Chapter 6
Mathematical Induction

Exercise 6.1

1. Find the n^{th} term and then the sum of the first n terms of each of the following series.

a. $1.3 + 2.4 + 3.5 + \ldots$
b. $1 + 4 + 9 + 16 + \ldots$
c. $1.3 + 3.5 + 5.7 + \ldots$
d. $1.2.3 + 2.3.4 + 3.4.5 + \ldots$
e. $1 + (1 + 2) + (1 + 2 + 3) + \ldots$

Solution:

a. Here,

Now, n^{th} term of given series

t_n = (n^{th} term of 1, 2, 3, ...) × (n^{th} term of 3, 4, 5, ...)

$= [1 + (n - 1). 1] \times [3 + (n-1).1]$

$= n \times (n + 2) = n(n + 2)$

$\therefore\ t_n = n(n + 2)$

Again, the sum of first n terms of the given series

$s_n = \sum t_n = \sum n(n + 2) = \sum(n^2 + 2n)$

$= \sum n^2 + 2\sum n$

$= \dfrac{n(n + 1)(2n + 1)}{6} + \dfrac{2n(n + 1)}{2}$

$= \dfrac{n(n + 1)(2n + 1 + 6)}{6}$

$= \dfrac{n(n + 1)(2n + 7)}{6}$

b. Here,

$1 + 4 + 9 + 16 + \ldots = 1^2 + 2^2 + 3^2 + 4^2 + \ldots$

n^{th} term of given series

$t_n = [a + (n-1)d]^2 = [1 + (n - 1).1]^2 = n^2$

Again, let the sum of n natural number

$s_n = \sum t_n = \sum n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

c. Here,

n^{th} term of given series

t_n = (nth term of 1, 3, 5, ...) × (n^{th} term of 3, 5, 7, ...)

$= [1 + (n - 1).2] + p3 + (n - 1).2]$

$= (2n - 1)(2n + 1) = 4n^2 - 1$

$\therefore\ t_n = 4n^2 - 1$

Again, the sum on of n natural number is

$s_n = \sum t_n = \sum(n^2 - 1) = 4\sum n^2 - \sum 1$

$= \dfrac{4n(n + 1)(2n + 1)}{6} - n$

$= n\left[\dfrac{2n(n + 1)(2n + 1) - 3}{3}\right]$

$= \dfrac{n}{3}[4n^2 + 6n - 1]$

d. Here,

n^{th} term of given series

t_n = (n^{th} term of 1, 2, 3, 4, ...) × (n^{th} term of 2, 3, 4, 5, ...) × (n^{th} term of 3, 4, 5, ...)

$= [1 + (n - 1).1] \times [2 + (n-1).1] \times [3 + (n-1).1]$

$= n(n + 1)(n + 2) = n(n^2 + 2n + n + 2) = n^3 + 3n^2 + 2n$

Again, the sum of first n natural number

$s_n = \sum t_n = \sum(n^3 + 3n^2 + 2n)$

$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n + 1)(2n + 1)}{6} + \frac{2n(n + 1)}{2}$

$= \frac{n^2(n + 1)^2}{4} + \frac{n(n + 1)(2n + 1)}{2} + n(n + 1)$

$= \frac{n^2(n + 1)^2 + 2n(n + 1)(2n + 1) + 4n(n + 1)}{4}$

$= \frac{n}{4}[(n + 1)(n + 2)(n + 3)]$

e. Here

$1 + (1 + 2) + (1 + 2 + 3) + ...$

The n^{th} term is $t_n = 1 + 2 + 3 + ...$

$= \frac{n(n + 1)}{2}$ (sum of As) $= \frac{n^2}{2} + \frac{n}{2}$

Now, sum of n term is

$S_n = \frac{1}{2}(\sum n^2 + \sum n) = \frac{1}{2}\left\{\frac{n(n + 1)(2n + 1)}{6} + \frac{n(n + 1)}{2}\right\} = \frac{n(n + 1)(n + 2)}{6}$

