Chapter 1 Permutation and Combination

Exercise 1.1

1. Solution:

Total no. of air flights (n_1) = 5

Total no. of buses (n_2) = 15

As from the addition rule

The no. of ways to travel from Bhairahawa to Kathmandu = 5 + 15 = 20

Hence, there are 20 ways to travel.

2. Solution:

The no. of girls and boys are 25 and 20 respectively. If a boy and a girl are to be chosen for debate competition, then

The no. of ways of selection would be 25×20 = 500 ways.

3. Solution:

If there are 5 routes from station A to station B and 4 routes from station B to C then, the no. of possible routes from A to B is 5 and from B to C is 4.

a. Here,

∴ The no. of possible routes from A to C is 5×4 = 20 routes.

b. Here,

The no. of possible routes from A to C is 20 and so as to return from C to A there is also 20 routes (i.e. 4×5).

Hence, the no. of required ways = 20×20 = 400 ways

c. Here,

The no. of routes to travel from A to C is 20. If the same route is not used more than once, the–n the no. of ways to travel and return back is 20×12 = 240 ways

4. Solution:

The no. of digits = 6

So, hundred place can be arranged in 6 ways

Tens place can be arranged in 5 ways

Units place can be arranged in 4 ways

∴ Required numbers = 6×5×4 = 120

Next, If these numbers formed must be even, the digit in the units place can be arranged in 3 ways

Ten's place can be arranged in 5 ways

Hundred place can be arranged in 4 ways

∴ Required number's place = 3×5×4 = 60 ways

5. Solution:

The no. of digits = 6

a. As we know, units place can never be filled by zero, so units place can be filled by 5 ways

Tens place can be filled by 5 ways

Hundred place can be filled by 4 ways

Thousand place can be filled by 3 ways

∴ The required no.s of 4 digit when repetition is not allowed

= 5×5×4×3 = 300

b. If the repetition is allowed, the unit place can be arranged/filled by 5 ways and then after all remaining places can be filled by 6 ways.

∴ The required no.s of 4 digit when repetition is allow = 5×6×6×6

= 1080 ways

6. Solution:

The given digits are 0, 1, 2, 3

If the digits may repeat: then

For 1 digits: For the units place, number of way = 4

For the ten's place, number of ways = 3

∴ Number of ways = 4 × 3 = 12

For 1 digit: The number of ways = 3

So, total number of ways = 12 + 3 = 15

If the digits may not repeat:

For 1 digit: Number of ways = 3

For two digits = Number of ways in tens place = 3

Number of ways in ones place = 3

∴ Number of ways = 3×3 = 9

So, total number of ways = 3 + 9 = 12

7. Solution:

The no. of digits = 5

The number must lies between 2000 and 3000 and so each no. should be started with 2. As the formed no. should be even each no. must be ended with 0,or 2 but here digits can be used only once.

So, units place can be filled by 1 ways

Tens place can be filled by 4 ways

Hundred place can be filled by 3 ways

Thousand place can be filled by 1 ways

∴ Required no. of digits = 1×4×3×1 = 12

8. Solution:

Here, the numbers are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and the telephone number starts with 562

i. If repetition is not allowed: Number of ways for remaining 3 places = 7×6×5 = 210

ii. If repetition is allowed: Number of ways for remaining 3 places = 10×10×10 = 1000

9. Solution:

The total digit is 10 and no. of choice for unit digit is 9. So the no. digits required = 8×9×9 = 648

Now, if there is only one zero given that the repeation allowed, then, the unit place filled by 9 ways

Tens place can be filled by 10 ways

Hundred place can be filled by 9 ways

∴ No. of total digits = 9×10×9 = 810

Hence, the required number = 810 – 648 = 162

Exercise 1.2

1. Solution

Given, $\frac{(n+1)!}{(n-1)!} = 12$

or, $\frac{(n+1)\,n\,(n-1)!}{(n-1)!} = 12$

or, $n(n+1) = 12$

or, $n^2 + n - 12 = 0$

or, $n^2 + 4n - 3n - 12 = 0$

or, $(n+4)(n-3) = 0$

either $n = -4$

or, $n = 3$

Since, $n \neq -4$, so $n = 3$

2. Solution:

Given, P(5, r) = 5

r = 1 [∵ if P(n, r) = n, then r = 1)

3. Solution:

The no. of digits (n) = 6

a. Units place can be filled only by 5 digits but the remaining 3 places can be filled 6 digits as the repetition is allowed.

∴ The required no. of 4 digits = 5×6×6×6
= 1080

b. Unit first place can be filled by 5 digit as the repletion not allowed.

2^{nd} first place can be filled by 5 digit
3^{rd} first place can be filled by 4 digit
4^{th} first place can be filled by 3 digit
∴ The required no. of 4 digit = $5\times5\times4\times3$
$= 300$

4. Solution:
There are 4 boys and 3 girls be seated in a row containing 7 seats.
∴ Required arrangement is $p(7, 7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = 5040$
Again,
If they seat alternatively, then 4 boys can set in 4! ways and 3 girls can seat in 3! ways.
∴ Required arrangement is $= 4! \times 3!$
$= 24\times6 = 144$ ways

5. Solution:
The total no. of digits = 10
The first digit can be chosen from only 1 to 9 so there is only 9 choices for first digit. The remaining 5 digits can be chosen from remaining 9 digits in p(95) ways
i.e. $\frac{9!}{(9-5)!} = \frac{15}{20}$ ways
∴ The total numbers of 6 digits is 9×15120 way = 136080
Next: For the divisible by 10. Last digit must be zero, so the last digit can be chosen from 0, so there is 1 choice for last digit. The remaining 5 digits can be chosen from 9 digits in p(95) way
i.e. $\frac{9!}{(9-5)!} = 15120$

6. Solution:
The numbers given in the question is 1, 2, 3, 4, 5
For one digit: No. of ways for even = 2
For two digits: No. of ways for ones place = 2
Number of ways for ten's place = 4
∴ Total no. of ways $2 + 2\times4 = 10$

7. Solution:
In a bracelet, beads are arrangement in circular form and the anticlockwise and clockwise arrangements are not different.
Here the total number of beds n = 9
They can be arranged in (n – 1)! ways $= \frac{1}{2} \times 8!$ ways = 20160

8. Solution:
The no. lying between 100 and 1000 is of 3 digit. In which at unit place can be chosen only from 5 digit and hundred place can only be chosen from 5 digit where as remaining tens place can be chosen from remaining 4 digit.
∴ The no. formed between 100 and 1000 $= \frac{5!}{(5-1)!} \times \frac{4!}{(4-1)!} \times \frac{5!}{(5-1)!}$
$= \frac{5\times4!}{4!} \times \frac{4\times3!}{3!} \times \frac{5\times4!}{4!} = 5\times4\times5 = 100$

9. Solution:
Each post cards can be posted in 4 ways
Hence, required number of ways $= n^r = 5^5 = 1024$

10. Solution:
a. Here,
PERMUTATION
Total no. of letters (n) = 11
No. of letter 'T' (p) = 2

$\therefore$ Total number of way of arrangement = $\frac{n!}{p!} = \frac{11!}{2!}$

b. INTERMEDIATE

Here, the total number of letters (n) = 12
No. of letter 'I' (p) = 2
No. of letter 'T' (q) = 2
No. of letter 'E' (r) = 3

$\therefore$ The total no. of arrangement = $\frac{n!}{p!\ q!\ r!} = \frac{12!}{2!\ 2!\ 3!}$

c. EXAMINATION

Here, the total number of letter (n) = 11
No. of letters 'A' (p) = 2
No. of letters 'I' (q) = 2
No. of letters 'N' (r) = 2

$\therefore$ Total no. of arrangement = $\frac{n!}{p!\ q!\ r!} = \frac{11!}{2!\ 2!\ 2!}$

d. CIVILIZATION

The total no. of letters (n) = 12
No. of letter 'I' (p) = 4

$\therefore$ Total no. of arrangement = $\frac{n!}{p!} = \frac{12!}{4!}$

11. Solution:

In 'ARRANGE'
Total no. of letters (n) = 7
No. of letter 'A' (p) = 2
No. off letter 'R' (q) = 2

$\therefore$ Total no. of ways of arrangement = $\frac{n!}{p!\ q!} = \frac{7!}{2!\ 2!} = 1260$

If we suppose (RR) as one letter, then the no. of letters will be 6

$\therefore$ The no. of ways of arrangement when R comes together = $\frac{n!}{p!} = \frac{6!}{2!} = 360$

Thus, the required no. of ways of arrangement when two R not comes together = 1260 – 360 = 900

12. Solution:

In UNIVERSITY'
The no. of letters (n) = 10
No. of letter 'I' (p) = 2

$\therefore$ Total no. of arrangement = $\frac{n!}{p!} = \frac{10!}{2!} = 1814400$

Since the arrangement begin with U there is only. Nine letters to arrange. So, the nine letters can be arranged in

$= \frac{n!}{p!} = \frac{9!}{2!} = 181440$

$\therefore$ Required no. of arrangement = $1\times181440 = 181440$

Next: The total no. of ways in which the arrangement begin with U but do not end with 'Y' = $4 \times p(88) = 4 \times \frac{8!}{0!} = 161280$

13. Solution:

Total no. of countries (n) = 8

If they sit in round table then they form a circle, so its arrangement is = $(n - 1)! = 7! = 5040$

If Nepali and Indian always sit together, then we take it as one. Then the total no. will be 6. So, the arrangement $(n–r+1)! = (7–2+1)! = 6! = 720$

If they sit together, then they also can interchange there seat between themselves in 2 ways.

Hence, the required no. of arrangement = 21×720 = 1440

14. Solution:

If there is 3 candidate for the president then election can be turned in 3 ways. Similarly for 5 secretary and 2 pressure the election can be turned in 5 ways and 2 ways. Hence, the required no. ways to conduct election = 3! × 5! × 2! = 1440

15. Solution:

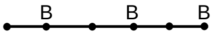
Total no. of digit = 4

The person can try his password in p(4, 4) ways = $\frac{4!}{(4-4)!} = \frac{4!}{0!} = 24$

16. Solution:

Since 6 persons are to be arranged in row with 6 seat, so that the girls and boys are in alternate, so girl are to be arranged in odd seats and boy in even seats.

∴ The total no. of arrangement = 6! = 720



Here, the no. of arrangement of boy restricted to occupy even seats is p(3, 3) = $\frac{3!}{0!} = 6$

The 3 boys can occupied seats in 3! by interchanging their seats.

Hence, the required no. of arrangement = 3! = 36

17. Solution:

In 'EQUATION'

The no. of total letters (n) = 8

∴ The total no. of arrangement = 8! = 40320

Next, The no. of vowels = 5

When we take all vowels as one then there will be total letters left = 4. Also the vowel letters be arranged themselves in 5 ways.

∴ The required no. of arrangement = 4! × 5! = 2880

Exercise 1.3

1. Solution:

Here, (ln, 10) = (ln, 12)

$\Rightarrow \frac{n!}{(n-10)!\,10!} = \frac{n!}{(n-12)!\,2!}$

$\Rightarrow \frac{(n-12)!\,12!}{(n-10)!\,10!} = \frac{n!}{n!}$

$\Rightarrow \frac{(n-12)!\,12\times11\times10!}{(n-10)\,(n-11)\,(n-12)!\,10!} = 1$

$\Rightarrow \frac{12\times11}{(n-10)\,(n-11)} = 1$

$\Rightarrow 132 = n^2 - 11n - 10n + 110$

$\Rightarrow n^2 - 21n - 22 = 0$

$\Rightarrow n^2 - 22n + n - 22 = 0$

$\Rightarrow n(n-22)\ 11(n-22) = 0$

$\Rightarrow (n-22)\,(n+1) = 0$

either n = 22

or, n = –1 (This is not possible, so rejected)

∴ n = 22

Next C(n, 6) = C(22, 6) = $\frac{22!}{(22-6)!\,6!} = \frac{22!}{16!\,16!}$

2. Solution:

Here, C(n, 8) = C(n, 6)

Then, n = 8 + 6 = 14

Now, $C(14, 2) = \frac{14!}{12!\ 2!} = 91$

3. Solution:

Given, C(n, 30) = C(n, 4)

$\Rightarrow$ $C(n, r) = C(n, r^1)$

$\Rightarrow$ $r + r^1 = n$

Then, 30 + 4 = n

$\therefore$ n = 34

Now, $C(n, 30) + C(n, 4) = \frac{34!}{(34-4)!\ 14!} + \frac{34!}{20!\ 14!}$

$C(n, 30) + C(n, 4) = \frac{34!}{4!30!} + \frac{34!}{30!\ 4!}$

= 46376 + 46376

= 92752

4. Solution:

Given, c(9, 2r) = c(9, 3r – 1)

$\Rightarrow$ $2r + 3r - 1 = 9$ [$\because C(n, r) = C(n, r^1) = r+r^1 = n$]

$\Rightarrow$ 5r = 10

$\Rightarrow$ r = 2 or, 2r = 3r – 1

$\Rightarrow$ r = 1

$\therefore$ r = 1 or 2

5. Solution:

The no. of workers required in 3 where total applicant is 10

$\therefore$ The selection be made as $c(10, 3) = \frac{10!}{7!\ 3!}$ $\left[\because c(n, r) = \frac{n!}{(n-n!\ r!)}\right]$ = 120 Ans.

6. Solution:

To invite 7 guests out of 12 friends

Relatives (8)	Non-relative (4)	Selection
5	2	$c(8, 5) \times c(4, 2)$

$\therefore$ The required selection is $6(8, 9) \times c(4, 2) = \frac{8!}{3!\ 5!} \times \frac{4!}{2!\ 2!}$

$= 56 \times 6$

= 336 Ans.

7. Solution:

Here,

Group A (10)	Group B(6)	Selection
6	4	$c(10, 6) \times (6, 4)$

$\therefore$ The required selection is $c(10, 6) \times c(6, 4) = \frac{10!}{4!\ 6!} \times \frac{6!}{2!\ 4!}$

$= 210 \times 15$

= 3150

8. Solution:

Total number of balls = 5 and maximum number of balls to select = 3.

Thus, we can select in C(5, 0) + C(5, 1) + C(5, 2) + C(5, 3) = 1 + 5 + 10 + 10 = 26 ways.

9. Solution:

From the 4 coins, the sum can be made in the following ways:

$C(4, 1) + C(4, 2) + C(4, 3) + C(4, 4) = \frac{4!}{3!\ 1!} + \frac{4!}{2!\ 2!} + \frac{4!}{1!\ 3!} + \frac{4!}{0!\ 4!}$

= 4 + 6 + 4 + 1

= 15 Ans

10. Solution:

The no. of player in class = 15

The no. of players taken in team (r) = 11

∴ Required no. of ways of selection = C(15, 11) = $\frac{15!}{4!\ 11!}$ = 1365

a. Here, if 2 particular persons are always included then there will be 13 players in class and 9 players required to be selected.

∴ Required selection is C(13, 9) = $\frac{13!}{4!\ 9!}$ = 715

b. Here, If 2 persons are always excluded then there will be 13 players in class and 11 players to be selected.

∴ Required selection = C(13, 11) = $\frac{13!}{2!\ 11!}$ = 78

11. Solution:

Party A(5)	Party B(6)	Selection
3 2	5 6	(15, 3) × C(6, 5) C(5, 2) × C(6, 6)

∴ Required selection =

C(5, 3) × C(6, 5) + C(5, 2) × C(6, 6)

$= \frac{5!}{2!\ 3!} \times \frac{6!}{1!\ 5!} = \frac{5!}{3!\ 2!} \times \frac{6!}{0!\ 6!}$

= 10 × 6 + 10 × 1

= 70 Ans.

12. Solution:

Group A(5)	Group B(5)	Selection
2 3 4	4 3 2	C(5, 2) × c(5, 4) C(5, 3) × c(5, 3) C(5, 4) × c(5, 2)

∴ The required selection = C(5, 2) × C(5,4) + C(5, 3) × C(5, 3) + C(5, 4) × C(5, 2)

$= \frac{5!}{3!\ 2!} \times \frac{5!}{1!\ 4!} + \frac{5!}{2!\ 3!} \times \frac{5!}{2!\ 3!} \times \frac{5!}{1!\ 4!} \times \frac{5!}{3!\ 2!}$

= 10 × 5 + 10 × 10 + 5 × 10

= 50 + 100 + 50

= 200 Ans.

13. Solution:

Ladies (6)	Gentle (8)	Selection
4	7	c(6, 4) × c(8, 7)

∴ Required selection is C(6, 4) × C(8, 7) = $\frac{6!}{2!\ 4!} \times \frac{8!}{1!\ 7!}$ = 15 × 8 = 120

Ladies (6)	Gentle (8)	Selection = 15×8 = 120
4 5 6	7 6 5	C(6, 4) × C(8, 7) C(6, 5) × C(8, 6) C(6, 6) × C(8, 5)

∴ Required selection = C(6, 4) × C(8, 7) + C(6, 5) × C(8, 6) + C(6, 6) × C(8, 5)

$= \frac{6!}{2!\ 4!} \times \frac{8!}{1!\ 7!} + \frac{6!}{1!\ 5!} \times \frac{8!}{2!\ 6!} + \frac{6!}{6!\ 1!} \times \frac{8!}{3!\ 5!}$

= 15 × 8 + 6 × 28 + 1 × 56

= 120 + 168 + 56 = 344 Ans.

14. Solution:

From the 6 question, a examinee can pass/solve one or more of questions in following ways.

A examinee can solve 1 or 2 or 3 or 4 or 5 or all

Thus, total no. of ways to solve = C(6, 1) + (6, 2) + C(6, 3) + C(6, 4) + C(6, 5) + C(6, 6)

$= \frac{6!}{5!\ 1!} + \frac{6!}{4!\ 2!} + \frac{6!}{3!\ 3!} + \frac{6!}{2!\ 4!} + \frac{6!}{1!\ 5!} + \frac{6!}{0!\ 6!}$

= 6 + 15 + 20 + 6 + 1 = 63 Ans.

15. Solution:

A candidate fails in an examination if he cannot pass either in 1 or 2 or 3 or 4 or 5 subjects

$\therefore$ Total no. of ways by which he falls = C(5, 1) + C(5, 2) + C(5, 3) + C(5, 4) + C(5, 5)

$= \frac{5!}{4!\ 1!} + \frac{5!}{3!\ 2!} + \frac{5!}{2!\ 3!} + \frac{5!}{1!\ 4!} + \frac{5!}{0!\ 5!}$

= 5 + 10 + 10 + 5 + 1 = 31 Ans.

Chapter 2 Binomial Theorem

Exercise 2.1

1.a. $(3x + 2y)^5$

We know that, $(a + x)^n = c(n, 0)\ a^n + c(n, 1)\ a^{n-1}\ x + c(n, 2)\ a^{n-2}\ x^2 + ... + c(n, r)\ a^{n-r}\ x^r + ... + c(n, n)\ x^n$.

$\therefore$ $(3x + 2y)^5 = c(5, 0)\ (3x)^5 + c(5, 1)\ (3x)4\ (2y) + c(5, 2)\ (3x)^3\ (2y)^2 + c(5, 3)\ (3x)^2\ (2y)^3 + c(5, 4)\ (3x)^1\ (2y)^4 + c(5, 5)\ (2y)^5$

$= 243x^5 + 810x^4y + 1080x^3y^2 + 270x^2y^3 + 240xy^4 + 32y^5$

b. $(2x - 3y)^6$

$= 6c_0(2x)^6 + 6c_1(2x)^5\ (-3y) + 6c_2(2x)^4\ (-3y)^2 + 6c_3(2x)^3\ (-3y)^3 + 6c_4(2x)^2\ (-3y)^4 + 6c_5(2x)\ (-3y)^5 + 6c_6(-3y)^6$

$= 64x^6 - 576x^5y + 2160x^4y^2 - 4320x^3y^3 + 4860x^2y^4 - 2916xy^5 + 729y^6$.

c. $\left(x + \frac{1}{y}\right)^{11} = x^{11} + 11x^{10}\frac{1}{y} + 55\frac{x^9}{y^2} + 165\frac{x^8}{y^3} + 330\frac{x^7}{y^4} + 462\frac{x^6}{y^5} + 462\frac{x^5}{y^6} + 330\frac{x^4}{y^7} + 165\frac{x^8}{y^8} + 55\frac{x^2}{y^9} + 11\frac{x}{y^{10}} + \frac{1}{y^{11}}$

d. $\left(x + \frac{1}{x}\right)^6 = 6c_0\ x^6 + 6c_1\ x^5\frac{1}{x} + 6c_2\ x^4.\frac{1}{x^2} + 6c_3\ x^3\frac{1}{x^3} + 6c_4\ x^2\frac{1}{x^4} + 6c_5\ x\frac{1}{x^5} + 6c_6\frac{1}{x^6}$

$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

e. $\left(x - \frac{1}{x}\right)^7$

$= x^7 + {}^7c_1\ x^6\left(-\frac{1}{x}\right) + {}^7c_2\ x^5\left(-\frac{1}{x}\right)^2 + {}^7c_3\ x^4\left(-\frac{1}{x}\right)^3 + {}^7c_4\ x^3\left(-\frac{1}{x}\right)^4 + {}^7c_5\ x^2\left(-\frac{1}{x}\right)^5 + {}^7c_6\ x\left(-\frac{1}{x}\right)^6 + \left(-\frac{1}{x}\right)^7$

$= x^7 + 7x^5 + 21x^3 - 35x + 35.\frac{1}{x} - 21.\frac{1}{x^3} + 7.\frac{1}{x^5} - \frac{1}{x^7}$

f. $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6$

$= \left(\frac{2x}{3}\right)^6 + {}^6c_1\left(\frac{2x}{3}\right)^5.\left(\frac{-3}{2x}\right) + {}^6c_2\left(\frac{2x}{3}\right)^4\ {}^6c_3\left(\frac{2x}{3}\right)^3.\left(\frac{-3}{2x}\right)^4 + {}^6c_4\left(\frac{2x}{3}\right)^2.\left(\frac{-3}{2x}\right)^4 + {}^6c_5\left(\frac{2x}{3}\right).\left(\frac{-3}{2x}\right)^5 + \left(\frac{-3}{2x}\right)^6$

$= \frac{64x^6}{729} - \frac{96}{81}x^4 + \frac{20}{3}x^2 - 21 + \frac{135}{4}.\frac{1}{x^2} - \frac{243}{8}.\frac{1}{x^4} + \frac{729}{64}.\frac{1}{x^6}$

g. $(1 + 2x - 3x^2)^5 = (1 + 2x)^5 + {}^5c_1\ (1 + 2x)^4\ .\ (-3x^2) + {}^5c_2\ (1 + 2x)^3$
$(-3x^2)^2 + {}^5c_3\ (1 + 2x)^2\ .\ (-3x^2)^3 + {}^5c_4$
$(1 + 2x)\ .\ (-3x^2)^4 + (-3x^2)^5$
$= 1 + 10x + 25x^2 - 40x^3 - 190x^5 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$

2.a. We know that the general term t_{r+1} of expansion of $(a + x)^n$ is given by $t_{r+1} = n_{C_r}\ a^{n-r}\ x^r$

Here,

$(a + x)^n \Rightarrow \left(\frac{2x}{3} + \frac{3}{2x}\right)^6$

$\therefore$ $a \Rightarrow \frac{2x}{3}$, $x \Rightarrow \frac{3}{2x}$ and $n \Rightarrow 6$

For 7^{th} term, put r = 6

$t_{6+1} = {}^6C_6\left(\frac{2x}{3}\right)^{6-6}\left(\frac{3}{2x}\right)^6$

$\therefore \quad t_7 = 1.\ 1.\ \frac{729}{64x^6}$

$\therefore \quad t_7 = \frac{729}{64x^6}$

b. The total number of terms of the expansion of $\left(\frac{x}{y} - \frac{2y}{x^2}\right)^6$ is 7.

So, there is no 10th term.

c. For 5th term, put r = 4.

$t_{r+1} = t_{4+1} = 12_{C_4}\ (2x)^{2-4}\ y^4 = 495 \times 2^8 x^8 y^4 = 126720\ x^8 y^4$

d. Given,

$\left(2x^2 + \frac{1}{x}\right)^8$

Here, $n \rightarrow 8$

$a \rightarrow 2x^2$ and $x \rightarrow \frac{1}{x}$

We know that, $t_{r+1} = n_{C_r}\ a^{n-r}\ x^r$

$t_5 = t_{4+1} = 8_{C_4}\ (2x^2)^4 \left(\frac{1}{x}\right)^4 = 1120x^4$

e. $\left(x - \frac{1}{x}\right)^7$

$t_6 = t_{5+1} = {}^7C_5\ x^{7-5} \left(-\frac{1}{x}\right)^5 = 21x^2 \left(-\frac{1}{x^5}\right) = -\frac{21}{x^3}$

3.a. $(x^2 - y)^6$

Here, n = 6

The general term $(t_{r+1}) = {}^6C_r\ (x^2)^{6-r}\ (-y)^r$

$= (-1)^r\ {}^6C_r\ x^{12-2r}\ y^r$

b. Given, $\left(x^2 - \frac{1}{x}\right)^{12}$

Here, n = 12

The general term $(t_{r+1}) = {}^{12}C_r\ (x^2) \left(-\frac{1}{x}\right)^r = (-1)^r\ {}^{12}C_r\ x^{24-3r}$

c. Here, n = 10

The general term $(t_{r+1}) = {}^{10}C_r \left(\frac{x}{b}\right)^{10-r} \left(-\frac{b}{x}\right)^r = (-1)^r\ {}^{10}C_r \left(\frac{x}{b}\right)^{10-2r}$

d. Given, $\left(x - \frac{1}{x}\right)^{12}$

The general term $(t_{r+1}) = {}^{12}C_r\ (x)^{12-r} \left(-\frac{1}{x}\right)^r = (-1)^r\ {}^{12}C_r\ x^{12-2r}$

4.a. The general term $(t_{r+1}) = {}^{11}C_r\ (x^2)^{11-r} \left(-\frac{1}{x}\right)^r = (-1)^r\ {}^{12}C_r\ x^{12-2r}$

For x^7, 22 – 3r = 7

15 = 3r

$\therefore \quad r = 5$

$\therefore \quad$ the coeff. of x^7 is 11_{C_r} i.e. ${}^{11}C_5 = 462$

b. The general term $(t_{r+1}) = {}^7C_r\ (x)^{7-r} \left(\frac{1}{2x}\right)^r = {}^7C_r \cdot \frac{1}{2^r}\ x^{7-2r}$

For x^5, we must have

7 – 2r = 5

r = 1

$\therefore \quad$ Coeff. $x^5 = 7_{C_1}.\ \frac{1}{2} = \frac{7}{2}$

c. We have,

$t_{r+1} = 9c_r\,(3x^2)^{9-r}\left(\frac{-1}{3x}\right)^r$

$= 9c_r\,.\,3^{9-2r}\,.\,x^{18-2r-r}.\,(-1)^r$

$= (-1)^r\,3^{9-2r}\,.\,9c_r\,x^{18-3r}$

Here, $18 - 3r = 6$

$\therefore\quad r = 4$

$\therefore$ The coeff. of x^6 is $(-1)^4\,3^{9-8}\,.\,9c_4$

$= 3 \times 9c_4 = 378$

d. The general term $(t_{r+1}) = {}^9C_r\,(ax^4)^{9-r}\,(-bx)^r$

$= {}^9C_r\,a^{9-r}\,(-b)^r\,x^{36-3r}$]

For x^{12}, we must have

$36 - 3r = 12$

$\therefore\quad r = 8$

The required coeff. of x^{12} is ${}^9C_8\,a^{9-8}\,(-b)^8$

$= 9ab^8$

e. We have, $t_{r+1} = {}^9C_r\,(2x)^{9-r}\left(-\frac{1}{3x^2}\right)^r = (-1)^r\,{}^9C_r\,\frac{2^{9-r}}{3^r}\,x^{9-3}$

For x^{-6}, $9-3r = -6$

$9 + 6 = 3r$

$\therefore\quad r = 5$

Coeff. of $x^{-6} = (-1)^5\,{}^9C_5\,\frac{2^{9-5}}{3^5} = -\frac{2016}{243} = -\frac{224}{27}$

5.a. The general term $(t_{r+1}) = {}^8C_r\,(2x)^{8-r}\left(-\frac{1}{3x^2}\right)^r$

$= {}^8C_r\left(-\frac{1}{3}\right)^r 2^{8-r}\,x^{8-3r}$

For the independent of x,
we must have $8 - 3r = 0$

$\therefore\quad r = \frac{8}{3}$ (not possible)

$\therefore$ There is no term which is free from x.

b. $\left(x + \frac{1}{x}\right)^{10}$

Here, $t_{r+1} = 10c_r\,x^{10-r}\left(\frac{1}{x}\right)^r = 10c_r\,x^{10-2r}$

For free from x, $10-2r = 0$

$\therefore\quad r = 5$

$\therefore\quad t_{r+1} = t_{5+1} = t_6$ is the required term.

c. $t_{r+1} = 15c_r\,(x^2)^{15-r}\left(-\frac{1}{x^3}\right)^r$

$= 15c_r\,(-1)^r\,x^{30-4r}$

$\therefore\quad 30 - 4r = 0$

$r = \frac{15}{2}$ (not possible)

$\therefore$ no term has free from x

d. $t_{r+1} = 10c_r\left(\frac{3x^2}{2}\right)^{10-r}\left(-\frac{1}{3x}\right)^r$

$= (-1)r\,10c_r\,\frac{3^{10-2r}}{2^{10-r}}\,x^{20-3r}$

For x^0, $20-3r = 0$

$r = \frac{20}{3}$ (not possible)

$\therefore$ No term is free from x.

e. The general term $(t_{r+1}) = 14c_r\,(x^2)^{14-r}\left(-\frac{1}{x^2}\right)^r$

$= (-1)^r\,14c_r\,x^{24-4r}$

$\therefore$ For free of x, we have $24 - 4r = 0$

$\therefore$ 7^{th} term is required term.

6.a. $(3 + x)^6$

Here n = 6, there is a single middle term.

$\therefore$ Middle term is

$t_{\frac{n}{2}+1} + t_{3+1} = t_4$

Using $t_{r+1} = nc_r\,a^{n-r}\,n^r$

$t_{3+1} = 6c_3\,3^{6-3}\,x^3 = 6c_3\,3^3\,x^3 = 540x^3$

b. $\left(x - \frac{1}{2y}\right)^{10}$

Since n = 10 (even), there is a single middle term

$\therefore$ Middle term is

$t_{\frac{n}{2}+1} = t_{5+1} = 10c_5\,(x)^{10-5}\left(-\frac{1}{2y}\right)^5 = 10c_5\,x^5\left(-\frac{1}{2}\right)^5\frac{1}{y^5} = -\frac{3}{8}\left(\frac{x}{y}\right)^5$

c. $\left(1 - \frac{x^2}{2}\right)^{14}$

Here, n = 14 (even), there is a single middle term

$t_{\frac{n}{2}+1} = t_{7+1}$

$\therefore$ $t_{7+1} = 14c_7\,(1)^{14-7}\left(-\frac{x^2}{2}\right)^7$ $\quad (\because t_{r+1} = nc_r\,a^{n-r}\,x^2)$

$= -14c_7\,\frac{x^{14}}{2^7}$

$= -\frac{429}{16}\,x^{14}$

d. Since n = 10, there is a middle term.

$t_{5+1} = 10c_5\,(x^2)^5\left(-\frac{2}{x}\right)^5 = -2^5\,.\,10c_3\,x^5 = -8064x^5$

e. Since 2n is even, there is single middle term

$t_{\frac{2n}{2}+1}$

i.e. t_{n+1}

$= 2nc_n\,(an)^n\left(-\frac{1}{an}\right)^n$

$= 2nc_n\,(-1)^n$

$= (-1)^n\,\frac{(2n)\,!}{n!\,n!})$

$= (-1)^n\,\frac{1.2.3....\,(2n-2)\,(2n-1).\,2n}{n!\,n!}$

$= \frac{(-1)^n\,2^n(1.2.3....\,n)\,(1.3.5....\,(2n-1))}{n!\,n!} = \frac{(-2)^n\,\{1.3.5....\,(2n-1)}{n!}$

f. There is a single middle term

$t_{\frac{n}{2}+1} = t_{4+1} = 8c_4\,(2x^2)^4\left(\frac{1}{x}\right)^4 = 8c_4\,2^4\,.\,x^4 = 1120x^4$

7.a. Here, n = 5 (odd), there are two middle terms.

i.e. $t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e. t_{2+1} and t_{3+1}

$t_{2+1} = 5c_2\, (x^2)^3\, (a^2)^2 = 10x^6a^4$

$t_{3+1} = 5c_3\, (x^2)^2\, (a^2)^3 = 10x^4a^6$

b. Here, n = 11 (odd), there are two middle terms.

i.e. $t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e t_{5+1} and t_{6+1}

Now, $t_{5+1} = 11c_5\, (x^4)^{11-5} \left(-\frac{1}{x^3}\right)^5 = (-1)^5\, 11c_5\, x^9 = -462x^9$

$t_{6+1} = 11c_6\, (x^4)^{11-6} \left(-\frac{1}{x^3}\right)^6 = 11c_6\, x^2 = -462x^2$

c. Here, n = 13 (odd), there are two middle terms.

$t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e. t_{6+1} and t_{7+1}

Now, $t_{6+1} = 13c_6\, (x)^7 \left(-\frac{1}{x}\right)^6 = 13c_6\, x = 1716x$

$t_{7+1} = 13c_7\, x^6 \left(-\frac{1}{x}\right)^7 = -\frac{1716}{x}$

d. Given, $\left(2x + \frac{1}{x}\right)^{17}$

Since n = 17, there are two middle terms.

$t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e. t_{8+1} and t_{9+1}

$t_{8+1} = 17c_8\, (2x)^9 \left(\frac{1}{x}\right)^8 = 2^9.c(17,8)x = 144446720x$

$t_{9+1} = 17c_9\, (2x)^8 \left(\frac{1}{x}\right)^9 = 2^8\, c(17, 9)\, .\, \frac{1}{x} = \frac{6223360}{x}$

e. Here, $\left(x + \frac{1}{x}\right)^{2n+1}$

Since (2n+1) is odd, there are two middle terms.

$t_{\frac{2n+1-1}{2}+1}$ and $t_{\frac{2n+1+1}{2}+1}$

i.e. t_{n+1} and $t_{(n+1)+1)}$

Now, $t_{n+1} = 2n+1c_n\, (x)^{2n+1-n} \left(\frac{1}{x}\right)^n = 2n+1c_n\, .\, x$

$t_{(n+1)+1} = 2n+1c_{n+1}\, x^{2n+1-n-1} \left(\frac{1}{x}\right)^{n+1} = 2n+1c_{n+1}\, .\, x^{-1}$

8.a. $(1+x)^{2n}$

Since 2n is even for any n, there is a single middle term.

