Set2

Group A **(3 x 10 = 30)**

Attempt any three questions.

1. (a) A function is defined by $f(x) = \begin{cases} 1+x, & if \quad x \le 1 \\ x^2 & if \quad x > 1 \end{cases}$ [5]

Evaluate f(3); f(1) and f(0) and sketch the graph.

(b) Prove that the limit $x \xrightarrow{\lim} 0 \ |x|$ exists then find its value [5]

2. (a) Sketch the curve : $f(x) = \dfrac{x^2}{\sqrt{x+1}}$. [5]

(b) Estimate the area between the curve $y = x^2$ and the lines x = 0 and x = 1, using rectangle method. [5]

3. (a) Find the volume of the solid obtained by rotating about the *x*-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. [4]

(b) Define order of a differential equation.

Solve: $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}$, $u(0) = -5$. [6]

4. (a) Find the the unit normal and binormal vectors for the circular helix $(\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$. [6]

(b) Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius. [4]

Group B **(10 x 5 = 50)**

Attempt any ten questions.

5. Determine whether each of the following functions is even, odd, or neither even nor odd.
(a) $y = x^5 + x$ (b) $y = 1 - x^4$ (c) $y = 2x - x^2$

6. Find an equation of the tangent line to the parabola $y = 2x - x^2$ at the point P(1, 1).

7. Where is the function $f(x) = |x|$ differentiable?

8. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $f(x) = x^3 - 2x - 5$.

9. State Net change theorem. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).
(a) Find the displacement of the particle during the time period $1 \le t \le 4$.
(b) Find the distance traveled during this time period.

10. Find the length of the arc of the semi-cubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8) .

11. Define homogeneous second-order linear differential equation.
Solve: $y'' + y = 0, \ x > 0$, $y'(0) = 3$ *and* $y(0) = 2$.

12. Find a vector perpendicular to the plane that passes through the points P(1, 4, 6), Q(-2, 5, -1) and R(1, -1, 1).

13. Determine whether the series $\sum_{n=1}^{\infty} \dfrac{5}{2n^2 + 4n + 3}$ converges or diverges.

14. Evaluate the iterated integrals. $\int_0^3 \int_1^2 x^2 y\,dy\,dx$ and $\int_1^2 \int_0^3 x^2 y\,dx\,dy$.

15. Find the length of the arc of the circular helix with vector equation $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$ from the point (1, 0, 0) to the point (1, 0, 2Π).

Set-2 Gr. A

(Q.N.1.a)

= solution,

here,
$$f(x) = \begin{cases} 1+x & \text{if } x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