2. Sum to n terms of the following series

a. $(x + a) + (x^2 + 2a) + (x^3 + 3a) + \ldots$

b. $5 + 55 + 555 + \ldots$ to n terms.

c. $0.3 + 0.33 + 0.333 + \ldots$ to n terms.

d. $1 + \frac{4}{3} + \frac{7}{3^2} + \frac{10}{3^3} + \ldots$

e. $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \ldots\ldots..$

f. $1\times n + 2 \times (n - 1) + 3 \times (n - 2) + \ldots.$

g. $1 + 3 + 6 + 10 + \ldots$

h. $3 + 6 + 11 + 18 + \ldots$

Solution:

a. $(x + a) + (x^2 + 2a) + (x^3 + 3a) + ...$

Here,

Let $s_n = (x + a) + (x^2 + 2a) + (x^3 + 3a) + ...$ to n term

$= (x + x^2 + x^3 + ... + x^n) + (a + 2a + 3a + ... na)$

$= \frac{n(x^n - 1)}{x -} + a(1 + 2 + 3 + ...$ th)

$= \frac{x(x^n - 1)}{x - 1} + \frac{a.n(n + 1)}{2}$

b. Let $s_n = 5 + 55 + 555 + ...$ to n

$= 5(1 + 11 + 111 + ...$ to n)

$= \frac{5}{9}(9 + 99 + 999 + ...$ to n)

$= \frac{5}{9}[(10 - 1) + (100 - 1) + (1000 - 1) + ...$ to n]

$= \frac{5}{9}[(10 + 100 + 1000 + ...$ to n) $- (1 + 1 + 1 ...$ to n)]

$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10 - L} - n\right]$

$S_n = \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

c. Here, Let $s_n = 0.3 + 0.33 + 0.333 + ...$ to n

$= \frac{3}{10} + \frac{33}{100} + \frac{333}{1000} + ...$ to n)

$= 3\left(\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + ... \text{ to n}\right)$

$$= \frac{3}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ to } n\right]$$

$$= \frac{1}{3}\left[\frac{(10-1)}{10} + \frac{(100-1)}{100} + \frac{(1000-1)}{1000} + \ldots \text{ to } n\right]$$

$$= \frac{1}{3}\left[(1 + 1 + 1 \ldots n) - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots n\right)\right]$$

$$= \frac{1}{3}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}\right]$$

$$= \frac{1}{3}\left[n - \frac{10}{90}\left(1 - \frac{1}{10^n}\right)\right]$$

$$= \frac{1}{3}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$$

$$= \frac{n}{3} - \frac{1}{27}\left(1 - \frac{1}{10^n}\right)$$

d. $1 + \frac{4}{3} + \frac{7}{3^2} + \frac{10}{3^3} + \ldots$

Solution:

The given series is $1 + \frac{4}{3} + \frac{7}{3^2} + \frac{10}{3^3} + \ldots..$

Let S_n be the sum of the series of first n terms

Then,

$$S_n = 1 + \frac{4}{3} + \frac{7}{3^2} + \frac{10}{3^3} + \ldots.. + \frac{3n-2}{3^{n-1}} \qquad \ldots. (1)$$

$$\text{Also, } \frac{1}{3}S_n = \frac{1}{3} + \frac{4}{3^2} + \frac{7}{3^3} + \ldots.. + \frac{3n-5}{3^{n-1}} + \frac{3n-2}{3^n} \qquad \ldots. (2)$$

