Chapter 22
MATHEMATICS FOR ECONOMICS AND FINANCE

Exercise 22.1

1. Find the quadratic supply function Qs = f(P) from the information given.

Price (P)	40	50	80
Quantity supplied (Q)	600	3300	15000

Solution:

Let $Q_s = ap^2 + bp + c$ ... ... (i) be a quadratic supply function.

Then acceleration to question, when p = 40 then $Q_s = 600$

$\therefore \quad a\times40^2 + b\times40 + c = 600 \Rightarrow 1600a - 40b + c = 600$ ... ... (i)

Similarly, other two points are (50, 3300) and (80, 1500)

Then,

$2500a + 50b + c = 3300$ ... ... (ii)

$6400a + 806 + c = 15000$ ... ... (iii)

From (i) and (ii)

$2500a + 50b + c = 3300$

$1600a + 40b + c = 600$

$\underline{-\quad\quad -\quad\quad -\quad\quad\quad}$

$900a + 10b = 2700$

$90a + b = 270$ ... ... (iv)

from (ii) and (iii)

$6400a + 80b + c = 15000$

$2500a + 50b + c = 3300$

$\underline{-\quad\quad -\quad\quad -\quad\quad\quad}$

$3900a + 30b = 11700$

$130a + b = 390$ ... ... (v)

from (iv) and (v)

$130a + b = 390$

$90a + b = 270$

$\underline{-\quad -\quad -\quad}$

$40a = 120$

$\therefore \quad a = 3$

from (iv) b = 0

Substituting the value of a and b in (i) we get c = –4200

Hence, required quadratic supply function is $Q_s = 3p^2 - 4200$

2. The supply and demand functions are given by $P = Q^2 + 12Q + 32$, $P = -Q^2 - 4Q + 200$ respectively. Find equilibrium price and quantity.

Solution:

Given, supply function $P_s = Q^2 + 12Q + 32$

Demand function $P_d = Q^2 - 4Q + 200$

For equilibrium, $P_d = P_s$

i.e. $Q^2 + 12Q + 32 = -Q^2 - 4Q + 200$

$2Q^2 + 16Q - 168 = 0$

$Q^2 + 8Q - 84 = 0$

$Q^2 + 14Q - 6Q - 84 = 0$

$\therefore$ Q = 6, Q = –14 (not possible)

When Q = 6

Then $p = 6^2 + 12\times 6 + 32 = 36 + 72 + 32 = 140$

$\therefore$ equilibrium price = 140

3. Given the supply and demand functions

$$Q_S = (P + 5)\sqrt{P + 10}$$

$$Q_d = \frac{210 - 9P - 3P^2}{\sqrt{P + 10}}$$

Calculate the equilibrium price and quantity.

Solution:

Given, $Q_s = (P+5)\sqrt{p + 10}$ and $Q_1 = \dfrac{210 - 9p - 3p^2}{\sqrt{p + 10}}$

For equilibrium condition,

$Qs = Q_d$

i.e. $(P + 5)\sqrt{p + 10} = \dfrac{210 - 9p - 3p^2}{\sqrt{p + 10}}$

or, $(p + 5)(p + 10) = 210 - 9p - 3p^2$

or, $p^2 + 15p + 50 + 3p^2 + 9p - 210 = 0$

or, $4p^2 + 24p - 160 = 0$

or, $p^2 + 6p - 40 = 0$

or, $p^2 + 10p - 4p - 40 = 0$

$\therefore$ p = 4

Then $Q = 9\sqrt{14}$

$\therefore$ equilibrium point $(4, 9\sqrt{14})$

4. The average cost of a product is given as

$$AC = 15Q - 3600 + \frac{486000}{Q}$$

Find the quantity for which the total cost is minimum. Also find the minimum cost.

Solution:

Given, average cost (AC) = $15Q - 3600 + \dfrac{486,000}{Q}$

Total cost function (TS) = AC $\times$ Q

$\therefore$ $TC = 15Q^2 - 3600Q + 486,000$

Comparing it with $y = ax^2 + bx + c$

a = 15, b = –3600 and c = 486,000

Since a > 0, TC represents a parabola concave upward. Being upward, TC has minimum value at $Q = -\dfrac{b}{2a}$ $\left(x = -\dfrac{b}{2a}\right)$

i.e. $Q = +\dfrac{3600}{30}$

Q = 120

$\therefore$ Total cost is minimum at Q = 120 units

Then the min. total cost is $TC = 15\times 120^2 - 3600 \times 120 + 486000$

$= 2,70,000$

5. For the price Rs. P, the quantity demanded is given by Q = 600,000 – 2,500P. Determine the total revenue function R = f(P).

a. What is the concavity of the revenue function?

b. What is the total revenue when price is Rs. 50?

c. Find the price for which the total revenue is maximized.

Solution:

Demand function is given by

$Q = 6,00,000 - 2,500P$

Total revenue function (TR) = $P \times Q$

$\therefore \quad R = 600,000p - 2,500p^2$

Comparing it with $y = ax^2 + bx + c$,

We get $a = -2500$, $b = 600,000$ and $c = 0$

Since $a < 0$, the graph is concave downward parabola.

