Chapter 12
Product of Vectors

Exercise 12.1

1. Solution

a. Given, $\vec{a} = 2\vec{i} - 3\vec{k} = (2, 0, -3)$

$\vec{b} = 2\vec{j} + 4\vec{k} = (0, 2, 4)$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -3 \\ 0 & 2 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & -3 \\ 2 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -3 \\ 0 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$

$= 6\vec{i} - 8\vec{j} + 4\vec{k}$

b. Given vectors

$\vec{a} = 2\vec{i} + 4\vec{k} = (2, 0, 4)$

$\vec{b} = 3\vec{j} - 2\vec{k} = (0, 3 -2)$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 4 \\ 0 & 3 & -2 \end{vmatrix} = -12\vec{i} + 4\vec{j} + 6\vec{k}$

c. Given,

$\vec{a} = 20\vec{i} + 3\vec{j} + \vec{k}$

$\vec{b} = -\vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ -1 & -2 & 3 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix}$

$= 11\vec{i} - 7\vec{j} - \vec{k}$

2. Given $\vec{a} = 3\vec{i} + 4\vec{j} - 5\vec{k}$

$\vec{b} = 7\vec{i} - 3\vec{j} + 6\vec{k}$

$\vec{a}+\vec{b} = (3\vec{i} + 4\vec{j} - 5k) + (7\vec{i} - 3\vec{j} + 6k) = 10\vec{i} + \vec{j} + \vec{k}$

$\vec{a} - \vec{b} = (3i + 4j - 5k) - (7\vec{i} - 3\vec{j} + 6\vec{k}) = -4\vec{i} + 7\vec{j} - 11\vec{k}$

$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 10 & 1 & 1 \\ -4 & 7 & -11 \end{vmatrix}$

$= -18\vec{i} + 106\vec{j} + 74\vec{k}$

$|(\vec{a}+\vec{b}) \times (\vec{a} - \vec{b})| = \sqrt{(-18)^2 + (106)^2 + (74)+} = \sqrt{17036}$

3. Here, $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}\,\vec{i}$

$(\vec{a} + \vec{b}) = (1 + 2)\vec{i} + (1 + 3)\vec{j} + (1 + 1)\vec{k} = 3\vec{i} + 4\vec{j} + 2\vec{k}$

$(\vec{a} - \vec{b}) = (1 - 2)\vec{i} + (1 - 3)\vec{j} + (1 - 1)\vec{k} = -\vec{i} - 2\vec{j} + 0\vec{k}$

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 2 \\ -1 & -2 & 0 \end{vmatrix}$$

$= (0 + 4)\vec{i} - \vec{j}(0 + 2) + \vec{k}(-6 + 4)$

$= 4\vec{i} - 2\vec{j} - 2\vec{k}$

$|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| = \sqrt{4^2 + (-2)^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$

4. Solution:

a. Given vectors $\vec{a} = 4\vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{b} = 5\vec{i} + \vec{j} - 4\vec{k}$

The vector orthogonal to each of given vectors is given by

$\vec{a}\times\vec{b}$

$$\text{Now, } \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 3 \\ 5 & 1 & -4 \end{vmatrix} = (8 - 3)\vec{i} - (-16 - 15)\vec{j} + (4 + 10)\vec{k}$$

$= 5\vec{i} + 31\vec{j} + 14\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{5^2 + 31^2 + 14^2} = \sqrt{1182}$

Unit vector is given as $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \dfrac{5\vec{i} + 31\vec{j} + 14\vec{k}}{\sqrt{1182}}$

b. Here,

$\vec{a} = (6, 3, -5)$ and $\vec{b} = (1, -4, 2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 3 & -5 \\ 1 & -4 & 2 \end{vmatrix}$$

$(6 - 20)\vec{i} - (12 + 5)\vec{j} + (-24 - 4)\vec{k}$

$= -14\vec{i} - 17\vec{j} - 27\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-14)^2 + (-17)^2 + (-27)^2} = \sqrt{1214}$

