Tutorial Sheet-2

PART-A

1. Define linear dependence and linear independence of vectors.

Ans A set of n vector in an m-dimensional space is linearly dependent if the rank of the matrix formed by these vectors is less than the number of vectors.

Rank(A) < Total number of vectors

A set of n vectors is linearly independent if the rank of the matrix formed by these vectors is equal to the number of vectors.

Rank(A) = Total number of vectors

2. Write the statement of Cayley-Hamilton Theorem.

Ans Every square matrix A satisfies its own characteristics equation i.e → If A be any square matrix of order n. Its characteristic eq^n is $|A-\lambda I| = 0$.

3. For which value of x will the matrix

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$ becomes singular.

$$|A| = 8(0-12) - x(0-24) + 0$$
$$-96 + 24x$$
$$24x = 96$$
$$x = \frac{96}{24} \quad = x = 4$$

4. Are the vectors $x_1 = (1, 3, 4, 2)$, $x_2 = (3, -5, 2, 2)$ and $x_3 = (2, -1, 3, 2)$ linear dependent?

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 3 & -5 & 2 & 2 \\ 2 & -1 & 3 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & -4 & -2 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - 2R_1$

$R_4 \to 10R_4 - 4R_3$

$$A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow 10R_2 - 14R_3$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $\rho(A) = 2 <$ No. of vectors (3)

so, $[x_1, x_2, x_3]$ are linearly dependent

5. Find the characteristic equation of the matrix :

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

2

$$\boxed{A - \lambda I} = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & +4 & 3-\lambda \end{vmatrix} = 0$$

$R_2 \to R_2 + 3R_3$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ 0 & -5-\lambda & 5-3\lambda \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)\left[(5+\lambda)(\lambda-3) + 4(5-3\lambda)\right] + 2\left[6(3\lambda-5) + 2(5+\lambda)\right] = 0$$

$$(8-\lambda)\left[\lambda^2 - 10\lambda + 5\right] + 2\left[20\lambda - 20\right]$$

$$18\lambda^2 - 45\lambda - \lambda^3 = 0$$

$$\boxed{\lambda^3 + 45\lambda - 18\lambda^2 = 0}$$ Ans

PART-B

1. Check the vectors $x_1 = (1,1,1,3)$, $x_2 = (1,2,3,4)$ and $x_3 = (2,3,4,9)$ are linear dependent or linear independent?

Ans Let a matrix $A = [x_1, x_2, x_3]$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{bmatrix}_{4\times 3}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 3R_1$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}_{4\times 3}$$

$\rho(A) = 3$ & No. of vectors $= 3$

Hence $\rho(A) =$ No. of vectors.

These 3 vectors x_1, x_2 & x_3 are linearly independent.

2. Find the eigen values of matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

7

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(3-\lambda)-2\right] - 1\left[2-2(2-\lambda)\right] = 0$$

$$(1-\lambda)\left[6-2\lambda-3\lambda+\lambda^2-2\right] - 1\left[2-4+2\lambda\right] = 0$$

$$(1-\lambda)\left[\lambda^2-5\lambda+4\right] + 2 - 2\lambda = 0$$

$$\lambda^2-5\lambda+4-\lambda^3+5\lambda^2-4\lambda+2-2\lambda = 0$$

$$6\lambda^2-11\lambda+6-\lambda^3 = 0$$

$$\lambda^3-6\lambda^2+11\lambda-6 = 0$$

$$\lambda^2(\lambda-1)-5\lambda(\lambda-1)+6(\lambda-1) = 0$$

$$(\lambda-1)(\lambda^2-5\lambda+6) = 0$$

$$(\lambda-1)(\lambda^2-3\lambda-2\lambda+6) = 0$$

$$(\lambda-1)(\lambda(\lambda-3)-2(\lambda-3)) = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\lambda = 1, 2, 3$$

Eigen values are 1, 2, 3 Ans

3. Find the eigen values of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Ans $|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

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$R_2 \to R_2 + R_3$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 2-\lambda & 2-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(6-\lambda)\left[(3-\lambda)+1\right] + 2\left[-2-2\right]\right] = 0$$

$$(2-\lambda)\left[(6-\lambda)\left[4-\lambda\right] - 8\right] = 0$$

$$(\lambda-2)\left[24 - 6\lambda - 4\lambda + \lambda^2 - 8\right] = 0$$

$$(\lambda-2)\left[\lambda^2 - 10\lambda + 16\right] = 0$$

$$(\lambda-2)\ (\lambda-8)\ (\lambda-2)=0$$

Eigen values are 2, 2, 8 Ans

PART-C

1. Find the eigen values and eigen vectors of the matrix.

$$A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$$

$$|A-\lambda I|=0$$

$$\begin{vmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{vmatrix}=0$$

$R_2 \to R_2 - R_3$

$$\begin{vmatrix}-\lambda & 1 & 1\\ 0 & -\lambda-1 & 1+\lambda\\ 1 & 1 & -\lambda\end{vmatrix}=0 = (\lambda+1)\begin{vmatrix}-\lambda & 1 & 1\\ 0 & -1 & 1\\ 1 & 1 & -\lambda\end{vmatrix}=0$$