so,

(i) $f(3) = (3)^2$ $[\because 3 > 1]$

$\therefore \boxed{f(3) = 9}$

(ii) $f(1) = 1 + (1)$ $[\because 1 = 1]$

$\therefore \boxed{f(1) = 2}$

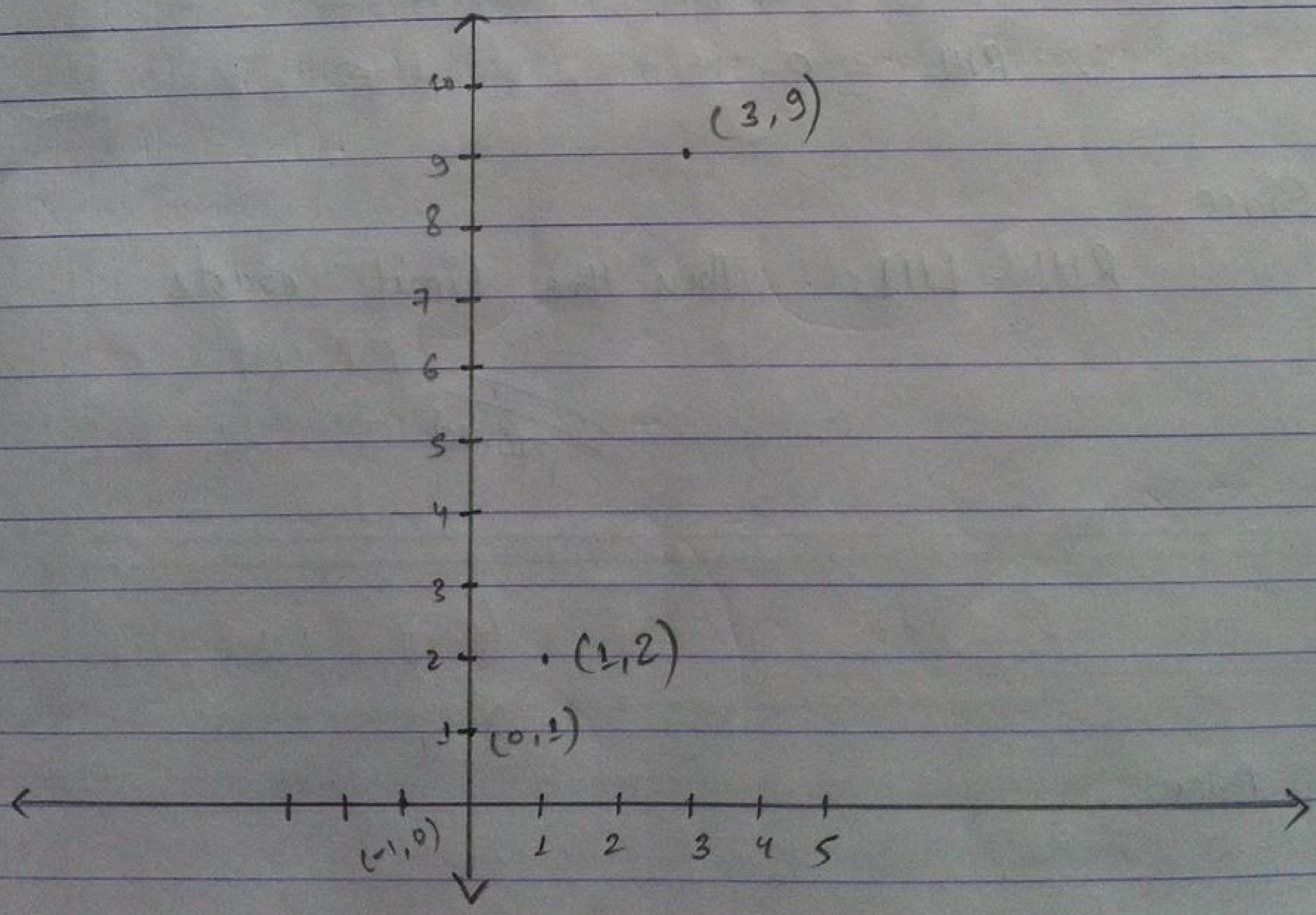
(iii) $f(0) = 1 + 0$ $[\because 0 < 1]$

$\therefore \boxed{f(0) = 1}$

Now,

Graph is :-

scale = 0.6 cm = 1 unit



(Q.N.1b)

= solution

here,

$$\lim_{x\to 0} |x|$$

we know,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

So,

left Hand Limit $\Rightarrow \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = 0$

$\therefore$ LHL = 0

Right Hand Limit $\Rightarrow \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$