$\therefore$ Unit vector is $\dfrac{-14\vec{i} - 17\vec{j} - 27\vec{k}}{\sqrt{1214}}$

5. Solution:

a. Given, $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$

$\vec{b} = 4\vec{i} - 7\vec{k}$

$a = |\vec{a}| = \sqrt{4 + 9 + 25} = \sqrt{38}$

$b = |\vec{b}| = \sqrt{4^2 + (-7)^2} = \sqrt{65}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ 4 & 0 & -7 \end{vmatrix} = 21\vec{i} + 34\vec{j} + 12\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{21^2 + 34^2 + 12^2} = \sqrt{1741}$

We know that $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|} = \dfrac{\sqrt{1741}}{\sqrt{38}\,.\,\sqrt{65}} = \sqrt{\dfrac{1741}{2470}}$

b. Given, $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 9)$

$a = |\vec{a}| = \sqrt{9+1+4} = \sqrt{14}$

$b = |\vec{b}| = \sqrt{4+4+81} = \sqrt{89}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 9 \end{vmatrix} = 13\vec{i} - 23\vec{j} - 8\vec{k}$$

$|\vec{a}\times\vec{b}| = \sqrt{13^2 + (-23)^2 + (-8)^2} = \sqrt{762}$

$\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \dfrac{\sqrt{762}}{\sqrt{14\times 89}} = \sqrt{\dfrac{762}{14\times 89}} = \sqrt{\dfrac{381}{623}}$

c. Given rectors are $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$

$a = |\vec{a}| = \sqrt{9+1+4} = \sqrt{14}$

$b = |\vec{b}| = \sqrt{4+4+16} = \sqrt{24}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 8\vec{i} - 8\vec{j} - \vec{k}$$

$|\vec{a}\times\vec{b}| = \sqrt{64+64+64} = \sqrt{192}$

$\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{ab} = \sqrt{\dfrac{192}{14\times 24}} = \dfrac{2}{\sqrt{7}}$

6. Let O be the origin

Given $\vec{OP} = \vec{i} + \vec{j} + 2\vec{k}$

$\vec{OQ} = 2\vec{i} + 3\vec{j} + \vec{k}$

$\vec{OP} = 3\vec{i} - \vec{j} + 4\vec{k}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \vec{OQ} = (2, 3, 1) - (1, 1, 2) = (1, 2, -1) = \vec{i} + 2\vec{j} - \vec{k}$

$\vec{QR} = \vec{OR} - \vec{OQ} = (3, -1, 4) - (2, 3, 1) = (1, -4, 3) = \vec{i} - 4\vec{j} + 3\vec{k}$

$$\overrightarrow{PQ}\times\overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 1 & -4 & 3 \end{vmatrix} = 2\vec{i} - 4\vec{j} - 6\vec{k}$$

Hence, $2\vec{i} - 4\vec{j} - 6\vec{k}$ is a vector perpendicular to both $\vec{PQ}$ and $\vec{QR}$ and hence perpendicular to the plane PQR.

7. Solution:

a. $\vec{a} = 3\vec{i} + \vec{j} + \vec{k}$

$\vec{b} = \vec{i} - 2\vec{j} - \vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \vec{i} + 4\vec{j} - 7\vec{k}$$

$|\vec{a}\times\vec{b}| = \sqrt{1+16+49} = \sqrt{66}$

$\therefore$ Area of triangle determined by $\vec{a}$ and $\vec{b}$ is given by

$\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{66}$ sq. units

b. Given vectors $\vec{a} = (3, 4, 0)$ and $\vec{b} = (-5, 7, 0)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ -5 & 7 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 41\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{0^2 + 0^2 + 41^2} = 41$

Area of triangle determined by the vectors $\vec{a}$ and $\vec{b}$ is given by

$\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2} \times 41 = 20\frac{1}{2}$ sq. unit

