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Set E

12. The principle of mathematical induction States that if $p(n)$ be the Statement
(i) $p(1)$ is true
(ii) $p(k+1)$ is true whenever $p(k)$ is true. then $p(n)$ is true for all $n \in N$.

proof:

$$2^n < (n+1)!$$

let $p(n)$ be the given Statement. Then
$p(n): 2^n < (n+1)!$
when $n=1$, $2^n = 2! = 2$
$\Rightarrow (n+1)! = (2+1) = 3! = 6$
which tends to $2^n < (n+1)!$
when $n=k$: let us suppose that $p(k+1)$ is true when $p(k)$ is true.
That is
$2^k < (k+1)!$ ——— ①
Now we have to Show that $p(k+1)$ is true when $p(k)$ is true. So, Add $(k+1)$ on both side.
That is $2^{k+1} < (k+1+1)! = (k+2)!$
$2^k \cdot 2 < (k+2)!$
$2^k \cdot 2 < (k+2) \cdot (k+1)!$
Which is true in relation to equation ① So, it shows that $p(k+1)$ is true whenever $p(k)$ is true. Hence the principal of mathematical induction $p(n)$ is true for all $n \in N$.

13 Any three properties of cubic root of unity are:-

(i) Each imaginary cubic root of unity is the square of one another.

(ii) The sum of the three cubic roots of unity is zero

(iii) The product of two imaginary cubic root of unity is equal to 1.

Proof: $\omega \cdot \omega^2 = \frac{-1+\sqrt{3}i}{2} \times \frac{-1-\sqrt{3}i}{2} = \frac{1-3i^2}{4}$

$= \frac{4}{4} = 1$

proof:-

$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8) = 9$

L.H.S

$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

$(1-\omega)(1-\omega^2)(1-\omega^3 \cdot \omega)(1-(\omega^3)^2 \cdot \omega^2)$

$(1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$(1-\omega)^2 \quad (1-\omega^2)^2$

$(1-2\omega+\omega^2)(1-2\omega^2+\omega^4)$

$(1-2\omega+\omega^2)(1-2\omega^2+\omega)$

$(-\omega-2\omega)(-\omega^2-2\omega^2)$

$-3\omega \times -3\omega^2$

$-3\omega \times -3\omega^2$

$(-3)(-3)[\omega \times \omega^2] \quad [\because \omega \times \omega^3 = 1]$

$\therefore 9$

14a.

proof:

let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

$|\vec{a}|^2 = (a_1^2 + a_2^2 + a_3^2)$ $|\vec{b}|^2 = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$\vec{a} \times \vec{b} = \begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{matrix}$

$(a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

$\vec{a} \cdot \vec{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$

$= (a_1 b_1 + a_2 b_2 + a_3 b_3)$

$= ab\cos\theta$

where θ is the angle between a and b.

$|\vec{a} \times \vec{b}|^2 = (\vec{a} \times \vec{b})(\vec{a} \times \vec{b})$

$= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

$= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$

$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + (b_2^2 + b_3^2)) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$

$= |\vec{a}|^2 \cdot |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

$= a^2 b^2 - (\vec{a} \cdot \vec{b})^2$

Again

$|\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}$

$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$

$= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)$

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 \sin^2\theta$$

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

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13. Solution:

let x and y represents commodity and price respectively

$\bar{x} = 36$, $\bar{y} = 85$

$\sigma_x = 11$, $\sigma_y = 8$

correlation coeff. $(r) = 0.66$

i) $b_{yx} = r\frac{\sigma_y}{\sigma_x} = 0.66 \times \frac{8}{11} = \frac{12}{25}$

$b_{xy} = r\frac{\sigma_x}{\sigma_y} = 0.66 \times \frac{11}{8} = \frac{363}{400}$

ii) Regression equation of y on x

$y - \bar{y} = b_{yx}(x - \bar{x})$

$y - 85 = \frac{12}{25}(x - 36)$

$25y - 2125 = 12x - 432$

$25y = 12x + 1693$

$y = 0.48x + 67.72$

Regression equation of x on y

$x - \bar{x} = b_{xy}(y - \bar{y})$

$x - 36 = \frac{363}{400}(y - 85)$

$400x - 14400 = 363y - 30855$

$400x = 363y - 16455$

$x = 0.9075y - 41.1375$

iii) When $x = 75$

$y = 0.48 \times 75 + 67.72$

$\therefore y = 103.72$

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Solution:

16. Introduce r and S slack variables in LPP

$$x + 2y + r = 20$$
$$x + y + S = 16$$

Formulating the LPP in standard form

$$x + 2y + r + 0 \cdot S + 0 \cdot Z = 20$$
$$x + y + 0 \cdot r + S + 0 \cdot Z = 16$$
$$-3x - 5y + 0 \cdot r + 0 \cdot S + Z = 0$$

Initial Simplex Tableau

B.V	x	y	r	S	Z	R.H.S
r	1	2	1	0	0	20
S	1	1	0	1	0	16
	-3	-5	0	0	1	0
		↑				

Since -5 is the most negative entry so, C_2 is pivot column

Since, $\frac{20}{2} < \frac{16}{1}$ so 2 is the pivot ~~column~~ number.