Subtracting (2) from (1) we get,

$$\frac{2}{3}S_n = 1 + 1 + \frac{3}{3^2} + \frac{3}{3^3} + \ldots \frac{3}{3^{n-1}} - \frac{3n-2}{3^n}$$

$$= 1 + \left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots.. + \frac{1}{3^{n-2}}\right) - \frac{3n-2}{3^n}$$

$$= 1 + 1\left(\frac{1 - \left(\frac{1}{3}\right)^{n-1}}{1 - \frac{1}{3}}\right) - \frac{3n-2}{3^n}$$

$$= 1 + \frac{3}{2} - \frac{1}{2.3^{n-2}} - \frac{3n-2}{3n}$$

$$= \frac{5}{2} - \frac{1}{2}\cdot\frac{1}{3^{n-2}} - \frac{3n-2}{3^n}$$

$$\text{or, } S_n = \frac{5}{2}\cdot\frac{3}{2} - \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{3^{n-2}} - \frac{3n-2}{3^n}\cdot\frac{3}{2}$$

$$= \frac{15}{4} - \frac{3}{4}\cdot\frac{9}{3^n} - \frac{1}{2}\cdot\frac{9n-6}{3^n}$$

$$= \frac{15}{4} - \left(\frac{27 + 18n - 12}{4.3^n}\right)$$

$$= \frac{15}{4} - \frac{15 + 18n}{4.3^n}$$

e. $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + ...$

Let $s_n = 1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + ... + \frac{n-1}{2^{n-2}} + \frac{n}{2^{n-1}}$

$\frac{1}{2} s_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + ... + \frac{n-1}{2^{n-1}} + \frac{n}{2^n}$

Subtracting these two, we get,

$$\left(1 - \frac{1}{2}\right) s_n = \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{n-1}}\right) - \frac{n}{2^n}$$

$$\Rightarrow \frac{1}{2} s_n = \frac{1.[1 - (1/2)^n]}{1 - \frac{1}{2}} - \frac{n}{2^n}$$

$$\Rightarrow \frac{1}{2} s_n = 2\left(1 - \frac{1}{2^n}\right) - \frac{n}{2^n}$$

$$\Rightarrow s_n = 4\left(1 - \frac{1}{2^n}\right) - \frac{2n}{2^n} = 4 - \frac{4}{2^n} - \frac{n}{2^{n-1}}$$

$$\therefore s_n = 4 - \frac{1}{2^{n-2}} - \frac{n}{2^{n-1}}$$

f. Here, r^{th} term of 1, 2, 3, ... = r and r^{th} term of n, n – 1, n – 2, ...
$= n - (r - 1) = n - r + 1$
So, the r^{th} term of the series is $r(n - r + 1)$
$\therefore t_r = nr - r^2 + r$

So, sum $S_n = \sum_{r=1}^{n} tr$

$$= n\sum r - \sum r^2 + \sum r$$

$$= \frac{n.n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}.\left\{n - \frac{2n+1}{3} + 1\right\}$$

$$= \frac{n(n+1)}{2} . \frac{3n - 2n - 1 + 3}{3}$$

$$= \frac{n(n+1)(n+2)}{6}$$

g. Let t_n be the n^{th} term and S_n the sum of the first n terms of 1 + 3 + 6 + 10 + ...
Then, $S_n = 1 + 3 + 6 + 10 + ... + t_{n-1} + t_n$
Also, $S_n = \quad 1 + 3 + 6 + ... + t_{n-2} + t_{n-1} + t_n$

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Subtraction yields, $0 = 1 + 2 + 3 + ... + (t_n - t_{n-1}) - t_n$
or, $t_n = 1 + 2 + 3 + ...$ to n terms

$$= \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

Hence, $S_n = \frac{1}{2}\sum n^2 + \frac{1}{2}\sum n$

$$= \frac{1}{2}(1^2 + 2^2 + 3^3 + ... + n^2) + \frac{1}{2}(1 + 2 + 3 + ... + n)$$

$$= \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2}$$

$$= \frac{1}{4}n(n+1)\left\{\frac{(2n+1)}{3} + 1\right\}$$

$$= \frac{1}{4}n(n+1)\frac{(2n+1+3)}{3}$$

$$= \frac{n(n + 1)(n + 2)}{6}$$

h. We have,

$3 + 6 + 11 + 18 + ...$

$= (2^2 - 1) + (3^2 - 3) + (4^2 - 5) + (5^2 - 7) + ...$

$= (2^2 + 3^2 + 4^2 + 5^2 + \text{to n terms}) - (1 + 3 + 5 + 7 + ... \text{ to n terms})$