8. Let 0 be the origin. Let A, B and C be vertices of triangle

Then $\vec{OA} = 3\vec{i} - \vec{j} + 2\vec{k}$

$\vec{OB} = \vec{i} - \vec{j} - 3\vec{k}$

$\vec{OC} = 4\vec{i} - 3\vec{j} + 2\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = -2\vec{i} + 0\vec{j} - 5\vec{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = 3\vec{i} - 2\vec{j} + 5\vec{k}$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & -5 \\ 3 & -2 & 5 \end{vmatrix} = -10\vec{i} - 5\vec{j} + 4\vec{k}$$

$|\vec{AB} \times \vec{BC}| = \sqrt{100 + 25 + 16} = \sqrt{141}$

$\therefore$ Area of triangle ABC $= \frac{1}{2}|\vec{AB} \times \vec{BC}| = \frac{1}{2}\sqrt{141}$ sq. units

9. Solution:

a. Given,

$\vec{a} = 7\vec{i} + 8\vec{j} - \vec{k}$

$\vec{b} = 10\vec{i} - 11\vec{j} + 12\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 8 & -1 \\ 10 & -11 & 12 \end{vmatrix} = (96 - 11)\vec{i} - (84 + 10)\vec{j} + (-77 - 80)\vec{k}$$

$= 85\vec{i} - 94\vec{j} - 157\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{85^2 + 94^2 + 157^2} = \sqrt{40710}$

$\therefore$ Area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\sqrt{40710}$ sq. units.

b. $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$

$\vec{b} = \vec{i} - 2\vec{j} + 4\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -2 & 4 \end{vmatrix} = (8+6)\vec{i} - (4 - 3)\vec{j} + (-2 - 2)\vec{k}$$

$= 14\vec{i} - \vec{j} - 4\vec{k}$

$|\vec{a}\times\vec{b}| = \sqrt{14^2 + 1^2 + 4^2} = \sqrt{196 + 1 + 16} = \sqrt{2131}$

$\therefore$ Area of parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{213}$ sq units.

c. Given, $\vec{a} = (1, -2, 3)$ and $\vec{b} = (3, 2, 2)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 3 & 2 & 2 \end{vmatrix} = (-4-6)\vec{i} - (2-9)\vec{j} + (2+6)\vec{k}\vec{k}$$

$$= -10\vec{i} + 7\vec{j} + 8\vec{k}$$

$|\vec{a}\times\vec{b}| = \sqrt{100+49+64} = \sqrt{213}$

$\therefore$ Area of parallelogram $= |\vec{a}\times\vec{b}| = \sqrt{213}$ sq units

d. Given,

$\vec{a} = (1, -2, 3)$ and $\vec{b} = (3, 2, 2)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 3 & 2 & 2 \end{vmatrix} = (-4-6)\vec{i} - (2-9)\vec{j} + (2+6)\vec{k}$$

$= -10\vec{i} + 7\vec{j} + 8\vec{k}$

$|\vec{a}\times\vec{b}| = \sqrt{100+49+64} = \sqrt{213}$

Area of parallelogram whose adjacent sides $\vec{a}$ and $\vec{b}$ is given by

$|\vec{a}\times\vec{b}| = \sqrt{213}$ square units.

10. Let $\vec{d_1} = \vec{i} + \vec{j} - \vec{k}$ and $\vec{d_2} = \vec{i} - \vec{j} + \vec{k}$ be two diagonals of a parallelogram.

$$\vec{d_1}\times\vec{d_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 0\vec{i} - 2\vec{j} - 2\vec{k}$$

$|\vec{d_1}\times\vec{d_2}| = \sqrt{0^2+2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

Area of parallelogram whose diagonals $\vec{d_1}$ and $\vec{d_2}$ is given by

$\frac{1}{2}|\vec{d_1}\times\vec{d_2}|$ sq. units

$= \frac{1}{2} . 2\sqrt{2}$ sq. units

$= \sqrt{2}$ sq. units

11. Solution:

a. Given $|\vec{a}| = 15$, $|\vec{b}| = 4$ and $|\vec{a}\times\vec{b}| = 36$

If θ be the angle between two vectors $\vec{a}$ and $\vec{b}$ then

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{36}{15\times4} = \frac{9}{15} = \frac{3}{5}$$

