Chapter 11
Co-ordinate in Space

Exercise 11.1

1.

a. A(–2, 1, 0) and B(3, 5, –2)

Here,

$x_1 = -2$ $\qquad$ $x_2 = 3$

$y_1 = 1$ $\qquad$ $y_2 = 5$

$z_1 = 0$ $\qquad$ $z_2 = -2$

Using distance formulae,

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(3 - (-2))^2 + (5 - 1)^2 + (-2 - 0)^2}$$
$$= \sqrt{5^2 + 4^2 + (-2)^2}$$
$$= \sqrt{25 + 16 + 4}$$
$$= \sqrt{45}$$
$$= 3\sqrt{5} \text{ units}$$

b. P(–4, 7, –7) and Q(–2, 1, –10)

Here,

$x_1 = -4$ $\qquad$ $x_2 = -2$

$y_1 = 7$ $\qquad$ $y_2 = 1$

$z_1 = -7$ $\qquad$ $z_2 = -10$

Using distance formula,

$$PQ = \sqrt{(x_2 - x_1)_2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(-2 - (-4))^2 + (1 - 7)^2 + (-10 - (-7))^2}$$
$$= \sqrt{(-2 + 4)^2 + (-6)^2 + (-10 + 7)^2}$$
$$= \sqrt{2^2 + (-6)^2 + (-3)^2}$$
$$= \sqrt{4 + 36 + 9}$$
$$= \sqrt{49}$$
$$= 7 \text{ units}$$

2. Show that the following points are collinear.

a. A(3, –2, 4), B(1, 1, 1) and C(–1, 4, –2)

Solution:

Using distance formula

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(1 - 3)^2 + (1 + 2)^2 + (1 - 4)^2}$$
$$= \sqrt{(-2)^2 + 3^2 + (-3)^2}$$
$$= \sqrt{4 + 9 + 9}$$
$$= \sqrt{22} \text{ units}$$

Again,

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(-1 - 1)^2 + (4 - 1)^2 + (-2 - 1)^2}$$
$$= \sqrt{(-2)^2 + (3)^2 + (-3)^2}$$
$$= \sqrt{4 + 9 + 9}$$
$$= \sqrt{22} \text{ units}$$

Finally,

AC $= \sqrt{(-1-3)^2 + (4+2)^2 + (-2-4)^2}$

$= \sqrt{(-4)^2 + (6)^2 + (-6)^2}$

$= \sqrt{16 + 36 + 36}$

$= \sqrt{88}$

$= 2\sqrt{22}$ units

Now,

AB + BC = $2\sqrt{22}$

Since, AB + BC = AC, the given points are collinear.

b. P(1, –2, 3), Q(2, 3, –4) and R(0, –7, 10)

Solution:

Using distance formula,

PQ $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(2-1)^2 + (3+2)^2 + (-4-3)^2}$

$= \sqrt{1^2 + 5^2 + (-7)^2}$

$= \sqrt{1 + 25 + 49}$

$= \sqrt{75}$

$= 5\sqrt{3}$ units

Again,

QR $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(0-2)^2 + (-7-3)^2 + (10+4)^2}$

$= \sqrt{(-2)^2 + (-10)^2 + (14)^2}$

$= \sqrt{4 + 100 + 196}$

$= \sqrt{300}$

$= 10\sqrt{3}$ units

Finally,

PR $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(0-1)^2 + (-7+2)^2 + (10-3)^2}$

$= \sqrt{(-1)^2 + (-5)^2 + 7^2}$

$= \sqrt{1 + 25 + 49}$

$= \sqrt{75}$

$= 5\sqrt{3}$ units

Now, PQ + PR = $10\sqrt{3}$

Since PQ + PR = QR, the given points are collinear.

c. x(1, 2, 3) y(4, 0, 4) and z(–2, 4, 2)

Solution:

Using distance formula,

xy $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(4-1)^2 + (0-2)^2 + (4-3)^2}$

$= \sqrt{(4-1)^2 + (0-2)^2 + (4-3)^2}$

$= \sqrt{3^2 + (-2)^2 + 1^2}$

$= \sqrt{9 + 4 + 1}$

$= \sqrt{14}$ units

Again,

$$yz = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 (z_2 - z_1)^2}$$
$$= \sqrt{(-2-4)^2 + (4-0)^2 + (z-4)^2}$$
$$= \sqrt{(-6)^2 + 4^2 + (-2)^2}$$
$$= \sqrt{36 + 16 + 4}$$
$$= \sqrt{56}$$
$$= 2\sqrt{14} \text{ units}$$

Finally,

$$xz = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 (z_2 - z_1)^2}$$
$$= \sqrt{(-2-1)^2 + (4-2)^2 + (2-3)^2}$$
$$= \sqrt{(-3)^2 + 2^2 + (-1)^2}$$
$$= \sqrt{9 + 4 + 1}$$
$$= \sqrt{14} \text{ units}$$

Since, yz = xz + xy

So, the points are collinear.

3. Find the mid-points

a. (–2, 6, –4) and (4, 0, 8)

Here,

$x_1 = -2$ $x_2 = 4$

$y_1 = 6$ $y_2 = 0$

$z_1 = -4$ $z_2 = 8$

Now,

Using mid-point formula,

$$M(x, y, z) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$
$$= \left(\frac{-2+4}{2}, \frac{6+0}{2}, \frac{-4+8}{2}\right)$$
$$= \left(\frac{2}{2}, \frac{6}{2}, \frac{4}{2}\right)$$

∴ Mid–points = (1, 3, 2)

b. (–1, –2, –1) and (4, 7, 6)

Here,

$x_1 = -1$ $x_2 = 4$

$y_1 = -2$ $y_2 = 7$

$z_1 = -1$ $z_2 = 6$

Now,

Using mid-point formula,

$$M(x, y, z) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$
$$= \left(\frac{-1+4}{2}, \frac{-2+7}{2}, \frac{-1+6}{2}\right)$$
$$= \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$$

$$\therefore \text{ Mid–point} = \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$$

4. Here,

Let the points of line be A(3, 3, 1) and B(3, –6, 4). And the ratio that divides the line is 2:1. So,

$x_1 = 3$ $\quad$ $x_2 = 3$

$y_1 = 3$ $\quad$ $y_2 = -6$

$z_1 = 1$ $\quad$ $z_2 = 4$

Also,

m : n = 2 : 1

Using section formula we get,

$$P(x, y, z) = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

$$= \left(\frac{2\times3 + 1\times3}{2 + 1}, \frac{2\times(-6) + 1 \times 3}{2 + 1}, \frac{2\times4 + 1\times1}{2 + 1}\right)$$

$$= \left(\frac{9}{3}, \frac{-9}{3}, \frac{9}{3}\right)$$

$\therefore$ P(x, y, z) = (3, – 3, 3)

5. Here,

Let the point of the line be M(3, 4, –5) and N(1, 3, –2) and the ratio that divides the time is 5 : 4. So,

$x_1 = 3$ $\quad$ $x_2 = 1$

$y_1 = 4$ $\quad$ $y_2 = 3$ $\quad$ m : n = 5 : 4

$z_1 = -5$ $\quad$ $z_2 = -2$

Using section formula we get

$$P(x, y, z) = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$$

$$= \left(\frac{5 \times 1 - 4 \times 3}{5 - 4}, \frac{5 \times 3 - 4 \times 4}{5 - 4}, \frac{5 \times (-2) - 4 \times (-5)}{5 - 4}\right)$$

= (5 – 12, 15 – 16, –10 + 20)

$\therefore$ P(x, y, z) = (–7, –1, 10)

6. Here,

The internal section of two points P(2, –4, 3) and Q(x, y, z) in the ratio 2 : 1 is A(–2, 2, –3). So,

