Chapter – 4
Complex Number

Exercise 4.1

1. Solution:

a. Let, z be the cube root of –1

$z^3 = -1$

or, $z^3 + 1 = 0$

or, $(z)^3 + (1)^3 = 0$

or, $(z + 1)\ (z)^2 - z + 1) = 0$

Either,

$z = -1$

$z^2 - z + 1 = 0$ ... ... (i)

Comparing equation (i) with $ax^2 + bx + c = 0$

$\therefore$ a = 1, $\therefore$ b = –1, c = +1

Now,

$x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$= 1 \pm \dfrac{\sqrt{1 - 4 \times 1 \times 1}}{2 \times 1}$

$= \dfrac{1 \pm \sqrt{-3}}{2}$

$= \dfrac{1 \pm \sqrt{3}i}{2}$

Taking positive

$x = \dfrac{1 + \sqrt{3}i}{2}$

Taking negative

$x = \dfrac{1 - \sqrt{3}i}{2}$

Here, z is the value of x.

Hence,

The cube root of unity is

$-1, \dfrac{1 + \sqrt{3}i}{2}$ and $\dfrac{1 - \sqrt{3}i}{2}$

b. Here,

Let, z be the cube root of 8

So, $z^3 = 8$

or, $(z)^3 = -(2)^3 = 0$

or, $(z - 2)\ (z^2 + 2z + 4) = 0$

Either,

$z = 2$

$z^2 + 22 + 4 = 0$ ... ... (i)

Comparing equation (i) with $az^2 + bz + c = 0$

So, a = 1, b = 2, c = 4

Now,

$z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-2 \pm \sqrt{4 - 4 \times 1 \times 4}}{2 \times 1}$

$= \dfrac{-2 \pm \sqrt{4 - 16}}{2}$

$= \dfrac{-2 \pm \sqrt{-12}}{2}$

$= \frac{-2 \pm 2\sqrt{3}i}{2}$

$= -1 \pm \sqrt{3}\, i$

Taking positive, Taking negative

$z = -1 + \sqrt{3}\, i$ $z = -1 - \sqrt{3}\, i$

Hence,

The required cube roots of 8 are 2, $-1 + \sqrt{3}i$ and $-1 - \sqrt{3}i$

2. Solution:

a. Here, $z^4 = 1$

or, $(z)^4 - (1)^4 = 0$

or, $(z^2)^2 - (1^2)^2 = 0$

or, $(z^2 - 1)(z^2 + 1) = 0$

or, $(z - 1)(z + 1)(z^2 + 1) = 0$

Either,

or, $z = 1$,

or, $z = -1$

or, $z^2 + 1 = 0$ ... ... (i)

or, Comparing equation (i) with $az^2 + bz + c = 0$

$\therefore$ $a = 1, b = 0, c = 1$

Now,

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{0 \pm \sqrt{0 - 4\times1\times1}}{2\times1}$

$= \frac{0 \pm \sqrt{-4}}{2}$

$= \frac{0 \pm 2i}{2}$

$= \pm i$

Taking positive Taking negative

$z = i$ $z = -i$

Hence,

The required value of z is ±1 and $\pm i$.

b. Here,

$z^4 = -1$

or, $z^4 = -1 + i \times 0$

or, $z^4 = \cos180^o + i\sin180^o$

or, $z^4 = \{\cos(k.360 + 180^o) + i\sin(k.360^o + 180)\}$

or, $z = \{\cos(k.360^o + 180^o) + i\sin(k.360^o + 180^o)\}^{1/4}$

$= \cos\left(\frac{k.\ 360 + 180}{4}\right) + i\sin\left(\frac{k.\ 360 + 180^o}{4}\right)$

where, k = 0, 1, 2, 3

When k = 0 then, $z = \cos(k.90^o + 45^o) + i\sin(k.90^o + 45^o)$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

when, k = 1, $z = \cos135^o + i\sin135^o$

$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

when, k = 2, $z = \cos225^o + i\sin 225^o$

$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

when, k = 3, $z = \cos 315^o + i\sin 315^o$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$$

$\therefore \quad z = \pm\left(\frac{1+i}{\sqrt{2}}\right), \pm\left(\frac{1-i}{\sqrt{2}}\right)$

c. Here, $z^6 = 1$

$z^6 = 1^6 = 0$

or, $(z^2)^3 - (12)^3 = 0$

or, $(z^2 - 1)(z^4 + z^2 + 1) = 0$

Either,

$z = \pm 1$

$z^4 + z^2 + 1 = 0$

or, $(z^2)^2 + (1)^2 + z^2 = 0$

or, $(z^2 + 1)^2 - 2z^2 + z^2 = 0$

or, $(z^2 + 1)^2 - (z)^2 = 0$

or, $(z^2 + 1 - z)(z^2 + 1 + z) = 0$

Either,

$z^2 + z + 1 = 0$ ... ... (i)

$z^2 - z + 1 = 0$ ... ... (ii)

Comparing equation (i) with $az^2 + bz + c = 0$

$\therefore \quad a = 1, b = 1, c = 1$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1 - 3\times1\times1}}{2\times1}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

Taking positive

$$z = \frac{-1 + \sqrt{3}i}{2}$$

Taking negative

$$z = \frac{-1 - \sqrt{3}i}{2}$$

Again, comparing equation (ii) with $az^2 + bz + c = 0$

$\therefore \quad a = 1, b = -1, c = 1$

Now,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1^2 - 4\times1\times1}}{2\times1} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

Taking positive

$$z = \frac{1}{2} + \frac{\sqrt{3}i}{2}$$

Taking negative

$$z = \frac{1}{2} - \frac{\sqrt{3}i}{2}$$

$\therefore \quad z = \pm 1, \pm\left(\frac{-1-\sqrt{3}i}{2}\right), \pm\left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$

3. Solution:

a. $(1 + \omega^2)^3 - (1 - \omega)^3$

$= (-\omega)^3 - (-\omega^2)^3$

$= -\omega^3 - (-\omega^6)$

$= -1 - (-(\omega^3)^3)$

$= -1 - (-1)$

$= -1 + 1$

$= 0$

b. $(2 + \omega)(2 + \omega^2)(2 - \omega^2)(2 - \omega^4)$

$= (1 + 1 + \omega)(1 + 1 + \omega^2)(1 + 1 - \omega^2)(1 + 1 - \omega^4)$

$= (1 - \omega^2)(1 - \omega)(1 + 1 - \omega^2)(1 + 1 - \omega) \quad (\because \omega^3 = 1)$

