

PART-A

Ques 1 Define Lagrange Auxiliary equation.

Ans Lagrange auxiliary eq<sup>n</sup> are the system of ordinary differential equation (ODE). So, to solve first order linear differential equation (PDES) of the form

$$P(x, y, z) \frac{dz}{dx} + Q(x, y, z) \frac{dz}{dy} = R(x, y, z)$$

This is written in the form of

$$\boxed{Pp + Qq = R}$$

$$p = \frac{dz}{dx}, \quad q = \frac{dz}{dy}$$

The Lagrange differential eq<sup>n</sup> is

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

Ques 2 Solve:  $xp + yq = z$

Ans form:  $\boxed{Pp + Qq = R}$ ,  $p = \frac{dz}{dx}$ ,  $q = \frac{dz}{dy}$

where  $P = x$ ,  $Q = y$ ,  $R = z$

Lagrange differential eq<sup>n</sup>

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$= \int \frac{dx}{x} = \int \frac{dy}{y} + \log c$$

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\boxed{\frac{x}{y} = c_1}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z} + \log c_2$$

$$\log y = \log z + \log c_2$$

$$\log y - \log z = \log c_2$$

$$\boxed{y = c_2 z}$$

So,

$$f(c_1, c_2) = 0$$

$$f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

Ans.

Ques 3 Solve the PDE:  $yzp + xq = xy$

Ans form:  $pp + qq = r$ ,  $p = \frac{dz}{dx}$ ,  $q = \frac{dz}{dy}$

where,

$$p = yz, \quad q = x, \quad r = xy$$

LDE:

$$\frac{dx}{yz} = \frac{dy}{x} = \frac{dz}{xy}$$



$$\frac{dx}{y^2} = \frac{dz}{xy} \Rightarrow \int x dx = \int z dz + C_1$$

$$\frac{x^2}{2} = \frac{z^2}{2} + C_1$$

$$\frac{x^2}{2} = \frac{z^2 + C_1}{2} \Rightarrow \boxed{x^2 - z^2 = C_1}$$

then,

$$\frac{dy}{x} = \frac{dz}{xy}$$

$$\int y dy = \int z dz + C_2$$

$$\frac{y^2}{2} = \frac{z^2}{2} + C_2$$

$$\frac{y^2 - z^2}{2} = C_2 \Rightarrow \boxed{y^2 - z^2 = C_2}$$

So,

$$f(C_1, C_2) = 0$$

$$f(x^2 - z^2, y^2 - z^2) = 0 = \text{Ans}$$

Ques Define Charpit's auxiliary equation.

Ans - Charpit method is a technique used to solve non-linear first order partial differential eqn of the form:

$$f(x, y, z, p, q) = 0$$

Charpit's Auxiliary equation!

$$\frac{df}{dx} + p \frac{df}{dz} = \frac{dq}{dy} + q \frac{df}{dz} = -p \frac{\partial f}{\partial p} = q \frac{\partial f}{\partial q}$$

$$\frac{\frac{\partial x}{-\partial f}}{\frac{\partial p}{-\partial f}} = \frac{\frac{\partial y}{-\partial f}}{\frac{\partial q}{-\partial f}} = \frac{\frac{\partial z}{0}}{\frac{\partial f}{0}}$$

Ques 5 Solve  $(p+q)(px+qy)=1$

Ans eq:  $p^2x + pqy + qpx + q^2y = 1$

$$\bullet \frac{\partial f}{\partial x} = p^2 + pq$$

$$\bullet \frac{\partial f}{\partial p} = 2p + qy + qx$$

$$\bullet \frac{\partial f}{\partial y} = pq + q^2$$

$$\bullet \frac{\partial f}{\partial q} = py + px + 2qy$$

$$\bullet \frac{\partial f}{\partial z} = 0$$

then,

$$\frac{\frac{\partial f}{\partial x}}{p^2 + pq} = \frac{\frac{\partial f}{\partial y}}{pq + q^2}$$

$$\frac{\frac{\partial p}{p(p+q)}}{\frac{\partial q}{q(p+q)}} = \frac{\frac{\partial p}{p}}{\frac{\partial q}{q}}$$



$$\log p = \log q + \log a$$

$$\boxed{p = qa}$$

put the value of  $p$  in equation.

$$(qa+a)(qax+ay)=1$$

$$q(a+1)q(ax+y)=1$$

$$q^2(a+1)(ax+y)=1$$

$$q^2 = \frac{1}{(a+1)(ax+y)} \Rightarrow \boxed{q = \frac{1}{\sqrt{(a+1)(ax+y)}}}$$