$= (n + 1)^2 - (2n - 1)$

$= n^2 + 2n + 1 - 2n + 1$

$\therefore \quad t_n = n^2 + 2$

Now, $\Sigma t_n = \Sigma n^2 + \Sigma 2$

$$s_n = \frac{n(n + 1)(2n + 1)}{6} + 2n$$

$$= \frac{(n^2 + n)(2n + 1) + 12n}{6} = \frac{2n^3 + n^2 + 2n^2 + n + 12n}{6}$$

$$= \frac{2n^3 + 3n^2 + 13n}{6} = \frac{n(2n^2 + 3n + 13)}{6}$$

Exercise 6.2

1. a. If P(n) is the statement "$n^3 + n$ is divisible by 2", prove that P(1), P(2), P(3) and P(4) are true.

b. If P(n) is the statement "$n^2 + n$ is even", Prove that P(1), P(2), P(3) and P(4) are true.

c. If P(n) is the statement "$n^3 \geq 2^n$" show that P(1) is false and P(2), P(3) are true.

d. Let P(n) denote the statement " $\frac{n(n + 1)}{6}$ is a natural number". Show that P(2) and P(3) are true but P(4) is not true.

Solution:

a. Here, $P(n) = (n^3 + n)$ is divisible by 2 ... (i)

Putting n =1, 2, 3, and 4 in (i) we get,

$P(1) = 1^3 + 1 = 2$

$P(2) = 2^3 + 1 = 9$

$P(3) = 3^3 + 1 = 28$

$P(4) = 4^3 + 1 = 65$

from above, P(n) is false.

b. Here, P(n) = '$n^2 + n$' is even

Put n = 1, 2, 3 and 4

$P(1) = 1^2 + L = 2$

$P(2) = 2^2 + 1 = 5$

$P(3) = 3^2 + 3 = 12$

$P(4) = 4^2 + 3 = 19$

$\therefore$ from above, P(n) is false.

c. Here, $P(n) = n^3 \geq 2^n$

Put $P(1) = 1^3 \geq 2^1 = 1 \geq 2$ which is false.

Put n = 2 and 3

$P(2) = 2^3 \geq 2^2 = 8 \geq 4$

$P(3) = 3^3 \geq 2^3 = 27 \geq 8$

From above P(1) is false and P(2) and P(3) is true.

d. Here,

$P(n) : \frac{n(n + 1)}{6}$ is natural number

Putting n = 28384

$\therefore P(2) = \frac{2(n + 1)}{6} = \frac{2\times3}{6} = 1$ true

$\therefore P(4) = \frac{4(4 + 1)}{6} = \frac{4\times6}{5} = \frac{10}{3}$ is false.

$\therefore$ P(3) = $\frac{3(3+1)}{6} = \frac{3\times4}{6}$ = 2 true

Hence, from above, P(n) is natural number.

2. Prove by the method of induction that

a. $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$ b. $1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

c. $4 + 8 + 12 + \ldots + 4n = 2n(n + 1)$ d. $1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n-1)}{2}$

e. $1.2 + 2.3 + 3.4 + \ldots$ to n terms = $\frac{n(n+1)(n+2)}{3}$

Solution:

a. If P(n) denotes the given statement, then;

$P(n) = 2 + 5 + 8 + ... + (3n - 1) = \frac{n(3n+1)}{2}$

When n = 1 then (HS : P(2) = 2

RHS : $\frac{1(3\times1+1)}{2} = 2$

$\therefore$ LHS = RSH i.e. P(1) is true.