$= (1 - \omega^2)(1 - \omega)(2 - \omega^2)(2 - \omega)$
$= 1 - \omega - \omega^2 + \omega^3)(4 - 2\omega - 2\omega^2 + \omega^3)$
$= (1 - \omega - \omega^2 + 1)(4 - 2\omega - 2\omega^2 + 1)$
$= (2 + 1)(4 + 1 + 2)$
$= 3 \times 7 = 21$

c. $(1 - \omega + \omega^2)^4 . (1 + \omega - \omega^2)^4$
$= (-2\omega)^4 . (-2\omega^2)^4$
$= 16\omega^3. \omega . 16\omega^3 . \omega^3 . \omega^2$
$= 16. \omega \times 16\omega^2$ $[\because \omega^3 = 1]$
$= 256 \times \omega^3$
$= 256 \times 1$
$= 256$

d. $(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6$
$= (-2\omega)^6 + (-2\omega^2)^6$
$= 64\,\omega^3. \omega^3 + 64\,\omega^3 . \omega^3 . \omega^3 . \omega^3$
$= 64 + 64$ $[\because \omega^3 = 1]$
$= 128$

e. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)$
$= (1 - \omega)(1 - \omega^2)(1 - \omega^3. \omega)(1 - \omega^3 . \omega^2)$
$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$
$= (1 - \omega^2)^2 (1 - \omega)^2$
$= (1 - 2\omega^2 + \omega^4)(1 - 2\omega + \omega^2)$
$= (1 - 2\omega^2 + \omega)(1 - 2\omega + \omega^2)$
$= (-3\omega^2)(-3\omega)$
$= 9\omega^3$
$= 9 \times 1$ $[\because \omega^3 = 1]$
$= 9$

f. $\dfrac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2}$
$= \dfrac{a\omega^3 + b\omega.\omega^3 + c\omega^2 . \omega^3}{a\omega^2 + c\omega + b}$
$= \dfrac{\omega(a\omega^2 + b\omega^3 + c\omega^4)}{(a\omega^2 + c\omega + b)}$
$= \omega \dfrac{(a\omega^2 + b + c\omega)}{(a\omega^2 + c\omega + b)}$ $[\because \omega^3 = 1]$
$= \omega$

g. $\dfrac{1}{1 + 2\omega} + \dfrac{1}{3 + \omega} - \dfrac{1}{1 + \omega}$
$= \dfrac{1}{1 + \omega + \omega} + \dfrac{1}{1 + 1 + \omega} - \dfrac{1}{1 + \omega}$
$= \dfrac{1}{-\omega^2 + \omega} + \dfrac{1}{-\omega^2 + 1} - \dfrac{1}{1 + \omega}$
$= \dfrac{1}{\omega(1 - \omega)} + \dfrac{1}{(1 - \omega^2)} - \dfrac{1}{1 + \omega}$
$= \dfrac{1}{\omega(1 - \omega)} + \dfrac{1}{(1 - \omega)(1 + \omega)} - \dfrac{1}{(1 + \omega)}$
$= \dfrac{1 + \omega + \omega - \omega + \omega^2}{\omega. (1 + \omega)(1 - \omega)}$
$= \dfrac{1 + \omega + \omega^2}{\omega (1 - \omega^2)}$
$= 0$ $[\because 1 + \omega + \omega^2 = 0]$

4. Solution:

a. If $\alpha = \omega$, $\beta = \omega^2$

$\alpha^4 + \beta^4 = \frac{1}{\alpha\beta}$

$= \omega^4 = (\omega^2)^4 + \frac{1}{\omega.\omega^2}$

$= \omega + (\omega^3)^2 . \omega^2 + 1 \quad [\because \omega^3 = 1]$

$= \omega + \omega^2 + 1$

$= 0 \ [\because 1 + \omega + \omega^2 = 0]$

b. Here,

$\alpha^4 + \alpha^2\beta^2 + \beta^4$

$= \omega^4 + \omega^2.(\omega^2)^2 + (\omega^2)^4$

$= \omega + \omega^2 . \omega^4 + \omega^8$

$= \omega + 1 + \omega^2 \qquad [\because \omega^3 = 1]$

$= 0 \qquad [\because 1 + \omega + \omega^2 = 0]$

5. Solution:

Given,

$x = a + b$

$y = a\omega + b\omega^2$

$z = a\omega^2 + b\omega$

a. xyz

$= (a + b)(aw + bw^2)(aw^2 + bw)$

$= (a + b)(a^2w^3 + abw^2 + abw^4 + b^2w^3)$

$= (a + b)\{a^2.1 + ab(w^2 + w^4) + b^2.1\}$

$= (a + b)\{a^2 + ab(w + w^2) + b^2\}$ [$\because w^4 = w^3. w = 1 . w = w$]

$= (a + b)\{a^2 - ab + b^2\} = a^3 + B^3$ [$\because w^2 + w = -1$)

b. $x + y + z$

$= (a + b) + (aw + bw^2) + (aw^2 + bw)$

$= a + b + aw + bw^2 + aw^2 + bw$

$= a(1 + w + w^2) + b(1 + w^2 + w)$

$= a \times 0 + b \times 0 = 0$

c. $x^3 + y^3 + z^3$

$= x^3 + y^3 + z^3 - 3xyz + 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) + 3xyz$

$= 0 + 3(a^3 + b^3)$ [$\because$ from (a) & (b)]

$= 3(a^3 + b^3)$

d. $x^2 + y^2 + z^2$

$= (x + y + z)^2 - 2xy - 2yz - 2xz$

$= 0 - 2(xy + yz + xz)$

$= -2\{(a + b)(aw + bw)^2 + (aw + bw^2). (aw^2 + bw) + (a + b). (aw^2 + bw)\}$

$= -2\{a^2w + abw^2 + abw + b^2w^2 + a^2 w^3 + abw^2 + abw + b^2 + a^2w^2 + abw + abw^2 + b^2w\}$

$= -2\{a^2w + a^2w^2 + a^2w^3 + 3abw^2 + 3abw + b^2w^2 + b^2 + b^2w\}$

$= -2\{a^2(w + w^2 + 1) + 3(-1)ab + b^2(w^2 + w + 1)\}$

$= -2\{a^2(w + w^2 + 1) + 3(-1)ab + b^2(w^2 + w + 1)\}$

$= -2\{0 - 3ab + 0\}$

$= 6ab$

6. Solution

a. We know,

$w = \frac{-1 + \sqrt{3}i}{2}$

$w^2 = \frac{-1 - \sqrt{3}i}{2}$

Now, $\left(\frac{-1+\sqrt{-3}}{2}\right)^6 + \left(\frac{-1-\sqrt{-3}}{2}\right)^{12}$

$= w^6 + (w^2)^{12}$

$= (w^3)^2 + w^{24}$

$= 1^2 + (w^3)^8$

$= 1 + 1^8 = 2$

b. $\left(\frac{-1+\sqrt{-3}}{2}\right)^8 + \left(\frac{-1-\sqrt{-3}}{2}\right)^8$

$= w^8 + (w^2)^8$

$= (w^3)^2.w^2 + (w^3)^5.\ w$

$= w^2 + w = -1$

c. Let, $w = \frac{-1+\sqrt{-3}}{2}$

$w^2 = \frac{-1-\sqrt{-3}}{2}$

Case-I: If n is multiple of 3 i.e. n = 3k, k is on integer.