So,

$$p = qa$$

$$\boxed{p = \frac{a}{\sqrt{(a+1)(ax+y)}}}$$

### PART-B

Ques Find the general integral of  $xzp + yzq = xy$

Ans equation:  $xzp + yzq = xy$

$$P = xz, Q = yz, R = xy$$

$$\text{Form: } Pp + Qq = R$$

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{xy}$$

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \int \frac{dx}{x^2} = \int \frac{dy}{y^2} + \log e,$$

$$= \log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\boxed{\frac{x}{y} = c_1}$$

$$\Rightarrow \frac{dx}{x^2} = \frac{dz}{xy}$$

$$\int y dx = \int z dz + c_2$$

$$xy = \frac{z^2}{2} + c_2$$

$$\boxed{xy - \frac{z^2}{2} = c_2}$$

$$f(c_1, c_2) = 0$$

$$\Rightarrow f\left(\frac{x}{y}, xy - \frac{z^2}{2}\right) = 0$$

Ans



Ques 2 Solve  $(1+y)p + (1+x)q = z + xh$

Ans. Eq<sup>n</sup>:  $(1+y)p + (1+x)q = z$

form:  $p p + q q = z$

Here,

$$p = 1+y, \quad q = 1+x, \quad r = z$$

$$\text{IDE: } \frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r}$$

$$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}$$

$$\frac{dx}{1+y} = \frac{dy}{1+x}$$

$$(1+x)dx = (1+y)dy + C_1$$

$$x + \frac{x^2}{2} = y + \frac{y^2}{2} + C_1$$

$$2x + x^2 = 2y + y^2 + C_1$$

$$\boxed{x^2 - y^2 + 2x - 2y = C_1}$$

Now put,  $(1, 1, 0)$  in each fraction.

$$\frac{dx+dy}{1+y+1+x} = \frac{dx+dy}{2+x+y} = \frac{dz}{z}$$

$$\frac{dx+dy}{2+x+y} = \frac{dz}{z}$$

$$\frac{d(2+x+y)}{2+x+y} = \frac{dz}{z}$$

$$\log(2+x+y) = \log z + \log c_1$$

$$2+x+y = z c_2$$

$$\boxed{\frac{2+x+y}{z} = c_2}$$

$$f(c_1, c_2) = 0$$

$$f(x^2 - y^2 + 2x - 2y, \frac{2+x+y}{z}) = 0$$

Ques 3 Find the complete integral of  $q = px + p^2$

Ans- eq<sup>n</sup>:  $p^2 + px = q$

let:  $f = p^2 + px - q = 0$

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial q} = -1$$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial p} = 2p + x$$



AE:

$$\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = \frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}$$

$$\frac{\partial p}{p + p(0)} = \frac{\partial q}{0 + 0}$$

$$\frac{\partial p}{p} = \frac{\partial q}{0}$$

$$\text{So, } dq = 0 \rightarrow \boxed{q = a}$$

Put the value of  $q$  in the equation

$$p^2 + px - q = 0$$

$$p^2 + px - a = 0$$

So, it is a quadratic equation

$$p = \frac{-x \pm \sqrt{x^2 - 4a}}{2}$$

$$dz = p dx + q dy$$

$$dz = \frac{-x \pm \sqrt{x^2 + 4a}}{2} dx + a dy$$

$$(dz = \int \frac{-x}{2} dx + \int \frac{\sqrt{x^2 + 4a}}{2} dx + \int a dy$$

$$z = \frac{-x^2}{4}$$

$$\therefore \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$Z = -\frac{x^2}{4} + \frac{1}{2} \left( \frac{x}{2} \sqrt{x^2 + 4a^2} + \frac{4a^2}{2} \log |x + \sqrt{x^2 + 4a^2}| \right) + ay + C$$

$$Z = -\frac{x^2}{4} + \frac{1}{2} \left( \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + 4a^2}| \right) + ay + C$$

### PART-C

Ques: Solve:  $\cos(x+y)p + \sin(x+y)q = z$

Ans: Eq:  $\cos(x+y)p + \sin(x+y)q = z$

form:  $Pp + Qq = R$

where,

$$P = \cos(x+y), Q = \sin(x+y), R = z$$

Compare Auxiliary eq<sup>n</sup>  $\left[ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \right]$

$$\Rightarrow \frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$$



$$= \frac{dx + dy}{\cos(x+y) + \sin(x+y)} = \frac{dx - dy}{\cos(x+y) - \sin(x+y)}$$

then,

$$\frac{dx + dy}{\cos(x+y) + \sin(x+y)} = \frac{dx - dy}{\cos(x+y) - \sin(x+y)}$$

$$dx - dy = \frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)} (dx + dy)$$

$$\text{let } \cos(x+y) + \sin(x+y) = t$$

$$-\sin(x+y) + \cos(x+y)(dx + dy) = dt$$

$$\Rightarrow dx - dy = \frac{dt}{t}$$

on integrating both sides.