Suppose that P(n) is true for some $n = k\in N$

Then $P(k) = 2 + 5 + 8 + ... (3k - 1) = \frac{k(3k+1)}{2}$ ... (i)

Here, we shall prove that P(k + 1) is true.

Whenever P(k) is true.

For this, adding 3(k + 1) – 1 = 3k + 2 on both sides of (i), we get

$$2 + 5 + 8 + ... + (3k - 1) + (3k + 2) = \frac{k(3k+1)}{2} + 3k + 2$$
$$= \frac{3k^2 + k + 6k + 4}{2}$$
$$= \frac{3k^2 + 7k + 4}{2} = \frac{3k^2 + 3k + 4k + 4}{2}$$
$$= \frac{(3k+4)(k+1)}{2}$$
$$= \frac{(k+1)[3(k+1)+1]}{2}$$

This shows that P(k+1) is true whenever P(k) is true. Hence by the principle of mathematical inclusion, P(n) is true for all $n\in N$.

b. Here, suppose P(n) denotes the given st.

Then, $P(n) = 1^2 + 3^2 + 5^2 + ... + (2n - 1)2 = \frac{n(2n-1)(2n+1)}{3}$

When, n = 1, then LHS = P(1) = 1

RHS = $\frac{L(2\times1 - 1)(2\times1 + 1)}{3} = \frac{3}{3} = 1$

Hence, LHS = RHS. This shows that P(n) is true for n = 1. So suppose P(n) is true for $n = k\in N$. so that

$P(k) = 1^2 + 3^2 + 5^2 + ... + (2k - 1)^2 = \frac{k(2k-1)(2k+1)}{3}$

Here, we shall prove that the statement P(k+1) is true whenever P(k) is true. For this, adding $(2k+1)^2$ on both sides of (1), we get

$$1^2 + 3^2 + 5^2 + ... + (2k-1)^2 + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{2} + (2k+1)^2$$
$$= \frac{(2k+1)(2k^2+5k+3)}{3}$$
$$= \frac{(2k+1)(2k+3)(k+1)}{3}$$

$$= \frac{(k + 1)(2k + 1)(2k + 3)}{3}$$

$$= \frac{(k + 1)[2(k + 1) - 1][2(k + 1) + 1]}{3}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

c. Suppose P(n) denotes the given st.

$P(n) = 4 + 8 + 12 + ... + 4n = 2n(n + 1)$

When n = 1 LHS: P(1) = 4 and RHS: $P(1) = 2 \times 1(1 + 1) = 4$

This shows that P(n) is true for n = 1, so suppose P(n) is true for some integer $n = k \in N$, then

$P(k) = 4 + 8 + 12 + ... + 4k = 2k(k + 1)$ ... ... (i)

Here, we shall show that P(k+1) is true whenever P(k) is true.

For this adding 4(k + 1) on both sides of (i), we get,

$$4 + 8 + 12 + ... + 4k + 4k(k + 1) = 2k(k + 1) + 4(k + 1)$$

$$= 2(k + 1)[k + 2]$$

$$= 2(k + 1)[(k + 1) + 1]$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

d. Here,

Suppose P(n) denotes the given st.

$P(n) = 1 + 4 + 7 + ... + (3n - 2) = \frac{n(3n - 1)}{2}$

When n=1, LHS : $P(1) = 3 \times 1 - 2 = 1$

RHS: $P(1) = \frac{1(3 \times 1 - 1)}{2} = 1$

$\therefore$ LHS = RHS

This shows that P(n) is true for n = L, so suppose P(n) is true for some integer $n = k \in N$, then

$P(k) = 1 + 4 + 7 + ... (3k - 2) = \frac{k(3k - 1)}{2}$ ... ... (i)

Here, we shall show that P(k + 1) is true whenever P(k) is true

$$1 + 4 + 7 + ... + (3k - 2) + (3k + 1) = \frac{k(3k - 1)}{2} + (3k + 1)$$

$$= \frac{k(3k - 1) + 2(3k + 1)}{2}$$

$$= \frac{3k^2 - k + 6k + 2}{2}$$

$$= \frac{3k^2 + 5k + 2}{2}$$

$$= \frac{3k^2 + 3k + 2k + 2}{2} = \frac{(3k + 2)(k + 1)}{2}$$

$$= \frac{(k + 1)[3(k + 1) - 1]}{2}$$

This shows that P(k+1) is true whenever P(k) is true. Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

e. Here,

Suppose P(n) denotes the given st.