$x_1 = 2$ $\quad$ $x_2 = x$ $\quad$ and internal point m : n

$y_1 = -4$ $\quad$ $y_2 = y$ $\quad$ p = –2 = 2 : 1

$z_1 = 3$ $\quad$ $z_2 = z$ $\quad$ q = 2

r = –3

Now,

Using section formula we get,

$$p = \frac{-mx_2 + nx_1}{m + n}$$

or, $-2 = \dfrac{2 \times x + 1 \times 2}{2 + 1}$

or, $-2 \times 3 = 2x + 2$

or, $-6 = 2x + 2$

or, $2x = -8$

$\therefore$ $x = -4$

Similarly,

$$q = \frac{my_2 + ny_1}{m + n}$$

or, $2 = \frac{2 \times y + 1 \times (-4)}{2 + 1}$

or, $6 = 2y - 4$

or, $2y = 6 + 4$

or, $y = \frac{10}{2}$

$\therefore$ $y = 5$

Again,

$$r = \frac{mz_2 + nz_1}{m + n}$$

or, $-3 = \frac{2 \times z + 1 \times 3}{2 + 1}$

or, $-3 \times 3 = 2z + 3$

or, $-9 = 2z + 3$

or, $2z = -9 - 3$

or, $z = \frac{-12}{2}$

$\therefore$ $z = -6$

Hence, $Q(x, y, z) = (-4, 5, -6)$

7. Solution:

Given, $A(x_1, y_1, z_1) = (3, 2, -4)$

$B(x, y, z) = (5, 4, -6)$

and $C(x_2, y_2, z_2) = (9, 8, -10)$ be three collinear points.

So, let B divide AC in the ratio x : 1

So,

$$x = \frac{kx_2 + x_1}{k + 1}$$

or, $5 = \frac{k \times 9 + 3}{k + 1}$

or, $5k + 5 = 9k + 3$

or, $5k - 9k = 3 - 5$

or, $-4k = -2$

$\therefore$ $k = \frac{1}{2}$ i.e. $k : 1 = 1 : 2$

$\therefore$ B divides AC in the ratio of 1 : 2

b. Solution:

Let, xy – plane divides the line joining the points (–2, 4, 7) and (3, –5, –8) in the ratio of k : 1.

At the xy – plane, z = 0

Now,

Using the section formula,

$$z = \frac{mz_2 + nz_1}{m + n}$$

or, $z = \frac{kz_2 + z_1}{k + 1}$

or, $0 = \frac{k(-8) + 7}{k + 1}$

or, $k8 + 7 = 0$

$\therefore$ $k = \frac{7}{8}$ or, $k : 1 = 7 : 8$

$\therefore$ Required ratio is 7:8

c. **Solution:**

Let xz plane divides the line jointing the points A(1, 2, 3) and B(4, –4, 9) in the ratio k : 1.

At the xz – plane, y = 0

Now,

Using the section formula,

$$y = \frac{my_2 + ny_1}{m + n}$$

or, $y = \frac{k\,x_2 + y_1}{k + 1}$

or, $0 = \frac{k(-4) + 2}{k + 1}$

or, $-4k + 2 = 0$

or, $4k = 2$

$\therefore$ $k = \frac{1}{2}$ or, k : 1 = 1 : 2

$\therefore$ Required ratio is 1:2

Then using $x = \frac{mx_2 + nx_1}{m + n} = 2$ and $z = \frac{mz_2 + nz_1}{m + n} = 5$

$\therefore$ The required point is (2, 0, 5).

8. **Solution**

a. Let P(x, y, z) be any point on the locus. Let

Let A(1, 2, 1) and B(3, –4, 2) be two points.

By the given condition

PA = PB

or, $PA^2 = PB^2$

or, $(x - 1)^2 + (y - 2)^2 + (2 - 1)^2 = (x - 3)^2 + (y + 4)^2 + (z - 2)^2$

or, $x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 2z + 1 = x^2 - 6x + 9 + y^2 + 8y + 16 + z^2 - 4z + 4$

or, $x^2 - x^2 + y^2 - y^2 + z^2 - z^2 - 2x + 6x - 4y - 8y - 2z + 4z + 1 + 4 + 1 - 9 - 16 - 4 = 0$

or, $4x - 12y + 2z - 23 = 0$ is the required equation of locus.

b. **Solution:**

Here, $PA^2 = (x + 1)^2 + (y - 2)^2 + (z - 1)^2$

$= x^2 + y^2 + z^2 + 2x - 4y + 2z + 6$

and $PB^2 = x^2 + (y - 3)^2 + (z + 2)^2$

$= x^2 + y^2 + z^2 - 6y + 4z + 113$

Since $PA^2 + PB^2 = 6$

So, we have

$x^2 + y^2 + z^2 + 2x - 4y + 2z + 6 + x^2 + y^2 + z^2 - 6y + 4z + 13 = 6$

$\Rightarrow$ $2x^2 + 2y^2 + 2z^2 + 2x - 10y + 6z + 13 = 0$

which is the required equation of the locus of a point.

9. **Solution**

Let (2, –3, 1) and (3, –4, 5) divides the plane 2x + y + z = 7 in the ratio k : 1.

So,

$$(x, y, z) = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

$$= \left(\frac{k.3 + 2}{k + 1}, \frac{-4k - 3}{k + 1}, \frac{-5k + 1}{k + 1}\right) \quad \ldots \text{(i)}$$

Also, (x, y, z) satisfies the plane 2x + y + z = 7

So, $2\left(\frac{3k + 2}{k + 1}\right) + \left(\frac{-4k - 3}{k + 1}\right) + \left(\frac{-5k + 1}{k + 1}\right) = 7$

or, $6k + 4 - 4k - 3 - 5k + 1 = 7k + 7$
or, $2k - 5k - 7k = 7 - 4 + 3 - 1$
$\therefore$ $k = \frac{-1}{2}$

Put the value of $k = \frac{-1}{2}$ in (i), we get the required point $(x, y, z) = (1, -2, 7)$.

10. Let P(x, y, z) be the centre of the sphere passing through the points A(3, 2, 2), B(–1, 1, 3), C(2, 1, 2) and D(1, 0, 4).

Here,
According to the question,
$P(x, y, z) = (1, 3, 4)$
Now,
Using distance formula,
$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ we have,
or, $AP^2 = (x - 3)^2 + (y - 2)^2 + (z - 2)^2$
$= x^2 - 6x + 9 + y^2 - 4y + 4 + z^2 - 4z + 4$ .... (i)
$BP^2 = (x + 1)^2 + (y - 1)^2 + (z - 3)^2$
$= x^2 + 2x + 1 + y^2 - 2y + 1 + z^2 - 6z + 9$ .... (ii)
$CP^2 = (x - 2)^2 + (y - 1)^2 + (z - 2)^2$
$= x^2 - 4x + 4 + y^2 - 2y + 1 + z^2 - 4z + 4$ .... (iii)
$DP^2 = (x - 1)^2 + (y - 0)^2 + (z - 4)^2$
$= x^2 - 2x + 1 + y^2 + z^2 - 8z + 16$ .... (iv)
$\therefore$ $AP = BP$
$\Rightarrow$ $AP^2 = BP^2$
$\Rightarrow$ $x^2 + y^2 + z^2 - 6x - 4y - 4z + 17 = x^2 + y^2 + z^2 + 2x - 2y - 6z + 11$
$\Rightarrow$ $2x + 6x - 2y + 4y - 6z + 4z = 17 - 11$
$\Rightarrow$ $8x + 2y - 2z = 6$
$\Rightarrow$ $4x + y - z = 3$
Similarly making (i) equal to (iii) and (i) equal to (iv)
We have,
$x^2 - 6x + 17 + y^2 - 4y + z2 - 4z = x^2 + y^2 + z^2 - 4x - 2y - 4z + 9$
or, $-2x - 2y = -8$
or, $x + y = 4$ .... (B)
Also, solving equation (iii) and (iv) we get
$-x - y + 2z = 4$ .... (C)
Hence, solving equation (A), (B) and (C) we get
$P(x, y, z) = (1, 3, 4)$
Hence proved.
Also, radius (r) = AP
$= \sqrt{x^2 + y^2 + z^2 - 6x - 4y - 4z + 17}$
$= \sqrt{1 + 9 + 16 - 6 - 12 - 16 + 17}$
$= \sqrt{9}$
= 3 units

11. Solution:

Let A(0, 7, 10), B(–1, 6, 6) and C(–4, 9, 6) be three points.
For AB
$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z^2 - z_1)^2$
$= (-1 - 0)^2 + (6 - 7)^2 + (6 - 70)^2$
$= 1 + 1 + 16$
$= 18$

AB $= \sqrt{18} = 3\sqrt{2}$ units

For BC

$BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$= (-4 + 1)2 + (9 - 6)2 + (6 - 6)2$

$= 9 + 9 + 0$

$= 18$

BC $= \sqrt{18} = 3\sqrt{2}$ units

For AC

$CA^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$= (0 + 4)2 + (7 - 9)^2 + (10 - 6)^2$

$= 16 + 4 + 16$

$= 36$

CA $= \sqrt{36}$

$= 6$ units

Now,

By Pythagoras theorem

We have,

$CA^2 = AB^2 + BC^2$

Also AB = AC

$\therefore$ A, B and C are the vertices of right angled isosceles triangle.