$= 1 + \left(\frac{-1+\sqrt{-3}}{2}\right)^n + \left(\frac{-1-\sqrt{-3}}{2}\right)^n$

$= 1 + w^{3k} + (w^2)^{3k}$

$= 1 + (w^3)^k + (w^3)^{2k}$

$= 1 + 1^k + 1^{2k}$

$= 1 + 1 + 1 = 2 + 1 = 3$ proved.

Case II: n is not a multiple of 3 i.e. n = 3k + 1

$= 1 + \left(\frac{-1+\sqrt{-3}}{2}\right)^n + \left(\frac{-1-\sqrt{-3}}{2}\right)^n$

$= 1 + (w)^{3k+1} + (w^2)^{3k+1}$

$= 1 + (w)^{3k+1} + w^{6k} - w^2$

$= 1 + (w^3)^k.w + (w^3)^{2k}.\ w^2$

$= 1 + w + w^2 = 0$ proved.

Exercise 4.2

1. Solution:

a. Here, 2 + 2i

$x = 2, y = 2$

$r = \sqrt{2^2 + 2^2}$

$= \sqrt{4+4}$

$= \sqrt{8}$

$= 2\sqrt{2}$

$\text{Tan}\theta = \frac{y}{x}$

$= \frac{2}{2} = 1$

$\therefore \quad \theta = 45^o$

It can be written in polar form as $2\sqrt{2}\,(\cos 45^o + i\sin 45^o)$

b. Here,

$-\sqrt{2} + \sqrt{2}i$

Here, $x = -\sqrt{2}$

$y = \sqrt{2}$

$r = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2}$

$\text{Tan}\theta = \frac{y}{x}$ $\qquad = \sqrt{2+2}$

$= \frac{\sqrt{2}}{-\sqrt{2}}$ $\quad = \sqrt{4}$

$= -1$ $\quad = 2$

$\therefore \quad \theta = 135^o$

In polar form = $2(\cos135^o + \text{is in } 135^o)$

c. Here,

Let, $z = -1 + 0i$

Here, $x = -1$

$y = 0$

$r = \sqrt{(-1)^2 + 0}$

$= -\sqrt{1}$

$= 1$

$\text{Tan}\theta = \frac{y}{x} = \frac{0}{-1} = 0$

$\therefore \theta = 180^o$

In polar form = $1(\cos180^o + i\sin180^o)$

$= \cos180^o + i\sin180^o$

d. Here,

Let, $z = 0 + 3i$

Here, $x = 0$, $y = 3$

$\text{Tan}\theta = \frac{y}{x}$ $\quad r = \sqrt{(0)^2 + (3)^2}$

$= \frac{3}{0}$ $\quad = \sqrt{9}$

$= \infty$ $\quad = 3$

$\theta = 90^o$

In polar form = $3(\cos90^o + i\sin90^o)$

e. Here,

Let $z = 0 - 5i$

Here, $x = 0$,

$y = -5$

$r = \sqrt{x^2 + y^2} = \sqrt{0 + 25} = \sqrt{25} = 5$

$\text{Tan}\theta = \frac{y}{x} = \frac{-5}{0} = \infty$

$\therefore \theta = 270^o$

Now, In polar form $-5i = 5(\cos270^o + i\sin270^o)$

f. Here,

Let, $z = -\sqrt{3} + i$

Here, $x = -\sqrt{3}$

$y = 1$

$r = \sqrt{(-\sqrt{3})^2 + (1)^2}$

$= \sqrt{4} = 2$

$\text{Tan}\theta = \frac{y}{x}$

or, $\text{Tan}\theta = \frac{1}{-\sqrt{3}}$

or, $\text{Tan}\theta = \text{Tan } 150^o$

$\therefore \quad \theta = 150^o$

In polar form $i-\sqrt{3} = 2(\cos150^o + i\sin150^o)$

g. Here,

Let, $z = -3 - \sqrt{3}i$

Here, $x = -3$

$y = -\sqrt{3}$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-3} = \frac{1}{\sqrt{3}}$

$\therefore \theta = 210^o$

$r = \sqrt{(-3)^2 + (-\sqrt{3})^2}$

$= \sqrt{9 + 3}$

$= \sqrt{12}$

$= 2\sqrt{3}$

In polar form, $-3-\sqrt{3}i = 2\sqrt{3}\,(\cos210^o + i\sin210^o)$

h. Here,

Let, $z = 1 - \sqrt{3}i$

Here, $x = 1$, $y = -\sqrt{3}$

$\tan\theta = \frac{y}{x}$

or, $\tan\theta = \frac{-\sqrt{3}}{1}$

or, $\tan\theta = -\sqrt{3}$

or, $\tan\theta = \tan300^o$

$r = \sqrt{(1)^2 + (-\sqrt{3})^2}$

$= \sqrt{1 + 3}$

$= \sqrt{4}$

$= 2$

In polar form, $1 - \sqrt{3}i = 2(\cos300^o + i\sin300^o)$

i. Here,

Let, $z = 2 + 2\sqrt{3}\,i$

Here, $z = 2$, $y = 2\sqrt{3}$

$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$\therefore \theta = 60^o$

$r = \sqrt{4 + 4\times3} = \sqrt{16} = 4$

In polar form, $(2, 2\sqrt{3}) = 4(\cos60^o + i\sin60^o)$

j. Here,

Let, $z = \frac{1}{1-i}$

$= \frac{1}{1-i} \times \frac{1+i}{1+i}$

$= \frac{1+i}{1+1}$

$\therefore z = \frac{1}{2} + \frac{1}{2}i$

Here, $x = \frac{1}{2}$, $y = \frac{1}{2}$

Now, $\tan\theta = \frac{y}{x}$, $\qquad r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \frac{\frac{1}{2}}{\frac{1}{2}}$ $\qquad = \sqrt{\frac{1}{4} + \frac{1}{4}}$