$\therefore$ RHL = 0

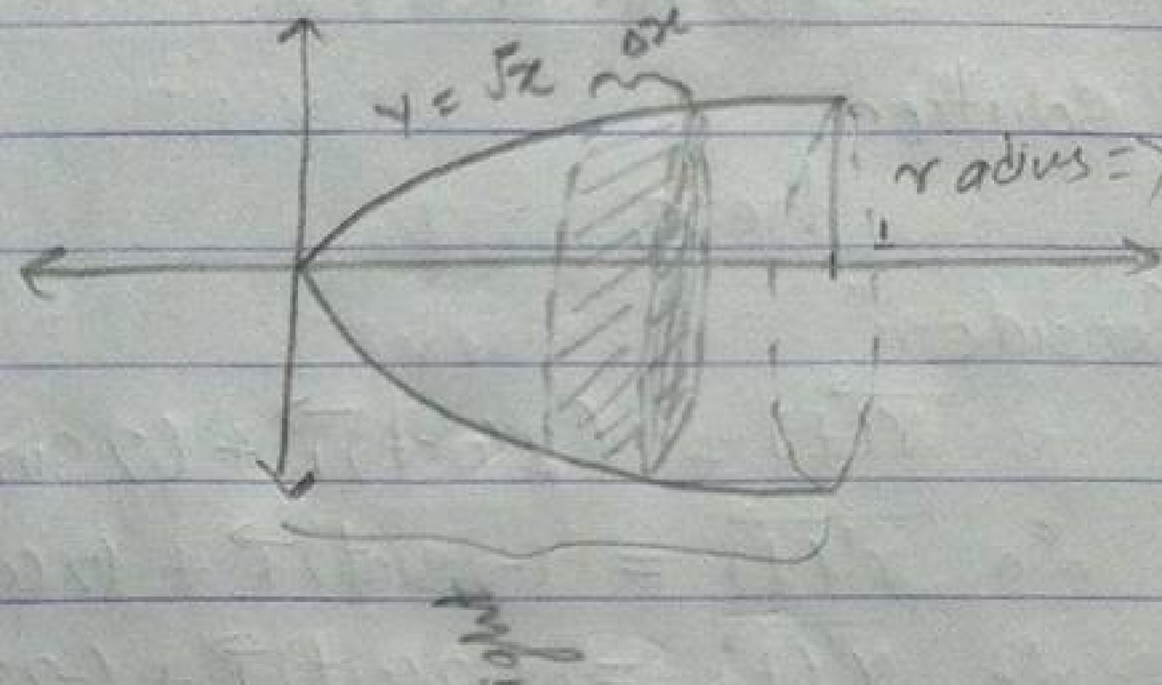
Since,

RHL = LHL . Thus the limit exists

proved

(Q. N. 3. a)

= solution,



The general sketch of graph shows that the radius of a shell is $\sqrt{x}$; so the area of it's base is πx.
The height of the shell is Δx.

So,
The volume of the approximate cylinder is:-

$$V = A(x) \times \Delta x$$
$$= \pi x \times \Delta x$$

The solid lies between 0 and 1. So, it's volume is:-

$$V = \int_0^1 \pi x \cdot dx = \pi\left[\frac{x^2}{2}\right]_0^1 = \pi\left[\frac{1}{2}\right]$$

$$\therefore V = \left(\frac{\pi}{2}\right) \text{ cubic units}$$

Ans

set 2's
(Remaining at the back)

Set 2

continued.. {$\leftarrow$ 1, 3.4 ahead}

(Q.N. 3b)

The order of a differential equation is the order of the highest derivative that occurs in the equation. eg:- in $y'' + 5y' + 6y = 0$, the order is two. It is a second order differential equation.

= solution,

$$\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}, \quad u(0) = -5$$

or, $du \times (2u) = (2t + \sec^2 t) \cdot dt$

Integrating both sides;

$$\int 2u \cdot du = \int (2t + \sec^2 t) \cdot dt$$

or, $2 \times \frac{u^2}{2} = 2 \times \frac{t^2}{2} + \tan t + k$

or, $u^2 = (t^2 + \tan t + k)$ —— (i)

where, k is an arbitrary constant.

We have,

$u(0) = -5$ —— (ii)

So, eq^n (i) and (ii) yield;

$$(-5)^2 = 0^2 + \tan 0 + k$$

or, $25 = 0 + k$

$\Rightarrow k = 25$

So, eq^n (i) can be written as:

$$u^2 = t^2 + \tan^2 t + 25$$

$$\Rightarrow u = \sqrt{t^2 + \tan t + 25}$$; which is the required general solution.

(Q.N. 2a)

= solution:

$$f(x) = \frac{x^2}{\sqrt{x+1}}$$

(i) The domain of the above function is : $(-1, \infty)$.

(ii) When $x = 0$; $f(x) = 0$; so no intercept exists.

(iii) $f(-x) \neq -f(x)$ nor $f(-x) = f(x)$. So, $f(x)$ is neither odd, nor even function.