$$\therefore \quad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

Also, we know that $\cos\theta = \dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}$

$$\therefore \quad \vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$= 15\times4\times\frac{4}{5}$$

$$= 48$$

$\therefore \quad \vec{a}.\vec{b} = 48$

b. Given, $|\vec{a}| = 9$, $|\vec{b}| = 5$ and $\vec{a}.\vec{b} = 36$

If θ be the angle between $\vec{a}$ and $\vec{b}$

Then, $\cos\theta = \dfrac{\vec{a}.\vec{b}}{ab} = \dfrac{36}{9\times5} = \dfrac{4}{5}$

$\therefore \quad \sin\theta = \sqrt{1 \cos^2\theta} = \dfrac{3}{5}$

Also, $|\vec{a}\times\vec{b}| = |\vec{a}|\ |\vec{b}| \sin\theta$

$= 9 \times 5 \times \dfrac{3}{5}$

$= 27$

c. LHS

$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a} + \vec{c}\times\vec{b}$

$= \vec{a}\times\vec{b} - \vec{c}\times\vec{a} + \vec{b}\times\vec{c} - \vec{a}\times\vec{b} + \vec{c}\times\vec{a} - \vec{b}\times\vec{c}$

$= 0$ RHS

d. Suppose

$\vec{a} + \vec{b} + \vec{c} = 0$

$\vec{a} + \vec{b} = -\vec{c}$ ... ... (i)

Taking cross product with $\vec{a}$ on both sides

$\vec{a} \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{c})$

or, $\vec{a}\times\vec{a} + \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$

or, $0 + \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$

or, $\vec{a}\times\vec{b} = \vec{c}\times\vec{a}$ ... ... (ii)

Again, taking cross product with $\vec{b}$ on equation (i) both sides

$\vec{b} \times (\vec{a} + \vec{b}) = \vec{b} \times (-\vec{c})$

$\vec{b} \times \vec{a} + \vec{b}\times\vec{b} = -\vec{b}\times\vec{c}$

or, $-\vec{a}\times\vec{b} = 0 = -\vec{b}\times\vec{c}$

$\therefore \quad \vec{a}\times\vec{b} = \vec{b}\times\vec{c}$ ... ... (iii)

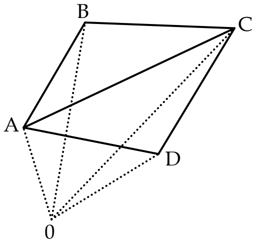
Combining (ii) and (iii) we get,

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ Proved.

12. Solution:

Let 0 be the origin suppose A, B, C, D are vertices of a quadrilateral ABCD.

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$ and $\vec{OD} = \vec{d}$



Now, $\vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

$\vec{BC} = \vec{OC} - \vec{OB} = \vec{c} - \vec{b}$

$\vec{AC} = \vec{OC} - \vec{OA} = \vec{c} - \vec{a}$

$\vec{CD} = \vec{OD} - \vec{OC} = \vec{d} - \vec{c}$

Vector area of $\Delta ABC = \dfrac{1}{2} \vec{AB} \times \vec{BC}$

$= \dfrac{1}{2} (\vec{b} - \vec{a}) \times (\vec{c} - \vec{b})$

$= \dfrac{1}{2} [\vec{b}\times\vec{c} - \vec{b}\times\vec{b} - \vec{a}\times\vec{c} + \vec{a}\times\vec{b}]$

$= \dfrac{1}{2} [\vec{b}\times\vec{c} - 0 + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}]$

$$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}]$$

Again, vector area of $\Delta ACD = \frac{1}{2}(\vec{AC}\times\vec{CD})$

$$= \frac{1}{2}[(\vec{c} - \vec{a}) \times (\vec{d} - \vec{c})]$$

$$= \frac{1}{2}[\vec{c}\times\vec{d} - \vec{c}\times\vec{c} - \vec{a}\times\vec{d} + \vec{a}\times\vec{c}]$$

$$= \frac{1}{2}[\vec{c}\times\vec{d} + \vec{d}\times\vec{a} + \vec{a}\times\vec{c}]$$

