Set3

Group A **(3 x 10 = 30)**

Attempt any three questions.

1. (a) A function is defined by $f(x)=\begin{cases} x+2, & if\ \ x<0 \\ 1-x & if\ \ x\geq 0 \end{cases}$ [5]

Find domain and sketch the graph.

(b) Evaluate the limit $\lim_{t\to 0}\dfrac{\sqrt{t^2+9}-3}{t^2}$. [5]

2. (a) Sketch the curve : $f(x)=xe^x$. [5]

(b) Estimate the area of the region bounded above by $y=e^x$, bounded below by y = x and bounded on the sides by x = 0 and x = 1. [5]

3. (a) Find the volume of the solid obtained by rotating the region bounded by , $y=x^3$, y = 8 and x = 0 about the y-axis. [4]

(b) Define degree of a differential equation. Solve the differential equation : $\dfrac{dy}{dx}=\dfrac{x^2}{y^2}$ and hence find the solution of this equation that satisfies the initial condition y(0) = 2. [6]

4. (a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\vec{i}+4\vec{j}-2\vec{k}$. Find two other points on the line. [6]

(b) If $\vec{a}=\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{b}=4\vec{i}+7\vec{k}$, find the unit vector along $2\vec{a}+3\vec{b}$. [4]

Group B **(10 x 5 = 50)**

Attempt any ten questions.

5. Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $y=x^5+x$ (b) $y=1-x^4$ (c) $y=2x-x^2$

6. Find the horizontal and vertical asymptotes of the graph of the function $f(x)=\dfrac{\sqrt{2x^2+1}}{3x-5}$.

7. Suppose that f(0) = -3 and $f'(x)\leq 5$ for all values of x . How large can f(2) possibly be?

8. Find the curvature of the parabola $y=x^2$ at the points (0, 0), (1, 1) and (2, 4)

9. Evaluate: $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx.$

10. Find the length of the arc of the parabola $y^2=x$ from the points (0, 0) and (1, 1) .

11. Define non-homogeneous second-order linear differential equation.

Solve: $y''+y'-2y=x^2$.

12. Find the Maclaurin series of the function $f(x)=e^x$ and its radius of convergence.

13. Find the distance between two parallel planes 10x + 2y - 2z = 5 and 5x + y - z = 1.

14. Evaluate: $\iint_R f(x,y)\,dA$ for $f(x,y)=y\sin(xy)$, $R=[1,2]\times[0,\pi]$.

15. Use the scalar triple product to show that the vectors, $\vec{a}=\vec{i}+4\vec{j}-7\vec{k}$, $\vec{a}=2\vec{i}-\vec{j}+4\vec{k}$ and $\vec{a}=-9\vec{j}+18\vec{k}$ are coplanar.

Set 3

Group-'A'

(Q.N.1.a)