When p = Rs. 50, then total revenue is $R = -2500 \times 50^2 + 600,000 \times 50$

Rs. 3,62,50,000

$\therefore$ total revenue is maximized at $P = -\dfrac{b}{2a}$

i.e. $P = \dfrac{-600,000}{-5,000} = 120$

$\therefore$ Revenue is maximized at P = Rs. 120

6. Fixed cost = 32

Variable cost = 5

$\therefore$ total cost for producing a units is given by,

$TS = 5Q + 32$

Also, demand function $p = 25 - 2Q$

$\therefore$ total revenue function $TR = 25Q - 2Q^2$

a. For break-even TR = TC

$25Q - 2Q^2 = 5Q + 32$

$20Q - 2Q^2 - 32 = 0$

or, $Q^2 - 10Q + 16 = 0$

$Q^2 - 8Q - 2Q + 16 = 0$

$Q = 2$ or 8

b. Profit function $\pi = TR - TC$

$= 25Q - 2Q^2 - 5Q - 32$

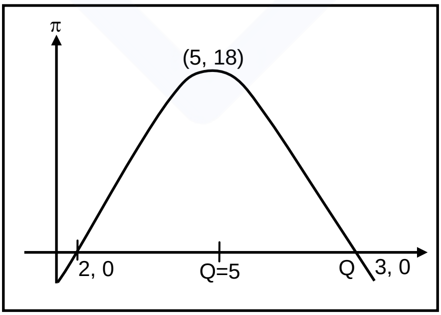
$\pi \;\; -2Q^2 + 20Q - 32$

Since coefficient of Q^2 is negative, parabola is concave downward,

Profit is maximum of $Q = -\dfrac{20}{-4} = 5$

c. Maximum profit $\pi_{max} = \dfrac{4ac - b^2}{4a} = \dfrac{4 \times (-2)\,(-32) - 400}{-8} = 18$

d.



7. Given the fixed cost as 32, variable cost per unit as 5 per unit and the demand function P = 25 – 2Q, express the profit function π in terms of Q.

a. Find the value(s) of Q for break even.

$TR = -2Q^2 + 14Q$

$TC = 2Q + 10$

For break-even, TR = TC

$-2Q^2 + 14Q = 2Q + 10$

$2Q^2 - 12Q + 10 = 0$

$Q^2 - 6Q + 5 = 0$

$Q^2 - 5Q - Q + 5 = 0$

Q = 1 or 5

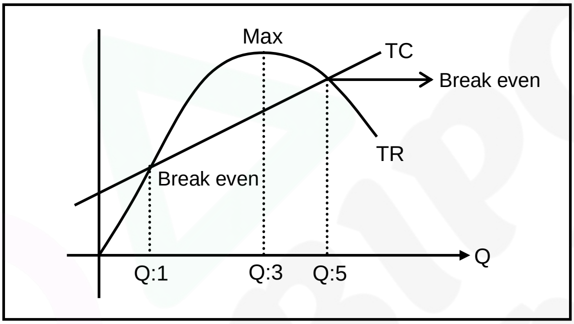
Now, profit function π = TR – TC

$\therefore \quad \pi = Q^2 - 6Q + 5$

It is quadratic function. Since a > 0 concave downward, gives max profit at

$Q = -\frac{b}{2a} = \frac{6}{2} = 3$

b. Find the value of Q for which π is maximum.



c. Max profit $\pi = 3^2 - 6\times3 + 5 = -4$

d. Sketch the graph of π.

Exercise 22.2

1. Solution:

a. Given,

$x_{11} = 200,\ x_{12} = 250,\ d_1 = 450$

$\therefore \quad x_1 = x_{11} + x_{12} + d_1 = 900$

$x_{21} = 125,\ x_{22} = 8,\ d_2 = 225$

$\therefore \quad x_2 = x_{21} + x_{22} + d_2 = 400$

$\therefore$ the consumption matrix or coefficient input matrix is given by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where

$a_{ij} = \frac{x_{ij}}{x_j}$ for all i, j

so, $a_{11} = \frac{x_{11}}{x_1} = \frac{200}{900} = \frac{2}{9}$

$a_{12} = \frac{x_{12}}{x_2} = \frac{125}{400} = \frac{5}{36}$

$a_{21} = \frac{x_{21}}{x_1} = \frac{125}{450} = \frac{5}{16}$

$$a_{22} = \frac{x_{22}}{x_2} = \frac{50}{450} = \frac{1}{9}$$

$\therefore$ Input coefficient matrix is $\begin{bmatrix} \frac{2}{9} & \frac{5}{16} \\ \frac{5}{36} & \frac{1}{9} \end{bmatrix}$

b. Given,

$x_{11} = 250$, $x_{11} = 140$, $x_{13} = 30$, $d_1 = 80$

Then total output $(x_1) = x_{11} + x_{12} + x_{13} + d_1$

$= 250 + 140 + 30 + 80 = 500$

$x_{21} = 100$, $x_{22} = 105$, $x_{23} = 15$, $d_2 = 130$

$\therefore$ Total output $(x_2) = x_{21} + x_{22} + x_{23} + d_2 = 350$

$x_{31} = 50$, $x_{32} = 35$, $x_{33} = 45$, $d_3 = 20$

$\therefore$ Total output $(x_3) = x_{31} + x_{32} + x_{33} + d_3 = 150$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Matrix where $a_{ij} = \frac{x_{ij}}{x_j}$