$\therefore$ Vector of quadrilateral ABCD = vector area of ΔABC + vector area of ΔACD

$$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{c}+\vec{d} + \vec{d}\times\vec{a} + \vec{a}\times\vec{b}\times\vec{c}]$$

$$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{c}+\vec{d} + \vec{d}\times\vec{a} - \vec{c}\times\vec{a}]$$

$$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}]$$

13. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be the position vectors of the vertices of a quadrilateral ABCD, then the vector area of this quadrilateral is given by

$\frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}]$ ... ... (i)

Now, $\vec{AC} = \vec{OC} - \vec{OA} = \vec{C} - \vec{a}$

$\vec{BD} = \vec{OD} - \vec{OB} = \vec{d} - \vec{b}$

$\overrightarrow{AC} \times \overrightarrow{BD} = (\vec{c} - \vec{a}) \times (\vec{d}-\vec{b})$

$= \vec{c}\times\vec{d} - \vec{c}\times\vec{b} - \vec{a}\times\vec{d} + \vec{a}\times\vec{b}$

$= \vec{c}\times\vec{d} + \vec{b}\times\vec{c} + \vec{d}\times\vec{a} + \vec{a}\times\vec{d}$

$= \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}$

$\therefore$ $\frac{1}{2}\vec{AC} \times \vec{BD} = \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}]$ ... ... (ii)

$\therefore$ Combining the result of (i) and (ii),

Vector area of quadrilateral ABCD = $\frac{1}{2}\vec{AC} \times \vec{BD}$ proved.

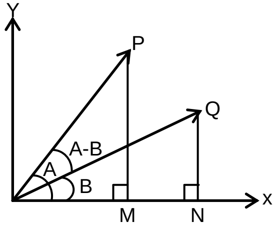
14. Solution:

Let OX and OY be two co-ordinate axes. Let P and Q be two points even that $\sphericalangle XOP = A$ and XOQ = B so that $\sphericalangle POQ = A–B$. Let $OP = r_1$ and $OQ = r_2$.
Draw 1_r PM and QN on x–axis.

$\vec{OP}$ = (OM, MP) = (OP cosA, OP sinA) = (r_1 cosA, r_1 sinA)

$\vec{OQ}$ = (ON, NQ) = (OQ cosB, OQ sinB) = (r_2 cosB, r_2 sinB)

Since, angle between OQ and OP is A–B.



$\therefore$ $|\vec{OP}\times\vec{OQ}| = |\vec{OP}||\vec{OQ}| \sin(A - B)$ ... ... (i)

Now, $\vec{OP}\times\vec{OQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_1\cos A & r_1\sin A & 0 \\ r_2\cos B & r_2\sin B & 0 \end{vmatrix}$

$= 0\vec{i} - 0\vec{j} + (r_1r_2 \cos A \sin B - r_1r_2 \sin A . \cos B)\vec{k}$

$= (0, 0, -r_1r_2 (\sin A \cos B - \cos A \sin B)$

$|\vec{OP}\times\vec{OQ}| = \sqrt{0^2 + 0^2 + (-r_1r_2)^2 (\sin A \cos B - \cos A.\sin B)^2}$

$= r_1r_2 (\sin A . \cos B - \cos A . \sin B)$

from (i)

$\frac{|\vec{OP}\times\vec{OQ}|}{|\vec{OP}||\vec{0Q}|} = \sin(A - B)$

$\therefore \quad \sin(A - B) = \frac{r_1r_2 (\sin A \cos B - \cos A - \sin B)}{r_1r_2}$

$\therefore \quad \sin (A-B) = \sin A. \cos B - \cos A . \sin B$

15. Suppose ABC be a triangle in which $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$

Now, by vector addition,

$\vec{AB} + \vec{Bc} = \vec{Ac}$

$\Rightarrow \quad \vec{c} + \vec{d} = -\vec{b}$

$\therefore \quad \vec{a} + \vec{b} + \vec{c} = \vec{o}$ ... ... (i)