Apply $R_1 = \frac{R_1}{2}$

B.V	x	y	r	S	Z	R.H.S
x	1/2	1	1/2	0	0	10
S	1	1	0	1	0	16
	-3	-5	0	0	1	0

Apply $R_2 = R_2 - R_1$ and $R_3 = 5R_1 + R_3$

B.V	x	y	r	s	z	R.H.S
	1/2	1	1/2	0	0	10
	1/2	0	-1/2	1	0	6
	-1/2	0	5/2	0	1	50

↑

Again C_1 is the pivot column since

$\frac{10}{1/2} > \frac{6}{1/2}$

$1/2$ of R_2 is the pivot element

Apply $R_2 = 2R_2$

B.V	x	y	r	s	z	R.H.S
x	1/2	1	1/2	0	0	10
y	1	0	-1	2	0	12
	-1/2	0	5/2	0	1	50

Apply $R_1 = -1/2\, R_2 + R_1$ and $R_3 = \frac{1}{2} R_2 + R_3$

B.V	x	y	r	s	z	R.H.S
x	0	1	1	-1	0	4
y	1	0	-1	2	0	12
	0	0	2	1	1	56

Hence required solution is $z = 56$ at $x = 4$ and $y = 12$

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17a. Solution:

put $f(x) = \tan^{-1} x$

$f(x+h) = \tan^{-1}(x+h)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

put $y = \tan^{-1} x$ and $y+k = \tan^{-1}(x+h)$

$k = \tan^{-1}(x+h) - \tan^{-1} x$ as $h \to 0$, $h \to 0$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y+k-y}{h}$$

or $$\lim_{h \to 0} \frac{k}{h}.$$

$$\lim_{h \to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$\lim_{k \to 0} \frac{k}{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}}$$

$$\lim_{k \to 0} \frac{k(\cos(y+k) - \cos y)}{\sin(y+k) - \sin y}$$

$$\lim_{k \to 0} \frac{k[\cos(y+k) \cdot \cos y]}{\sin(y+k) \cdot \cos y - \cos(y+k) \cdot \sin y}$$

$$\lim_{k \to 0} \frac{k \cos y \cdot \cos(y+k)}{\sin(y+k-y)}$$

$$\lim_{k \to 0} \frac{\cos y \cdot \cos(y+k)}{\frac{\sin k}{k}}$$

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$\cos x \cdot \cos(x+c)$

$\cos^2 x = \frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x}$

$= \frac{1}{1+x^2}$

17.b. Solution:

Given equation is $x^2 - y^2 = 7$

$y^2 = x^2 - 7$

diff. both side w.r.to x

$\frac{dy^2}{dy} \times \frac{dy}{dx} = \frac{dx^2}{dx} - \frac{d(7)}{dx}$

$2y \cdot \frac{dy}{dx} = 2x - 0$

$2y\, dy = 2x \cdot dx$

$\frac{dy}{dx} = \frac{x}{y}$

At $(4,3)$, $\frac{dy}{dx} = \frac{4}{3}$

Slope $(m) = \frac{4}{3}$

Equation of tangent $= y - y_1 = m(x - x_1)$

$y - 3 = \frac{4}{3}(x - 4)$

$3y - 9 = 4x - 16$

$4x - 3y = 7$

Eqn of normal:

$y - y_1 = \frac{-1}{m}(x - x_1)$

$y - 3 = \frac{-3}{4}(x - 4)$

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$$4y - 12 = -3x + 12$$
$$3x + 4y = 24$$