11. Solution:

b. Let A(2, 0, –4) B(4, 2, 4) and C(10, 2, –2) be three points.

$AB^2 = 14 - 2)^2 + (2 - 0)^2 + (4 + 4)^2 = 4 + 4 + 64 = 72$

Again,

$BC^2 = (10 - 4)^2 + (2 - 2)^2 + (-2 - 4)^2 = 36 + 0 + 36 = 72$

Similarly,

$CA^2 = (2 - 10)^2 + (0 - 2)^2 + (-4 + 2)^2 = 64 + 4 + 4 = 72$

So, AB = BC = CA

$\therefore$ A, B and C are the vertices of an equilateral triangle.

12. Solution

a. Let A(1, 2, 3), B(–1, –2, –1), C(2, 3, 2)and D(4, 7, 6) be four points

$AB^2 = (-1 - 1)^2 + (-2 - 2)^2 + (-1 - 3)^2$

$= 36$

$\therefore$ AB $= 6$

$BC^2 = (2 + 1)^2 + (3 + 2)^2 + (2 + 1)^2$

$= 43$

$\therefore$ BC $= \sqrt{43}$

$CD^2 = (4 - 2)^2 + (7 - 3)^2 + (6 - 2)^2$

$= 36$

$\therefore$ CD $= 6$

$DA^2 = (1 - 4)^2 + (2 - 7)^2 + (3 - 6)^2$

$= 43$

$\therefore$ DA $= \sqrt{43}$

Hence, AB = CD and BC = DA so, A, B, C and D are the vertices of a parallelogram.

Here,

AC $= \sqrt{(2 - 1)^2 + (3 - 2)^2 + (2 - 3)^2}$

$= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ units

Again,

BD $= \sqrt{(4+2)^2 + (7+2)^2 + (6+1)^2}$

$= \sqrt{36 + 81 + 49} = \sqrt{166}$ units

Since, the two diagonals AC and BD are not equal.

$\therefore$ The points A, B, C and D do not represent a rectangle.

b. Solution

Let, $D(\overline{x}, \overline{y}, \overline{z})$ be the point of intersection of the diagonals AC and BD.

For AC : A (–5, 5, 2) and C(–3, –3, 0)

The coordinates of the mid-point AC

$$= \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2}\right) = \left(\frac{-3-5}{2}, \frac{5-3}{2}, \frac{2+0}{2}\right) = (-4, 1, 1)$$

For BD : B(–9, –1, 2) and $D(\overline{x}, \overline{y}, \overline{z})$

Since mid point of AC = midpoint of BD

$$x = \frac{x_1 + x_2}{2}$$

or, $-4 = \frac{-9 + x}{2}$ or, $-8 = -9 + \overline{x}$ $\therefore \overline{x} = -8 + 9 = 1$

$$y = \frac{y_1 + y_2}{2}$$

or, $1 = \frac{-1 + \overline{y}}{2}$ or, $2 = -1 + \overline{y}$ $\therefore \overline{y} = 2 + 1 = 3$

$$z = \frac{z_1 + z_2}{2}$$

or, $1 = \frac{2 + z}{2}$ or, $2 = 2 + \overline{z}$ $\therefore \overline{z} = 2 - 2 = 0$

$\therefore$ The coordinates of $D = (\overline{x}, \overline{y}, \overline{z}) = (1, 3, 0)$

13. Solution:

Let $(x_1, y_1, z_1) = (2, 6, -4)$

$(x_2, y_2, z_2) = (15, -10, 16)$

$(x, y, z) = (7, -2, 5)$ and $(x_3, y_3, z_3) = ?$

By the centroid formula,

$$x = \frac{x_1 + x_2 + x_3}{3}$$

or, $7 = \frac{2 + 15 + x_3}{3}$

or, $21 = 17 + x_3$

$\therefore$ $x_3 = 21 - 17 = 4$

$$y = \frac{y_1 + y_2 + y_3}{3}$$

or, $-6 = -4 + y_3$

$\therefore$ $y_3 = -6 + 4 = -2$

$$z = \frac{z_1 + z_2 + z_3}{3}$$

or, $5 = \frac{-4 + 16 + z_3}{3}$

or, $15 = 12 + z_3$

or, $z_3 = 15 - 12$

$\therefore$ $z_3 = 3$

$\therefore$ $(x_3, y_3, z_3) = (4, -2, 3)$

Exercise 11.2

1. Solution:

$\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{4}$

But $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or, $\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} + \cos^2\gamma = 1$

or, $\frac{1}{2} + \frac{1}{2} + \cos^2\gamma = 1$

or, $1 + \cos^2\gamma = 1$

or, $\cos^2\gamma = 0$

or, $\cos\gamma = 0$

$\therefore$ $\gamma = \cos^{-1}(0)$

$\therefore$ $\gamma = \frac{\pi}{2}$

$\therefore$ The angle is $\frac{\pi}{2}$.

2. Solution:

Let the angle made by a line with 3 axes be α, α, α.

Now, we know

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or, $\cos^2\alpha + \cos^2\alpha \cos^2\alpha = 1$

or, $3\cos^2\alpha = 1$

or, $\cos\alpha = \pm\frac{1}{\sqrt{3}}$

Similarly, for $\cos^2\alpha$ and $\cos^2\alpha$

$\cos\alpha = \pm\frac{1}{\sqrt{3}}$

and $\cos\alpha = \pm\frac{1}{\sqrt{3}}$

$\therefore$ The direction cosines = $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

3. Solution:

Given,f α, β and γ be the angles made by the line with the co-ordinates axis.

$\text{Cos}^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Multiplying by 2 on both sides we get,

$2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma = 2$

or, $1 + \cos^2\alpha + 1 + \cos^2\beta + 1 + \cos^2\gamma = 2$

or, $3 + \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2$

$\therefore$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 1 = 0$

4. Solution:

a. Here, a = 6, b = 2 and c = –3

The direction cosines are

$l = \frac{1}{\sqrt{a^2 + b^2 + c^2}} = \frac{6}{\sqrt{6^2 + 2^2 + (-3)^2}} = \frac{6}{\sqrt{49}} = \frac{6}{7}$

$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{2}{\sqrt{49}} = \frac{2}{7}$

and, $n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{-3}{\sqrt{49}} = \frac{-3}{7}$

$\therefore$ The direction cosines $(l, m, n) = \left(\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}\right)$

b. Solution:

Here, $a = -1$, $b = -2$, and $c = -3$

Now,

The direction cosines are:

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{-1}{\sqrt{(-1)^2 + (-2)^2 + (-3)^2}} \quad = \frac{-2}{\sqrt{(-1)^2 + (-2)^2 + (-3)^2}} \quad = \frac{-3}{\sqrt{(-1)^2 + (-2)^2 + (-3)^2}}$$

$$= \frac{-1}{\sqrt{14}} \quad = \frac{-2}{\sqrt{14}} \quad = \frac{-3}{\sqrt{14}}$$

