Chapter 16: Anti-derivatives

Exercise 16.1

1. Solution

a. $\int \frac{dx}{4x^2 + 9}$

$\int \frac{dx}{(2x)^2 + 3^2}$

Put y = 2x

$\frac{dy}{2} = dx$

Now, $= \frac{1}{2} \int \frac{dy}{y^2 + 3^2}$

$= \frac{1}{2} \frac{1}{3} \tan^{-1} \frac{y}{3} + c$

$= \frac{1}{6} \tan^{-1} \frac{2x}{3} = c$

b. $\int \frac{xdx}{x^4 + 3}$

$= \int \frac{xdx}{(x^2)^2 \ (\sqrt{3})^2}$

Put $y = x^2$

$\frac{dy}{2} = xdx$

$= \frac{1}{2} \int \frac{dy}{y^2 + (\sqrt{3})^2}$

$= \frac{1}{2} . \frac{1}{\sqrt{3}} \tan^{-1} \frac{4}{\sqrt{3}} + c$

$= \frac{1}{2\sqrt{3}} \tan^{-1} \frac{x^2}{\sqrt{3}} + c$

c. $\int \frac{2x + 3}{4x^2 + 1} dx$

$I = I_1 + I_2$

$I_1 = \int \frac{2x}{4x^2 + 1} dx \qquad I_2 = 3 \int \frac{dx}{4x^2 + 1}$

where,

$I_1 = \int \frac{2x}{4x^2 + 1} dx$

Put $y = 4x^2 + 1$

$\frac{dy}{4} = 2x . dx$

Now, $= \frac{1}{4} \int \frac{dy}{y} = \frac{1}{4} \log y + c$

$= \frac{1}{4} \log (4x^2 + 1) + c$

$I_2 = 3\int \frac{dx}{4x^2 + 1} = 3\int \frac{dx}{(2x)^2 + 1^2}$

Put $2x = y = dx = \frac{dy}{2}$

Then, $\frac{3}{2} \int \frac{dy}{y^2 + 1^2}$

$= \frac{3}{2} \tan^{-1} y + c_2 = \frac{3}{2} \tan^{-1} 2x + c_2$

Hence, $J = J_1 + J_2$

$= \frac{1}{4} \log (4x^2 + 1) + \frac{3}{2} \tan^{-1} 2x + c$

d. $\int \frac{x^2dx}{x^6 - 8} = \int \frac{x^2dx}{(x^3)^2 - 3^2}$

Put $y = x^3$

$\frac{dy}{3} = x^2 \, dx$

Now, $\frac{1}{3} \int \frac{dy}{y^2 - 3^2} = \frac{1}{3} \frac{1}{2.3} \log \frac{y - 3}{y + 3} + c = \frac{1}{18} \log \frac{x^3 - 3}{x^3 + 3} + c$

e. $\int \frac{2x}{1 - x^4} dx = \int \frac{2x}{1 - (x^2)^2} dx$

Let $x^2 = y$

$2x.dx = dy$

$= \int \frac{dy}{1 - y^2} = \frac{1}{2.1} \log \frac{1 + y}{1 - y} + c = \frac{1}{2} \log \left(\frac{1 + x^2}{1 - x^4}\right) + C$

f. $\int \frac{dx}{x^2 + 6x + 8} = \int \frac{dx}{x^2 + 2.3x + 9 - 9 + 8}$

$= \int \frac{dx}{(x + 3)^2 - 1^2}$

Put $y = x + 3$

$dy = dx$

Now, $\int \frac{dy}{y^2 - 1^2} = \frac{1}{2.1} \log \frac{y - 1}{y + 1} + c$

$= \frac{1}{2} \log \frac{x + 3 - 1}{x + 3 + 1} + c = \frac{1}{2} \log \frac{x + 2}{x + 4} + c$

g. $\int \frac{\cos x dx}{\sin^2 x + 4\sin x + 5}$

Put $y = \sin x$, $dy = \cos x.dx$

$= \int \frac{dy}{y^2 + 4y + 5}$

$= \int \frac{dy}{y^2 + 2.2y + 4 - 4 + 5}$

$$= \int \frac{dy}{(y+2)^2 + 1^2}$$

$$= \frac{1}{2} \tan^{-1} \frac{y+2}{1} + c$$

$\text{Tan}^{-1} (\sin x + 2) + c$

h. $$\int \frac{x dx}{x^4 - x^2 - 1}$$

$$\int \frac{x dx}{(x^2)^2 - x^2 - 1}$$

Put $y = x^2$

$dy = 2x.dx$

$\frac{dy}{2} = xdx$

Now, $$= \frac{1}{2} \int \frac{dy}{y^2 - y - 1}$$

$$= \frac{1}{2} \int \frac{dy}{y^2 - 2.y.\frac{1}{2} + \frac{1}{4} - \frac{1}{4} - 1}$$

$$= \frac{1}{2} \int \frac{dy}{\left(y - \frac{1}{2}\right)^2 - \frac{5}{4}}$$

$$= \frac{1}{2} \int \frac{dy}{\left(y - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \frac{1}{2} \frac{1}{2.\frac{\sqrt{5}}{2}} \log \frac{y - \frac{1}{2} - \frac{\sqrt{5}}{2}}{y - \frac{1}{2} + \frac{\sqrt{5}}{2}} + c$$

$$= \frac{1}{2\sqrt{5}} \log \frac{2x^2 - 1 - \sqrt{5}}{2y - 1 + \sqrt{5}} + c$$

$$= \frac{1}{2\sqrt{5}} \log \frac{2x^2 - 1 - \sqrt{5}}{2x^2 - 1 + \sqrt{5}} + c$$

i. $$\int \frac{e^x dx}{e^{2x} + 2e^x + 5}$$

Put $e^x = y$

$dy = e^x . dx$

Now, $$= \int \frac{dy}{y^2 + 2y + 5}$$

$$= \int \frac{dy}{y^2 + 2.y.1 + 1 - 1 + 5}$$

$$= \int \frac{dy}{(y+1)^2 + (2)^2}$$

$$= \int \frac{dy}{(y+1)^2 + (2)^2}$$

$$= \frac{1}{2} \tan^{-1} \frac{y+1}{2} + c$$

$$= \frac{1}{2} \tan^{-1} \frac{e^x + 1}{2} + c$$

j. $\int \frac{dx}{9x^2 + 12x + 13}$

$$= \int \frac{dx}{(3x)^2 + 2.3x \,.\, 2 + 4 - 4 + 13}$$

$$= \int \frac{dx}{(3x+2)^2 + 3^2}$$

Put y = 3x +2

$\frac{dy}{3} = dx$

$$= \frac{1}{3} \int \frac{dy}{y^2 + 3^2}$$

$$= \frac{1}{3} \cdot \frac{1}{3} \tan^{-1} \frac{y}{3} + c$$

$$= \frac{1}{9} \tan^{-1} \frac{3x+2}{3} + c$$

k. $\int \frac{dx}{1 - 6x - 9x^2} = -\int \frac{dx}{(3x)^2 + 6x - 1}$

$$= -\int \frac{dx}{(3x)^2 + 2.3x \,.\, 1 + 1 - 1 - 1}$$

$$= -\int \frac{dx}{(3x)^2 + 2.3x \,.\, 1 + 1 - 1 - 1}$$

$$= -\int \frac{dx}{(3x+1)^2 - (\sqrt{2})^2}$$

Put y = 3x +1

$\frac{dy}{3} = dx$

Now, $-\frac{1}{3} \int \frac{dy}{y^2 - (\sqrt{2})^2} = -\frac{1}{3} \cdot \frac{1}{2.\sqrt{2}} \log \frac{y - \sqrt{2}}{y + \sqrt{2}} + c$

$$= -\frac{1}{6\sqrt{2}} \log \frac{3x + 1 - \sqrt{2}}{3x + 1 + \sqrt{2}} = c$$

l. $\int \frac{3x+5}{x^2 + 4x + 20} dx$

$$p \int \frac{2x+4}{x^2 + 4x + 20} dx + q \int \frac{dx}{x^2 + 4x + 20}$$

$\therefore$ 2p = 3 and 4p + q = 5

or, $p = \frac{3}{2}$ and q = –1

Now, $\int \frac{3x+5}{x^2+4x+20}\,dx = \frac{3}{2}\int \frac{(2x+4)\,dx}{x^2+4x+20} - \int \frac{dx}{x^2+4x+20}$

$= \frac{3}{2}\log(x^2+4x+20) - \int \frac{dx}{(x+2)^2+4^2}$

$= \frac{3}{2}\log(x^2+4x+20) - \frac{1}{4}\tan^{-1}\frac{x+2}{4} + c$

m. $\int \frac{(2x+2)}{(3+2x-x^2)}\,dx$

$I = \int \frac{(2x+2)}{(3+2x-x^2)}\,dx = -\int \frac{-2x+2-4}{3+2x-x^2}\,dx$

$= -\int \frac{2-2x}{3+2x-x^2}\,dx + 4\int \frac{1}{3+2x-x^2}\,dx$

$= -\log(3+2x-x^2) + 4\int \frac{1}{(2)^2-(x-1)}\,dx$

$= -\log(3+2x-x^2) + \frac{4}{2.2}\log\frac{2+x-1}{2-x-1} + c$

$= -\log(3+2x-x^2) + \log\frac{x+1}{3-x} + c$

$= \text{Ln}\left(\frac{x+1}{3-x}\right) - \text{Ln}(3+2x-x^2) + c$

n. $\int \frac{6x+2}{9x^2+6x+26}\,dx$

$= \frac{1}{3}\int \frac{18x+6}{9x^2+6x+26}\,dx$

$= \frac{1}{3}\log(9x^2+6x+26) + c$

2. Solution:

a. $\int \frac{dx}{\sqrt{x^2-4}}$

$= \int \frac{dx}{\sqrt{x^2-2^2}}$

$= \log(x+\sqrt{x^2-4}) + c$

b. $\int \frac{dx}{\sqrt{x^2+x-2}} = \int \frac{dx}{\sqrt{x^2+2x\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}-2}}$