Multiplying (i) vectorically by $\vec{a}$, we get

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{o}$

$\vec{a}\times\vec{a} + \vec{a}\times\vec{b} + \vec{a}\times\vec{c} = 0$

$\therefore \quad \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$ ... ... (ii) $(\because \vec{a}\times\vec{a} = 0$ and $\vec{a}\times\vec{c} = -\vec{c}\times\vec{a})$

Similarly, multiplying (i) vectorially by $\vec{b}$, we get

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c}$ ... ... (iii)

Combining (ii) and (iii) we get

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

or, $|\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$

ab $\sin(\pi - c)$ = bc $\sin(\pi - A)$ = ca $\sin(\pi - B)$

or, absinC = bcsinA = casinB

Dividing by abc, we get

$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ proved.

16. Given,

$\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$

$\vec{b} = 2\vec{i} + \vec{j} - 3\vec{k}$

$\vec{c} = \vec{i} - 2\vec{j} + 2\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \vec{i} + 13\vec{j} + 5\vec{k}$$

$$(\vec{a}\times\vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 13 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 36\vec{i} + 3\vec{j} - 15\vec{k} \text{ ... ... (i)}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & -2 & 2 \end{vmatrix} = -4\vec{i} - 7\vec{j} - 5\vec{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ -4 & -7 & -5 \end{vmatrix} = 19\vec{i} + 7\vec{j} - 25\vec{k} \text{ ... ... (ii)}$$

From (i) and (ii),

Hence $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

17. Given,

$\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{c} = \vec{i} - \vec{j}$

Let $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{a}.\vec{b} = (\vec{i} + \vec{j} + \vec{k}) . (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$

$3 = b_1 + b_2 + b_3$ ... ... (i)

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} = (b_3 - b_2)\vec{i} + (b_1 - b_3)\vec{j} + (b_2 - b_1)\vec{k}$$

or, $\vec{c} = (b_3 - b_2)\vec{i} + (b_1 - b_3)\vec{j} + (b_2 - b_1)\vec{k}$

$\vec{i} - \vec{j} = (b_3 - b_2)\vec{i} + (b_1 - b_3)\vec{j} + (b_2 - b_1)\vec{k}$

Equating corresponding vectors

$b_3 - b_2 = 1$, $b_1 - b_3 = -1$ and $b_2 - b_1 = 0$

i.e. $b_2 - b_1 = 0$

$\therefore$ $b_1 = b_2$ ... ... (ii)

$b_3 = 1 + b_2$ ... ... (iii)

$b_3 = 1 + b_1$ ... ... (iv)

$b_1 + b_2 + b_3 = 3$

$b_1 + b_1 + 1 + b_1 = 3$

$3b_1 = 2$

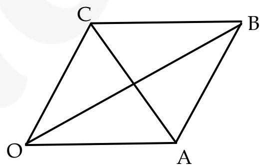
$\therefore$ $b_1 = \frac{2}{3}$

$\therefore$ from (ii) $b_2 = \frac{2}{3}$

from (iv) $b_3 = 1 + \frac{2}{3} = \frac{5}{3}$

$\therefore$ $\vec{b} = (b_1, b_2, b_3) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{5}{3}\vec{k} = \frac{1}{3}(2\vec{i} + 2\vec{j} + 5\vec{k})$

18. Let $\vec{OA} = \vec{a}$ and $\vec{OC} = \vec{b}$



Then, $\vec{OB} = \vec{OA} + \vec{AB}$

$= \vec{A} = \vec{Oc} = \vec{a} + \vec{b}$

and $\vec{AC} = \vec{AO} + \vec{OC}$

$= -\vec{a} + \vec{b} = \vec{b} - \vec{a}$

Now, the vector equation of the line OB is

$\vec{r} = t(\vec{a} + \vec{b})$ ... ... (i)

Where t is a scalar.

Again, the vector equation of the straight line AC is

$\vec{r} = (1 - 5)\,\vec{a} + 5\vec{b}$ ... ... (ii) where s is a scalar.