$$a_{11} = \frac{x_{11}}{x_1} = \frac{250}{500} = 0.5$$

$$a_{12} = \frac{x_{12}}{x_2} = \frac{140}{350} = 0.4$$

$$a_{13} = \frac{x_{13}}{x_3} = \frac{30}{150} = 0.2$$

$$a_{21} = \frac{x_{21}}{x_1} = \frac{100}{500} = 0.2$$

$$a_{22} = \frac{x_{22}}{x_2} = \frac{105}{350} = 0.3$$

$$a_{23} = \frac{x_{23}}{x_3} = \frac{15}{150} = 0.1$$

$$a_{31} = \frac{x_{31}}{x_1} = \frac{50}{500} = 0.1$$

$$a_{32} = \frac{x_{32}}{x_1} = \frac{35}{200} = 0.1$$

$$a_{33} = \frac{x_{33}}{x_3} = \frac{45}{150} = 0.3$$

Therefore, $A = \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{pmatrix}$

2. Given, consumption input coefficient matrix is sector I, Sector II and Sector III

$$A = \begin{array}{c} \begin{pmatrix} 0.1 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.2 \\ 0.1 & 0.0 & 0.2 \end{pmatrix} \end{array} \begin{array}{l} \text{Sector I} \\ \text{Sector II} \\ \text{Sector III} \end{array}$$

If first sector decides to produce 200 units, then it consumes 0.1×200 units

$= 20$ units of itself

and 0.4×200 units $= 80$ units of sectors 2

and 0.1×200 units $= 20$ units of sector 3

3. Given,

Coefficient matrix A = $\begin{bmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{bmatrix}$ and final demand vector D = $\begin{bmatrix} 18 \\ 11 \end{bmatrix}$

Then technology matrix T = I – A

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{pmatrix}$$

$$T = \begin{bmatrix} 0.9 & -0.6 \\ -0.5 & 0.8 \end{bmatrix}$$

$$|T| = \begin{vmatrix} 0.9 & -0.6 \\ -0.5 & 0.8 \end{vmatrix} = 0.72 - 0.30 = 0.42$$

$$\therefore T^{-1} = \frac{\text{Adj. (T)}}{|T|} = \frac{\begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{bmatrix}}{0.42}$$

Let x = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the gross output to meet the final demand then,

$$x = T^{-1}D = \frac{1}{0.42}\begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{bmatrix}\begin{bmatrix} 18 \\ 11 \end{bmatrix} = \frac{1}{0.42}\begin{bmatrix} 21 \\ 18.9 \end{bmatrix} = \begin{bmatrix} 50 \\ 45 \end{bmatrix}$$

$\therefore$ The production level is 50 units and 45 units respectively.

4. Given, consumption input coefficient matrix be $\begin{vmatrix} 0.2 \\ 0.1 \end{vmatrix}\begin{vmatrix} 0.05 \\ 0.1 \end{vmatrix}$

i.e. A = $\begin{pmatrix} 0.2 & 0.05 \\ 0.1 & 0.1 \end{pmatrix}$

Also given market demand vector D = $\begin{bmatrix} 750 \\ 500 \end{bmatrix}$

Now, technology matrix T = I – A = $\begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{bmatrix}$

$$T^{-1} = \frac{\text{Adj (T)}}{|T|}$$

$$|T| = \begin{vmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{vmatrix} = 0.72 - 0.005 = 0.715$$

$$T^{-1} = \frac{1}{|T|}\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.8 \end{bmatrix}$$

$$= \frac{1}{0.715}\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.8 \end{bmatrix}$$

Using $x = T^{-1}D = \dfrac{1}{0.715}\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} 750 \\ 500 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 979.02 \\ 664.34 \end{bmatrix}$$

$\therefore$ x_1 = Rs. 979.02, x_2 = Rs. 664.34

5. Given,

$$A = \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.2 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

$$D = \begin{bmatrix} 35 \\ 0 \\ 100 \end{bmatrix}$$

Technology matrix T = I – A

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.2 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.9 & -0.2 & -0.5 \\ -0.2 & 0.9 & -0.2 \\ -0.3 & -0.4 & 0.8 \end{bmatrix}$$

$$|T| = \begin{vmatrix} 0.9 & -0.2 & -0.5 \\ -0.2 & 0.9 & -0.2 \\ -0.3 & -0.4 & 0.8 \end{vmatrix}$$

$$= 0.9\begin{vmatrix} 0.9 & -0.2 \\ -0.4 & 0.8 \end{vmatrix} + 0.2\begin{vmatrix} -0.2 & -0.2 \\ -0.3 & 0.8 \end{vmatrix} - 0.5\begin{vmatrix} -0.2 & 0.9 \\ -0.3 & -0.4 \end{vmatrix}$$

$$= 0.9\,(0.72 - 0.08) + 0.2\,(-0.16 - 0.06) - 0.5(0.08 + 0.27)$$

$$= 0.576 - 0.044 - 0.175$$

$$= 0.357$$

Let $\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$ be a cofactor matrix of T

Then

T_{11} = cofactor of 0.9 = $\begin{vmatrix} 0.9 & -0.2 \\ -0.4 & 0.8 \end{vmatrix} = 0.64$

T_{12} = Cofactor of –0.2 = $-\begin{vmatrix} -0.2 & -0.2 \\ -0.3 & 0.8 \end{vmatrix} = 0.22$

T_{13} = Cofactor of –0.5 = $\begin{vmatrix} -0.2 & 0.9 \\ -0.3 & -0.\!\cdot\!4 \end{vmatrix} = 0.35$

T_{21}= Cofactor of –0.2 = $-\begin{vmatrix} -0.2 & -0.5 \\ -0.4 & 0.8 \end{vmatrix} = 0.36$