$\therefore$ The direction cosines are $(l, m, n) = \left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$

5. Solution:

a. $(x_1, y_1, z_1) = (-2, 1, -8)$ and $(x_2, y_2, z_2) = (4, 3, -5)$

$$PQ = r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(4 + 2)^2 + (3 - 1)^2 (-5 + 8)^2}$$
$$= \sqrt{6^2 + 2^2 + 3^2}$$
$$= \sqrt{36 + 4 + 9}$$
$$= \sqrt{49} = 7$$

$\therefore$ The direction cosines of PQ are

$$l = \frac{x_2 - x_1}{r}, \quad m = \frac{y_2 - y_1}{r}, \quad n = \frac{z_2 - z_1}{r}$$

$$= \frac{4 + 2}{7} \quad = \frac{3 - 1}{7} \quad = \frac{-5 + 8}{7}$$

$$= \frac{6}{7} \quad = \frac{2}{7} \quad = \frac{3}{7}$$

$\therefore$ The direction cosines are $(l, m, n) = \left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

b. Solution:

$(x_1, y_1, z_1) = (5, 2, 8)$ and $(x_2, y_2, z_2) = (7, -1, 9)$

$$AB = r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(7 - 5)^2 + (-1 - 2)^2 + (9 - 8)^2}$$
$$= \sqrt{2^2 + (-3)^2 + 1^2}$$
$$= \sqrt{4 + 9 + 1}$$
$$= \sqrt{14}$$

$\therefore$ The direction cosines of AB are:

$$l = \frac{x_2 - x_1}{r}, \quad m = \frac{y_2 - y_1}{r}, \quad n = \frac{z_2 - z_1}{r}$$

$$= \frac{7 - 5}{\sqrt{14}} = \frac{-1 - 3}{\sqrt{14}} = \frac{9 - 8}{\sqrt{14}}$$

$$= \frac{2}{\sqrt{14}} = \frac{-3}{\sqrt{14}} = \frac{1}{\sqrt{14}}$$

$\therefore$ The direction cosines (l, m, n) = $\left(\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$

6. Solution:

a. $a_1 = 1$ $b_1 = 2$, $c_1 = 2$

$a_2 = 2$, $b_2 = 3$, $c_2 = 6$

We have,

$$\text{Cos}\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{1\times2 + 2\times3 + 2\times6}{\sqrt{1^2 + 2^2 + 2^2}\sqrt{2^2 + 3^2 + 6^2}} = \frac{20}{3\times7}$$

or, $\theta = \cos^{-1}\left(\frac{20}{21}\right)$

$\therefore$ $\theta = \cos^{-1}\left(\frac{20}{21}\right)$

b. For AB :

$AB = \sqrt{(1 + 2)^2 + (4 - 1)^2 + (2 - 2)^2}$

$= \sqrt{9 + 9 + 0} = 3\sqrt{2}$ units

and $(l_1, m_1, n_1) = \left(\frac{-2-1}{3\sqrt{2}}, \frac{1-4}{3\sqrt{2}}, \frac{2-2}{3\sqrt{2}}\right) = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

For BC

$BC = \sqrt{(2 + 2)^2 + (-3 - 1)^2 + (4 - 2)^2}$

$= \sqrt{16 + 16 + 4} = \sqrt{36} = 6$ units

and $(l_2, m_2, n_2) = \left(\frac{2+2}{6}, \frac{-3-1}{6}, \frac{4-2}{6}\right) = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

We know,

$\cos B = l, l_2 + m_1m_2 + n_1n_2$

$= \frac{-1}{\sqrt{2}} \times \frac{2}{3} + -\frac{1}{\sqrt{2}} \times \frac{-2}{3} + 0 \times \frac{1}{3}$

$= \frac{-\sqrt{2}}{3} + \frac{\sqrt{2}}{3} + 0 = 0$

or, $\cos B = 0$

$\therefore$ $B = \frac{\pi}{2}$

$\therefore$ $B = \frac{\pi}{2} = 90^o$

Hence, the lines are perpendicular.

For AC

$AC = \sqrt{(2 - 1)^2 + (-3 - 4)^2 + (4 - 2)^2}$

$= \sqrt{1 + 49 + 4}$

$= \sqrt{54} = 3\sqrt{6}$ units.

Again, $(l_2, m_2, n_2) = \frac{2-1}{3\sqrt{6}}, \frac{-3-4}{3\sqrt{6}}, \frac{4-2}{3\sqrt{6}} = \left(\frac{1}{3\sqrt{6}}, \frac{-7}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}\right)$

Similarly,

$\text{Cos } A = l_1 l_2 + m_1m_2 + n_1n_2$

$= \frac{-1}{\sqrt{2}} \times \frac{1}{3\sqrt{6}} + \frac{-1}{\sqrt{2}} \times \frac{-7}{3\sqrt{6}} + 0 \times \frac{2}{3\sqrt{6}}$

$= \frac{-\sqrt{3}}{18} + \frac{7\sqrt{3}}{18} + 0$

$= \frac{1}{\sqrt{3}}$

$\therefore$ CosA $= \frac{1}{\sqrt{3}}$

7. Solution:

a. For the live joining the points (1, 2, 3) and (–1, –2, –3) $(x_1, y_1, z_1) = (1, 2, 3)$ and $(x_2, y_2, z_2) = (-1, -2, -3)$

$a_1 = x_2 - x_1 = -1 - 1 = -2$
$b_1 = y_2 - y_1 = -2 - 2 = -4$
$c_1 = z_2 - z_1 = -3 - 3 = -6$

For the live joining the points (2, 3, 4) and (5, 9, 13) $(x_1, y_1, z_1) = (2, 3, 4)$ and $(x_2, y_2, z_2) = (5, 9, 13)$

$a_2 = x_2 - x_1 = 5 - 2 = 3$
$b_2 = y_2 - y_1 = 9 - 3 = 6$
$c_2 = z_2 - z_1 = 13 - 4 = 9$

Now,

$\frac{a_1}{a_2} = \frac{-2}{3}, \frac{b_1}{b_2} = \frac{-4}{6} = \frac{-2}{3}, \frac{c_1}{c_2} = \frac{-6}{9} = \frac{-2}{3}$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Hence, the two lines are parallel.

7. Solution:

b. For the line joining the points (0, 4, 1) and (2, 6, 2) $(x_1, y_1, z_1) = (0, 4, 1)$ and $(x_2, y_2, z_2) = (2, 6, 2)$

$a_1 = x_2 - x_1 = 2 - 0 = 2$
$b_1 = y_2 - y = 6 - 4 = 2$
$c_1 = 2_2 - 2_1 = 2 - 1 = 1$

For the line joining the points (4, 5, 0) and (2, 6, 2) $(x_1, y_1, z_1) = (4, 5, 0)$ and $(x_2, y_2, z_2) = (2, 6, 2)$

$a_2 = x_2 - x_1 = 2 - 4 = -2$
$b_2 = y_2 - y_1 = 6 - 5 = 1$
$c_2 = z_2 - z_1 = 2 - 0 = 2$

Now,

$a_1\ a_2 = -4 \quad b_1\ b_2 = 2 \quad c_1\ c_2 = 2$

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 0$

or, $-4 + 2 + 2 = 0$

$-4 + 4 = 0$

$\therefore 0 = 0$

8.a. Here,

The two points of a line (1, 2, 3) and (4, 5, k) is parallel to two points of a line (–4, 3, –6) and (2, 9, 2).