$t_{\frac{n}{2}+1}$

i.e. $t_{\frac{2n}{2}+1}$ t_{n+1}

$\therefore\quad t_{n+1} = 2nc_n\, x^n = n(2n, n)\, x^n.$

$= (2x)^n\, .\, \frac{1.3.5.\, ...\, (2n-1)}{n!}$

b. Since (2n+1) is an odd number, there is only one middle term given by

$t_{\frac{2n}{2}+1}$ i.e. t_{n+1}

We know $t_{r+1} = nc_r\ a^{n-r}\ n^r$ where $a \Rightarrow x\ x - \frac{1}{x}$

$n \Rightarrow 2n$

$\therefore$ Using r = n is above formula, we get

$$t_{n+1} = 2nc_n\ a^{2n-n}\left(-\frac{1}{x}\right)^n = \frac{2n!}{n!\ n!}(-1)^n$$

$$= \frac{1\times2\times3\times4\times5\times6\times\ldots\times(2n-3)\times(2n-2)\times(2n-1)\times 2n}{n!\ n!}(-1)^n$$

$$= \frac{\{1\times3\times5\times\ldots\times(2n-1)\}\ [2\times4\times6\times\ldots\times(2n-2)\times 2n]}{n!\times n!}(-1)^n$$

$$= \frac{\{1\times3\times5\times\ldots\times(2n-1)\}\times 2^n\{1\times2\times3\times\ldots(n-1)\times n\}}{n!\ n!}(-1)^n$$

$$= \frac{1.3.5.\ \ldots\ (2n-1)}{n!}(-2)^n$$

c. $\left(\frac{x}{y} - \frac{y}{x}\right)^{2n+1}$

Since (2n+1) is odd for any n, the number of terms of the expansion is (2n+2), which is even, so there are two middle terms, given by

$t_{\frac{2n+1-1}{2}+1}$ and $t_{\frac{2n+1+1}{2}+1}$

i.e. t_{n+1} and $t_{(n+1)+1}$

Now, $t_{n+1} = 2n+1c_n\left(\frac{x}{y}\right)^{2n+1-n}\left(-\frac{-y}{x}\right)$

$= 2n+1c_n\left(\frac{x}{y}\right)$

$= c(2n+1, n)\frac{x}{y}$

$t_{(n+1)+1} = c(2n+1, n+1)\frac{y}{x}$

9. Let ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ be the three consecutive coefficients in the expansion of $(1+x)^n$. Then,

$nc_{r-1} = 165$ .... (i)

$nc_r = 330$ .... (ii)

nc_{r+1} 462 .... (iii)

Dividing (i) by (ii), we get,

$$\frac{nc_{r-1}}{nc_r} = \frac{165}{330}$$

$$\Rightarrow \frac{n!}{(n-r+1)!\ (r-1)!}\times\frac{(n-r)!\ r!}{n!} = \frac{1}{2}$$

or, $\frac{(n-r)!\ r!}{(r-1)!\ (n-r+1)!} = \frac{1}{2}$

or, $\frac{(n-r)!\ (r-1)!\ r}{(r-1)!\ (n-r+1)\ (n-r)!} = \frac{1}{2}$

$\therefore$ $\frac{r}{n-r+1} = \frac{1}{2}$

or, $2r = n-r+1$

$3r = n+1$ .... (iv)

Again, dividing (ii) by (iii) we get

$$\frac{nc_r}{nc_{r+1}} = \frac{330}{462} \Rightarrow \frac{n!}{(n-r)!\ r!}\times\frac{(n-r-1)!\ (r+1)!}{n!} = \frac{5}{7}$$

or, $\frac{(n-r-1)!\ (r+1)!}{(n-r)!\ r!} = \frac{5}{7}$

or, $\dfrac{(n-r-1)!\ r!\ (r+1)}{(n-r)\ (n-r-1)!\ r!} = \dfrac{5}{7}$

or, $\dfrac{r+1}{n-r} = \dfrac{5}{7}$

or, $7r + 7 = 5n - 5r$

$12r = 5n - 7$ .... (v)

from (iv) and (v), we get

$4(n+1) = 5n - 7$

$4n + 4 = 5n - 7$

$\therefore$ $n = 11$ and $r = 4$

10. Let $n_{c_{r-1}}$, n_{c_r} and $n_{c_{r+1}}$ be three consecutive coefficients of the expansion of $(1+n)^n$.

Acc^r to question $n_{c_{r-1}} : n_{c_r} : n_{c_{r+1}} = 1 : 7 : 42$

Let $n_{c_{r-1}}$ k .... (i) $n_{c_r} = 7k$ .... (ii) and $n_{c_{r+1}} = 42k$ .... (iii)

Dividing (ii) by (i), $\dfrac{n_{c_r}}{n_{c_{r-1}}} = \dfrac{7k}{k}$

$\dfrac{n!}{(n-r)!\ r!} \times \dfrac{(n-r+1)\ !\ (r-1)!}{n!} = 7$

or, $\dfrac{(n-r+1)\ (n-r)!}{(n-r)!\ (r-1)!} = 7$

or, $n-r+1 = 7r$

$n+1 = 8r$ .... (iv)

Again, dividing (iii) by (ii)

$\dfrac{n_{c_{r+1}}}{n_{c_r}} = \dfrac{42k}{7k}$

$\dfrac{n!}{(n-r-1)!\ (r+1)!} \times \dfrac{(n-r)!\ r!}{n!} = 6$

or, $\dfrac{(n-r)\ (n-r-1)!\ r!}{(r+1)\ r!\ (n-r-1)!} = 6$

or, $n-r = 6r + 6$

$n = 7r+6$ .... (v)

From (iv) and (v)

$7r + 6 + 1 = 8r$

$7 = r$

$\therefore$ $r = 7$

$\therefore$ $n = 55$

11. Here, $(x+y)^n$

$\therefore$ $t_{r+1} = n_{c_r}\, x^{n-r}\, y^r$

4^{th} term $(t_4) = t_{3+1} = n_{c_3}\, x^{n-3}\, y^3$

13^{th} term $(t_{13}) = t_{12+1} = n_{c_{12}}\, x^{n-12}\, y^{12}$

Acc^r to question,

Coeff. of 4^{th} term = coeff. of 13^{th} terms

$n_{c_3} = n_{c_{12}}$

$\dfrac{n!}{(n-3)!\ 3!} = \dfrac{n!}{(n-12)!\ 12!}$

or, $(n-12)!\ 12! = (n-3)!\ 3!$

or, $(15-12)!\ 12! = (15-3)!\ 3!$

or, $3!\ 12! = 12!\ 3!$

$\therefore$ $n = 15$

12. We have $(1+x)^{20}$.

r^{th} term $(t_r) = t_{(r-1)+1} = {}^{20}c_{r-1}\, x^{r-1}$

$\therefore$ Coeff. of r^{th} term is ${}^{20}c_{r-1}$

$(r+4)^{th}$ term $(t_{r+4}) = t_{(r+3)+1} = {}^{20}c_{r+3}\, x^{r+3}$

$\therefore$ Coeff. of $(r+4)^{th}$ term is ${}^{20}c_{r+3}$

Acc^r to question,

${}^{20}c_{r-1} = {}^{20}c_{r+3} \Rightarrow r-1+r+3 = 20$

$r = 9$

13. We have $(1+x)^{m+n}$

Comparing it with $(a+x)^n$, we get $a\Rightarrow 1$, $x \Rightarrow x$ $n\rightarrow m+n$.

We know that the general term of expansion of $(a+n)^n$ is $t_{r+1} = {}^nc_r\, a^{n-r}\, x^r$.

Here, $t_{r+1} = {}^{m+n}c_r\, (1)^{m+n-r} \,.\, x^r$

$\therefore$ $t_{r+1} = {}^{m+n}c_r\, x^r$ .... (i)

for x^m, put r = m, in (i)

Then $t_{m+1} = {}^{m+n}c_m\, x^m$

for x^n, put r = n in (i)

Then $t_{n+1} = {}^{m+n}c_n\, x^n$

Now, ${}^{m+n}c_m = \dfrac{(m+n)!}{(m+n-m)!\, m!} = \dfrac{(m+n)!}{n!\, m!} = \dfrac{(m+n)!}{n!\, m!}$

$= \dfrac{(m+n)!}{(m+n-n)!\, n!} = {}^{m+n}c_n$

Hence, ${}^{m+n}c_m = {}^{m+n}c_n$

This proves that the coeff. of x^m and x^n are equal.

14. Here, $(1+x)^{2n}$

The general term $t_{r+1} = {}^{2n}c_r\, x^r$ .... (i)

for 2^{nd} term, put r=1 in (i)

$\therefore$ $t_2 = {}^{2n}c_1\, x^1$

When r = 2 in (i) $\quad t_3 = {}^{2n}c_2\, x^2$

When r = 3 in (i) $\quad t_4 = {}^{2n}c_3\, x^3$

Acc^r to question ${}^{2n}c_1$, ${}^{2n}c_2$ and ${}^{2n}c_3$ are in AP.

Then ${}^{2n}c_2 = \dfrac{{}^{2n}c_1 + {}^{2n}c_3}{2}$

or, $2.\dfrac{(2n)!}{(2n-2)!\, 2!} = \dfrac{(2n)!}{(2n-1)!} + \dfrac{(2n)!}{(2n-3)!\, 3!}$

or, $\dfrac{(2n)!}{(2n-2)!} = 2n! \left[\dfrac{1}{(2n-1)!} + \dfrac{1}{(2n-3)!\, 3!}\right]$

or, $\dfrac{1}{(2n-2)!} = \dfrac{3! + (2n-2)\,(2n-3)}{(2n-1)!\, 3!}$

or, $\dfrac{1}{(2n-2)!} = \dfrac{6 + (2n-2)\,(2n-3)}{(2n-1)\,(2n-2)!\, 6}$

or, $6(2n-1) = 6 + (2n-2)\,(2n-3)$

or, $12n-6 = 6 + 4n^2 - 6n - 4n + 6$

or, $4n^2 - 10n - 12n + 18 = 0$

$4n^2 - 22n + 18 = 0$

$2(2n^2 - 11n + 9) = 0$

$2n^2 - 11n + 9 = 0$

15. The general term of the expansion of $(1+x)^n$ is $t_{r+1} = {}^nC_r\, x^r$

The 2^{nd}, 3^{rd} and 4^{th} terms are respectively,

${}^nc_1\, x^1$, ${}^nC_2\, x^2$ and ${}^nc_3\, x^3$

Acc^r to question, ${}^nC_1$, ${}^nC_2$ and ${}^nC_3$ are in AP.

Then $2.\,{}^nC_2 = {}^nC_1 + {}^nC_3$

$2.\dfrac{n!}{(n-2)!\, 2!} = \dfrac{n!}{(n-1)!} + \dfrac{n!}{(n-3)!\, 3!}$

or, $\frac{n!}{(n-2)!} = n!\left(\frac{1}{(n-1)!} + \frac{1}{6(n-3)!}\right)$

or, $\frac{1}{(n-2)!} = \frac{6 + (n-2)(n-1)}{6(n-1)!}$

or, $\frac{1}{(n-2)!} = \frac{6 + n^2-n-2n+2}{6(n-1)(n-2)!}$

or, $6n-6 = 6+n^2-3n+2$

or, $n^2-9n+14 = 0$

or, $n^2-7n-2n+14 = 0$

$(n-7)(n-2) = 0$

Either $n = 7$ or $n = 2$ (not possible)

$\therefore$ $n = 7$

16. The coeff. of $(2r+1)^{th}$ term is ${}^{21}c_{2r}$ .... (i)

The coeff. of $(3r+2)^{th}$ term is $21c_{3r+1}$ .... (ii)

Acc^r to question,

$21c_{3r+1} = 21c_{2r}$

$\therefore$ Then $3r + 1 + 2r = 21$

$5r = 20$

$\therefore$ $r = 4$

17. Let us suppose that x^r and x^{r+1} occurs in the $(r+1)^{th}$ and $(r+2)^{th}$ terms in the expansion of $(1+x)^{2n+1}$

Then,

$t_{r+1} = nc_r\, a^{n-r}\, x^r$ and $t_{r+2} = nc_{r+1}\, a^{n-r-1}\, x^{r+1}$

where $a \Rightarrow 1$, $x \Rightarrow x$ $n \Rightarrow 2n+1$

$\therefore$ $t_{r+1} = {}^{2n+1}c_r\, 1^{2n+1-r}\, x^r$ and $t_{r+2} = {}^{2n+1}c_{r+1}\, (1)^{2n+1-r-1}\, x^{r+1}$

$\Rightarrow$ $t_{r+1} = {}^{2n+1}c_r\, x^r$ and $t_{r+2} = {}^{2n+1}c_{r+1}\, x^{r+1}$ .... (i)

Now, by question, coefficient x^r = coefficient of x^{r+1}

$\Rightarrow$ ${}^{2n+1}c_r = {}^{2n+1}c_{r+1}$

or, $\frac{(2n+1)!}{r!\,(2n+1-r)!} = \frac{(2n+1)!}{(r+1)!\,(2n+1-r-1)!}$

or, $r!(2n-r+1)! = (r+1)!\,(2n-r)!$

or, $r!\,(2n-r)!\,(2n-r+1) = r!(r+1)\,(2n-r)!$

or, $2n-r+1 = r+1$

$2n = 2r$

$\therefore$ $r = n$

18. Since the number of terms in the expansion of $(1+x)^{2n}$ is $2n+1$, odd number. So there is only one middle term given by $t_{\frac{2n}{2}+1}$ i.e. t_{n+1}.

Now, coefficient of $(n+1)^{th}$ term $= {}^{2n}c_3$ m

Again, the number of terms in the expansion of $(1+x)^{2n-1}$ is $2n-1+1 = 2n$, even number. So, there are two middle terms given by $t_{\frac{2n-1+1}{2}}$, $t_{\frac{2n-1+1}{2}+1}$ i.e. t_n, t_{n+1}

Now, the coefficients of two middle terms are ${}^{2n-1}c_{n-1}$ and ${}^{2n-1}c_n$

$$\therefore\ {}^{2n-1}c_{n-1} + {}^{2n-1}c_n = \frac{(2n-1)!}{(n-1)!\,n!} + \frac{(2n-1)!}{n!(n-1)!}$$

$$= \frac{2(2n-1)!}{n!(n-1)!}$$

$$= \frac{2n(2n-1)!}{n!\,n(n-1)!}$$

$$= \frac{(2n)!}{n!n!}$$

$$= 2nc_n$$

Hence proved

19. Since $(1+x)^n = nc_0 + nc_1\, x + nc_2\, x^2 + ... + nc_n\, x^n$

Using x = 1 and –1, we get

$(1+1)^n = nc_0 + nc_1 + nc_2 + ... + nc_n$ .... (*)

$(1–n)^n = nc_0 – nc_1 + nc_2 – .... + nc_n$ .... (**)

Again, $(1+x)^{n-1} = {}^{n-1}c_0 + {}^{n-1}c_1\, x^1 + {}^{n-1}c_2\, x^2 + ... + {}^{n-1}c_{n-1}\, x^{n-1}$

Using x=1 and –1, we get

$(1+1)^{n-1} = {}^{n-1}c_0 + {}^{n-1}c_1 + {}^{n-1}c_2 + ... + {}^{n-1}c_{n-1}$ .... (***)

$(1–1)^{n-1} = {}^{n-1}c_0 – {}^{n-1}c_1 + {}^{n-1}c_2 + ... + (–1)^{n-1}\, {}^{n-1}c_{n-1}$ .... (****)

a. $C_1 – 2.C_2 + 3.C_3 – ..... + n(–1)^{n-1}\, C_n$

$= n – 2\dfrac{n(n–1)}{2!} + 3.\dfrac{n(n–1)\,(n–2)}{3!} – ..... + n.\,(–1)^{n-1}.\,1$

$= n\left[1 – \dfrac{(n–1)}{1!} + \dfrac{(n–1)\,(n–2)}{2!} – ...... + (–1)n–1\right]$

$= n[{}^{n-1}c_0 – {}^{n-1}c_1 + {}^{n-1}c_2 – ...... + (–1)^{n-1}\, {}^{n-1}c_{n-1}]$

$= n(1–1)^{n-1}$ (By using formula (****) above)

$= n\times 0$

= 0 Hence, proved

b. $C_1 + 2.C_2 + 3.C_3 + ... + n.\, C_n$

$n + 2.\dfrac{n(n–1)}{2!} + 3.\dfrac{n(n–1)\,(n–2)}{3!} + ... + n.1$

$= n[\, {}^{n-1}c_0 + {}^{n-1}c_1 + {}^{n-1}c_2 + ... + {}^{n-1}c_{n-1}]$

$= n(1+1)^{n-1}$ (Using formula (***) above)

$= n.2^{n-1}$ Hence proved

c. $C_0 + 2.C_1 + 3.C_2 + ... + (n+1)\,.\,C_n$

$= (C_0 + C_1 + C_2 + ... + C_n) + (C_1 + 2C_2 + 3C_3 + ... + n.\,C_n)$

$= (1+1)^n + \left[n + \dfrac{n(n–1)}{1!} + \dfrac{n(n–1)\,(n–2)}{2!} + ... + n\right]$

$= 2^n + n\left[1 + \dfrac{(n–1)}{1!} + \dfrac{(n–1)(n–2)}{2!} + ... + 1\right]$

$= 2^n + n[\, {}^{n-1}c_0 + {}^{n-1}c_1 + {}^{n-1}c_2 + ... + {}^{n-1}c_{n-1}]$

$= 2^n + n.\,(1+1)^{n-1}$ (By using formula *** above)

$= 2^n + n\,.\,2^{n-1}$

$= 2^{n-1}.\,2 + n.2^{n-1}$

$= (n+2)2^{n-1}$ Hence proved

d. $C_0 + 2.C_1 + 4.C_2 + ... + 2n.\,C_n\ \boxed{1+n.2^n}$

Similar as above

e. Since $(1+x)^n = C_0 + C_1\, x + C_2\, x^2 + ... + C_n\, x^n$

$\therefore\ (1+x)^{2n} = (C_0 + C_1x + C_2x^2 + ...\ C_nx^n)\,(C_0 + C_1x + C^2x^2 + ... + C_nx^n)$

$\therefore\ {}^{2n}c_0 + {}^{2n}c_1\, x + {}^{2n}c_2\, x^2 + ... + {}^{2n}c_n\, x^n + ... + {}^{2n}c_{2n}\, x^{2n}$ = (") (")

Equating the coefficient of x^n in both sides.

Coefficient of x^n in LHS $= {}^{2n}c_n = \dfrac{(2n)!}{n!\ n!}$ .... (i)

Coefficient of x^n in RHS $= C_0C_n + C_1C_{n-1} + C_2C_{n-2} + ... + ... + C_nC_0$ ...... (ii)

Equating (i) and (ii), we get

$C_0C_n + C_1C_{n-1} + ... + C_nC_0 = \dfrac{(2n)!}{n!\ n!}$ Hence proved

f. Since $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ .... (i)

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n$ .... (ii)

Multiplying (i) and (ii), we get

$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + ... + C_nx^n)\,(C_0x^n + C_1x^{n-1} + ... + C_n)$

Equating the coefficient of x^n both sides, we get

Coefficient of x^n in LHS = ${}^{2n}c_n = \frac{(2n)!}{n!\ n!}$ .... (iii)

Coefficient of x^n in RHS = $C^2 + C_1^2 + C_2^2 + ... + C_n^2$ .... (iv)

Equating (iii) and (iv), we get,

$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \frac{(2n)!}{n!\ n!}$ Hence proved

g. Since $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ .... (i)

$(x+1)^n = C_0x^n = C_1x^{n-1} + C_2x^{n-2} + ... + C_n$ ..... (ii)

Multiplying (i) and (ii), we get

$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + ... + C_nx^n)\ (C_0x^n + C_1x^{n-1} + ... + C_n)$

Equating the coefficient of x^{n-1} both sides,

Coeff. of x^{n-1} in LHS = ${}^{2n}c_{n-1} = \frac{(2n)!}{(2n-n+1)!\ (n-1)!} = \frac{(2n)!}{(n+1)!\ (n-1)!}$ ...... (iii)

Coefficient of x^{n-1} in RHS = $C_0C_1 + C_1C_2 + ... + C_{n-2}\ C_{n-1} + C_{n-1}\ C_n$ .... (iv)

Equating (iii) and (iv), we get

$C_0C_1 + C_1C_2 + ... + C_{n-2}\ C_{n-1} + C_{n-1}\ C_n = \frac{(2n)!}{(n+1)!\ (n-1)!}$ Proved.

h. Since, $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ ...... (i)

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n$ .... (ii)

Multiplying (i) and (ii)

$(1+x)^{2n} = (C_0+C_1x + C_2x^2 + ... + C_nx^n)\ (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n)$

This is identify, so coeff. of any power of x in LHS and coeff. of same power of x in RHS must be equal.

Coeff. of x^{n-2} in LHS = ${}^{2n}c_{n-2} = \frac{(2n)!}{(n+2)!\ (n-2)!}$ .... (iii)

Coeff. of x^{n-2} in RHS = $C_0C_2 + C_1C_3 + ... + C_{n-2}\ C_n$ .... (iv)

Equating (iii) and (iv) we get

$C_0C_2 + C_1C_3 + ... + C_{n-2}\ C_n = \frac{(2n)!}{(n-2)!\ (n+2)!}$ Hence proved

i. Same as (h)

Equating the coeff. of x^{n-r} in both sides.

Coeff. of x^{n-r} in LHS = ${}^{2n}c_{n-r} = \frac{(2n)!}{(n+r)!\ (n-r)!}$ .... (iii)

Coeff. of x^{n-r} in RHS = $C_0C_r + C_1C_{r+1} + ... + C_{n-r}\ C_n$ ...... (iv)

Equating (iii) and (iv), we get

$C_0C_r + C_1C_{r+1} + ... + C_{n-r}\ C_n = \frac{(2n)!}{(n-r)!\ (n+r)!}$ Proved.

20. Let ${}^nc_{r-1}$, ${}^nc_r$, ${}^nc_{r+1}$ and ${}^nc_{r+2}$ be four consecutive coefficients in the expansion of $(1+x)^n$ such that ${}^nc_{r-1} = a$, ${}^nc_r = b$ ${}^nc_{r+1} = c$ and ${}^nc_{r+2} = d$.

Now, $\frac{a}{a+b} = \frac{1}{1+\frac{b}{a}} = \frac{1}{1+\frac{{}^nc_r}{{}^nc_{r-1}}} = \frac{1}{1+\frac{n!}{(n-r)!\ r!}\times\frac{(n-r+1)!\ (r-1)}{n!}} = \frac{1}{1+\frac{(n-r+1)}{r}} = \frac{r}{n+1}$

$\therefore\ \frac{a}{a+b} = \frac{r}{n+1}$ .... (i)

Similarly, we can prove

$\frac{b}{b+c} = \frac{r+1}{n+1}$ .... (ii) and $\frac{c}{c+d} = \frac{r+2}{n+1}$ .... (iii)

$\therefore\ \frac{a}{a+b} + \frac{c}{c+d} = \frac{r}{n+1} + \frac{r+2}{n+1} = \frac{2(r+1)}{n+1} = \frac{2b}{b+c}$

$\therefore\ \frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}$ proved.

Exercise 2.2

1.a. $(2 - 3x)^{-3}$

$= 2^{-3}\left(1 - \frac{3}{2}x\right)^{-3}$

The expansion is valid when $\left|\frac{3}{2}x\right| < 1$ i.e. $|x| < \frac{2}{3}$

Now, $(2 - 3x)^{-3} = 2^{-3}\left(1 - \frac{3}{2}x\right)^{-3}$

$$= \frac{1}{2^3}\left[1 + (-3)\left(-\frac{3}{2}x\right) + \frac{(-3)(-3-1)}{2!}\left(-\frac{3}{2}x\right)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}\left(-\frac{3}{2}x\right)^3 + \ldots \text{ to } \infty\right]$$

$$= \frac{1}{2^3}\left[1 + \frac{9}{2}x + \frac{27}{2}x^2 + \frac{135}{4}x^3 + \ldots \text{ to } \infty\right]$$

$$= \frac{1}{8}\left(1 + \frac{9x}{2} + \frac{27}{2}x^2 + \frac{135}{4}x^3 + \ldots \text{ to } \infty\right)$$

b. Here $(2 + 3x)^{5/2}$

$e^{5/2}\left(1 + \frac{3}{2}x\right)^{5/2}$

The expansion is valid when $\left|\frac{3x}{2}\right| < 1$ i.e. $|x| < \frac{2}{3}$

We know that,

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}n^2 + \frac{n(n-1)(n-2)}{3!}n^3 + \ldots$

Now, $2^{5/2}\left(1 + \frac{3}{2}x\right)^{5/2}$

$$= 2^{5/2}\left[1 + \left(\frac{5}{2}\right)\left(\frac{3}{2}x\right) + \frac{\frac{5}{2}\left(\frac{5}{2} - 1\right)}{2!}\left(\frac{3x}{2}\right)^2 + \frac{\frac{5}{2}\left(\frac{5}{2} - 1\right)\left(\frac{5}{2} - 2\right)}{3!}\left(\frac{3}{2}x\right)^3 + \ldots\right]$$

$$= 2^{5/2}\left[1 + \frac{15}{4}x + \frac{135x^2}{64} + \frac{135x^3}{128} + \ldots \text{ to } \infty\right]$$

c. $(5 + 4x)^{-1/2}$

$5^{-1/2}\left[1 + \frac{4}{5}x\right]^{-1/2}$

This expansion is valid when $\left|\frac{4x}{5}\right| < 1$ i.e. $|x| < \frac{5}{4}$

$5^{-1/2}$

$$\left[1 + \left(-\frac{1}{2}\right)\frac{4}{5}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)}{2!}\left(\frac{4x}{5}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\left(\frac{-1}{2} - 2\right)}{3!}\left(\frac{4x}{5}\right)^3 + \ldots\right]$$

$$= \frac{1}{\sqrt{5}}\left[1 - \frac{2x}{5} + \frac{6x^2}{25} - \frac{4x^3}{25} + \ldots \text{ to } \infty\right]$$

d. $(3 - 2x^2)^{-2/3}$

$3^{-2/3}\left(1 - \frac{2}{3}x^2\right)^{-2/3}$

The expansion is valid when $\left|\frac{2x^2}{3}\right| < 1$ i.e. $|x^2| < \frac{3}{2}$

Now, $e^{-2/3}\left[1 - \frac{2}{3}x^2\right]^{-2/3}$

$= e^{-2/3}\left[1 + \left(\frac{-2}{3}\right)\left(-\frac{2}{3}x^2\right)\right] + \frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3} - 1\right)}{2!}\left(\frac{-2}{3}x^2\right)^2 + \text{... to } \infty)$

$= e^{-2/3}\left[1 + \frac{4x^2}{9} + \frac{20x^4}{81} + \text{... to } \infty\right]$

2.a. $(1+x)^{1/2}$

$1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)x^2}{2!} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3!}x^3 + \ldots$

$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}x^3 \text{ ... to } \infty$

b. $(1 + x^2)^{-1/2}$

$1 + \left(\frac{-1}{2}\right)x^2 + \frac{\left(\frac{-1}{2}\right)\left(-\frac{1}{2} - 1\right)}{2!}(x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\left(\frac{-1}{2} - 2\right)}{3!}(x^2)^3 + \ldots$

$= 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \text{... to } \infty$

c. $(1 + x)^{1/4}$

$1 + \left(+\frac{1}{4}\right)x + \frac{\left(+\frac{1}{4}\right)\left(+\frac{1}{4} - 1\right)}{2!}x^2 + \frac{\left(\frac{+1}{4}\right)\left(+\frac{1}{4} - 2\right)\left(\frac{1}{4} - 2\right)}{3!}x^3 + \ldots$

$= 1 + \frac{x}{4} - \frac{3x^2}{32} + \frac{7x^3}{128} - \text{... to } \infty$

d. $(1 - x^2)^{-1/3}$

$1 + \left(-\frac{1}{3}\right)(-x^2) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3} - 1\right)}{2!}(-x^2)^2 + \text{... to } \infty$

$= 1 + \frac{1}{3}x^2 + \frac{2x^4}{9} + \text{... to } \infty$

3.a. $(1.03)^{-5}$

$(1 + 0.03)^{-5}$

$= 1 + (-5)(0.03) + \frac{(-5)(-5-1)(0.03)^2}{2!} + \frac{(-5)(-5-1)(-5-2)}{3!}(0.03)^3 + \ldots$

$= 0.915$

b. $(0.01)^{1/2}$

$(1 - 0.99)^{1/2}$

$= 1 + \frac{1}{2}(-0.99) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}(-0.99)^2 + \ldots$

$= 0.1$

c. $(28)^{1/3}$

$(27 + 1)^{1/3}$

$= 3\left(1 + \frac{1}{27}\right)^{1/3}$

$= 3\left[1 + \frac{1}{3}\cdot\frac{1}{27} + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{1}{27}\right)^2 + \ldots\right]$

$= 3.0.37$

d. $\sqrt{17} = (16 + 1)^{1/2} = 4\left(1 + \frac{1}{16}\right)^{1/2}$

$$4\left[1 + \frac{1}{2}\frac{1}{16} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(\frac{1}{16}\right)^2 + ...\right]$$

$= 4.123$

e. $\left(\frac{96}{101}\right)^{1/3}$

$\left(1 - \frac{5}{101}\right)^{1/3}$

$$1 + \frac{1}{3}\left(\frac{-5}{101}\right) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{-5}{101}\right)^2 + ... = 0.983$$

4.a. Let $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.2} + ...$ to $\infty = (1 + x)^n$

Then, $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + ...$ to $\infty = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

Equating corresponding term, we get,

$nx = \frac{1}{4}$

$\therefore \quad x = \frac{1}{4n}$ ... ... (i) and $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{4.8}$

or, $\frac{n(n-1)}{2}\frac{1}{(4n)^2} = \frac{1.3}{4.8}$

$\frac{n(n-1)}{2.16\, n^2} = \frac{3}{32}$

$n - 1 = 3n$

$-2n = 1$

$\therefore \quad n = -\frac{1}{2}$

from (i) $x = \frac{1}{4(-1/2)} = -\frac{1}{2}$

Hence, $(1 + x)^n = \left(1 - \frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2} = 2^{1/2} = \sqrt{2}$

$\therefore \quad 1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + ...$ to $\infty = \sqrt{2}$

b. Let $(1+x)^n$ be equal to $1 + \frac{1.2}{2.3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + ...$ to ∞

i.e. $1 + nx + \frac{n(n-1)}{2!}x^2 + ...$ to $\infty = 1 + \frac{1.2}{2.3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + ...$ to ∞

Equating corresponding term, we get

$nx = \frac{1.2}{2.3} \Rightarrow x = \frac{1}{3n}$ ... ... (i) and $\frac{n(n-1)}{2}x^2 = \frac{1.3}{3.6}$

$\frac{n(n-1)}{2}\frac{1}{n^2} = \frac{1}{6}$

$n - 1 = 3n$

$-2n = 1$

$\therefore \quad n = -\frac{1}{2}$

from (i) $x = \frac{1}{3\left(\frac{-1}{2}\right)} = -\frac{2}{3}$

$\therefore \quad (1 + x)^n = \left(1 - \frac{2}{3}\right)^{-1/2} = \left(\frac{1}{3}\right)^{-1/2} = 3^{1/2} = \sqrt{3}$

c. Let $1 - \frac{1}{6} + \frac{1.3}{6.12} + \frac{1.3.5}{6.12.18} + \ldots$ to $\infty = (1 + x)^n$

$1 - \frac{1}{6} + \frac{1.3}{6.12} + \frac{1.3.5}{6.12.18} + \ldots$ to $\infty = (1 + x)^n$

Equating corresponding term

$nx = -\frac{1}{6}$

$\therefore \quad x = -\frac{1}{6n} \ldots \ldots \ldots$ (i)

$\frac{n(n-1)}{2} x^2 = \frac{1.3}{6.12}$

or, $\frac{n(n-1)}{2} \left(\frac{-1}{6n}\right)^2 = \frac{1}{24}$

$\frac{n(n-1)}{2} \frac{1}{36.n^2} = \frac{1}{24}$

or, $n - 1 = 3n,\ n = -\frac{1}{2}$

from (i) $x = -\frac{1}{6\left(-\frac{1}{2}\right)} = \frac{1}{3}$

$\therefore \quad (1 + x)^n = \left(1 + \frac{1}{3}\right)^{-1/2} = \left(\frac{4}{3}\right)^{-1/2} = \left(\frac{3}{4}\right)^{1/2} = \frac{\sqrt{3}}{2}$

d. Let $1 + \frac{1}{4} + \frac{1.4}{4.8} + \ldots$ to $\infty = (1 + x)^n$

$1 + \frac{1}{4} + \frac{1.4}{4.8} + \ldots$ to $\infty = 1 + nx + \frac{n(n-1)}{2} x^2 + \ldots$

$\therefore \quad nx = \frac{1}{4}$ and $\frac{n(n-1)}{2} x^2 = \frac{1.4}{4.8}$

$x = \frac{1}{4n} \ldots \ldots \ldots$ (i)

or, $\frac{n(n-1)}{2} . \frac{1}{16n^2} = \frac{1}{8}$

or, $n - 1 = 4n$

$3n = -1$

$\therefore \quad n = -\frac{1}{3}$

$\therefore \quad x = -\frac{3}{4}$

$\therefore \quad (1 + x)^n = \left(1 - \frac{3}{4}\right)^{-1/3} = \left(\frac{1}{4}\right)^{-1/3} = 4^{1/3} = 2^{2/3}$ proved.

e. Let $1 + \frac{1}{4} - \frac{1.1}{4.8} + \frac{1.1.3}{4.8.12} - \ldots = 1 + nx + \frac{n(n-1)}{2} x^2 + \ldots$

Equating, $nx = \frac{1}{4}$

$\therefore \quad x = \frac{1}{4n} \ldots \ldots \ldots$ (i)

$\frac{n(n-1)}{2} x^2 = -\frac{1}{32}$

$\frac{n(n-1)}{2} . \frac{1}{16n^2} = -\frac{1}{32}$

$n - 1 = -n$

$2\,n = 1$

$\therefore \quad n = \frac{1}{2}$

$\therefore \quad x = \frac{1}{2}$

$\therefore \quad (1 + n)^n = \left(1 + \frac{1}{2}\right)^{1/2} = \left(\frac{3}{2}\right)^{1/2} = \sqrt{\frac{3}{2}}$

Hence, $1 + \frac{1}{4} - \frac{1.1}{4.8} + \frac{1.1.3}{4.8.12} - \frac{1.1.3.5}{4.8.12.16} + ... = \sqrt{\frac{3}{2}}$

5. Given, $y = 2x + 3x^2 + 4x^3 + ...$ to ∞

$1 + y = 1 + 2x + 3x^2 + 4x^3 + ...$ to ∞

or, $1 + y = (1 - x)^{-2}$

or, $(1 - x) = (1 + y)^{-1/2}$

or, $1 - x = 1 - \frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{10}y^3 + ...$

$\therefore$ $x = \frac{1}{2}y - \frac{3}{8}y^2 + \frac{5}{16}y^3 - ...$ to ∞

6. Here, $(1 - x + x^2 - x^3 + ...$ to $\infty)$ $(1 + x + x^2 + x^3 + ...$ to $\infty)$

$= (1 + x)^{-1} . (1 - x)^{-1}$

$= (1 - x^2)^{-1}$

$= 1 + (-1)(-x^2) + \frac{(-1)(-1-1)}{2}(-x^2)^2 + ...$ to ∞

$= 1 + x^2 + x^4 + ...$ to ∞ proved.

7. Here,

$(1 + x + x^2 + x^3 + ...$ to $\infty)$ $(1 + 2x + 3x^2 + ...$ to $\infty)$

$= (1 - x)^{-1}(1 - x)^{-2}$

$= (1 - x)^{-3}$

$= 1 + (-3)(-x) + \frac{(-3)(-3-1)}{2!}(-x)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}(-x)^3 + ...$

$= 1 + 3x + 6x^2 + ...$ to ∞

Exercise 2.3

1.a. $\frac{e^{5x} + e^x}{e^{3x}} = e^{2x} + e^{-2x}$

We know that, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ to ∞

$\therefore$ $e^{2x} + e^{-2x} = \left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(3x)^3}{3!} + ... \text{ to } \infty\right) + \left(1 - \frac{2x}{1!} + \frac{(2x)^2}{2!}\right) - \frac{(2x)^3}{3!} + ... \text{ to } \infty)$

$= 2 + 2\frac{(2x)^2}{2!} + 2\frac{(2x)^4}{4!} + ...$ to ∞

$= 2\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + ...\right) = 2\left(1 + \frac{2^2.x^2}{2!} + \frac{2^4.x^4}{4!} + \frac{2^6.x^6}{6!} ... + \infty\right)$

b. $\frac{e^{7x} + e^x}{2e^{4x}}$

$= \frac{1}{2}[e^{3x} + e^{-3x}]$

$= \frac{1}{2}\left[\left(1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + ...\right)\right] + \left(1 - \frac{3x}{1!} + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + ...\right)$

$= \frac{1}{2}\left[2 + \frac{2(3x)^2}{2!} + 2((3x)^4,4!) + ...\right]$

$= 1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!}$

$= 1 + \frac{3^2.x^2}{2!} + \frac{3^4.x^4}{4!} + \frac{3^6\,x^6}{6!} + ... \infty$

c. $\frac{e^{4x} - 1}{e^{2x}} = e^{2x} - e^{-2x} = \left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + ..\right) - \left(1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + ...\right)$

$= 2\left(\frac{2x}{1!} + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + ...\right)$

2.a. $(e^1) . (e^{-1})$

$e^0 = 1$ proved.

b. $\frac{(1 + 1)}{1!} + \left(\frac{3 + 1}{3!}\right) + \frac{(5 + 1)}{5!} + ...$ to ∞

$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + ... = e$

c. n^{th} term of given series $(t_n) = \frac{1 + 2 + 3 + ... + n}{(n-1)!} = \frac{\frac{n}{2}(n + 1)}{(n-1)!} = \frac{n(n + 1)}{2(n-1)!}$

d. $t_n = \frac{1 + 3 + 5 + ... (2n - 1)}{n!} = \frac{n^2}{n!} = \frac{n.n}{n!} = \frac{n.n}{n(n-1)!}$

$t_n = \frac{n}{(n-1)!} = \frac{n - 1 + 1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$

$S_n = \Sigma t_n = \Sigma \frac{1}{(n-2)!} + \Sigma \frac{1}{(n-1)!}$

$= \left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...\right)$

$= \left(\frac{1}{0} + 1 + \frac{1}{1!} + \frac{1}{2!} + ...\right) + \left(1 + \frac{1}{1!} + \frac{1}{2!} + ...\right)$

$= e + e$

$= 2e$

e. $t_n = \frac{2 + 4 + 6 + ... + 2n}{n!} = \frac{n(n + 1)}{n!}$

$= \frac{n(n + 1)}{n(n-1)!} = \frac{n - 1 + 2}{(n-1)!} = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$

$t_n = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$

$s_n = \Sigma t_n = \sum_{n=1}^{\infty} \frac{1}{(n-2)!} + 2\sum_{n=1}^{\infty} \frac{1}{(n-1)!}$

$= e + 2e$

$= 3e$

f. $3 + \frac{5}{1!} + \frac{7}{2!} + \frac{9}{3!} + ...$ to ∞

Let t_n be the n^{th} term of above series.

Then, $t_n = \frac{3 + (n-1)d}{(n-1)!} = \frac{3 + (n-1) . 2}{(n-1)!} = \frac{2n + 1}{(n-1)!}$

$t_n = \frac{2(n-1) + 3}{(n-1)!} = \frac{2}{(n-2)!} + \frac{3}{(n-1)!}$

$s_n = \sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} \frac{2}{(n-2)!} + 3\sum_{n=1}^{\infty} \frac{1}{(n-1)!}$

$= 2\left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{!} + \frac{1}{2!} + ...\right) + 3\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...\right)$

$= 2e + 3e$

$= 5e$

g. Here, $\frac{2 - 1}{2!} + \frac{3-1}{3!} + \frac{4 - 1}{4!} + ...$

$1 - \frac{1}{2!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{3!} - \frac{1}{4!} + ...$

$= 1$ proved.

h. We know, $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to $\infty = e^x$

Putting x = 1 and x = –1, we get

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = e \ldots \ldots \ldots$ (i)

and $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots = e^{-1} \ldots \ldots \ldots$ (ii)

Adding (i) and (ii), we get

$$2 + \frac{2}{2!} + \frac{2}{4!} + \ldots = e + e^{-1}$$

or, $2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\right) = e + e^{-1}$

or, $1 + \frac{1}{2!} + \frac{1}{4!} + \ldots = \frac{e + e^{-1}}{2}$

$\therefore\ 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots = \frac{e^2 + 1}{2e} \ldots \ldots \ldots$ (iii)

Subtracting (ii) from (i), we get

$2 + \frac{2}{3!} + \frac{2}{5!} + \frac{2}{7!} + \ldots = e - e^{-1}$

$1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots = \frac{e^2 - 1}{2e} \ldots \ldots \ldots$ (iv)

Dividing (iii) by (iv), we get,

$$\frac{1 + \frac{1}{2!} + \frac{1}{4!} + \ldots}{1 + \frac{1}{3!} + \frac{1}{5!} + \ldots} = \frac{e^2 + 1}{e^2 - 1}$$ Hence proved.

i. $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots$

$\frac{3 - 1}{3!} + \frac{5 - 1}{5!} + \frac{7 - 1}{7!} + \ldots$

$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \ldots$

$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots = e^{-1}$ proved.

j. We know,

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \ldots \ldots$ (*)

$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \ldots \ldots \ldots$ (**)

Adding (*) and (**),

$e^x + x^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right)$

or, $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \ldots \ldots$ (***)

Subtracting (**) from (*), we get,

$e^x - e^{-x} = 2\left(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$

$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \ldots \ldots$ (****)

Now, $\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \text{ to } \infty\right)^2 - \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)^2$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right)$$

$= e^x \cdot e^{-x} = 1$

k. We know $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$\therefore \quad \frac{1}{2} e^x = \frac{1}{2}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$

$\frac{1}{2} e^2 = \frac{1}{2}\left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \ldots\right)$

$= \frac{1}{2} + 1 + \frac{2}{2!} + \frac{2^2}{3!} + \frac{2^3}{4!} + \ldots$

$\therefore \quad 1 + \frac{2}{2!} + \frac{2^2}{3!} + \frac{2^3}{2!} + \ldots = \frac{1}{2} e^2 - \frac{1}{2} = \frac{1}{2}(e^2 - 1)$

Also, $e - e^{-1} = 1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots$

$\therefore \quad \dfrac{1 + \frac{2}{2!} + \frac{2^2}{3!} + \frac{2^3}{4!} + \ldots}{1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots} = e$ proved.