= solution,

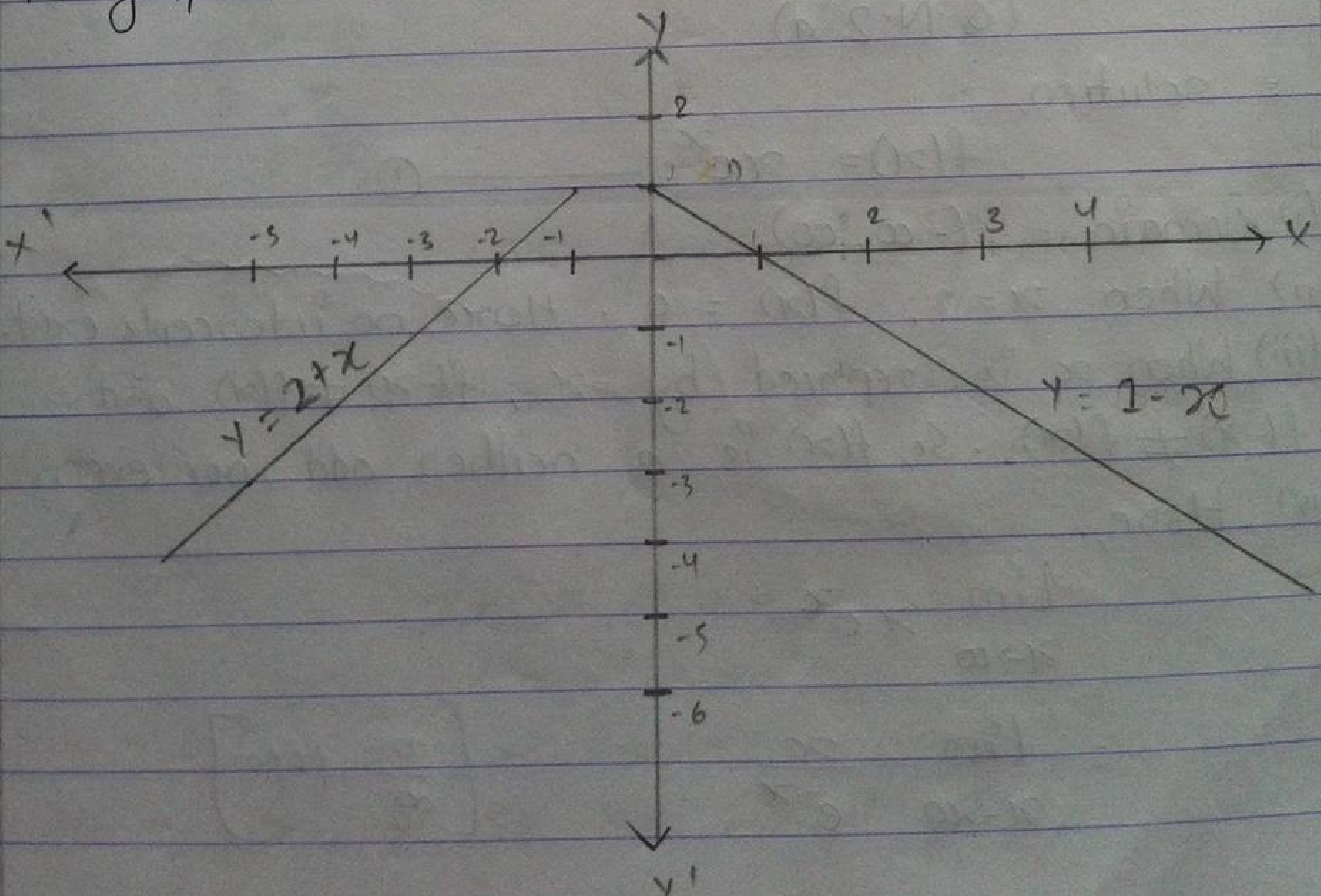
here,

$$f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 1-x & \text{if } x \geq 0 \end{cases}$$

The domain is a set of all real numbers, i.e. $(-\infty, \infty)$. Few of them are :-

x	0	1	2	3	4	5	-1	-2	-3	-4	-5
$f(x)$	1	0	-1	-2	-3	-4	1	0	-1	-2	-3

The graph based on above characteristics is :-



(Q.N.1.b)

= solution,

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \quad \left[\because \frac{0}{0} \text{ form}\right]$$

$$= \lim_{t\to 0} \frac{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)}{t^2 \times (\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{t^2+\cancel{9}-\cancel{9}}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{1}{(\sqrt{t^2+9}+3)}$$

$$= \frac{1}{3+3}$$

$$= \left(\frac{1}{6}\right) \quad \underline{\underline{\text{Ans}}}$$

(Q.N.2.a)

= solution,

$$f(x) = xe^{x} \quad \text{——} \quad (1)$$

(i) Domain $= (-\infty, \infty)$

(ii) When $x=0$; $f(x)=0$. Hence, no intercepts exist.

(iii) When x is replaced by $-x$; $f(-x) \neq -f(x)$ and $f(-x) \neq f(x)$. So, $f(x)$ is a neither odd, nor even.

(iv) Here,

$$\lim_{x\to\infty} xe^{x}$$

$$= \lim_{x\to\infty} \frac{x}{e^{-x}} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{1}{-e^{-x}}$$

$$= \frac{1}{-e^{-\infty}}$$

$$= 0$$

$\therefore$ $Y = 0$ is a horizontal asymptote.

We don't have vertical asymptote here.

(v) And,

$$f'(x) = (xe^{x} + e^{x})$$

$$\text{or, } f'(x) = e^{x}(x+1)$$

When $x = -1$; $f'(x) = 0$. So, $x = -1$ is the critical point. Here,

Table ①: Rise and Fall :-

	$(-\infty, -1)$	$(-1, \infty)$
$(x+1)$	$-$	$+$
e^{x}	$+$	$+$
$f'(x)$	$-$	$+$
	Fall $\swarrow$	$\uparrow$ Rise

$\therefore$ $f(x)$ is increasing in $(-1, \infty)$ and decreasing in $(-\infty, -1)$.

(v) $f''(x) = xe^{x} + e^{x} + e^{x}$

$$= 2e^{x} + xe^{x}$$

$$\therefore f''(x) = e^{x}(x+2)$$

Here, $x = -2$ is the critical point.