$= \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}} = \log\left[\left(\frac{2x+1}{2}\right)+\sqrt{x^2+x-2}\right] + c$

c. $\int \frac{dx}{\sqrt{2x^2+3x+4}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+\frac{3}{2}x+2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+2\cdot\frac{3}{4}\cdot x+\frac{9}{16}-\frac{9}{16}+2}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{4}\right)}}^2 = \frac{1}{\sqrt{2}} \log \left(x + \frac{3}{4} + \sqrt{x^2 + \frac{3}{2}x + 2}\right) + c$$

d. $\displaystyle\int \frac{dx}{\sqrt{2ax + x^2}} = \int \frac{dx}{\sqrt{a^2 + 2ax + n^2 - a^2}}$

$$= \int \frac{dx}{(a + x)^2 - a^2} = \log (a + x + \sqrt{2ax + x^2}) + c$$

e. $\displaystyle\int \frac{dx}{\sqrt{5 - x + x^2}} = \int \frac{dx}{\sqrt{x^2 - x + 5}}$

$$= \int \frac{dx}{\sqrt{x^2 - 2.x \, . \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + 5}}$$

$$= \int \frac{dx}{\sqrt{x^2 - 2.x \, . \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + 5}}$$

$$= \int \frac{dx}{\sqrt{\left(x \;\frac{1}{2}\right)^2 + \left(\frac{\sqrt{19}}{2}\right)^2}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{19}}{2}\right)^2}}$$

$$= \log \left(\frac{2x - 1}{2} + \sqrt{x^2 - x + 5}\right) + c$$

f. $\displaystyle\int \frac{dx}{\sqrt{6 + x - x^2}} = -\int \frac{dx}{\sqrt{x^2 - x - 6}}$

$$= -\int \frac{dx}{\sqrt{x^2 - 2.x \, . \frac{1}{2} + \frac{1}{4} - \frac{1}{4} - 6}}$$

$$= -\int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{5}{2}\right)^2}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}}$$

$$= \sin^{-1} \frac{\frac{2x - 1}{2}}{\frac{5}{2}} + c$$

$$= \sin^{-1} \frac{2x - 1}{5} + c$$

g. $\displaystyle\int \frac{dx}{\sqrt{2 - 2x - x^2}}$

$$= -\int \frac{dx}{\sqrt{x^2 + 2x - 2}}$$

$$= -\int \frac{dx}{\sqrt{x^2 + 2x - 2}}$$

$$= -\int \frac{dx}{\sqrt{x^2 + 2x + 1 - 1 - 2}}$$

$$= -\int \frac{dx}{\sqrt{(x + 1)^2 - (\sqrt{3})^2}}$$

$$= \int \frac{dx}{\sqrt{(\sqrt{3})^2 - (x + 1)^2}}$$

$$= \sin^{-1} \frac{x + 1}{\sqrt{3}} + c$$

h. $\displaystyle\int \frac{dx}{\sqrt{9x^2 + 6x + 10}}$

$$= \int \frac{dx}{\sqrt{(3x)^2 + 6x + 10}}$$

$$= \int \frac{dx}{\sqrt{(3x)^2 + 2.3x.1 + 1 - 1 + 10}}$$

$$= \int \frac{dx}{\sqrt{(3x + 1)^2 + 3^2}}$$

Put $y = 3x + 1$

$\frac{dy}{3} = dx$

Now, $\displaystyle\frac{1}{3} \int \frac{dx}{\sqrt{y^2 + 3^2}}$

$$= \frac{1}{3} . \frac{1}{3} \tan^{-1} \frac{3x + 1}{3} + c$$

$$= \frac{1}{9} \tan^{-1} \frac{3x + 1}{3} + c$$

i. $\displaystyle\int \frac{xdx}{\sqrt{a^4 + x^4}}$

$$\int \frac{xdx}{\sqrt{(a^2)^2 + (x^2)^2}}$$

Put $x^2 = y$

$xdx = \frac{dy}{2}$

Now, $\displaystyle\frac{1}{2} \int \frac{dy}{\sqrt{a^2 + y^2}}$

$= \frac{1}{2} \log \left(y + \sqrt{a^2 + y^2}\right) + c$

$= \frac{1}{2} \log \left(x^2 + \sqrt{a^4 + x^4}\right) + c$

j. $\int \frac{xdx}{\sqrt{x^4 + 2x^2 + 10}}$

$= \int \frac{dx}{\sqrt{(x^2)^2 + 2x^2 + 10}}$

Put $x^2 = y$

$x.dx = \frac{dy}{2}$

Now, $\frac{1}{2} \int \frac{dy}{\sqrt{y^2 + 2y + 10}}$

$= \frac{1}{2} \int \frac{dy}{\sqrt{y^2 + 2.y.1 + 1 - 1 + 10}}$

$= \frac{1}{2} \int \frac{dy}{\sqrt{(y + 1)^2 + 3^2}}$

$= \frac{1}{2} \log \left(y + 1 + \sqrt{x^4 + 2x^2 + 10}\right) + c$

$= \frac{1}{2} \log \left(x^2 + 1 + \sqrt{x^4 + 2x^2 + 10}\right) + c$

k. $\int \frac{2x + 3}{\sqrt{x^2 + 4x + 20}} dx$

$= \int \frac{2x + 3 + 1 - 1}{\sqrt{x^2 + 4x + 20}} dx$

$= \int \frac{2x + 4}{\sqrt{x^2 + 4x + 20}} dx - \int \frac{dx}{\sqrt{x^2 + 4x + 20}}$

$=2\sqrt{x^2 + 4x + 20} - \int \frac{dx}{\sqrt{x^2 + 2.2x + 4 - 4 + 20}}$

$= 2\sqrt{x^2 + 4x + 20} - \int \frac{dx}{\sqrt{(x + 2)^2 + 4^2}}$

$= 2\sqrt{x^2 + 4x + 20} - \log \left(x + 2 + \sqrt{x^2 + 4x + 20}\right) + c$

l. $\int \frac{x - 2}{\sqrt{2x^2 - 8x + 5}} dx$

$= \frac{1}{2} \int \frac{2x - 8}{\sqrt{2x^2 - 8x + 5}} dx$

$= \frac{1}{2} 2\sqrt{2x^2 - 8x + 5} + c$

$= \frac{1}{2} \sqrt{2x^2 - 8x + 5} + c$

m. $\int \frac{xdx}{\sqrt{7 + 6x - x^2}}$

This equation can be written as

$$p\int \frac{6 - 2x}{\sqrt{7 + 6x - x^2}} dx + q\int \frac{1}{\sqrt{7 + 6x - x^2}} dx \quad \text{... .... (i)}$$

By comparing

$-2p = 1 \qquad 6p+q = 0$

$\therefore \quad p = \frac{-1}{2} \qquad 6 \times \left(\frac{-1}{2}\right) + q = 0$

$-3 + q = 0$

$\therefore \quad q = 3$

Put the value of p and q in equation (i)

$$= \frac{-1}{2} \int \frac{6 - 2x}{\sqrt{7 + 6x - x^2}} dx + 3 \int \frac{1}{\sqrt{7 + 6x - x^2}} dx$$

$$= \frac{-1}{2} \int (6 - 2x) (7 + 6x - x^2)^{-1/2} dx + 3 \int \frac{1}{\sqrt{-(x^2 - 6x - 7)}} dx$$

$$= \frac{-1}{2} \times \frac{2}{1} (7 + 6x - x^2)^{1/2} + 3 \int \frac{1}{\sqrt{\{x^2 - 2.3x + 9 - 9 - 7}} dx$$

$$= -\sqrt{7 + 6x - x^2} + 3 \int \frac{dx}{\sqrt{-\{(x - 3)^2 - (4)^2\}}}$$

$$= -\sqrt{7 + 6x - x^2} + 3 \times \sin^{-1} \left(\frac{x - 3}{4}\right) + c$$

n. $I = \int \frac{dx}{\sqrt{(x + a)(x + b)}}$

$$= \int \frac{1}{\sqrt{x^2 + bx + ax + ab}} dx$$

$$= \int \frac{1}{\sqrt{x^2 + (a + b) . x + ab}} dx$$

$$= \int \frac{1}{\sqrt{(x)2 + 2.x \left(\frac{a + b}{2}\right) + \left(\frac{a + b}{2}\right)^2 - \left(\frac{a + b}{2}\right)^2 + ab}} dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{a + b}{2}\right)^2 - \left(\frac{a^2 + 2ab + b^2 - 4ab}{4}\right)}} dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{a + b}{2}\right)^2 - \left(\frac{a^2 - 2ab + b^2}{4}\right)}}$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{a + b}{2}\right)^2 - \left(\frac{a - b}{2}\right)^2}} dx$$