T_{22} = Cofactor 0.9 = $\begin{vmatrix} 0.9 & -0.5 \\ -0.3 & 0.8 \end{vmatrix} = 0.57$

T_{23} = Cofactor of –0.2 = $-\begin{vmatrix} 0.9 & -0.2 \\ -0.3 & -0.4 \end{vmatrix} = 0.42$

T_{31} = Cofactor of –0.3 = $\begin{vmatrix} 0.2 & -0.5 \\ 0.9 & -0.2 \end{vmatrix} = 0.49$

T_{32} = Cofactor of –0.4 = $-\begin{vmatrix} 0.9 & -0.5 \\ -0.2 & -0.2 \end{vmatrix} = 0.28$

T_{33} = Cofactor of 0.8 = $\begin{vmatrix} 0.9 & -0.2 \\ -0.2 & 0.9 \end{vmatrix} = 0.77$

∴ Cofactor matrix is $\begin{bmatrix} 0.64 & 0.22 & 0.35 \\ 0.36 & 0.57 & 0.42 \\ 0.49 & 0.28 & 0.77 \end{bmatrix}$

$$\text{Adj (T)} = \begin{bmatrix} 0.64 & 0.36 & 0.49 \\ 0.22 & 0.57 & 0.28 \\ 0.35 & 0.42 & 0.77 \end{bmatrix}$$

$$T^{-1} = \frac{1}{0.357}\begin{bmatrix} 0.64 & 0.36 & 0.49 \\ 0.22 & 0.57 & 0.28 \\ 0.35 & 0.42 & 0.77 \end{bmatrix}$$

Using $x = T^{-1}D = \frac{1}{0.357}\begin{bmatrix} 35 \\ 0 \\ 100 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \\ 250 \end{bmatrix}$$

$\therefore$ $X_1 = 200$, $X_2 = 100$, $X_3 = 250$

6. From, 1(b),

Input coefficient matrix $A = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$ and given,

Demand vector $D = \begin{bmatrix} 400 \\ 110 \\ 250 \end{bmatrix}$

Exercise 22.3

1. Solution:

a. Given, Demand function $p = 100 - Q^2$

at $Q = 8$, $p = 100 - 64 = 36$

Consumer's surplus (C.S.) = $\int_0^8 (100 - Q^2)\, dQ - 36 \times 8$

$= \left[100Q - \frac{Q^3}{3}\right]_0^8 - 288$

$= 341.33$

b. Given demand function

$p = \frac{80}{\sqrt[3]{Q}}$

$p = 80Q^{-1/3}$

When $Q = 64$ then $p = 80(64)^{1/3} = 20$

Consumers' surplus (c.s.) = $\int_0^Q p\,dQ - p\times Q$

C.S. $\int_0^{64} 80\, Q^{-1/3}\, dQ - 20 \times 64$

$= 80\,\frac{3}{2}\left[Q^{\frac{2}{3}}\right]_0^{64} - 1280$

$= 120 \times 16 - 1280$

$= 640$

c. $Q = \frac{10 - p}{2p}$ at $p = 2$

$Q = \frac{5}{p} - \frac{1}{2} \Rightarrow p = \frac{10}{2Q+1}$

When $p = 2$ then $Q = 2$

Consumer's demand (C.S.) = $\int_0^Q pdQ - p\times Q$

$$= \int_0^2 \frac{10}{2Q+1}\, dQ - 2\times 2$$

$$= \frac{10}{2}\,[\ln(2Q+1)]_0^Q - 4$$

$$= 5 \ln 5 - 4$$

d. Given, $p = \frac{2Q}{Q^2+1}$

When Q = 10, then $p = \frac{20}{101}$

Consumer's surplus (C.S.) = $\int_0^{10} p\, dQ - p \times Q$

$$= \int_0^{10} \frac{2Q}{Q^2+1}\, dQ - \frac{20}{101} \times 10$$

$$[\ln(Q^2+1)]_0^{10} - \frac{200}{101}$$

$$= \ln 101 - \frac{200}{101}$$

$$= 2.63$$

2. Solution:

a. Given, supply sunction p = 12 + 2Q

When Q = 5 then p = 22

Producer surplus (P.S.) = $P\times Q - \int_0^Q pdQ$

$$= 22\times 5 - \int_0^5 (121 + 2Q)\, dQ$$

$$= 110 - [12Q + Q^2]_0^5$$

$$= 110 - 85$$

$$= 25$$

b. Given,

$P = 20\sqrt{Q} + 15$ at Q = 25

When Q = 25 then p = 115

Then P.S. = $P \times Q - \int_0^Q pdQ$

$$= 115 \times 25 - \int_0^{25} (20\sqrt{Q} + 15)\, dQ$$

$$= 2875 - \left[20\,\frac{Q^{3/2}}{3/2} + 15Q\right]_0^{25}$$

$$= 2675 - \left(\frac{40}{3} \times 125 + 375\right)$$

$$= 2875 - 2041.67$$

$$= 833.33$$

3. Solution:

a. Given, Demand function $p = \frac{4000}{Q + 20}$

Supply function p = Q + 50
For equilibrium
Supply = demand
i.e. (Q + 50) (Q + 20) = 4000
$Q2 + 70Q - 3000 = 0 \Rightarrow Q = 30$
When Q = 30 then p = 80