3.a. We know that,

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to ∞

$\therefore \quad e^{x^2} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots$ to ∞ ... ... (i)

$e^{y^2} = 1 + \frac{y^2}{1!} + \frac{y^4}{2!} + \frac{y^6}{3!} + \ldots$ to ∞ ... ... (ii)

Equation (i) – equation (ii)

$e^{x^2} - e^{y^2} = \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots\right) - \left(1 + \frac{y^2}{1!} + \frac{y^2}{2!} + \frac{y^6}{3!} + \ldots\right)$

$= (x^2 - y^2) + \frac{(x^4 - y^4)}{2!} + \frac{1}{3!}(x^6 - y^6) + \ldots$ Proved.

b. RHS $\dfrac{1}{\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots \text{ to } \infty}$

$= \dfrac{1}{\frac{3-1}{3!} + \frac{5-1}{5!} + \frac{7-1}{7!} + \ldots \text{ to } \infty}$

$= \dfrac{1}{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \ldots \text{ to } \infty}$

$= \dfrac{1}{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots}$

$= \frac{1}{e^{-1}} = e$

LHS $\frac{1+1}{1!} + \frac{3+1}{3!} + \frac{5+1}{5!} + \ldots$ to ∞

$= 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$ to ∞

$= e$ Hence proved

c. We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ to ∞

$\therefore \quad e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \ldots$

$$e^2 - 1 = 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \ldots$$

4.a. $\sqrt{e} = e^{1/2}$

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

Put $x = \frac{1}{2}$

$$e^{1/2} = 1 + \frac{\frac{1}{2}}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \ldots$$

$$= 1 + 0.5 + 0.125 + 0.0208 + \ldots$$

$$= 1.6458$$

b. $\frac{1}{\sqrt{e}} = e^{-1/2}$

$$= 1 - \frac{\frac{1}{2}}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} - \frac{\left(\frac{1}{2}\right)^3}{3!} + \ldots$$

$$= 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \ldots$$

$$= 1 - 0.5 + 0.125 - 0.0208 + \ldots$$

$$= 0.6042$$

5.a. Let n^{th} term of above series be t_n

Then $t_n = \frac{1 + 2 + 2^2 + \ldots + 2^{n-1}}{n!} = \frac{\frac{1(2^n - 1)}{2 - 1}}{n!} = \frac{2^n - 1}{n!}$

$$t_n = \frac{2^n}{n!} - \frac{1}{n!}$$

Let s_∞ be the required sum of the series.

Then, $s_\infty = \Sigma t_n = \Sigma\left(\frac{2^n}{n!} - \frac{1}{n!}\right) = \sum_1^\infty \frac{2^n}{n!} - \sum_1^\infty \frac{1}{n!}$

$$= \left(\frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots\right) - \left(\frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

$$= (e^2 - 1) - (e - 1)$$

$$= e^2 - e$$

b. $1 + \frac{1}{2!} + \frac{1.3}{4!} + \frac{1.3.5}{6!} + \ldots$

$$= 1 + \frac{1}{2} + \frac{1.3}{1.2.3.4} + \frac{1.3.5}{1.2.3.4.5.6} + \ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \ldots$$

$$= 1 + \frac{\left(\frac{1}{2}\right)}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \ldots$$

$$= e^{1/2}$$

c. Let t_n be the n^{th} term of the given series

Then $t_n = \frac{n(n + 1)}{n!} = \frac{n^2 + n}{n!} = \frac{n}{(n - 1)!} + \frac{1}{(n - 1)!}$

$$= \frac{n+1}{(n-1)!} = \frac{(n-1)+2}{(n-1)!} = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$$

$\therefore$ The given series

$$\Sigma t_n = \Sigma \frac{1}{(n-2)!} + \Sigma \frac{2}{(n-1)!}$$

$$= \left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right) + 2\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

but $(-1)! = \infty$ and $0! = 1$

$$\therefore \Sigma t_n = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) + 2\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

$$= e + 2e$$

$$= 3e$$

d. Let t_n be the n^{th} term of the series then,

$$t_n = \frac{1+(n-1)^2}{(n-1)!} = \frac{2n-1}{(n-1)!} = \frac{2(n-1)+1}{(n-1)!}$$

$$= \frac{2}{(n-2)!} + \frac{1}{(n-1)!}$$

$\therefore$ Sum of the given series

$$\sum_1^\infty t_n = 2\sum_1^\infty \frac{1}{(n-2)!} + \sum_1^\infty \frac{1}{(n-1)!}$$

$$= 2\left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \ldots\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

But $(-1)! = \infty$ and $0! = 1$

$$\therefore \sum_1^\infty t_n = 2e + e = 3e$$

e. Let t_n be the n^{th} term

$$t_n = \frac{n^2}{n!} = \frac{n}{(n-1)!} = \frac{n-1+1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$$

$\therefore$ Sum of the series

$$\sum_1^\infty t_n = \sum_n^\infty \frac{1}{(n-2)!} + \Sigma\frac{1}{(n-1)!} = e+e = 2e$$

6.a. $t_n = \dfrac{n^2}{(n+1)!} = \dfrac{n^2-1+1}{(n+1)!} = \dfrac{(n^2-1)}{(n+1)!} + \dfrac{1}{(n+1)!}$

$$= \frac{(n-1)}{n!} + \frac{1}{(n+1)!}$$

$$= \frac{n}{n!} - \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$= \frac{1}{(n-1)!} - \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$\Sigma t_n = \Sigma\frac{1}{(n-1)!} - \Sigma\frac{1}{n!} + \Sigma\frac{1}{(n+1)!}$$

$$= \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) - \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) + \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\right)$$

$$= e - (e-1) + (e-2)$$

$$= e - e + 1 + e - 2$$

$$= e - 1$$

b. $t_n = \dfrac{1}{(n+1)!}$

$$\Sigma t_n = \sum_1^{\infty} \frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$
$$= e - 2$$

c. $t_n = \frac{1}{(n+2)!}$

$$\Sigma t_n = \sum_1^{\infty} \frac{1}{(n+2)!} = \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$
$$= \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots\right) - \frac{5}{2}$$
$$= e - \frac{5}{2}$$

d. $t_n = \frac{n^3}{n!} = \frac{n^2}{(n-1)!} = \frac{(n-1)(n+1)}{(n-1)!} + \frac{1}{(n-1)!}$

$$= \frac{n+1}{(n-2)!} + \frac{1}{(n-1)!} = \frac{(n-2)}{(n-2)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$$
$$= \frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$$
$$\Sigma t_n = \Sigma \frac{1}{(n-3)!} + 3\Sigma\frac{1}{(n-2)!} + \Sigma\frac{1}{(n-1)!}$$
$$= \left(\frac{1}{(-2)!} + \frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right) + 3\left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \ldots\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$
$$= e + 3e + e$$
$$= 5e$$

e. $t_n = \frac{n(n+1)}{n!} = \frac{n+1}{(n+1)!} = \frac{n-1}{(n-1)!} + \frac{2}{(n-1)!} = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$

$$\Sigma t_n = \Sigma\frac{1}{(n-2)!} + \Sigma \frac{2}{(n-1)!}$$
$$= e + 2e$$
$$= 3e$$

7.a. We know that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ to ∞

$$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$
$$\ln 2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots$$
$$= \left(\frac{2-1}{1.2}\right) + \frac{(4-3)}{3.4} + \left(\frac{6-5}{5.6}\right) + \ldots$$
$$= \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$$

b. $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots$ to $\infty = 2 - 2\ln 2$

We have,

$\ln(1 + x) = x - x\frac{2}{2} + x\,\frac{3}{3} - x\frac{4}{4} + x\frac{5}{5} - x\,\frac{6}{6} + x\,\frac{7}{7} - \ldots$

Putting x = 1, we get,

$\ln(1 + 1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots$

$\Rightarrow\ \ln^2 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \ldots$

$\Rightarrow\ \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \ldots = 1 - \ln 2$

$\Rightarrow\ \left(\frac{3-2}{2.3}\right) + \left(\frac{5-4}{4.5}\right) + \left(\frac{7-6}{6.7}\right) + \ldots = 1 - \ln 2$

$\Rightarrow\ \frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots = 1 - \ln 2$

Multiplying by 2 on both sides, we get

$\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots = 2(1 - \ln 2)$

Hence, $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots = 2 - 2\ln 2$

c. We know that,

$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$ to ∞ ... ... (i)

$\log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$ to ∞ ... ... (ii)

Subtracting (ii) from (i)

$\log_e(1 + x) - \log(1 - x) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + 2\frac{x^7}{7} + \ldots$

or, $\log_e\left(\frac{1 + x}{1 - x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$

$\frac{1}{2}\log_e\left(\frac{1 + x}{1 - x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots$ to ∞

Put $x = \frac{1}{3}$

$$\frac{1}{2}\log_e\left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right) = \frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \frac{\left(\frac{1}{3}\right)^7}{7} + \ldots$$

$$\frac{1}{2}\log_e\left(\frac{\frac{4}{3}}{\frac{2}{3}}\right) = \frac{1}{3} + \frac{1}{3^3.3} + \frac{1}{3^5.5} + \frac{1}{3^7.7} + \ldots$$

$\therefore\ \frac{1}{2}\log_e(2) = \frac{1}{3} + \frac{1}{3^3.3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \ldots$

d. We know,

$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$$

$$\therefore \quad \log(1+x) = \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$$

Put $x = \frac{1}{2}$

$$\log\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) = 2\left(\frac{1}{2} + \frac{1}{3.2^3} + \frac{1}{5.2^5} + \frac{1}{7.2^7} + \ldots\right)$$

$$\log 3 = 1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots$$

e. Given,

$2\ln x - \ln(x+1) - \ln(x-1)$

$= 2\ln x - \{\ln(x+1) + \ln(x-1)\}$

$= \ln x^2 - \ln\{(x+1)(x-1)\}$

$= \ln x^2 - \ln(x^2 - 1)$

$= -[\ln(x^2 - 1) - \ln^2]$

$$= -\ln\left(\frac{x^2-1}{x^2}\right)$$

$$= -\ln\left(1 - \frac{1}{x^2}\right)$$

$$= -\left[-\frac{1/x^2}{1} - \frac{(1/x^2)^2}{2} - \frac{(1/x^2)^3}{3} - \ldots \text{ to } \infty\right]$$

$$= \frac{1}{x^2} + \frac{1}{2.x^2} + \frac{1}{3.x^6} + \ldots \text{ to } \infty$$

f. LHS. $\left(\frac{1}{3} - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{3^2} - \frac{1}{2^2}\right) + \frac{1}{3}\left(\frac{1}{3^3} - \frac{1}{2^3}\right) + \ldots$

$$= \left(\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1}{3}\,\frac{1}{3^3} + \ldots\right) - \left(\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\cdot\frac{1}{2^3} - \ldots\right)$$

$$= -\left[-\left(\frac{1}{3}\right) - \frac{\left(\frac{1}{3}\right)^2}{2} - \frac{\left(\frac{1}{3}\right)^3}{3} - \ldots\right] - \left[\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \ldots\right]$$

$$= -\log_e\left(1 - \frac{1}{3}\right) - \log_e\left(1 + \frac{1}{2}\right)$$

$$= -\log\frac{2}{3} - \log\frac{3}{2}$$

$$= -\left[\log\frac{2}{3} + \log\frac{3}{2}\right]$$

$$= -\log\left(\frac{2}{3}, \frac{3}{2}\right)$$

$= -\log 1$

$= 0$ Hence proved.

8.a. Sum to infinity the following series:

$$\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots \text{ to } \infty$$

$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \ldots$ to ∞

$= \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \ldots$ to ∞

$= 1 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \ldots\right)$

$= 1 - \text{Ln}(1+1) = 1 - \ln^2$

b. We know that,

$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$ to ∞

Putting x = 1 on both sides, we get

$\ln(1 + 1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

or, $\ln 2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots$

$= \left(\frac{2-1}{2}\right) + \left(\frac{4-3}{3.4}\right) + \left(\frac{6-5}{5.6}\right) + \ldots$

$\ln 2 = \frac{1}{2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$ to ∞

or, $\ln 2 - \frac{1}{2} = \frac{1}{3.4} + \frac{1}{5.6} + \ldots$

or, $\ln 2 - \frac{1}{2} = \frac{1}{3.4} + \frac{1}{5.6} + \ldots$

$\therefore \frac{1}{3.4} + \frac{1}{5.6} + \ldots$ to $\infty = \ln 2 - \frac{1}{2}$

9. Here, $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ to ∞

or, $y = \log_e(1 + x)$

$\therefore e^y = 1 + x$

$1 + x = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$ to ∞

$\therefore x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$ to ∞

10. Here, $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$ to ∞

$y = -\left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \text{ to } \infty\right]$

or, $y = -[\log_e(1 - x)]$

or, $-y = \log_e(1 - x)$

$\therefore 1 - x = e^{-y}$

$1 - x = 1 - \frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} - \ldots$

$x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \ldots$ to ∞

11. Here, $y = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to ∞

$1 + y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to ∞

$1 + y = e^x$

Taking 'ln' on both sides

$\ln(1 + y) = x$

$\therefore \quad x = \ln(1 + y)$

$$x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \text{ to } \infty$$

Chapter 3
Elementary Group Theory

Exercise 3.1

1. a. No, the operation $*$ on the set of positive odd numbers 0^+ defined by $x * y = x+y$ is not a binary operation because for all $x, y \in 0^+$, $x * y = x + y \notin 0^+$.
e.g. $1, 3 \in 0^+$ but $1*3 = 1 + 3 = 4 \notin 0^+$.

b. Yes, since $\forall$ x, y $\in$ R, $x + y = 2^{xy} \in R$

c. Yes, Here Z = {..., –3, –2, –1, 0, 1, 2, 3, ...}
$\forall$ x, y $\in$ Z, 2x – y is also an integer and uniquely belongs to Z. So, it is a binary operation.

d. No, let $2, 3 \in N$ then $2*3 = 2+3 - 2.3$
$= 5 - 6$
$= -1 \notin N$
Therefore $*$ defined by $x * y = x + y - xy$ on the set of natural number is not a binary.

e. Yes, $\forall$ A, B $\in$ M = {set of 2×2 matrix}
A*B = AB is also a 2×2 matrix and uniquery belongs to M. So, it is a binary.

3. Let S = {–1, 0, 1}
For any $a, b \in S$ $\quad a*b = a.b \in S$
$\therefore$ multiplication operation on S = {–1, 0, 1} is a binary operation.

4. Given S = {–1, 0, 1}
Operation $*$ defined by $a*b = a\times b$
a. $\forall a, b \in S,$ $\quad a*b = a.b = b.a = b*a$
$\therefore$ $*$ is commutative on S.

b. $\forall a, b, c \in S$
$(a*b) * c = (a\times b) * c$
$= a\times b\times c$
$= a\times(b*c)$
$= a*(b*c)$
$\therefore$ '$*$' is an associative on S.

5. Let e be an identify element of $a \in Z$ then
$a*e = a$ $\quad$ and $e*a = a$
$2a + e = a$ $\quad$ $2e + a = a$
$e = -a \in z$ $\quad$ $e = 0 \in z$
identify is not uniquely.
Let a' be inverse of $a \in z$ then $a * a' = e$
$2a + a' = -a$
$a^1 = -3a \in z$

6. Let $a, b, c \in Q$ be any elements.
Then, $(a * b) * c = (a + b + ab) * c$
$= a + b + ab + c + (a + b + ab) c$
$= a + b + ab + c + ac + bc + abc$
$= a + b + c + bc + ca + ab + abc$
$= a + (b + c + bc) + a(c + b + bc)$
$= (b + c + bc) + a + (b + c + bc) a$
$= a + (b + c + bc) + (b + c + bc) a$
$= a * (b + c + bc)$
$= a * (b*c)$
$\therefore$ '$*$' is an associative.

7. $\forall$ a, b $\in$ z

   $a*b = 3a + 2b$ is also an integers and uniquely belongs to z. So, $*$ is a binary operation.

   But $a*b = 3a + 2b \neq 3b + 2a = b*a$

   $\therefore$ $a*b \neq b*a$

   $\therefore$ '$*$' is not a commutative.

8. Given, P = power set of a non–empty set X.

a. Let A, B $\in$ P with $A*B = A\cup B$

   Here, $A\cup B$ must belong to set P. So, union operation on P is a binary.

b. Let A, B $\in$ P with $A*B = A–B$

   Here, A–B or B–A must belong to the power set P.

   $\therefore$ difference operation is a binary.

c. $\forall$ A, B $\in$ P $A\cap B \in$ P. So, intersection is a binary.

9. Given, set S = $\{1, \omega, \omega^2\}$ where ω is the cube root of unity operation; multiplication.

   $1 \times \omega = \omega \in S$

   $\omega \times \omega^2 = \omega^3 = 1 \in S$

   $1 \times 1 = 1 \in S$

   $\omega^2 \times \omega^2 = \omega^4 = \omega^3. \omega = 1. \omega = \omega \in S$

   So, $\forall$ a, b $\in$ S $\quad$ $a * b = a, b \in S$

   $\therefore$ multiplication operation is binary on S.

a. Commutative

   $1 \times \omega = \omega \times 1$

   $\omega^2 \times \omega = \omega^3 = \omega \times \omega^2$

   $\therefore$ $\forall$ a, b $\in$ S $\quad$ $a*b = ab = ba = b*a$

   $\therefore$ multiplication is commutative on S.

b. Associative

   $$1 \times (\omega \times \omega^2) = 1\times\omega^3$$
   $$= 1\times\omega\times\omega^2$$
   $$= (1 \times \omega) \times \omega^2$$

   $\therefore$ $\forall$ a, b, c $\in$ s. $(a*b)*c = (ab) * c$

   $$= abc$$
   $$= a(bc)$$
   $$= a(b*c)$$
   $$= a*(b*c)$$

   $\therefore$ multiplication operation is association.

Exercise 3.2

1. If x, y $\in$ z and n is positive integer

   Then, x is said to be the congruent to y with modulo n if x–y is exactly divisible by n.

   It can be expressed as $x \equiv y$ modulo n.

   e.g. $7 \equiv 1$ modulo 3 $\Rightarrow$ 7 – 1 is divisible by 3

   i.e. when 7 is divided by 3 the remainder is 1.

   Similarly, $9 \equiv 1$ modulo 4 $\Rightarrow$ 9 – 1 is divisible by 4 i.e. when 9 is divided by 4 leaves remainder 1.

   $a \equiv b$ modulo n $\Rightarrow$ a – b is divisible by n.

   i.e. when a is divided by n, remainder is b.

2. **Addition Modulo 'n'**

   Let x, y $\in$ z and n be a positive integer. The addition modulo 'n' is written as $(+_n)$, defined as $x +_n y = r$ $(0 \leq r < n)$ where r is remainder when x + y is divided by n.

   e.g.

   $4 +_2 3 = 1$ i.e. when 4+3 = 7 is divided by 2, leaves remainder 1.

   $12 +_3 4 = 1$ i.e. when 12+4 = 16 is divided by 3, remainder 1.

   $18 +_4 4 = 2$ i.e. when 18+4 = 22 is divided by 4 remainder 2.

Multiplication Modulo 'n'

Let x, y $\in$ z and n is a positive integer. Then multiplication modulo n is denoted by (x_n), is defined by

$x \times_n y = r$, $(0 \leq r < n)$ where r is remainder when $x \times y$ is divided by n.

e.g. $3x_2\ 2 = 0$ i.e. when $3\times2 = 6$ is divided by 2, remainder is 0.

$7 \times_3 5 = 2$ when $7\times5 = 35$ is divided by 3, reminder is 2.

3.

x	1	–1	i	–i
1	1	–1	i	–i
–1	–1	1	–i	i
i	i	–i	–1	1
–i	–i	i	1	–1

$(i^2 = -1)$

Since, it is closed, the operation is a binary operation.

4. Given $z_3 = \{0, 1, 2\}$

We need to prepare a Caleys table for multiplication modulo 3.

x_3	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Since, it is closed, the multiplication modulo 3 on the set z = {0, 1, 2,} is a binary.

5. An operation '$*$' is said to be a binary on a set S if $\forall$ a, b $\in$ S then $a * b \in S$. In other words, an operation $*$ is said to be binary if it is closed.

x	**0**	**1**
0	0	0
1	0	1

Here, $0\times0 = 0$

$0 \times 1 = 0$

$1 \times 0 = 0$

$1 \times 1 = 1$

i.e. $\forall$ a, b $\in$ S $\qquad$ $a\times b \in S$

'x' is a binary operation on S.

6. $\forall$ x, y $\in$ Z

$x * y = x + y - 2$ also belongs to Z

i.e. $\forall x, y \in Z \Rightarrow x * y = x + y - 2 \in Z$

$\therefore$ '$*$' is closed.

Since it is closed, it is a binary.

$x * y = x + y - 2$
$= y + x - 2$
$= y * x$

$\therefore$ $\forall$ x, y $\in$ Z, $x*y = y*x$ is proved.

'$*$' is commutative.

Finally, let x, y, z $\in$ Z then

$x*(y*z) = x * (y + z - 2)$
$= x + y + z - 2 - 2$
$= x + y + z - 4$
$= x + y - 2 + z - 2$
$= (x*y) + z - 2$
$= (x*y) * z$

$\therefore$ This proves that '$*$' also associative.

7. Given, M = {set of all 3×2 matrices}

Operation: addition

$\forall$ A, B $\in$ M, A + B $\in$ M

because addition of two matrices of order 3×2 is also 3×2 matrix.

Addition operation on set M is closed. It means it is a binary.

Let A, B, C $\in$ M then,

(A + B) + C = A + (B + C)

$\therefore$ Associative

Let I be an identify element of $A\in M$. Then,

A + I = A

I = A = A

I = null matrix

$$\therefore \quad I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ $A^1 = -A \in M$

8. $2x + 1 = 6$ in Z_7

or, $2\times_7 x +_7 1 = 6$

or, $2\times_7 x +_7 1 +_7 6 = 6 +_7 6$

or, $2\times_7 x = 5$

or, $7\times_7 (2\times_7 x) = 4\times_7 5$

or, $(4\times_7 2) \times_7 x = 4\times_7 5$ (By associative law)

or, $1\times_7 x = 6$

or, $x = 6$

Exercise 3.3

1.a. Set N (Natural number) Operation: Multiplication 'x'

(N, x) is not a group because there doesn't exist inverse element.

b. (Z, +) is a group

c. (Q – {0}, X) is a group

d. Yes

e.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

It is closed, so binary

$(1*2)$ r $*\ 4 = 3*4 = 2$

$1*(2*4) = 1*1 = 2$

$\forall$ a, b, c $\in$ S, $(a*b) \times c = a*(b*c$

$\therefore$ associative holds

$0\in s$ is an identity element $\forall\ a\in S$.

$\forall$ x $\in$ S, $\exists$ inverse element y$\in$ S such that

$x +_5 y = 0$

Here, Inverse of 0 is 0

Inverse of 1 is 4

Inverse of 2 is 3

Inverse of 3 is 2

Inverse 4 is 1

$\therefore$ $(S, +_5)$ is a group

f. S = {1, –1, i, –i}

(S, ×) is a group

g. yes
h. yes
i. no, identity does not exist.
j. yes
k. yes
2.

$\times_3$	**0**	**1**	**2**
0	0	0	0
1	0	1	2
2	0	2	1

Closure Property: $\forall$ a, b $\in$ s $\quad$ $a x_3 b \in s$

$0x_2\ 0 = 9$ $\quad$ $0x_3\ 1 = 1$ $\quad$ $0x_3\ 2 = 0$

$1x_3\ 0 = 0$ $\quad$ $1x_3\ 1 = 1$ $\quad$ $1x_3\ -2 = -2$

$2x_3\ 0 = 0$ $\quad$ $2x_3\ 1 = 2$ $\quad$ $2x_3\ 2 = 1$

Associative Property

$0x_3\ (1\times_3 2) = 0\times_3 2 = 0$

$(0x_3 1)\times_3 2 = (0\times_3 2) = 0$

$(2x_3\ 1)\times_3 2) = 2\times_3 2 = 1$

$\therefore\quad \forall$ a, b, c $\in$ S, $(a*b)*c = a*(b*c)$

Existence of identity

$\forall$ a $\in$ S, $\exists$ e$\in$ S s.t. $a\times_3 e = a$

Existence of inverse:

$\forall$ a$\in$S, $\exists$ a' $\in$ S s.t. a$*$a' = e

$a * a^1 = e$

$\therefore\quad (S, \times_3)$ is a group.

3. Given (S, ×) where s = {1, –1, 1, –1}

For closure

For any a, b $\in$ s $a*b = ab \in s$

e.g. $1\times1 = 1\in s$

$-1\times-1 = 1 \in s$

$-1\times i = -i\in s$

$i\times i = i^2 = -1\in s$

$-i\times i = -i^2 = 1\in s$

So, for any two elements of S, the new element after operating also must belong to sets. So it is closed.

For associatively,

$(1\times1 \times - i = 1\times-i = -i$

$1\times(1 \times -i) = 1 \times - i = -1$

$\therefore (1 \times 1) \times -i = 1\times (1 \times - i)$

Similarly others follows

That is $\forall$ a, b, c $\in$ s $\Rightarrow (a\times b) \times c = a\times(b\times c)$

$\therefore\quad$ It is associative.

For existence of identify:

Let $1\in s$ then $1\times1 = 1$

Let $-i\in s$ then $-i\times1 = -i$

Let $i\in s$ then $i\times1 = i$

Let $-1\in s$ then $-1\times1 = -1$

$\therefore\quad -1$ is an identify element of any element $\in$s.

For existence of inverse:

For $1 \in s$	$1 \times 1 = 1$	$\therefore$ 1 is inverse of 1
For $-1 \in s$	$-1 \times -1 = 12$	$\therefore$ -1 is inverse of -1
For $i \in s$	$i \times -i = 1$	$\therefore$ $-i$ is inverse of i
For $-i \in s$	$-i \times i = 1$	$\therefore$ i is inverse of $-i$

Therefore, $\forall\, a \in s\ \exists\, a' \in$ s.t. $a \times a' = e$

Hence, the algebraic structure $(S, \times)$ satisfies all the properties (i.e. closure, associativity, existence of identity and existence of inverse)

$\therefore$ $(S, \times)$ is a group

4.a. Algebraic Structure:

An structure of the form $(G, *)$ is known as an algebraic structure. Where G is a non-empty set and '$*$' is a binary operation.

e.g. (G, +), (G, x), (Z, –) (Q, +) etc are some examples of an algebraic structure.

b. Semi-group:

An algebraic structure $(G, *)$ is said to be a semi–group. It satisfies the associative property.

e.g. $(z^+, +)$ is a semi-group but (z, +) is not.

c. Group: An algebraic structure $(G, *)$ is said to be a group if it satisfies the following for properties.

- Closure
- Associative
- Existence of identity
- Existence of inverse

d. Monoid: An algebraic structure $(G, *)$ is said to be a monoid if it satisfies associativity and existence of an identity. e.g. $(Z, \times)$

e. Abelian group: A group $(G, *)$ is said to be an abelian of it satisfies the commutative property.

f. Trivial group: A group $(G, *)$ is said to be a trivial group if G consists of a single element.

5. Let $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \text{ and } ad - bc \neq 0 \right\}$ be the set of 2×2 real non-singular matrices.

i. $\forall$ A, B $\in$ M, AB is again 2×2 real non-singular matrix. So, M is closed.

ii. $\forall$ A, B, C $\in$ M, A(BC) = (AB) C by matrix algebra. So M is associative under multiplication.

iii. $\forall$ A$\in$M, we get $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ even that AI = IA = A. So the identify matrix I exists.

iv. $\forall$ A$\in$M, we get A^{-1} (Since A is non–singular) set. $AA^{-1} - A^{-1}A = I$ where $A^{-1} = \dfrac{\text{Adj.}(A)}{|A|}$, is known as inverse of A. Hence M is a group.

6. Show that (Z, +) is a group.

i. Closure property: $\forall$ a, b $\in$ z, a + b $\in$ Z

$\therefore$ z is closed

ii. Associative: $\forall$ a, b, c $\in$ Z

(a + b) + c = a + (b + c)

$\therefore$ z is associative

iii. Existence of identify: $\forall a \in z$, the must exist $0 \in z$ s.t. a+0 = a

$\therefore$ $0 \in Z$ is an identify element.

iv. Existence of inverse: $\forall a \in Z$ there must $-a \in Z$ s.t. a + (–a) = 0

$\therefore$ –a is inverse of a

Hence (Z, +) is a group.

7.

×	**1**	**ω**	ω^2
1 ω ω^2	1 ω ω^2	ω ω^2 1	ω^2 1 ω

From the above table, s is closed

$(1\times\omega) \times \omega^2 = \omega\times\omega^2 = \omega^3 = 1$

$1\times(\omega\times\omega^2) = 1\times\omega^3 = 1\times1 = 1$

$\therefore \quad (1\times\omega) \times \omega^2 = 1\times(\omega\times\omega^2)$

Again, $\omega\times(\omega^2 \times \omega^2) = \omega \times \omega^4 = \omega\times\omega = \omega^2$

$(\omega\times\omega^2) \times \omega^2 = \omega^3\times\omega^2 = 1\times\omega^2 = \omega^2$

$\therefore \quad \omega\times(\omega^2 \times \omega^2) = (\omega \times \omega^2) \times \omega^2$

That is

$\forall$ a, b, c $\in$ S

$(a\times b) \times c = a\times(b\times c)$

$\therefore$ S is associative.

From the table, 1 is an identity element of any element of S.

i.e. $1\times1 = 1$

$\omega\times1 = \omega$

$\omega^2\times1 = \omega^2$

$\therefore$ identity element 1 exist.

Since, $1\times1 = 1$

$\omega\times\omega^2 = 1$

$\omega^2\times\omega = 1$

$\therefore$ Inverse of 1 is 1

Inverse of w is ω^2

Inverse of w^2 is ω

So, there is an existence of an inverse element.

Finally, $1\times\omega = \omega\times1 = \omega$

$\forall$ a, b $\in$ s $\qquad$ $a\times b = b \times a$

Comutative property satisifies.

Hence, (S, ×) is an abelian group.

8. The set: Z

Operation '∗' defined by $a*b = a+b+2ab$

a. Since, a, b $\in$ Z

a + b + 2ab is also belongs to Z.

$\therefore$ z is closed

b. $\forall$ a, b, c $\in$ Z

$a * (b * c) = a * (b + c + 2bc)$
$= a + b + c + 2bc + 2a(b + c + 2bc)$
$= a + b + c + 2bc + 2ab + 2ca + 4abc$
$= a + b + c + 2ab + 2bc + 2ca + 4abc$

Again, $(a*b) * c = (a + b + 2ab) * c$
$= a + bb + 2ab + c + 2(a + b + 2ab) c$
$= a + b + 2ab + c + 2ca + 2bc + 4abc$
$= a + b + c + 2ab + 2bc + 2ca + 4abc$

$\therefore \quad a*(b*c) = (a*b) * c$

$\therefore$ z is associative

c. Since $a*0 = a + 0 + 2a0 = a$

$\forall$ a$\in$z, the identity element 0$\in$ Z exists.

d. Let d be inverse of a such that $a*d = 0$ (identify)

$a + d + 2ad = 0$
$d + 2ad = -a$
$d(1 + 2a) = -a$
$d = \frac{-a}{1 + 2a} \notin Z$
Even through
$a \neq -\frac{1}{2}$, if a = 1 then
$d = \frac{1}{3} \notin Z$
$\therefore$ inverse element may not exist. Therefore, $(Z, *)$ is not a group.

9. For definition of group look at 4(c)
From the given Calyes table,
S is closed.
$\forall$ a, b, c $\in$ S
$a * (b * c) = a * a = a$
$(a * b) * c = b * c = a$
$\therefore$ $a * (b * c) = (a * b) * c$
$\therefore$ S is associative
From the table, $a * a = a$
$b * a = b$
$c * a = c$
$\therefore$ a is identity element.
From the table,
$a * a = a$
$b * c = a$
$c * b = a$
$\therefore$ inverse of a is itself a
inverse of b is itself c
Therefore, inverse elements exists.
Since, S satisfies closure property, associative property, existence of identity and existence of inverse, $(S, *)$ is a group.

10.
a. Set: Z
Operation: –
Now, we check $(Z, -)$ is a group or not.
$\forall$ a, b $\in$ Z, $a * b = a - b \in Z$
$\therefore$ z is closed.
$\forall$ a, b, c $\in$ Z, $(a - b) - c \neq a - (b - c)$
e.g. let $a = -1$, $b = -3$ and $c = 5$
Then, $(a - b) - c = (-1 + 3) - 5 = 2 - 5 = -3$
$a - (b - c) = -1 - (-3 - 5) = -1 + 8 = 7$
$\therefore$ $(a - b) - c \neq a - (b - c)$
$\therefore$ Z is not associative
Since associative property is not satisfied.
The set of integers with subtraction operation is not a group.

b. (z, x) $\Rightarrow$ Group (check)
$\forall$ a, b $\in$ z, $a \times b \in$ z so, closure is satisfied.
$\forall$ a, b, c $\in$ z, $(a \times b) \times c = a \times (b \times c)$
$\therefore$ z is associative.
Let a $\in$z then $a \times 1 = a$
$\therefore$ so there must exist $1 \in z$ s.t. $a \times 1 = a$ so identify element 1 exists.

If b is inverse of $a \in z$ then $a \times b = 1$

$b = \frac{1}{a} \notin Z$

Since, if a = 2 then $b = \frac{1}{2} \notin Z$

Therefore, there is no existence of inverse element.

$\therefore$ $(Z, \times)$ is not a group.

11. Let $V = \{(a_1, a_2, a_3) : a_1\, a_2\, a_3 \in R\}$ be a set of 3 dimensional vectors.

Now, we have to show that (v, +) is a group.

$\forall v_1\, v_2 \in v \qquad v_1 + v_2 \in V$

Since addition of two 3–dimention vectors is also 3–dimentional

$\therefore$ v is closed.

$\forall\, v_1, v_2, v_3 \in V$, then it is obvious that $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$

$\therefore$ associative property also holds.

$\forall\, v_1 \in v$ of 3 dimensional null vector

(0, 0, 0) s.t. $v_1 + (0, 0, 0) = v_1$

$\therefore$ Identity element (0, 0, 0) exists.

$\forall v_1 \in V, \exists -v_1 \in V$ s.t. $v_1 + (-v_1) = (0, 0, 0)$

$\therefore$ inverse element also exists.

Hence, (V, +) is a group.

12. Solution:

i. Closure property:

$\forall\, a, b \in Q^+, \qquad a*b = \frac{ab}{4} \in Q^+$

$\therefore$ Q^+ is closed.

ii. Associative property:

$\forall\, a, b, c \in Q^+$ then $(a*b) * c = \left(\frac{ab}{4}\right) * c = \frac{abc}{\frac{4}{4}} = \frac{abc}{16}$

$a*(b*c) = a*\left(\frac{bc}{4}\right) = \frac{abc}{\frac{4}{4}} = \frac{abc}{16}$

$\therefore$ $(a*b) * c = (b * c)$

$\therefore$ associative property holds

iii. Existence of identity

Let e be an identify of $a \in Q^+$

Then, $a*e = a$

$\frac{ae}{4} = a$

$ae = 4a$

$ae - 4a = 0$

$a(e - 4) = 0$

$e = 4 \in Q^+$ since $a \neq 0$

Identify element exists.

iv. Existence of inverse:

let b be an inverse of $a \in Q^+$

such that, $a * b = e$

$\frac{ab}{4} = 4$

$ab = 16$

$b = \frac{16}{a} \in Q^+$

$\therefore$ inverse element $b \in Q^+$ exists.

Hence, $(Q^+, *)$ is a group.

Where $*$ is defined by $a*b = \frac{ab}{4}$

Further, $\forall$ a, $b \in Q^+$

$a*b = \frac{ab}{4} = \frac{ba}{4} = b*a$

$\therefore$ commutative property is also satisfied. Therefore, $(Q^+, *)$ is an abelian group.

13. Given,

P = {non empty subsets of X}

Is (P, U) is a group?

$\forall P_1 P_2 \in P$ then $P_1*P_2 = P_1UP_2 \in P$

$\therefore$ P is closed.

$\forall P_1, P_2, P_3 \in P$, $(P_1UP_2) \text{ U } P_3 = P_1U(P_2UP_3)$

$\therefore$ P is associative.

$\forall P_1 \in P$ then $P_1U\phi = P_1$ but $\phi \notin P$.

$\therefore$ identity element does not exist.

$\therefore$ this is not a group.

