On the correctness of the Pesin's Identity in turbulent flow (too ideal theoretical context)

Perfect — this is a deep and elegant topic, one of the most beautiful bridges between **dynamical systems** and **statistical mechanics**.
Let's develop the full **thermodynamic analogy** step by step — mapping each concept from statistical mechanics to its dynamical counterpart.

⚛️ 1. The Big Picture

The **thermodynamic formalism**, introduced by Ruelle, Sinai, and Bowen (1970s), treats a chaotic dynamical system as a kind of **thermodynamic ensemble**.

- Each **orbit segment** of length n plays the role of a **microstate**.
- The **potential function** $\phi(x)$ acts like the **energy** of that microstate.
- The **topological pressure** $P(\phi)$ plays the role of the **free energy per step**.
- The **equilibrium measure** is analogous to the **Gibbs canonical ensemble**.

🧩 2. The Correspondence Table

Thermodynamics	Dynamical Systems
Microstate	Finite orbit segment $(x, f(x), \dots, f^{n-1}(x))$
Energy of state E_i	Potential $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(f^k(x))$
Temperature $T = 1/\beta$	Scaling of potential: $\beta\phi$
Partition function $Z(\beta) = \sum_i e^{-\beta E_i}$	$Z_n(\phi, \varepsilon) = \sum_{x \in E} e^{S_n\phi(x)}$
Free energy $F = -\frac{1}{\beta} \ln Z$	Pressure $P(\phi) = \lim(1/n) \ln Z_n(\phi)$
Entropy $S = -\sum p_i \ln p_i$	Metric entropy $h_\mu(f)$
Internal energy $U = \sum p_i E_i$	$\int \phi \, d\mu$
Equilibrium state	Invariant Gibbs measure μ_ϕ
Variational principle: $F = U - TS$	$P(\phi) = h_\mu(f) + \int \phi \, d\mu$

🧠 3. Physical Intuition

In thermodynamics, a system distributes itself among microstates with Boltzmann weights $e^{-\beta E_i}$, maximizing entropy subject to energy constraints.

In a chaotic map f:

- Orbit segments play the role of microstates.
- The potential $\phi(x)$ assigns an "energy density" to each part of phase space.
- The dynamics "explores" microstates with frequency given by an invariant measure μ.
- The **equilibrium measure** μ_ϕ maximizes

$$h_\mu(f) + \int \phi \, d\mu,$$

exactly like maximizing $S - \beta U$ in statistical mechanics.

📈 4. The Partition Function Analogy

For a compact dynamical system, define the **partition function**

$$Z_n(\phi, \varepsilon) = \sum_{x \in E} e^{S_n \phi(x)},$$

where E is a maximal (n, ε)-separated set (like counting distinguishable microstates at resolution ε).

This mimics the canonical partition function

$$Z(\beta) = \sum_i e^{-\beta E_i}.$$

Then the **topological pressure** is the growth rate:

$$P(\phi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln Z_n(\phi, \varepsilon),$$

just like the **free energy per step**.

🧮 5. Variational Principle (Free Energy Relation)

The most fundamental identity:

$$P(\phi) = \sup_{\mu \in \mathcal{M}_f} \left(h_\mu(f) + \int \phi \, d\mu \right).$$

Compare with thermodynamics:

$$\text{Free energy} = \text{Entropy term} + \text{Energy term}.$$

At equilibrium, the measure μ_ϕ that attains this supremum satisfies

$$h_{\mu_\phi}(f) + \int \phi \, d\mu_\phi = P(\phi),$$

analogous to $F = U - TS$.

🔥 6. Temperature, Scaling, and Phase Transitions

If we scale the potential:

$$P(t\phi),$$

then t plays the role of **inverse temperature** ($t = \beta$).

- As $t \to 0$: high temperature, all orbits contribute equally → maximal entropy.
- As $t \to \infty$: low temperature, only orbits with minimal energy (maximal ϕ) dominate.

If $P(t\phi)$ is nonanalytic in t, the system exhibits a **phase transition**,
analogous to thermodynamic ones (this happens in multifractal or symbolic systems).

🌌 7. Example: Dimension from Pressure

For a conformal map f, let $\phi_t(x) = -t \ln |f'(x)|$.

The Hausdorff dimension D of the invariant set satisfies

$$P(\phi_D) = 0.$$

That is, D is the unique "temperature" at which the free energy (pressure) vanishes —
a beautiful **thermodynamic interpretation of fractal dimension**.

🧭 8. Conceptual Summary

Concept	Role
$h_{\text{top}} = P(0)$	Pure disorder (entropy)
$P(\phi)$	Weighted disorder (free energy rate)
$\phi(x)$	Energy density (potential)
$S_n\phi(x)$	Energy of orbit segment
$Z_n(\phi)$	Partition function over orbit segments
$P(\phi) = \lim(1/n) \ln Z_n$	Free energy per step
μ_ϕ	Gibbs-equilibrium measure
t in $P(t\phi)$	Inverse temperature
$P(t\phi) = 0$	Defines fractal dimension or critical temperature

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In essence:

> The **thermodynamic formalism** translates the language of **chaos and orbits** into that of **energy, entropy, and equilibrium**.
> Topological pressure is the **free energy** of the dynamical system, unifying entropy, dimension, and measure theory under one umbrella.