Now, consumer's surplus = $\int_0^Q$ demand function – P×Q

$$= \int_0^{30} \frac{4000}{Q + 20} dQ - 80 \times 30$$

$$= 4000 [\ln (Q + 20)]_0^{30} - 2400$$

$$= 400 \ln\left(\frac{50}{20}\right) - 2400$$

$$= 1265.16$$

Producer's surplus (P.S.) = P × Q – $\int$supply function

$$= 80 \times 30 - \int_0^{30} (Q + 50)\, dQ$$

$$= 2400 - \left[\frac{Q^2}{2} + 50Q\right]_0^{30}$$

$$= 2400 - (450 + 1500)$$

$$= 450$$

b. Given, $p_d = 74 - Q^2{}_d$
$p_s = Q^2{}_s + 2$
For equilibrium, $p_d - p_s = p$
$Q_d = Q_s = Q$
$Q^2 + Q = 74 - Q^2$
$2Q^2 = 72$
Q = 6
When Q = 6 then p = 38

Consumer's surplus (C.S.) = $\int_0^6 p_d\, dQ - p \times Q$

$$\int_0^6 (74 - Q^2{}_d)\, dQ - 38 \times 6$$

$$\left[74Q - \frac{Q^3}{3}\right]_0^6 - 228 = 144$$

And, P.S. = P × Q – $\int_0^Q p_s\, dQ$

$$= 228 - \int_0^6 (Q^2 + 2)\, dQ$$

$= 228 - \left[\frac{Q^3}{3} + Q\right]_0^6$

$= 144$

c. Given, demand function $p = 100\ e^{-Q/5}$

Supply function $p = 20\ e^{2Q/5}$

For market equilibrium,

Supply function = Demand function

i.e. $20\ e^{2Q/5} = 100\ e^{-Q/5}$

$e^{3Q/5} = 5$

$\frac{3Q}{5} = \ln 5$

$Q = 2.68$

When $Q = 2.68$ then $p = 20\ e^{1.073} = 58.48$

Now, consumer surplus (C.S.) $= \int_0^6 100e^{-Q/5}\, dQ - 58.48 \times 2.68$

$= 100\left[\frac{e^{-Q/5}}{-1/5}\right]_0^{2.68} - 156.73$

$= -500\,(e^{-2.68/5} - e^0) - 156.73$

$= 50.73$

Producer surplus (P.S.) $= P \times Q - \int_0^Q 20\ e^{2Q/5}\, dQ$

$= 156.73 - \int_0^{2.68} 20\ e^{2Q/5}\, dQ$

$= 156.73 - 20 \times \frac{5}{2}\,[e^{2Q/5}]_0^{2.68}$

$= 156.73 - 50\,(e^{0.4\times 2.68} - e^0)$

$= 60.7$

4. Given, Supply function $p = 3 + 4Q$

Producer's surplus at $Q = \propto$ is 72

When $Q = \propto$ then $p = 4 \propto 3$

Now, using

P.S. $= p \times Q - \int_0^Q (3 + 4Q)\, dQ$

$72 = (4\alpha + 3)\,\alpha - [3Q + 2Q^2]_0^{\alpha}$

$72 = (4\alpha^2 + 3\alpha) - (3\alpha + 2\alpha^2)$

$72 = 4\alpha + 3\alpha - 2\alpha 2$

$2\alpha^2 = 72$

$\therefore\ \alpha = 6$

5. Given, $Q_d = \gamma - \delta p$, $Q_s = \beta p - \alpha$

At equilibrium, $Q_d = Q_s$

i.e. $\gamma - \delta p = \beta p - \alpha$

$p = \frac{\propto + \gamma}{\beta + \delta}$

When $p = \frac{\propto + \gamma}{\beta + \delta}$ then $Q = \gamma - \delta\left(\frac{\propto + \gamma}{\beta + \delta}\right) = \frac{\beta\gamma + \delta\gamma - \delta\propto - \delta\gamma}{\beta + \delta} = \frac{\beta\gamma - \delta\propto}{\beta + \delta}$

Producer's surplus (P.S.) = $P \times Q - \int_0^Q$ supply function

$$= \left(\frac{\propto + \gamma}{\beta + \delta}\right)\left(\frac{\beta\gamma - \delta\propto}{\beta + \delta}\right) - \int_0^{\beta\gamma - \delta\propto/\beta+\delta} \left(\frac{Q + \propto}{\beta}\right) dQ$$

$$= \frac{(\propto + \gamma)(\beta\gamma - \delta\propto)}{(\beta + \delta)^2} - \frac{1}{-\beta}\int_0^{\beta\gamma - \delta\propto/\beta+\delta} (Q + \alpha)\, dQ$$

$$= \frac{(\propto + \gamma)(\beta\gamma - \delta\propto)}{(\beta + \delta)^2} - \frac{1}{\beta}\left[\frac{Q^2}{2} + \alpha Q\right]_0^{\beta\gamma - \delta\propto/\beta+\delta}$$

$$= \frac{(\propto + \gamma)(\beta\gamma - \delta\propto)}{(\beta + \delta)^2} - \frac{1}{\beta}\left[\frac{\left(\frac{\beta\gamma - \propto\delta}{\beta + \delta}\right)^2}{2} + \frac{\propto(\beta\gamma - \propto\delta)}{\beta + \delta}\right]$$