$= 1$ $\qquad = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$

$\therefore 45^o$

In polar form, $\frac{1}{1-i} = \frac{1}{\sqrt{2}}(\cos 45^o + i\sin 45^o)$

k. Here,

Let, $z = \sqrt{\frac{1+i}{1-i}} = \sqrt{\frac{1+i}{1-i} \times \frac{1+i}{1+i}} = \frac{1+i}{\sqrt{2}}$

$\therefore z = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} i$

Here, $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$

$r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

Now, $\text{Tan}\theta = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

$\therefore \quad \theta = 45^o$

In polar form, $\sqrt{\frac{1+i}{1-i}} = \cos 45^o + i\sin 45^o$

2. Solution:

a. Here,

Let, $2(\cos 30^o + i\sin 30^o) = x + iy$

Equating real and imaginary parts;

$x = 2\cos 30^o$ $\qquad y = 2\sin 30^o$

$= 2 \times \frac{\sqrt{3}}{2}$ $\qquad = 2 \times \frac{1}{2}$

$= \sqrt{3}$ $\qquad = 1$

$\therefore 2(\cos 30^o + i\sin 30^o) = \sqrt{3} + i.$

b. Here,

Let, $3(\cos 150^o + i\sin 150^o) = x + iy$

Equating real and imaginary parts;

$x = 3\cos 150^o$, $\qquad y = 3\sin 150^o$

$= 3 \times \left(\frac{-\sqrt{3}}{2}\right)$ $\qquad = 3 \times \frac{1}{2}$

$= \frac{-3\sqrt{3}}{2}$ $\qquad = \frac{3}{2}$

$\therefore 3(\cos 150^o + i\sin 150^o) = \frac{-3\sqrt{3}}{2} + \frac{3}{2} i$

c. Here,

Let, $4(\cos 240^o + i\sin 240^o) = x + iy$

Equating real is imaginary parts;

$x = y\cos 240^o$, $\qquad y = 4\sin 240^o$

$= 4 \times \left(\frac{-1}{2}\right)$ $\qquad = 4 \times \left(\frac{-\sqrt{3}}{2}\right)$

$= -2$ $\qquad = \frac{-\sqrt{3} \times 2}{2}$

$= -2\sqrt{3}$

$\therefore$ $4(\cos240^o + i\sin240^o) = -2 - 2\sqrt{3}i$

d. Here,

Let, $2\sqrt{2}\,(\cos 270^o + i\sin270^o) = x + iy$

Equating real and imaginary parts;

$x = 2\sqrt{2}\cos270^o$

$= 2\sqrt{2} \times 0 = 0$

$y = 2\sqrt{2}\sin270^o$

$= -2\sqrt{2}$

$\therefore$ $2\sqrt{2}\,(\cos270^o + i\sin270^o) = -2\sqrt{2}i$

3. Solution:

a. Here,

$2(\cos 53^o + i\sin53^o) . 3(\cos7^o + i\sin7^o)$

$= 2\times3\,\{\cos(53^o + 7) + i\sin\,(53^o + 7^o)\}$

$= 6P\{\cos60^o + i\sin60^o\}$

$= 6\left(\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right)$

$= 3 + 3\sqrt{3}i$

b. $(\cos5\theta + i\sin\theta)\,(\cos3\theta + i\sin3\theta)$

$= \cos(5\theta + 3\theta) + i\sin(5\theta + 3\theta)$

$= \cos8\theta + i\sin8\theta$

c. $(\cos72^o + i\sin72^o)\,(|\cos12 - i\sin12^o)$

$= (\cos72^o + i\sin72^o)\,\{\cos(-12) + i\sin(-12)\}$

$= \cos(72 - 12) + i\sin\,(72 - 12)$

$= \cos60^o + i\sin60^o$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

d. $\frac{\cos50^o + i\sin50^o}{\cos20^o + i\sin20^o}$

$= \cos(50 - 20) + i\sin\,(50 - 20)$

$= \cos30^o + i\sin30^o$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

e. $\frac{(\cos4\theta + i\sin4\theta)\,(\cos3\theta - i\sin3\theta)}{\cos3\theta + i\sin3\theta}$

$= \frac{(\cos4\theta + i\sin4\theta)\,(\cos(-3\theta) + i\sin(-3\theta))}{\cos3\theta + i\sin3\theta}$

$= \frac{\cos(4\theta - 3\theta) + i\sin(4\theta - 3\theta)}{\cos3\theta + i\sin3\theta}$

$= \frac{\cos\theta = i\sin\theta}{\cos3\theta + i\sin3\theta}$

$= \cos(\theta - 3\theta) + i\sin(\theta - 3\theta)$

$= \cos2\theta - i\sin2\theta$

f. $\frac{\cos5\theta + i\sin5\theta}{(\cos2\theta + i\sin2\theta)^2}$

$= \frac{\cos5\theta + i\sin5\theta}{(\cos4\theta + i\sin4\theta)}$

$= \cos(5\theta - 4\theta) + i\sin\,(5\theta - 4\theta)$

$= \cos\theta + i\sin\theta$

g. $\frac{(\cos3\theta + i\sin3\theta)^5}{(\cos\theta + i\sin\theta)^7}$

$= \dfrac{(\cos 15\theta + i\sin 15\theta)}{(\cos 7\theta + i\sin 7\theta)}$

$= \cos(15\theta - 7\theta) + i\sin(15\theta - 7\theta)$

$= \cos 8\theta + i\sin 8\theta$

4. Solution:

a. Here, $\left[3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{16}$

$= 3^{16} \times \left[\cos\left(16 \times \frac{\pi}{4}\right) + i\sin\left(16 \times \frac{\pi}{4}\right)\right]$