$$= \log\left(x + \frac{a+b}{2} + \sqrt{(x+a)(x+b)}\right) + c$$

o. $\displaystyle\int \frac{dx}{(11+x)\sqrt{2+x}}$

Put $z^2 = 2 + x$

$2z \,.\, dz = dx$

Then $I = \displaystyle\int \frac{2z \,.\, dz}{(9+z^2)z}$

$$= 2\int \frac{dz}{z^2 + 3^2}$$

$$= \frac{2}{3}\tan^{-1}\frac{2}{3} + c$$

$$= \frac{2}{3}\tan^{-1}\frac{2+x}{3} + c$$

p. $J = \displaystyle\int \frac{dx}{(4x+3)\sqrt{x+3}}$

Put $x + 3 = y^2$

$\therefore \quad dx = 2y.dy$

$\therefore \quad x = y^2 - 3$

$$J = \int \frac{2ydy}{[4(y^2-3)+3]y}$$

$$= 2\int \frac{dy}{4y^2 - 9}$$

$$= 2\int \frac{dy}{(2y)^2 - 3^2}$$

$$= 2 \,.\, \frac{1}{2} \,.\, \frac{1}{2.3}\log\frac{2y-3}{2y+3} + c$$

$$= \frac{1}{6}\log\frac{2\sqrt{x+3}-3}{2\sqrt{x+3}+3} + c$$

q. $I = \displaystyle\int \sqrt{\frac{1+x}{1-x}}\, dx$

$$= \int \sqrt{\frac{1+x}{1-x} \times \frac{1+x}{1+x}}\, dx$$

$$= \int \sqrt{\frac{(1+x)^2}{1-x^2}}\, dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\, dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\, dx + \int \frac{x}{\sqrt{1-x^2}}\, dx$$

$= \int \frac{1}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}}$

$= \sin^{-1} x - \frac{1}{2} 2\sqrt{1-x^2} + c$

$= \sin^{-1} x - \sqrt{1-x^2} + c$

Exercise 16.2

1. $\int \sqrt{x^2 - 36}\,dx$

$I = \int \sqrt{x^2 - 36}\,dx$

$= \sqrt{(x) - (6)^2}\,dx$

$= \frac{x\sqrt{x^2-36}}{2} + \frac{(6)^2}{2} \log \left(x + \sqrt{x^2 - 36}\right) + c$

$= \frac{1}{2} x\sqrt{x^2-36} - 18 \log \left(x + \sqrt{x^2 - 36}\right) + c$

2. $\int \sqrt{1 - 4x^2}\,dx$

$I = \int \sqrt{1 - 4x^2\,dx} = \int \sqrt{(1)^2 - (2x)^2\,dx}$

Put $y = 2x$

$\frac{dy}{2} - dx$

Now, $I = \frac{1}{2} \int \sqrt{1^2 - y^2}\,dx$

$= \frac{1}{2} \int \sqrt{1^2 - y^2}\,dx$

$= \frac{1}{2} \cdot \frac{1}{2} \left[y\sqrt{1^2 - y^2 + \frac{1}{2} 12 \sin^{-1} \frac{y}{1}} \right] + c$

$= \frac{1}{4} y\sqrt{1^2 - y^2} + \frac{1}{4} \sin^{-1} \frac{y}{1} + c$

$= \frac{2x}{4} \sqrt{1^2 - 4x^2} + \frac{1}{4} \sin^{-1} 2x + c$

$= \frac{x\sqrt{1-4x^2}}{2} + \frac{1}{4} \sin^{-1} 2x + c$

3. $I = \int \sqrt{3x^2 + 5}\,dx = \int \sqrt{(\sqrt{3}x)^2 + (\sqrt{5})^2}\,dx$

Put $y = \sqrt{3}\,x$

$\therefore \quad \frac{dy}{\sqrt{3}} = dx$

Now, $I = \frac{1}{\sqrt{3}} \int \sqrt{y^2 + (\sqrt{5})^2}\,dx$

$I = \frac{1}{\sqrt{3}} \int \sqrt{y^2 + (\sqrt{5})^2\,dx}$

$$= \frac{1}{\sqrt{3}}\left[\frac{y\sqrt{y^2 + (\sqrt{5})^2}}{2} + \frac{\sqrt{(5)^2}}{2}\log\left(y + \sqrt{y^2 + (\sqrt{5})^2}\right)\right]$$

$$= \frac{\sqrt{3}x}{\sqrt{3}2}\sqrt{(\sqrt{3}x)^2 + (\sqrt{5}^2)} + \frac{5}{2\sqrt{3}}\log\left(\sqrt{3}x + \sqrt{(\sqrt{3}x^2) + \sqrt{(5)^2}}\right) + c$$

$$= \frac{x\sqrt{3x^2 + 5}}{2} + \frac{5}{2\sqrt{3}}\log\left(\sqrt{3}\,x + \sqrt{3x + 5}\right) + c$$

4. $I = \int\sqrt{3 - 2x - x^2}\,dx$

$$= -\int\sqrt{x^2 + 2x - 3}\,dx$$

$$= -\int\sqrt{x^2 + 2.x.1 + 1 - 1 - 3}\,dx$$

$$= \int\sqrt{-(x + 1)^2 + (2)^2}\,dx$$

$$= \frac{1}{2}(n+1)\left((x + 1)^2 - 2^2\right) + \frac{1}{2}(2)^2\log\left(x + 1 + \sqrt{(x + 1)^2 - 2^2}\right) + c$$

$$= \int\sqrt{(2)^2 - (x + 1)^2}$$

$$= \frac{1}{2}(x + 1)\left(\sqrt{3 - 2x - x^2}\right) + \frac{4}{2}\sin^{-1}\left(\frac{x + 1}{2}\right) + c$$

$$= \frac{(x + 1)\sqrt{3 - 2x - x^2}}{2} + 2\sin^{-1}\left(\frac{x + 1}{2}\right) + c$$

5. $I = \int\sqrt{5 - 2x + x^2}\,dx$

$$= \int\sqrt{4 + 1 - 2x + x^2}\,dx$$

$$= \int\sqrt{(2)^3 + (x - 1)^2}\,dx$$

$$= \frac{(x - 1)\sqrt{(2)2 + (x - 1)^2}}{2} + \frac{(2)^2}{2}\log\left(x - 1 + \sqrt{(2)^2 + (x - 1)^2}\right) + c$$

$$= \frac{1}{2}(x - 1)\sqrt{5 - 2x + x^2} + 2\log\left(x - 1 + \sqrt{5 - 2x + x^2}\right) + c$$

6. $\int\sqrt{18x - x^2 - 65}\,dx$

$$= \int\sqrt{81 - 81 + 2.9x - x^2 - 65}\,dx$$

$$= \int\sqrt{81 - 65 - (x^2 - 18x + 9^2)}\,dx$$

$$= \int\sqrt{(4)^2 - (x - 9)^2)}\,dx$$

$$= \frac{1}{2}(x - 9)\sqrt{16 - (x - 9)^2} + \frac{1}{2}16\sin^{-1}\frac{x - 9}{4} + c$$

$$= \frac{1}{2}(x - 9)\sqrt{18x - x^2 - 65} + 8\sin^{-1}\frac{x - 9}{4} + c$$

7. $\int \sqrt{5x^2 + 8x + 4}\, dx$

$= \int \sqrt{5\left(x^2 + \frac{8x}{5} + \frac{4}{5}\right)}\, dx$

$= \sqrt{5} \int \sqrt{\left(x2 + 2.x.\frac{4}{5} + \frac{16}{25} - \frac{16}{25} + \frac{4}{5}\right)}\, dx$

$= \sqrt{5} \int \sqrt{(x + 4/5)^2 + (2/5)^2}\, dx$

$= \sqrt{5}\left[\frac{\left(x + \frac{4}{5}\right)}{2}\sqrt{(x + 4/5)^2} + \left(\frac{2}{5}\right)^2 + \frac{\left(\frac{2}{5}\right)^2}{2} \text{Ln}\left\{\left(x + \frac{4}{5}\right) + \sqrt{(x + 4/5)^2 + (2/5)^2}\right\}\right]$

$= \frac{(5x + 4)\sqrt{5x^2 + 8x + 4}}{10} + \frac{2}{5\sqrt{2}} \text{Ln}\left[(5x + 4) + \sqrt{5x^2 + 8x + 4}\right] + c$

8. $I = \int \sqrt{(x - \alpha)(\beta - x)}\, dx$

Put $x - \alpha = y$

$\therefore \quad dx = dy$

$\therefore \quad x = y + \alpha$

$I = \int \sqrt{y(\beta - y\ \alpha)}\, dy$

$= \int \sqrt{(\beta - \alpha)\, y - y^2}\, dy$

$= \int \sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - \left(y - \frac{\beta - \alpha}{2}\right)^2} dy$

$= \left(y - \frac{\beta - \alpha}{2}\right)\sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - \left(y - \frac{\beta - \alpha}{2}\right)^2} + \frac{1}{2}\frac{(\beta - \alpha)^2}{4} \sin^{-1}\frac{y - \left(\frac{\beta - \alpha}{2}\right)}{\left(\frac{\beta - \alpha}{2}\right)} + c$

$= \left(x - \alpha - \frac{\beta - \alpha}{2}\right)\sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - \left(x - \alpha - \frac{\beta - \alpha}{2}\right)^2} + \frac{1}{8}(\beta - \alpha)^2 \sin^{-1}\frac{x - \alpha - \frac{\beta-\alpha}{2}}{\frac{\beta-\alpha}{2}} + c$