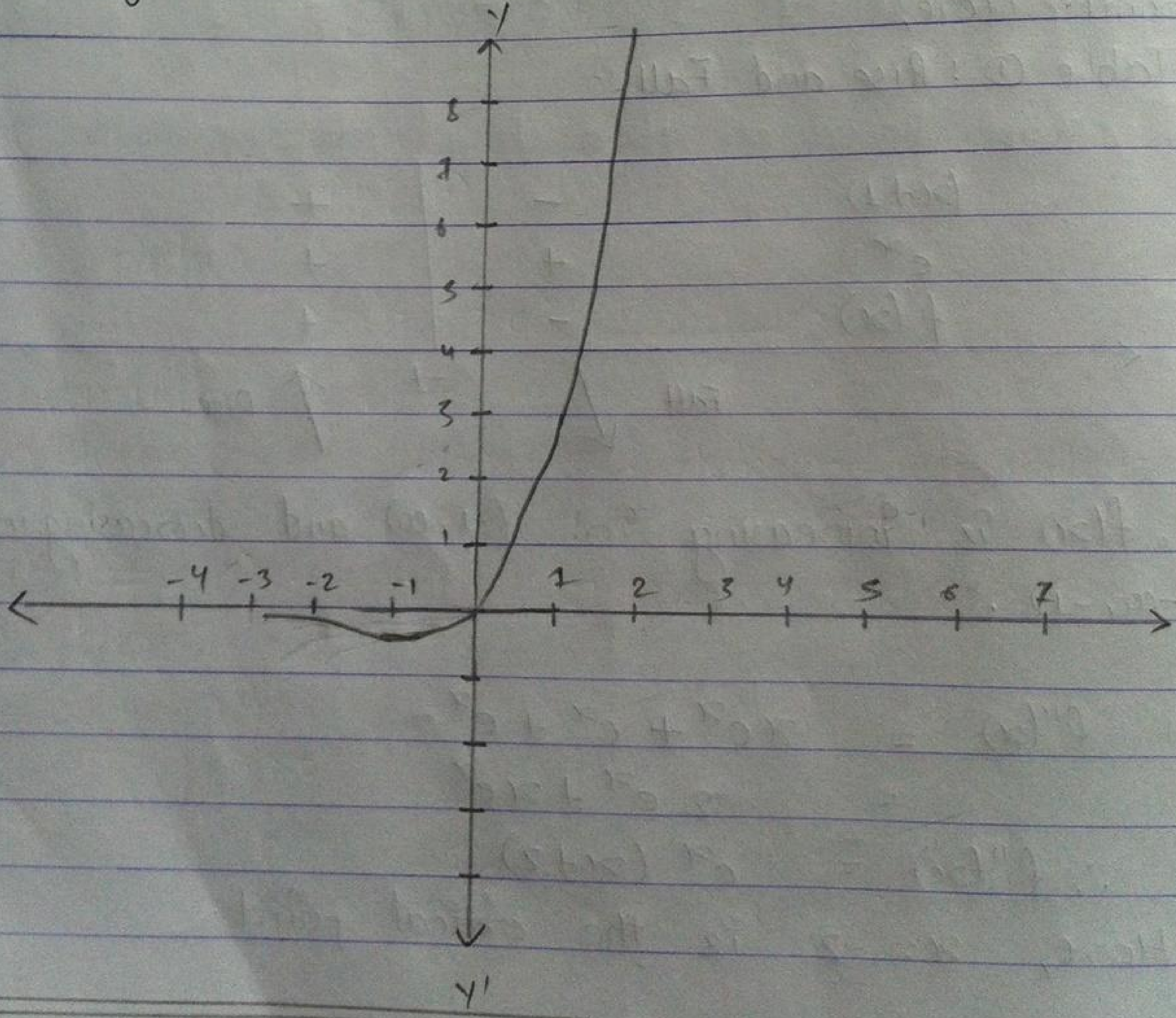
Table: (ii) :- Concavity :-

e^x	+	+
$(x+2)$	-	+
$f''(x)$	-	+
	concave downward	concave upwards

(the dividing point is -2)

$\therefore f(x)$ is concave downward in $(-\infty,-2)$ and concave upward in $(-2,\infty)$.