$= 3^{16} [\cos 4\pi + i\sin 4\pi]$

$= 3^{16} [1 + 0]$

$= 3^{16}$

b. Here, $[2(\cos 50^o + i\sin 50^o)]^3$

$= 2^3 [\cos 150^o + i\sin 150^o)$

$= 8\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -4\sqrt{3} + 4i$

c. $[4(\cos 6^o + i\sin 6^o)]^{30}$

$= 4^{30} [\cos(6\times30) + i\sin(6\times30)]$

$= 4^{30} [-1+0]$

$= -4^{30}$

d. $(\cos 70^o + i\sin 70^o)^6$

$= \cos(70\times6) + i\sin(70\times6)$

$= \cos 420^o + i\sin 420^o$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

e. $(1 + i)^{15}$

Here, $x = 1$, $y = 1$

$\tan\theta = \frac{y}{x}$, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

or, $\tan\theta = \frac{1}{2}$

or, $\tan\theta = 1$

$\therefore \theta = 45^o$

In polar form, $(1 + i) = \sqrt{2}(\cos 45^o + i\sin 45^o)$

Now,

$= \sqrt{2}(\cos 45^o + i\sin 45^o)\}^{15}$

$= 128 \times \sqrt{2}\{\cos(45\times15) + i\sin(45\times15)\}$

$= 128\sqrt{2}(\cos 675^o + i\sin 675^o)$

$= 128\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= 128(1 - i)$

f. $(1 - i)^{10}$

Let, $z = 1 - i$

Here, $x = 1$, $y = -1$, $r = \sqrt{(1)^2 + (-1)^2}$

$\tan\theta = \frac{y}{x}$ $\qquad = \sqrt{1 + 1} = \sqrt{2}$

or, $\tan\theta = \frac{-1}{1}$

or, $\tan\theta = -1$

or, $\theta = 315^o$

In polar form, $(1 - i) = \sqrt{2}\,(\cos315^o + i\sin 315^o)$

Now,

$$\left\{\begin{matrix}\sqrt{2}\,(\cos315^o + i\sin315^o) \\ 25\,(\cos630^o + i\sin630^o)\end{matrix}\right\}^{10}$$

$= 2^5\,(0 + (-1)i)$

$= 25 \times (-1)i$

$= -32i$

g. $(2i)^4$

Let, $z = 0 + 2i$

Here, $x = 0$, $y = 2$

$\text{Tan}\theta = \dfrac{y}{x} = \dfrac{2}{0} = \infty$

$\therefore \quad \theta = 90^o$

$r = \sqrt{0 + 2^2} = \sqrt{4} = 2$

In polar form, $2i = 2(\cos90^o + i\sin90^o)$

Now,

$= \{2(\cos90^o + i\sin90^o\}^4$

$= 2^4\,(\cos360^o + i\sin360^o)$

$= 16\,(1 + 0)$

$= 16$

h. Here,

Let, $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\,i$

Here, $x = \dfrac{1}{x}$, $y = \dfrac{\sqrt{3}}{2}$

$\text{Tan}\theta = \dfrac{y}{x}$, $\qquad r = \sqrt{\left(\dfrac{1}{1}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2}$

$= \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \qquad = \sqrt{\dfrac{1}{4} + \dfrac{3}{4}}$

$= \sqrt{3} \qquad = \sqrt{\dfrac{4}{4}} = 1$

$\therefore \quad \theta = 60^o$

In polar form, $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\,i = \cos60^o + i\sin60^o$

Now, $(\cos60^o + i\sin60^o)^7$

$= \cos420^o + i\sin720^o = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\,i$

5. Solution:

Let, $z = -2 - 2\sqrt{3}i$

Here, $x = -2$

$y = -2\sqrt{3}$

$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2}$

$= \sqrt{4 + 12}$

$= \sqrt{16} = 4$

$\text{Tan}\theta = \dfrac{y}{x}$

$= \frac{-2\sqrt{3}}{-2} = \sqrt{3}$

$\therefore\ \theta = 240^o$

In polar form, $z = 4(\cos 24^o + i\sin 240^o)$

In general polar form;

$\sqrt{z} = 4\,\{\cos(240 + 360.k) + i\sin(240 + 360.k)\}^{½}$

$= 2\left\{\cos\left(\frac{240 + 360.k}{2}\right) + i\sin\left(\frac{240 + 360.k}{2}\right)\right\}$

where, k = 0 and 1

When, k = 0

$\sqrt{z} = 2(\cos 120^o + i\sin 120^o)$

$= 2\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$

when k = 1

$\sqrt{z} = 2\,(\cos 300^o + i\sin 300^o)$

$= 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}\,i$

$\therefore\ \sqrt{-2 - 2\sqrt{3}i} = \pm(-1 + \sqrt{3}i)$

b. Let, $z = 4 + 4\sqrt{3}i$

Here, $x = 4$, $y = 4\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x} = \frac{4\sqrt{3}}{4} = \sqrt{3}$

$\therefore\ \theta = 60^o$

$r = \sqrt{(4)^2 + (4\sqrt{3})^2}$

$= \sqrt{16 + 48}$

$= \sqrt{64}$

$= 8$

In polar form, $4 + 4\sqrt{3}\,i = 8(\cos 60^o + i\sin 60^o)$

In general polar form i

$z = 8\{\cos(60 + 360.k) + i\sin(60 + 360.k)\}$

where, k = 0 and 1

when, k = 0

$\sqrt{z} = 2\sqrt{2}\left\{\cos\left(\frac{60 + 360k}{2}\right) + i\sin\left(\frac{60 + 360k}{2}\right)\right\}$

$= 2\sqrt{2}\,(\cos 30^o + i\sin 30^o)$

$= 2\sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\,i\right)$

$= \sqrt{6} + \sqrt{2}\,i$

when, k = 1

$\sqrt{z} = 2\sqrt{2}\,\{\cos 210^o + i\sin 210^o\}$

$= 2\sqrt{2}\,(\cos 210^o + i\sin 210^o)$

$= 2\sqrt{2}\left(\frac{-\sqrt{3}}{2} \frac{-1}{2}\,i\right)$

$= -\sqrt{6} - \sqrt{2}i$

$= -(\sqrt{6} + \sqrt{2}i)$

$\therefore\ \sqrt{4 + 4\sqrt{3}\,i} = \pm(\sqrt{6} + \sqrt{2}i)$

c. Let, $z = 0 + 4i$

Here, x = 0, y = 4

$\text{Tan}\theta = \frac{y}{x} = \frac{4}{0} = \infty$

$\therefore\ \theta = 90^o$

$r = \sqrt{0 + (4)^2}$

$= \sqrt{16}$

$= 4$

In polar form, $z = 4\ (\cos 90^o + i\sin 90^o)$

In general polar formi

$z = 4\{\cos(90 + 360.k) + i\sin(90 + 360k)\}$

where, k = 0, 1

$\sqrt{z} = \sqrt{4}\ \left\{\cos\left(\frac{90 + 360.k}{2}\right) + i\sin\left(\frac{90 + 360k}{2}\right)\right\}$