$P(n) = 1.2 + 2.3 + 3.4 + ... + n(n+1) = \frac{n(n + 1)(n + 2)}{3}$

When n = 1, then LHS: P(1) = 1(1 + 1) = 2

RHS: $P(1) = \frac{1(1 + 1)(1 + 2)}{3} = 2$

$\therefore$ LHS = RHS

This shows that P(n) is true for n = 1. So suppose P(n) is true for some integer $n = k \in N$. then,

$P(k) = 1.2 + 2.3 + 3.4 + ... + k(k + 1) = \frac{k(k + i)(k + 2)}{3}$ ... ... (i)

Here, we shall show that P(k + 1) is true whenever P(k) is true for $k \in N$ for this purpose, adding, (k+1) (k+2) on both sides (i) we get

$$1.2 + 2.3 + 3.4 + ... + (k+1)(k+2) + (k+1) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= (k + 1)(k - 12)\left[1 + \frac{k}{3}\right]$$

$$= \frac{(k + 1)(k + 2)(k + 3)}{3}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathematical induction P(n) is true for all $n \in N$.

3. Prove by the method of induction that

a. $\frac{1}{1.3} + \frac{1}{3.5} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

b. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

c. $2 + 2^2 + \ldots + 2^n = 2(2^n - 1)$

d. $3 + 3^2 + \ldots + 3^n = \frac{3(3^n - 1)}{2}$

e. $\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots$ to n terms $= \frac{1}{4}\left(1 - \frac{1}{5^n}\right)$.

Solution:

a. Suppose P(n) denotes the given st. then

$P(n) = \frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

When n = 1, then LHS: $P(1) = \frac{1}{(2\times1 - 1)(2\times1 + 1)} = \frac{1}{3}$

RHS: $P(1) = \frac{1}{3} \Rightarrow$ LHS = RHS

This show that P(n) is true for n=1, so suppose P(n) is true for some integer $n = k \in N$. then

$P(k) = \frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

Here, we shall show that P(k + 1) is true whenever P(k) is true.

For this adding $\frac{1}{(2k+1)(2k+3)}$ on both sides of (i), we get

$$\frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3) + 1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2 + 3k + L}{(2k+1)(2k+3)} = \frac{2k^2 + 2k + k + 1}{4k^2 + 8k + 3} = \frac{(2k+1)(k+1)}{(2k+1)[2(k+1)+1]} = \frac{k+1}{2(k+1)+1}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

b. Here, Suppose P(n) denotes the given st. then

$P(n) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

When n = 1, then LHS: $P(1) = \frac{1}{2'} = \frac{1}{2}$ RHS: $1 - \frac{1}{2'} = \frac{1}{2}$

This shows that P(n) is true for n = 1, so suppose P(n) is true for some integer $n = k \in N$.