Chapter – 4
Complex Number

Exercise 4.1

1. Solution:

a. Let, z be the cube root of –1

$z^3 = -1$

or, $z^3 + 1 = 0$

or, $(z)^3 + (1)^3 = 0$

or, $(z + 1)\ (z)^2 - z + 1) = 0$

Either,

$z = -1$

$z^2 - z + 1 = 0$ ... ... (i)

Comparing equation (i) with $ax^2 + bx + c = 0$

$\therefore$ a = 1, $\therefore$ b = –1, c = +1

Now,

$$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= 1 \pm \frac{\sqrt{1 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

Taking positive

$$x = \frac{1 + \sqrt{3}i}{2}$$

Taking negative

$$x = \frac{1 - \sqrt{3}i}{2}$$

Here, z is the value of x.

Hence,

The cube root of unity is

$-1, \dfrac{1 + \sqrt{3}i}{2}$ and $\dfrac{1 - \sqrt{3}i}{2}$

b. Here,

Let, z be the cube root of 8

So, $z^3 = 8$

or, $(z)^3 = -(2)^3 = 0$

or, $(z - 2)\ (z^2 + 2z + 4) = 0$

Either,

z = 2

$z^2 + 22 + 4 = 0$ ... ... (i)

Comparing equation (i) with $az^2 + bz + c = 0$

So, a = 1, b = 2, c = 4

Now,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 4}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$= \frac{-2 \pm 2\sqrt{3}i}{2}$

$= -1 \pm \sqrt{3}\, i$

Taking positive, Taking negative

$z = -1 + \sqrt{3}\, i$ $z = -1 - \sqrt{3}\, i$

Hence,

The required cube roots of 8 are 2, $-1 + \sqrt{3}i$ and $-1 - \sqrt{3}i$

2. Solution:

a. Here, $z^4 = 1$

or, $(z)^4 - (1)^4 = 0$

or, $(z^2)^2 - (1^2)^2 = 0$

or, $(z^2 - 1)(z^2 + 1) = 0$

or, $(z - 1)(z + 1)(z^2 + 1) = 0$

Either,

or, $z = 1$,

or, $z = -1$

or, $z^2 + 1 = 0$ ... ... (i)

or, Comparing equation (i) with $az^2 + bz + c = 0$

$\therefore$ $a = 1, b = 0, c = 1$

Now,

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{0 \pm \sqrt{0 - 4\times1\times1}}{2\times1}$

$= \frac{0 \pm \sqrt{-4}}{2}$

$= \frac{0 \pm 2i}{2}$

$= \pm i$

Taking positive Taking negative

$z = i$ $z = -i$

Hence,

The required value of z is ±1 and $\pm i$.

b. Here,

$z^4 = -1$

or, $z^4 = -1 + i \times 0$

or, $z^4 = \cos180^o + i\sin180^o$

or, $z^4 = \{\cos(k.360 + 180^o) + i\sin(k.360^o + 180)\}$

or, $z = \{\cos(k.360^o + 180^o) + i\sin(k.360^o + 180^o)\}^{1/4}$

$= \cos\left(\frac{k.\,360 + 180}{4}\right) + i\sin\left(\frac{k.\,360 + 180^o}{4}\right)$

where, $k = 0, 1, 2, 3$

When $k = 0$ then, $z = \cos(k.90^o + 45^o) + i\sin(k.90^o + 45^o)$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, i$

when, $k = 1$, $z = \cos135^o + i\sin135^o$

$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, i$

when, $k = 2$, $z = \cos225^o + i\sin 225^o$

$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\, i$

when, k = 3, $z = \cos 315^o + i\sin 315^o$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

$\therefore \quad z = \pm\left(\frac{1+i}{\sqrt{2}}\right), \pm\left(\frac{1-i}{\sqrt{2}}\right)$

c. Here, $z^6 = 1$

$z^6 = 1^6 = 0$

or, $(z^2)^3 - (12)^3 = 0$

or, $(z^2 - 1)(z^4 + z^2 + 1) = 0$

Either,

$z = \pm 1$

$z^4 + z^2 + 1 = 0$

or, $(z^2)^2 + (1)^2 + z^2 = 0$

or, $(z^2 + 1)^2 - 2z^2 + z^2 = 0$

or, $(z^2 + 1)^2 - (z)^2 = 0$

or, $(z^2 + 1 - z)(z^2 + 1 + z) = 0$

Either,

$z^2 + z + 1 = 0$ ... ... (i)

$z^2 - z + 1 = 0$ ... ... (ii)

Comparing equation (i) with $az^2 + bz + c = 0$

$\therefore \quad a = 1, b = 1, c = 1$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1 - 3\times1\times1}}{2\times1}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

Taking positive

$$z = \frac{-1 + \sqrt{3}i}{2}$$

Taking negative

$$z = \frac{-1 - \sqrt{3}i}{2}$$

Again, comparing equation (ii) with $az^2 + bz + c = 0$

$\therefore \quad a = 1, b = -1, c = 1$

Now,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1^2 - 4\times1\times1}}{2\times1} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

Taking positive

$$z = \frac{1}{2} + \frac{\sqrt{3}i}{2}$$

Taking negative

$$z = \frac{1}{2} - \frac{\sqrt{3}i}{2}$$

$\therefore \quad z = \pm 1, \pm\left(\frac{-1-\sqrt{3}i}{2}\right), \pm\left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$

3. Solution:

a. $(1 + \omega^2)^3 - (1 - \omega)^3$

$= (-\omega)^3 - (-\omega^2)^3$

$= -\omega^3 - (-\omega^6)$

$= -1 - (-(\omega^3)^3)$

$= -1 - (-1)$

$= -1 + 1$

$= 0$

b. $(2 + \omega)(2 + \omega^2)(2 - \omega^2)(2 - \omega^4)$

$= (1 + 1 + \omega)(1 + 1 + \omega^2)(1 + 1 - \omega^2)(1 + 1 - \omega^4)$

$= (1 - \omega^2)(1 - \omega)(1 + 1 - \omega^2)(1 + 1 - \omega) \quad (\because \omega^3 = 1)$

$= (1 - \omega^2)(1 - \omega)(2 - \omega^2)(2 - \omega)$
$= 1 - \omega - \omega^2 + \omega^3)(4 - 2\omega - 2\omega^2 + \omega^3)$
$= (1 - \omega - \omega^2 + 1)(4 - 2\omega - 2\omega^2 + 1)$
$= (2 + 1)(4 + 1 + 2)$
$= 3 \times 7 = 21$

c. $(1 - \omega + \omega^2)^4 . (1 + \omega - \omega^2)^4$
$= (-2\omega)^4 . (-2\omega^2)^4$
$= 16\omega^3 . \omega . 16\omega^3 . \omega^3 . \omega^2$
$= 16. \omega \times 16\omega^2$ $[\because \omega^3 = 1]$
$= 256 \times \omega^3$
$= 256 \times 1$
$= 256$

d. $(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6$
$= (-2\omega)^6 + (-2\omega^2)^6$
$= 64\ \omega^3 . \omega^3 + 64\ \omega^3 . \omega^3 . \omega^3 . \omega^3$
$= 64 + 64$ $[\because \omega^3 = 1]$
$= 128$

e. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)$
$= (1 - \omega)(1 - \omega^2)(1 - \omega^3 . \omega)(1 - \omega^3 . \omega^2)$
$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$
$= (1 - \omega^2)^2 (1 - \omega)^2$
$= (1 - 2\omega^2 + \omega^4)(1 - 2\omega + \omega^2)$
$= (1 - 2\omega^2 + \omega)(1 - 2\omega + \omega^2)$
$= (-3\omega^2)(-3\omega)$
$= 9\omega^3$
$= 9 \times 1$ $[\because \omega^3 = 1]$
$= 9$

f. $\dfrac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2}$
$= \dfrac{a\omega^3 + b\omega.\omega^3 + c\omega^2 . \omega^3}{a\omega^2 + c\omega + b}$
$= \dfrac{\omega(a\omega^2 + b\omega^3 + c\omega^4)}{(a\omega^2 + c\omega + b)}$
$= \omega \dfrac{(a\omega^2 + b + c\omega)}{(a\omega^2 + c\omega + b)}$ $[\because \omega^3 = 1]$
$= \omega$

g. $\dfrac{1}{1 + 2\omega} + \dfrac{1}{3 + \omega} - \dfrac{1}{1 + \omega}$
$= \dfrac{1}{1 + \omega + \omega} + \dfrac{1}{1 + 1 + \omega} - \dfrac{1}{1 + \omega}$
$= \dfrac{1}{-\omega^2 + \omega} + \dfrac{1}{-\omega^2 + 1} - \dfrac{1}{1 + \omega}$
$= \dfrac{1}{\omega(1 - \omega)} + \dfrac{1}{(1 - \omega^2)} - \dfrac{1}{1 + \omega}$
$= \dfrac{1}{\omega(1 - \omega)} + \dfrac{1}{(1 - \omega)(1 + \omega)} - \dfrac{1}{(1 + \omega)}$
$= \dfrac{1 + \omega + \omega - \omega + \omega^2}{\omega. (1 + \omega)(1 - \omega)}$
$= \dfrac{1 + \omega + \omega^2}{\omega (1 - \omega^2)}$
$= 0$ $[\because 1 + \omega + \omega^2 = 0]$

4. Solution:

a. If $\alpha = \omega$, $\beta = \omega^2$

$\alpha^4 + \beta^4 = \frac{1}{\alpha\beta}$

$= \omega^4 = (\omega^2)^4 + \frac{1}{\omega.\omega^2}$

$= \omega + (\omega^3)^2 . \omega^2 + 1 \quad [\because \omega^3 = 1]$

$= \omega + \omega^2 + 1$

$= 0 \ [\because 1 + \omega + \omega^2 = 0]$

b. Here,

$\alpha^4 + \alpha^2\beta^2 + \beta^4$

$= \omega^4 + \omega^2.(\omega^2)^2 + (\omega^2)^4$

$= \omega + \omega^2 . \omega^4 + \omega^8$

$= \omega + 1 + \omega^2 \qquad [\because \omega^3 = 1]$

$= 0 \qquad [\because 1 + \omega + \omega^2 = 0]$

5. Solution:

Given,

$x = a + b$

$y = a\omega + b\omega^2$

$z = a\omega^2 + b\omega$

a. xyz

$= (a + b)(aw + bw^2) (aw^2 + bw)$

$= (a + b) (a^2w^3 + abw^2 + abw^4 + b^2w^3)$

$= (a + b) \{a^2.1 + ab(w^2 + w^4) + b^2.1\}$

$= (a + b) \{a^2 + ab(w + w^2) + b^2\} [\because w^4 = w^3. w = 1 . w = w]$

$= (a + b) \{a^2 - ab + b^2\} = a^3 + B^3 [\because w^2 + w = -1)$

b. $x + y + z$

$= (a + b) + (aw + bw^2) + (aw^2 + bw)$

$= a + b + aw + bw^2 + aw^2 + bw$

$= a(1 + w + w^2) + b(1 + w^2 + w)$

$= a \times 0 + b \times 0 = 0$

c. $x^3 + y^3 + z^3$

$= x^3 + y^3 + z^3 - 3xyz + 3xyz$

$= (x + y + z) (x^2 + y^2 + z^2 - xy - yz - zx) + 3xyz$

$= 0 + 3(a^3 + b^3)$ [$\because$ from (a) & (b)]

$= 3(a^3 + b^3)$

d. $x^2 + y^2 + z^2$

$= (x + y + z)^2 - 2xy - 2yz - 2xz$

$= 0 - 2(xy + yz + xz)$

$= -2\{(a + b)(aw + bw)^2 + (aw + bw^2). (aw^2 + bw) + (a + b). (aw^2 + bw)\}$

$= -2\{a^2w + abw^2 + abw + b^2w^2 + a^2 w^3 + abw^2 + abw + b^2 + a^2w^2 + abw + abw^2 + b^2w\}$

$= -2\{a^2w + a^2w^2 + a^2w^3 + 3abw^2 + 3abw + b^2w^2 + b^2 + b^2w\}$

$= -2\{a^2(w + w^2 + 1) + 3(-1)ab + b^2(w^2 + w + 1)\}$

$= -2\{a^2(w + w^2 + 1) + 3(-1)ab + b^2(w^2 + w + 1)\}$

$= -2\{0 - 3ab + 0\}$

$= 6ab$

6. Solution

a. We know,

$w = \frac{-1 + \sqrt{3}i}{2}$

$w^2 = \frac{-1 - \sqrt{3}i}{2}$

Now, $\left(\frac{-1+\sqrt{-3}}{2}\right)^6 + \left(\frac{-1-\sqrt{-3}}{2}\right)^{12}$

$= w^6 + (w^2)^{12}$

$= (w^3)^2 + w^{24}$

$= 1^2 + (w^3)^8$

$= 1 + 1^8 = 2$

b. $\left(\frac{-1+\sqrt{-3}}{2}\right)^8 + \left(\frac{-1-\sqrt{-3}}{2}\right)^8$

$= w^8 + (w^2)^8$

$= (w^3)^2.w^2 + (w^3)^5.\ w$

$= w^2 + w = -1$

c. Let, $w = \frac{-1+\sqrt{-3}}{2}$

$w^2 = \frac{-1-\sqrt{-3}}{2}$

Case-I: If n is multiple of 3 i.e. n = 3k, k is on integer.

$= 1 + \left(\frac{-1+\sqrt{-3}}{2}\right)^n + \left(\frac{-1-\sqrt{-3}}{2}\right)^n$

$= 1 + w^{3k} + (w^2)^{3k}$

$= 1 + (w^3)^k + (w^3)^{2k}$

$= 1 + 1^k + 1^{2k}$

$= 1 + 1 + 1 = 2 + 1 = 3$ proved.

Case II: n is not a multiple of 3 i.e. n = 3k + 1

$= 1 + \left(\frac{-1+\sqrt{-3}}{2}\right)^n + \left(\frac{-1-\sqrt{-3}}{2}\right)^n$

$= 1 + (w)^{3k+1} + (w^2)^{3k+1}$

$= 1 + (w)^{3k+1} + w^{6k} - w^2$

$= 1 + (w^3)^k.w + (w^3)^{2k}.\ w^2$

$= 1 + w + w^2 = 0$ proved.

Exercise 4.2

1. Solution:

a. Here, 2 + 2i

$x = 2, y = 2$

$r = \sqrt{2^2 + 2^2}$

$= \sqrt{4+4}$

$= \sqrt{8}$

$= 2\sqrt{2}$

$\text{Tan}\theta = \frac{y}{x}$

$= \frac{2}{2} = 1$

$\therefore \quad \theta = 45^o$

It can be written in polar form as $2\sqrt{2}\ (\cos 45^o + i\sin 45^o)$

b. Here,

$-\sqrt{2} + \sqrt{2}i$

Here, $x = -\sqrt{2}$

$y = \sqrt{2}$

$r = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2}$

$\text{Tan}\theta = \frac{y}{x}$ $\qquad = \sqrt{2+2}$

$= \frac{\sqrt{2}}{-\sqrt{2}}$ $\quad\quad = \sqrt{4}$

$= -1$ $\quad\quad = 2$

$\therefore \theta = 135^o$

In polar form = $2(\cos135^o + \text{is in } 135^o)$

c. Here,

Let, $z = -1 + 0i$

Here, $x = -1$

$y = 0$

$r = \sqrt{(-1)^2 + 0}$

$= -\sqrt{1}$

$= 1$

$\text{Tan}\theta = \frac{y}{x} = \frac{0}{-1} = 0$

$\therefore \theta = 180^o$

In polar form = $1(\cos180^o + i\sin180^o)$

$= \cos180^o + i\sin180^o$

d. Here,

Let, $z = 0 + 3i$

Here, $x = 0$, $y = 3$

$\text{Tan}\theta = \frac{y}{x}$ $\quad\quad r = \sqrt{(0)^2 + (3)^2}$

$= \frac{3}{0}$ $\quad\quad = \sqrt{9}$

$= \infty$ $\quad\quad = 3$

$\theta = 90^o$

In polar form = $3(\cos90^o + i\sin90^o)$

e. Here,

Let $z = 0 - 5i$

Here, $x = 0$,

$y = -5$

$r = \sqrt{x^2 + y^2} = \sqrt{0 + 25} = \sqrt{25} = 5$

$\text{Tan}\theta = \frac{y}{x} = \frac{-5}{0} = \infty$

$\therefore \theta = 270^o$

Now, In polar form $-5i = 5(\cos270^o + i\sin270^o)$

f. Here,

Let, $z = -\sqrt{3} + i$

Here, $x = -\sqrt{3}$

$y = 1$

$r = \sqrt{(-\sqrt{3})^2 + (1)^2}$

$= \sqrt{4} = 2$

$\text{Tan}\theta = \frac{y}{x}$

or, $\text{Tan}\theta = \frac{1}{-\sqrt{3}}$

or, $\text{Tan}\theta = \text{Tan } 150^o$

$\therefore \theta = 150^o$

In polar form $i - \sqrt{3} = 2(\cos150^o + i\sin150^o)$

g. Here,

Let, $z = -3 - \sqrt{3}i$

Here, $x = -3$

$y = -\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-3} = \frac{1}{\sqrt{3}}$

$\therefore\ \theta = 210^o$

$r = \sqrt{(-3)^2 + (-\sqrt{3})^2}$

$= \sqrt{9 + 3}$

$= \sqrt{12}$

$= 2\sqrt{3}$

In polar form, $-3-\sqrt{3}i = 2\sqrt{3}\,(\cos 210^o + i\sin 210^o)$

h. Here,

Let, $z = 1 - \sqrt{3}i$

Here, $x = 1$, $y = -\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x}$

or, $\text{Tan}\theta = \frac{-\sqrt{3}}{1}$

or, $\text{Tan}\theta = -\sqrt{3}$

or, $\text{Tan}\theta = \text{Tan}300^o$

$r = \sqrt{(1)^2 + (-\sqrt{3})^2}$

$= \sqrt{1 + 3}$

$= \sqrt{4}$

$= 2$

In polar form, $1 - \sqrt{3}i = 2(\cos 300^o + i\sin 300^o)$

i. Here,

Let, $z = 2 + 2\sqrt{3}\ i$

Here, $z = 2$, $y = 2\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$\therefore\ \theta = 60^o$

$r = \sqrt{4 + 4\times 3} = \sqrt{16} = 4$

In polar form, $(2, 2\sqrt{3}) = 4(\cos 60^o + i\sin 60^o)$

j. Here,

Let, $z = \frac{1}{1 - i}$

$= \frac{1}{1 - i} \times \frac{1 + i}{1 + i}$

$= \frac{1 + i}{1 + 1}$

$\therefore\ z = \frac{1}{2} + \frac{1}{2}i$

Here, $x = \frac{1}{2}$, $y = \frac{1}{2}$

Now, $\text{Tan}\theta = \frac{y}{x}$, $\qquad r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \frac{\frac{1}{2}}{\frac{1}{2}} \qquad = \sqrt{\frac{1}{4} + \frac{1}{4}}$

$= 1$ $\qquad\qquad = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$

$\therefore\ 45^o$

In polar form, $\frac{1}{1-i} = \frac{1}{\sqrt{2}}(\cos 45^o + i\sin 45^o)$

k. Here,

Let, $z = \sqrt{\frac{1+i}{1-i}} = \sqrt{\frac{1+i}{1-i} \times \frac{1+i}{1+i}} = \frac{1+i}{\sqrt{2}}$

$\therefore\ z = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} i$

Here, $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$

$r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

Now, $\mathrm{Tan}\theta = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

$\therefore\quad \theta = 45^o$

In polar form, $\sqrt{\frac{1+i}{1-i}} = \cos 45^o + i\sin 45^o$

2. Solution:

a. Here,

Let, $2(\cos 30^o + i\sin 30^o) = x + iy$

Equating real and imaginary parts;

$x = 2\cos 30^o$ $\qquad\qquad y = 2\sin 30^o$

$= 2 \times \frac{\sqrt{3}}{2}$ $\qquad\qquad = 2 \times \frac{1}{2}$

$= \sqrt{3}$ $\qquad\qquad = 1$

$\therefore\ 2(\cos 30^o + i\sin 30^o) = \sqrt{3} + i.$

b. Here,

Let, $3(\cos 150^o + i\sin 150^o) = x + iy$

Equating real and imaginary parts;

$x = 3\cos 150^o,$ $\qquad\qquad y = 3\sin 150^o$

$= 3 \times \left(\frac{-\sqrt{3}}{2}\right)$ $\qquad\qquad = 3 \times \frac{1}{2}$

$= \frac{-3\sqrt{3}}{2}$ $\qquad\qquad = \frac{3}{2}$

$\therefore\ 3(\cos 150^o + i\sin 150^o) = \frac{-3\sqrt{3}}{2} + \frac{3}{2} i$

c. Here,

Let, $4(\cos 240^o + i\sin 240^o) = x + iy$

Equating real is imaginary parts;

$x = y\cos 240^o,$ $\qquad\qquad y = 4\sin 240^o$

$= 4 \times \left(\frac{-1}{2}\right)$ $\qquad\qquad = 4 \times \left(\frac{-\sqrt{3}}{2}\right)$

$= -2$ $\qquad\qquad = \frac{-\sqrt{3} \times 2}{2}$

$= -2\sqrt{3}$

$\therefore$ $4(\cos240^o + i\sin240^o) = -2 - 2\sqrt{3}i$

d. Here,

Let, $2\sqrt{2}\,(\cos 270^o + i\sin270^o) = x + iy$

Equating real and imaginary parts;

$x = 2\sqrt{2}\cos270^o$

$= 2\sqrt{2} \times 0 = 0$

$y = 2\sqrt{2}\sin270^o$

$= -2\sqrt{2}$

$\therefore$ $2\sqrt{2}\,(\cos270^o + i\sin270^o) = -2\sqrt{2}i$

3. Solution:

a. Here,

$2(\cos 53^o + i\sin53^o)\,.\,3(\cos7^o + i\sin7^o)$

$= 2\times3\,\{\cos(53^o + 7) + i\sin(53^o + 7^o)\}$

$= 6P\{\cos60^o + i\sin60^o\}$

$= 6\left(\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right)$

$= 3 + 3\sqrt{3}i$

b. $(\cos5\theta + i\sin\theta)\,(\cos3\theta + i\sin3\theta)$

$= \cos(5\theta + 3\theta) + i\sin(5\theta + 3\theta)$

$= \cos8\theta + i\sin8\theta$

c. $(\cos72^o + i\sin72^o)\,(|\cos12 - i\sin12^o)$

$= (\cos72^o + i\sin72^o)\,\{\cos(-12) + i\sin(-12)\}$

$= \cos(72 - 12) + i\sin(72 - 12)$

$= \cos60^o + i\sin60^o$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

d. $\dfrac{\cos50^o + i\sin50^o}{\cos20^o + i\sin20^o}$

$= \cos(50 - 20) + i\sin(50 - 20)$

$= \cos30^o + i\sin30^o$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

e. $\dfrac{(\cos4\theta + i\sin4\theta)\,(\cos3\theta - i\sin3\theta)}{\cos3\theta + i\sin3\theta}$

$= \dfrac{(\cos4\theta + i\sin4\theta)\,(\cos(-3\theta) + i\sin(-3\theta))}{\cos3\theta + i\sin3\theta}$

$= \dfrac{\cos(4\theta - 3\theta) + i\sin(4\theta - 3\theta)}{\cos3\theta + i\sin3\theta}$

$= \dfrac{\cos\theta = i\sin\theta}{\cos3\theta + i\sin3\theta}$

$= \cos(\theta - 3\theta) + i\sin(\theta - 3\theta)$

$= \cos2\theta - i\sin2\theta$

f. $\dfrac{\cos5\theta + i\sin5\theta}{(\cos2\theta + i\sin2\theta)^2}$

$= \dfrac{\cos5\theta + i\sin5\theta}{(\cos4\theta + i\sin4\theta)}$

$= \cos(5\theta - 4\theta) + i\sin(5\theta - 4\theta)$

$= \cos\theta + i\sin\theta$

g. $\dfrac{(\cos3\theta + i\sin3\theta)^5}{(\cos\theta + i\sin\theta)^7}$

$= \frac{(\cos 15\theta + i\sin 15\theta)}{(\cos 7\theta + i\sin 7\theta)}$

$= \cos(15\theta - 7\theta) + i\sin(15\theta - 7\theta)$

$= \cos 8\theta + i\sin 8\theta$

4. Solution:

a. Here, $\left[3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{16}$

$= 3^{16} \times \left[\cos\left(16 \times \frac{\pi}{4}\right) + i\sin\left(16 \times \frac{\pi}{4}\right)\right]$

$= 3^{16}[\cos 4\pi + i\sin 4\pi]$

$= 3^{16}[1 + 0]$

$= 3^{16}$

b. Here, $[2(\cos 50^o + i\sin 50^o)]^3$

$= 2^3[\cos 150^o + i\sin 150^o)$

$= 8\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -4\sqrt{3} + 4i$

c. $[4(\cos 6^o + i\sin 6^o)]^{30}$

$= 4^{30}[\cos(6\times 30) + i\sin(6\times 30)]$

$= 4^{30}[-1+0]$

$= -4^{30}$

d. $(\cos 70^o + i\sin 70^o)^6$

$= \cos(70\times 6) + i\sin(70\times 6)$

$= \cos 420^o + i\sin 420^o$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

e. $(1 + i)^{15}$

Here, $x = 1, y = 1$

$\tan\theta = \frac{y}{x}$, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

or, $\tan\theta = \frac{1}{2}$

or, $\tan\theta = 1$

$\therefore \theta = 45^o$

In polar form, $(1 + i) = \sqrt{2}(\cos 45^o + i\sin 45^o)$

Now,

$= \sqrt{2}(\cos 45^o + i\sin 45^o)\}^{15}$

$= 128 \times \sqrt{2}\{\cos(45\times 15) + i\sin(45\times 15)\}$

$= 128\sqrt{2}(\cos 675^o + i\sin 675^o)$

$= 128\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= 128(1 - i)$

f. $(1 - i)^{10}$

Let, $z = 1 - i$

Here, $x = 1, y = -1, r = \sqrt{(1)^2 + (-1)^2}$

$\tan\theta = \frac{y}{x}$ $\qquad = \sqrt{1 + 1} = \sqrt{2}$

or, $\tan\theta = \frac{-1}{1}$

or, $\tan\theta = -1$

or, $\theta = 315^o$

In polar form, $(1 - i) = \sqrt{2}\,(\cos315^o + i\sin 315^o)$

Now,

$\left\{\begin{matrix}\sqrt{2}\,(\cos315^o + i\sin315^o)\\ 25\,(\cos630^o + i\sin630^o)\end{matrix}\right\}^{10}$

$= 2^5\,(0 + (-1)i)$

$= 25 \times (-1)i$

$= -32i$

g. $(2i)^4$

Let, $z = 0 + 2i$

Here, $x = 0$, $y = 2$

$\text{Tan}\theta = \frac{y}{x} = \frac{2}{0} = \infty$

$\therefore \quad \theta = 90^o$

$r = \sqrt{0 + 2^2} = \sqrt{4} = 2$

In polar form, $2i = 2(\cos90^o + i\sin90^o)$

Now,

$= \{2(\cos90^o + i\sin90^o\}^4$

$= 2^4\,(\cos360^o + i\sin360^o)$

$= 16\,(1 + 0)$

$= 16$

h. Here,

Let, $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Here, $x = \frac{1}{x}$, $y = \frac{\sqrt{3}}{2}$

$\text{Tan}\theta = \frac{y}{x}$, $\qquad r = \sqrt{\left(\frac{1}{1}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \qquad = \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= \sqrt{3} \qquad = \sqrt{\frac{4}{4}} = 1$

$\therefore \quad \theta = 60^o$

In polar form, $\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos60^o + i\sin60^o$

Now, $(\cos60^o + i\sin60^o)^7$

$= \cos420^o + i\sin720^o = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

5. Solution:

Let, $z = -2 - 2\sqrt{3}i$

Here, $x = -2$

$y = -2\sqrt{3}$

$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2}$

$= \sqrt{4 + 12}$

$= \sqrt{16} = 4$

$\text{Tan}\theta = \frac{y}{x}$

$= \dfrac{-2\sqrt{3}}{-2} = \sqrt{3}$

$\therefore\ \theta = 240^o$

In polar form, $z = 4(\cos 24^o + i\sin 240^o)$

In general polar form;

$\sqrt{z} = 4\ \{\cos(240 + 360.k) + i\sin\ (240 + 360.k)\}^{½}$

$= 2\left\{\cos\left(\dfrac{240 + 360.k}{2}\right) + i\sin\left(\dfrac{240 + 360.k}{2}\right)\right\}$

where, k = 0 and 1

When, k = 0

$\sqrt{z} = 2(\cos 120^o + i\sin 120^o)$

$= 2\left(\dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$

when k = 1

$\sqrt{z} = 2\ (\cos 300^o + i\sin 300^o)$

$= 2\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}\ i$

$\therefore\ \sqrt{-2 - 2\sqrt{3}i} = \pm\ (-1 + \sqrt{3}i)$

b. Let, $z = 4 + 4\sqrt{3}i$

Here, $x = 4$, $y = 4\sqrt{3}$

$\text{Tan}\theta = \dfrac{y}{x} = \dfrac{4\sqrt{3}}{4} = \sqrt{3}$

$\therefore\quad \theta = 60^o$

$r = \sqrt{(4)^2 + (4\sqrt{3})^2}$

$= \sqrt{16 + 48}$

$= \sqrt{64}$

$= 8$

In polar form, $4 + 4\sqrt{3}\ i = 8(\cos 60^o + i\sin 60^o)$

In general polar form i

$z = 8\{\cos(60 + 360.k) + i\sin(60 + 360.k)\}$

where, k = 0 and 1

when, k = 0

$\sqrt{z} = 2\sqrt{2}\left\{\cos\left(\dfrac{60 + 360k}{2}\right) + i\sin\left(\dfrac{60 + 360k}{2}\right)\right\}$

$= 2\sqrt{2}\ (\cos 30^o + i\sin 30^o)$

$= 2\sqrt{2}\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\ i\right)$

$= \sqrt{6} + \sqrt{2}\ i$

when, k = 1

$\sqrt{z} = 2\sqrt{2}\ \{\cos 210^o + i\sin 210^o\}$

$= 2\sqrt{2}\ (\cos 210^o + i\sin 210^o)$

$= 2\sqrt{2}\left(\dfrac{-\sqrt{3}}{2}\ \dfrac{-1}{2}\ i\right)$

$= -\sqrt{6} - \sqrt{2}i$

$= -(\sqrt{6} + \sqrt{2}i)$

$\therefore\ \sqrt{4 + 4\sqrt{3}\ i} = \pm\ (\sqrt{6} + \sqrt{2}i)$

c. Let, $z = 0 + 4i$

Here, x = 0, y = 4

$\text{Tan}\theta = \frac{y}{x} = \frac{4}{0} = \infty$

$\therefore\ \theta = 90^o$

$r = \sqrt{0 + (4)^2}$

$= \sqrt{16}$

$= 4$

In polar form, $z = 4\ (\cos 90^o + i\sin 90^o)$

In general polar formi

$z = 4\{\cos(90 + 360.k) + i\sin(90 + 360k)\}$

where, k = 0, 1

$\sqrt{z} = \sqrt{4}\ \left\{\cos\left(\frac{90 + 360.k}{2}\right) + i\sin\left(\frac{90 + 360k}{2}\right)\right\}$

$= 2\left\{\cos\left(\frac{90 + 360k}{2}\right) + i\sin\left(\frac{90 + 360k}{2}\right)\right\}$

where, k = 0

$\sqrt{z} = 2\{\cos 45^o + i\sin 45^o\}$

$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i\right) = \sqrt{2} + \sqrt{2}\,i$

when k = 1

$\sqrt{z} = 2\{\cos 225^o + i\sin 22.5^o\}$

$= 2\left(\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i\right) = \sqrt{2} - \sqrt{2}\,i$

$\therefore\ \sqrt{4i} = \pm\sqrt{2}\ (1 + i)$

d. Let, $z = -i$

Here, x = 0, y = –1

$\text{Tan}\theta = \frac{y}{x} = \frac{-1}{0}\ = -\infty$

$\therefore\quad \theta = 270^o$

$r = \sqrt{0 + (-1)^2}$

$= \sqrt{1} = 1$

In polar form;

$z = \cos(270^o + i\sin 270^o$

In general polar form;

$z = \cos(270 + 360.k) + i\sin\ (270 + 360.k)$

Now, k = 0 and 1

$\sqrt{z} = \cos\left(\frac{270 + 360.k}{2}\right) + i\sin\left(\frac{270 + 360.k}{2}\right)$

when k = 0

$\sqrt{z} = \cos 135^o + i\sin 135^o$

$= \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i$

$= \frac{-1}{\sqrt{2}}\,(1 - i)$

when, k = 1

$\sqrt{z} = \cos 315^o + i\sin 315^o$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i$

$= \frac{1}{\sqrt{2}}(1 - i)$

$\therefore \sqrt{-i} = \pm\frac{1}{\sqrt{2}}(1 - i)$

e. Let, $z = -1 + 0i$

Here, $x = -1, y = 0$

$\text{Tan}\theta = \frac{y}{x}$, $r = \sqrt{(-1)2 + 0}$

$= \frac{0}{-1}$ $= \sqrt{1}$

$= 0$ $= 1$

$\therefore \theta = 180^o$

In polar form,

$z = \cos 180^o + i\sin 180^o$

In general polar form;

$z = \cos(180 + 360k) + i\sin(180 + 360.k)$

where, k = 0 and 1

$\sqrt{z} = \cos\left(\frac{180 + 360.k}{2}\right) + i\sin\left(\frac{180 + 360.k}{2}\right)$

when, k = 0

$\sqrt{z} = \cos 90^o + i\sin 90^o$

$= 0 + i$

when, k = 1

$\sqrt{z} = a\cos 270^o + i\sin 270^o$

$= 0 - 1i$

$\therefore \sqrt{-1} = \pm i$

f. Let, $z = 2 + 2\sqrt{3}\,i$

Here, $x = 2$,

$y = 2\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x}$, $r = \sqrt{(2)^2 + (2\sqrt{3})^2}$

$= \frac{2\sqrt{3}}{2}$ $= \sqrt{4 + 12}$

$= \sqrt{3}$ $= \sqrt{16} = 4$

$\therefore \theta = 60^o$

In polar form,

$z = 4(\cos 60^o + i\sin 60^o)$

In general polar form;

$z = 4\{\cos(60 + 360.k) + i\sin(60 + 360.k)\}$

where, k = 0 and 1

$\sqrt{2} = 2\left\{\cos\left(\frac{60 + 360.k}{2}\right) + i\sin\left(\frac{60 + 360.k}{2}\right)\right\}$

when, $k = 0 = 2(\cos 30^o + i\sin 30^o)$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= \sqrt{3} + i$

when, k = 1

$\sqrt{z} = 2(\cos 210^o + i\sin 210^o)$

$= 2\left(\frac{-\sqrt{3}}{2}\frac{-1}{2}i\right)$

$= -\sqrt{3} - 1i$

$\therefore \sqrt{2 + 2\sqrt{3}i} = \pm (\sqrt{3} + i)$

g. Let, $z = 4 - 4\sqrt{3}$

Here, $x = y$, $y = -4\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{(4)^2 + (-4\sqrt{3}^2)}$

$= \frac{-4\sqrt{3}}{4} \quad = \sqrt{16 + 48}$

$= -\sqrt{3} \quad = 8$

$\therefore \theta = 300^o$

In polar form,

$z = 8(\cos 300^o + i\sin 300^o)$

In general form;

$z = 8\{\cos(300 + 360.k) + i\sin(300 + 360.k)\}$

where, k = 0 and 1

$$\sqrt{z} = 2\sqrt{2}\left\{\cos\left(\frac{300 + 360k}{2}\right) + i\sin\left(\frac{300 + 360.k}{2}\right)\right\}$$

when, k = 0

$\sqrt{z} = 2\sqrt{2}\,(\cos 150^o + i\sin 150^o)$

$= 2\sqrt{2}\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -\sqrt{6} + \sqrt{2}\,i$

when, k = 1

$\sqrt{z} = 2\sqrt{2}\,(\cos 330^o + i\sin 330^o)$

$= 2\sqrt{2}\left(\frac{\sqrt{3}}{2}\frac{-1}{2}i\right)$

$= \sqrt{6} - \sqrt{2}\,i$

$\therefore \sqrt{4 - 4\sqrt{3}i} = (\sqrt{6} - \sqrt{2}\,i)$

6. Solution:

a. Let, $z = 1 + 0i$

Here, $x = 1$, $y = 0$

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{(1)^2 + 0}$

$= \frac{0}{1} \quad = \sqrt{1}$

$= 0 \quad = 1$

$\therefore \theta = 0^o$

In polar form,

$z = \cos 0^o + i\sin 0^o$

In general polar form;

$z = \cos(360.k + 0^o) + i\sin(0 + 360.k)$

where, k = 0 and 2

$z^{1/3} = \{\cos(0 + 360.k) + i\sin(0 + 360.k)\}^{1/3}$

$= \cos(0 + 120k) + i\sin(0 + 120k)$

when, k = 0

$z^{1/3} = \cos 0 + i\sin 0$

$= 1 + 0$

$= 1$

when, k = 1

$z^{1/3} = \cos 120 + i\sin 120^o$

$= \frac{-1}{2} + \frac{\sqrt{3}}{2} i$

when, k = 2

$z^{1/3} = \cos 240^o + i\sin 240^o$

$= \frac{-1}{2} \frac{-\sqrt{3}}{2} i$

$\therefore$ Cube roots of 1 = 1, $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right), \left(\frac{-1 - \sqrt{3}}{2} i\right)$

b. Let, z = –1 + 0i

Here, x = –1, y = 0

$\tan\theta = \frac{y}{x}$, $r = \sqrt{(-1)^2 + 0}$

$= \frac{0}{-1}$ $= \sqrt{1}$

$= 0$ $= 1$

$\therefore \theta = 180^o$

In polar form, $z = \cos 180^o + i\sin 180^o$

In general polar form;