$= \frac{1}{2}(2x - 2\alpha - \beta + \alpha)\sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - \left(x - \frac{\alpha + \beta}{2}\right)^2} + \frac{1}{8}(\beta - \alpha)^2 \sin^{-1}\frac{2x - \alpha - \beta}{\beta - \alpha} + c$

$= \frac{1}{4}(2x - \alpha - \beta)\sqrt{(x - \alpha)(\beta - x)} + \frac{1}{8}(\beta - \alpha)^2 \sin^{-1}\frac{2x - \alpha - \beta}{\beta - \alpha} + c$

9. $I = \int \sqrt{2ax - x^2}\, dx$

$= \int \sqrt{a^2 - (a^2 - 2ax + x^2)}\, dx$

$= \int \sqrt{a^2 - (x - a)^2}\, dx$

$$= \frac{1}{2}(x-a)\sqrt{(a)^2-(x-a)^2} + \frac{(a)^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$$

$$= \frac{1}{2}(x-a)\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$$

10. $I = \int (2x-5)\sqrt{x^2-4x+5}\,dx$

$$= \int (2x-4-1)\sqrt{x^2-4x+s}\,dx$$

$$I_1 = \int (2x-4)\sqrt{x^2-4x+5}\,dx - I_2 = \int \sqrt{x^2-4x+5}\,dx$$

Put $y = x^2 - 4x + 5$

$dy = (2x-4)\,dx \quad \therefore\ I = I_1 + I_2$

$$= \int \sqrt{y}\,.\,dy - \int \sqrt{x^2-2.2x+4-4+5}\,dx$$

$$= \frac{2}{3}(y)^{3/2} - \int \sqrt{(x-2)^2+(1)^2}\,dx$$

$$= \frac{2}{3}(y)^{3/2} - \left[\frac{(x-2)\sqrt{x^2-4x+5}}{2} + \frac{1}{2}\log\left(x-2+\sqrt{x^2-4x+5}\right)\right] + c$$

$$= \frac{2}{3}(x^2-4x+5)^{3/2} - \frac{(x-2)\sqrt{x^2-4x+5}}{2} - \frac{1}{2}\log[(x-2)+\sqrt{x^2-4x+5}) + c$$

11. $\int (2-x)\sqrt{16-6x-x^2}\,dx$

$$I = \int (2-x)\sqrt{16-6x-x^2}\,dx$$

$$= \frac{1}{2}\int (4-2x)\sqrt{16-6x-x^2}\,dx$$

$$= \frac{1}{2}\int \{10+(-6-2x)\}\sqrt{16-6x-x^2}\,dx$$

$$5\int \sqrt{16-6x-x^2}\,dx + \frac{1}{2}\int (-6-2x)\sqrt{16-6x-x^2}$$

$I = I_1 + I_2$

$$I_1 = 5\int \sqrt{16-6x-x^2}\,dx$$

$$= 5\int \sqrt{25-(9+6x+x^2)}\,dx$$

$$= 5\int \sqrt{(5)^2-(x+3)^2}\,dx$$

$$= 5\left\{\frac{1}{2}(x+3)\sqrt{(5)^2-(x+3)^2} + \frac{52}{2}\sin^{-1}\left(\frac{x+3}{5}\right)\right\} + c_1$$

$$= \frac{5}{2}(X+3)\sqrt{16-6X-X^2} + \frac{125}{2}\mathrm{Sin}^{-1}\left(\frac{x+3}{5}\right) C_1$$

$I_2 = \frac{1}{2}\int (-6-2x)\sqrt{16-6x-x^2}\,dx$

Put $16-6x-x^2 = y$

$\therefore\ dy = (-6-2x)\,dx$

$I_2 = \frac{1}{2}\int y^{1/2}\,dy = \frac{1}{3}y^{3/2} + C_2\ z\ \frac{1}{3}(16-6x-x^2)^{3/2} + C_2$

$I = I_1 + I_2 = \frac{5}{2}(x+3)\sqrt{16-6x-x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x+3}{2}\right) + \frac{1}{3}(16-6x-x^2)^{3/2} + C$

12. $\int (2x+1)\sqrt{4x^2+20x+21}\,dx$

$I = \int (2x+1)\sqrt{4x^2+20x+21}\,dx$

$= \frac{1}{4}\int (8x+4)\sqrt{4x^2+20x+21}\,dx$

$= \frac{1}{4}\int \{(8x+20)-16\}\sqrt{4x^2+20x+21}\,dx$

$= \frac{1}{4}\int (8x+20)\sqrt{4x^2+20x+21}\,dx - 4\int\sqrt{4x^2+20x+21}\,dx$

$= I_1 - I_2$

$I_1 = \frac{1}{4}\int (8x+20)\sqrt{4x^2+20x+21}\,dx$

Put $4x^2+20x+21 = y$

$(8x+20)\,dx = dy$

$I_1 = \frac{1}{4}\int y^{1/2}\,dy = \frac{1}{4}\frac{y^{3/2}}{3/2} + C_1$

$= \frac{1}{6}(4x^2+20x+21)^{3/2} + C_1$

$I_2 = 4\int\sqrt{4x^2+20x+21}\,dx$

$= 4\int\sqrt{(4x^2+20x+25)-4}\,dx$

$= 4\int\sqrt{(2x+5)^2-(2)^2}\,dx$

Put $2x+5 = y$

$\therefore\quad dx = \frac{1}{2}y$

$I_2 = 2\int\sqrt{(y)^2-(2)^2}\,dy$

$= 2\left\{\frac{1}{2}y\sqrt{(y)^2-(2)^2} - \frac{(2)^2}{2}\log\left(y+\sqrt{y^2-4}\right)\right\} + C_2$

$= y\sqrt{y^2-4} - 4\log\left(y+\sqrt{y^2-4}\right) + C_2$

$= (2x+5)\sqrt{(2x+5)^2-4} - 4\log\left(2x+5+\sqrt{(2x+5)^2-4}\right)$

$= (2x+5)\sqrt{2x^2+20x+21} - 4\log\left(2x+5+\sqrt{4x^2+20x+2}\right)$

$I = I_1 - I_2$

$= \frac{1}{6}(4x^2 + 20x + 21)^{3/2} - (2x + 5)\sqrt{4x^2 + 20x + 21} + 4\log(2x + 5 + \sqrt{4x^2 + 20x + 21} + C$

13. $I = \int (2x + 3)\sqrt{x^2 - 2x - 3}\, dx$

$= \int (2x + 3 - 5 + 5)\sqrt{x^2 - 2x - 3}\, dx$

$= \int (2x - 2)\sqrt{x^2 - 2x - 3}\, dx + 5\int \sqrt{x^2 - 2x - 3}\, dx$

$I = I_1 + I_2$

$I_1 = \int (2x - 2)\sqrt{x^2 - 2x - 3}\, dx$

Put $y = x^2 - 2x - 3$

$\therefore \quad dy = (2x - 2)\, dx$

$= \int \sqrt{y}\, dy = \frac{y^{3/2}}{3/2} + C = \frac{2}{3}\sqrt{x^2 - 2x - 3} + C$

$I_2 = 5\int \sqrt{x^2 - 2x - 3}\, dx$

$= 5\int \sqrt{x^2 - 2x + 1 - 1 - 3}\, dx$

$= 5\int \sqrt{(x - 1)^2 - (2)^2}\, dx$

$= 5\left[\frac{1}{2}(x - 1)\sqrt{x^2 - 2x - 3} - \frac{4}{2}\log(x - 1 + \sqrt{x^2 - 2x - 3})\right] + C$

$= \frac{5}{2}(x - 1)\sqrt{x^2 - 2x - 3} - \frac{20}{2}\log(x - 1 + \sqrt{x^2 - 2x - 3}) + C$

$= \frac{5}{2}(x - 1)\sqrt{x^2 - 2x - 3} - 10\log(x - 1 + \sqrt{x^2 - 2x - 3}) + C$

$\therefore I = I_1 + I_2$

$= \frac{2}{3}(x^2 - 2x - 3)^{3/2} + \frac{5}{2}(x - 1)\sqrt{x^2 - 2x - 3} - 10\log(x - 1 + (x^2 - 2x - 3) + C$

14. $I = \int e^{3x} . \sin^5 x\, dx$

We suppose $u = \sin 5x$ $v = e^{3x}$

By using formula $\int e^{ax} \sin bx dx = \int \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$

We have,

$I = \frac{1}{3^2 + 5^2} . e^{3x}(3\sin 5x - 5\cos 5x) + C$

$\therefore \quad I = \frac{1}{34} e^{3x}(3\sin 5x - 5\cos 5x) + C$

15. $I = \int e^x \cos 3x\, dx$

By using formula $\int e^{ax} . \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

We have,

$$I = \frac{1}{1^2 + 3^2} e^x (1.\cos 3x + 3\sin 3x) + C$$

$$= \frac{1}{10} e^x (\cos 3x + 3\sin 3x) + C$$

16. $I = \int e^{2x} \sin(x + 1)\, dx$

By using formula $\int e^{ax} . \sin bx = e^{ax} \frac{(a\sin bx - b\cos bx)}{a^2 + b^2}$

We have,

$$I = \frac{1}{(2)^2 + 1^2} e^{2x} [2\sin(x + 1) - \cos(x + 1)] + C$$

$$= \frac{1}{5} e^{2x} [2\sin(x + 1) - \cos(x + 1)] + C$$

Exercise 16.3

1. $I = \int \frac{dx}{1 + 2\sin^2 x}$

Dividing number and de no. by $\cos^2 x$

$$\int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{2\sin^2 x}{\cos^2 x}} dx$$

$$= \int \frac{\sec^2 x}{\sec^2 x + 2\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 + \tan^2 x + 2\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 + (\sqrt{3}\tan x)^2} dx$$