Then

$P(k) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^k} = 1 - \frac{1}{2^k}$ ... ... (i)

We shall show that P(k + 1) is true whenever P(k) is true for this adding $\frac{1}{2^{(k+1)}}$ on both side of (i), we get

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$

$$= 1 - \frac{1}{2^k}\left(1 - \frac{1}{2}\right) = 1 - \frac{1}{2^k} : \frac{1}{2} = 1 - \frac{1}{2^{k+1}}$$

This show that P(k+1) is true whenever P(k) is true. Hence by the principle of mathematical induction P(n) is true for all $n \in k$.

c. Here, Suppose P(n) denotes the given st. then

$P(n) = 2 + 2^n + 2^3 + ... + 2^n = 2(2^n - 1)$

When, n = 1, then LHS = P(1) = 2 and RHS = 2

$\therefore$ LHS = RHS. This shows that P(n) is true for n = 1. So suppose P(n) is true for some integer $n=k \in N$. Then,

$P(k) = 2 + 2^2 + 2^3 + ... + 2k = 2(2k - 1)$ ... ... (i)

Here, we shall prove that P(k + 1) is true whenever P(k) is true.

For this, adding 2^{k+1} on both sides of (i), we get

$$\begin{aligned} 2 + 2^2 + 2^3 + ... + 2^k + 2^{k+1} &= 2(2^k - 1) + 2^{k+1} \\ &= 2^k.2 - 2 + 2^k.2 \\ &= 2.2^{k+1} - 2 \\ &= 2(2^{k+1} - 1) \end{aligned}$$

This shows that P(k + 1) is true whenever P(k) is true for all $k \in N$. Hence by the principle of mathematical induction P(n) is true for all $n \in N$.

d. Here, Suppose P(n) denotes the given st. then

$P(n) = 3 + 3^2 + ... 3^n = \frac{3(3^2 - 1)}{2}$

When, n = 1, LHS = 3 and RHS 3

$\therefore$ LHS = RHS. This shows that P(n) is true for n = 1. So, suppose P(n) is true for some integer $n = k \in N$. then

$P(k) = 3 + 3^2 + ... + 3^k = \frac{3(3^k - 1)}{2}$ ... ... (i)

Here, we shall prove that P(k+1) is also true whenever P(k) is true for this adding 3^{k+1} on both side of (i) $3+3^2 + ... 3^k + 3^{k+1} = \frac{3(3^k - 1)}{2} + 3^{k+1}$

$$= \frac{3.3^k - 3 + 2.3^k.3}{2}$$

$$= k.\frac{3.3^{k+1} - 3}{2}$$

$$= \frac{3(3^{k+1} - 1)}{2}$$

This shows that P(k + 1) is true whenever P(k) is true for all $k \in N$. Hence by the principle of mathematical induction P(n) is true for all $n \in N$.

e. Suppose P(n) denotes the given st. then

$P(n) = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ...$ to n terms $= \frac{1}{4}\left(1 - \frac{1}{5^n}\right)$

i.e. to = ar^{n-1}

$= \frac{1}{5}\left(\frac{1}{5}\right)^{n-1} = \frac{1}{5^n}$

When, n = 1, LHS = $\frac{1}{5}$ RHS = $\frac{1}{4}\left(1 - \frac{1}{5'}\right) = \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$

$\therefore$ LHS = RHS this show that P(n) is true for n = 1. So suppose P(n) is true for some integer n = k$\in$N. Then,

$$P(k) = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ... + \frac{1}{5^k} = \frac{1}{4}\left(1 - \frac{1}{5^k}\right)$$

Here, we shall prove that P(k+1) is true whenever P(k) is true. For this adding 5^{k+1} on both sides of (i)

$$\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ... + \frac{1}{5^k} + \frac{1}{5^{k+1}} = \frac{1}{4}\left(1 - \frac{1}{5^k}\right) + \frac{1}{5^{k+1}}$$

$$= \frac{1}{4} - \frac{1}{4.5^k} + \frac{1}{5.5^k}$$

$$= \frac{1}{4} + \frac{1}{5.5^k} - \frac{1}{4.5^k}$$

$$= \frac{1}{4} + \frac{4-5}{4.5.5^k}$$

$$= \frac{1}{4} + \frac{-1}{4.5^{k+1}}$$

$$= \frac{1}{4}\left[1 - \frac{1}{5^{k+1}}\right]$$

This shows that P(k+1) is also true whenever P(k) is true for all k$\in$N. Hence by the principle of mathematical induction P(n) is true for all n$\in$N.