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19. Solution:

i) coefficient of output matrix

$$\begin{pmatrix} \frac{45}{250} & \frac{50}{100} \\ \frac{30}{250} & \frac{40}{100} \end{pmatrix} = \begin{pmatrix} 0.18 & 0.5 \\ 0.12 & 0.4 \end{pmatrix}$$

let x_1 and x_2 be total output of industry

ii) Demand vector (D) = $\begin{pmatrix} 72 \\ 48 \end{pmatrix}$

$$I-A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.18 & 0.5 \\ 0.12 & 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.82 & -0.5 \\ -0.12 & 0.6 \end{pmatrix}$$

$$|I-A| = \begin{vmatrix} 0.82 & -0.5 \\ -0.12 & 0.6 \end{vmatrix} = (0.492 - 0.06)$$

$$= 0.432 \neq 0$$

$\therefore (I-A)^{-1}$ exist

$$(I-A)^{-1} = \frac{1}{|I-A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\frac{1}{0.432} \begin{pmatrix} 0.6 & 0.5 \\ 0.12 & 0.82 \end{pmatrix}$$

$$x = (I-A)^{-1} D$$

$$= \frac{1}{0.432}\begin{pmatrix} 0.6 & 0.5 \\ 0.12 & 0.82 \end{pmatrix}\begin{pmatrix} 72 \\ 48 \end{pmatrix}$$

$$= \frac{1}{0.432}\begin{pmatrix} 67.2 \\ 48 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 155.55 \\ 111.11 \end{pmatrix}$$

$\therefore$ For the fulfillment of 72 unit of A and 48 unit of of industry the total output of 155.55 unit and 111.11 unit of Industry are required.

Group 'C'

10a. Solution

If t_{r+1}, t_{r+2}, t_{r+3} be three consecutive terms in the expansions of $(1+x)^n$ then their respective coefficent are $C(n,r)$, $C(n,r+1)$, $C(n,r+1)$ respectively

$C(n,r) = 45$ — (i)

$C(n,r+1) = 120$ — (ii)

$C(n,r+1) = 210$ — (iii)

Dividing eqn (i) by (ii)

$$\frac{C(n,r)}{C(n,r+1)} = \frac{45}{120}$$

$$\frac{\frac{n!}{(n-r)!\,r!}}{\frac{n!}{(n-r-1)!\,(r+1)!}} = \frac{45}{120}$$

$$\frac{45}{120} = \frac{(n-r-1)!\,(r+1)!}{(n-r)!\,r!}$$

$$\frac{45}{120} = \frac{\cancel{(n-r-1)!}\,(r+1)\cancel{(r)!}}{(n-r)\cancel{(n-r-1)!}\,\cancel{r!}}$$

$$\frac{3}{8} = \frac{45}{120} = \frac{r+1}{n-r}$$

$$3n-3r = 8r+8$$

$$3n-11r = 8 \quad \text{— (iv)}$$

Again Dividing eqn (ii) & (iii)

$$\frac{120}{210} = \frac{C(n,r+1)}{C(n,r+2)}$$

$$\frac{4}{7} = \frac{\frac{n!}{(n-r-1)!\,(r+1)!}}{\frac{n!}{(n-r-2)!\,(r+2)!}}$$

$$\frac{4}{7} = \frac{(n-r-2)!\,(r+2)!}{(n-r-1)!\,(r+1)!}$$

$$\frac{4}{7} = \frac{\cancel{(n-r-2)!}\,(r+2)\,\cancel{(r+1)!}}{(n-r-1)\,\cancel{(n-r-2)!}\,\cancel{(r+1)!}}$$

$$\frac{4}{7} = \frac{(r+2)}{(n-r-1)}$$

$$4x - 4y - 4 = 7y + 14$$
$$4x - 11y = 10 \quad \text{--- (v)}$$

Solving (iv) & (v)

$$3x - 11y = 8$$
$$4x - 11y = 10$$
$$- \quad + \quad -$$
$$-x = -2$$
$$x = 2$$

put $x = 2$ eqn (i)

$$3 \times 2 - 11y = 8$$
$$y = -2$$

? 20b: Solution:

We have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

$$e + e^{-1} = 2 + \frac{2}{2!} + \frac{2}{4!} + \cdots$$

$$\frac{e+1}{e} = \frac{e^2+1}{e} = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots\right)$$

Similarly,

$$\frac{e^2-1}{e} = \left(\frac{2}{1!} + \frac{2}{3!} + \frac{2}{5!} + \cdots\right).$$

$$= 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right)$$

Dividing eqn (i) & (ii) we get

$$\frac{\left(\frac{e^2+1}{e}\right)}{\left(\frac{e^2-1}{e}\right)} = \frac{2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots\right)}{2\left(1 + \frac{1}{3!} + \frac{1}{5!} + \cdots\right)}$$

$$\frac{e^2+1}{e^2-1} = \frac{1 + \frac{1}{2!} + \frac{1}{4!} + \cdots}{1 + \frac{1}{3!} + \frac{1}{5!} + \cdots}$$

21.a. An ellipse is the locus of the point in a plane such that its distance from a fixed point (called the focus) bears a constant ratio (called eccentricity) to its distance from a fixed st. line (called the directrix).