$R_1 \to R_1 + R_2$

$$(\lambda+1) \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & -1 & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$(\lambda+1)\left[(-\lambda)[\lambda-1] + 2(1)\right] = 0$$

$$(\lambda+1)\left[-\lambda^2 + \lambda + 2\right] = 0$$

$$(\lambda+1)\left[\lambda^2 - \lambda - 2\right) = 0$$

$$(\lambda+1)(\lambda-2)(\lambda+1) = 0$$

$\boxed{\lambda = -1, -1, 2}$ → eigen values

To find eigen vectors:

$$AX = 0$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} X = 0$$

① For $\lambda_1 = 2$

$$(A - \lambda_1 I)\, x_1 = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} X_1 = 0$$

$$-2x_a + x_b + x_c = 0$$
$$x_a - 2x_b + x_c = 0$$
$$x_a + x_b - 2x_c = 0$$

$$\frac{x_a}{1+2} = \frac{-x_b}{-2-1} = \frac{x_c}{4-1}$$

$$\frac{x_a}{3} = \frac{x_b}{3} = \frac{x_c}{3}$$

$$\frac{x_a}{1} = \frac{x_b}{1} = \frac{x_c}{1}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ Ans

(II) For $\lambda_2 = -1$

$$(A - \lambda_2 I) X_2 = 0$$

$$\begin{bmatrix} +1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} X_2 = 0$$

$x_a + x_b + x_c = 0$

Let $x_a = -1$, $x_b = 1$ then $x_c = 0$

$$X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$ Ans

(iii) For $\lambda_3 = -1$

$(A - \lambda_3 I) X_3 = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} X_3 = 0$$

$x_a + x_b + x_c = 0$

Let $x_a = 1$, $x_b = 1$ then $x_c = -2$

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$ Ans

2 Find the eigen values and eigen vectors of the matrix.

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$R_3 \to R_3 + R_2$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ 0 & -1-\lambda & -1-\lambda \end{vmatrix}$$

$$(-1-\lambda) \begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$C_2 = C_2 - C_3$

$$(\lambda+1) \begin{vmatrix} 4-\lambda & 0 & 6 \\ 1 & 1-\lambda & 2 \\ 0 & 0 & 2 \end{vmatrix} = 0$$

$(\lambda+1)(1-\lambda)[4-\lambda] = 0$

$\boxed{\lambda = -1, 1, 4} \rightarrow$ 3 eigen values

To find eigen vectors

$AX = 0$

$$\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} x = 0$$

(1) For $\lambda = -1$

$[A - \lambda I] X_1 = 0$

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} X_1 = 0$$

$5x_1 + 6x_2 + 6x_3 = 0$

$x_1 + 4x_2 + 2x_3 = 0$

$-x_1 - 4x_2 - 2x_3 = 0$

$$\frac{x_1}{12-24} = \frac{-x_2}{10-6} = \frac{x_3}{20-6}$$

$$\frac{x_1}{-12} = \frac{x_2}{-4} = \frac{x_3}{14}$$

$$\frac{x_1}{-6} = \frac{x_2}{-2} = \frac{x_3}{7}$$

$$X_1 = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$$

(ii) For $\lambda_2 = 1$

$$(A - \lambda_2 I) X_2 = 0$$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} X_2 = 0$$

$$3x_1 + 6x_2 + 6x_3 = 0$$
$$x_1 + 2x_2 + 2x_3 = 0$$
$$-x_1 - 4x_2 - 4x_3 = 0$$

$$\frac{x_1}{-24+24} = \frac{-x_2}{-12+6} = \frac{x_3}{-12+6}$$

$$\frac{x_1}{0} = \frac{x_2}{6} = \frac{x_3}{-6} = \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(iii) For $\lambda_3 = 4$

$$(A - \lambda_3 I) x_3 = 0$$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} X_3 = 0$$

$$0x_1 + 6x_2 + 6x_3 = 0$$
$$x_1 - x_2 + 2x_3 = 0$$
$$-x_1 - 4x_2 - 7x_3 = 0$$

$$\frac{x_1}{12+6} = \frac{-x_2}{-6} = \frac{x_3}{-6}$$

$$\frac{x_1}{\cancel{18}\,3} = \frac{x_2}{\cancel{6}\,1} = \frac{x_3}{-\cancel{6}\,2}$$

$$\frac{x_1}{3} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_3 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \text{ Ans}$$

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③ Verify Cayley Hamilton theorem for the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$ and hence find A^{-1}

$$|A - \lambda I| = 0$$

$$= \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$R_2 = R_2 + R_3$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 0 & 1-\lambda & 1-\lambda \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)\left[(\lambda-1)(\lambda-2) + (1-\lambda)\right] + 1\left[(\lambda-1) + (\lambda-2)\right] = 0$

$\Rightarrow (2-\lambda)\left[\lambda^2 - 4\lambda + 3\right] + 2\lambda - 2 = 0$

$\Rightarrow 6\lambda^2 - 9\lambda + 4 - \lambda^3 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

By Cayley Hamilton theorem:

$A^3 - 6A^2 + 9A - 4I = 0$

Multiply above equation by A^{-1}

$A^2 - 6A + 9I_3 - 4A^{-1} = 0$

$4A^{-1} = A^2 - 6A + 9I_3$

$A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I_3\right]$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

Substituting all values in Cayley's equation

$$A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9I_3\right]$$

$$A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right]$$

$$\boxed{A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}}$$ Ans

6/6