$= 2\left\{\cos\left(\frac{90 + 360k}{2}\right) + i\sin\left(\frac{90 + 360k}{2}\right)\right\}$

where, k = 0

$\sqrt{z} = 2\{\cos 45^o + i\sin 45^o\}$

$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, i\right) = \sqrt{2} + \sqrt{2}\, i$

when k = 1

$\sqrt{z} = 2\{\cos 225^o + i\sin 22.5^o\}$

$= 2\left(\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\, i\right) = \sqrt{2} - \sqrt{2}\, i$

$\therefore\ \sqrt{4i} = \pm\sqrt{2}\ (1 + i)$

d. Let, z = –i

Here, x = 0, y = –1

$\text{Tan}\theta = \frac{y}{x} = \frac{-1}{0} = -\infty$

$\therefore\quad \theta = 270^o$

$r = \sqrt{0 + (-1)^2}$

$= \sqrt{1} = 1$

In polar form;

$z = \cos(270^o + i\sin 270^o$

In general polar form;

$z = \cos(270 + 360.k) + i\sin\ (270 + 360.k)$

Now, k = 0 and 1

$\sqrt{z} = \cos\left(\frac{270 + 360.k}{2}\right) + i\sin\left(\frac{270 + 360.k}{2}\right)$

when k = 0

$\sqrt{z} = \cos 135^o + i\sin 135^o$

$= \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, i$

$= \frac{-1}{\sqrt{2}}\ (1 - i)$

when, k = 1

$\sqrt{z} = \cos 315^o + i\sin 315^o$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\, i$

$= \frac{1}{\sqrt{2}}(1 - i)$

$\therefore \quad \sqrt{-i} = \pm \frac{1}{\sqrt{2}}(1 - i)$

e. Let, z = –1 + 0i

Here, x = –1, y = 0

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{(-1)2 + 0}$

$= \frac{0}{-1} \quad = \sqrt{1}$

$= 0 \quad = 1$

$\therefore \theta = 180^o$

In polar form,

$z = \cos 180^o + i\sin 180^o$

In general polar form;

z = cos(180 + 360k) + isin (180 + 360.k)

where, k = 0 and 1

$\sqrt{z} = \cos\left(\frac{180 + 360.k}{2}\right) + i\sin\left(\frac{180 + 360.k}{2}\right)$

when, k = 0

$\sqrt{z} = \cos 90^o + i\sin 90^o$

$= 0 + i$

when, k = 1

$\sqrt{z} = a\cos 270^o + i\sin 270^o$

$= 0 - 1i$

$\therefore \sqrt{-1} = \pm i$

f. Let, $z = 2 + 2\sqrt{3}\,i$

Here, x = 2,

$y = 2\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{(2)^2 + (2\sqrt{3})^2}$

$= \frac{2\sqrt{3}}{2} \quad = \sqrt{4 + 12}$

$= \sqrt{3} \quad = \sqrt{16} = 4$

$\therefore \theta = 60^o$

In polar form,

$z = 4(\cos 60^o + i\sin 60^o)$

In general polar form;

z = 4{cos(60 + 360.k) + isin (60 + 360.k)}

where, k = 0 and 1

$\sqrt{2} = 2\left\{\cos\left(\frac{60 + 360.k}{2}\right) + i\sin\left(\frac{60 + 360.k}{2}\right)\right\}$

when, k = 0 = $2(\cos 30^o + i\sin 30^o)$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= \sqrt{3} + i$

when, k = 1

$\sqrt{z} = 2(\cos 210^o + i\sin 210^o)$

$= 2\left(\frac{-\sqrt{3}}{2} \frac{-1}{2}i\right)$

$= -\sqrt{3} - 1i$

$\therefore \sqrt{2 + 2\sqrt{3}i} = \pm (\sqrt{3} + i)$

g. Let, $z = 4 - 4\sqrt{3}$

Here, $x = y$, $y = -4\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{(4)^2 + (-4\sqrt{3^2})}$

$= \frac{-4\sqrt{3}}{4}$ $\quad = \sqrt{16 + 48}$

$= -\sqrt{3}$ $\quad = 8$

$\therefore \theta = 300^o$

In polar form,

$z = 8(\cos300^o + i\sin300^o)$

In general form;

$z = 8\{\cos(300 + 360.k) + i\sin(300 + 360.k)\}$

where, k = 0 and 1

$$\sqrt{z} = 2\sqrt{2}\left\{\cos\left(\frac{300 + 360k}{2}\right) + i\sin\left(\frac{300 + 360.k}{2}\right)\right\}$$

when, k = 0

$\sqrt{z} = 2\sqrt{2}\ (\cos150^o + i\sin150^o)$

$= 2\sqrt{2}\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -\sqrt{6} + \sqrt{2}\ i$

when, k = 1

$\sqrt{z} = 2\sqrt{2}\ (\cos330^o + i\sin330^o)$

$= 2\sqrt{2}\left(\frac{\sqrt{3}}{2}\frac{-1}{2}i\right)$

$= \sqrt{6} - \sqrt{2}\ i$

$\therefore \sqrt{4 - 4\sqrt{3}i} = (\sqrt{6} - \sqrt{2}\ i)$

6. Solution:

a. Let, $z = 1 + 0i$

Here, $x = 1$, $y = 0$

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{(1)^2 + 0}$

$= \frac{0}{1}$ $\quad = \sqrt{1}$

$= 0$ $\quad = 1$

$\therefore \theta = 0^o$

In polar form,

$z = \cos0^o + i\sin0^o$

In general polar form;

$z = \cos(360.k + 0^o) + i\sin(0 + 360.k)$

where, k = 0 and 2

$z^{1/3} = \{\cos(0 + 360.k) + i\sin(0 + 360.k)\}^{1/3}$

$= \cos(0 + 120k) + i\sin(0 + 120k)$

when, k = 0

$z^{1/3} = \cos 0 + i\sin0$

$= 1 + 0$

$= 1$

when, k = 1

$z^{1/3} = \cos120 + i\sin 120^o$

$= \frac{-1}{2} + \frac{\sqrt{3}}{2} i$

when, k = 2

$z^{1/3} = \cos 240^o + i\sin 240^o$

$= \frac{-1}{2} \frac{-\sqrt{3}}{2} i$

$\therefore$ Cube roots of 1 = 1, $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right), \left(\frac{-1-\sqrt{3}}{2} i\right)$

b. Let, z = –1 + 0i

Here, x = –1, y = 0

$\tan\theta = \frac{y}{x}$, $r = \sqrt{(-1)^2 + 0}$

$= \frac{0}{-1}$ $= \sqrt{1}$

$= 0$ $= 1$

$\therefore \theta = 180^o$

In polar form, $z = \cos 180^o + i\sin 180^o$

In general polar form;