The graph based on above characteristics is :-



Q.N.2.b

= solution,

Consider the given equations as :-

$$Y = e^x \quad \text{—— (i)}$$

$$Y = x \quad \text{—— (ii)}$$

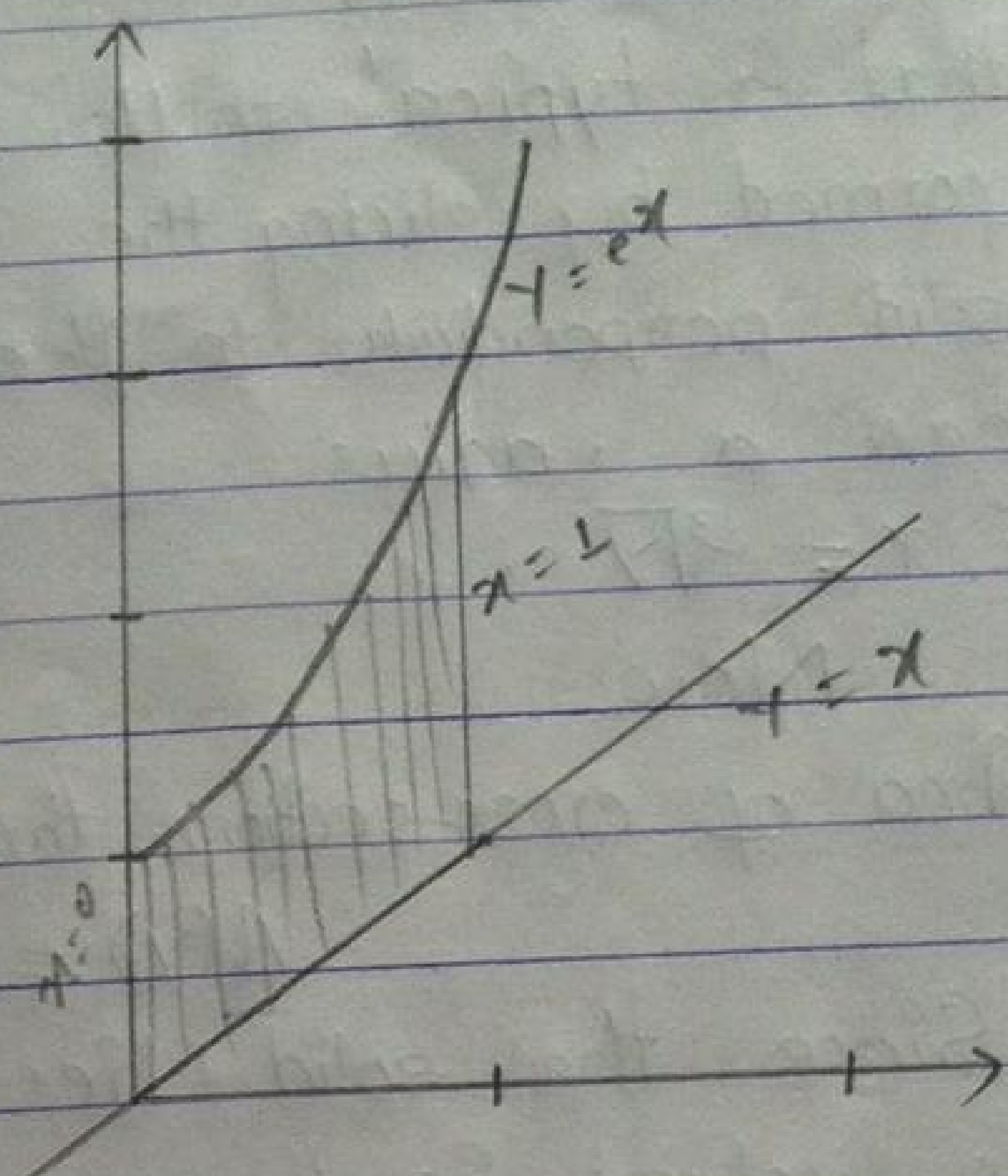
$$x = 0 \quad \text{—— (iii)}$$

$$x = 1 \quad \text{—— (iv)}$$

solving eq^n ~~(i) and (ii);~~

$$\cancel{x = e^x}$$

$$\text{or, } \cancel{x = \ln x}$$



The curves are shown in graph alongside.