$$\int dx - \int dy = \int \frac{dt}{t}$$

$$x - y = \log[\cos(x+y) - \sin(x+y)] - \log C_1$$

$$x - y = \log(\cos(x+y) + \sin(x+y))$$

$$e^{x-y} = \cos(x+y) + \sin(x+y)$$

$$\Rightarrow C_1 = [\cos(x+y) + \sin(x+y)] \cdot e^{-(x+y)} \quad \text{--- (1)}$$

then,

$$\frac{dz}{z} = \frac{dx + dy}{\cos(x+y) + \sin(x+y)}$$

divide and multiply by  $\frac{1}{\sqrt{2}}$ 

$$= \frac{1}{\sqrt{2}} \frac{dx + dy}{\frac{1}{\sqrt{2}} \cos(x+y) + \frac{1}{\sqrt{2}} \sin(x+y)}$$

$$\text{put } \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4} \text{ and } \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} \frac{(dx + dy)}{\sin \frac{\pi}{4} \cos(x+y) + \cos \frac{\pi}{4} \sin(x+y)}$$

$$\therefore \sin A \cos B + \cos A \sin B = \sin(A+B)$$

$$\Rightarrow \sqrt{2} \frac{dz}{z} = \frac{dx + dy}{\sin(x+y + \frac{\pi}{4})}$$

$$\sqrt{2} \int \frac{dz}{z} = \int \operatorname{cosec}(x+y + \frac{\pi}{4}) (dx + dy)$$

$$\text{let } x+y + \frac{\pi}{4} = t$$

$$dx + dy = dt$$

$$\sqrt{2} \log z = \int \operatorname{cosec} t \, dt$$



$$\sqrt{2} \log z = \log \tan \frac{t}{2} + \log \frac{c_2}{2}$$

$$\log z^{\sqrt{2}} = \log c_2 \tan \left( \frac{x+y+\pi/4}{2} \right)$$

$$z^{\sqrt{2}} = \frac{c_2 \tan \left( \frac{x+y+\pi/4}{2} \right)}{2}$$

$$z^{\sqrt{2}} \cot \left( \frac{x+y}{2} + \frac{\pi}{8} \right) = c_2$$

So final answer  $f(c_1, c_2) = 0$

$$f \left( z^{\sqrt{2}} \cot \left( \frac{x+y}{2} + \frac{\pi}{8} \right), (\cos(x+y) + \sin(x+y)) \right) = 0$$

$$e^{-(x+y)} = 0$$

Ans

Ques-3 Find the complete integral of  $z^2 = pxy$

Ans  $\frac{df}{dx} = -pxy, \frac{df}{dy} = -pyx$

$$\frac{df}{dz} = 2x, \frac{df}{dp} = -axy, \frac{df}{dz} = -pxy$$

$$\frac{dx}{axy} = \frac{dy}{pyx} = \frac{dz}{paxy + axyp} = \frac{dp}{-pxy + 2pz}$$

$$= \frac{dq}{-pqx + 2qz}$$

$$\frac{p dx + x dp}{2xpz} = \frac{q dy + y dq}{2yqz}$$

$$\log(px) = \log qy + \log q$$

$$p = \frac{qy}{x}$$

put  $\frac{qy}{x}$  in eq (i)

$$z^2 - \left(\frac{qy}{x}\right) x y = 0$$

$$q = \frac{z}{y\sqrt{a}} \Rightarrow p = \frac{z\sqrt{a}}{x}$$

$$dz = p dx + q dy$$

$$dz = \frac{z\sqrt{a}}{x} dx + \frac{z}{y\sqrt{a}} dy$$

$$\int \frac{dz}{z} = \int \sqrt{a} \frac{dx}{x} + \int \frac{1}{\sqrt{a}} \frac{dy}{y}$$

$$\log z = \sqrt{a} \log x + \frac{1}{\sqrt{a}} \log y + \log c$$

$$z = x^{\sqrt{a}} y^{\frac{1}{\sqrt{a}}}$$

where

$$\sqrt{a} = c$$

$$z = x^c y^{1/c}$$

Ans