$z = \cos(180^o + 360.k) + i\sin(180^o + 360.k)$

where, k = 0, 1, 2

$z^{1/3} = \cos\left(\frac{180 + 360.k}{3}\right) + i\sin\left(\frac{180 + 360.k}{3}\right)$

when, k = 0

$z^{1/3} = \cos 60^o + i\sin 60^o$

$= \frac{1}{2} + \frac{\sqrt{3}}{2} i$

when, k = 1,

$z^{1/3} = \cos 180^o + i\sin 180^o$

$= -1 + 0$

$= -1$

when, k = 2

$z^{1/3} = \cos 300^o + i\sin 300^o$

$= \frac{1}{2} - \frac{\sqrt{3}}{2} i$

$\therefore$ Hence, the required cube roots of unity are

$-1, \frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2} i$,

c. Let, z = 0 + 1i

Here, z = 0, y = 1

$\tan\theta = \frac{y}{x} = \frac{1}{0} = \infty$

$\therefore \theta = 90^o$

$r = \sqrt{0 + 1^2} = 1$

In polar form, $z = \cos 90^o + i\sin 90^o$

In general polar form;

$z = \cos(90 + 360.k) + i\sin(90 + 360.k)$

$z^{1/3} = \cos\left(\frac{90 + 360.k}{3}\right) + i\sin\left(\frac{90 + 360.k}{3}\right)$

when k = 0, where, k = 0, 1 and 2

$z^{1/3} = \cos 30^o + i\sin 30^o$

$= \frac{\sqrt{3}}{2} + \frac{1}{2} i$

when k = 1

$z^{1/3} = \cos 150^o + i\sin 150^o$

$= \frac{-\sqrt{3}}{2} + \frac{1}{2} i$

when k = 2

$z^{1/3} = \cos 270^o + i\sin 270^o$

$= 0 - 1i$

$= -i$

Hence, the required cube roots of unity are $-i$, $\frac{\sqrt{3}}{2} + \frac{1}{2} i$ and $\frac{-\sqrt{3}}{2} + \frac{1}{2} i$

d. Let, $z = 0 - i$

Here, $x = 0$, $y = -1$

$\tan\theta = \frac{y}{x} = \frac{-1}{0} = \infty$

$\therefore\ \theta = 270^o$

$r = \sqrt{0 + (-1)^2}$

$= \sqrt{1}$

$= 1$

In polar form, $z = \cos 270^o + i\sin 270^o$

In general polar form;

$z = \cos(270 + 360.k) + i\sin(270 + 360.k)$

$z^{1/3} = \cos\left(\frac{270 + 360.k}{3}\right) + i\sin\left(\frac{270 + 360.k}{3}\right)$

where, k = 0, 1 and 2

when, k = 0

$z^{1/3} = \cos 90^o + i\sin 90^o$

$= 0 + 1i$

$= i$

when k = 1,

$z^{1/3} = \cos 210^o + i\sin 210$

$= \frac{-\sqrt{3}}{2} - \frac{1}{2} i$

$= \frac{-\sqrt{3} - 1i}{2}$

when k = 2

$z^{1/3} = \cos.330^o + i\sin 330^o$

$= \frac{\sqrt{3}}{2} - \frac{1i}{2}$

$= \frac{\sqrt{3} - 1i}{2}$

Hence, the required cube roots of i are i, $\frac{-\sqrt{3} - 1i}{2}$ and $\frac{\sqrt{3} - 1i}{2}$

e. Let, $z = 1 + 0i$

Here, $x = 1$, $y = 0$

$\tan\theta = \frac{y}{x}$, $\qquad r = \sqrt{1 + 0}$

$= \frac{0}{1} \qquad = \sqrt{1}$

$= 0 \qquad = 1$

$\therefore\ \theta = 0^o$

In polar form;

In general polar form, z = cos(0 + 360.k) + isin (0 + 360k)

$$z^{1/4} = \cos\left(\frac{0 + 360.k}{4}\right) + i\sin\left(\frac{0 + 360.k}{4}\right)$$

when k = 0, where, k = 0, 1, 2 and 3

$z^{1/4} = \cos 0^o + i\sin 0^o$

$= 1 + 0$

$= 1$

when k = 1

$z^{1/4} = \cos 90^o + i\sin 90^o$

$= 0 + 1i$

$= i$

when k = 2

$z^{1/4} = \cos 180^o + i\sin 180^o$

$= -1 + 0$

$= -1$

when k = 3

$z^{1/4} = \cos 270^o + i\sin 270^o$

$= 0 - 1i$

$= -i$

Hence, the required forth roots of unity are ±1 and 1i

f. Let, $z = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$

Here, $x = \frac{-1}{2}$, $y = \frac{\sqrt{3}}{2}$

$\text{Tan}\theta = \frac{y}{x}$, $\qquad r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}}$ $\qquad = \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= \frac{\sqrt{3}}{2} \times \frac{2}{-1}$ $\qquad = \sqrt{\frac{4}{4}}$

$= -\sqrt{3}$ $\qquad = 1$

$\therefore \theta = 120^o$

In polar form,

$z = \cos 120^o + i\sin 120^o$

In general polar form;

$z = \cos(120 + 360.k) + i\sin(120^o + 360.k)$

$$z^{1/4} = \cos\left(\frac{120 + 360.k}{4}\right) + i\sin\left(\frac{120 + 360.k}{4}\right)$$

where, k = 0, 1, 2, 3

when k = 0

$z^{1/4} = \cos 30^o + i\sin 30^o$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

when k = 1

$z^{1/4} = \cos 120^o + i\sin 120^o$

$= \frac{-1}{2} + \frac{\sqrt{3}}{2}i$

$= \frac{-1 + \sqrt{3}i}{2}$

when k = 2

$z^{1/4} = \cos 210^o + i\sin 210^o$

$= \frac{-\sqrt{3}}{2} - \frac{1}{2}i$

$= \frac{-\sqrt{3} - 1i}{2}$

when k = 3

$z^{1/4} = \cos 300^o + i\sin 300^o$

$= \frac{1}{2} \frac{-\sqrt{3}i}{2}$

$= \frac{1 - \sqrt{3}i}{2}$

Hence, the required fourth roots of unity are $\pm\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$ and $\pm\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

7. a. Here, $z^3 + 8i = 0$

Let,

$z^3 = -8i$

$z^3 = 0 - 8i$

Here, x = 0, y = –8

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{x^2 + y^2}$

$= \frac{-8}{0} \quad = \sqrt{0 + (-8)^2}$

$= \frac{-8}{0} \quad = \sqrt{64}$

$= \infty \quad = 8$

$\therefore\ \theta = 270^o$

In polar form, $z = 8(\cos 270^o + i\sin 270^o)$

In general polar form;

$z = 8\{\cos(270 + 360.k) + i\sin(270 + 360.k)\}$

$z^{1/3} = 2\left\{\left(\frac{270 + 360.k}{3}\right) + i\sin\left(\frac{270 + 360.k}{3}\right)\right\}$

where, k = 0, 1, 2

when k = 0

$z^{1/3} = \{\cos 90^o + i\sin 90^o\}$

$= 2(0 + 1i)$

$= 2i$

when k = 1

$z^{1/3} = 2(\cos 210^o + i\sin 210^o)$

$= 2\left(\frac{-\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - 1i$

when k = 2

$z^{1/3} = 2(\cos 330^o + i\sin 330^o)$

$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= \sqrt{3} - 1i$

Hence, the required cube roots of –8i are 2i, $\sqrt{3} - 1i$ and $-(\sqrt{3} + 1i)$

b. Let, $z^4 = -1$

$= -1 + 0i$

Here, $x = -1, y = 0$

$\tan\theta = \frac{y}{x}$, $\quad r = \sqrt{(-1)^2 + 0}$

$= \frac{0}{-1} \quad = \sqrt{1}$

$= 0 \quad = 1$

$\therefore \theta = 180^o$

In polar form;

$z = \cos(180^o + i\sin 180^o)$

In general polar form;

$z = \cos(180 + 360.k) + i\sin(180 + 360.k)$

$z^{1/4} = \cos\left(\frac{180 + 360.k}{4}\right) + i\sin\left(\frac{180 + 360.k}{4}\right)$

when $k = 0$

$z^{1/4} = \cos 45^o + i\sin 45^o$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

when $k = 1$

$z^{1/4} = \cos 135^o + i\sin 135^o$

$= \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

when $k = 2$

$z^{1/4} = \cos 225^o + i\sin 225^o$

$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

when $k = 3$

$z^{1/4} = \cos 315^o + i\sin 315^o$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

Hence, the required fourth roots of -1 is

$\pm\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$ and $\pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$

c. $z^6 = 1$

We have,

$z^6 = 1 = 1 + i(0) = \cos 30^o + i\sin 0^o$

$\Rightarrow \quad z^6 = \cos 2n\pi + i\sin 2n\pi$

$\Rightarrow \quad z = [\cos 2n\pi + i\sin 2n\pi]^{1/6}$

By De-moivre's theorem

$z = \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$

where $n = 0, 1, 2, 3, 4, 5$

When $n = 0$ then the first root of z is,

$z = \cos 0 + i\sin 0 = 1 + 0 = 1$

When $n = 1$ then the 2^{nd} root of z is,

$z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1 + i\sqrt{3}}{2}$

When $n = 2$ then the 3^{rd} root of z is,

$z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{-1 + i\sqrt{3}}{2}$

When $n = 3$ then the 4^{th} root of z is

$z = \cos\pi + i\sin\pi = -1 + i.0 = -1$

When $n = 4$ then the 5^{th} root of z is,

$z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1 - i\sqrt{3}}{2}$

When n = 5 then the 6^{th} root of z is,

$z = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1 - i\sqrt{3}}{2}$

Hence, the required six roots of z are

$1, -1, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$

8. Solution:

a. Here, $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos n\theta - i\sin n\theta$

LHS

$z^n + \frac{1}{z^n}$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$ proved.

b. Here, $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta - i\sin n\theta$

LHS

$z^n - \frac{1}{z^n}$

$= z^n - z^{-n}$

$= \cos n\theta + i\sin\theta - \cos n\theta + i\sin n\theta$

$= 2i\sin n\theta$ RHS

9. Solution:

a. Let, z_1 and z_2 be $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ respectively.

Then,

$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) . r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\therefore \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$ proved.

b. Let z_1 and z_2 be $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ respectively with $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

Now, $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

So, $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$ proved

c. Let, $z = r(\cos\theta + i\sin\theta)$

where, Ag . $z = \theta$

Then, $\overline{z} = r(\cos\theta - i\sin\theta)$

$\overline{z} = r\{\cos(2\pi - \theta) + i\sin(2\pi - \theta)\}$

$\therefore \text{Arg}(\overline{z}) = 2\pi - \theta$

$= 2\pi - \text{Arg}(z)$

10. a. $e^{i\pi/2} = \cos^{\pi/2} + i\sin^{\pi/2} = 0 + i(1) = i$

b. $e^{-i\pi/6} = \cos^{\pi/6} - i\sin^{\pi/6} = \frac{\sqrt{3}}{2} - i\frac{1}{2}$

c. $-5e^{-i\pi/3} = -5[\cos^{\pi/3} - i\sin^{\pi/3}] = -5\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = -\frac{5}{2} + i\frac{\sqrt{3}}{2}$

11. Solution:

a. To express the complex form into re^{ix} form firstly, we change into polar form,

Let $3 + 4i = r(\cos\theta + i\sin\theta)$ ... ... (i)

$\Rightarrow$ $r\cos\theta = 3$ and $ir\sin\theta = 4i \Rightarrow r\sin\theta = 4$

Squaring and adding these two

We get,

$r^2 = 25$ $\therefore$ $r = 5$

Also, $\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927$

$\therefore$ The complex number in exponential form is $re^{i\theta}$ i.e. $5e^{0.927i}$

b. 3i

Let $0 + 3i = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $r\cos\theta = 0$ and $r\sin\theta = 3$

$r^2 = 9$

$\therefore$ $r = 3$

And, $\tan\theta = \frac{3}{0} = \infty = \tan\frac{\pi}{2}$

$\therefore$ The complex number in exponential form is $re^{i\theta}$ i.e. $3e^{i\pi/2}$

c. $-2 - 2i$

Let $-2 - 2i = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $r\cos\theta = -2$ and $r\sin\theta = -2$

$r^2 = 4 + 4 \Rightarrow r^2 = 8$

$\therefore$ $r = 2\sqrt{2}$

And, $\tan\theta = \frac{-2}{-2} = 1 = \tan\frac{5\pi}{4}$

$\therefore$ $\theta = 5\frac{\pi}{4}$

$\therefore$ The complex number is exponential form is, $re^{i\theta}$ i.e. $2\sqrt{2}\, e^{i5\pi/4}$

d. $1 + i\sqrt{3}$

Let $1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $r\cos\theta = 1$ and $r\sin\theta = \sqrt{3}$

$r^2 = 4 \Rightarrow r = 2$

And, $\tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$

$\therefore$ The complex number in exponential form is $re^{i\theta}$ i.e. $2e^{i\pi/3}$

Chapter–5
Quadratic Equations

Exercise 5.1

1. Solution:

a. Here, $x^2 - 12x + 40 = 0$ ... ... (i)
Comparing equation (i) with $ax^2 + bx + c = 0$, we get
$\therefore$ a = 1, b = –12, c = 40
Now,
$b^2 - 4ac = (-12)2 - 4 \times 1 \times 40 = 144 - 160 = -16 < 0$
Here, Roots are imaginary and unequal.

b. Here,
$x^2 - 14x - 3 = 0$ ... ... (i)
Comparing equation (i) with $ax^2 + bx + c = 0$, we get
$\therefore$ a = 1, b = –4, c = –3
Now,
$b^2 - 4ac = (-4)^2 - 4\times1\times(-3) = 16 + 12 = 28 > 0$
Hence, Roots are unequal, real and irrational.

c. Here, $2x^2 - 12x + 18 = 0$ ... ... (i)
Comparing equation (i) with $ax^2 + bx + = 0$, we get
$\therefore$ a = 2, b = –12, c = 18
Now,
$b^2 - 4ac = (-12)^2 - 4\times2\times18 = 144 - 144 = 0$
Hence, roots are real, equal and rational.

d. Here, $4x^2 + 8x - 5 = 0$ ... ... (i)
Comparing equation (i) with $ax^2 = bx + c = 0$, we get,
$\therefore$ a = 4, b = 8, c = –5
Now,
$b^2 - 4ac = (8)2 - 4 \times 4 \times (-5) = 64 + 80 = 144 > 0$ and perfect square
Hence,
Roots are real, unequal is rational.

e. Here, $x^2 - 16 = 0$ ... ... (ii)
Comparing equation (i) with $ax^2 + bx + c = 0$
$\therefore$ a = 1, b = 0, c = –16
Now,
$b^2 - 4ac = 0 - 4 \times 1 \times (-16) = 64 > 0$ and perfect square
Hence, roots are real, unequal is rational.

2. Solution:

Given equation is $5x^2 - px + 45 = 0$ ... (i)
Comparing equation (i) with
$ax^2 + bx + c = 0$
$\therefore$ a = 5, b = –p, c = 45
Now, for being equal roots;
$b^2 = 4ac = 0$
or, $(-p)^2 - 4\times5\times45 = 0$
or, $p^2 = 900$
or, $(p)^2 = (\pm 30)^2$
$\therefore$ $p = \pm 30$

3. Solution:

a. Here,
Comparing equation $x^2 + (k + 2)x + 2k = 0$ with $ax^2 + bx + c = 0$
$\therefore$ a = 1, b = k+2, c = 2k

Now, for being equal roots;

$b^2 - 4ac = 0$

or, $(k+2)^2 - 4\times1\times2k = 0$

or, $k^2 + 4k + 4 - 8k = 0$

or, $k^2 - 4k + 4 = 0$

or, $(k - 2)^2 = 0$

$\therefore$ $k = 2$

b. Here, Comparing equation $x^2 - (2k-1).\ x - (k - 1) = 0$ with $ax^2 + bx + c = 0$. we get,

$\therefore$ $a = 1, b = -(2k - 1), c = -(k - 1)$

Now, for being equal roots;

$b^2 - 4ac = 0$

or, $\{-(2k - 1)\}^2 - 4\times1\times\{-(k - 1)\} = 0$

or, $4k^2 - 4k + 1 + 4k - 4 = 0$

or, $4k^2 - 3 = 0$

or, $k^2 = \frac{3}{4}$

$\therefore$ $k = \pm\frac{\sqrt{3}}{2}$

4. Solution:

a. Here, comparing equation $(1 + m^2).\ x^2 + 2mc.\ x + (c^2 - a^2) = 0$ with $ax^2 + bx + c = 0$, we get,

$\therefore$ $a = 1+m^2, b = 2mc, c = c^2 - a^2$

Now,

For being equal roots;

$b^2 - 4ac = 0$

or, $(2mc)^2 - 4(1 + m^2)\ .\ (c^2 - a^2) = 0$

or, $4m^2c^2 - 4\{1(c^2 - a^2) + m^2\ (c^2 - a^2)\} = 0$

or, $m^2c^2 - (c^2 - a^2) - m^2c^2 + m^2a^2 = 0$

or, $-(c^2 - a^2) = - m^2a^2$

or, $c^2 - a^2 = m^2a^2$

or, $c^2 = m^2a^2 + a^2$

or, $c^2 = a^2(1 + m^2)$ proved.

5. Solution:

Here, comparing $(a^2 - bc)\ .\ x^2 + 2(b^2 - ca)\ .\ x + c^2 - ab = 0$ with $Ax^2 + Bx + C = 0$

$\therefore$ $A = a^2 - bc, B = 2(b^2 - ca), c = c^2 - ab$

For equal roots,

$B^2 - 4AC = 0$

or, $\{2(b^2 - ca)\}^2 - 4(a^2 - bc)\ .\ (c^2 - ab) = 0$

or, $(b^2 - ca)^2 - (a^2 - bc)\ (c^2 - ab) = 0$

or, $b^4 = 2ab^2c + c^2a^2 - a^2c^2 + a^3b + bc^3 - ab^2c = 0$

or, $a^3b + b^4 + bc^3 - 3ab^2c = 0$

or, $b(a^3 + b^3 + c^3 - 3abc) = 0$

Either, $b = 0$,

$a^3 + b^3 + c^3 - 3abc = 0$

6. Solution:

Here, given equation is

$(x - a)\ (x - b) + (x - b)\ (x - c) + (x - c)\ (x - a) = 0$

or, $x^2 - bx - ax + ab + x^2 - cx - bx + bc + x^2 - ax - cx + ca = 0$

or, $3x^2 - 2(a + b + c).\ x + (ab + bc + ca) = 0$ ... ... (i)

Comparing equation (i) with $Ax^2 + Bx + C = 0$

$A = 3, B = -2\ (a + b + c), c = ab + bc + ca$

Now,

$B^2 - 4ac = 0$

or, $\{-2(a + b + c)\}^2 - 4\times3 (ab + bc + ca) = 0$
or, $4(a^2 + b^2 + c^2 + ab + bc + ca) - 12 (ab + bc + ca) = 0$
or, $(a^2 + b^2 + c^2 + ab + bc + ca - 3ab - 3bc - 3ca) = 0$
or, $(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca) = 0$
or, $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$
Either,
$a = b, b = c, c = a$
$\therefore$ $a = b = c$

7. Solution:
Here, comparing $(a^2 + b^2) x^2 - 2(ac + bd) x + (c^2 + d^2) = 0$ with $Ax^2 + BX + C = 0$, we get,
$A = a^2 + b^2$
$B = -2(ac + bd)$
$C = c^2 + d^2$
The roots are equal if
$B^2 - 4AC = 0$
or, $\{-2(ac + bd)\}^2 - 4\times(a^2 + b^2) (c^2 + d^2) = 0$
or, $4(ac + bd)^2 - 4(a^2 + b^2) (c^2 + d^2) = 0$
or, $a^2c^2 + 2abcd + b^2d^2 - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2 = 0$
or, $-a^2d^2 + 2abcd - b^2c^2 = 0$
or, $-(ad - bc)^2 = 0$
or, $ad - bc = 0$
or, $ad = bc$
or, $\frac{a}{b} = \frac{c}{d}$
$\therefore$ $\frac{a}{b} = \frac{c}{d}$ proved.

8. Solution:
Here, given equation is $(b + c - a) . x^2 + (c + a - b) . x + (a + b - c) = 0$ ... ... (i)
If $a + b + c = 0$
Comparing equation (i) with $Ax^2 + Bx + C = 0$
$A = (b + c - a)$
$B = (c + a - b)$
$C = (a + b - c)$
Now,
$B^2 - 4AC$
$= (c + a - b)^2 - 4(b + c - a) . (a + b - c)$
$= (-b - b)^2 - 4(-2a) . (-2c)$
$= 4b^2 - 16ac$
$= 4(b^2 - 4ac)$
$= 4\{b^2 - 4a (-a - b)\}$
$= 4(b^2 + 4a^2 + 4ab)$
$= 4(b + 2a)^2 > 0$ and a perfect square
Hence, roots are rational.

9. Solution:
Here, given equation is $(x - a) (x - b) = k^2$ ... ... (i)
or, $x^2 - bx - ax + ab - k^2 = 0$
or, $x^2 - (a + b) . x + (ab - k^2) = 0.$
Comparing equation (i) with $Ax^2 = BX + C = 0$, we get,
$A = 1, B = -(a + b), C = ab - k^2$
Now,
$B^2 - 4AC$
$= \{-(a + b)\}^2 - 4\times1(ab - k^2)$
$= a^2 + 2ab + b^2 - 4(ab - k^2)$
$= a^2 - 2ab + b^2 + 4ab + 4k^2$

$= a^2 - 2ab + b^2 + 4k^2$
$= (a - b)^2 + 4k^2 > 0$ for all k

Hence, roots are real.

10. Solution:

Comparing equation $(b - c) x^2 + 2(c - a) . x + (a - b) = 0$ with $Ax^2 + BX + C = 0$.

$A = (b - c)$
$B = 2(c - a)$
$C = (a - b)$

Now,

$B^2 - 4AC$
$= 4(c -)^2 - 4(b - c) (a - b)$
$= 4\{(c - a)^2 - (b - c) (a - b)^2\}$
$= 4\{c^2 + a^2 - 2ca - ab + b^2 + ca - bc\}$
$= 4\{a^2 + b^2 + c^2 - ab - bc - ca^2\}$
$= 2\{2a^2 + 2b^2 + 2c - 2ab - 2bc - 2ca^2\}$
$= 2\{(a - b)^2 + (b - c)^2 + (c - a)^2\} > 0$

Hence, roots are always real.

11. Solution:

Here, given equation is $x^2 + (2m - 1). x + m^2 = 0$ .. .. (i)

Comparing equation (i) with $ax^2 + bx + c = 0$, we get,

$a = 1, b = (2m - 1), c = m^2$

Now,

$b^2 - 4ac$

or, $(2m - 1)^2 - 4\times1\times m^2$

or, $4m^2 - 4m + 1 - 4m^2$

or, $-(um - 1)$

or, $-(4m - 1)$

The roots will be real if $b^2 -4ac$

or, $- 4m + 1 \geq 0$

or, $1 \geq 4m$

$\therefore \quad m \leq \frac{1}{4}$

12. Solution:

Comparing $x^2 + 4abx + (a^2 + 2b^2)^2 = 0$ with $Ax^2 + Bx + C = 0$. We get,

$A = 1, B = 4ab, C = (a^2 + 2b^2)^2$

Now,

$B^2 - 4AC$
$= (4ab)^2 = 4\times1\times(a^2 + 2b^2)^2$
$= 16a^2b^2 - 4(a^4 + 2a^2b^2 + 4b^4)$
$= 4(4a^2b^2 - a^4 - 2a^2b^2 - 4b^4)$
$= 4(-a^4 + 2a^2b^2 - 4b^4)$
$= -4(a^4 - 2a^2b^2 + 4b^4)$
$= -4(a^2 - 2b^2)^2 < 0$

Hence, roots are imaginary.

13. Solution:

Here, $qx^2 + 2px + 2q = 0$

$b^2 - 4ac = (2p)^2 - 4.q. 2q]$
$= 4p^2 - 8q^2$
$= 4(p^2 - 2q)^2 > 0$ ... ... (i)

$(p + q) x^2 + 2qx + (p - q) = 0$

$b^2 - 4ac = (2q)^2 - 4(p + q) . (p - q)$
$= 4q^2 - 4(p^2 - q^2)$
$= 4(q^2 - p^2 + q^2)$
$= -4(p^2 - 2q^2) < 0$ ... ... (ii)

The roots of second equation (ii) are imaginary if the roots of first equations are real.

14. Solution:

Here, comparing (ab – ac) x^2 + (bc - ab) x + ca - ab = 0 with $Ax^2 + BX + C = 0$

$\therefore$ A = (ab – ac)

$\therefore$ B = (bc – ab)

$\therefore$ c = (ca – ab)

Now, $B^2 - 4AC = 0$

or, $(bc - ab)^2 - 4(ab - ac)(ca - ab) = 0$

or, $b^2c^2 - 2ab^2c + a^2b^2 - 4\{a^2bc - a^2b^2 - a^2c^2 + a^2bc\} = 0$

or, $b^2c^2 - 2ab^2c + a^2b^2 - 4a^2bc + 4a^2b^2 + 4a^2c^2 - 4a^2bc$

Exercise: 5.2

1. Solution:

a. Let, α and β be the two roots i.e. $\alpha = 3$, $\beta = -5$

Now, x^2 – (sum of roots) . x + product of roots = 0

or, $x^2 - (3 - 5) . x + 3 \times (-5) = 0$

or, $x^2 + 2x - 15 = 0$

Hence, The required quadratic equation is $x^2 + 2x - 15 = 0$.

b. Here, let, α and β be the two roots i.e. $\alpha = 2$, $\beta = \frac{1}{2}$.

Now, x^2 – (sum of roots) x + product of roots = 0

or, $x^2 - \left(2 + \frac{1}{2}\right) x + 2 \times \frac{1}{2} = 0$

or, $x^2 - \frac{5x}{2} + 1 = 0$

or, $2x^2 - 5x + 2 = 0$

Hence, the required equation is $2x^2 - 5x + 2 = 0$

c. Here, let α and β be the two roots i.e. $\alpha = 2 - 3i$, $\beta = 2 + 3i$

Now, x^2 – (sum of roots) x + product of roots = 0

or, $x^2 - (2 - 3i + 2 + 3i) x + (2 - 3i)(2 + 3i) = 0$

or, $x^2 - 4x + 4 + 9 = 0$

or, $x^2 - 4x + 13 = 0$

Hence, the required quadratic equation is $x^2 - 4x + 13 = 0$

2. Solution:

a. Here, one root (α) = $3 - \sqrt{5}$

Other root (β) = $3 + \sqrt{5}$

Quadratic equation is x^2 – (sum of roots) x + product of roots = 0

or, $x^2 - (3 - \sqrt{5} + 3 + \sqrt{5}) x + 9 - 5 = 0$

or, $x^2 - 6x + 4 = 0$

$\therefore$ $x^2 - 6x + 4 = 0$

b. Here, one root (α) = $-2i$

Other root (β) = $2i$

The required equation is $x^2 - (\alpha + \beta) x + \alpha\beta = 0$

or, $x^2 - (2i - 2i) . x - 4i^2 = 0$

or, $x^2 + 4 = 0$

c. Here, one root (α) = $1 + \sqrt{3}i$

Other root (β) = $1 - \sqrt{3}i$

The required quadratic equation is

$x^2 - (\alpha + \beta) x + \alpha\beta = 0$

or, $x^2 - (1 + \sqrt{3}i + 1 - \sqrt{3}i) x + 1^2 + (\sqrt{3})^2 = 0$

or, $x^2 - 2x + 1 + 3 = 0$

or, $x^2 - 2x + 4 = 0$

d. Here,

One root $(\alpha) = \frac{1}{3+\sqrt{5}}$

Other root $(\beta) = \frac{1}{3-\sqrt{5}}$

The required quadratic equation is,

$x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$

or, $x^2 - \left(\frac{1}{3+\sqrt{5}} + \frac{1}{(3-\sqrt{5})}\right) x + \frac{1}{3+\sqrt{5}} \frac{1}{3-\sqrt{5}} = 0$

or, $x^2 - \frac{(3-\sqrt{5}+3+\sqrt{5})}{9 - 5} . x + \frac{1}{9-5} = 0$

or, $x^2 - \frac{6x}{4} + \frac{1}{4} = 0$

or, $4x^2 - 6x + 1 = 0$

e. Here,

One root $(\alpha) = \frac{1}{3!}$

Other root $(\beta) = -\frac{1}{3!}$

The required quadratic equation is,

$x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$

or, $x^2 - \left(\frac{1}{3i} - \frac{1}{3i}\right) . x - \frac{1}{3i}\frac{1}{3i} = 0$

or, $x^2 + \frac{1}{9} = 0$

or, $9x^2 + 1 = 0$

3. Solution:

a. Here, let α and β be the roots of $4x^2 + 8x - 5 = 0$

$\alpha + \beta = \frac{-8}{4} = -2,\ \alpha\beta = \frac{-5}{4}$

and α^2 and β^2 be the roots of required equation.

$\alpha^2 + \beta^2$	$\alpha^2\,\beta^2$
$= (\alpha+\beta)^2 - 2\alpha\beta$	$= (\alpha\beta)2$
$= 4 - 2. \frac{-5}{2} = \frac{13}{2} = 6.5$	$= \left(\frac{-5}{4}\right)^2 = \frac{25}{16}$

The required quadratic equation is,

$x^2 - \frac{13}{2} . x + \frac{25}{16} = 0$

or, $16x^2 - 104x + 25 = 0$

b. Here, let α and β be the two roots of $3x^2 - 5x - 2 = 0$

$\alpha + \beta = \frac{5}{3},\ \alpha\beta = \frac{-2}{3}$

and $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ be the two roots of required equation,

$\frac{1}{\alpha} + \frac{1}{\beta}$	$\frac{1}{\alpha} \times \frac{1}{\beta}$
$= \frac{\beta+\alpha}{a\beta} = \frac{1}{\alpha\beta}$	
$= \frac{\frac{5}{3}}{\frac{-2}{3}}$	$= \frac{1}{\frac{-2}{3}}$

$= \frac{-5}{3} \times \frac{3}{2} = \frac{1 \times 3}{-2}$

$= -\frac{5}{2}$ $\quad\quad$ $= -\frac{3}{2}$

The required quadratic equation is, $x^2 - \frac{5}{2} . x - \frac{3}{2} = 0$

or, $2x^2 + 5x - 3 = 0$

c. Here, let, α and β be the two roots of $x^2 - bx + c = 0$

$\alpha + \beta = \frac{b}{1}$, $\alpha\beta = c$

or, $\alpha + \beta = b$

and $m\alpha$ and $m\beta$ be the two roots of required equation,

$m\alpha + m\beta$ $\quad\quad$ $m\alpha . m\beta$

$= m(\alpha + \beta)$ $\quad\quad$ $= m^2\alpha\beta$

$= mb$ $\quad\quad$ $= m^2 \times c$

The required quadratic equation is $x^2 - mbx + m^2c = 0$

d. Here, let α and β be the roots of $x^2 - px + q = 0$

Then,

sum of roots = $\alpha + \beta$

$= \frac{-p}{-1} = p$

Product of roots = $\alpha\beta$

$= \frac{q}{1} = q$

Since,

The roots of the required equation are by h, so

Sum of roots = $(\alpha + h) + \beta + h$

$= (\alpha + \beta) + 2h = p + 2h$

Product of roots = $(\alpha + h)(\beta + h)$

$= \alpha\beta + (\alpha + \beta) h + h^2$

$= q + ph + h^2$

The required equation is $x^2 - (p + 2h) . x + (q + ph + h^2) = 0$

4. Solution:

a. Here, let α and 3α be the two roots of $ax^2 + bx + c = 0$

Sum of roots, $\alpha + 3\alpha = \frac{b}{a}$

or, $4\alpha = \frac{b}{a}$ $\quad\quad$ or, $\alpha = \frac{b}{4a}$

Production of roots, $\alpha . 3\alpha = \frac{c}{a}$

or, $3\alpha^2 = \frac{c}{a}$

or, $\frac{3b^2}{16a} = \frac{c}{a}$

or, $3b^2 = 16ac$

$\therefore$ $3b^2 = 16ac$ proved.

5. Solution:

Here,

α and β be the roots of $ax^2 + bx + b = 0$

Then, $\alpha + \beta = \frac{-b}{a}$

$\alpha\beta = \frac{c}{a}$

Now,

$$\text{or, } \frac{\alpha+\beta}{\sqrt{\alpha\beta}} = \frac{-b/a}{\sqrt{\frac{c}{a}}}$$

$$\text{or, } \frac{\alpha}{\sqrt{\alpha\beta}} = \frac{\beta}{\sqrt{\alpha\beta}} = -\sqrt{\frac{b}{a}}$$

$$\text{or, } \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = -\sqrt{\frac{b}{a}}$$

$$\text{or, } \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}} = 0$$

$$\text{i.e., } \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{b}{a}} = 0 \qquad (\because \text{ roots } \alpha, \beta \text{ are in the ratio p:q})$$

6. Solution:

a. Here,

If α, β be the roots of $px^2 + qx + q = 0$

Then,

$$\alpha + \beta = \frac{-q}{p}$$

$$\alpha\beta = \frac{q}{p}$$

$$\text{L.H.S. } \frac{1}{\alpha} + \frac{1}{\beta} + 1$$

$$= \frac{\beta+\alpha}{\alpha\beta} + 1$$

$$= \frac{\frac{-q}{p}}{\frac{q}{p}} + 1$$

$$= -1 + 1$$

$$= 0 \text{ R.H.S.}$$

b. Here, $\alpha + \beta = -\frac{q}{p}$

$$\sqrt{\alpha\beta} = \sqrt{\frac{q}{p}}$$

$$\text{Now, } \frac{\alpha+\beta}{\sqrt{\alpha\beta}} = \frac{\frac{-q}{p}}{\sqrt{\frac{q}{p}}}$$

$$\text{or, } \frac{\alpha}{\sqrt{\alpha\beta}} + \frac{\beta}{\sqrt{\alpha\beta}} = -\sqrt{\frac{q}{p}}$$

$$\text{or, } \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$$

7. Solution:

Let, α be the one root of $ax^2 + bx + c = 0$ then other root be α^2.

$$\alpha + \alpha^2 = \frac{-b}{a} \text{ .... (i)}$$

$$\text{or, } \alpha \cdot \alpha^2 = \frac{c}{a}$$

$$\text{or, } \alpha^3 = \frac{c}{a} \text{ .... (ii)}$$

Cubing on both side of equation (i)

$(\alpha + \alpha^2)^3 = \left(\frac{-b}{a}\right)^3$

or, $\alpha^3 + (\alpha^2)^3 + 3\alpha \,.\, \alpha^2 (\alpha + \alpha^2) = \frac{-b^3}{a^3}$

or, $\alpha^3 + (\alpha^3)^2 + 3\alpha^3 (\alpha + \alpha^2) = \frac{-b^3}{a^3}$

or, $\frac{c}{a} + \frac{c^2}{a^2} \frac{-3bc}{a^2} = \frac{-b^3}{a^3}$

Multiplying each term by a^3

Then,

$a^2c + ac^2 - 3abc = -b^3$

or, $b^3 + a^2c + ac^2 = 3abc$ proved.

