whose frequencies and wave vectors satisfy the conditions

$$
\begin{aligned}
\omega_0 = \omega_1(k_1) + \omega_2(k_2) \\
k_0 = k_1 + k_2
\end{aligned}
\tag{10}
$$

i.e., the conditions of spatiotemporal synchronization.

In accordance with the above, we will seek the solution (9) in the form $V_k = a(t)\exp\left[-\mathrm{i}\omega\left(k\right)t\right],\quad$ where $a(t)$-slowly changing amplitudes of the coupled waves. Then

$$
\begin{cases}
-2\mathrm{i}\omega_1 \dfrac{\partial a_1}{\partial t} = -\left(\dfrac{\varepsilon}{2}\right) k_2^2 v_A^2 a_2^* \exp(-\mathrm{i}\Delta\omega t) \\
-2\mathrm{i}\omega_2 \dfrac{\partial a_2^*}{\partial t} = -\left(\dfrac{\varepsilon}{2}\right) k_1^2 v_A^2 a_1 \exp(\mathrm{i}\Delta\omega t)
\end{cases}
\tag{11}
$$

where

$$
\Delta\omega = \omega_0 - \omega_1 - \omega_2.
$$

It is easy to see that the solution to (11) is:

$$
\begin{aligned}
a_1 \sim \exp\left(-\mathrm{i}\frac{\Delta\omega}{2}t + \nu t\right); \\
a_2^* \sim \exp\left(\mathrm{i}\frac{\Delta\omega}{2}t + \nu t\right),
\end{aligned}
\tag{12}
$$

where

$$
\nu = \sqrt{\gamma_D^2 - (\Delta\omega)^2/4}; \quad \gamma_D^2 \equiv \frac{\varepsilon^2 k_1^2 k_2^2 v_A^4}{16\omega_1\omega_2}
\tag{13}
$$

This solution describes a first-order parametric decay instability. It follows from (13) that at zero frequency detuning, i.e., at $\Delta\omega = 0$ [this means strict fulfillment of the resonance conditions (10)], the amplitudes of the waves a_1 and a_2 grow exponentially with an increment $\gamma = \gamma_D$. In this case, the relationship $\omega_1\omega_2 > 0$ must be fulfilled, which, together with the resonance conditions, gives $\omega_0 > \omega_1, \omega_2$.

In other words, in the case of parametric resonance instability, waves with frequencies less than the pump wave frequency are excited (red satellites). It should be noted that in the absence of dissipation the increment of the decay instability is proportional to the first power of the pump wave amplitude:

$$
\gamma_D \sim \varepsilon
$$

Equation (13) determines the width of the first-order instability zone $n = 1$. For detuning $|\Delta\omega/2| > \gamma_D$, the instability disappears. **This means that the width of the first PI zone is proportional to the first power of the pump wave amplitude.**

Knowing the theory of parametric resonance in oscillatory systems, this conclusion could have been made immediately after reducing the problem to solving the shortened equations (9), which describe a system of two parametrically coupled oscillators.

It should be emphasized that the system of shortened equations is obtained using the conditions of not only temporal ($\omega_0 = \omega_1 + \omega_2$), but also spatial $\mathbf{k_0} = \mathbf{k_1} + \mathbf{k_2}$ resonance. The similarity of (1) with Mathieu's equation, as well as the method of obtaining systems of shortened equations (based on spatiotemporal resonance of modes), allow qualitative conclusions to be drawn about higher-order parametric resonance and the corresponding instability zones.

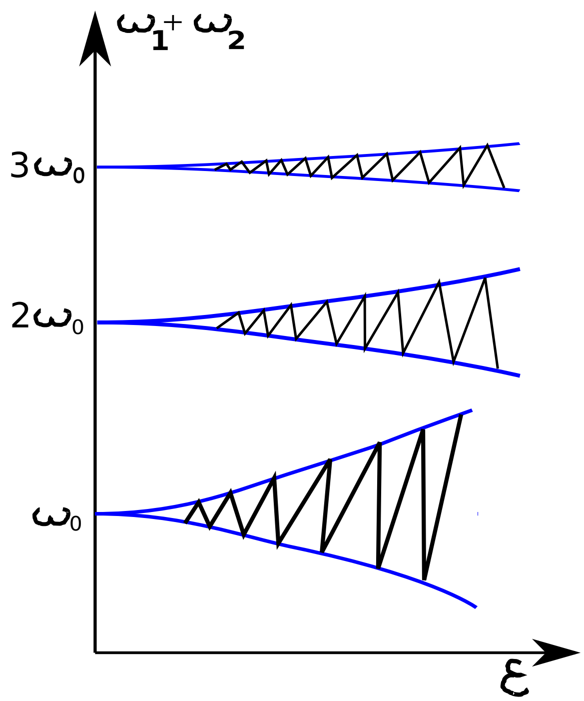

Figure 1: Stability - instability chart of the parametric interaction. (Mathieu equation stability chart).

Obviously, for waves of relatively small amplitude (in our example, $\tilde{\rho} \ll \rho_0$), the increment of the n-th order PI $\gamma_n \approx \varepsilon^n$.

Accordingly, the instability zone narrows with increasing n, since $|\Delta\omega_n|/2 = \gamma_n$, where $\Delta\omega_n = n\omega_0 - \omega_1 - \omega_2$.

Figure 1 shows the zones of PI of the n-th order.

Instabilities of the first and second orders are of mos practical importance due to decrease in increments and the narrowing of instability zones with increasing n.

PI of the second order manifest themselves in those cases when PI of the first order do not arise due to the impossibility of fulfilling conditions (10).

In systems where PI of the first order are absent, the conditions for the occurrence of PI of the second order are usually met.

3 Thresholds of PDI

Parametric decay instabilities arise when the amplitudes exceed certain values. In the approximation of a homogeneous plasma, these thresholds are determined by the decrements of the excited wave doublet, which can be shown by introducing dissipation terms into equation (9). This is not difficult to do if small imaginary additions to the natural frequencies ω_i are introduced according to the scheme $\omega_i \to \omega_i + i\gamma_i$, where γ_i are the decrements of the corresponding waves. Assuming $\Delta\omega = 0$ and performing simple calculations and **taking dissipation into account**, we obtain the following expression for the **growth rate of the parametrically excited wave** ν_D :

$$\nu_D = -\frac{\gamma_1 + \gamma_2}{2} + \sqrt{\gamma_D^2 + \frac{(\gamma_1 - \gamma_2)^2}{4}}, \tag{14}$$

The requirement $\nu_D > 0$ gives the expression for the instability threshold in general form:

$$\hat{\gamma}^2{}_D > \gamma_1\gamma_2 \tag{15}$$

or for the model problem (9):

$$\varepsilon_{\text{thr}} = \frac{16\omega_1\omega_2\gamma_1\gamma_2}{k_1^2 k_2^2 v_A^4} \tag{16}$$

From (15), it follows that the threshold disappears when at least one of the decrements of the excited wave pair tends to zero.