Put $y = \sqrt{3}\tan x$

$$\therefore \quad I = \frac{1}{\sqrt{3}} \int \frac{dy}{1^2 + y^2}$$

$$= \frac{1}{\sqrt{3}} \frac{1}{1} \tan^{-1} \frac{y}{1} + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} (\sqrt{3}\tan x) + C$$

2. $I = \int \frac{dx}{5 + 4\cos x}$

$$\int \frac{dx}{5\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}\right) + 4\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)} dx$$

$$= \int \frac{1}{5\sin^2 \frac{x}{2} + 5\cos^2 \frac{x}{2} + 4\cos^2 \frac{x}{2} - 4\sin^2 \frac{x}{2}} dx$$

$$\int \frac{1}{9\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}\,dx$$

Dividing deno. and nu no. by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{9 + \tan^2\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = y$

or, $\sec^2\frac{x}{2}\frac{1}{2} = dy$

$\therefore$ $2dy = \sec^2 x$

Now, $I = 2\int \frac{dy}{(3)^2 + y^2}$

$$= \frac{2}{3}\tan^{-1}\frac{\left(\tan\frac{x}{2}\right)}{3} + C$$

$$= \frac{2}{3}\tan^{-1}\frac{1}{3}\left(\tan\frac{x}{2}\right) + C$$

3. $I = \int \frac{dx}{1 - 3\sin x}$

$$= \int \frac{dx}{\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) - 2.3\sin\frac{x}{2}\cos\frac{x}{2}}$$

Dividing by $\cos^2\frac{x}{2}$ in deno-and num.

$$\int \frac{\sec^2\frac{x}{2}}{ran^2\frac{x}{2} + - 6\tan\frac{x}{2}}\,dx$$

Put $y = \tan\frac{x}{2}$

$2dy = \sec^2\frac{x}{2}\,dx$

Now,

$\therefore$ $I = 2\int \frac{dy}{y^2 - 6y + 1}$

$$= 2\int \frac{dy}{y^2 - 2.3y + 9 - 9 + 1}$$

$$= 2\int \frac{dy}{(y-3)^2 - (2\sqrt{2}^2)}$$

$$= 2\frac{1}{2.2\sqrt{2}}\log\frac{y - 3 - 2\sqrt{2}}{y - 3 + 2\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}} \log \frac{\tan\frac{x}{2} - 3 - 2\sqrt{2}}{\text{Tan}\,\frac{x}{2} - 3 + 2\sqrt{2}} + C$$

4. $I = \int \frac{dx}{a^2 \sin^2 x - b^2 \cos^2 x}$

Dividing number and deno. by $\cos^2 x$ then

$I = \int \frac{\sec^2 x}{a^2 \text{Tan}^2 x - b^2} dx$

Put $y = a\text{Tan}x$

$\frac{dy}{a} = \sec^3 x \,.\, dx$

Now,

$\therefore \quad I = \frac{1}{a} \int \frac{dy}{y^2 - b^2}$

$= \frac{1}{2.ba} \log \frac{y - b}{y + b} + C$

$= \frac{1}{2ba} \log \left(\frac{a \tan x - b}{a \tan x + b}\right) + C$

5. $I = \int \frac{dx}{4 \cos x - 1}$

$$= \int \frac{dx}{4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) - \left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{4\cos^2\frac{x}{2} - 4\sin^2\frac{x}{2} - \sin^2\frac{x}{2} - \cos^2\frac{x}{2}}$$

$$= \int \frac{dx}{3\cos^2\frac{x}{2} - 5\sin^2\frac{x}{2}}$$

Dividing deno. and num. by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{3 - (\sqrt{5})^2 \text{Tan}^2\frac{x}{2}}$$

Put $y = \sqrt{5}\,\text{Tan}\,\frac{x}{2}$

$\therefore \quad \frac{2dy}{\sqrt{5}} = \sec^2\frac{x}{2}$

Now, $I = \frac{2}{\sqrt{5}} \int \frac{dy}{(\sqrt{3})^2 - \left(\sqrt{5}\,\text{Tan}\,\frac{x}{2}\right)^2}$

$$= \frac{2}{\sqrt{5}} \int \frac{dy}{(\sqrt{3})^2 - y^2}$$

$$= \frac{2}{\sqrt{5}} \frac{1}{2\sqrt{3}} \log \frac{\sqrt{3}+y}{\sqrt{3}-y} + C$$

$$= \frac{1}{\sqrt{15}} \log \left(\frac{\sqrt{3} + \sqrt{5}\,\text{Tan}\,\frac{x}{2}}{\sqrt{3} - \sqrt{5}\,\text{Tan}\,\frac{x}{2}} \right) + C$$

6. $I = \int \frac{dx}{2 + 3\cos x}$

$$= \int \frac{dx}{2\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + 3\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{5\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

Dividing deno. and num. by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{(\sqrt{5})^2 - \text{Tan}^2\frac{x}{2}}\,dx$$

Put $y = \text{Tan}\,\frac{x}{2}$

$2dy = \sec^2\frac{x}{2}\,dx$

Now,

$$\therefore \quad I = 2\int \frac{dy}{(\sqrt{5})^2 - y^2} = 2\frac{1}{2\sqrt{5}} \log \frac{\sqrt{5}+y}{\sqrt{5}-y} + C = \frac{1}{\sqrt{5}} \log \left(\frac{\sqrt{5} + \text{Tan}\,\frac{x}{2}}{\sqrt{5} - \text{Tan}\,\frac{x}{2}} \right) + C$$

7. $I = \int \frac{\sin x \cos x}{(\sin x + \cos x)^2}\,dx$

$$= \frac{1}{2} \int \frac{2\sin x - \cos x}{(\sin x + \cos x)^2}\,dx$$

$$= \frac{1}{2} \int \frac{\sin^2 x\,dx}{(\sin x + \cos x)^2}$$

$$= \frac{1}{2} \int \frac{(1 + \sin 2x) - 1}{(\sin x + \cos x)^2}\,dx$$

$$= \frac{1}{2} \int \frac{(\sin x + \cos x)^2 - 1}{(\sin x + \cos x)^2}\,dx$$

$$= \frac{1}{2} \int \frac{(\sin x + \cos x)^2}{(\sin x + \cos x)^2}\,dx - \int \frac{dx}{(\sin x + \cos x)^2}$$

$$I_1 = \frac{1}{2} x - I_2 = \int \frac{dx}{(\sin x + \cos x)^2}$$

Dividing deno. and num. by $\cos^2 x$

$I = I_1 + I_2$

$= \frac{x}{2} - \int \frac{\sec^2 dx}{(1 + \text{Tan}x)^2}$ Put Tan x + 1 = y, dy = $\sec^2 x.dx$

$= \frac{x}{2} - \int \frac{dy}{y^2} = \frac{x}{2} + \frac{1}{y} + C = \frac{x}{2} + \frac{1}{\text{Tan } x + 1} + C$

8. $I = \int \frac{dx}{\sin x + \cos x}$

Put $1 = r\cos\theta$ $\quad$ $1 = r\sin\theta$

So that $r^2 = 2$

$\therefore \quad r = \sqrt{2}$

$\text{Tan}\theta = 1$

$\therefore \quad \theta = \frac{\pi}{4}$

$I = \int \frac{dx}{r\cos\theta \,.\, \sin x + r\sin\theta \,.\, \cos x} = \frac{1}{r} \int \frac{dx}{\sin(x + \theta)} = \frac{1}{r} \int \text{cosec}\,(x + \theta)\, dx$

$= \frac{1}{r} \log \text{Tan} \frac{1}{2}(x + \theta) + C = \frac{1}{\sqrt{2}} \log \text{Tan} \frac{1}{2}\left(x + \frac{\pi}{4}\right) + C$

$= \frac{1}{\sqrt{2}} \log \text{Tan} \left(\frac{x}{2} + \frac{\pi}{8}\right) + C$

9. $I = \int \frac{dx}{1 + \sin x + \cos x}$

$= \int \frac{dx}{(1 + \cos x) + \sin x}$

$= \int \frac{dx}{2\cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}$

Dividing by $\cos^2 \frac{x}{2}$ in deno. and num.