6. Given, Demand $p = 80 - 6\sqrt{Q}$

when p = 62 then $62 = 80 - 6\sqrt{Q}$

$6\sqrt{Q} = 18$

$\therefore \quad Q = 9$

$$C.S. = \int_0^9 (80 - 6\sqrt{Q})\, dQ - P \times Q$$

$$= \left[80Q - \frac{6Q^{3/2}}{3/2}\right]_0^9 - 62 \times 9$$

$= (720 - 4 \times 27) - 558$

$= 54$

Again, when p = 56 then $56 = 80 - 6\sqrt{Q}$

$6r\sqrt{Q} = 24$

$\therefore \quad Q = 16$

$$C.S. = \int_0^{16} (80 - 6\sqrt{Q})dQ - 56 \times 16$$

$= [80Q - 4Q^{3/2}]_0^{16} - 896$

$= (1280 - 256) - 896$

$= 128$

Change in C.S. is $128 - 54 = 74$

7. Given, supply function $aP - bQ = 1$

$$P = \frac{1 + bQ}{a}$$

Price = 12 and quantity = 6

Producer's surplus (P.S.) = $P \times Q - \int_0^Q$ supply function.

$$18 = 12 \times 6 - \int_0^6 \left(\frac{1 + bQ}{a}\right) dQ$$

or, $\frac{1}{a}\left[Q + b\frac{Q^2}{2}\right]_0^6 = 54$

or, $\frac{1}{a}(6 + 18b) = 54$

$18b + 6 = 54a$

$\therefore$ 54a – 18b = 6 ... ... (i)

Since, ap – bQ = 1

a12 – b6 = 1

12a – 6b = 1 ... ... (ii)

Solving (i) and (ii),

54a – 18b = 6

36a – 18b = 3

– + –

18a = 3

$a = \frac{1}{6}$

Substituting the alue of a in (ii) we get,

12a – 6b = 1

$12 \times \frac{1}{6} - 6b = 1$

2 – 6b = 1

6b = 1

$\therefore \quad b = \frac{1}{6}$

Hence, $a = \frac{1}{6}$ and $b = \frac{1}{6}$

Exercise 22.4

1. Form the difference equations from

a. $Y_t = 8(4^t) - 7 \Rightarrow Y^t + 7 = 8(4)^t$

Replacing T by T+1

$Y_{T+1} = 8(4^{t+1}) - 7$

$= 84^t.4 - 7$

$= 4(4_t + 7) - 7$

$= 4Y_t + 28 - 7$

$= 4Y_t + 21$

Hence, the required difference equation is,

$Y_t + = 4Y_t + 21$

Equivalent $Y_t = 4Y_{t+1} + 21$

$\therefore \quad Y_T = 4Y_{t+1} + 21)$

b. $Y_t = A4^t + B.7^t$

We have,

$Y_t = A(4^t) + B(7^t)$

Replacing t by t+1

$Y_{t+1} = A(4^{t+1}) + B(7^{t+1})$

$A4^t.(4) + B7^t.(7)$

From the given equation $(Y_t - A4^t) = B7^t$ so

$Y_{t+1} = 4A4^t + 7(Y_t - A4^t)$

$= 4A4^t + 7Y_t - 7A4^t$

$Y_{t+1} = 7Y_t - 3A4^t$

again,

Replacing t by t+1

$Y_{t+2} = 7Y_{t+1} - 3A(4^{t+1})$

From the given equation $(7Y_t - Y_{t+1}) = 4A4^{t+1})$

$= 7Y_{t+1} - 3(7Y_t - Y_{t+1})$

$= 7Y_{t+1} - 21Y_t + 3Y_{t+1} = 0$

$= 10Y_{t+1} - 21Y_t$

so required difference equation is

$\therefore\ Y_{t+2} - 10Y_{t+1} - 21Y_t = 0$

2. Solve the following difference equation

a. $Y_T = Y_{T-1} + 2$, $Y_o = 2$

Comparing the equation with $Y_T = aY_{T-1} + b$, we have

$a = 1$

$b = 2$

Since $a \neq 1$

Another method,

The complementary function (c.f.) $= Aa^T = A(1)^T$

Let $Y_T = KT$ be a particulars solution

Then $Y_{T-1} = K.(T-1)$

Substituting the value Y_T and Y_{T-1}

$K_T = K(T - 1) + 2$

$K_T - K(T-1) = 2$

$K_T - K_T + K = 2$

$\therefore\quad K = 2$

So, PS = 2

The required general solution is

$Y_T = CF + PS$

$= A(1)^T + 2 \Rightarrow Y_T = A(1)^T + 2$

As given, $Y_o = 2$, then

$2 = A + 2$

$A = 0$

Now, $(Y_T = (1)^T + 2)$

b. $Y_{T+1} = -Y_T + 6$, $Y_o = 4$

$Y_T = -Y_{T+1} + 6$

Comparing the equation with $Y_T = aY_{T+1} + b$, we have

$a = -1$,

$b = 6$

Since $a \neq 1$, the required solution is

$Y_T = Aa^T + \dfrac{b}{1 - a}$, where A is constant.

Substituting the value of a, b

$Y_T = A(-1)^T + \dfrac{6}{1 + 1}$

$Y_T = A(-1)^T + 3$

Putting,

Given $Y_0 = 4$, $T = 0$

Then,

$4 = A(-1)^0 + 3$

$4 - 3 = A$

$A = 1$,

Now,

$Y_T = 1(-1)^T + 3$, is the required solution,

c. $4Y_T = Y_{T-1} + 24$

$Y_T = \dfrac{1}{4} Y_{T-1} + 6$

Comparing the equation with $Y_T = aY_{T+1} + b$, we have

$a = \dfrac{1}{4}$, $b = 6$

Since $a \neq 1$, the required solution is

$Y_T = A4^T + \frac{b}{1-a}$ where A is constant.