Given eqn of ellipse is

$$x^2 + 5y^2 + 3x - 10y - \frac{71}{4} = 0$$

$$x^2 + 3x + 5y^2 - 10y = \frac{71}{4}$$

$$x^2 + 2 \times \frac{3}{2}x + \left(\frac{3}{2}\right)^2 - \frac{9}{4} + 5[y^2 - 2y + 1 - 1]$$

$$= \frac{71}{4}$$

$$\left(x + \frac{3}{2}\right)^2 + 5[y-1]^2 - 5 = \frac{71}{4} + \frac{9}{4}$$

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$$\left(x+\frac{3}{2}\right)^2 + 5(y-1)^2 = 20+5 = 25$$

$$\frac{\left(x+\frac{3}{2}\right)^2}{25} + \frac{(y-1)^2}{5} = 1 \quad \text{--- } ①$$

comparing eqn ① with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

i) centre $= (h, k) = \left(\frac{-3}{2}, 1\right)$

ii) vertices $= (h \pm a, k) = \left(\frac{-3}{2} \pm 5, 1\right)$

iii) eccentricity $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{25}}$

$= \frac{\sqrt{20}}{5} = \frac{2\sqrt{5}}{5} = \frac{2}{\sqrt{5}}$

iv) foci $= (h \pm ae, k) = \left(\frac{-3}{2} \pm 5 \times \frac{2}{\sqrt{5}}, 1\right)$

$= \left(\frac{-3}{2} \pm 2\sqrt{5}, 1\right)$

v. Eqn of directrix is $x = \pm \frac{a}{e}$

$= \pm \frac{\sqrt{5}}{2/\sqrt{5}}$

$= \pm\frac{5}{2}$

$$n = \pm\frac{5}{2}$$

6. Solution

let the direction cosine of line be l, m, n

Now,

D.C.S of line with drs $3, -1, 1$ is

$$\frac{3}{\sqrt{3^2+(-1)^2+(1)^2}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$$

ie. $l_1, m_1, n_1 = \frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$

Again

DCS of line with drs $-3, 2, 4$

$$\frac{-3}{\sqrt{9+4+16}}, \frac{2}{\sqrt{29}}, \frac{4}{\sqrt{29}}$$

$$l_2, m_2, n_2 = \frac{-3}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{4}{\sqrt{29}}$$

Now, Acc. to question

For perpendicularity

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\frac{3l}{\sqrt{11}} - \frac{m}{\sqrt{11}} + \frac{n}{\sqrt{11}} = 0$$

ie. $3l - m + n = 0$ —— (1)

Again,

$$l l_2 + m_1 m_2 + n_1 n_2 = 0$$

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$$\frac{-3l}{\sqrt{29}} + \frac{2m}{\sqrt{29}} + \frac{4n}{\sqrt{29}} = 0$$

$$-3l + 2m + 4n = 0 \quad \text{—(ii)}$$

Solving eqn (i) & (ii) by cross multiplication

$$\begin{array}{ccccc} 3 & -1 & 1 & 3 & -1 \\ -3 & 2 & 4 & -3 & 2 \end{array}$$

$$\frac{l}{-4-2} = \frac{m}{-3-12} = \frac{n}{6-3}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{36+225}}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} = \frac{1}{\sqrt{270}} = \frac{1}{3\sqrt{30}}$$

$$l = \frac{-6}{3\sqrt{30}} = \frac{-2}{\sqrt{30}}, \quad m = \frac{-5}{\sqrt{30}}$$

$$n = \frac{1}{\sqrt{30}}$$

Either it can be written as

$$l, m, n = \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}$$

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22b. $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

$\frac{dy}{dx} = \frac{-1+\cos 2y}{1+\cos 2y} = \frac{-2\cos^2 y}{2\sin^2 x}$

$\frac{dy}{\cos^2 y} = \frac{-dx}{\sin^2 x}$

$\sec^2 y\, dy = -\sin^2 x\, dy$

on integration

$\int \sec^2 y\, dy = -\int \operatorname{cosec}^2 x\, dx$

$\tan y = -(-\cot x) + c$

$\cot x - \tan y + c = 0$

ii) $y(1+xy)dx - x\,dy = 0$

$y\,dx + xy^2\,dx - x\,dy = 0$

$y\,dx - x\,dy + xy^2\,dx = 0$

$\frac{y\,dx - x\,dy}{y^2} = \frac{-xy^2\,dx}{y^2}$

$d\left(\frac{x}{y}\right) = -x\,dx$

On integration

$\frac{x}{y} = \frac{-x^2}{2} + c$

$\frac{x^2}{y} + \frac{x^2}{2}$

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