8. Solution:

Let, α and β be the two roots of $x^2 + px + q = 0$ $\alpha + \beta = -p$, $\alpha\beta = q$

a. The roots of required equation are

$\alpha\beta - 1$ and $\beta\alpha - 1$

Sum of roots = $\alpha\beta - 1 + \beta\alpha - 1$

$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{p^2 - 2q}{q}$

Product of roots = $\alpha\beta - 1 \times \beta\alpha - 1$

$= \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$

The required equation is

$x^2 -$ (sum of roots) . x + product of roots = 0

or, $x^2 - \frac{(p^2 - 2q)}{q} \,.\, x + 1 = 0$

or, $qx^2 - (p^2 - 2q)\, x + q = 0$

b. Here,

$(\alpha - \beta)^2$ and $(\alpha + \beta)^2$ are the roots of required equation,

Sum of roots = $(\alpha - \beta)^2 + (\alpha + \beta)^2$

$= (\alpha + \beta)^2 - 4\alpha\beta + (\alpha + \beta)^2$

$= 2(\alpha + \beta)^2 - 4\alpha\beta$

$= 2p^2 - 4q$

Product of roots = $(\alpha - \beta)^2 \,.\, (\alpha + \beta)^2$

$= \{(\alpha + \beta)^2 - 4\alpha\beta\} (\alpha + \beta)^2$

$= (p^2 - 4q) \,.\, p^2$

The required equation is

$x^2 -$ (Sum of roots) x + product of roots = 0

or, $x^2 - (2p^2 - 4q) \,.\, x + (p^4 - 4p^2q) \,.\, (2p^2 - 4q) = 0$

or, $x^2 - (2p^2 - 4q) \,.\, x + p^2 (p^2 - 4q) = 0$

or, $x^2 - 2(p^2 - 2q) \,.\, x + p^2 (p^2 - 4q) = 0$

c. $\alpha^2\beta - 1$ and $\beta^2\alpha - 1$ be the two roots of required equation,

Sum of roots = $\alpha^2\beta - 1 + \beta^2\alpha - 1$

$= \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

$= \frac{\alpha^3 + \beta^3}{\alpha\beta}$

$= \frac{(\alpha + \beta)\,(\alpha^2 + \beta^2 - \alpha\beta)}{\alpha\beta}$

$= \frac{-p\,\{(\alpha + \beta)^2 - 3\alpha\beta}{q}$

$= \frac{-p\{p^2 - 3q\}}{q}$

$= \frac{-p^3 + 3pq}{q}$

Product of roots $= \frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha} = \frac{(\alpha\beta)^2}{\alpha\beta} = \alpha\beta = q$

The required equation is,

x^2 – (sum of roots) x + product of roots = 0

or, $x^2 + \frac{(p3 - 3pa) - x}{q} + q = 0$

or, $qx^2 - (p^3 - 3pq)x + q^2 = 0$

9. Solution:

a. Let, the other root be α then,

$\alpha \cdot 3 =$ Product of roots $= \frac{-15}{2}$

or, $\alpha = -\frac{5}{2}$

Sum of roots $= \frac{-k}{2}$

or, $\alpha + 3 = \frac{-k}{2}$

or, $\frac{-k}{2} + 3 = \frac{-k}{2}$

or, $\frac{-5 + 6}{2} = \frac{-k}{2}$

or, $k = -1$

$\therefore$ $k = -1$

b. Given, equation is $3x^2 + kx - 2 = 0$

Sum of roots $= \frac{-k}{3}$

or, $6 = \frac{-k}{3}$

$\therefore$ $k = -18$

c. Given, equation is $2x^2 + (4 - k).\ x - 17 = 0$

If one root = α then other root = $-\alpha$

So that sum of the roots = 0

Sum of roots $= -\frac{4 - k}{2}$

or, $0 = -\frac{4 - k}{2}$

or, $0 = -4 - k$

$\therefore$ $k = 4$

d. Let, one root = α

Another root $= \frac{1}{\alpha}$

Product of roots $= \frac{c}{a}$

or, $\alpha \cdot \frac{1}{\alpha} = \frac{-21}{7k}$

or, $7k = -21$

$\therefore$ $k = -3$

e. Let, α and α^2 be the two roots of $x^2 - kx + 1 = 0$

Sum of roots = $\frac{k}{1}$

or, $\alpha + \alpha^2 = k$

Products of roots = $\frac{1}{1}$

or, $\alpha \,.\, \alpha^2 = 1$

or, $\alpha^3 = 1$, or, $\alpha = 1$, i.e. $1 \times 1^2 = k \Rightarrow k = 2$

10. Solution:

Here, if α and β be the two roots of equations, then,

$\alpha + \beta = 1$, and $\alpha^2 + \beta^2 = 13$

or, $(\alpha + \beta)^2 - 2\alpha\beta = 13$

or, $1 - 2\alpha\beta = 13$

or, $2\alpha\beta = -12$

or, $\alpha\beta = -6$

Now,

Sum of roots = $\alpha + \beta = 1$

Products of roots = -6

The required equation is

x^2 – (sum of roots) . x + product of roots = 0

or, $x^2 - 1.x - 6 = 0$

$\therefore$ $x^2 - x - 6 = 0$

11. Solution:

Let, α and β be the equation of $x^2 + px + q = 0$

$\alpha + \beta = -p$

$\alpha\beta = q$

If the roots of $x^2 + lx + m = 0$ are in same ratio. Let $k\alpha$ and $k\beta$ be the roots of $x^2 + \ell x + m = 0$

Then,

$k\alpha + k\beta = -\ell, \Rightarrow k = \frac{-\ell}{-p} = \frac{\ell}{p}$

$k\alpha \,.\, k\beta = m$

or, $k^2 = \frac{m}{q}$

Now, $\frac{\ell^2}{p^2} = \frac{m}{q}$

or, $p^2m = \ell^2q$

$\therefore$ $p^2m = \ell^2q$ proved.

12. Solution:

Let, α and β be the roots of $\ell x^2 + mx + n = 0$

Then, $\alpha + \beta = \frac{-m}{\ell}$

$\alpha\beta = \frac{n}{\ell}$

Again, Let, α' and β' be the roots of $\ell_1x^2 + m_1x + n_1 = 0$

Then,

$\alpha'\beta' = \frac{-m_1}{\ell_1}$ $\qquad$ $\alpha'\beta' = \frac{n_1}{\ell_1}$

By the question,

$\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}$

By componendo and dividendo,

$\dfrac{\alpha + \beta}{\alpha - \beta} = \dfrac{\alpha' + \beta'}{\alpha' - \beta'}$

or, $\dfrac{(\alpha + \beta)^2}{(\alpha - \beta)^2} = \dfrac{(\alpha' + \beta')^2}{(\alpha' - \beta')^2}$

or, $\dfrac{(\alpha + \beta)^2}{(\alpha + \beta)^2 - (\alpha - \beta)^2} = \dfrac{(\alpha' + \beta')^2}{(\alpha' + \beta')^2 - (\alpha' - \beta')^2}$

or, $\dfrac{(\alpha + \beta)^2}{4\alpha\beta} = \dfrac{(\alpha' + \beta')^2}{4\alpha'\beta'}$

or, $\dfrac{\left(\dfrac{-m}{\ell}\right)^2}{4\dfrac{n}{\ell}} = \dfrac{\left(\dfrac{-m_1}{\ell_1}\right)^2}{\dfrac{n_1}{\ell_1}}$

or, $\dfrac{m^2}{4\ell n} = \dfrac{m1^2}{4\ell_1 n_1}$

or, $\dfrac{m^2}{ln} = \dfrac{m\ell^2}{\ell_1 n_1}$, or, $\dfrac{m^2}{m\ell^2} = \dfrac{\ell n}{\ell_1 n_1}$ proved.

Exercise 5.3

1. Solution:

a. Given, equations are

$2x^2 + x - 3 = 0$ and $3x^2 - 4x + 1 = 0$

Writing the coefficients of order and repeating the first one.

$\begin{matrix} 2 & 1 & 3 & 2 \\ 3 & -4 & 1 & 3 \end{matrix}$

The left hand expression of the condition

$(2 \times (-4) - 3\times1) . (1\times1 - (-4) \times (-3))$

$= (-8 - 3) . (1 - 12)$

$= -11 -11 = 121$

The right hand expression of the condition,

$\{(-3\times3) - 1\times2\}^2 = (-9 - 2)$

$= (-11)^2 = 121$

Since, two results are equal, they have common root.

b. Here, given equations are

$3x^2 - 8x + 4 = 0$ and $4x^2 - 7x - 2 = 0$

Writing the coefficients of order and repeating the first one

$\begin{matrix} 3 & -8 & 4 & 3 \\ 4 & -7 & -2 & 4 \end{matrix}$

The left hand expression of the condition,

$= (3\times(-7) - 4\times(-8) . (-8) . (-2) - (-7) \times 4$

$= (-21 + 32) . (16 + 28)$

$= 11.44 = 484$

The right hand expression of the condition

$(4\times4 - (-2) \times 3)^2$

$= (16 + 6)^2 = (22)^2 = 484$

Since, two results are equal, they have common root.

2. Solution:

Here, given equations are

$3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$

Writing the coefficients of order and repeating the first one

$\begin{matrix} 3 & 4m & 2 & 3 \\ 2 & 3 & -2 & 2 \end{matrix}$

The left hand expression of the condition,

$= (3\times3 - 2\times4m) . (4m . (-2)) - (3\times2)$

$= (9 - 8m) . (-8m - 6)$

$= -72m - 54 + 64m^2 + 48m$
$= -24m + 64m^2 - 54$
The right hand expression of the condition,
$(2\times2 - (-2) \times 3)^2 = (4 + 6)^2 = 100$
$\therefore\quad 64m^2 - 24m - 54 = 100$
or, $\quad 64m^2 - 24m - 154 = 0$
or, $\quad 32m^2 - 12m - 77 = 0$ ... ... (i)
or, Comparing equation (i) with $ax^2 + bx + c = 0$
$\therefore\quad a = 32, b = -12, c = -77$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{12 \pm \sqrt{144 - 4 \times 32 \times (-77)}}{2\times32}$$
$$= \frac{12 \pm 100}{64}$$

Taking +ve,
$$x = \frac{12 + 100}{64} = \frac{7}{4}$$

Taking –ve,
$$x = \frac{12 - 100}{64} = \frac{-11}{8}$$

Here, x is the value of m
So, $m = \frac{7}{4}$ and $\frac{-11}{8}$

3. Solution:

a. Here, given equation are
$4x^2 + px - 12 = 0$ and $4x^2 + 3px - 4 = 0$
Writing the coefficients of order and repeating the first one.

$$\begin{array}{cccc} 4 & p & -12 & 4 \\ 4 & 3p & -4 & 4 \end{array}$$

The left hand expression of the condition,
$= (4 \times 3p - 4p) . (-4p + 36p)$
$= (12p - 4p) . (32p)$
$= 8p . 32p = 256p^2$
The right hand expression of the condition,
$= (-12\times4 - (-4) \times 4)^2$
$= (-48 + 16)^2 = (32)^2 = 1024$
Now, $256p^2 = 1024$
or, $\quad p^2 = 4$
$\therefore\quad p = \pm2$

b. Here,
Given equations are
$2x^2 + px - 1 = 0$ and $3x^2 - 2x - 5 = 0$
Writing the coefficients of order and repeating the first one,

$$\begin{array}{cccc} 2 & p & -1 & 2 \\ 3 & -2 & -5 & 3 \end{array}$$

The left hand expression of the condition,
$= (2\times(-2) - 3p) . (-5p - 2)$
$= (-4 - 3p) . (-5p - 2)$
$= 20p + 8 + 15p^2 + 6p$
$= 26p + 8 + 15p^2$
The right hand expression of the condition,
$= ((-1) \times 3 - (-5) \times 2)^2$
$= (-3 + 10)^2 = 49$

Now,

$15p^2 + 26p + 8 = 49$

or, $15p^2 + 26p - 41 = 0$

or, $15p^2 + 41p - 15p - 41 = 0$

or, $p(15p + 41) - 1p(15p + 41) = 0$

or, $(15p + 41)(p - 1) = 0$

either,

$p = 1,$

or, $p = \frac{-41}{15}$

4. Solution:

Let, α be the common root of the given equations,

$\alpha^2 + p\alpha + q = 0$

$\alpha^2 + p'\alpha + q' = 0$

By using cross multiplication method;

$$\frac{\alpha^2}{pq' - qp'} = \frac{\alpha}{q - q'} = \frac{1}{p' - p}$$

$\therefore \quad \alpha = \frac{pq' - p'q}{q - q'}$, $\alpha = \frac{q - q'}{p' - p}$

$\therefore$ The common root is $\frac{pq' - p'q}{q - q'}$ or $\frac{q - q'}{p' - p}$

5. Solution:

Let, α be the common root of the given equations,

$\alpha^2 + q\alpha + pr = 0$

$\alpha^2 + r\alpha + pq = 0$

By the rule of cross–multiplication method'

$$\frac{\alpha^2}{pq^2 - pr^2} = \frac{\alpha}{pr - pq} = \frac{1}{r - q}$$

$\alpha = \frac{pq^2 - pr^2}{pr - pq}$, $\qquad \alpha = \frac{p(r - q)}{(r - q)}$

$= \frac{p(q - r)(q + r)}{p(r - q)}$ $\qquad \alpha = p$

$= -q - r$

Now,

$-q - r = p$

or, $p = -q - r$

or, $p + q + r = 0$

$\therefore \quad p + q + r = 0$

6. Solution:

Let, α be the common root of the given equations,

$\alpha^2 + b\alpha + ca = 0$

$\alpha^2 + c\alpha + ab = 0$

By the rule of cross multiplication method,

$$\frac{\alpha^2}{ab^2 - ac^2} = \frac{\alpha}{ca - ab} = \frac{1}{c - b}$$

or, $\frac{\alpha^2}{a(b + c)(b - c)} = \frac{\alpha}{-a(b - c)} = \frac{1}{-(b - c)}$

$\therefore \quad \alpha = \frac{-a(b - c)}{-(b - c)}$

Also, $\alpha = \frac{a(b + c)(b - c)}{-a(b - c)} = -(b + c)$

$\therefore \quad a = -(b + c)$

or, $a + b + c = 0$

If β be the other roots of $x^2 + bx + ca = 0$, then $\alpha\beta = \frac{ca}{1}$.

or, $a\beta = ca$

$\therefore$ $\beta = c$

Again,

If γ be the other root of $x^2 + cx + ab = 0$, then $\alpha . \gamma = \frac{ab}{1} = ab$

or, $a . \gamma = \frac{1}{ab}$

$\therefore$ $\gamma = b$

The quadratic equation whose roots are β and γ is

$x2 - (\beta + \gamma) x + \beta\gamma = 0$

or, $x^2 - (c + b) . x + cb = 0$

or, $x^2 - (-a) . x + bc = 0$ $[\because a + b + c = 0]$

or, $x^2 + ax + bc = 0$

$\therefore$ $x^2 + ax + bc = 0$

7. Solution:

Let, α be the common root of the equation,

$\alpha^2 + 2b\alpha + c = 0$

$a\alpha^2 + 2c\alpha + b = 0$

By the rule of cross multiplication method,

$$\frac{\alpha 2}{2b^2 - 2c^2} = \frac{\alpha}{ac - ab} = \frac{1}{2ac - 2ab}$$

$\alpha = \frac{2(b - c)(b + c)}{a(c - b)}$, $\alpha = \frac{a(c - b)}{2a(c - b)}$

$= \frac{2(-b - c)}{a \times 1}$ $\alpha = \frac{1}{2}$

$= \frac{2(-b - c)}{a}$

Now, $\frac{-2b - 2c}{a} = \frac{1}{2}$

or, $-4b - 4c = a$

or, $a = -4b - 4c$

or, $a + 4b + 4c = 0$

8. Solution:

Here, α be the common roots of the given equations, then

$\alpha^2 + p\alpha + q = 0$

$\alpha^2 + q\alpha + p = 0$

By the rule of cross multiplication,

$$\frac{\alpha^2}{p^2 - q^2} = \frac{\alpha}{q - p} = \frac{1}{q - p}$$

$\alpha = \frac{q - p}{q - p}$ and $\alpha = \frac{p^2 - q^2}{q - p}$

$\therefore$ $\frac{p^2 - q^2}{p - p} = \frac{q - p}{q - p}$

or, $p^2 - q^2 = -(p - q)$

or, $(p + q)(p - q) + (p - q) = 0$

or, $(p - q)(p + q + 1) = 0$

either,

$p - q = 0 \therefore p = q$

$p + q + 1 = 0$ P

Chapter 6
Mathematical Induction

Exercise 6.1

1. Find the n^{th} term and then the sum of the first n terms of each of the following series.

a. $1.3 + 2.4 + 3.5 + \ldots$
b. $1 + 4 + 9 + 16 + \ldots$
c. $1.3 + 3.5 + 5.7 + \ldots$
d. $1.2.3 + 2.3.4 + 3.4.5 + \ldots$
e. $1 + (1 + 2) + (1 + 2 + 3) + \ldots$

Solution:

a. Here,

Now, n^{th} term of given series

$t_n = (n^{th}$ term of 1, 2, 3, ...) $\times$ (n^{th} term of 3, 4, 5, ...)

$= [1 + (n - 1). 1] \times [3 + (n-1).1]$

$= n \times (n + 2) = n(n + 2)$

$\therefore \quad t_n = n(n + 2)$

Again, the sum of first n terms of the given series

$s_n = \sum t_n = \sum n(n + 2) = \sum(n^2 + 2n)$

$= \sum n^2 + 2\sum n$

$= \frac{n(n + 1)(2n + 1)}{6} + \frac{2n(n + 1)}{2}$

$= \frac{n(n + 1)(2n + 1 + 6)}{6}$

$= \frac{n(n + 1)(2n + 7)}{6}$

b. Here,

$1 + 4 + 9 + 16 + \ldots = 1^2 + 2^2 + 3^2 + 4^2 + \ldots$

n^{th} term of given series

$t_n = [a + (n-1)d]^2 = [1 + (n - 1).1]^2 = n^2$

Again, let the sum of n natural number

$s_n = \sum t_n = \sum n^2 = \frac{n(n + 1)(2n + 1)}{6}$

c. Here,

n^{th} term of given series

t_n = (nth term of 1, 3, 5, ...) $\times$ (n^{th} term of 3, 5, 7, ...)

$= [1 + (n - 1).2] + p3 + (n - 1).2]$

$= (2n - 1)(2n + 1) = 4n^2 - 1$

$\therefore t_n = 4n^2 - 1$

Again, the sum on of n natural number is

$s_n = \sum t_n = \sum(n^2 - 1) = 4\sum n^2 - \sum 1$

$= \frac{4n(n + 1)(2n + 1)}{6} - n$

$= n\left[\frac{2n(n + 1)(2n + 1) - 3}{3}\right]$

$= \frac{n}{3}[4n^2 + 6n - 1]$

d. Here,

n^{th} term of given series

$t_n = (n^{th}$ term of 1, 2, 3, 4, ...) $\times$ (n^{th} term of 2, 3, 4, 5, ...) $\times$ (n^{th} term of 3, 4, 5, ...)

$= [1 + (n - 1).1] \times [2 + (n-1).1] \times [3 + (n-1).1]$

$= n(n + 1)(n + 2) = n(n^2 + 2n + n + 2) = n^3 + 3n^2 + 2n$

Again, the sum of first n natural number

$s_n = \sum t_n = \sum(n^3 + 3n^2 + 2n)$

$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n + 1)(2n + 1)}{6} + \frac{2n(n + 1)}{2}$

$= \frac{n^2(n + 1)^2}{4} + \frac{n(n + 1)(2n + 1)}{2} + n(n + 1)$

$= \frac{n^2(n + 1)^2 + 2n(n + 1)(2n + 1) + 4n(n + 1)}{4}$

$= \frac{n}{4}[(n + 1)(n + 2)(n + 3)]$

e. Here

$1 + (1 + 2) + (1 + 2 + 3) + ...$

The n^{th} term is $t_n = 1 + 2 + 3 + ...$

$= \frac{n(n + 1)}{2}$ (sum of As) $= \frac{n^2}{2} + \frac{n}{2}$

Now, sum of n term is

$S_n = \frac{1}{2}(\sum n^2 + \sum n) = \frac{1}{2}\left\{\frac{n(n + 1)(2n + 1)}{6} + \frac{n(n + 1)}{2}\right\} = \frac{n(n + 1)(n + 2)}{6}$

2. Sum to n terms of the following series

a. $(x + a) + (x^2 + 2a) + (x^3 + 3a) + \ldots$

b. $5 + 55 + 555 + \ldots$ to n terms.

c. $0.3 + 0.33 + 0.333 + \ldots$ to n terms.

d. $1 + \frac{4}{3} + \frac{7}{3^2} + \frac{10}{3^3} + \ldots$

e. $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \ldots\ldots..$

f. $1\times n + 2 \times (n - 1) + 3 \times (n - 2) + \ldots.$

g. $1 + 3 + 6 + 10 + \ldots$

h. $3 + 6 + 11 + 18 + \ldots$

Solution:

a. $(x + a) + (x^2 + 2a) + (x^3 + 3a) + ...$

Here,

Let $s_n = (x + a) + (x^2 + 2a) + (x^3 + 3a) + ...$ to n term

$= (x + x^2 + x^3 + ... + x^n) + (a + 2a + 3a + ... na)$

$= \frac{n(x^n - 1)}{x -} + a(1 + 2 + 3 + ...$ th$)$

$= \frac{x(x^n - 1)}{x - 1} + \frac{a.n(n + 1)}{2}$

b. Let $s_n = 5 + 55 + 555 + ...$ to n

$= 5(1 + 11 + 111 + ...$ to n$)$

$= \frac{5}{9}(9 + 99 + 999 + ...$ to n$)$

$= \frac{5}{9}[(10 - 1) + (100 - 1) + (1000 - 1) + ...$ to n$]$

$= \frac{5}{9}[(10 + 100 + 1000 + ...$ to n$) - (1 + 1 + 1 ...$ to n$)]$

$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10 - L} - n\right]$

$S_n = \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

c. Here, Let $s_n = 0.3 + 0.33 + 0.333 + ...$ to n

$= \frac{3}{10} + \frac{33}{100} + \frac{333}{1000} + ...$ to n)

$= 3\left(\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + ... \text{ to n}\right)$

$$= \frac{3}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \dots \text{ to } n\right]$$

$$= \frac{1}{3}\left[\frac{(10-1)}{10} + \frac{(100-1)}{100} + \frac{(1000-1)}{1000} + \dots \text{ to } n\right]$$

$$= \frac{1}{3}\left[(1 + 1 + 1 \dots n) - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots n\right)\right]$$

$$= \frac{1}{3}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}\right]$$

$$= \frac{1}{3}\left[n - \frac{10}{90}\left(1 - \frac{1}{10^n}\right)\right]$$

$$= \frac{1}{3}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$$

$$= \frac{n}{3} - \frac{1}{27}\left(1 - \frac{1}{10^n}\right)$$

d. $1 + \frac{4}{3} + \frac{7}{3^2} + \frac{10}{3^3} + \dots$

Solution:

The given series is $1 + \frac{4}{3} + \frac{7}{3^2} + \frac{10}{3^3} + \dots..$

Let S_n be the sum of the series of first n terms

Then,

$$S_n = 1 + \frac{4}{3} + \frac{7}{3^2} + \frac{10}{3^3} + \dots.. + \frac{3n-2}{3^{n-1}} \quad \dots. (1)$$

Also, $\frac{1}{3} S_n = \frac{1}{3} + \frac{4}{3^2} + \frac{7}{3^3} + \dots.. + \frac{3n-5}{3^{n-1}} + \frac{3n-2}{3^n}$ .... (2)

Subtracting (2) from (1) we get,

$$\frac{2}{3} S_n = 1 + 1 + \frac{3}{3^2} + \frac{3}{3^3} + \dots \frac{3}{3^{n-1}} - \frac{3n-2}{3^n}$$

$$= 1 + \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots.. + \frac{1}{3^{n-2}}\right) - \frac{3n-2}{3^n}$$

$$= 1 + 1\left(\frac{1 - \left(\frac{1}{3}\right)^{n-1}}{1 - \frac{1}{3}}\right) - \frac{3n-2}{3^n}$$

$$= 1 + \frac{3}{2} - \frac{1}{2.3^{n-2}} - \frac{3n-2}{3n}$$

$$= \frac{5}{2} - \frac{1}{2} \cdot \frac{1}{3^{n-2}} - \frac{3n-2}{3^n}$$

or, $S_n = \frac{5}{2} \cdot \frac{3}{2} - \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{3^{n-2}} - \frac{3n-2}{3^n} \cdot \frac{3}{2}$

$$= \frac{15}{4} - \frac{3}{4} \cdot \frac{9}{3^n} - \frac{1}{2} \cdot \frac{9n-6}{3^n}$$

$$= \frac{15}{4} - \left(\frac{27 + 18n - 12}{4.3^n}\right)$$

$$= \frac{15}{4} - \frac{15 + 18n}{4.3^n}$$

e. $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \ldots$

Let $s_n = 1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \ldots + \frac{n-1}{2^{n-2}} + \frac{n}{2^{n-1}}$

$\frac{1}{2} s_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots\ldots\ldots\ldots + \frac{n-1}{2^{n-1}} + \frac{n}{2^n}$

Subtracting these two, we get,

$\left(1 - \frac{1}{2}\right) s_n = \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}}\right) - \frac{n}{2^n}$

$\Rightarrow \frac{1}{2} s_n = \frac{1.[1 - (1/2)^n]}{1 - \frac{1}{2}} - \frac{n}{2^n}$

$\Rightarrow \frac{1}{2} s_n = 2\left(1 - \frac{1}{2^n}\right) - \frac{n}{2^n}$

$\Rightarrow s_n = 4\left(1 - \frac{1}{2^n}\right) - \frac{2n}{2^n} = 4 - \frac{4}{2^n} - \frac{n}{2^{n-1}}$

$\therefore s_n = 4 - \frac{1}{2^{n-2}} - \frac{n}{2^{n-1}}$

f. Here, r^{th} term of 1, 2, 3, ... = r and r^{th} term of n, n – 1, n – 2, ...
= n – (r – 1) = n – r + 1
So, the r^{th} term of the series is r(n – r + 1)
$\therefore t_r = nr - r^2 + r$

So, sum $S_n = \sum_{r=1}^{n} tr$

$= n\sum r - \sum r^2 + \sum r$

$= \frac{n.n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}. \left\{n - \frac{2n+1}{3} + 1\right\}$

$= \frac{n(n+1)}{2} . \frac{3n - 2n - 1 + 3}{3}$

$= \frac{n(n+1)(n+2)}{6}$

g. Let t_n be the n^{th} term and S_n the sum of the first n terms of 1 + 3 + 6 + 10 + ...
Then, $S_n = 1 + 3 + 6 + 10 + \ldots \ldots \ldots + t_{n-1} + t_n$
Also, $S_n = \quad 1 + 3 + 6 + \ldots \ldots \ldots + t_{n-2} + t_{n-1} + t_n$
– – – – – – – –

Subtraction yields, $0 = 1 + 2 + 3 + \ldots \ldots \ldots + (t_n - t_{n-1}) - t_n$
or, $t_n = 1 + 2 + 3 + \ldots \ldots \ldots$ to n terms

$= \frac{n(n+1)}{2} = \frac{1}{2} n^2 + \frac{1}{2} n$

Hence, $S_n = \frac{1}{2} \sum n^2 + \frac{1}{2} \sum n$

$= \frac{1}{2}(1^2 + 2^2 + 3^3 + \ldots \ldots + n^2) + \frac{1}{2}(1 + 2 + 3 + \ldots \ldots + n)$

$= \frac{1}{2} \frac{n(n+1)(2n+1)}{6} + \frac{1}{2} \frac{n(n+1)}{2}$

$= \frac{1}{4} n(n+1) \left\{\frac{(2n+1)}{3} + 1\right\}$

$= \frac{1}{4} n(n+1) \frac{(2n+1+3)}{3}$

$$= \frac{n(n+1)(n+2)}{6}$$

h. We have,

$3 + 6 + 11 + 18 + ...$

$= (2^2 - 1) + (3^2 - 3) + (4^2 - 5) + (5^2 - 7) + ...$

$= (2^2 + 3^2 + 4^2 + 5^2 + \text{to n terms}) - (1 + 3 + 5 + 7 + ... \text{ to n terms})$

$= (n + 1)^2 - (2n - 1)$

$= n^2 + 2n + 1 - 2n + 1$

$\therefore \quad t_n = n^2 + 2$

Now, $\Sigma t_n = \Sigma n^2 + \Sigma 2$

$$s_n = \frac{n(n+1)(2n+1)}{6} + 2n$$

$$= \frac{(n^2 + n)(2n + 1) + 12n}{6} = \frac{2n^3 + n^2 + 2n^2 + n + 12n}{6}$$

$$= \frac{2n^3 + 3n^2 + 13n}{6} = \frac{n(2n^2 + 3n + 13)}{6}$$

Exercise 6.2

1. a. If P(n) is the statement "$n^3 + n$ is divisible by 2", prove that P(1), P(2), P(3) and P(4) are true.

b. If P(n) is the statement "$n^2 + n$ is even", Prove that P(1), P(2), P(3) and P(4) are true.

c. If P(n) is the statement "$n^3 \geq 2^n$" show that P(1) is false and P(2), P(3) are true.

d. Let P(n) denote the statement " $\frac{n(n+1)}{6}$ is a natural number". Show that P(2) and P(3) are true but P(4) is not true.

Solution:

a. Here, $P(n) = (n^3 + n)$ is divisible by 2 ... (i)

Putting n =1, 2, 3, and 4 in (i) we get,

$P(1) = 1^3 + 1 = 2$

$P(2) = 2^3 + 1 = 9$

$P(3) = 3^3 + 1 = 28$

$P(4) = 4^3 + 1 = 65$

from above, P(n) is false.

b. Here, $P(n) = 'n^2 + n'$ is even

Put n = 1, 2, 3 and 4

$P(1) = 1^2 + L = 2$

$P(2) = 2^2 + 1 = 5$

$P(3) = 3^2 + 3 = 12$

$P(4) = 4^2 + 3 = 19$

$\therefore$ from above, P(n) is false.

c. Here, $P(n) = n^3 \geq 2^n$

Put $P(1) = 1^3 \geq 2^1 = 1 \geq 2$ which is false.

Put n = 2 and 3

$P(2) = 2^3 \geq 2^2 = 8 \geq 4$

$P(3) = 3^3 \geq 2^3 = 27 \geq 8$

From above P(1) is false and P(2) and P(3) is true.

d. Here,

$P(n) : \frac{n(n+1)}{6}$ is natural number

Putting n = 28384

$\therefore P(2) = \frac{2(n+1)}{6} = \frac{2\times3}{6} = 1$ true

$\therefore P(4) = \frac{4(4+1)}{6} = \frac{4\times6}{5} = \frac{10}{3}$ is false.

$\therefore$ P(3) = $\frac{3(3 + 1)}{6} = \frac{3\times4}{6}$ = 2 true

Hence, from above, P(n) is natural number.

2. Prove by the method of induction that

a. $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n + 1)}{2}$ b. $1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{n(2n - 1)(2n + 1)}{3}$

c. $4 + 8 + 12 + \ldots + 4n = 2n(n + 1)$ d. $1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n - 1)}{2}$

e. $1.2 + 2.3 + 3.4 + \ldots$ to n terms = $\frac{n(n + 1)(n + 2)}{3}$

Solution:

a. If P(n) denotes the given statement, then;

$P(n) = 2 + 5 + 8 + ... + (3n - 1) = \frac{n(3n + 1)}{2}$

When n = 1 then (HS : P(2) = 2

RHS : $\frac{1(3\times1+1)}{2} = 2$

$\therefore$ LHS = RSH i.e. P(1) is true.

Suppose that P(n) is true for some n = k$\in$N

Then $P(k) = 2 + 5 + 8 + ... (3k - 1) = \frac{k(3k + 1)}{2}$ ... (i)

Here, we shall prove that P(k + 1) is true.

Whenever P(k) is true.

For this, adding 3(k + 1) – 1 = 3k + 2 on both sides of (i), we get

$$2 + 5 + 8 + ... + (3k - 1) + (3k + 2) = \frac{k(3k + 1)}{2} + 3k + 2$$

$$= \frac{3k^2 + k + 6k + 4}{2}$$

$$= \frac{3k^2 + 7k + 4}{2} = \frac{3k^2 + 3k + 4k + 4}{2}$$

$$= \frac{(3k + 4)(k + 1)}{2}$$

$$= \frac{(k + 1)[3(k + 1) + 1]}{2}$$

This shows that P(k+1) is true whenever P(k) is true. Hence by the principle of mathematical inclusion, P(n) is true for all n$\in$N.

b. Here, suppose P(n) denotes the given st.

Then, $P(n) = 1^2 + 3^2 + 5^2 + ... + (2n - 1)2 = \frac{n(2n - 1)(2n + 1)}{3}$

When, n = 1, then LHS = P(1) = 1

RHS = $\frac{L(2\times1 - 1)(2\times1 + 1)}{3} = \frac{3}{3} = 1$

Hence, LHS = RHS. This shows that P(n) is true for n = 1. So suppose P(n) is true for n = k$\in$N. so that

$P(k) = 1^2 + 3^2 + 5^2 + ... + (2k - 1)^2 = \frac{k(2k - 1)(2k + 1)}{3}$

Here, we shall prove that the statement P(k+1) is true whenever P(k) is true. For this, adding $(2k+1)^2$ on both sides of (1), we get

$$1^2 + 3^2 + 5^2 + ... + (2k-1)^2 + (2k+1)^2 = \frac{k(2k - 1)(2k + 1)}{2} + (2k + 1)^2$$

$$= \frac{(2k + 1)(2k^2 + 5k + 3)}{3}$$

$$= \frac{(2k + 1)(2k + 3)(k + 1)}{3}$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$= \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

c. Suppose P(n) denotes the given st.

P(n) = 4 + 8 + 12 + ... + 4n = 2n(n + 1)

When n = 1 LHS: P(1) = 4 and RHS: P(1) = $2 \times 1(1 + 1) = 4$

This shows that P(n) is true for n = 1, so suppose P(n) is true for some integer $n = k \in N$, then

P(k) = 4 + 8 + 12 + ... + 4k = 2k(k + 1) ... ... (i)

Here, we shall show that P(k+1) is true whenever P(k) is true.

For this adding 4(k + 1) on both sides of (i), we get,

$$4 + 8 + 12 + ... + 4k + 4k(k + 1) = 2k(k + 1) + 4(k + 1)$$
$$= 2(k + 1)[k + 2]$$
$$= 2(k + 1)[(k + 1) + 1]$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

d. Here,

Suppose P(n) denotes the given st.

$$P(n) = 1 + 4 + 7 + ... + (3n - 2) = \frac{n(3n-1)}{2}$$

When n=1, LHS : P(1) = $3 \times 1 - 2 = 1$

$$\text{RHS: } P(1) = \frac{1(3 \times 1 - 1)}{2} = 1$$

$\therefore$ LHS = RHS

This shows that P(n) is true for n = L, so suppose P(n) is true for some integer $n = k \in N$, then

$$P(k) = 1 + 4 + 7 + ... (3k - 2) = \frac{k(3k-1)}{2} \text{ ... ... (i)}$$

Here, we shall show that P(k + 1) is true whenever P(k) is true

$$1 + 4 + 7 + ... + (3k - 2) + (3k + 1) = \frac{k(3k-1)}{2} + (3k + 1)$$
$$= \frac{k(3k-1) + 2(3k+1)}{2}$$
$$= \frac{3k^2 - k + 6k + 2}{2}$$
$$= \frac{3k^2 + 5k + 2}{2}$$
$$= \frac{3k^2 + 3k + 2k + 2}{2} = \frac{(3k+2)(k+1)}{2}$$
$$= \frac{(k+1)[3(k+1)-1]}{2}$$

This shows that P(k+1) is true whenever P(k) is true. Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

e. Here,

Suppose P(n) denotes the given st.

$$P(n) = 1.2 + 2.3 + 3.4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

When n = 1, then LHS: P(1) = 1(1 + 1) = 2

$$\text{RHS: } P(1) = \frac{1(1+1)(1+2)}{3} = 2$$

$\therefore$ LHS = RHS

This shows that P(n) is true for n = 1. So suppose P(n) is true for some integer n = k$\in$N. then,

$$P(k) = 1.2 + 2.3 + 3.4 + ... + k(k + 1) = \frac{k(k + i)\ (k + 2)}{3} \quad ...\ ...\ ...\ (i)$$

Here, we shall show that P(k + 1) is true whenever P(k) is true for k$\in$N for this purpose, adding, (k+1) (k+2) on both sides (i) we get

$$1.2 + 2.3 + 3.4 + ... + (k+1)\ (k+2) + (k+1) = \frac{k(k+1)\ (k+2)}{3} + (k+1)\ (k+2)$$

$$= (k + 1)\ (k - 12)\left[1 + \frac{k}{3}\right]$$

$$= \frac{(k + 1)\ (k + 2)\ (k + 3)}{3}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathematical induction P(n) is true for all n$\in$N.

3. Prove by the method of induction that

a. $\frac{1}{1.3} + \frac{1}{3.5} + \dots + \frac{1}{(2n-1)\ (2n+1)} = \frac{n}{2n+1}$

b. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

c. $2 + 2^2 + \dots + 2^n = 2\ (2^n - 1)$

d. $3 + 3^2 + \dots + 3^n = \frac{3(3^n - 1)}{2}$

e. $\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots$ to n terms $= \frac{1}{4}\left(1 - \frac{1}{5^n}\right)$.

Solution:

a. Suppose P(n) denotes the given st. then

$$P(n) = \frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2n-1)\ (2n+1)} = \frac{n}{2n+1}$$

When n = 1, then LHS: $P(1) = \frac{1}{(2\times1 - 1)\ (2\times1 + 1)} = \frac{1}{3}$

RHS: $P(1) = \frac{1}{3} \Rightarrow$ LHS = RHS

This show that P(n) is true for n=1, so suppose P(n) is true for some integer n = k$\in$N. then

$$P(k) = \frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2k-1)\ (2k+1)} = \frac{k}{2k+1}$$

Here, we shall show that P(k + 1) is true whenever P(k) is true.

For this adding $\frac{1}{(2k+1)\ (2k+3)}$ on both sides of (i), we get

$$\frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)\ (2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)\ (2k+3)}$$

$$= \frac{k(2k+3) + 1}{(2k+1)\ (2k+3)}$$

$$= \frac{2k^2 + 3k + L}{(2k+1)\ (2k+3)} = \frac{2k^2 + 2k + k + 1}{4k^2 + 8k + 3} = \frac{(2k+1)\ (k+1)}{(2k+1)\ [2(k+1)+1]} = \frac{k+1}{2(k+1)+1}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all n$\in$N.

b. Here, Suppose P(n) denotes the given st. then

$$P(n) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

When n = 1, then LHS: $P(1) = \frac{1}{2'} = \frac{1}{2}$ RHS: $1 - \frac{1}{2'} = \frac{1}{2}$

This shows that P(n) is true for n = 1, so suppose P(n) is true for some integer n = k$\in$N.

Then

$P(k) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^k} = 1 - \frac{1}{2^k}$ ... ... (i)

We shall show that P(k + 1) is true whenever P(k) is true for this adding $\frac{1}{2^{(k+1)}}$ on both side of (i), we get

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$

$$= 1 - \frac{1}{2^k}\left(1 - \frac{1}{2}\right) = 1 - \frac{1}{2^k} : \frac{1}{2} = 1 - \frac{1}{2^{k+1}}$$

This show that P(k+1) is true whenever P(k) is true. Hence by the principle of mathematical induction P(n) is true for all $n \in k$.

c. Here, Suppose P(n) denotes the given st. then

$P(n) = 2 + 2^n + 2^3 + ... + 2^n = 2(2^n - 1)$

When, n = 1, then LHS = P(1) = 2 and RHS = 2

$\therefore$ LHS = RHS. This shows that P(n) is true for n = 1. So suppose P(n) is true for some integer $n=k \in N$. Then,

$P(k) = 2 + 2^2 + 2^3 + ... + 2k = 2(2k - 1)$ ... ... (i)

Here, we shall prove that P(k + 1) is true whenever P(k) is true.