Substituting the values of a and b

$Y_T = A\left(\frac{1}{4}\right)^T + \frac{6}{1-\frac{1}{4}}$

$Y_T = A\left(\frac{1}{4}\right)^T + 8 = A(0.25)^t + 8$

d. $Y_T = -0.5Y_{T-1} + 1$

Comparing with the equation $Y_T = aY_{T+1} + b$, we have

$a = -0.5$

$b = 1$

Since $a \neq 1$ the required

Solution is

$Y_T = Aa^T + \frac{b}{1-a}$, where A is constant.

Substituting the value of a and b

$Y_T = A(-0.5)^T + \frac{1}{1+0.5}$

$Y_T = A(-0.5)^T + 0.66$

3. Solution

Given, $Y_T = 3Y_{T-1} + 7$, $Y_o = 2$

a. Find the value of Y_1, Y_2, Y_3 without solving d.e. when, Y_1 then, Y_2 then,

$Y_1 = 3Y_{1-1} + 7$	$Y_2 = 3Y_{2-2} + 7$
$= 3Y + 7$	$Y_2 = 3Y_1 + 7$
$3Y = -7$	$= 3\left(\frac{-7}{3}\right) + 7$
$Y = \frac{-7}{3}$	$= -7 + 7 = 0$
$\therefore \quad Y_1 = \frac{-7}{3}$	$\therefore Y_2 = 0$

Y_3, then,

$Y_3 = 3Y_{3-1} + 7$

$= 3Y_2 + 7$

$= 3(0) + 7$

$\therefore \quad Y_3 = 7$

$\therefore \quad Y_1 = \frac{-7}{3}$, $Y_2 = 0$ and $Y_3 = 7$

b. Find Y_1, Y_2, Y_3 using this solution.

Comparing with the equation $Y_T = aY_{T-1} + b$, we have

$a = 3$

$b = 7$

Since, $a \neq 1$, the required solution is,

$Y_T = Aa^T + \frac{b}{1-a}$

Substituting the value and a, b

$Y_T = A(3)^T + \frac{7}{1-3}$

$Y_T = A(3)^T - 3.5$

When, $Y_0 = 2$, then

$2 = A(3)^0 - 3.5$

$2 + 3.5 = A$

$\therefore \quad A = 5.5$

Now, $Y_T = 5.5(3)^T - 3.5$

When, Y_1, Y_2, Y_3, 300

$Y_1 = 5.5(3)^1 - 3.5$	$Y_2 = 5.5(3)^2 - 3.5$	$Y_3 = 5.5(3)^3 - 3.5$
$Y_1 = 13$	$= 46$	$= 145$

$$\therefore \begin{pmatrix} Y_1 = 13 \\ Y_2 = 46 \\ Y_3 = 145 \end{pmatrix}$$

4. Given, $Y_T = 0.3Y_{T-1} + 0.4T + 5$

a. $y_t = 1.008\, y_{t-1} - 4{,}000$

$y_c = m = 1.008$

$y_c = A(1.008)^t$

Particular integral

(y_p) = let $y_t = k$ be

$y_{t-1} = k$

$k - 1.008k = -4{,}000$

$k = 5{,}00{,}000$

$\therefore \quad y_t = A(1.008)^t + 5{,}00{,}000$

$y_0 = 1{,}50{,}000$

$1{,}50{,}000 = A + 5{,}00{,}000$

$A = -3{,}50{,}000$

$y_t = -3{,}50{,}000\,(1.008)^t + 5{,}00{,}000$

$y_{12} = 114881.46$

c. We have, $y_t = A(1.008)^t + 5{,}00{,}000$

To pay the loan, $y_t = 0$

$3{,}50{,}000\,(1.008)^t = 5{,}00{,}000$

$(1.008)^t = 1.43$

$$t = \frac{\ln(1.43)}{\ln(1.008)}$$

$= 45$ months

.

Exercise 22.5

1. Given,

$Q_{ST} = P_{T-1} - 8$

$Q_{dT} = -2P_T + 22$

For equilibrium

$Q_{ST} = Q_{dT}$

So, $P_T - 1 - 8 = -2P_T + 22$

$P_{T-1} - 8 + 2P_T - 22 = 0$

$2P_T = -P_{T-1} + 30$

$$P_T = -\frac{1}{2}P_{T-1} + 15$$

Comparing with $P_T = aP_{T-1} + b$

Now, $a = \frac{-1}{2}$, $b = 13$

The general solution Rs. $P_T = AaT + \frac{b}{1-a}$

Where, A is constant.

Substituting the values,

$$P_T = A\left(\frac{-1}{2}\right)^T + \frac{15}{1 + \frac{1}{2}}$$

$P_T = A\left(\frac{-1}{2}\right)^T + 10$ ... ... (i)

When, $P_o = 11$,

Now, $11 = A + 10$

$11 - 10 = A$

$\therefore \quad A = 1$

Now, $P_T = 1\left(1 - \frac{1}{2}\right)^T + 10$

Putting this expression in $Q_{dT} = -2_{PT} + 22$

$= -2\left[1\left(\frac{-1}{2}\right)^T + 10\right] + 22$

$= -2\left(1 - \frac{1}{2}\right)^T + 2$

Since, $|a| = |-\frac{1}{2}| = | = \frac{1}{2} > 0$. So it is stable.