$z = \cos(180^o + 360.k) + i\sin(180^o + 360.k)$

where, k = 0, 1, 2

$z^{1/3} = \cos\left(\frac{180 + 360.k}{3}\right) + i\sin\left(\frac{180 + 360.k}{3}\right)$

when, k = 0

$z^{1/3} = \cos 60^o + i\sin 60^o$

$= \frac{1}{2} + \frac{\sqrt{3}}{2} i$

when, k = 1,

$z^{1/3} = \cos 180^o + i\sin 180^o$

$= -1 + 0$

$= -1$

when, k = 2

$z^{1/3} = \cos 300^o + i\sin 300^o$

$= \frac{1}{2} - \frac{\sqrt{3}}{2} i$

$\therefore$ Hence, the required cube roots of unity are

$-1, \frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2} i$,

c. Let, z = 0 + 1i

Here, z = 0, y = 1

$\tan\theta = \frac{y}{x} = \frac{1}{0} = \infty$

$\therefore \theta = 90^o$

$r = \sqrt{0 + 1^2} = 1$

In polar form, $z = \cos 90^o + i\sin 90^o$

In general polar form;

$z = \cos(90 + 360.k) + i\sin(90 + 360.k)$

$z^{1/3} = \cos\left(\frac{90 + 360.k}{3}\right) + i\sin\left(\frac{90 + 360.k}{3}\right)$

when k = 0, where, k = 0, 1 and 2

$z^{1/3} = \cos 30^o + i\sin 30^o$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

when k = 1

$z^{1/3} = \cos150^o + i\sin150^o$

$= \frac{-\sqrt{3}}{2} + \frac{1}{2}i$

when k = 2

$z^{1/3} = \cos270^o + i\sin270^o$

$= 0 - 1i$

$= -i$

Hence, the required cube roots of unity are $-i$, $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $\frac{-\sqrt{3}}{2} + \frac{1}{2}i$

d. Let, $z = 0 - i$

Here, $x = 0$, $y = -1$

$\text{Tan}\theta = \frac{y}{x} = \frac{-1}{0} = \infty$

$\therefore \theta = 270^o$

$r = \sqrt{0 + (-1)^2}$

$= \sqrt{1}$

$= 1$

In polar form, $z = \cos270^o + i\sin270^o$

In general polar form;

$z = \cos(270 + 360.k) + i\sin(270 + 360.k)$

$z^{1/3} = \cos\left(\frac{270 + 360.k}{3}\right) + i\sin\left(\frac{270 + 360.k}{3}\right)$

where, k = 0, 1 and 2

when, k = 0

$z^{1/3} = \cos 90^o + i\sin90^o$

$= 0 + 1i$

$= i$

when k = 1,

$z^{1/3} = \cos210^o + i\sin210$

$= \frac{-\sqrt{3}}{2} - \frac{1}{2}i$

$= \frac{-\sqrt{3} - 1i}{2}$

when k = 2

$z^{1/3} = \cos.330^o + i\sin330^o$

$= \frac{\sqrt{3}}{2} - \frac{1i}{2}$

$= \frac{\sqrt{3} - 1i}{2}$

Hence, the required cube roots of i are i, $\frac{-\sqrt{3} - 1i}{2}$ and $\frac{\sqrt{3} - 1i}{2}$

e. Let, $z = 1 + 0i$

Here, $x = 1$, $y = 0$

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{1 + 0}$

$= \frac{0}{1} \quad = \sqrt{1}$

$= 0 \quad = 1$

$\therefore \theta = 0^o$

In polar form;

In general polar form, z = cos(0 + 360.k) + isin (0 + 360k)

$z^{1/4} = \cos\left(\frac{0 + 360.k}{4}\right) + i\sin\left(\frac{0 + 360.k}{4}\right)$

when k = 0, where, k = 0, 1, 2 and 3

$z^{1/4} = \cos0^o + i\sin0^o$

$= 1 + 0$

$= 1$

when k = 1

$z^{1/4} = \cos90^o + i\sin90^o$

$= 0 + 1i$

$= i$

when k = 2

$z^{1/4} = \cos180^o + i\sin180^o$

$= -1 + 0$

$= -1$

when k = 3

$z^{1/4} = \cos270^o + i\sin270^o$

$= 0 - 1i$

$= -i$

Hence, the required forth roots of unity are ±1 and 1i

f. Let, $z = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$

Here, $x = \frac{-1}{2}$, $y = \frac{\sqrt{3}}{2}$

$\tan\theta = \frac{y}{x}$, $r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}}$ $= \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= \frac{\sqrt{3}}{2} \times \frac{2}{-1}$ $= \sqrt{\frac{4}{4}}$

$= -\sqrt{3}$ $= 1$

$\therefore \theta = 120^o$

In polar form,

$z = \cos120^o + i\sin120^o$

In general polar form;

$z = \cos(120 + 360.k) + i\sin(120^o + 360.k)$

$z^{1/4} = \cos\left(\frac{120 + 360.k}{4}\right) + i\sin\left(\frac{120 + 360.k}{4}\right)$

where, k = 0, 1, 2, 3

when k = 0

$z^{1/4} = \cos30^o + i\sin30^o$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

when k = 1

$z^{1/4} = \cos120^o + i\sin120^o$

$= \frac{-1}{2} + \frac{\sqrt{3}}{2}i$

$= \dfrac{-1+\sqrt{3}i}{2}$

when k = 2

$z^{1/4} = \cos 210^o + i\sin 210^o$

$= \dfrac{-\sqrt{3}}{2} - \dfrac{1}{2}i$

$= \dfrac{-\sqrt{3} - 1i}{2}$

when k = 3

$z^{1/4} = \cos 300^o + i\sin 300^o$

$= \dfrac{1}{2} \dfrac{-\sqrt{3}i}{2}$

$= \dfrac{1-\sqrt{3}i}{2}$

Hence, the required fourth roots of unity are $\pm\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)$ and $\pm\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)$

7. a. Here, $z^3 + 8i = 0$

Let,

$z^3 = -8i$

$z^3 = 0 - 8i$

Here, x = 0, y = –8

$\tan\theta = \dfrac{y}{x}$, $\quad r = \sqrt{x^2 + y^2}$

$= \dfrac{-8}{0} \quad = \sqrt{0 + (-8)^2}$

$= \dfrac{-8}{0} \quad = \sqrt{64}$

$= \infty \quad = 8$

$\therefore \theta = 270^o$

In polar form, $z = 8(\cos 270^o + i\sin 270^o)$

In general polar form;