For this, adding 2^{k+1} on both sides of (i), we get

$2 + 2^2 + 2^3 + ... + 2^k + 2^{k+1} = 2(2^k-1) + 2^{k+1}$

$= 2^k.2 - 2 + 2^k.2$

$= 2.2^{k+1} - 2$

$= 2(2^{k+1} - 1)$

This shows that P(k + 1) is true whenever P(k) is true for all $k \in N$. Hence by the principle of mathematical induction P(n) is true for all $n \in N$.

d. Here, Suppose P(n) denotes the given st. then

$P(n) = 3 + 3^2 + ... 3^n = \frac{3(3^2 - 1)}{2}$

When, n = 1, LHS = 3 and RHS 3

$\therefore$ LHS = RHS. This shows that P(n) is true for n = 1. So, suppose P(n) is true for some integer $n = k \in N$. then

$P(k) = 3 + 3^2 + ... + 3^k = \frac{3(3^k - 1)}{2}$ ... ... (i)

Here, we shall prove that P(k+1) is also true whenever P(k) is true for this adding 3^{k+1} on both side of (i) $3+3^2 + ... 3^k + 3^{k+1} = \frac{3(3^k - 1)}{2} + 3^{k+1}$

$= \frac{3.3^k - 3 + 2.3^k.3}{2}$

$= k.\frac{3.3^{k+1} - 3}{2}$

$= \frac{3(3^{k+1} - 1)}{2}$

This shows that P(k + 1) is true whenever P(k) is true for all $k \in N$. Hence by the principle of mathematical induction P(n) is true for all $n \in N$.

e. Suppose P(n) denotes the given st. then

$P(n) = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ...$ to n terms $= \frac{1}{4}\left(1 - \frac{1}{5^n}\right)$

i.e. to = ar^{n-1}

$= \frac{1}{5}\left(\frac{1}{5}\right)^{n-1} = \frac{1}{5^n}$

When, n = 1, LHS = $\frac{1}{5}$ RHS = $\frac{1}{4}\left(1 - \frac{1}{5'}\right) = \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$

∴ LHS = RHS this show that P(n) is true for n = 1. So suppose P(n) is true for some integer n = k∈N. Then,

$$P(k) = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ... + \frac{1}{5^k} = \frac{1}{4}\left(1 - \frac{1}{5^k}\right)$$

Here, we shall prove that P(k+1) is true whenever P(k) is true. For this adding 5^{k+1} on both sides of (i)

$$\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ... + \frac{1}{5^k} + \frac{1}{5^{k+1}} = \frac{1}{4}\left(1 - \frac{1}{5^k}\right) + \frac{1}{5^{k+1}}$$

$$= \frac{1}{4} - \frac{1}{4.5^k} + \frac{1}{5.5^k}$$

$$= \frac{1}{4} + \frac{1}{5.5^k} - \frac{1}{4.5^k}$$

$$= \frac{1}{4} + \frac{4-5}{4.5.5^k}$$

$$= \frac{1}{4} + \frac{-1}{4.5^{k+1}}$$

$$= \frac{1}{4}\left[1 - \frac{1}{5^{k+1}}\right]$$

This shows that P(k+1) is also true whenever P(k) is true for all k∈N. Hence by the principle of mathematical induction P(n) is true for all n∈N.

4. Prove by the method of induction that

a. $4^n - 1$ is divisible by 3.

b. $3^{2n} - 1$ is divisible by 8.

c. $10^{2n-1} + 1$ is divisible by 11.

d. $x^n - y^n$ is divisible by x – y.

e. n(n + 1) (n + 2) is a multiple of 6.

Solution:

a. Here, suppose P(n) denotes the given st. then

P(n) = $4^n - 1$ is divisible by 3

When n = 1, P(1) = $4^1 - 1 = 3$ is divisible by 3. So P(1) is true

Let P(k) be true for k∈N. That is

P(k) : $4_k - 1$ is divisible by 3 ... ... (i)

Now we shall show that P(k+1) is true when P(k) is true.

P(k+1): $4^{k+1} - 1$ is divisible by 3

Now, $(4^{k+1} - 1)$ is divisible by 3. Therefore P(k+1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all n∈N.

$= 1^k . 4 - 4 + 3 = 4(4^k - 1) + 3$

b. Here,

Suppose P(n) be the given st. then P(n): $3^{2n} - 1$ is divisible by 8.

If n = 1. P(1) : $3^2 - 1 = 8$ which is divisible by 8.

So, the statement P(n) is true for n = 1

Let P(k) be true for k∈N, that is

P(k) = $3^{2k} - 1$ is divisible by 8 ... ... (i)

Now, we shall show that P(k+1) is true when P(k) is true i.e. P(k+1): $3^{2(k+1)} - 1$

$= 3^{2k+2} - 1$

$= 3^{2k} . 3^2 - 1$

$= 9.3^{2k} - 1 = 9.32k - 9+8$

$= 9(3^{2k} - 1) + 8$ is divisible by 8.

Thus, P(k + 1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all n∈N.

c. Here,

Let P(n) be given st. then

P(n) : $10^{2n-1} + 1$ is divisible by 11

When n = 1, P(L): $10^{2-1} + 1 = 11$ which is divisible by 11. So P(1) is true.

Let P(k) be true for K∈N. That is

P(k) : $10^{2k-1} + 1$ ... ... (i)

We shall show that P(k+1) is true when P(k) is true i.e. P(k+1): $10^{2(k+1)-1} + 1$
$= 10^{2k+1} + 1 = 10^{2k-1} \cdot 10^2 + 1 = (10^{2k-1} + 1 - 1)10^2 + 1 = 100(10^{2k-1} + 1) - 99$ which is divisible by 11.

d. Here, let P(n) be given st.
i.e. P(n): $x^n - y^n$ is divisible by $x-y$
When n = 1 P(1): $x - y$ is divisible y $x - y$. So P(1) is true.
Let P(k) be true for $k \in N$. i.e.
P(k) : $x^k - y^k$ is divisible by $x-y$ ... ... (i)
Now, we shall show that P(k+1) is true when P(k) is true i.e. P(k+1): $x^{k+1} - y^{k+1}$
$= x(x^k - y^k) + y(x^k - y^k) - xy(x^{k-1} - y^{k-1})$
$= (x + y)(x^k - y^k) - xy(x^{k-1} - y^{k-1})$ is divisible by $x - y$.
Therefore, P(k+1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all $n \in N$.

e. Here,
Let P(n) be given st. then
P(n): n(n+1) (n+2) is multiple of 6.
When n=1, P(1): 1(1+1) (1+2) = 6 is multiple of 6. So P(1) is true
Let P(k) is true for $k \in N$. i.e.
$\therefore$ P(k): k(k+1) (k+2) is multiple of 6 ... ... (i)
Now, we shall show that P(k+1) is true when P(k) is true i.e. P(k+1) : (k+1) (k+2) (k+3) i.
= k(k+1) (k+2) +3(k+1) (k+2) is multiple of 6.
Therefore, P(k+1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all $n \in N$.

Chapter – 7
Matrix Based System of Linear Equations

Exercise 7.1

1. By drawing graph or otherwise, classify each of the following system of the equations.

a. Here,

Given equations are $4x - 3y = -6$ ... (i) and $-4x + 2y = 16$ ... (ii)

Adding equation (i) and (ii), we get

$4x - 3y = -6$

$\underline{-4x + 2y = 16}$

$-y = 10$

$\therefore \quad y = -10$

Putting in equation (i),

$4x - 3x - 10 = -6$

or, $4x = -6 - 30$

$\therefore \quad x = -9$

Hence, (–9, –10) is the solution of the system. This kind of system where we get only one solution is known as consistent and independent.

b. Here,

Given equation of system are,

$2x - y = 3$ ... ... (i)

$-4x + 2y = 6$ ... ... (ii)

Multiplying by 2 in equation (i) and adding with (ii), we get

$4x - 2y = 6$

$\underline{-4x + 2y = 6}$

$0 = 12$

This is impossible result. In other word, the system has no solution. This is an inconsistent and independent.

c. Here,

Given, $-6x + 4y = 10$ ... ... (i)

$3x - 2y = -5$ ... ... (ii)

Multiplying by 2 in equation (ii)and adding with (i), we get

$6x + 4y = 10$

$\underline{6x - 4y = -10}$

$0 = 0$

So, we do not get particular value of x and y. However, the result 0 = 0 is true. In this situation, whatever be the solution of one equation satisfies the other equation as well. This kind of system, where we get infinitely many solution is known as consistent and dependent.

d. Here,

Given, $7x + 2y = 15$ ... ... (i)

$x + y = 5$ .. .. (ii)

Multiplying with 7 in equation (ii) and subtracting (i) from (ii),

$7x + 2y = 15$

$\underline{(-)\ (-)\quad (-)}$

$5y = 20$

$\therefore \quad y = 4$

Putting y = 4 in equation (ii), we get

$x + 4 = 5$

$\therefore \quad x = 1$

Hence, (1, 4) is the solution of the system. This kind of system of solution where only one solution we get is known consistent and independent.

2. Solve the following systems by using row – equivalent matrix method

a. Here,

$x + y = 5$
$2x + 3y = 12$
The augmented matrix is

$$\begin{bmatrix} 1 & 1 & : & 5 \\ 2 & 3 & : & 12 \end{bmatrix}$$

Multiplying by 2 in R_1 and subtracting from R_2.

$$\sim \begin{bmatrix} 1 & 1 & : & 5 \\ 0 & 1 & : & 2 \end{bmatrix}$$

Applying $R_2 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & : & 3 \\ 0 & 1 & : & 2 \end{bmatrix}$$

Hence the solution is $x = 3$ and $y = 2$

b. Here,
Augmented matrix is

$$\begin{bmatrix} 2 & 12 & : & 16 \\ 3 & 10 & : & 8 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 3 & 10 & : & 8 \\ 2 & 12 & : & 16 \end{bmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & : & -8 \\ 2 & 12 & : & 16 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & -2 & : & -8 \\ 0 & 16 & : & 32 \end{bmatrix}$$

Applying $R_2 \to \frac{1}{16} R_2$

$$\sim \begin{bmatrix} 1 & -2 & : & -8 \\ 0 & 1 & : & 2 \end{bmatrix}$$

Applying $R_1 \to R_1 + 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & : & -4 \\ 0 & 1 & : & 2 \end{bmatrix}$$

Hence, the required solution is $x = -4$ and $y = 2$

c. Here,
Augmented matrix is

$$\begin{bmatrix} 1 & -3 & : & -1 \\ 4 & -1 & : & 7 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_1$

$$\sim \begin{bmatrix} 1 & -3 & : & -1 \\ 0 & 11 & : & 11 \end{bmatrix}$$

Applying $R_2 \to \frac{1}{11} R_2$

$$\sim \begin{bmatrix} 1 & -3 & : & -1 \\ 0 & 1 & : & 1 \end{bmatrix}$$

Applying $R_1 \to R_1 + 3R_2$

$$\begin{bmatrix} 1 & 0 & : & 2 \\ 0 & 1 & : & 1 \end{bmatrix}$$

Hence, the required solution is $x = 2$ and $y = 1$

d. Here,

The augmented matrix is

$$\left[\begin{array}{cc:c} 8 & -3 & -31 \\ 2 & 6 & 26 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{cc:c} 2 & 6 & 26 \\ 8 & -3 & -31 \end{array}\right]$$

Applying $R_1 \to \frac{1}{2} R_1$

$$\sim \left[\begin{array}{cc:c} 1 & 3 & 13 \\ 8 & -3 & -31 \end{array}\right]$$

Applying $R_2 \to R_2 - 8R_1$

$$\sim \left[\begin{array}{cc:c} 1 & 3 & 13 \\ 0 & -27 & -13 \end{array}\right]$$

Applying $R_2 \to -\frac{1}{27} R_2$

$$\sim \left[\begin{array}{cc:c} 1 & 3 & 13 \\ 0 & 1 & 5 \end{array}\right]$$

Applying $R_1 \to R_1 - 3R_2$

$$\sim \left[\begin{array}{cc:c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$$

Hence, the required solution is $x = -2$ and $y = 5$

e. Here,

The augmented matrix is

$$\left[\begin{array}{cc:c} 5 & -3 & -2 \\ 4 & 2 & 5 \end{array}\right]$$

Applying $R_1 \to \frac{1}{5} R_1$

$$\sim \left[\begin{array}{cc:c} 1 & -3/5 & -2/5 \\ 4 & 2 & 5 \end{array}\right]$$

Applying $R_2 \to R_2 - 4R_1$

$$\left[\begin{array}{cc:c} 1 & -3/5 & -2/5 \\ 0 & 22/5 & 33/5 \end{array}\right]$$

Applying $R_2 \to \frac{5}{22} R_2$

$$\sim \left[\begin{array}{cc:c} 1 & -3/5 & -2/5 \\ 0 & 1 & 3/2 \end{array}\right]$$

Applying $R_1 \to R_1 + \frac{3}{5} R_2$

$$\sim \left[\begin{array}{cc:c} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \end{array}\right]$$

Hence the required solution is $x = \frac{1}{2}$ and $y = \frac{3}{2}$

f. Here,

The augmented matrix is

$$\left[\begin{array}{cc:c} 2 & 3 & 2 \\ 4 & -5 & 7 \end{array}\right]$$

Applying $R_1 \to \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 1 & 3/2 & : & 1 \\ 4 & -5 & : & 7 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_1$

$$\sim \begin{bmatrix} 1 & 3/2 & : & 1 \\ 0 & -11 & : & 3 \end{bmatrix}$$

Applying $R_2 \to -\frac{1}{11} R_2$

$$\sim \begin{bmatrix} 1 & 3/2 & : & 1 \\ 0 & 1 & : & -3/11 \end{bmatrix}$$

Applying $R_1 \to R_1 - \frac{3}{2} R_1$

$$\sim \begin{bmatrix} 1 & 0 & : & 31/22 \\ 0 & 1 & : & -3/11 \end{bmatrix}$$

Hence, the required solution is $\frac{1}{x} = \frac{31}{22} \Rightarrow x = \frac{22}{31}$

and $\frac{1}{y} = \frac{-3}{11} \Rightarrow y = \frac{-11}{3}$

3. Use the row equivalent matrix method to solve the system of equations:

a. Here,

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 3 & : & -1 \\ 2 & -1 & 2 & : & -4 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - {}_2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 2 & : & -2 \\ 0 & -3 & 0 & : & -6 \end{bmatrix}$$

Applying $R_3 \to R_3 + 3R_2$ and $R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & -1 & : & 3 \\ 0 & 1 & 2 & : & -2 \\ 0 & 0 & 6 & : & -12 \end{bmatrix}$$

Applying $R_3 \to \frac{1}{6} R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 & : & 3 \\ 0 & 1 & 2 & : & -2 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Applying $R_1 \to R_1 + R_3$ and $R_2 \to R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Hence, the required solution is $x = 1$, $y = 2$ and $z = -2$

b. Here,

The augmented matrix is

$$\begin{bmatrix} 1 & 4 & 1 & : & 18 \\ 3 & 3 & -2 & : & 2 \\ 0 & -4 & 1 & : & -7 \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$

$$\sim\begin{bmatrix} 1 & 4 & 1 & : & 18 \\ 0 & -9 & -5 & : & -52 \\ 0 & -4 & 1 & : & -7 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 4 & 1 & : & 18 \\ 0 & 1 & 5/9 & : & 52/9 \\ 0 & -4 & 1 & : & -7 \end{bmatrix}$$

Applying $R_3 \to 4R_2 + R_3$, we get

$$\sim\begin{bmatrix} 1 & 4 & 1 & : & 18 \\ 0 & 1 & 5/9 & : & 52/9 \\ 0 & 0 & 29/9 & : & 145/9 \end{bmatrix}$$

Applying $R_3 \to \frac{9}{29} \times R_3$ we get

$$\sim\begin{bmatrix} 1 & 4 & 1 & : & 18 \\ 0 & 1 & 5/9 & : & 52/9 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Applying $R_2 \to R_2 - \frac{5}{9} R_3$ we get,

$$\sim\begin{bmatrix} 1 & 4 & 1 & : & 18 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Applying $R_1 \to R_1 - R_3$ we get

$$\sim\begin{bmatrix} 1 & 4 & 0 & : & 13 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Applying $R_1 \to R_1 - 4R_2$ we get

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Hence, $x = 1$, $y = 3$, $z = 5$

c. The augmented matrix is

$$\begin{bmatrix} -6 & 9 & 0 & : & 3 \\ 1 & 0 & 1 & : & 1 \\ 0 & 2 & 1 & : & 2 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$

$$\sim\begin{bmatrix} 1 & 0 & 1 & : & 1 \\ -5 & 9 & 0 & : & 3 \\ 0 & 2 & 1 & : & 2 \end{bmatrix}$$

Applying $R_2 \to R_2 + 5R_1$

$$\sim\begin{bmatrix} 1 & 0 & 1 & : & 1 \\ 0 & 9 & 5 & : & 8 \\ 0 & 2 & 1 & : & 2 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_3$

$$\sim\begin{bmatrix} 1 & 0 & 1 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 2 & 1 & : & 2 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$\sim\begin{bmatrix} 1 & 0 & 1 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & -1 & : & 2 \end{bmatrix}$$

Applying $R_3 \to -1R_3$

$$\sim\begin{bmatrix} 1 & 0 & 1 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_1 - R_3$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Hence the solution is x = 3, y = 2 and z = –2

d. The augmented matrix is

$$\begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 1 & -2 & 3 & : & -1 \\ 2 & -2 & 1 & : & -3 \end{bmatrix}$$

Applying $R_{12} \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$\sim\begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & -1 & 1 & : & -1 \\ 0 & 0 & -3 & : & -3 \end{bmatrix}$$

Applying $R_2 \to -1R_2$

$$\sim\begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & -1 & : & 1 \\ 0 & 0 & -3 & : & -3 \end{bmatrix}$$

Applying $R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & -1 & : & -1 \\ 0 & 1 & -1 & : & 1 \\ 0 & 0 & -3 & : & -3 \end{bmatrix}$$

Applying $R_3 \to -\frac{1}{3}R_3$

$$\sim\begin{bmatrix} 1 & 0 & -1 & : & -1 \\ 0 & 1 & -1 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Applying $R_1 \to R_1 + R_3$ and $R_2 \to R_2 + R_3$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Hence, the required solution is x = 0, y = 2, and z = 1

e. The augmented matrix is

$$\begin{bmatrix} 2 & -1 & 4 & : & -3 \\ 1 & 0 & -4 & : & 5 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$

$$\sim\begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 2 & -1 & 4 & : & -3 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 6R_3$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & -1 & 26 & : & -20 \end{bmatrix}$$

Applying $R_2 \to 1R_2$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & -1 & 26 & : & -20 \end{bmatrix}$$

Applying $R_3 \to R_2 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & 0 & 14 & : & -7 \end{bmatrix}$$

Applying $R_3 \to \frac{1}{14} R_3$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & 0 & 1 & : & -1/2 \end{bmatrix}$$

Applying $R_2 \to R_2 + 12R_3$ and $R_1 \to R_1 + 4R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 7 \\ 0 & 0 & 1 & : & -1/2 \end{bmatrix}$$

Hence, $x = 3$, $y = 7$, $z = -\frac{1}{2}$

f. The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & : & 9 \\ 2 & -1 & 2 & : & -8 \\ 3 & -1 & -4 & : & 3 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & : & 9 \\ 0 & -5 & 8 & : & -26 \\ 0 & -7 & 5 & : & -24 \end{bmatrix}$$

Applying $R_2 \to \frac{1}{5} R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 & : & 9 \\ 0 & 1 & -8/5 & : & 26/5 \\ 0 & -7 & 5 & : & -24 \end{bmatrix}$$

Applying $R_3 \to R_3 + 7R_2$ and $R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1/5 & : & -7/5 \\ 0 & 1 & -8/5 & : & 26/5 \\ 0 & 0 & -31/5 & : & 62/5 \end{bmatrix}$$

Applying $R_3 \to -\frac{5}{31} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 1/5 & : & -7/5 \\ 0 & 1 & -8/5 & : & 26/5 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Applying $R_1 \to R_1 - \frac{1}{5} R_3$ and $R_2 \to R_2 + \frac{1}{8} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Hence, the required solution is x = –1, y = 2 and z = –2

g. The augment matrix is

$$\begin{bmatrix} 3 & -2 & -3 & : & -3 \\ 2 & 1 & 1 & : & 6 \\ 1 & 3 & -2 & : & 13 \end{bmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & -3 & -4 & : & -9 \\ 2 & 1 & 1 & : & 6 \\ 1 & 3 & -2 & : & 13 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & -3 & -4 & : & -9 \\ 0 & 7 & 9 & : & 24 \\ 0 & 6 & 2 & : & 22 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 1 & -3 & -4 & : & -9 \\ 0 & 1 & 7 & : & 2 \\ 0 & 6 & 2 & : & 22 \end{bmatrix}$$

Applying $R_1 \to R_1 + 3R_2$ and $R_3 \to R_3 - bR_2$

$$\sim \begin{bmatrix} 1 & 0 & 17 & : & -3 \\ 0 & 1 & 7: & 0 & 2 \\ 0 & 0 & -40 & : & 10 \end{bmatrix}$$

Applying $R_3 \to -\frac{1}{40} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 17 & : & -3 \\ 0 & 1 & 7 & : & 2 \\ 0 & 0 & 1 & : & -1/4 \end{bmatrix}$$

Applying $R_1 \to R_1 - 17R_3 \to R_2 \to R_2 - 7R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 5/4 \\ 0 & 1 & 0 & : & 15/4 \\ 0 & 0 & 1 & : & -1/4 \end{bmatrix}$$

$\therefore \quad x = \frac{5}{4}$, $y = \frac{15}{4}$ and $z = -\frac{1}{4}$

h. The augmented matrix is

$$\sim \begin{bmatrix} 3 & 0 & -5 & : & -7 \\ 3 & 5 & 0 & : & 3 \\ 0 & -3 & 3 & : & 2 \end{bmatrix}$$

Applying $R_1 \to \frac{1}{3} R_1$

$$\sim \begin{bmatrix} 1 & 0 & -5/3 & : & -7/3 \\ 3 & 5 & 0 & : & 3 \\ 0 & -3 & 3 & : & 2 \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$

$$\sim \begin{bmatrix} 1 & 0 & -5/3 & : & -7/3 \\ 0 & 5 & 5 & : & 10 \\ 0 & -3 & 3 & : & 2 \end{bmatrix}$$

Applying $R_2 \to \frac{1}{5} R_2$

$$\sim \begin{bmatrix} 1 & 0 & -5/3 & : & -7/3 \\ 0 & 1 & 1 & : & 2 \\ 0 & -3 & 3 & : & 2 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & -5/3 & : & -7/3 \\ 0 & 1 & 1 & : & 2 \\ 0 & 0 & 6 & : & 8 \end{bmatrix}$$

Applying $R_3 \to \frac{1}{6} R_3$

$$\sim \begin{bmatrix} 1 & 0 & -5/3 & : & -7/3 \\ 0 & 1 & 1 & : & 2 \\ 0 & 0 & 1 & : & 4/3 \end{bmatrix}$$

Applying $R_1 \to R_1 + \frac{5}{3} R_3$ and $R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -1/9 \\ 0 & 1 & 0 & : & 2/3 \\ 0 & 0 & 1 & : & 4/3 \end{bmatrix}$$

Hence, $x = -\frac{1}{9}$, $y = \frac{2}{3}$ and $z = \frac{4}{3}$

Exercise: 7.2

1. Solution:

a. $x + y = 4$

$3x - 2y = 17$

Coe. of x	Coe. of y	Constant
1	1	4
3	–2	17

Now,

$$D = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2 - 3 = -5$$

$$D_1 = \begin{vmatrix} 4 & 1 \\ 17 & -2 \end{vmatrix} = -8 - 17 = -25$$

$$D_2 = \begin{vmatrix} 1 & 4 \\ 3 & 17 \end{vmatrix} = 17 - 12 = 5$$

The solution is $x = \frac{D_1}{D} = \frac{-25}{-5} = 5$

$$y = \frac{D_2}{D} = \frac{5}{-5} = -1$$

b. Let,

Coe. of x	Coe. of y	Constant
2	–1	5
1	–2	1

Now,

$$D = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -4 + 1 = -3$$

$$D_1 = \begin{vmatrix} 5 & -1 \\ 1 & -2 \end{vmatrix} = -10 + 1 = -9$$

$D_2 = \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 2 - 5 = -3$

The solution is, $x = \frac{D_1}{D} = \frac{-9}{-3} = 3$

$y = \frac{D_2}{D} = \frac{-3}{-3} = 1$

c. Let,

Coe. of x	Coe. of y	Constant
3	4	–2
15	20	24

Now,

$D = \begin{vmatrix} 3 & 4 \\ 15 & 20 \end{vmatrix} = 60 - 60 = 0$

$\therefore$ D is negative, the solution does not exist.

d. Let,

Coe. of x	Coe. of y	Constant
5	–3	20
2	5	8

Now,

$D = \begin{vmatrix} 5 & -3 \\ 2 & 5 \end{vmatrix} = 25 + 6 = 31$

$D_1 = \begin{vmatrix} 20 & -3 \\ 8 & 5 \end{vmatrix} = 100 + 24 = 124$

$D_2 = \begin{vmatrix} 5 & 20 \\ 2 & 8 \end{vmatrix} = 40 - 40 = 0$

Now, the solution is, $x = \frac{D_1}{D} = \frac{124}{31} = 4$

$y = \frac{D_2}{D} = \frac{0}{32} = 0$

e. Let,

Coe. of x	Coe. of y	Constant
$\frac{2}{3}$	1	16
1	$\frac{1}{4}$	14

Now,

$D = \begin{vmatrix} \frac{2}{3} & 1 \\ 1 & \frac{1}{4} \end{vmatrix} = \frac{1}{6} - 1 = \frac{-5}{6}$

$D_1 = \begin{vmatrix} 16 & 1 \\ 14 & \frac{1}{4} \end{vmatrix} = 4 - 14 = -10$

$D_2 = \begin{vmatrix} \frac{2}{3} & 16 \\ 1 & 14 \end{vmatrix} = \frac{28}{3} - 16 = \frac{-20}{3}$

The solution is $x = \frac{D_1}{D} = \frac{-10}{-\frac{5}{6}} = 12$

$$y = \frac{D_2}{D} = \frac{-\frac{20}{3}}{-\frac{5}{6}} = 8$$

f. Let,

Coe. of x	Coe. of y	Constant
3	4	10
–2	3	–1

$$D = \begin{vmatrix} 3 & 4 \\ -2 & 3 \end{vmatrix} = 9 + 8 = 17$$

$$D_1 = \begin{vmatrix} 10 & 4 \\ -1 & 3 \end{vmatrix} = 30 + 4 = 34$$

$$D_2 = \begin{vmatrix} 3 & 10 \\ -2 & -1 \end{vmatrix} = -3 + 20 = 17$$

The solution is $x = \frac{D_1}{D} = \frac{34}{17} = 2$

$$\frac{1}{y} = \frac{D_2}{D} = \frac{1}{1} = 1$$

$$\therefore \quad \frac{1}{y} = 1$$

$$y = 1$$

g. Let,

Coe. of x	Coe. of y	Constant
3	–4	–11
2	5	31

$$D = \begin{vmatrix} 3 & -4 \\ 2 & 5 \end{vmatrix} = 15 + 8 = 23$$

$$D_1 = \begin{vmatrix} -11 & -4 \\ 31 & 5 \end{vmatrix} = -55 + 124 = 69$$

$$D_2 = \begin{vmatrix} 3 & -11 \\ 2 & 31 \end{vmatrix} = 93 + 22 = 115$$

The solution is $x = \frac{D_1}{D} = \frac{69}{23} = 3$

$$y = \frac{D_2}{D} = \frac{115}{23} = 5$$

h. Let,

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Coe. of x	Coe. of y	Constant
2	1	7
1	3	11

Now,

$$D = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5$$

$$D_1 = \begin{vmatrix} 7 & 1 \\ 11 & 3 \end{vmatrix} = 21 - 11 = 10$$

$$D_2 = \begin{vmatrix} 2 & 7 \\ 1 & 11 \end{vmatrix} = 22 - 7 = 15$$

Now, the solution is, $x = \frac{D_1}{D} = \frac{10}{5} = 2$

$$y = \frac{D_2}{D} = \frac{15}{5} = 3$$

2. Solution

a. The matrix equation of given system is Ax = B

Where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$

Now,

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & -3 \end{bmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= 1(-3 - 1) + 1(-3 - 2) + 1(1 - 2)$$

$$= -4 - 5 - 1 = -10$$

$\therefore$ $|A| \neq 0$, so A^{-1} exist

Let cofactor of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{11} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -3 - 1 = -4$

$A_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -(-3 - 2) = 5$

$A_{13} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$

$A_{21} = -\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = -(3 - 1) = -2$

$A_{22} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5$

$A_{23} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -(1 + 2) = -3$

$A_{31} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1 - 1) = -2$

$A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -(1 - 1) = 0$

$A_{33} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2$

Co. factor of $A = \begin{bmatrix} -4 & 5 & -1 \\ -2 & -5 & -3 \\ -2 & 0 & 2 \end{bmatrix}$

Adj. of $A = \begin{bmatrix} -4 & 5 & -1 \\ -2 & -5 & -3 \\ -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -2 & -2 \\ 5 & -5 & 0 \\ -1 & -3 & 2 \end{bmatrix}$

The solution given by,

$x = A^{-1} B$

$$= \frac{1}{-10}\begin{bmatrix} -4 & -2 & -2 \\ 5 & -5 & 0 \\ -1 & -3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{-10}\begin{bmatrix} -20 \\ 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ $x = 2, y = -1, z = 1$

b. The matrix equation of system is AX = B

Where $A = \begin{bmatrix} 2 & -3 & -1 \\ 1 & -2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$

Now,

$$|A| = \begin{bmatrix} 2 & -3 & -1 \\ 1 & -2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 2\begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$= 2(-4 -1) +3 (2 + 1) -1 (-1 + 2)$

$= 2x - 5 + 3 \times 3 - 1\times1$

$= -2$

$\therefore$ $|A| \neq 0$, A^{-1} exist

Cofactor of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{11} = \begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} = -4 - 1 = -5$

$A_{12} = -\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = -(2 + 1) = -3$

$A_{13} = \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = -1 + 2 = 1$

$A_{21} = \begin{vmatrix} -3 & -1 \\ -1 & 2 \end{vmatrix} = -(-6 - 1) = 7$

$A_{22} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = (4 \times 1) = 5$

$A_{23} = -\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} = -(-2 + 3) = -1$

$A_{31} = \begin{vmatrix} -3 & -1 \\ -2 & -1 \end{vmatrix} = (3 - 2) = 1$

$A_{32} = -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -(-2 + 1) = 1$

$A_{33} = \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = -4 + 3 = -1$

Now,

Co factor of $A = \begin{bmatrix} -5 & -3 & 1 \\ 7 & 5 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\text{Adj of A} = \begin{bmatrix} -5 & -3 & 1 \\ 7 & 5 & -1 \\ 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -5 & 7 & 1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

The solution given by,

$x = A^{-1} B$

$$= \frac{1}{-2}\begin{bmatrix} -5 & 7 & 1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$\therefore$ $x = 2, y = -1, z = 3$

c. The matrix equation of given system is $Ax = B$

$$\text{where } A = \begin{bmatrix} 3 & 5 & 0 \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ -7 \\ 2 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 3 & 5 & 0 \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{vmatrix} = 3\begin{vmatrix} 0 & -3 \\ 4 & 2 \end{vmatrix} - 5\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}$$

$= 3(12) - 5(4 \neq 0) + 0 = 16$

$\therefore$ $|A| \neq 0$, A^{-1} exist

$$\text{Let cofactor of A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 0 & -3 \\ 4 & 2 \end{vmatrix} = 12$$

$$A_{12} = -\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} = -(4 - 0) = -4$$

$$A_{13} = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8$$

$$A_{21} = \begin{vmatrix} 5 & 0 \\ 4 & 2 \end{vmatrix} = -(10) = -10$$

$$A_{22} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$$

$$A_{23} = \begin{vmatrix} 3 & 5 \\ 0 & 4 \end{vmatrix} = -12$$

$$A_{31} = \begin{vmatrix} 5 & 0 \\ 0 & -3 \end{vmatrix} = -15$$

$$A_{32} = \begin{vmatrix} 3 & 0 \\ 2 & -3 \end{vmatrix} = -(9) = 9$$

$$A_{33} = \begin{vmatrix} 3 & 5 \\ 2 & 0 \end{vmatrix} = 0 - 10 = -10$$

$$\therefore \text{Cofactor of A} = \begin{bmatrix} 12 & -4 & 8 \\ -10 & 6 & -12 \\ -15 & 9 & 10 \end{bmatrix}$$

$$\text{Adj of } A = \begin{bmatrix} 12 & -4 & 8 \\ -10 & 6 & -12 \\ -15 & 9 & 10 \end{bmatrix}^T = \begin{bmatrix} 12 & -10 & -15 \\ -4 & 6 & 9 \\ 8 & -12 & 10 \end{bmatrix}$$

The solution given by,

$X = A^{-1}B$

$$= \frac{1}{10}\begin{bmatrix} 12 & -10 & -15 \\ -4 & 6 & 9 \\ 8 & -12 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ -7 \\ 2 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 64 \\ -32 \\ 120 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$$

$\therefore \quad x = 4, y = -2, z = 5$

d. The matrix equation of given system is AX = B

$$\text{where } A = \begin{bmatrix} 1 & -3 & -7 \\ 2 & 3 & 1 \\ 4 & 1 & 0 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 1 & -3 & -7 \\ 2 & 3 & 1 \\ 4 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$

$= -1 + 3\times-4 - 7(2 - 12) = 57$

$\therefore \quad |A| \neq 0, A^{-1}$ exist

$$\text{Let cofactor of } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$A_{12} = -\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = 4$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 2 - 12 = -10$$

$$A_{21} = \begin{vmatrix} -3 & -7 \\ 1 & 0 \end{vmatrix} = -(7) = -7$$

$$A_{22} = \begin{vmatrix} 1 & -7 \\ 4 & 0 \end{vmatrix} = 28$$

$$A_{23} = \begin{vmatrix} 1 & -3 \\ 4 & 1 \end{vmatrix} = -(1 + 12) - 13$$

$$A_{31} = \begin{vmatrix} -3 & -7 \\ 3 & 1 \end{vmatrix} = -3 + 21 = 18$$

$$A_{32} = -\begin{vmatrix} 1 & -7 \\ 2 & 1 \end{vmatrix} = -(1 + 14) = -15$$

$$A_{33} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 3 + 6 = 9$$

$$\therefore \text{ Cofactor of } A = \begin{bmatrix} -1 & 4 & -10 \\ -7 & 28 & -13 \\ 18 & -15 & 9 \end{bmatrix}$$

$$\text{Adj of A} = \begin{bmatrix} -1 & 4 & -10 \\ -7 & 28 & -13 \\ 18 & -15 & 9 \end{bmatrix}^T = \begin{bmatrix} -1 & -7 & 18 \\ 4 & 28 & -15 \\ -10 & -13 & 9 \end{bmatrix}$$

Now, the solution is given by,

$x = A^{-1} B$

$$= \frac{1}{57}\begin{bmatrix} -1 & -7 & 18 \\ 4 & 28 & -15 \\ -10 & -13 & 9 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix} = \frac{1}{57}\begin{bmatrix} 57 \\ 171 \\ -114 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$\therefore$ x = 1, y = 3, z = –2.15

e. The matrix equation of given system is AX = B.

$$\text{where } A = \begin{bmatrix} 2 & -5 & 0 \\ 0 & 3 & 2 \\ 7 & 0 & -3 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 2 & -5 & 0 \\ 0 & 3 & 2 \\ 7 & 0 & -3 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 0 & -3 \end{vmatrix} + 5\begin{vmatrix} 0 & 2 \\ 7 & -3 \end{vmatrix} + 0$$

$= 2(-9) + 5(-14)$

$= -88$

$\therefore \quad |A| \neq 0, A^{-1}$ exist

$$\text{Let cofactor of A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -3 \end{vmatrix} = -9$$

$$A_{12} = \begin{vmatrix} 0 & 2 \\ 7 & -3 \end{vmatrix} = 14$$

$$A_{13} = \begin{vmatrix} 0 & 3 \\ 7 & 0 \end{vmatrix} = -21$$

$$A_{21} = -\begin{vmatrix} -5 & 0 \\ 0 & -3 \end{vmatrix} = -15$$

$$A_{22} = \begin{vmatrix} 2 & 0 \\ 7 & -3 \end{vmatrix} = -6$$

$$A_{23} = -\begin{vmatrix} 2 & -5 \\ 7 & 0 \end{vmatrix} = -35$$

$$A_{31} = \begin{vmatrix} -5 & 0 \\ 3 & 2 \end{vmatrix} = -10$$

$$A_{32} = -\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = -4$$

$$A_{33} = \begin{vmatrix} 2 & -5 \\ 0 & 3 \end{vmatrix} = 6$$

$$\text{Cofactor of A} = \begin{bmatrix} -9 & 14 & -21 \\ -15 & -6 & -35 \\ -10 & -4 & 6 \end{bmatrix}$$

$$\text{Adj of A} = \begin{bmatrix} -9 & 14 & -21 \\ -15 & -6 & -35 \\ -10 & -4 & 6 \end{bmatrix}^T = \begin{bmatrix} -9 & -15 & -10 \\ 14 & -6 & -4 \\ -21 & -35 & -6 \end{bmatrix}$$

The solution is given by,

$x = A^{-1}B$

$$= \frac{1}{-88}\begin{bmatrix} -9 & -15 & -10 \\ 14 & -6 & -4 \\ -21 & -35 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{88} \\ -\frac{-35}{44} \\ \frac{-15}{88} \end{bmatrix}$$

$\therefore\ x = \frac{1}{88},\ y = \frac{-35}{44},\ z = \frac{-15}{88}$

f. The matrix equation of system is AX = B.

$$\text{where, } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix},\ x = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 7 \\ 3 \\ 8 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}$$

$= (-2 - 1) - 2(4 - 3) + 1(2 + 3)$

$= -3 - 2 + 5$

$= 0$

$\therefore\ |A| = 0$, A^{-1} does not exist.

3. Solution:

a.