4. Prove by the method of induction that

a. $4^n - 1$ is divisible by 3.
b. $3^{2n} - 1$ is divisible by 8.
c. $10^{2n-1} + 1$ is divisible by 11.
d. $x^n - y^n$ is divisible by x – y.
e. n(n + 1) (n + 2) is a multiple of 6.

Solution:

a. Here, suppose P(n) denotes the given st. then

P(n) = $4^n - 1$ is divisible by 3

When n = 1, P(1) = $4^1 - 1 = 3$ is divisible by 3. So P(1) is true

Let P(k) be true for k$\in$N. That is

P(k) : $4_k - 1$ is divisible by 3 ... ... (i)

Now we shall show that P(k+1) is true when P(k) is true.

P(k+1): $4^{k+1} - 1$ is divisible by 3

Now, $(4^{k+1} - 1)$ is divisible by 3. Therefore P(k+1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all n$\in$N.

$= 1^k . 4 - 4 + 3 = 4(4^k - 1) + 3$

b. Here,

Suppose P(n) be the given st. then P(n): $3^{2n} - 1$ is divisible by 8.

If n = 1. P(1) : $3^2 - 1 = 8$ which is divisible by 8.

So, the statement P(n) is true for n = 1

Let P(k) be true for k$\in$N, that is

P(k) = $3^{2k} - 1$ is divisible by 8 ... ... (i)

Now, we shall show that P(k+1) is true when P(k) is true i.e. P(k+1): $3^{2(k+1)} - 1$

$= 3^{2k+2} - 1$

$= 3^{2k} . 3^2 - 1$

$= 9.3^{2k} - 1 = 9.32k - 9+8$

$= 9(3^{2k} - 1) + 8$ is divisible by 8.

Thus, P(k + 1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all n$\in$N.

c. Here,

Let P(n) be given st. then

P(n) : $10^{2n-1} + 1$ is divisible by 11

When n = 1, P(L): $10^{2-1} + 1 = 11$ which is divisible by 11. So P(1) is true.

Let P(k) be true for K$\in$N. That is

$P(k) : 10^{2k-1} + 1$ ... ... (i)

We shall show that P(k+1) is true when P(k) is true i.e. P(k+1): $10^{2(k+1)-1} + 1$

$= 10^{2k+1} + 1 = 10^{2k-1} . 10^2 + 1 = (10^{2k-1} + 1 - 1)10^2 + 1 = 100(10^{2k-1} + 1) - 99$ which is divisible by 11.

d. Here, let P(n) be given st.

i.e. P(n): $x^n - y^n$ is divisible by x–y

When n = 1 P(1): x – y is divisible y x – y. So P(1) is true.

Let P(k) be true for $k \in N$. i.e.

P(k) : $x^k - y^k$ is divisible by x–y ... ... (i)

Now, we shall show that P(k+1) is true when P(k) is true i.e. P(k+1): $x^{k+1} - y^{k+1}$

$= x(x^k - y^k) + y(x^k - y^k) - xy(x^{k-1} - y^{k-1})$

$= (x + y) (x^k - y^k) - xy (x^{k-1} - y^{k-1})$ is divisible by x – y.

Therefore, P(k+1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all $n \in N$.

e. Here,

Let P(n) be given st. then

P(n): n(n+1) (n+2) is multiple of 6.

When n=1, P(1): 1(1+1) (1+2) = 6 is multiple of 6. So P(1) is true

Let P(k) is true for $k \in N$. i.e.

$\therefore$ P(k): k(k+1) (k+2) is multiple of 6 ... ... (i)

Now, we shall show that P(k+1) is true when P(k) is true i.e. P(k+1) : (k+1) (k+2) (k+3) i.

= k(k+1) (k+2) +3(k+1) (k+2) is multiple of 6.

Therefore, P(k+1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all $n \in N$.