(iv) here,

$$\lim_{x \to \infty} \frac{x^2}{\sqrt{x+1}} \qquad \left[\because \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{2 \times 2x \times \sqrt{x+1}}{1}$$

$$= \infty$$

$\therefore$ Horizontal asymptote does not exist.

When $x = (-1)$, the denominator becomes zero. So, $x = -1$ is the vertical asymptote.

(v)

$$f'(x) = \frac{(\sqrt{x+1})\, 2x - x^2 \cdot \frac{1}{2} \cdot (x+1)^{-1/2}}{(x+1)}$$

$$= \frac{2(x+1) \cdot 2x - x^2}{2(x+1)\,(x+1)^{1/2}}$$

$$= \frac{4x^2+4x-x^2}{2(x+1)^{3/2}}$$

$$= \frac{(3x^2+4x)}{2(x+1)^{3/2}}$$

$$\therefore f'(x) = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

When $x=0$; $f'(x)=0$. (when $x=-4/3$, $f'(x)=0$; but $-4/3$ does not lie in the domain). So, the only critical number is 0. So, The function is increasing in the interval $(0,\infty)$ and is decreasing in the interval $(-1,0)$

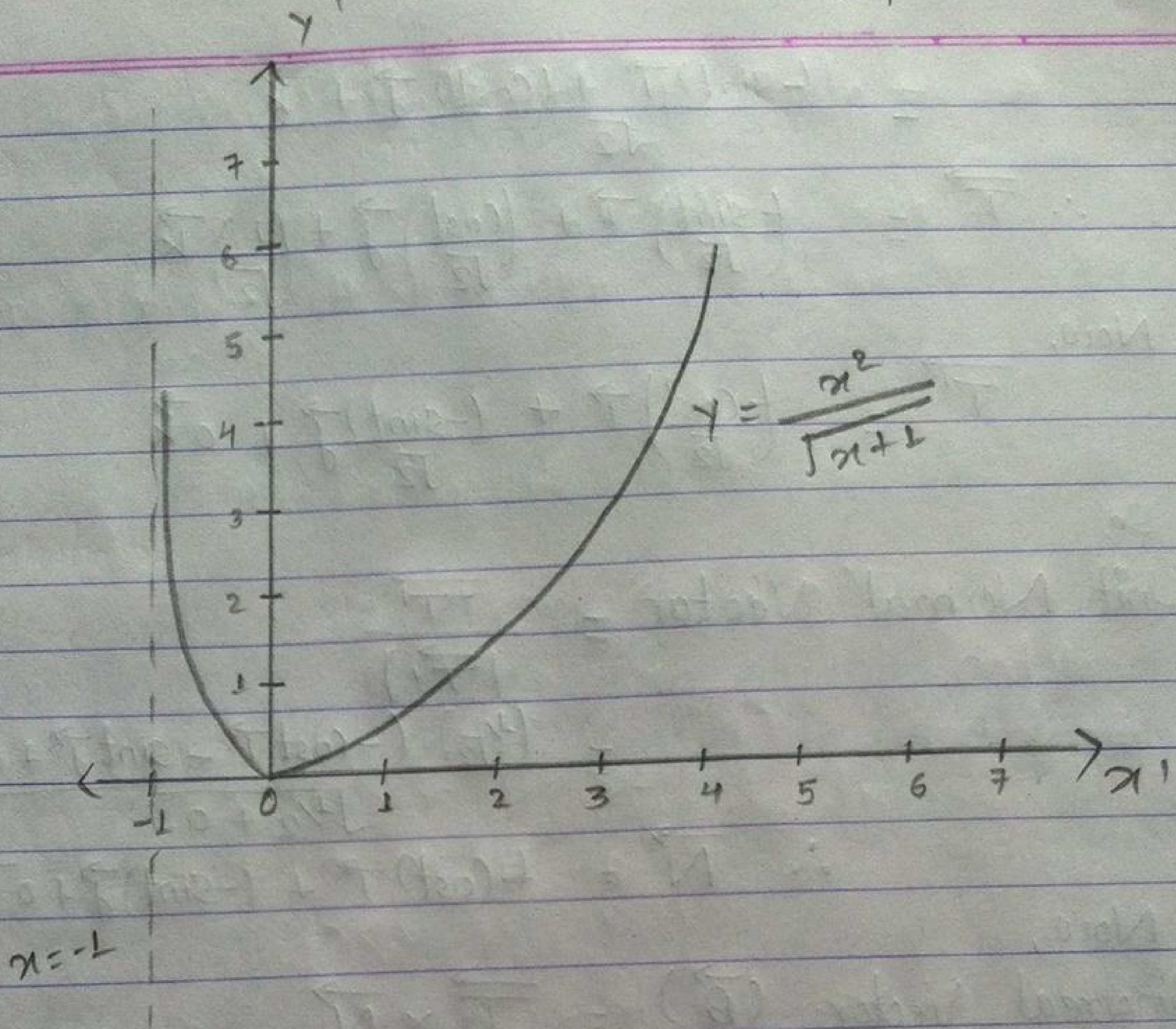
(vi) $$f''(x) = \frac{2\times(x+1)^{3/2}\cdot(6x+4)-(3x^2+4x)\cdot 3\times(x+1)^{1/2}}{4(x+1)^3}$$