$= \frac{1}{2} \int \frac{\sec^2 \frac{x}{2}}{1 + \text{Tan} \frac{x}{2}} dx$

Put $1 + \text{Tan} \frac{x}{2} = y$

$\sec^2 \frac{x}{2} \frac{1}{2} dx = dy$

$\therefore \quad \sec^2 \frac{x}{2} dx = 2dy$

$I = \int \frac{dy}{y}$

$= \log y + C$

$= \log \left(1 + \text{Tan} \frac{x}{2}\right) + C$

10. $I = \int \frac{dx}{3 + 2\sin x + \cos x}$

$$= \int \frac{1}{3\cos^2\frac{x}{2} + 3\sin^2\frac{x}{2} + 4\sin\frac{x}{2}.\cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}} dx$$

$$= \int \frac{1}{4\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2} + 4\sin\frac{x}{2}\cos\frac{x}{2}} dx$$

Dividing denominator and numerator by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{4 + 2\mathrm{Tan}^2\frac{x}{2} + 4\mathrm{Tan}\frac{x}{2}} dx$$

$$= \frac{1}{4}\int \frac{\sec^2\frac{x}{2}}{1 + \frac{1}{2}\tan^2\frac{x}{2} + \tan\frac{x}{2}}$$

Put $\tan\frac{x}{2} = y$

$\Rightarrow \sec^2\frac{x}{2}\, dx = 2dy$

from (i)

$$I = \frac{1}{4}\int \frac{2dy}{1 + \frac{1}{2}y^2 + y} = \int \frac{dy}{y^2 + 2y + z} = \int \frac{dy}{(y+1)^2 + 1^2}$$

$$= \frac{1}{1}\tan^{-1}(y+1) + C = \tan^{-1}\left(\tan\frac{x}{2} + 1\right) + C$$

11. $I = \int \frac{1}{1 - \sin x + \cos x}$

$$= \int \frac{1}{(1 + \cos x) - \sin x}$$

$$= \int \frac{1}{2\cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}} dx$$

$$= \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2} - \sin\frac{x}{2}\cos\frac{x}{2}}$$

Dividing by deno. and num $\cos^2\frac{x}{2}$

$$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{1 - \mathrm{Tan}\frac{x}{2}} dx$$

Put $1 - \mathrm{Tan}\frac{x}{2} = y$

$\frac{1}{2}\sec^2\frac{x}{2}\, dx = dy$

$\sec^2\frac{x}{2}\, dx = 2dy$

$$= -\frac{2}{2}\int \frac{dy}{y}$$

$$= -\log y + C$$

$$= -\log\left(1 - \text{Tan}\,\frac{x}{2}\right) + C$$

12. $$\int \frac{dx}{2 + \cos x - \sin x}$$

$$= \int \frac{dx}{2\sin^2\frac{x}{2} + 2\cos^2\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{1}{\sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} - 2\tan\frac{x}{2} + 1}dx$$

Put $\tan\frac{x}{2} = y\ \frac{1}{2}\sec^2\frac{x}{2}\ dx = dy\ \sec^2\frac{x}{2}\,dx = 2dy$

$$= \int \frac{2dy}{y^2 - 2y + 1}$$

$$= 2\int \frac{1}{(y-1)^2}\,dy = -\frac{2}{(y-1)} + c$$

$$= -\frac{2}{(\tan\frac{x}{2} - 1)}$$

13. $$\int \frac{dx}{\cos x - \sqrt{3}\sin x}$$

Put $1 = r\sin\theta$, $\sqrt{3} = r\cos\theta$
So that $r^2 = 3 + 1 = 4$
$\therefore\quad r = 2$

Also, $\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\therefore\quad \theta = \frac{\pi}{6}$

Now, $$\int \frac{dx}{r\sin\theta\,.\,\cos x - r\cos\theta\,\sin x}$$

$$= \int \frac{dx}{r[\sin(\theta - x)]}$$

$$= \frac{1}{2}\int \cos(\theta - x)\,dx$$

$$= \frac{1}{2}\left|\text{Ln}\tan\left(\frac{\theta - x}{2}\right)\right| + c$$

$= \frac{1}{2} \text{Ln} \tan\left(\frac{\pi}{12} - \frac{x}{2}\right) + c$

14. $I = \int \frac{1}{1 + 2\sin x} dx$

$= \int \frac{dx}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2.2\sin\frac{x}{2}\cos\frac{x}{2}}$

Dividing by $\cos^2\frac{x}{2}$ in deno and num

$= \int \frac{\sec^2\frac{x}{2}}{1 + \tan^2\frac{x}{2} + 4\text{ Tan}\frac{x}{2}}$

$= \int \frac{\sec^2\frac{x}{2}}{\left(\tan\frac{x}{2} + 2\right)^2 - 3}$

Put $\tan\frac{x}{2} + 2 = y$

$\text{Sec}^2\frac{x}{2} dx = 2dy$

$\therefore \quad I = 2\int \frac{dy}{(y)^2 - (\sqrt{3})^2}$

$= 2\frac{1}{2\sqrt{3}} \log\frac{y - \sqrt{3}}{y + \sqrt{3}} + C$

$= \frac{1}{\sqrt{3}} \log\frac{\text{Tan}\frac{x}{2} + 2 - \sqrt{3}}{\text{Tan}\frac{x}{2} + 2 + \sqrt{3}} + C$

15. $I = \int \frac{dx}{2 + \sin x}$

$= \int \frac{dx}{2\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$

Dividing by deno and num by $\cos^2\frac{x}{2}$

$= \frac{1}{2} \int \frac{\sec^2\frac{x}{2}}{1 + \text{Tan}^2\frac{x}{2} + \text{Tan}\frac{x}{2}} dx$

Put $\tan\frac{x}{2} = y$

$\text{Sec}^2\frac{x}{2} dx = 2dy$

$I = \frac{2}{2}\int \frac{dy}{1 + y^2 + y}$

$$= \int \frac{dy}{\left(y + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{y + \frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2y + 1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2\text{Tan}\,\frac{x}{2} + 1}{\sqrt{3}} + C$$

16. $I = \int \frac{dx}{4 + 3\sin x}$

or, $I = \int \frac{dx}{4\left(\cos x^2 \frac{x}{2} - \sin x^2 \frac{x}{2}\right) + 3.2 \sin x \frac{x}{2} \cos x \frac{x}{2}}$

$$= \int \frac{\sec^2 \frac{x}{2}}{4\left(1 - \text{Tan}x^2 \frac{x}{2}\right) + 6\ \text{Tan}x \frac{x}{2}}\, dx$$

Let $\tan x \frac{x}{2} = y$

$\text{Sec}x^2 \frac{x}{2} = 2dy$

Now,

$\therefore \quad I = 2\int \frac{dy}{4(1 - y^2) + 6y}$

$$= 2\int \frac{dy}{4 - \left(2y - \frac{3}{2}\right)^2 + \frac{9}{4}}$$

$$= 2\int \frac{dy}{\left(\frac{5}{2}\right)^2 - \left(2y + \frac{3}{2}\right)^2}$$

$$= 2.\frac{1}{2}\,\frac{1}{2.\frac{5}{2}} \log \left\{\frac{\frac{5}{2} + 2y - \frac{3}{2}}{\frac{5}{2} - \left(2y - \frac{3}{2}\right)}\right\} + C$$

$$= \frac{1}{5} \log \left(\frac{2y + 1}{4 - 2y}\right) + C$$

$$= \frac{1}{5} \log \left[\frac{2\ \text{Tan}x \frac{x}{2} + 1}{4 - 2\ \text{Tan}x \frac{x}{2}}\right] + C$$

17. $I = \int \frac{dx}{4 + 3\cosh x}$

$$I = \int \frac{dx}{4\left(\cos h^2 \frac{x}{2} + \sin h^2 \frac{x}{2}\right) + 3\left(\cos h^2 \frac{x}{2} - \sin h^2 \frac{x}{2}\right)}$$

$$= \int \frac{dx}{7\cos h^2 \frac{x}{2} - \sin h^2 \frac{x}{2}}$$

Dividing by deno and num by $\cos^2 \frac{x}{2}$

$$= \int \frac{\sec h^2 \frac{x}{2}}{7 - \tan h^2 \frac{x}{2}}$$

Let $\tan h \frac{x}{2} = y$

$\sec h^2 \frac{x}{2} = 2dy$

$$\therefore\ I = 2\int \frac{dy}{7 - y^2}$$

$$= 2\int \frac{dy}{((\sqrt{7})^2 - (y)^2)}$$

$$= 2.\frac{1}{2.\sqrt{7}} \log\left(\frac{\sqrt{7} + y}{\sqrt{7} - y}\right) + C$$

$$= \frac{1}{\sqrt{7}} \log\left[\frac{\sqrt{7} + \tan h \frac{x}{2}}{\sqrt{7} - \tan h \frac{x}{2}}\right] + C$$

18. $I = \int \frac{\tan hx}{36 \sec hx + \cos hx}\, dx$

Multiplying by cosx deno. and num.

$$= \int \frac{\cos hx. \frac{\sin hx}{\cos hx}}{36 \frac{\cos hx}{\cos hx} + \cos hx \, . \cos hx}$$

$$= \int \frac{\sin hx}{(6)^2 + \cos h^2 x}\, dx$$

Put $\cos hx = y$

$\therefore\ \sin hx \, . dx = dy$

$$\therefore\ \int \frac{dy}{(6)^2 + (y)^2}$$

$$= \frac{1}{6} \tan^{-1} \frac{y}{6} + C$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{\cos h\, x}{6}\right) + C$$