From the figure,

Area of region enclosed ⇒

$$A = \int_0^1 (e^x - x) \cdot dx$$

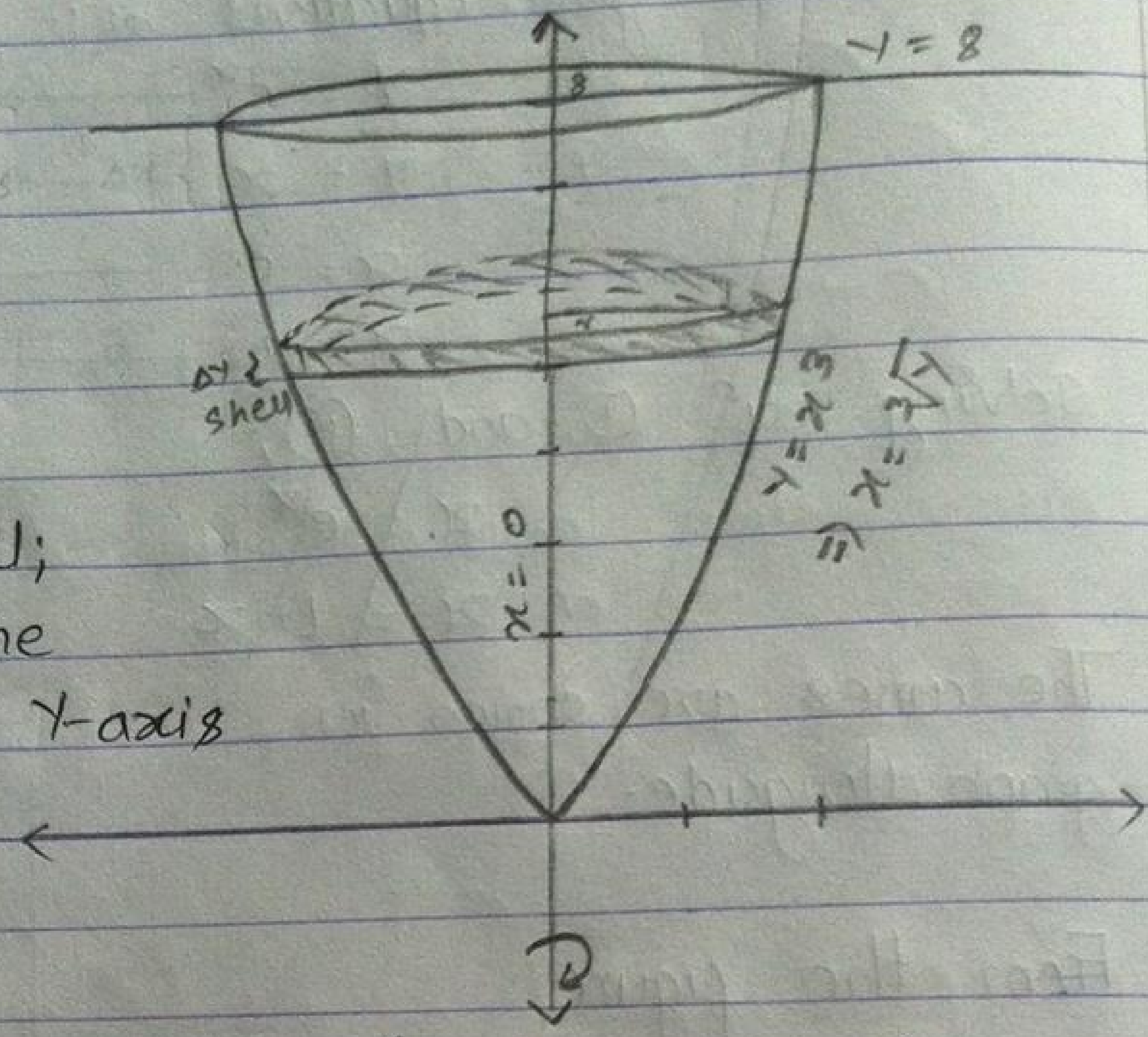
$$= \left[e^x - \frac{x^2}{2} \right]_0^1$$

$$= \left[e - 1 - \frac{1}{2} + 0 \right]$$

$$= 1.5 \text{ sq. units} \quad \underline{\underline{\text{Ans}}}$$

(Q.N. 3.a)

= solution,



From the sketch alongside, we understand that a typical shell; formed by slicing the solid perpendicular to Y-axis has a radius

$\therefore r = \sqrt[3]{y}$.

Now,

Area of cross-section through y is :-

$$A(y) = (\pi y^{2/3})$$

Since the solid lies between $y = 0$ to $y = 8$; it's volume is :-

$$V = \int_0^8 A(y) \cdot dy$$

$$= \int_0^8 \pi y^{2/3} \cdot dy$$

$$= \pi \left[\frac{3y^{5/3}}{5} \right]_0^8$$

$$= \pi \times \frac{3}{5} \times 8^{5/3}$$

$$= \pi \times 3/5 \times 2^5$$

$= \frac{96\pi}{5}$ cubic units Ans

(Q.N.3.b)

The degree of a differential equation is the highest power of the highest order differential coefficient that the equation contains after it has been rationalised.

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^3 + 4y = 4e^x \cos x$$ $\left[\begin{array}{l} \therefore - 3^{rd} \text{ ODE} \\ \therefore - 1^{st} \text{ degree ODE} \end{array}\right.$

$$\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^3 + 5y = \sec x$$; 1^{st} degree ODE

$$\left(\frac{dy}{dx}\right)^2 = x\left(\frac{dy}{dx}\right) + k$$; degree = 2

= solution,

here,

$$\frac{dy}{dx} = \frac{x^2}{y^2} \quad \text{——— (I)}$$

$$y(0) = 2 \quad \text{——— (II)}$$

Solving eq^n (I);

$$y^2 . dy = x^2 . dx$$

Integrating both sides;

$$\int y^2 . dy = \int x^2 . dx$$

unit vector along $\vec{x} = \frac{\vec{x}}{|\vec{x}|}$

or, $\frac{y^3}{3} = \frac{x^3}{3} + c$

where, c is an arbitrary constant.