$$= \frac{(x+1)^{1/2}\left[(12x+8)(x+1)-9x^2-12x\right]}{4(x+1)^3}$$

$$f''(x) = \frac{[3x^2+8x+8]}{4(x+1)^{5/2}}$$

Note that the denominator is always positive. The numerator is the quadradic $3x^2+8x+8$; since it's discriminant is negative (i.e. -32); and the coefficient of x^2 is positive. Thus; $f''(x)$ is positive in the domain of x. Therefore the curve is concave upward in the domain.

confuse



(Q.N. 4.a)

= solution,

Consider $\overrightarrow{r(t)} = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$

$\therefore \overrightarrow{r'(t)} = (-\sin t)\vec{i} + (\cos t)\vec{j} + \vec{k}$

so,

$$|\overrightarrow{r'(t)}| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{2}$$

so,

Unit Tangent Vector $(\vec{T}) = \frac{\overrightarrow{r'(t)}}{|\overrightarrow{r'(t)}|}$

$$= \frac{(-\sin t)\vec{i} + (\cos t)\vec{j} + \vec{k}}{\sqrt{2}}$$

$$\therefore \vec{T} = \left(\frac{-\sin t}{\sqrt{2}}\right)\vec{i} + \left(\frac{\cos t}{\sqrt{2}}\right)\vec{j} + \left(\frac{1}{\sqrt{2}}\right)\vec{k}$$

Now,

$$\vec{T'} = \left(\frac{-\cos t}{\sqrt{2}}\right)\vec{i} + \frac{(-\sin t)}{\sqrt{2}}\vec{j} + 0\vec{k}$$

So,

$$\text{Unit Normal Vector} = \frac{\vec{T'}}{|\vec{T'}|}$$

$$= \frac{(1/\sqrt{2})(-\cos t\,\vec{i} - \sin t\,\vec{j} + 0\vec{k})}{\sqrt{1/2 + 0}}$$

$$\therefore \vec{N} = (-\cos t)\vec{i} + (-\sin t)\vec{j} + 0\vec{k}$$

Now,

$$\text{Binormal Vector } (\vec{B}) = \vec{T} \times \vec{N}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\cos t}{\sqrt{2}} & -\frac{\sin t}{\sqrt{2}} & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= (0\vec{i} + 0\vec{j} + 0\vec{k})$$

Ans

(Q.N.4b)

= solution,

we have:-

$$x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0 \quad \text{——— (1)}$$

Solving eqⁿ ① ;

$$x^2+y^2+z^2+4x-6y+2z+6=0$$
or, $x^2+4x+4-4+y^2-6y+9-9+z^2+2z+1-1+6=0$
or, $(x+2)^2+(y-3)^2+(z+1)^2-4-9+6-1=0$
or, $(x+2)^2+(y-3)^2+(z+1)^2=8$
or, $(x+2)^2+(y-3)^2+(z+1)^2=(2\sqrt{2})^2$

This equation is of the form $(x-h)^2+(y-k)^2+(z-l)^2=r^2$; So, the given equation is the equation of sphere; whose center is $(-2,3,-1)$ and radius $2\sqrt{2}$ units.