So,

$a_1 = x_2 - x_1, \quad b_1 = y_2 - y_1, \quad c_1 = k - 3$
$= 4 - 1 \quad = 5 - 2 \quad = k - 3$
$= 3 \quad = 3$

Again,

$a_2 = x_2 - x_1, \quad b_2 = 9 - 3, \quad c_2 = z_2 - z_1$
$= 2 + 4 \quad = 6 \quad = 2 + 6$
$= 6 \quad = 8$

Since the two lines are parallel to each other we know,

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

or, $\frac{3}{6} = \frac{3}{6} = \frac{k-3}{8}$

or, $\frac{3}{6} = \frac{k-3}{8}$

or, $24 = 64 - 18$

or, $24 + 18 = 64$

or, $k = \frac{42}{6}$

$\therefore$ $k = 7$

b. Solution

Here, the line joining the points (k, 2, 3) and (–1, –2, –3) is perpendicular to the line joining the points (–2, 1, 5) and (3, 3, 2)

Now,

The direction ratios are:

$a_1 = x_2 - x_1$, $b_1 = y_2 - y_1$, $c_1 = z_2 - z_1$

$= -1 - k$ $= -2 - 2$ $= -3 - 3$

$= -4$ $= -6$

Again,

$a_2 = x_2 - x_1$, $b_2 = y_2 - y_1$, $c_2 = z_2 - z_1$

$= 3 + 2$ $= 3 - 1$ $= 2 - 5$

$= 5$ $= 2$ $= -3$

Since the two lines are perpendicular to each other. We know,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

or, $(-1 - k) \times 5 + (-4) \times 2 + (-6) \times (-3) = 0$

or, $18 - 13 = 5k + 0$

or, $5 - 0 = 5k$

$\therefore$ $k = 1$

9. Here,

Let l, m and n be the direction cosines of the line perpendicular to the given line.

$l + 2m + 3n = 0$ .... (i)

$-l + 3m + 5n = 0$ .... (ii)

By cross = multiplication method

$\frac{l}{10-9} = \frac{m}{-5-3} = \frac{n}{3+2}$

or, $\frac{l}{1} = \frac{m}{-8} = \frac{n}{5}$

$= \frac{1}{\sqrt{1^2 + 8^2 + 5^2}} = \frac{1}{\sqrt{90}}$

$\therefore$ $l = \frac{1}{\sqrt{90}}$, $m = \frac{-8}{\sqrt{90}}$, $n = \frac{5}{\sqrt{90}}$

10. Solution

Suppose l, m, n are the direction cosines of a line and its length

Then, $lr = 1$

$mr = 2$

$nr = 2$

Squaring and adding there relations

$r^2(l^2 + m^2 + n^2) = 12 + 22 + 22 = 9$

or, $r^2 = 9$

$\therefore$ $r = 3$ units

So length of the line = 3

Hence, $l = \frac{1}{r} = \frac{1}{3}$

$m = \frac{2}{r} = \frac{2}{3}$

$n = \frac{2}{r} = \frac{2}{3}$

11. Solution

$(x_1, y_1, z_1) = (3, -1, 2)$ and $(x_2, y_2, z_2) = (5, -7, 4)$

Here,

a. $a = 1, b = -1, c = 2$

$\therefore \quad l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{1 + 1 + 4}} = \frac{1}{\sqrt{6}}$

$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{-1}{\sqrt{1 + 1 + 4}} = \frac{-1}{\sqrt{6}}$

$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{2}{\sqrt{1 + 1 = 4}} = \frac{2}{\sqrt{6}}$

The required projection on the line

$= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1) n$

$= (5 - 3) \times \frac{1}{\sqrt{6}} + (-7 + 1) \times -\frac{1}{\sqrt{6}} + (4-2) \times \frac{2}{\sqrt{6}} = 2\sqrt{6}$

b. Here,

$r = \sqrt{(1 - 0)^2 + (3-1)^2 + (7-0)^2}$

$= \sqrt{54}$

$= 3\sqrt{6}$

Now,

$l = \frac{x_2 - x}{r} = \frac{1 - 0}{3\sqrt{6}} = \frac{1}{3\sqrt{6}}$

$m = \frac{y_2 - y_1}{r} = \frac{3 - 1}{3\sqrt{6}} = \frac{2}{3\sqrt{6}}$

$n = \frac{z_2 - z_1}{r} = \frac{7 - 0}{3\sqrt{6}} = \frac{7}{3\sqrt{6}}$

The projection $= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$

$= (5-3) \times \frac{1}{3\sqrt{6}} + (-7 + 1) \times \frac{2}{3\sqrt{6}} + (4-2) \times \frac{7}{3\sqrt{6}}$

$= \frac{4}{3\sqrt{6}}$

12. Solution

a. Here, $4l + 3m - 2n = 0$ .... (i)

$lm + mn + nl + 0$ .... (ii)

From the equation (i), $n = \frac{4l + 3m}{2}$

From the equation (ii)

$lm - m\frac{(4l + 3m)}{2} + l\frac{(4l + 3m)}{2} = 0$

or, $lm - \frac{4ml}{2} - \frac{3m^2}{2} + \frac{4l^2}{2} + \frac{3ml}{2} = 0$

or, $lm - 2ml - \frac{3}{2}m^2 + 2l^2 + \frac{3ml}{2} = 0$

or, $2l^2 + \frac{ml}{2} - \frac{3}{2}m^2 = 0$

or, $4l^2 + ml - 3m^2 = 0$

or, $4l^2 + 4ml - 3ml - 3m^2 = 0$

or, $4l\,(4l + m) - 3m\,(l + m) = 0$

or, $(l + m)(4l - 3m) = 0$

$\therefore$ $l + m = 0$ .... (iii)

$4l - 3m = 0$ .... (iv)

From equation (i) and (iii)

$4l + 3m - 2n = 0$ and $l + m + 0n = 0$

$$\therefore \frac{l}{0+2} = \frac{m}{-2-0} = \frac{n}{4-3}$$

$$\text{or, } \frac{l}{2} = \frac{m}{-2} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{4 + k + 1}} = \frac{1}{3}$$

$$\therefore l = \frac{2}{3}, m = -\frac{2}{3}, n = \frac{1}{3}$$

From equation (i) and (iv)

$4l + 3m - 2n = 0$ and $4l - 3m + 0n = 0$

$$\therefore \frac{l}{0-6} = \frac{m}{-8-0} = \frac{n}{-12-12}$$

$$\text{or, } \frac{l}{6} = \frac{m}{8} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{36 + 64 + 576}} = \frac{1}{\sqrt{676}} = \frac{1}{26}$$

$$\therefore l = \frac{6}{26} = \frac{3}{13}$$

$$m = \frac{8}{26} = \frac{4}{13}$$

$$n = \frac{24}{26} = \frac{12}{13}$$

b. Solution

Here,

$2l + 2m - n = 0$ .... (i)

$mn + nl + lm = 0$ .... (ii)

Using equation (i) in equation (ii) we have

$m(2l + 2m) + l(2l + 2m) + lm = 0$

or, $2lm + 2m^2 + 2l^2 + 2lm + lm = 0$

or, $2m^2 + 5lm + 2l^2 = 0$

or, $2m^2 + (4 + 1)\, lm + 2l^2 = 0$

or, $2m^2 + 4lm + lm + 2l^2 = 0$

or, $2m(m + 2l) + l(m + 2l) = 0$

or, $(m + 2l)(2m + l) = 0$

$\therefore$ $m + 2l = 0$ .... (iii)

$2m + l = 0$ .... (iv)

from (i) and (iii)

$2l + 2m - n = 0$ and $2l + m + 0.n = 0$

$$\therefore \frac{l}{0+1} = \frac{m}{0+2} = \frac{n}{2-4}$$

$$\text{or, } \frac{l}{1} = \frac{m}{2} = \frac{n}{-2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{1 + k + 4}} = \frac{1}{3}$$

$$\therefore l = \frac{1}{3}, m = \frac{2}{3}, n = -\frac{2}{3}$$

from (i) and (iv)

$2l + 2m - n = 0$ and $l + 2m + 0.n = 0$

$$\therefore \frac{l}{0+2} = \frac{m}{0+1} = \frac{n}{4-2}$$

$$\text{or, } \frac{l}{2} = \frac{m}{1} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{4 + 1 + 4}} = \frac{1}{3}$$