If two diagonals OB and AC meet at M, then for M, we have

$t(\vec{a} + \vec{b}) = (1 - 5)\,\vec{a} + 5\vec{b}$

Equating the coeff. of like vectors,

t = 1–5 and t = 5

Solving $t = 5 = \frac{1}{2}$

$\therefore$ The position vector of M i.e. $\overrightarrow{OM} = \frac{1}{2}(\vec{a}+\vec{b}) = \frac{1}{2}\,\vec{OB}$

Also, $\vec{AM} = \vec{AO} + \vec{OM}$

$= -\vec{a} + \frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{2}(\vec{b} - \vec{a}) = \frac{1}{2}\vec{AC}$

$\therefore$ Hence the diagonals bisect each other.

Again, $\vec{OB}.\vec{AC} = (\vec{a} + \vec{b}) . (\vec{b} - \vec{a})$

$= (\vec{b})^2 - (\vec{a})^2$

$= b^2 - a^2$

$= OC^2 - OA^2$

$= 0 \ (\because OC = OA)$

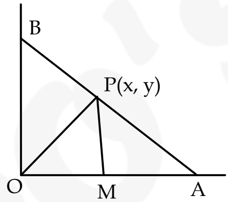
So, the diagonals of a rhombus are right angles.

$\therefore$ The diagonals of a rhombus bisect each other at right angles.

19. Let $\vec{i}$ and $\vec{j}$ be the unit vectors along two mutually perpendicular straight lines OX and OY respectively. Let OA = a and OB = b.

Then $\vec{OA} = a\vec{i}$

$\vec{OB} = b\vec{j}$



Let P(x, y) be a point on the line AB.

From P, draw PM $\perp_r$ to OA.

Then $\vec{OM} = x\vec{i}$ and $\vec{MP} = y\vec{j}$

Join OP

By vector addition,

$\vec{OP} = \vec{OM} + \vec{MP}$

$x\vec{i} + y\vec{j}$ ... ... (i)

Again, the vector equation of the straight line AB is

$\vec{r} = (l - t)\, a\vec{i} + tb\vec{j}$ ... ... (ii)

For P, the point of intersection of OP and AB, we have

$x\vec{i} + y\vec{j} = (1-t)a\vec{i} + tb\vec{j}$

Equating the coeff. of like vectors,

x = (1 – t)a and y = tb

$\frac{x}{a} = 1 - t \qquad \frac{y}{b} = t$

$\therefore \frac{x}{a} = 1 - \frac{y}{b}$

$\therefore \boxed{\frac{x}{a} + \frac{y}{b} = 1}$

Exercise 12.2

1. Solution:

a. Here, $\vec{a} = (1, 2, 0)$, $\vec{b}\ (2, 0, 3)$ and $\vec{c} = (2, -1, 2)$

Then, $\vec{b}\times\vec{c} = (2, 0, 3) \times (1, -1, 2)$

$$\begin{array}{ccccc} 2 & 0 & 3 & 2 & 0 \\ 1 & -1 & 2 & 1 & -1 \end{array}$$

$= (0\times2 - (-1) \times 3, 3\times1 - 2\times2, 2\times-1 - 1\times0$

$= (3, -1, -2)$

Now, $\vec{a}(\vec{b} \times \vec{c}) = (1, 2, 0) . (3, -1, -2)$

$= 1 \times 3 + 2 \times -1 + 0 \times -2$

$= 1$

b. Here, $\vec{a} = (-1, 2, 3)$, $\vec{b} = (0, 1, -2)$ and $\vec{c} = (3, 0, -1)$

Then $\vec{b}\times\vec{c} = (0, 1, -2) \times (3, 0, -1)$

$$= \begin{array}{cccccc} 0 & 1 & -2 & 0 & 1 \\ 3 & 0 & -1 & 3 & 0 \end{array}$$

$= (-1 + 0, -6 + 0, 0 - 3)$

$= (-1, -6, -3)$

and $\vec{a} \times \vec{b} = (-1, 2, 3) \times (0, 1, -2)$

$$= \begin{array}{cccccc} -1 & 2 & 3 & -1 & 2 \\ 0 & 1 & -2 & 0 & 1 \end{array}$$