'Group- B'

(Q.N. 6)

= solution,
Consider the given equation as:-
$$y=2x-x^2 \quad \text{———} \quad (1)$$
The slope at any point on the parabola is:

$$m=\frac{dy}{dx}=\frac{d(2x-x^2)}{dx}$$
$$\therefore m=(2-2x)$$
At (1,1), the slope is :-
$$m=2-2\times1=0$$

So, the equation of tangent at $(1,1)$ is :-

$$y - 1 = m(x-1)$$

or, $y = 1$; which is the required equation.

Ans

(Q.N.7)

= solution,

we know,

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

(i) Consider $x > 0$. Let h be a very small quantity. So, $|x+h| > 0$ and $|x+h| = (x+h)$. Now,

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h} \qquad [\because x > 0]$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= 1$$

So, $f(x) = |x|$ is differentiable at all points greater than 0.

(ii) When $x = 0$; let h be a very small quantity. so,

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{|h| - 0}{h} \qquad [\because \text{ if } x = 0]$$

~~whe~~ and,

$$\left.\begin{array}{l} \lim_{h \to 0^+} \frac{|h|}{h} = 1 \\ \text{while } \lim_{h \to 0^-} \frac{|h|}{h} = -1 \end{array}\right\} \text{limit does not exist}$$

Since LHL ≠ RHL, $f(x) = |x|$ is not differentiable at $x = 0$

(iii) Consider $x < 0$. If 'h' be a very small quantity, $|x+h| < 0$ and $|x+h| = -(x+h)$. Here,

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-x - h + x}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h}$$

$$= -1 \quad \text{(Limit exists)}$$

$\therefore f(x) = |x|$ is differentiable for $x < 0$.

Hence $f(x) = |x|$ is differentiable for all real values of x except 0.

(Q. 8)

= solution,

$f(x) = x^3 - 2x - 5$

$x_0 = 2$

$x_3 = ?$

$f'(x) = (3x^2 - 2)$

Here, The general formula of Newton's method is:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Accordingly,

First Approximation $\Rightarrow$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{(8-4-5)}{10}$$

$$\therefore x_1 = 2.1$$

Next,

Second Approximation $\Rightarrow$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.1 - \frac{0.061}{11.23}$$

$$= 2.094568121$$

Now,

Third approximation $\Rightarrow$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.094568121 - \frac{1.857220104 \times 10^{-4}}{11.16164684}$$

$$\therefore x_3 = 2.094408244$$

Ans

(Q.N. 9)

Net Change Theorem:-

The integral of a rate of change is the net change :-

$$\int_a^b f'(x) \cdot dx = F(b) - F(a)$$

eg:-

$$\int_{t_1}^{t_2} V'(t) \cdot dt = V(t_2) - V(t_1)$$

solution,

$$v(t) = (t^2 - t - 6)$$

displacement during time period: $1 \leq t \leq 4 = ?$

we know,

By Net Change Theorem;

$$s(4) - s(1) = \int_1^4 v(t) \cdot dt$$

$$= \int_1^4 (t^2 - t - 6) \cdot dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^4$$

$$= -4.5 \text{ m}$$

$= 4.5$ m towards left.

we know,
$v(t) = (t^2 - t - 6) = (t-3)(t+2)$. So, $v(t) \leq 0$ on the interval $[1, 3]$ and $v(t) \geq 0$ on $[3, 4]$. So, The distance travelled is :-

$$\int_1^4 |v(t)| \cdot dt = \int_1^3 [-v(t)] \cdot dt + \int_3^4 v(t) \cdot dt$$

$$= \int_1^3 (-t^2 + t + 6) \cdot dt + \int_3^4 (t^2 - t - 6) \cdot dt$$

$$= 10.167 \text{ m}$$

<u>Ans</u>

(Q.N. 10)

= solution,

$$y^2 = x^3$$

or, $$y = (x^{3/2}) \text{ ——— (i)}$$

We know,

$$\text{Arc-length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

So, Differentiating eqn (i) w.r.t. x;

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$; which is continuous for all $x \in [1,4]$.