$\therefore \quad l = \frac{2}{3}, m = \frac{1}{3}, n = \frac{2}{3}$

13. Solution

Here, $l + m + n = 0$ .... (i)

or, $n = -1 - m$

Putting the value of n in

$l^2 + m^2 - n^2 = 0$ .... (ii)

or, $l^2 + m^2 - (-1 - m)^2 = 0$

or, $2lm = 0$

or, $lm = 0$

$\therefore \quad l = 0$ .... (iii)

and $m = 0$ .... (iv)

from (i) and (iii)

$l + m + n = 0$ and $l + 0.m + 0.n = 0$

or, $\frac{l}{0-0} = \frac{m}{1-0} = \frac{n}{0-1}$

or, $\frac{l}{0} = \frac{m}{1} = \frac{n}{-1} = \frac{1}{\sqrt{0+1+1}} = \frac{1}{\sqrt{2}}$

$\therefore \quad l = 0, m = \frac{1}{\sqrt{2}}, n = -\frac{1}{\sqrt{2}}$

from (i) and (iv)

$l + m + n = 0$ and $0.l + m + 0.n = 0$

$\therefore \quad \frac{l}{0-1} = \frac{m}{0-0} = \frac{n}{1-0}$

or, $\frac{l}{-1} = \frac{m}{0} = \frac{n}{1} = \frac{1}{\sqrt{1+0+1}} = \frac{1}{\sqrt{2}}$

$\therefore \quad l = -\frac{1}{\sqrt{2}}, m = 0, n = \frac{1}{\sqrt{2}}$

$\therefore \quad \cos\theta = 0\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right).0 + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

$\therefore \quad \theta = \frac{2\pi}{3} = 120^o$

14. Solution

Given, relations are

$al + bm + cn = 0$ .... (i)

$fmn + gnl + klm = 0$ .... (ii)

Eliminating n between (i) and (ii) we have

$fm\left(-\frac{al + bm}{c}\right) + g\left(-\frac{al + bm}{c}\right) l + hlm = 0$

$\Rightarrow \quad agl2 + (af + bg - ch)lm + bfm2 = 0$

$\Rightarrow \quad ag\left(\frac{l}{m}\right)^2 + (ch - af + bg)\left(\frac{l}{m}\right) + bf = 0$

which is quadratic in $\frac{l}{m}$. Let the two roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$

$\frac{l_1}{m_1}\ \frac{l_2}{m_2} = \frac{bf}{ag}$

$\Rightarrow \quad \frac{l_1\, l_2}{bf} = \frac{m_1 m_2}{ag}$

$\Rightarrow \quad \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b}$ .... (iii)

Similarly, if we eliminate l between (i) and (ii) have,

$\frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c}$ .... (iv)

From equation (iii) and (iv)

$\frac{l_1 l_2}{f/a} = \frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c}$

If each ratio be equal to k, then

$l_1l_2 = k\frac{f}{a}$, $m_1m_2 = k\frac{g}{b}$, $n_1n_2 = k\frac{h}{c}$

The two lines will be perpendicular if

$l_1l_2 + m_1m_2 + n_1n_2 + 0$

$k\frac{f}{a} + k\frac{g}{b} + k\frac{h}{c} = 0$

i.e. $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$ proved.

Exercise 11.3

1. Solution:

a. The equation of the plane is $2x + 3y + 4z = 24$

Dividing both sides by 24, $\frac{2x}{24} + \frac{3y}{24} + \frac{4z}{24} = \frac{24}{24}$

or, $\frac{x}{12} + \frac{4}{8} + \frac{z}{6} = 1$

The intercepts on the x–axis, y–axis and z–axis are 12, 8 and d 6 respectively.

b. To reduce the equation of the plane $2x – y + 2z = 4$ into normal form,

Divide each term by $\sqrt{2^2 + (–1)^2 + 2^2} = 3$

$\therefore \frac{24}{3} – \frac{y}{3} + \frac{2z}{3} = \frac{4}{3}$ is in normal form where length of perpendicular from origin is $\frac{4}{3}$ units.

The dc's are

$$\frac{2}{\sqrt{2^2 + (–1)^2 + 2^2}}, \frac{–1}{\sqrt{2^2 + (–1)^2 + 2^2}}, \frac{2}{\sqrt{2^2 + (–1)^2 + 2^2}}$$

i.e. $\frac{2}{3}, –\frac{1}{3}, \frac{2}{3}$

2. Solution:

a. The equation of plane which cuts intercepts 2, 3, 4 on the coordinate axes is

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

or, $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$

$\therefore$ $6x + 4y + 3z = 12$

b. Here, $a = b = c$

The equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$x + y + z = a$ .... (i)

If the plane (i) passes through (2, 3, 4) then $2 + 3 + 4 = a$

$\therefore$ $a = 9$

From equation (i) $x + y + z = 9$ which is the equation of the plane.

3. Solution:

a.(i) The equation of the plane through (2, 3, –3) $a(x – 2) + b(y –3) + c(z + 3) = 0$ .... (i)

If the plane passes through (1, 1, –2) and (–1, 1, 2) then,

$a(1 – 2) + b(1 – 3) + c(–2 + 3) = 0$

or, $–a –2b + c = 0$ .... (ii)

Again,

$a(–1 – 2) + b(1 – 3) + c(2 + 3) = 0$

or, $-3a - 2b + 5c = 0$ .... (iii)

From (ii) and (iii) and cross multiplication gives

$$\frac{a}{-10+2} = \frac{b}{-3+5} = \frac{c}{2-6}$$

$$\frac{a}{-8} = \frac{b}{2} = \frac{c}{-4} = k \text{ (say)}$$

$\therefore$ $a = -8k$, $b = 2k$ $c = -4k$

Substituting the values of a, b and c in equation (i) we get,

$-8h(x - 2) + 2k(y - 3) - 4k(2 + 3) = 0$

or, $-8x + 16 + 2y - 6 - 4 - 12 = 0$

or, $-8x + 2y - 4z - 2 = 0$

or, $-4x + y + 4z - 2 = 0$

or, $4x - y - 4z + 2 = 0$

(ii). The equation of the plane through (2, 2, 2) is

$a(x - 2) + b(y - 2) + c(z - 2) = 0$ .... (i)

If the plane passes through (3, 1, 1) and (6, –4, –6) then,

$a(3 - 2) + b(1 - 2) + c(1 - 2) = 0$

or, $a - b - c = 0$ .... (ii)

Again,

$a(6 - 2) + b(-4 - 2) + c(-6 - 2) = 0$

or, $4a - 6b - 8c = 0$ .... (iii)

From (ii) and (iii) and cross multiplication gives

$$\frac{a}{8-6} = \frac{b}{-4+8} = \frac{c}{-6+4}$$

$$\frac{a}{2} = \frac{b}{4} = \frac{c}{-2} = k \text{ (say)}$$

$\therefore$ $a = 2k$, $b = 4k$, $c = -2k$

Substituting the values of a, b and c in equation (i)

$2k(x - 2) + (4k)\, y-2) + (-2)(z- 2) = 0$

or, $2k - 4 + 4y - 8 - 2z + 4 = 0$

or, $2x + 4y - 2z = 8$

or, $2x + 4y - 2z = 8$

or, $x + 2y - z = 4$

b. The points (1, 3, –1), (3, 5, 1), (0, 2, –2) and (2, 1, –2) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-1 & 5-3 & 1+1 \\ 0-1 & 2-3 & -2+1 \\ 2-1 & 1-3 & -2+1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 1 & -2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow -2\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = 0$$

$\Rightarrow$ $-2 \times 0 = 0 \Rightarrow 0 = 0$ (true)

Hence, the given four points are coplanar.

4. Solution:

The equation of the plane parallel to the plane $ax + by + cz = 0$ is

$ax + by = cz + k = 0$ .... (i)

If the plane (i) passes through (α, β, γ) then

$a\alpha = b\beta = c\gamma + k = 0$

$\therefore \quad k = -a\alpha - b\beta - c\gamma$

Substituting the value of k in (i) we have

$ax + by + cz = a\alpha + b\beta + c\gamma$

which is the required equation of the plane.