$= (-4, -3, 0, -2, -1, -0)$

$= (-7, -2, -1)$

Now, $\vec{a} \cdot (\vec{b} \times \vec{c}) = (-1, 2, 3) \cdot (-1, -6, -3)$

$= (-1\times-1 + 2\times-6 + 3\times-3)$

$= -20$

and $(\vec{a}\times\vec{b}) \cdot \vec{c} = (-7, -2, -1) \cdot (3, 0, -1)$

$= -7 \times 3 + (-2) \times 0 + (-1) \times -1$

$= -21 + 0 + 1$

$= -20$

2. We have,

$\vec{a} = \vec{i} - \vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j}$ and $c = 3\vec{k}$

Now, $\vec{a}\ (\vec{b} \times \vec{c}) = (\vec{i} - \vec{j} - \vec{k}) \cdot [(2\vec{i} + \vec{j}) \times (3\vec{k})]$

$= (\vec{i} - \vec{j} - \vec{k}) \cdot (-6\vec{j} + 3\vec{i})$

$= (\vec{i} - \vec{j} - \vec{k}) \cdot (3\vec{i} - 6\vec{j})$

$= 3 + 6 - 0 = 9$

3. Solution:

a. Here, the adjacent edges of a parallelepiped are represented by the vectors $\vec{i}+\vec{j}$, $2\vec{i} - \vec{j} + \vec{k}$ and $\vec{i}+2\vec{j}-3\vec{k}$

$= (1, 1, 0), (2, -1, 1)$ and $(1, 2, -3)$

i.e., $\vec{a}.(\vec{b} \times \vec{c})$ ... ... (i)

Where, $\vec{b}\times\vec{c} = (2, -1, 1) \times (1, 2, -3)$

$$= \begin{array}{cccccc} 2 & -1 & 1 & 2 & -1 \\ 1 & 2 & -3 & 1 & 2 \end{array}$$

$= (3 - 2, 1 + 6, 4 + 1)$

$= (1, 7, 5)$

$\therefore$ from (i) volume $= (1, 1, 0) \cdot (1, 7, 5)$

$= (1.1 + 1.7 + 0.5)$

$= (1 + 7 + 0)$

$= 8$

b. Let $\vec{a} = 5\vec{i} + 2\vec{j} - 3\vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} - 4\vec{k}$ and $\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k}$

The volume of the parallelepiped represented by the given three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (5\vec{i} + 2\vec{j} - 3\vec{k}) \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -4 \\ 3 & -1 & 2 \end{vmatrix}$$

$= (5\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{i} - 16\vec{j} - 11\vec{k})$
$= 10 - 32 + 33 = 11$ cu units

4. Here, the three concurrent edges of a parallelogpiped are given by $2\vec{i} + 3\vec{j} + m\vec{k}$, $\vec{i} - 2\vec{j}$ and $3\vec{i} + \vec{j} - 2\vec{k}$ and volume = 26 cu.unit.
   Let the three edges be denoted by
   $\vec{a} = (2, 3, -m)$, $\vec{b} = (1, -2, 0)$ and $\vec{c} = (3, 1, -2)$
   Now, $\vec{b}\times\vec{c} = (1, -2, 0) \times$
   $$= \begin{matrix} 1 & -2 & 0 & 1 & -2 \\ 3 & 1 & -2 & 3 & 1 \end{matrix}$$
   $= (4 - 1, 0 + 2, 1 + 6) = (3, 2, 7)$
   Using the volume of parallelepiped $= \vec{a} \cdot (\vec{b}\times\vec{c})$
   or, $26 = (2, 3, -m) \cdot (3, 2, 7)$
   or, $26 = (6 + 6 - 7m)$
   or, $26 = 12 - 7m$
   or, $7m = 12 = 26 = -14$
   $\therefore$ $m = -2$