So,

The arc-length of semi-cubal parabola is :-

$$L = \int_1^4 \sqrt{1+\left(\frac{3}{2}\right)^2 \cdot x} \cdot dx$$

$$= \int_1^4 \sqrt{1+\frac{9}{4}x} \cdot dx$$

$$= \frac{4}{9} \times \left[\left[1+\frac{9x}{4}\right]^{3/2}\right]_1^4$$

$$= \frac{4}{9} \times \left[\left[1+\frac{9}{4}\times 4\right]^{3/2} - \left[1+\frac{9}{4}\times 1\right]^{3/2}\right]$$

$$= 7.634 \text{ units}$$

Ans

(Q.N. 11)

A differential equation of the form: $y'' + P(x)y' + Q(x)y = 0$ is called ~~sec~~ homogeneous second order differential equation.

The solution of such equation can be obtained by calculating the complementary function only.

= solution,

here,

$$y'' + 0y' + y = 0 \quad \text{——} (1)$$

$$x > 0,$$

$$y'(0) = 3$$

$$y(0) = 2$$

eq^n (1) is a homogeneous second order differential equation, whose solution is a complementary function only.

For Complementary function; y_c;

Consider an auxiliary eq^n :- $m^2 + 0m + 1 = 0$

$$\Rightarrow m^2 = -1$$

$$\text{or, } m = \pm i$$

$$\Rightarrow m = (0 \pm i)$$

So,

Complementary function $(y_c) = e^0(c_1 \cos x + c_2 \sin x)$

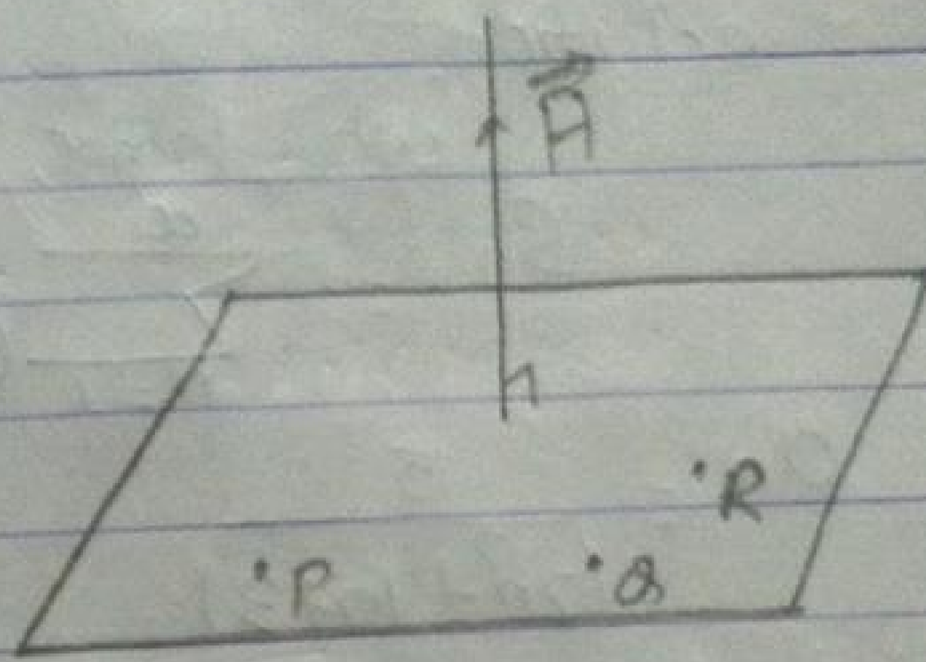
$$y = c_1 \cos x + c_2 \sin x$$

$\therefore$ This is the required solution, where c_1 and c_2 are arbitrary constants.