2. Given,

$Q_{2t} = -5P_T + 35$

$Q_{ST} = 4P_{T-1} - 10$

For equation

$Q_{dT} = Q_{ST}$

$4P_{T-1} - 10 = -5P_T + 35$

$5P_T = -4P_{T-1} + 45$

$P_T = \frac{-4}{5}P_{T-1} + 9$

Comparing with $P_T = aP_{T-1} + b$,

so, $a = \frac{-4}{5}$, $b = 9$

The general solution is $P_T = Aa^T \neq \frac{Aa}{1 - a}$ (A is constants)

Substitution the values

$P_T = A\left(\frac{-4}{5}\right)^T + \frac{9}{1 + \frac{4}{5}} = A\left(\frac{-4}{5}\right)^T + 5$

When, $P_o = 6$ then

$6 = A + 5$

$\therefore \quad A = 1$

$\therefore \quad P_T = 1\left(\frac{-4}{5}\right)^T + 5$

Putting this expression is $Q_{dT} = -5P_T + 35$

$= -5\left[1\left(\frac{-4}{5}\right)^T + 3\right] + 33$

$= -5\left(\frac{-4}{5}\right)^T + 10$

$\therefore \quad Q_{dT} = -5\left(\frac{-4}{5}\right)^T + 10$

$P_T = 1\left(\frac{-4}{5}\right)^T + 5 = (-0.8)^t + 5$

3. Given,

$Q_d = -4p + 10$

$Q_s = 6p - 10$

a. For equilibrium, $Q_d \neq Q_s$

$-4p + 10 = 6p - 10$
$-10p = -20$
$p = 2$
Substituting the value of $Q_s = 6p - 10$
$6\times2 - 10$
$\therefore\ Q = 2$
$\therefore\ \begin{pmatrix} p = 2 \\ Q = 2 \end{pmatrix}$

4. Given,
$y_t = c_t + I_t$
$= 0.75y_{t-1} + 400 + 200$
$\therefore\ y_t = 0.75y_{t-1} + 600$
If $t = 1$,
$y_1 = 0.75\, y_0 + 600$
$= 0.75 \times 400 + 600$
$= 900$
So, from $c_t = 0.75\, y_{t-1} + 400$
$c_2 = 0.75y_1 + 400 = 1075$

5. We have,
$y_t = c_t + I_t$
$y_t = 0.7y_{t-1} + 400 + 0.1y_{t-1} + 100$
or, $y_t = 0.8y_{t-1} + 500$
$y_t - 0.8y_{t-1} = 500$ ... ... (i)
Solution of (i) is $y_t = y_c + y_p$ where
y_c = complementary function
y_p = particular integral
For complementary function (y_c) : Reduce (i) into homogeneous form as
$y_t - 0.8y_{t-1} = 0$ ... ... (ii)
Let $y_t = A(m)^t$ be a trial solution.
Then $y_{t-1} = Am^{t-1}$
from (ii)
$Am^t - 0.8\, Am^{t-1} = 0$
$Am^t\,(1 - 0.8m^{-1}) = 0$
$m = 0.8$ since $Am^t \neq 0$
$\therefore\ y_c = A(0.8)^t$
For particular integral (y_p) :
Let $y_t = k$ be a trial solution of (i).
Then $y_{t-1} = k$
$\therefore$ (i) becomes
$k - 0.8k = 500$
$0.2k = 500$
$k = \dfrac{500}{0.2} = 2500$
$\therefore\ y_p = 2500$
$\therefore\ y_t = A(0.8)^t + 2500$ is general solution.
When $t = 0$ then $y_0 = A(0.8)^0 + 2500$
$300 = A + 2500$
$A = 500$
$\therefore\ y_t = 500\,(0.8)^t + 2500$ is required particular solution for y_t.

6. Given,
$y_t = c_t + I_t$

$y_t = (0.8y_{t-1} + 200) + 1000$
$y_t = 0.8y_{t-1} + 1200$
$y_t - 0.8y_{t-1} = 1200$ ... ... (i)

a. When t = 1 then
$y_1 - 0.8y_0 = 1200$
$y_1 - 0.8 \times 5000 = 1200$
$y_1 = 5200$
When t = 2
Then $y_2 = 0.8y_1 = 1200$
$y_2 \quad = 1200 + 0.8\ 5200 = 5360$
We have,
$c_t \quad = 0.8y_{t-1} + 200$
When t = 2
$c_2 \quad = 0.8 \times y_1 + 200$
$\quad = 0.8 \times 5200 + 200$
$\quad = 4360$

b. The difference equation relating $y_t + y_{t-1}$ is $y_t - 0.8\ y_{t-1} = 1200$ ... ... (i)

c. Its solution is $y_t = y_c + y_p$
For y_c:
$y_t - 0.8y_{t-1} = 0$
$Am^t\ (1 - 0.8\ m^{-1}) = 0$
$\therefore \quad m = 0.8$ since $Am^t \neq 0$
$y_c = A(0.8)^t$
For y_p:
Let $y_t = k$ be a solution
Then $y_{t-1} = k$
from (i)
$0.2k = 1200$
$k = 6000$
$\therefore \quad y_p = 6000$
Hence $y_t = A(0.8)^t + 6000$
When t = 0
$y_0 = A + 6,000$
$\therefore \quad A = -1,000$ since $y_0 = 5,000$
$\therefore \quad y_t = -1,000\ (0.8)^t + 6,000$
when t = 2
$y_2 = -1,000\ (0.8)^2 + 6,000$
5,340 which is in (a).