19. $I = \int \frac{\tan hx\, dx}{\cos hx + \sec hx\ 64}$

Multiplying cosx x in deno. and num

$$= \int \frac{\cos hx \dfrac{\sin hx}{\cos hx}}{\cos hx \,.\, \cos hx + 64 \,.\, \cos hx \dfrac{1}{\cos hx}}$$

$$= \int \frac{\sin hx}{\cos h^2 x + 64} dx$$

Let cos hx = y

$$\sin h\, x = \frac{dy}{dx}$$

sin h x . dx = dy

$$\therefore \quad I = \int \frac{dy}{y^2 + 64}$$

$$= \int \frac{dy}{y^2 + (8)^2}$$

$$= \frac{1}{8} \text{Tan}^{-1} \frac{y}{8} + C$$

$$= \frac{1}{8} \text{Tan}^{-1} \left(\frac{\cosh x}{8}\right) + C$$

20. $$I = \int \frac{\sin h\, x}{4\tan hx - \text{cosec } hx \,.\, \text{sech}} dx$$

$$= \int \frac{\sin x\, x}{4 \dfrac{\sin hx}{\cos hx} - \dfrac{1}{\text{cosec } hx \,.\, \sec hx}} dx$$

$$= \int \frac{\sin h^2 x \cos hx}{4\sin h^2 x - 1}$$

Let sin hx = y

cos hx.dx = dy

$$\therefore \quad I = \int \frac{y^2\, dy}{4y^2 - 1} = \frac{1}{4} \int \frac{y^2\, dy}{4y^2 - 1} = \frac{1}{4} \int \frac{(14y^2 - 1) + 1}{(4y^2 - 1)} dy$$

$$= \frac{1}{4} \int \left(1 - \frac{1}{4y^2 - y}\right) dy = \frac{1}{4} \left\{ y + \int \frac{1}{(2y)^2 - (1)^2} dy \right\}$$

$$= \frac{1}{4} \left\{ y + \frac{1}{2} \frac{1}{2.1} \log \left(\frac{2y - 1}{2y + 1}\right) \right\} + C$$

$$= \frac{1}{4} \left\{ \sin hx + \frac{1}{4} \log \left(\frac{2\sin hx - 1}{2\sin hx + 1}\right) \right\} + C$$

Exercise 16.4

1. $$\int \frac{2x}{(2x + 3)(3x + 5)} dx$$

Let $\dfrac{2x}{(2x + 3)(3x + 5)} = \dfrac{A}{2x + 3} + \dfrac{B}{3x + 5}$

2x = A(3x + 5) + B(2x + 3)

Equating the coefficient of x and constant terms we to get

3A + 2B = 2

5A + 3B = 0

A = –6 and B = 10

$$\therefore \quad \frac{2x}{(2x+3)(3x+5)} = \frac{-6}{2x+3} + \frac{10}{3x+5}$$

So, we have by integration

$$\int \frac{dx}{(2x+3)(3x+5)} = -3 \log(2x+3) + \frac{10}{3} \log(3x+5) + C$$

2. $\displaystyle\int \frac{3x}{(x-a)(x-b)} dx$

Let $\displaystyle\frac{3x}{(x-a)(x-b)} = \frac{A}{(2x+3)} + \frac{B}{x-b}$

Put x = a

$3a = A(x-b)$

$\therefore \quad A = \dfrac{3a}{a-b}$

$x = b,\ 3b = B(b-a)$

$B = \dfrac{-3b}{a-b}$

$$\therefore \quad \frac{3x}{(x-a)(x-b)} = \frac{3a}{(x-a)(a-b)} - \frac{3b}{(a-b)(x-b)}$$

$$\text{or,} \quad \int \frac{3x}{(x-a)(x-b)} dx = \frac{3}{a-b} \int \left\{ \frac{a}{x-a} - \frac{b}{x-b} \right\} dx$$

$$= \frac{3}{a-b} \ [a \log(x-a) - b \log(x\ \ b)] + C$$

3. $\displaystyle\int \frac{1}{(x+2)(x+3)^2} dx$

Let $\displaystyle\frac{1}{(x+x2)(x+3)^2} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

$1 = A(x+3)^2 + B(x+2)(x+3) + C(x+2)$

Put x = –2

1 = A.1

$\therefore$ A = 1

x = –3

I = C(–1)

$\therefore$ C = –1

x = 0, 1 – 9A + B.2.3 + C.2

6B = –6

$\therefore$ B = –1

$$\therefore \quad \frac{1}{(x+2)(x+3)^2} = \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{(x+3)^2}$$

$$\int \frac{1}{(x+2)(x+3)^2} dx = \int \left\{ \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{(x+3)^2} \right\} dx$$

$$\text{Log}(x+2) - \log(x+3) + \frac{1}{x+3} + C = \log \frac{x+2}{x+3} + \frac{1}{x+3} + C$$

4. $\displaystyle\int \frac{x^2\, dx}{(x-a)(x-b)(x-c)}$

Let $\displaystyle\frac{x^2}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

$x^2 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$

$\therefore$ Put x = b

$b^2 = B(b - a)(b - c) + c(b - c)(b - b)$

Similarly,

Again put x = a

$\frac{a^2}{(a - b)(a - c)} = A$

$\frac{c^2}{(c - b)(c - a)} = C$

Then,

$$= \frac{a^2}{(a - b)(a - c)} \cdot \int \frac{1}{x - a}dx + \frac{b^2}{(b - a)(b - c)} \int \frac{1}{x - b}dx + \frac{c^2}{(c - b)(c - a)} \int \frac{1}{x - c}dx$$

$$= \frac{a^2}{(a - b)(a - c)} \log(x - a) + \frac{b^2}{(b - a)(b - c)} \log(x - c) + \frac{c^2}{(c - b)(c - a)} \log(x - c) + c$$

5. $\int \frac{x^2 + 1}{x - 1} dx$

$$\int \frac{x^2}{x - 1} dx + \int \frac{1}{x - 1} dx$$

$$= \int \frac{x^2 - 1 + 1}{x - 1} dx + \int \frac{1}{x - 1} dx$$

$$= \int x + 1dx + \int \frac{1}{x - 1} + \int \frac{1}{x + 1} dx$$

$$= \frac{x^2}{2} + x + 2\log(x - 1) + C$$

6. $\int \frac{dx}{1 + x + x^2 + x^3}$

$$\int \frac{dx}{x^3 + x^2 + x + 1}$$

$$\int \frac{1}{x^2(x + 1) + (x + 1)} dx$$

$$\int \frac{1}{(x^2 + 1)(x + 1)} dx$$

Let, $\frac{1}{(x + 1)(x^2 + 1)} = \frac{A}{(x + 1)} + \frac{Bx + c}{x^2 + 1}$

$1 = A(x^2 + 1) + (Bx + c)(x + 1)$

Put x = 0

$1 = A + C$ ... ... (i)

Put the value of $A = \frac{1}{2}$

$x = -1$

$1 = 2A$

$\therefore \frac{1}{2} = A \Rightarrow C = \frac{1}{2}$

Put x + 1

$1 = 2A + (B + C) 2$

$1 = 2 \times \frac{1}{2} + \left(B + \frac{1}{2}\right) 2$

$1 = 1\frac{1}{2}\left(B + \frac{1}{2}\right)$

$0 = 2\left(B + \frac{1}{2}\right)$

$\therefore \quad -\frac{1}{2} = B$

$= \frac{1}{2}\int \frac{1}{(x+1)} + \int \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1}dx$

$= \frac{1}{2}\log(x+1) - \frac{1}{2} \times \frac{1}{2}\int \frac{2x+2}{x^2-1}dx$

$= \frac{1}{2}\log(x+1) - \frac{1}{4}\log(x^2+1) + 2\int \frac{1}{x^2+1}dx$

$= \frac{1}{2}\log(x+1) - \frac{1}{4}\log(x^2+1) + 2\,\mathrm{Tan}^{-1}x + c$

7. $\int \frac{7x^2 - 18x + 13}{(x-3)(x^2+2)}dx$

Let, $\frac{7x^2 - 18x + 13}{(x-3)(x^2+2)} = \frac{A}{(x-3)} + \frac{Bx + C}{(x^2+2)}$

$7x^2 - 18x + 13 = (x^2+2)A + (Bx + C)(x-3)$

Put $x = 0$

$13 = 2A - 3C$ ... ... (i)

Again put $x = 3$

$7\times3^2 = -18 \times 3 + 13 = 11A$

$63 - 54 + 13 = 11A$

or, $22 = 11A$

$\therefore \quad A = 2$

Put the value of A in equation (i)

$13 = 2\times2 - 3c$

or, $\frac{13-2}{3} = -C$

$\therefore \quad C = -3$

Put $x = 1$

$2 = 3A - 2(B + C)$

$2 = 6 - 2(B - 3)$

$-4 = -2(B - 3)$

or, $2 = B - 3$

$\therefore \quad B = 5$

$= \int \frac{7x^2 - 18x + 13}{(x-3)(x^2+2)}dx = 2\int \frac{1}{x-3}dx + \int \frac{5x - 3dx}{x^2+2}$

$= 2\log(x-3) + \frac{5}{2}\int \frac{2x - \frac{3}{5}\times 2}{x^2+2}dx$

$= 2\log(x-3) + \frac{5}{2}\int \frac{1}{x^2+2}dx - \frac{6}{5}\times\frac{5}{2}\int \frac{1}{x^2+2}dx$

$= 2\log(x-3) + \frac{5}{2}(x^2+2) - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

8. $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

Let I = $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

$$= \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 1} dx$$

Put $x + \frac{1}{x} = y$

$\therefore \quad \left(1 - \frac{1}{x^2}\right) dx = dy$

$I = \int \frac{dy}{y+-1}$

$= \frac{1}{2.1} \log \frac{y - 1}{y + 1} + C$

$= \frac{1}{2} \log \frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} + C$

$= \frac{1}{2} \log \frac{x^2 - x + 1}{x^2 + x + 1} + C$

9. $\int \frac{1}{x^4 - 1} dx$

Let $\frac{1}{x^4 - 1} dx = \frac{1}{(x - 1)(x + 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{x + D}{x^2 + 1}$