(Q.N. 12)

= solution,

P(1, 4, 6), Q(−2, 5, −1) and R(1, −1, 1) are the points in the plane given aside. Consider $\vec{A}$ as the vector perpendicular to plane. Then,



$$\vec{PQ} = (1+2)\vec{i} + (4+5)\vec{j} + (-1-6)\vec{k}$$
$$= 3\vec{i} + \vec{j} - 7\vec{k}$$
$$\vec{QR} = (1+2)\vec{i} + (-1-5)\vec{j} + (1+1)\vec{k}$$
$$= (3\vec{i} - 6\vec{j} + 2\vec{k})$$

We know;

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -7 \\ 3 & -6 & 2 \end{vmatrix}$$

$$= (2-42)\vec{i} - \vec{j}(6+21) + \vec{k}(-18-3)$$
$$= (-40\vec{i} - 27\vec{j} - 21\vec{k})$$; which is

a vector perpendicular to the plane of $\vec{PQ}$ & $\vec{QR}$. Since $\vec{PQ}$ & $\vec{QR}$ lie on the same plane, the required vector perpendicular to the plane is none other than : $(-40\vec{i} - 27\vec{j} - 21\vec{k})$

Ans

(Q.N13)

= solution,

we have,

$$S_n = \sum_{n=1}^{\infty} \frac{5}{(2n^2+4n+3)}$$

$$a_n = \frac{5}{(2n^2+4n+3)}$$

let b_n be the n^{th} term of corresponding auxiliary series. Then,

$$b_n = \frac{1}{(n^2+n)}$$

now, by limit comparison test;

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{5}{\cancel{n^2}(2+4/n+3/n^2)} \times \frac{\cancel{n^2}(1+1/n)}{1}$$

$$= \lim_{n\to\infty} \frac{5(1+1/n)}{(2+4/n+3/n^2)}$$

$$= \frac{5(1+0)}{(2+0+0)}$$

$$= \left(\frac{5}{2}\right) \neq 0$$

Here, $b_n > a_n$

Now,

$$\sum_{n=1}^{\infty} b_n = \frac{1}{n(n+1)} = \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

or, $\sum_{1}^{n} = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \cdots$

$\cdots \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$$= \left(1 - \frac{1}{n+1}\right)$$

Also,

$$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

$\therefore \sum_{1}^{\infty} b_n$ is convergent; and therefore the given series is also convergent.

(Q.N.14)

= solution,

here,

$$\int_0^3 \int_1^2 x^2 y . dy . dx$$

$$= \int_0^3 \left[\frac{x^2 y^2}{2}\right]_1^2 . dx$$

$$= \int_0^3 \left[\frac{4x^2}{2} - \frac{x^2}{2}\right] . dx$$

$$= \int_0^3 \frac{3x^2}{2} . dx = \frac{3}{2} \times \left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{\cancel{8}}{2}\left[\frac{27}{\cancel{8}}\right]$$

$$= \left(\frac{27}{2}\right)$$ Ans

According to Fubini's theorem;

$$\int_0^3\int_1^2 x^2y\,.dy.dx = \int_1^2\int_0^3 x^2y\,.dx.dy$$

$$\therefore \int_0^3\int_1^2 x^2y\,.dy.dx = \int_1^2\int_0^3 x^2y\,.dx.dy = \left(\frac{27}{2}\right)$$

Ans

(Q.N.15)

= solution

$$\overrightarrow{r(t)} = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$$

$$\therefore \overrightarrow{r'(t)} = (-\sin t)\vec{i} + (\cos t)\vec{j} + \vec{k}$$

$$\therefore |\overrightarrow{r'(t)}| = \sqrt{1+1} = \sqrt{2} \text{ units}$$

The arc from $(1,0,0)$ to $(1,0,2\pi)$ is described by the parameter interval $0 \le t \le 2\pi$ and so,

$$\text{length of arc} = \int_0^{2\pi} |\overrightarrow{r'(t)}|.dt = \int_0^{2\pi} \sqrt{2}\,.dt = 2\sqrt{2}\pi$$

$$\therefore l = 2\pi\sqrt{2} \text{ units}$$

Ans