Can one analyse the chaotic behaviour of the continuous system , like dumped-driven pendulum in terms of the Lyapunov exponent, in the same way as the Logistic map ?

Yes, one can analyze the chaotic behavior of a continuous system like the damped-driven pendulum using Lyapunov exponents, but the method of calculation differs significantly from that used for a discrete map like the logistic map.

The core concept remains the same: a positive Lyapunov exponent indicates chaos because it quantifies the exponential divergence of nearby trajectories. However, the mathematical framework to calculate them is different for continuous versus discrete systems.

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1. Lyapunov Exponents for Discrete Systems (e.g., Logistic Map)

A **discrete system** is defined by a map or a recurrence relation, where the state of the system at the next step (x_{n+1}) is a function of its current state (x_n).

The logistic map is a prime example: $x_{n+1} = rx_n(1 - x_n)$. To find its Lyapunov exponent (λ), you calculate the average of the logarithm of the absolute value of the derivative of the map function over many iterations.

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln |f'(x_n)|$$

Here, $f(x) = rx(1 - x)$, and $f'(x) = r(1 - 2x)$. The derivative is what measures the local stretching or compression of the phase space. A positive value for λ means that on average, the trajectories are separating exponentially, which is the hallmark of chaos.

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2. Lyapunov Exponents for Continuous Systems (e.g., Damped-Driven Pendulum)

A **continuous system** is described by a set of ordinary differential equations (ODEs) that govern its evolution over time.

The equation of motion for a damped-driven pendulum is a second-order nonlinear ODE, which can be rewritten as a system of first-order ODEs to define its phase space:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\frac{g}{L}\sin(\theta) - \beta\omega + A\cos(\Omega t)$$

Here, θ is the angular displacement and ω is the angular velocity. The phase space is the (θ, ω) plane, but since the forcing term depends on time, the system is often considered a 3D autonomous system in $(\theta, \omega, \Omega t)$.

To calculate the Lyapunov exponents for this system, you don't just take the derivative of the function. Instead, you analyze how a small perturbation vector changes over time. This involves solving the **variational equations**, which are a linearized version of the original system's ODEs. You must track how a set of orthogonal vectors in the phase space evolve, stretching and contracting with the flow. The Lyapunov exponents are the asymptotic growth rates of these vectors. For an n-dimensional system, there will be n Lyapunov exponents, which form the Lyapunov spectrum.

The **largest Lyapunov exponent** (λ_{max}) is the one that determines if the system is chaotic. If $\lambda_{max} > 0$, the system is chaotic. For a dissipative system like the damped-driven pendulum, the sum of all Lyapunov exponents is negative, meaning that the phase space volume contracts over time, but chaos can still exist if at least one exponent is positive.