Put

x = 1 $\qquad$ 1 = A.2.2 $\qquad$ $\therefore$ A = $\frac{1}{4}$

x = –1 $\qquad$ 1 = B.(–2).2 $\qquad$ $\therefore$ B = $\frac{-1}{4}$

x = 0 $\qquad$ 1 = A + B(–1) + (–1) D

$1 = \frac{1}{4} + \frac{1}{4} - D$

$\therefore \quad D = -\frac{1}{2}$

Equating the coefficients of x^3

0 = A + B + C

or, $\frac{1}{4} - \frac{1}{4} + C$

$\therefore \quad C = 0$

$\therefore \quad \frac{1}{x^4 - 1} = \frac{1}{4} \frac{1}{x - 1} - \frac{1}{4} \frac{1}{x + 1} - \frac{1}{2} \frac{1}{x^2 + 1}$

$$\int \frac{1}{x^4-1}\,dx = \int \left\{\frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x^2+1}\right\} dx$$

$$= \frac{1}{4}\log(x-1) - \frac{1}{4}\log(x+1) - \frac{1}{2}\mathrm{Tan}^{-1} x + c$$

$$= \frac{1}{4}\log\frac{x-1}{x+1} - \frac{1}{2}\tan^{-1} x + c$$

10. $\displaystyle\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$

$$\frac{x^2}{(x^2+a^2)(x^2+b^2)} = \frac{y}{(y+a^2)(y+b^2)} = \frac{A}{y+a^2} + \frac{B}{y+b^2}$$

where $y = x^2$

$y = A(y + b^2) + B(y + a^2)$

When $y = -a^2$

$-a^2 = A(-a^2 + b^2)$

$\therefore\quad A = \dfrac{a^2}{a^2-b^2}$

When $y = -b^2$

$-b^2 = B(-b^2 + a^2)$

$\therefore\quad B = -\dfrac{b^2}{a^2-b^2}$

$$\frac{y}{(y+a^2)(y+b^2)} = \frac{a^2}{a^2-b^2}\frac{1}{y+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{y+b^2}$$

or, $\displaystyle\frac{x^2}{(x^2+a^2)(x^2+b^2)} = \frac{a^2}{a^2-b^2}\frac{1}{x^2+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{x^2+b^2}$

or, $\displaystyle\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \int \left[\frac{-a^2}{a^2-b^2}\frac{1}{x^2+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{x^2+b^2}\right] dx$

$$= \frac{a^2}{a^2-b^2}\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{b^2}{a^2-b^2}\frac{1}{b}\mathrm{Tan}^{-1}\frac{x}{b} + c$$

$$= \frac{a}{a^2-b^2}\tan^{-1}\frac{x}{a} - \frac{b}{a^2-b^2}\tan^{-1}\frac{x}{b} + C$$

11. $\displaystyle\int \frac{x^2+4}{x^4+16}\,dx$

$$= \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}\,dx = \int \frac{1+\frac{4}{x^2}}{\left(x-\frac{4}{x^2}\right)^2+8}$$

Put $x - \dfrac{4}{x} = y$

$\Rightarrow 1 + \dfrac{4}{x^2}\,dx = dy$

$$I = \int \frac{dy}{y^2 + (2\sqrt{2})^2} = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{y}{2\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x^2-4}{2\sqrt{2}} + C$$

12. $\displaystyle\int \frac{x^3}{(x-a)(x-b)(x-c)}$

Let $\frac{x^3}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{c}{(x-c)}$

$\Rightarrow \quad x^3 = A(x-b)(x-c) + B(x-a)(x-c) + c(x-a)(x-b)$ ... (i)

Putting x = a, x = b, x = c turn by turn, we get,

$A = \frac{a^3}{(a-b)(a-c)}$, $B = \frac{b^3}{(b-a)(b-c)}$, $c = \frac{c^3}{(c-a)(c-b)}$

Now, $\int \frac{x^3}{(x-a)(x-b)(x-c)}$

$= \frac{a^3}{(a-b)(a-c)} \int \frac{1}{x-a} dx + \frac{b^3}{(b-a)(b-c)} \int \frac{1}{x-b} dx + \frac{c^3}{(c-a)(c-a)} \int \frac{1}{x-c} dx$

$= \frac{a^3}{(a-b)(a-c)} \ln(x-a) + \frac{b^3}{(b-a)(b-c)} \ln(x-b) + \frac{c^3}{(c-a)(c-b)} \ln(x-c) + c$

13. $\int \frac{x^3 \, dx}{2x^4 - x^2 - 10}$

Put $x^2 = y$

$2xdx = dy$

$\therefore \quad dx = \frac{dy}{2x}$

Now, $\int \frac{x^3 dx}{2x^4 - x^2 - 10} = \int \frac{x^2 \, xdx}{2(x^2)^2 = x^2 - 10} = \int \frac{y \, . \, x}{2y^2 - y - 10} \cdot \frac{dy}{2x}$

$= \frac{1}{2} \int \frac{y \, dy}{2y^2 - 5y + 4y - 10} = \frac{1}{2} \int \frac{y \, dy}{y(2y-5) + 2(2y-5)}$

$= \frac{1}{2} \int \frac{y \, dy}{(y+2)(2y-5)}$ ... (i)

Let $\frac{y}{(y+2)(2y-5)} = \frac{A}{(y+2)} + \frac{B}{(2y-5)}$

$\Rightarrow \quad y = A(2y-5) + B(y+2)$ ... (ii)

Putting y = –2 in (ii), we get,

$-2 = A(-4-5) + B \times 0 \Rightarrow A = \frac{2}{9}$

Again, putting $y = \frac{5}{2}$ in (ii), we get

$\frac{5}{2} = A \times 0 + B\left(\frac{5}{2} + 2\right) \Rightarrow B = \frac{5}{9}$

$\therefore \quad \frac{y}{(y+2)(2y-5)} = \frac{2}{9(y+2)} + \frac{5}{9(2y-5)}$

from (i) $\frac{1}{2} \int \frac{y \, dy}{(y+2)(2y-5)} = \frac{1}{9} \int \frac{1}{y+2} dy + \frac{5}{18} \int \frac{1}{2y-5} dy$

$= \frac{1}{9} \ln(y+2) + \frac{5}{36} \ln(2y-5) + c$

$= \frac{5}{36} \ln(2x^2 - 5) + \frac{1}{9} \ln(x^2 + 2) + c$

14. $\int \frac{dx}{(x-1)^2 (x-3)^2}$

Put $x - 1 = z(x-3)$

$\Rightarrow \quad x - zx = 1 - 3z \Rightarrow x = \frac{1-3z}{1-z}$

$$dx = \frac{(1-z) - 3 - (1-3z) \times (-1)}{(1-z)^2} dz$$

$$\Rightarrow dx = \frac{-3 + 3z + 1 - 3z}{(1-z)^2} dz = \frac{-2}{(1-z)^2} dz$$

Here, $\frac{1}{(x-1)^2 (x-3)^2} = \frac{1}{z^2(x-3)^2 (x-3)^2} = \frac{1}{z^2\left[\frac{1-3z}{1-z} - 3\right]^4}$

$$= \frac{1}{z^2 \left[\frac{1-3z-3+\_\ 3z}{1-z}\right]^4} = \frac{(1-z)^4}{-2z^2}$$

Now, $\int \frac{dx}{(x-1)^2 (x-3)^2} = \int \frac{-2}{(1-z)^2} \times \frac{(1-z)^4}{-2z^2}$

$$= \int \frac{(1-z)^2}{z^2} dz = \int \frac{1-2z+z^2}{z^2} dz$$

$$= \int \left(\frac{1}{z^2} - \frac{2}{z} + 1\right) dz = -\frac{1}{z} - 2\ln z + z + c$$

$$= -\left(\frac{x-3}{x-1}\right) - 2\ln\left(\frac{x-1}{x-3}\right) + \left(\frac{x-1}{x-3}\right) + c$$

15. $I = \int \frac{dx}{(x-1)^2 (x-4)^3}$

Put $x - 1 = z(x-3)$

$\therefore \quad x = \frac{3z-1}{z-1}$

or, $dx = \frac{3(z-1) - (3z-1)}{(z-1)^2} dz = \frac{3z - 3 - 3z + 1}{(z-\ 1)^2} dz = \frac{-2}{(z-1)^2} dz$

Also, $\frac{1}{(x-1)^2 (x-3)^3} = \frac{1}{z^2(x-3)^5} = \frac{1}{z^2\left(\frac{3z-1}{z-1} - 3\right)^5} = \frac{(z-1)^5}{32z^2}$

So, $I = \int \frac{(z-1)^5}{32z^2} \cdot \frac{-2}{(z-1)^2} dz$

$$= \frac{-1}{16} \int \frac{(z-1)^3}{z^2} dz$$

$$= \frac{-1}{16} \int \frac{z^3 - 3z^2 + 3z - 1}{z^2} dz$$

$$= \frac{-1}{16} \int \left(z - 3 + \frac{3}{z} - \frac{1}{z^2}\right) dz$$

$$= \frac{-1}{16} \left(\frac{z^2}{2} - 3z + 3\ln z + \frac{1}{z}\right) + C$$

$$= \frac{-1}{32} z^2 + \frac{3}{16} z - \frac{3}{16} \ln z - \frac{1}{16} z + C$$

$$= \frac{-1}{32} \left(\frac{x-1}{x-3}\right)^2 + \frac{3}{16} \frac{x-1}{x-3} - \frac{3}{16} \ln \left|\frac{x-1}{x-3}\right| - \frac{1}{16} \left(\frac{x-1}{(x-3)}\right) + C$$