$\therefore \quad y^3 = x^3 + 3c$

$\Rightarrow \quad y = \sqrt[3]{x^3 + 3c}$

We have;

When $x = 0;\ y = 2$

so;

$2 = \sqrt[3]{0 + 3c}$

or, $8 = 3c$

$\Rightarrow c = 8/3$

$\therefore \quad y = \sqrt[3]{x^3 + 3 \times 8/3}$

or, $y^3 = x^3 + 8$

$\Rightarrow \quad x^3 - y^3 + 8 = 0$; which is the required solution. Ans

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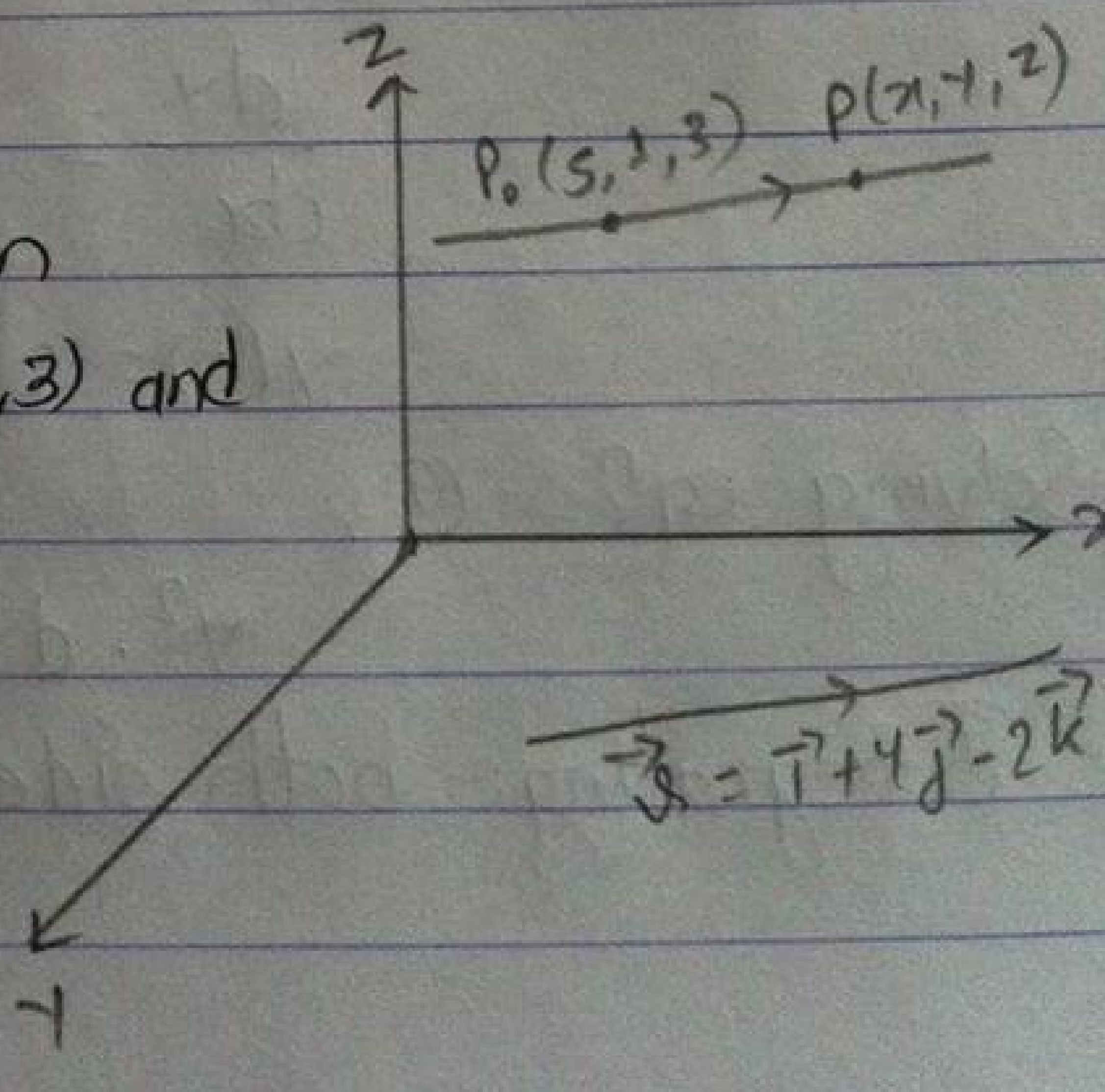
(Q.N. 4.a)

= solution,

let $P(x, y, z)$ be any point in the line passing through $P_0(5, 1, 3)$ and parallel to the vector $\vec{v} = (\vec{i} + 4\vec{j} - 2\vec{k})$



Now;

$\vec{P_0P} = t\vec{v}$ ——— (1)

or, $\overrightarrow{OP} - \overrightarrow{OP_0} = t\vec{v}$

or, $\overrightarrow{OP} = t(1,4,-2) + (5,1,3)$

or, $\overrightarrow{OP} = (5+t)\vec{i} + (1+4t)\vec{j} + (3-2t)\vec{k}$

This is the required equation in vector form.

Also;

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$$

So;

$$x = (5+t), \quad y = (1+4t), \quad z = (3-2t)$$

; which is the required equation in parametric form.