$z = 8\{\cos(270 + 360.k) + i\sin(270 + 360.k)\}$

$z^{1/3} = 2\left\{\left(\dfrac{270+360.k}{3}\right) + i\sin\left(\dfrac{270+360.k}{3}\right)\right\}$

where, k = 0, 1, 2

when k = 0

$z^{1/3} = \{\cos 90^o + i\sin 90^o\}$

$= 2(0 + 1i)$

$= 2i$

when k = 1

$z^{1/3} = 2(\cos 210^o + i\sin 210^o)$

$= 2\left(\dfrac{-\sqrt{3}}{2} - \dfrac{1}{2}i\right) = -\sqrt{3} - 1i$

when k = 2

$z^{1/3} = 2(\cos 330^o + i\sin 330^o)$

$= 2\left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)$

$= \sqrt{3} - 1i$

Hence, the required cube roots of –8i are 2i, $\sqrt{3} - 1i$ and $-(\sqrt{3} + 1i)$

b. Let, $z^4 = -1$

$= -1 + 0i$

Here, $x = -1$, $y = 0$

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{(-1)^2 + 0}$

$= \frac{0}{-1}$ $\quad = \sqrt{1}$

$= 0$ $\quad = 1$

$\therefore \theta = 180^o$

In polar form;

$z = \cos(180^o + i\sin180^o)$

In general polar form;

$z = \cos(180 + 360.k) + i\sin(180 + 360.k)$

$z^{1/4} = \cos\left(\frac{180 + 360.k}{4}\right) + i\sin\left(\frac{180 + 360.k}{4}\right)$

when $k = 0$

$z^{1/4} = \cos45^o + i\sin 45^o$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

when $k = 1$

$z^{1/4} = \cos135^o + i\sin135^o$

$= \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

when $k = 2$

$z^{1/4} = \cos225^o + i\sin225^o$

$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

when $k = 3$

$z^{1/4} = \cos315^o + i\sin315^o$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

Hence, the required fourth roots of -1 is

$\pm\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$ and $\pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$

c. $z^6 = 1$

We have,

$z^6 = 1 = 1 + i(0) = \cos30^o + i\sin0^o$

$\Rightarrow \quad z^6 = \cos2n\pi + i\sin2n\pi$

$\Rightarrow \quad z = [\cos2n\pi + i\sin2n\pi]^{1/6}$

By De-moivre's theorem

$z = \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$

where $n = 0, 1, 2, 3, 4, 5$

When $n = 0$ then the first root of z is,

$z = \cos0 + i\sin0 = 1 + 0 = 1$

When $n = 1$ then the 2^{nd} root of z is,

$z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1 + i\sqrt{3}}{2}$

When $n = 2$ then the 3^{rd} root of z is,

$z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{-1 + i\sqrt{3}}{2}$

When $n = 3$ then the 4^{th} root of z is

$z = \cos\pi + i\sin\pi = -1 + i.0 = -1$

When $n = 4$ then the 5^{th} root of z is,

$z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1 - i\sqrt{3}}{2}$

When n = 5 then the 6^{th} root of z is,

$z = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1 - i\sqrt{3}}{2}$

Hence, the required six roots of z are

$1, -1, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$

8. Solution:

a. Here, $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos n\theta - i\sin n\theta$

LHS

$z^n + \frac{1}{z^n}$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$ proved.

b. Here, $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta - i\sin n\theta$

LHS

$z^n - \frac{1}{z^n}$

$= z^n - z^{-n}$

$= \cos n\theta + i\sin\theta - \cos n\theta + i\sin n\theta$

$= 2i\sin n\theta$ RHS

9. Solution:

a. Let, z_1 and z_2 be $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ respectively.

Then,

$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\ r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\therefore \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$ proved.

b. Let z_1 and z_2 be $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ respectively with $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

Now, $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

So, $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$ proved

c. Let, $z = r(\cos\theta + i\sin\theta)$

where, $Ag.\ z = \theta$

Then, $\overline{z} = r(\cos\theta - i\sin\theta)$

$\overline{z} = r\{\cos(2\pi - \theta) + i\sin(2\pi - \theta)\}$

$\therefore \text{Arg}(\overline{z}) = 2\pi - \theta$

$= 2\pi - \text{Arg}(z)$

10. a. $e^{i\pi/2} = \cos^{\pi/2} + i\sin^{\pi/2} = 0 + i(1) = i$

b. $e^{-i\pi/6} = \cos^{\pi/6} - i\sin^{\pi/6} = \frac{\sqrt{3}}{2} - i\frac{1}{2}$

c. $-5e^{-i\pi/3} = -5[\cos^{\pi/3} - i\sin^{\pi/3}] = -5\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = -\frac{5}{2} + i\frac{\sqrt{3}}{2}$

11. Solution:

a. To express the complex form into re^{ix} form firstly, we change into polar form,

Let $3 + 4i = r(\cos\theta + i\sin\theta)$ ... ... (i)

$\Rightarrow$ $r\cos\theta = 3$ and $ir\sin\theta = 4i \Rightarrow r\sin\theta = 4$

Squaring and adding these two

We get,

$r^2 = 25$ $\therefore$ $r = 5$

Also, $\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927$

$\therefore$ The complex number in exponential form is $re^{i\theta}$ i.e. $5e^{0.927i}$

b. 3i

Let $0 + 3i = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $r\cos\theta = 0$ and $r\sin\theta = 3$

$r^2 = 9$

$\therefore$ $r = 3$

And, $\tan\theta = \frac{3}{0} = \infty = \tan\frac{\pi}{2}$

$\therefore$ The complex number in exponential form is $re^{i\theta}$ i.e. $3e^{i\pi/2}$

c. $-2 - 2i$

Let $-2 - 2i = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $r\cos\theta = -2$ and $r\sin\theta = -2$

$r^2 = 4 + 4 \Rightarrow r^2 = 8$

$\therefore$ $r = 2\sqrt{2}$

And, $\tan\theta = \frac{-2}{-2} = 1 = \tan\frac{5\pi}{4}$

$\therefore$ $\theta = 5\frac{\pi}{4}$

$\therefore$ The complex number is exponential form is, $re^{i\theta}$ i.e. $2\sqrt{2}\ e^{i5\pi/4}$

d. $1 + i\sqrt{3}$

Let $1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $r\cos\theta = 1$ and $r\sin\theta = \sqrt{3}$

$r^2 = 4 \Rightarrow r = 2$

And, $\tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$

$\therefore$ The complex number in exponential form is $re^{i\theta}$ i.e. $2e^{i\pi/3}$