5. Solution:

a. $x + 2y + 3z = 6$ and $3x - 3y + z = 1$

Here,

$x + 2y + 3z - 6 = 0$ and $3x - 3y + 2 - 1 = 0$

$a_1 = 1$	$a_2 = 3$
$b_1 = 2$	$b_2 = -3$
$c_1 = 3$	$c_2 = 1$

Now,

Angle between the two planes

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{1 \times 3 + 2 \times (-3) + 3 \times 1}{\sqrt{1^2 + 2^2 + 3^2}\sqrt{3^2 + (-3)^2 + 1^2}}$$

$$= \frac{0}{\sqrt{1 + 4 + 9}\sqrt{9 + 9 + 1}}$$

$\therefore \quad \cos\theta = 0$

$\therefore \quad \theta = \frac{\pi}{2}$

b. $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$

Here,

Repeating the same procedure as No. 5a

$\therefore \quad \theta = \frac{\pi}{2}$

6. Solution:

a. The equation of the two planes is given:

$x - 6y + 8z = 4$ .... (i)

$4x + ky + z = 7$ .... (ii)

Now,

$a_1 = 1$	$a_2 = 4$
$b_1 = -6$	$b_2 = k$
$c_1 = 8$	$c_2 = 1$

We know, the equation for two planes being perpendicular is given by,

$a_1a_2 = b_1b_2 + c_1c_2 = 0$

or, $1.4 + (-6).\ k + 8.1 = 0$

or, $4 - 6k + 8 = 0$

or, $-6k = -12$

$\therefore \quad k = 2$

b. The two equations of the plane is given

$2x - ky + 5z = 9$ .... (i)

$2x + 3y + 4z = 7$ .... (ii)

We know,

The equation of two planes being perpendicular is given by

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

or, $2 \times 2 + (-4) \times 3 + 5 \times 4 = 0$

or, $-3x = -24$

$\therefore \quad k = 8$

7. Solution:

For parallel

Two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel, if

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Here,

$a_1 = 2$, $b_1 = 3$ $c_1 = -4$

$a_2 = 10$, $b_2 = 15$, $c_2 = -20$

$\frac{a_1}{a_2} = \frac{2}{10} = \frac{1}{5}, \frac{b_1}{b_2} = \frac{3}{15} = \frac{1}{5}, \frac{c_1}{c_2} = \frac{-4}{-20} = \frac{-1}{-5} = \frac{1}{5}$

$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\therefore$ The planes $2x + 3y - 4z = 3$ and $10x + 15y - 202 = 12$ are parallel.

For perpendicular

Two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$

are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Here, $a_1 = 2$, $b_1 = 3$, $c_1 = -4$

$a_2 = 3$, $b_2 = 2$, $c_2 = 3$

And

$a_1a_2 + b_1b_2 + c_1c_2$

$= 2 \times 3 + 3 \times 2 - 4 \times 3$

$= 6 + 6 - 12$

$= 12 - 12$

$= 0$

$\therefore$ The two planes $2x + 3y - 4z = 3$ and $3x + 2y + 3z = 5$ are perpendicular.

8. Solution:

a. An equation of a plane passing through $(-2, 3, 4)$ so, the equation is

$a(x + 2) + b(y - 3) + c(z - 4) = 0$ .... (i)

Now,

It is perpendicular to the equation $2x + 3y + 4z = 6$ then,

$a \times 2 + b \times 3 + c \times 4 = 0$

or, $2a + 3b + 4c = 0$ .... (ii)

Again,

It is perpendicular to the equation $3x + 2y + 2 = 9$ then,

$a \times 3 + 2 \times b + 2 \times c = 0$

or, $3a + 2b + 2c = 0$ .... (iii)

By cross multiplication

$2a + 3b + 4c = 0$

$3a + 2b + 2c = 0$

$\therefore \quad \frac{a}{6 - 8} = \frac{b}{12 - 4} = \frac{c}{4 - 9} = k \text{ (say)}$

$\frac{a}{-2} = \frac{b}{8} = \frac{c}{-5} = k \text{ (say)}$

$\Rightarrow \quad a = -2k, b = 8k, c = -5k$

Substituting the values of a, b, c in equal (i) we have,

$-2k(x + 2) + 8k(y - 3) + (-5k)(z - 4) = 0$

or, $-2x - 4 + 8y - 24 - 5z + 20 = 0$

or, $-2x + 8y - 5z - 28 + 20 = 0$

or, $-2x + 8y - 5z - 8 = 0$

or, $2x - 8y + 5z + 8 = 0$ is the required equation of the plane.

b. Solution:

Here,
Repeating the same procedure as in No. 8a
So, the required equation of the plane is $2x - y + 3z = 9$

9. Solution:

a. The equation of the plane through P(a, b, c) is
$A(x - a) + B(y - b) + C(z - c) = 0$ .... (i)
The direction cosines of OP are proportional to $a - 0, b - 0, c - 0$
i.e., a, b, c
Since the plane (i) is perpendicular to OP,
$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = k$ (let)
$A = ak, B = bk, C = ck$
Substituting the values of A, B, C in (i)
$ak(x - a) + bk(y - b) + ck(z - c) = 0$
or, $a(x - a) + b(y - b) + c(z - c) = 0$
$\therefore$ $ax + by + cz = a^2 + b^2 + c^2$

b. **Solution:**
The equation of the plane through P(3, 2, 1) is
$a(x - 3) + b(y - 2) + c(z - 1) = 0$ .... (i)
The direction cosines of MN are proportional to $2 + 5, -4 - 3, 5 - 7$
i.e, 7, –7, –2
Since the plane (i) is perpendicular to MN,
$\frac{a}{7} = \frac{b}{-7} = \frac{c}{-2} = k$ (say)
$\therefore$ $a = 7k, b = -7k, c = -2k$
Now,
Substituting the value of a, b, c in equation (i) we have,
$7k(x - 3) - 7k(y - 2) - 2k(z - 1) = 0$
or, $7x - 21 - 7y + 14 - 2z + 2 = 0$
or, $7x - 7y - 2z - 5 = 0$ is the required equation of the plane.

10. Solution:

a. Any plane passing through (–1, 1, 2) is
$a(x + 1) + b(y - 1) + c(z - 2) = 0$ .... (i)
But, it passes through (1, –1, 1) so
$a(1 + 1) + b(-1 - 1) + c(1 - 2) = 0$
or, $2a - 2b - c = 0$ .... (ii)
The plane (i) is perpendicular to the given plane $x + 2y + 2z = 5$.
i.e. if $a + 2b + 2c = 0$ ... ... (iii)
From equation (ii) and (iii) we have
$\frac{a}{-4+2} = \frac{b}{-1 - 4} = \frac{c}{4 + 2}$
$\therefore$ $\frac{a}{-2} = \frac{b}{-5} = \frac{c}{6} = k$ (say)
$\Rightarrow$ $a = -2k, b = -5k, c = 6k$
From equation (i) we get
$-2k(x + 1) - 5k(y - 1) + 6k(z - 2) = 0$
or, $-2x - 2 - 5y + 5 + 6z - 12 = 0$
or, $-2x - 5y + 6z - 9 = 0$
or, $2x + 5y - 6z + 9 = 0$

b. Here, two planes are
$x + y + z = 5$ .... (i)
and $2x + 3y + 4z - 5 = 0$ .... (ii)
Then the equation of plane through intersection of (i) and (ii) is
$x + y + z - 5 + \lambda(2x + 3y + 4z - 5) = 0$

or, $x + y + z - 5 + 2\lambda x + 3\lambda y + 4\lambda z - 5\lambda = 0$

or, $(1 + z\lambda), c + (1 + 3\lambda) y + (1 + 4\lambda) z - 5 - 5\lambda = 0$ .... (iii)

and the plane (iii) passes through (0, 0, 0) so

$(1 + 2 \lambda) 0 + (1 + 3\lambda) 0 + (1 + 4\lambda) 0 - 5 - 5\lambda = 0$

or, $-5 - 5\lambda = 0$

or, $-5 = 5\lambda$

$\Rightarrow$ $\lambda = -1$

So, the required equation of plane is

$x + y + z - 5 - 1(2x + 3y + 4z - 5) = 0$

or, $x + y + z - 5 - 2x - 3y - 4z + 5 = 0$

or, $-x - 2y - 3z = 0$

$\therefore$ $x + 2y + 3z = 0$

c. Here, two planes are:

$x + 2y + 3z + 4 = 0$ ... ... (i)

$4x + 3y + 4z + 1 = 0$ ... ... (ii)

The equation of the plane through the intersection is given by,

$x + 2y + 3z + 4 + \lambda(4x + 3y + 4z + 1) = 0$

or, $x + 2y + 3z + 4 + 4\lambda x + 3\lambda y + 4\lambda z + \lambda = 0$

or, $x(1 + 4\lambda) + y(2 + 3\lambda) + z(3 + 4\lambda) + 4 + \lambda = 0$ ... ... (ii)

And, the plane (iii) passes through the point (1, –3, –1)

$1(1 + 4\lambda) + (-3) (2 + 3\lambda) + (-1) (3 + 4\lambda) + 4 + \lambda = 0$

or, $1 + 4\lambda - 6 - 9\lambda - 3 - 4\lambda + 4 + \lambda = 0$

or, $1 - 6 - 3 + 4 - 9\lambda + \lambda = 0$

or, $-9 + 5 - 8\lambda = 0$

or, $-4 = 8\lambda$

$\therefore$ $\lambda = -\frac{4}{8}$

$= -\frac{1}{2}$

So, the required equation of the plane is $x + 2y + 32 + 4 + \left(-\frac{1}{2}\right) (4x + 3y + 4z + 1) = 0$

or, $x + 2y + 3z + 4 - \frac{4x}{2} - \frac{3y}{2} - \frac{4z}{2} - \frac{1}{2} = 0$

or, $x + 2y + 3z + 4 - 2x - \frac{3y}{2} - 2z = \frac{1}{2}$

or, $-x - \frac{3y}{2} + 2y + z + 4 - \frac{1}{2} = 0$

or, $-x - \frac{3y + 4y}{2} + z + \frac{8 - 7}{2} = 0$

or, $-2x + y + 2z + 7 = 0$

or, $2x - y - 2z = 7$ is the required equation of the plane.

11. Solution:

The equation of the plane through the intersection of the given planes is

$x + 2y + 3z - 4 + \lambda(2x + y - z) = 0$ .... (i)

or, $x + 2y + 3z - 4 + 2\lambda x + \lambda y - \lambda z = 0$

or, $(1 + 2\lambda) x + (2 + \lambda) y + (3 - \lambda) z - 4 = 0$

Since the plane (i) is perpendicular to the plane: $5x + 3y + 6z + 8 = 0$

So, $(1 + 2\lambda) 5 + (2 + \lambda) 3 + (3 - \lambda). 6 = 0$ $\quad [\because a_1a_2 + b_1b_2 + c_1c_2 = 0]$

or, $5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$

or, $29 + 7\lambda = 0$

or, $\lambda = -\frac{29}{7}$

Substituting the value of λ in equation (i) we get

or, $x + 2y + 3z - 4 - \frac{29}{7}(2x + y - z) = 0$

or, $x + 2y + 3z - 4 - \frac{58x}{7} - \frac{29y}{7} + \frac{29z}{7} = 0$

or, $7x - 58x + 14y - 29y + 21z + 29z - 28 = 0$

or, $-51x - 15y + 50z - 28 = 0$

$\therefore$ $51x + 15y - 50z + 28 = 0$ is the required equation of the planes.

12.a. Solution:

i. The given plane is,

$2x - 3y + 3z + 27 = 0$

The distance from the point (3, 4, –5) to the plane $2x - 3y + 3z + 27 = 0$ is,

$\pm \frac{2 \times 3 - 3 \times 4 + 3 \times -5 + 27}{\sqrt{2^2 + (-3)^2 + 3^2}}$

$= \pm \frac{(6 - 12 - 15 + 27)}{\sqrt{4 + 9 + 9}}$

$= \pm \frac{6}{\sqrt{22}} = \frac{6}{\sqrt{22}}$ (in magnitude)

ii. Similar to No. 12a(i)

b. Here, the two points (1, –1, 3) and (3, 3, 3) are equidistance from the equation of the plane $5x + 2y - 7z + 9 = 0$.

Firstly,

$x_1 = 1, y_1 = -1, z_1 = 3$

and the equation of the plane is given by $ax_1 + by_1 + cz_1 + d = 0$

So,

$$\text{Distance} = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{5 \times 1 + 2 \times (-1) + (-7) \times 3 + 9}{\sqrt{5^2 + 2^2 + (-7)^2}}\right|$$

$$= \left|\frac{5 - 2 - 21 + 9}{\sqrt{25 + 4 + 49}}\right|$$

$$= \left|-\frac{9}{\sqrt{78}}\right|$$

Distance $= \frac{9}{\sqrt{78}}$ units

Similarly, $x_2 = 3, y_2 = 3, z_2 = 3$

$$\text{Distance} = \left|\frac{ax_2 + by_2 + cz_2 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{5 \times 3 + 2 \times 3 + (-7) \times 3 + 9}{\sqrt{5^2 + 2^2 + (-7)^2}}\right|$$

$$= \left|\frac{15 + 6 - 21 + 9}{\sqrt{25 + 4 + 49}}\right|$$

$$= \left|\frac{9}{\sqrt{78}}\right|$$

$\therefore$ Distance $= \frac{9}{\sqrt{78}}$ units.

Since the two point (1, –1, 3) and (3, 3, 3) are in at the same distance from the given plane.

$\therefore$ They are at equidistance from the plane.

c. Solution:

The equation of two parallel planes is given by,

$3x + 3y - 6z + 1 = 0$ .... (i)

$6x + 4y - 12z + 9 = 0$ .... (ii)

Now,

In equation (i) let y = z = 0 then, $x = -\frac{1}{3}$

i.e, $\left(-\frac{1}{3}, 0, 0\right)$ is the point which lies in equation (i) plane.

i.e, $(x_1, y_1, z_1) = \left(-\frac{1}{3}, 0, 0\right)$

and

$$\text{Distance} = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{6 \times \left(-\frac{1}{3}\right) + 4 \times 0 - 12 \times 0 + 9}{\sqrt{6^2 + 4^2 + (-12)^2}}\right| = \left|\frac{7}{\sqrt{36 + 16 = 144}}\right| = \left|\frac{7}{\sqrt{196}}\right| = \left|\frac{7}{14}\right|$$

$\therefore$ Distance $= \frac{1}{2}$

Hence, according to the qn, the distance between the two planes is $\frac{1}{2}$ units

13. Solution:

Let the plane (ΔABC) whose vertices are

A(a, 0, 0), B (0, b, 0), and (0, 0, c) at a distance of 3p units from the origin.

So, the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

or, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$ .... (i)

Since, 1^v distance of the equation (i) from (0, 0, 0) is 3p,

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{\frac{1}{a} . 0 + \frac{1}{b} . 0 + \frac{1}{c} . 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right|$$

or, $3p = \dfrac{1}{\sqrt{\frac{1}{q^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$

or, $\frac{1}{(3p)^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ .... (ii)

Let (α, β, α) be the centroid of ΔABC

So,

$\alpha = \frac{a + 0 + 0}{3}$

$\therefore$ $\alpha = \frac{a}{3} \Rightarrow a = 3\alpha$

Similarly, $b = 3\beta$ and $c = 3\nu$

Now,

From equation (ii)

$\frac{1}{9p^2} = \frac{1}{(3\alpha)^2} + \frac{1}{(3\beta)^2} + \frac{1}{(3\nu)^2}$

or, $\frac{1}{9p^2} = \frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\nu^2}$

or, $\frac{1}{p^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\nu^2}$

$\therefore \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\nu^2} = \frac{1}{p^2}$ proved.

Hence, the locus of the centroid of ΔABC is

$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$