Next,

Chosing the parameter value as $t=1$; we get; $x=6$, $y=5$ and $z=1$, so, $(6,5,1)$ is a point on the line.

Similarly, choosing the parameter value as $t=-1$; we get $x=4$, $y=-3$ and $z=5$. So, $(4,-3,5)$ is another point on the line.

Group-'B'

(Q.N.6)

= solution,

$$f(x) = \frac{\sqrt{2x^2+1}}{(3x-5)}$$

For horizontal asymptote;

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{x(3-5/x)}$$

$$= \lim_{x\to\infty} \sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}} \cdot \frac{1}{(3-5/x)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{(3-5/x)}$$

$$= \left(\frac{\sqrt{2}}{3}\right)$$

$\therefore$ $Y = \frac{-\sqrt{2}}{3}$ is the horizontal asymptote.

And;
The denominator of $f(x)$ becomes zero when $x = (5/3)$. So, $x = 5/3$ is the vertical asymptote.

(Q. N. 7)

= solution,

$$f(0) = -3$$
$$f'(x) \leq 5$$

maximum$(f(2)) = ?$

here,
Since $f(x)$ is differentiable at all points. So, it is continuous everywhere. Now, we can use the mean value theorem.

Suppose, there exists a point 'c' between 0 and 2 where:

$$f'(c) = \frac{f(b)-f(a)}{(b-a)}$$

or, $f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$

or, $2 \cdot f'(c) = f(2) + 3$

or, $f(2) = 2 \times f'(c) - 3$

The maximum value of $f'(x)$ is 5. Since $c \in x$; $f'(c)$ can have a max. value of 5. So,

$$f(2) = 2 \times 5 - 3 = 7$$

Ans

$\therefore f(2)$ can have a max. value of 7.

(Q.N. 8)

= solution,

here,

$$y = x^2$$

$$\therefore y' = 2x$$

$$\therefore y'' = 2$$

Now,

The general formula for curvature is :-

$$k = \frac{|y''|}{(1 + [y']^2)^{3/2}}$$

$$= \frac{2}{[1 + (2x)^2]^{3/2}}$$

$$\therefore k = \frac{2}{(1 + 4x^2)^{3/2}}$$

At $(0,0)$;

$$k = \frac{2}{(1+0)^{3/2}}$$

$\therefore k = 2$

At $(1,1)$;

$$k = \frac{2}{(1+4)^{3/2}}$$

$$= \frac{2}{\sqrt{5^3}}$$

$$= \left(\frac{2}{5\sqrt{5}}\right)$$

At $(2,4)$;

$$k = \frac{2}{(1+4\times4)^{3/2}}$$

$$= \frac{2}{(17)^{3/2}}$$

$$= \left(\frac{2}{17\sqrt{17}}\right) \text{ \underline{Ans}}$$

(Q.N. 9)

= solution,

$$I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)} \cdot dx$$

here,

$$I = \lim_{a\to-\infty} \int_{a}^{0} \frac{1 \cdot dx}{(1+x^2)} + \lim_{b\to\infty} \int_{0}^{b} \frac{1 \cdot dx}{(1+x^2)}$$

$$= \lim_{a \to -\infty^+} \left[\operatorname{Tan}^{-1} x \right]_a^0 + \lim_{b \to \infty^-} \left[\operatorname{Tan}^{-1} x \right]_0^b$$

$$= \lim_{a \to -\infty^+} \left[\operatorname{Tan}^{-1} 0 - \operatorname{Tan}^{-1} a \right] + \lim_{b \to +\infty^-} \left[\operatorname{Tan}^{-1} b - \operatorname{Tan}^{-1} 0 \right]$$

$$= 0 + \frac{\pi}{2} + \frac{\pi}{2} - 0$$

$$= \pi$$ Ans

(8·N·10)

= solution,

here,

$$y^2 = x$$

$$\Rightarrow x = y^2$$

so;

$$\frac{dx}{dy} = 2y$$

The limits of integration are: $y = 0$ to $y = 1$. Now,

$$\text{length of Arc} = \int_0^1 \sqrt{1 + (2y)^2} \cdot dy$$

$$= \int_0^1 \sqrt{1 + (2y)^2} \cdot dy$$

$$= \frac{1}{2} \left[\frac{2y}{2} \sqrt{1 + 4y^2} + \frac{1}{2} \ln \left| 2y + \sqrt{1 + 4y^2} \right| \right]_0^1$$

$$= \frac{1}{2} \left[\sqrt{5} + \frac{1}{2} \ln |2 + \sqrt{5}| - 0 - 0 \right]$$

$$= \frac{1}{2} \left[\sqrt{5} + \frac{1}{2} \ln |2 + \sqrt{5}| \right]$$ Ans

(Q.N. 11)

The differential equation of the form :-
$P(x)y'' + Q(x)y' + R(x)y = G(x)$; where $G(x) \neq 0$
is called ~~second~~ non-homogeneous second order linear differential equation.

It's solution is :-

$$\boxed{Y = \underbrace{Y_c}_{\text{Complementary function}} + \underbrace{Y_p}_{\text{particular solution of } G(x)}}$$

= solution,

here,

$$y'' + y' - 2y = x^2 \quad \text{——— (1) (Non-homogeneous)}$$

Now,

For Complementary function, the auxiliary eq? is :-

$$m^2 + m - 2 = 0$$

or, $(m-1)(m+2) = 0$

$\Rightarrow m = -2, +1$

so,

Complementary Function $(Y_c) = (c_1 e^{-2x} + c_2 e^{x})$

For Particular Solution,

let $Y_p = (ax^2 + bx + c)$

$\therefore y' = (2ax + b)$

$y'' = 2a$

so, eq? (1) gives;

$$2a + 2ax + b - 2(ax^2 + bx + c) = x^2$$

or, $(-2a)x^2 + (2a-2b)x + (2a+b-2c) = x^2 + 0x + 0$

Equating the corresponding coefficients;

$-2a = 1$ $\Rightarrow a = (-\frac{1}{2})$, $2a - 2b = 0$ $\Rightarrow a = b$ $\therefore b = -\frac{1}{2}$, $2a + b - 2c = 0$ $\Rightarrow 2a + b = 2c$ $\Rightarrow -\cancel{2} \times \frac{1}{\cancel{2}} - \frac{1}{2} = 2c$

$\Rightarrow -\frac{3}{4} = c$

$\therefore c = (-\frac{3}{4})$

$\therefore Y_p = \left(\frac{-x^2}{2} - \frac{x}{2} - \frac{3}{4}\right)$

so,

The general solution is :-

$$Y = Y_c + Y_p$$
$$= \left(c_1 e^{-2x} + c_2 e^{x} - \frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}\right) \quad \underline{\text{Ans}}$$

(Q.N.12)

= solution,

here,

$$f(x) = e^x$$
$$f^n(x) = e^x$$

At $x = 0$; $f^n(0) = e^0 = 1$ (for all n)

So,

The Maclaurian series is :-

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence, we let $a_n = \frac{x^n}{n!}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| = \frac{|x|}{|n+1|} \to 0 < 1$$

So, by the ratio test, the series converges for all x and the radius of convergence is $R = \infty$

(Q.N. 13)

= solution,

Consider the given planes as :-

$$10x + 2y - 2z = 5 \text{ ——— (i)}$$
$$5x + y - z = 1 \text{ ——— (ii)}$$

Now, Consider plane (ii). When $x = 0$ and $y = 0$;

$$z = -1$$

$\therefore$ $(0, 0, -1)$ is a point on plane (ii)

So, the distance between planes (i) and (ii) can be calculated by finding the distance betn point $(0, 0, -1)$ and plane (i) ; i.e.

$$\text{Distance between planes} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|10 \times 0 + 2 \times 0 + (-2) \times (-1) - 5|}{\sqrt{100 + 4 + 4}}$$

$$= \frac{|-3|}{\sqrt{108}} = \frac{3}{6\sqrt{3}} = \frac{1}{2\sqrt{3}} \text{ units}$$

(Q.N. 14)

= solution,

$$f(x,y) = y\sin(xy) \quad , R = [1,2] \times [0,\pi]$$

so;

$$\iint_R f(x,y) \cdot dA$$

$$= \int_0^{\pi}\int_1^{2} y\sin(xy) \cdot d\cancel{y} \cdot d\cancel{x}y$$

$$= \int_0^{\pi} \Big[-\cancel{y}\cos(xy)\Big]_1^2 \cdot dy$$

$$= \int_0^{\pi} \Big[-\cos 2y + \cos y\Big] \cdot dy$$

$$= \Big[\frac{-\sin 2y}{2} + \sin y\Big]_0^{\pi}$$

$$= \Big[-\frac{\sin 2\pi}{2} + 0 + \sin\pi - 0\Big]$$

$$= [0]$$

$$= 0$$

Ans

(Q.N. 15)

= solution,

here,

$$\vec{a} = (\vec{i} + 4\vec{j} - 7\vec{k})$$

$\vec{b} = (2\vec{i} - \vec{j} + 4\vec{k})$
$\vec{c} = (0\vec{i} - 9\vec{j} + 18\vec{k})$

Now, we know,
the scalar triple product: $\vec{a}.(\vec{b} \times \vec{c})$ = volume of a parallelepiped

so,

$$\vec{a}.(\vec{b}\times\vec{c}) = (1, 4, -7).\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= (\vec{i} + 4\vec{j} - 7\vec{k}).(18\vec{i} - 36\vec{j} + (-18)\vec{k})$$

$$= 18 - 144 + 126$$

$$= 144 - 144$$

$$= 0$$

$\therefore$ The volume of parallelepiped formed by a, b, c is 0. This means: $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

proved