Coe. of x	Coe. of y	coe. of z	Constant
2	–3	–1	4
1	–2	–1	1
1	–1	2	9

$$D = \begin{vmatrix} 2 & -3 & -1 \\ 1 & -2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$= 2(-4 - 1) + 3(2 + 1) - 1(-1 + 2)$

$= 2x - 5 + 3\times3 - 1 \times 1$

$= -2$

$$D_1 = \begin{vmatrix} 4 & -3 & -1 \\ 1 & -2 & -1 \\ 9 & -1 & 2 \end{vmatrix} = 4\begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 9 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 9 & -1 \end{vmatrix}$$

$= 4(-4 - 1) + 3(2 + 9) - 1(-1 + 18)$

$= -4$

$$D_2 = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 1 & -1 \\ 1 & 9 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 9 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix}$$

$= 2(2 + 9) -4(2 + 1) -1(9 - 1)$
$= 2$

$$D_3 = \begin{vmatrix} 2 & -3 & 4 \\ 1 & -2 & 1 \\ 1 & -1 & 9 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ -1 & 9 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$= 2(-18+1) + 3(9 - 1) + 4(-1 + 2)$
$= -34 + 24 + 4 = - 6$

The solution is $x = \frac{D_1}{D} = \frac{-4}{-2} = 2$

$y = \frac{D_1}{D} = \frac{2}{-2} = -1$

$z = \frac{D_3}{D} = \frac{-6}{-2} = 3$

b. Let,

Coe. of x	Coe. of y	coe. of z	Constant
1	1	1	–1
3	1	1	1
4	–2	2	0

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 4 & -2 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix}$$

$= (2 + 2) -1 (6 - 4) + (-6 - 4)$
$= -8$

$$D_1 = \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix}$$

$= -1(2 + 2) -1(2 - 0) + 1(-2 - 0)$
$= -4 - 2 - 2$
$= -8$

$$D_2 = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & 1 \\ 4 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix}$$

$= 2 + (6 - 4) + (-4)$
$= 2 + 2 - 4$
$= 0$

$$D_3 = \begin{vmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 4 & -2 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix}$$

$= (0 + 2) -1(-4) -1(-6 - 4)$
$= 2 + 4 + 10$
$= 16$

The solution is $x = x = \frac{D_1}{D} = \frac{-8}{-8} = 1$

$y = \frac{D_2}{D} = \frac{0}{-8} = 0$

$z = \frac{D_3}{D} = \frac{16}{-8} = -2$

c. Let,

Coe. of x	Coe. of y	coe. of z	Constant
0	6	6	–1

8	0	6	–1
4	9	0	8

$$D = \begin{vmatrix} 0 & 6 & 6 \\ 8 & 0 & 6 \\ 4 & 9 & 0 \end{vmatrix} = -6\begin{vmatrix} 8 & 6 \\ 4 & 0 \end{vmatrix} + 6\begin{vmatrix} 8 & 0 \\ 4 & 9 \end{vmatrix}$$

$= -6(-24) + 6\times72$

$= 144 + 432$

$= 576$

$$D_1 = \begin{vmatrix} -1 & 6 & 6 \\ -1 & 0 & 6 \\ 8 & 9 & 0 \end{vmatrix} = -1\begin{vmatrix} 0 & 6 \\ 9 & 0 \end{vmatrix} - 6\begin{vmatrix} -1 & 6 \\ 8 & 0 \end{vmatrix} + 6\begin{vmatrix} -1 & 0 \\ 8 & 9 \end{vmatrix}$$

$= -1(-54) - 6(-48) + 6(-9)$

$= 288$

$$D_2 = \begin{vmatrix} 0 & -1 & 6 \\ 8 & -1 & 6 \\ 4 & 8 & 0 \end{vmatrix} = +1\begin{vmatrix} 8 & 6 \\ 4 & 0 \end{vmatrix} + 6\begin{vmatrix} 8 & -1 \\ 4 & 8 \end{vmatrix}$$

$= -24 + 6(64 + 4)$

$= 384$

$$D_3 = \begin{vmatrix} 0 & 6 & -1 \\ 8 & 0 & -1 \\ 4 & 9 & 8 \end{vmatrix} = -6\begin{vmatrix} 8 & -1 \\ 4 & 8 \end{vmatrix} - 1\begin{vmatrix} 8 & 0 \\ 4 & 9 \end{vmatrix}$$

$= -6(64 + 4) - 1(72 - 0)$

$= -480$

The solution is $x = \frac{D_1}{D} = \frac{288}{576} = \frac{1}{2}$

$y = \frac{D_2}{D} = \frac{288}{576} = \frac{2}{3}$

$z = \frac{D_3}{D} = \frac{-480}{570} = \frac{-5}{6}$

d. Let,

Coe. of x	Coe. of y	coe. of z	Constant
1	–3	–7	6
2	3	1	9
4	1	0	7

$$D = \begin{vmatrix} 1 & -3 & -7 \\ 2 & 3 & 1 \\ 4 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$

$= -1 + 3(-4) - 7(2 - 12)$

$= 57$

$$D_1 = \begin{vmatrix} 6 & -3 & -7 \\ 9 & 3 & 1 \\ 7 & 1 & 0 \end{vmatrix} = 6\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 9 & 1 \\ 7 & 0 \end{vmatrix} + -7\begin{vmatrix} 9 & 3 \\ 7 & 1 \end{vmatrix}$$

$= 6(-1) + 3(0 - 7) - 7(9 - 21)$

$= 57$

$$D_2 = \begin{vmatrix} 1 & 6 & -7 \\ 2 & 9 & 1 \\ 4 & 7 & 0 \end{vmatrix} = 1\begin{vmatrix} 9 & 1 \\ 7 & 0 \end{vmatrix} - 6\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} - 7\begin{vmatrix} 2 & 9 \\ 4 & 7 \end{vmatrix}$$

$= (-7) - 6(-4) - 7(14 - 36)$

$= 171$

$$D_3 = \begin{vmatrix} 1 & -3 & 6 \\ 2 & 3 & 9 \\ 4 & 1 & 7 \end{vmatrix} = 1\begin{vmatrix} 3 & 9 \\ 1 & 7 \end{vmatrix} + 3\begin{vmatrix} 2 & 9 \\ 4 & 7 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$

$= (21 - 9) + 3(14 - 36) + 6(2 - 12)$

$= -114$

The solution is, $x = \frac{D_1}{D} = \frac{57}{57} = 1$

$y = \frac{D_2}{D} = \frac{171}{57} = 3$

$z = \frac{D_3}{D} = \frac{-114}{57} = -2$

4. Solution

a. Let,

Coe. of x	Coe. of y	coe. of z	Constant
1	1	1	6
2	3	5	23
7	5	–2	11

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 7 & 5 & -2 \end{vmatrix} = 1\begin{vmatrix} 3 & 5 \\ 5 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ 7 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 7 & 5 \end{vmatrix}$$

$= (-6 - 25) - 1(-4 - 35) + 2(10 - 21)$

$= -3$

$$D_2 = \begin{vmatrix} 6 & 1 & 1 \\ 23 & 3 & 5 \\ 11 & 5 & -2 \end{vmatrix} = 6\begin{vmatrix} 3 & 5 \\ 5 & -2 \end{vmatrix} - 1\begin{vmatrix} 23 & 5 \\ 11 & -2 \end{vmatrix} + 1\begin{vmatrix} 23 & 3 \\ 11 & 5 \end{vmatrix}$$

$= 6(-6 - 25) - 1(-46 - 55) + 1(115 - 33)$

$= -3$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 23 & 5 \\ 7 & 11 & -2 \end{vmatrix} = 1\begin{vmatrix} 23 & 5 \\ 11 & -2 \end{vmatrix} - 6\begin{vmatrix} 2 & 5 \\ 7 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & 23 \\ 7 & 11 \end{vmatrix}$$

$= (-46 - 55) - 6(-4 - 35) + 2(22 - 16)$

$= -6$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & 23 \\ 7 & 5 & 11 \end{vmatrix} = 1\begin{vmatrix} 3 & 23 \\ 5 & 11 \end{vmatrix} - 1\begin{vmatrix} 2 & 23 \\ 7 & 11 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 7 & 5 \end{vmatrix}$$

$= (33 - 115) - 1(22 - 161) + 6(10 - 21)$

$= -9$

The solution is $x = \frac{D_1}{D} = \frac{-3}{-3} = 1$

$y = \frac{D_2}{D} = \frac{-6}{-3} = 2$

$z = \frac{D_3}{D} = \frac{-9}{3} = 3$

b. Let,

Coe. of x	Coe. of y	coe. of z	Constant
1	4	1	18
3	3	–2	2
0	–4	1	–7

Now,

$$D = \begin{vmatrix} 1 & 4 & 1 \\ 3 & 3 & -2 \\ 0 & -4 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} - 4\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 3 \\ 0 & -4 \end{vmatrix}$$

$= (3 - 8) - 4(3) + (-12)$

$= -29$

$$D_1 = \begin{vmatrix} 18 & 4 & 1 \\ 2 & 3 & -2 \\ -7 & -4 & 1 \end{vmatrix} = 18\begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} - 4\begin{vmatrix} 32 & -2 \\ -7 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ -7 & -4 \end{vmatrix}$$

$= 18(3 - 8) - 4(2 - 14) + 1(-8 + 21)$

$= -29$

$$D_2 = \begin{vmatrix} 1 & 18 & 1 \\ 3 & 2 & -2 \\ 0 & -7 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & -2 \\ -7 & 1 \end{vmatrix} - 18\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 0 & -7 \end{vmatrix}$$

$= (2 - 14) - 18(3) + 1(-21)$

$= -87$

$$D_3 = \begin{vmatrix} 1 & 4 & -18 \\ 3 & 3 & 2 \\ 0 & -4 & -7 \end{vmatrix} = 1\begin{vmatrix} 3 & 2 \\ -4 & -7 \end{vmatrix} - 4\begin{vmatrix} 3 & 2 \\ 0 & -7 \end{vmatrix} + 28\begin{vmatrix} 3 & 3 \\ 0 & -4 \end{vmatrix}$$

$= (-21 + 8) - 4(-21) + 18(-12)$

$= -145$

The solution is $x = \dfrac{D_1}{D} = \dfrac{-29}{-29} = 1$

$y = \dfrac{D_2}{D} = \dfrac{-87}{-29} = 3$

$z = \dfrac{D_3}{D} = \dfrac{-145}{-29} = 5$

Chapter 8: Inverse Circular Functions

Exercise 8

1. Solution:

a. Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$

Then, $\sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\theta}{4}$

$\Rightarrow \quad \theta = \frac{\theta}{4}$

$\therefore \quad \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\theta}{4}$

b. Let $\text{cosec}^{-1}(2) = \theta$

Then $\text{cosec}\theta = 2$

$\frac{1}{\sin\theta} = 2$

$\sin\theta = \frac{1}{2}$

$\sin\theta = \sin\frac{\theta}{6}$

$\Rightarrow \quad \theta = \frac{\theta}{6}$

$\therefore \quad \text{cosec}^{-1}(2) = \frac{\theta}{6}$

c. Let $\cot^{-1}(-\sqrt{3}) = \theta$

Then $\cot\theta = -\sqrt{3}$

$\therefore \quad \tan\theta = -\frac{1}{\sqrt{3}}$

$\tan\theta = \tan\frac{5\theta}{6}$

$\therefore \quad \theta = \frac{5\pi}{6}$

d. Let arc $\tan\left(\frac{2}{\sqrt{3}}\right) = \theta$

$\tan^{-1}\left(\frac{2}{\sqrt{3}}\right) = \theta$

$\Rightarrow \quad \tan\theta = \frac{2}{\sqrt{3}}$

e. Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \theta$

Then, $\sec\theta = \frac{2}{\sqrt{3}}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = \cos\frac{\pi}{6}$

$\therefore \quad \theta = \frac{\pi}{6}$

$\therefore \quad \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

2. Evaluate:

a. $\cos\left(\tan^{-1}\frac{3}{4}\right)$

Let $\tan^{-1}\frac{3}{4} = \theta$

$\tan\theta = \frac{3}{4}$

$\cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right) = \cos\theta = \frac{4}{5}$

b. $\sin(\cot^{-1} x)$

Let $co^{-1} x = \theta$

$\therefore \quad \cot\theta = x$

$\sin(\cot^{-1} x)$

$\sin\theta = \frac{p}{h} = \frac{1}{\sqrt{1 + x^2}}$

c. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

d. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

e. $\cos(2\cot^{-1} x)$

Let $\cot^{-1}x = \theta$

$\therefore \quad \cot\theta = x$

Now, $\cos(2\cot^{-1} x)$

$= \cos2\theta$

$= \frac{\cot^2\theta - 1}{\cot^2\theta + 1} = \frac{x^2 - 1}{x^2 + 1}$

f. $\sin(2 \text{ arc } \tan x)$

$= \sin(2\tan^{-1}x)$

Let $\tan^{-1}x = \theta$

$\tan\theta = x$

$\sin(2\tan^{-1}x) = \sin2\theta$

$$= \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2x}{1 + x^2}$$

3. Solution:

a. $\cos\left[\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right]$

Let $\sin^{-1}\frac{4}{5} = 4$

$\therefore \quad \sin A = \frac{4}{5}$

$\Rightarrow \quad \cos A = \sqrt{1 - \sin^2 A} = \frac{3}{5}$

and $\tan^{-1}\frac{5}{12} = B$

$\therefore$ $\tan B = \frac{5}{12}$

Now, $\cos\left[\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right]$

$= \cos(A + B)$

$= \cos A.\cos B - \sin A.\sin B$

$= \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13}$

$= \frac{36-20}{65}$

$= \frac{16}{65}$

b. $\tan[\tan^{-1}x - \tan^{-1}2y]$

$= \tan\left[\tan^{-1}\left(\frac{x-2y}{1+x.2y}\right)\right]$ $\left[\tan^{-1}A \;\; \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$

$= \frac{x-2y}{1+2xy}$

c. $\sin^{-1}x - \cos^{-1}(-x)$

$\sin^{-1}x - (\pi - \cos^{-1}x)$

$= \sin^{-1}x - \pi + \cos^{-1}x$

$= \sin^{-1}x + \cos^{-1}x - \pi$

$= \frac{\pi}{2} - \pi \Rightarrow -\frac{\pi}{2}$

d. Let $\sin^{-1}\frac{4}{5} = A$

$\sin A = \frac{4}{5}$ $\therefore$ $\cos A = \frac{3}{5}$

and $\cot^{-1}3 = B$

$\cot B = 3$

$\therefore$ $\sin B = \frac{1}{\sqrt{10}}$ $\cos B = \frac{3}{\sqrt{10}}$

Now, $\sin\left(\sin^{-1}\frac{4}{5} + \cot^{-1}3\right)$

$\sin(A + B)$

$= \sin A.\cos B + \cos A.\sin B$

$= \frac{4}{5}\cdot\frac{3}{\sqrt{10}} + \frac{3}{5}\cdot\frac{1}{\sqrt{10}}$

$= \frac{15}{5\sqrt{10}}$

$= \frac{3}{\sqrt{10}}$

e. Let $\cos^{-1}\frac{4}{5} = A$ and $\tan^{-1}\frac{2}{3} = B$

$\therefore$ $\cos A = \frac{4}{5}$ $\tan B = \frac{2}{3}$

$\sin A = \frac{3}{5}$ $\sin B = \frac{2}{\sqrt{13}}$ and $\cos B = \frac{3}{\sqrt{13}}$

Now, $\tan\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right]$

$$= \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}} = \frac{\frac{9+8}{12}}{\frac{12-6}{12}} = \frac{17}{6}$$

4. Solution:

a. Prove that $\sin^{-1}(3x - 4x^3) = 3\sin^{-1}x$

Let $x = \sin\theta$ then $\sin^{-1}x = \theta$

LHS $\sin^{-1}(3x - 4x^3) = \sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin(\sin 3\theta) = 3\theta$

$= 3\theta$

$= 3\sin^{-1}x$ RHS

b. $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x$

Let $x = \cos\theta$

$\therefore \quad \cos^{-1}x = \theta$

Taking LHS:

$\cos^{-1}(4x^3 - 3x) = \cos{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos^3\theta)$

$= 3\theta$

$= 3\cos^{-1}x$ RHS

c. $\tan^{-1}2 - \tan^{-1}1 = \tan^{-1}\left(\frac{1}{3}\right)$

LHS $\tan^{-1}2 = \tan^{-1}1$

$= \tan^{-1}\left(\frac{(2-1)}{1 + 2.1}\right) \qquad \left[\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$

$= \tan^{-1}\left(\frac{1}{3}\right)$ RHS

d. $\sec^{-1}x + \mathrm{cosec}^{-1}x = \frac{\pi}{2}$

Let $\sec^{-1}x = \theta$ then $x = \sec\theta$

$x = \mathrm{cosec}\left(\frac{\pi}{2} - \theta\right)$

$\mathrm{cosec}^{-1}x = \frac{\pi}{2} - \theta$

$\Rightarrow \quad \theta + \mathrm{cosec}^{-1}x = \frac{\pi}{2}$

$\therefore \quad \sec^{-1}x + \mathrm{cosce}^{-1}x = \frac{\pi}{2}$

e. $\mathrm{Tan}^{-1}x = \frac{1}{2}\sin^{-1}x\frac{2x}{1+x^2}$

Let $x = \tan\theta$

$\therefore \quad \tan^{-1}x = \theta$

Now, $\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\sin^{-1}(\sin^2\theta)$

$= \frac{1}{2} \cdot 2\theta$

$= \theta$

$= \tan^{-1}x$

f. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

LHS $\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}\right)+\left(\tan+\frac{1}{7}+\tan^{-1}\frac{1}{8}\right)$

$\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}}\right)+\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}\right)$

$=\tan^{-1}\left(\frac{8}{14}\right)+\tan^{-1}\left(\frac{15}{55}\right)$

$=\tan^{-1}\left(\frac{4}{7}\right)+\tan^{-1}\left(\frac{3}{11}\right)$

$=\tan^{-1}\left(\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{12}{77}}\right)\Rightarrow\tan^{-1}\left(\frac{65}{65}\right)=\tan^{-1}1=\frac{\pi}{4}$

g. $\cot^{-1}3+\text{cosec}^{-1}\sqrt{4}$

$=\tan^{-1}\left(\frac{1}{3}\right)+\text{cosec}^{-1}\sqrt{(5)}$ ... ... (i)

Let $\text{cosec}^{-1}\sqrt{5}=\theta$ then, $\text{cosec}\theta=\sqrt{5}$

$\therefore\ \frac{h}{p}=\frac{\sqrt{5}}{1}$

$\therefore\ h=\sqrt{5}$, $p=1$ then $b=2$

from fig.

$\tan\theta=\frac{1}{2}$

$\therefore\ \theta=\tan^{-1}\frac{1}{2}$

$\text{cosec}^{-1}\sqrt{5}=\tan^{-1}\left(\frac{1}{2}\right)$

from (i),

$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{2}\right)$

$=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{6}}\right)=\tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)=\tan^{-1}1=\frac{\pi}{4}$

h. $\tan^{-1}\frac{m}{n}-\tan^{-1}\left(\frac{m-n}{m+n}\right)$

$\tan^{-1}\left(\frac{\frac{m}{n}-\frac{m-n}{m+n}}{1+\frac{m}{n}\cdot\left(\frac{m-n}{m+n}\right)}\right)=\tan^{-1}\left(\frac{\frac{m^2+mn-mn+n^2}{n(m+n)}}{\frac{mn+n^2+m^2-mn}{n(m+n)}}\right)=\tan^{-1}1=\frac{\pi}{4}$

i. Let $\cos^{-1}x=A\Rightarrow\cos A=x$

$\therefore\ \sin A=\sqrt{1-x^2}$

$\cos^{-1}y=B\Rightarrow\cos B=y$

$\therefore\ \sin B=\sqrt{1-y^2}$

We know that,

$\cos(A - B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B$

$\cos(A - B) = x.y + \sqrt{1 - x^2} \,.\, \sqrt{1 - y^2}$

$\therefore \quad A–B = \cos^{-1}\left(xy + \sqrt{1–x^2} \,.\, \sqrt{1 - y^2}\right)$

$\cos^{-1}x = \cos^{-1}y = \cos^{-1}(xy + \sqrt{1–x^2} \,.\, \sqrt{1–y^2})$

j. We have,

$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}$

$= \sin^{-1}\left\{\frac{3}{5}\sqrt{1 - \frac{25}{169}} + \frac{5}{13}\sqrt{1 - \frac{9}{25}}\right\}$

$= \sin^{-1}\left\{\frac{3}{5} \cdot \frac{12}{13} + \frac{5}{13} \cdot \frac{4}{5}\right\}$

$= \sin^{-1}\left(\frac{36}{65} + \frac{20}{65}\right)$

$= \sin^{-1}\left(\frac{56}{65}\right)$

k. Let $\sin^{-1}\frac{12}{13} = \theta$ and $\cos^{-1}\frac{4}{5} = \beta$

$\therefore \quad \sin\theta = \frac{12}{13}$ then $\cos\beta = \frac{4}{5}$

$\therefore \quad \tan\theta = \frac{12}{5}$ $\therefore \quad \tan\beta = \frac{3}{4}$

$\theta = \tan^{-1}\left(\frac{12}{5}\right)$ $\therefore \quad \beta = \tan^{-1}\left(\frac{3}{4}\right)$

$\therefore \quad \sin^{-1}\left(\frac{12}{13}\right) = \tan^{-1}\left(\frac{12}{5}\right)$ $\cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

Now, $\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$\tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= \tan^{-1}\left(\dfrac{\frac{12}{5} + \frac{3}{4}}{1 - \frac{36}{20}}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= \tan^{-1}\left(\frac{63}{-16}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= \tan^{-1}\left(\frac{-63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= -\tan^{-1}\left(\frac{63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right)$ $[\because \tan^{-1}(-x) = -\tan^{-1}(x)]$

$= 0$

l. $\tan^{-1}\left(\frac{1}{3}\right) + \sec^{-1}\frac{\sqrt{5}}{2}$

Let $\sec^{-1}\left(\frac{\sqrt{5}}{2}\right) = \theta$

$\therefore \quad \sec\theta = \frac{\sqrt{5}}{2}$

$\therefore \quad \frac{h}{b} = \frac{\sqrt{5}}{2}$

$\therefore \quad p = 1$

$\tan\theta = \frac{p}{b} = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right)$

or, $\sec^{-1}\left(\frac{\sqrt{5}}{2}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Therefore, $\tan^{-1}\left(\frac{1}{3}\right) + \sec^{-1}\left(\frac{\sqrt{5}}{2}\right)$

$= \tan^{-1}\left(\frac{1}{3}\right) + \sec^{-1}\left(\frac{\sqrt{5}}{2}\right)$

$= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{6}}\right)$

$= \tan^{-1}1 = \frac{\pi}{4}$

5. Solution:

a. $\cos^{-1}x = \sin^{-1}x = 0$

$\cos^{-1}x = \sin^{-1}x$

or, $\sin^{-1}\sqrt{1 - x^2} = \sin^{-1}x$

or, $\sqrt{1 - x^2} = x$

Squaring both sides

$1 - x^2 = x^2$

$1 = 2x^2$

$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$

b. $\sin^{-1}\frac{x}{2} = \cos^{-1}x$

or, $\sin^{-1}\frac{x}{2} = \sin^{-1}\sqrt{1 - x^2})$

$\therefore \quad \frac{x}{2} = \sqrt{1 - x^2}$

Squaring both sides

or, $\frac{x^2}{4} = 1 - x^2$

or, $5x^2 = 4$

$\therefore \quad x = \pm\frac{2}{\sqrt{5}}$

c. $\cos^{-1}x = \cos^{-1}\frac{1}{2x}$

$\therefore \quad x = \frac{1}{2x}$

$x^2 = \frac{1}{2}$

$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$

d. $\tan^{-1}x - \cot^{-1}x = 0$

$\tan^{-1}x = \cot^{-1}x$

or, $\tan^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad x = \frac{1}{x}$

$\therefore \quad x = \pm 1$

e. $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}1$

or, $\tan^{-1}\left\{\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right\} = \tan^{-1} 1$

or, $\tan^{-1}\left(\frac{x^2 + 2x - x - 2 + x^2 + x - 2x - 2}{x^2 - 4 - x^2 + 1}\right) = \tan^{-1}1$

tan–1 $\left(\frac{2x^2 - 4}{-3}\right) = \tan^{-1}1$

$\therefore \quad \frac{2x^2 - 4}{-3} = 1$

$2x^2 - 4 = -3$

$2x^2 = 1$

$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$

f. $\sin^{-1}2x - \sin^{-1}\sqrt{3}x = \sin^{-1}x$

or, $\sin^{-1} 2x - \sin^{-1}x = \sin^{-1}x\sqrt{3}x$

$\sin^{-1}\left\{2x \,.\, \sqrt{1-x^2} - x \,.\, \sqrt{1 - 4x^2}\right\} = \sin^{-1}\sqrt{3}x$

$2x\sqrt{1 - x^2} - x\sqrt{1 - 4x^2} = \sqrt{3}\,x$

$x\left(2\sqrt{1-x^2} - \sqrt{1 - 4x^2}\right) = \sqrt{3}\,x$

$\therefore \quad x\left(2 - \sqrt{1 - x^2} - \sqrt{1 - 4x^2} - \sqrt{3}\right) = 0$

Either $x = 0$ Or, $2\sqrt{1 - x^2} - \sqrt{1 - 4x^2} - \sqrt{3} = 0$

$2\sqrt{1 - x^2} - \sqrt{3} = \sqrt{1 - 4x^2}$

Squaring both sides, we get,

$x = \frac{1}{2}$

Hence, $x = 0, \frac{1}{2}$. Since $x > 0$, therefore required $x = \frac{1}{2}$

g. The given equation is

$3\tan^{-1}\left(\frac{1}{2 + \sqrt{3}}\right) - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$

or, $3\tan^{-1}(2 - \sqrt{3}) - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$

or, $\tan^{-1}\left\{\dfrac{3(2-\sqrt{3})\ (2-\sqrt{3}^3)}{1-3(2-\sqrt{3})^2}\right\} - \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{x}$

$\therefore$ $3\tan^{-1}x = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

or, $\tan^{-1}\left(\dfrac{12\sqrt{3}-20}{12\sqrt{3}-20}\right) - \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{x}$

or, $\tan^{-1}1 - \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{x}$

or, $\tan^{-1}\left(\dfrac{1-\dfrac{1}{3}}{1+1\dfrac{1}{3}}\right) = \tan^{-1}\dfrac{1}{x}$

or, $\tan^{-1}\left(\dfrac{1}{2}\right) = \tan^{-1}\dfrac{1}{x}$

$\therefore$ $\dfrac{1}{2} = \dfrac{1}{x}$

$\therefore$ $x = 2$

h. $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right) - \cos^{-1}\left(\dfrac{1-b^2}{1+b^2}\right) = 2\tan^{-1}x$

or, $2\text{tn}^{-1}a - 2\tan^{-1}b = 2\tan^{-1}x$

or, $\tan^{-1}a - \tan^{-1}b = \tan^{-1}x$

or, $\tan^{-1}\left(\dfrac{a-b}{1+ab}\right) = \tan^{-1}x$

$\therefore$ $x = \dfrac{a-b}{1+ab}$

i. $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\dfrac{8}{31}\right)$

or, $\tan^{-1}\left\{\dfrac{x+1+x-1}{1-(x+1)(x-1)}\right\} = \tan^{-1}\left(\dfrac{8}{31}\right)$

$\tan^{-1}\left(\dfrac{2x}{1-x^2+1}\right) = \tan^{-1}\left(\dfrac{8}{31}\right)$

$\therefore$ $\dfrac{2x}{2-x^2} = \dfrac{8}{31}$

or, $8 - 4x^2 = 31x$

or, $4x^2 + 31x - 8 = 0$

or, $4x^2 + 32x - x - 8 = 0$

or, $4x(x+8) - 1(x+8) = 0$

$\therefore$ $x = -8$ or $\dfrac{1}{4}$

6. Solution:

a. Let $x = \tan\theta$ then $2\tan^{-1}x = 2\tan^{-1}\tan\theta = 2\theta$

$\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin^2\theta) = 2\theta$

$\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos^2\theta) = 2\theta$

$\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}\tan^2\theta) = 2\theta$

Combining the above results, we get the required result.

Hence, $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

b. $\tan(2\tan^{-1} x) = \tan\left(\tan^{-1}\frac{2x}{1-x^2}\right) = \frac{2x}{1-x^2}$

$2\tan(\tan^{-1}x + \tan^{-1} x^3)$

$= 2\tan \tan^{-1}\left(\frac{x + x^3}{1 - x^4}\right)$

$= 2\frac{x(1 + x^2)}{(1 - x^2)(1 + x^2)}$

$= \frac{2x}{1 - x^2}$

Hence, $\text{tn}(2\tan^{-1} x) = 2\tan(\tan^{-1}x + \tan^{-1} x^3)$

c. LHS $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$

$= \tan^{-1}1 + \tan^{-1}\left(\frac{2+3}{1-6}\right)$

$= \tan^{-1}1 + \tan^{-1}(-1)$

$= \frac{\pi}{4} + \frac{3\pi}{4}$

$= \pi$

RHS, $2\left(\tan^{-1} 1 + \tan^{-1} \text{eq} \frac{1}{2} + \tan^{-1}\frac{1}{3}\right)$

$= 2\left\{\tan^{-1}1 + \tan{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}\right)\right\}$

$= 2\{\tan^{-1}1 + \tan^{-1}1\}$

$= 2\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$

$= \pi$

Hence, LHS = RHS

7. We have,

$\cot^{-1}x + \cot^{-1}y + \cot^{-1}z = \pi$

$\cot^{-1}x + \cot^{-1}y = \pi - \cot^{-1}z$

$\cot^{-1}\left(\frac{xy - 1}{x + y}\right) = \pi - \cot^{-1}z$

or, $\frac{xy - 1}{x + y} = \cot(\pi - \cot^{-1}z)$

or, $\frac{xy - 1}{x + y} = -\cot \cot^{-1}z$

or, $\frac{xy - 1}{x + y} = -z$

$xy - 1 = -xz - yz$

$\therefore \quad xy + yz + zx = 1$

8. Given,

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

$\tan^{-1}\left(\frac{x + y}{1 - x}\right) = \pi - \tan^{-1}z$

or, $\frac{x + y}{1 - xy} = \tan(\pi - \tan^{-1}z)$

or, $\frac{x + y}{1 - xy} = -\tan\tan^{-1}z$ $\quad (\because \tan(\pi - \theta) = -\tan\theta)$

$\frac{x + y}{1 - xy} = -z$

$x + y = -z + xyz$

$\therefore \quad x+y+z = xyz$

9. Let $\sin^{-1}x = A \Rightarrow \sin A = x \quad \therefore \cos A = \sqrt{1 - x^2}$

$\sin^{-1}y = B \Rightarrow \sin B = y \quad \therefore \cos B = \sqrt{1 - y^2}$

$\sin^{-1}z = C \Rightarrow \sin C = z \quad \therefore \cos C = \sqrt{1 - z^2}$

Since,

$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$

i.e. $A + B + C = \pi$

$\therefore \quad A + B = \pi - C$

$\therefore \quad \text{Sin}(A + B) = \sin(\pi - c) = \sin C$

$\cos(A + B) = \cos(\pi - C) = -\cos C$

Now,

Taking, LHS, $x\sqrt{1 - x^2} + y\sqrt{1 - y^2} + z\sqrt{1 - z^2}$

$= \sin A.\ \cos A + \sin B.\ \cos B + \sin C\ .\ \cos C$

$= \frac{1}{2}(\sin 2A + \sin 2B) + \sin C\ .\ \cos C$

$= \frac{1}{2}\ 2\sin\frac{2A + 2B}{2}\ .\ \cos\frac{2A - 2B}{2} + \sin C.\ \cos C$

$= \sin(A + B)\ .\ \cos(A - B) + \sin C\ .\ \cos C$

$= \sin C.\ \cos(A - B) + \sin C.\ \cos C$

$= \sin C\,\{\cos(A - B) + \cos C\}$

$= \sin C\,\{\cos(A - B) - \cos(A + B)\} \quad [\because \cos(A+B) = -\cos C]$

$= \sin C\ .\ 2\sin A.\ \sin B$

$= 2\sin A.\ \sin B.\ \sin C$

$= 2\ x.\ y.\ z = 2xyz$

Hence, $x\sqrt{1 - x^2} + y\sqrt{1 - y^2} + z\sqrt{1 - z^2} = 2xyz$

10. We have,

$\cos^{-1}a + \cos^{-1}b + \cos^{-1}c = \pi$

or, $\cos^{-1}a + \cos^{-1}b = \pi - \cos^{-1}c$

or, $\cos^{-1}\left\{ab - \sqrt{1-a^2}\ .\ \sqrt{1-b^2}\right\} = \pi - \cos^{-1}c$

or, $ab - \sqrt{1 - a^2}\ .\ \sqrt{1 - b^2} = \cos(\pi - \cos^{-1}c)$

$ab - \sqrt{1 - a^2}\ .\ \sqrt{1 - b^2} = -\cos\cos^{-1}c$

$ab - \sqrt{1 - a^2}\sqrt{1 - b^2} = -c$

$ab + c = \sqrt{(1 - a+)(1 - b^2)}$

Squaring both sides

$a^2b^2 + 2ac + c^2 = (1 - a^2)(1 - b^2)$

$a^2b^2 + 2abc + c^2 = 1 - b^2 - a^2 + a^2b^2$

$a^2 + b^2 + c^2 + 2abc = 1$

11. LHS $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c$

$= \tan^{-1}\left(\frac{a + b}{1 - ab}\right) + \tan^{-1}c$

$$= \tan^{-1}\left\{\frac{\frac{a+b}{1-ab}+c}{1-\left(\frac{a+b}{1-ab}\right).c}\right\}$$

$$= \tan^{-1}\left\{\frac{\frac{a+b+c-abc}{1-ab}}{\frac{1-ab-ac-bc}{1-ab}}\right\}$$

$$= \tan^{-1}\left(\frac{a+b+c-abc}{1-ab+bc-ca}\right) \text{ RHS.}$$

12. LHS $\sin^{-1}(\cos \sin^{-1}x) + \cos^{-1}(\sin \cos^{-1}x)$

$$= \sin^{-1}\left(\cos \cos^{-1}\sqrt{1-x^2}\right) + \cos^{-1}\left(\sin \sin^{-1}\sqrt{1-x^2}\right)$$

$$= \sin^{-1}\sqrt{1-x^2} + \cos^{-1}\sqrt{1-x^2}$$

$$= \frac{\pi}{2} \qquad \left(\because \sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}\right) \text{ RHS}$$

13. LHS $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$$= \tan^{-1}\left\{\frac{\cos + \frac{x}{2} - \sin^2\frac{x}{2}}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\right\}$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right) \qquad (\because \text{Dividing by } \cos x/2)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1+\tan\frac{\pi}{4} . \tan\frac{x}{2}}\right)$$

$$= \tan^{-1} \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$= \frac{\pi}{4} - \frac{x}{2} \text{ RHS}$$

Chapter 9
Trigonometric Equations and General Values

Exercise 9

1. Solution:

a. $2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$\cos x = \cos\frac{\pi}{3}$

$\therefore\quad x = \frac{\pi}{3}, \frac{5\pi}{3}$

b. $\sqrt{3}\sec x = 2$

$\sec x = \frac{1}{\sqrt{3}}$

$\cos x = \frac{\sqrt{3}}{2}$

$\therefore\quad \cos x = \cos\frac{\pi}{6}, \cos\left(2\pi - \frac{\pi}{6}\right)$

$\therefore\quad x = \frac{\pi}{6}, \frac{11\pi}{6}$

c. $\tan x = -\frac{1}{\sqrt{3}}$

$\tan x = \tan\left(\pi - \frac{\pi}{6}\right), \tan\left(2\pi - \frac{\pi}{6}\right)$

$\tan x = \tan\frac{5\pi}{6}, \tan\frac{11\pi}{6}$

$\therefore\quad x = \frac{5\pi}{6}, \frac{11\pi}{6}$

d. $\sin x = \frac{1}{\sqrt{2}}$

$\sin x = \sin\frac{\pi}{4}, \sin\left(\pi - \frac{\pi}{4}\right)$

$\sin x = \sin\frac{\pi}{4}, \sin\frac{3\pi}{4}$

$\therefore\quad x = \frac{\pi}{4}, \frac{3\pi}{4}$

2. Solution:

a. $\cos^2 x = \frac{1}{2}$

$\cos^2 x = \cos^2\frac{\pi}{4}$

$\therefore\quad x = n\pi \pm \frac{\pi}{4}$ (Since $\cos^2\theta = \cos^2\propto \Rightarrow \theta = n\pi \pm \propto$)

b. $\cos 3x = -\frac{1}{\sqrt{2}}$

$\cos 3x = \cos\frac{3\pi}{4}$

$\therefore$ The general solution is

$3x = 2n\pi \pm \frac{3\pi}{4} \Rightarrow x = \frac{2n\pi}{3} \pm \frac{\pi}{4}$

c. $\cos 3x = \sin 2x$

$\cos 3x = \cos\left(\frac{\pi}{2} - 2x\right)$

$\therefore$ $3x = 2n\pi \pm \left(\frac{\pi}{2} - 2x\right)$ ($\because \cos\theta = \cos\propto \Rightarrow \theta = 2n\pi \pm \propto$)

$3x = 2n\pi + \frac{\pi}{2} - 2x$ or, $3x = 2n\pi - \frac{\pi}{2} + 2x$

$5x = 2x\pi + \frac{\pi}{2}$ $x = 2n\pi - \frac{\pi}{2}$

$5x = (4n + 1)\frac{\pi}{2}$ $\therefore$ $x = (4n - 1)\frac{\pi}{2}$

$5x = (4n + 1)\frac{\pi}{2}$

$\therefore$ $x = (4x + 1)\frac{\pi}{10}$

Hence, $x = (4n + 1)\frac{\pi}{10}, (4n - 1)\frac{\pi}{10}$

d. $\tan^2 x = \frac{1}{3}$

$\tan^2 x = \left(\frac{1}{\sqrt{3}}\right)^2$

or, $\tan^2 x = \left(\tan\frac{\pi}{6}\right)^2$

$\therefore$ $x = n\pi \pm \frac{\pi}{6}$ ($\because \tan^2\theta = \tan^2\propto \Rightarrow \theta = n\pi \pm \propto$)

3.a. $\sin 2x + \cos x = 0$

or, $2\sin x.\cos x + \cos x = 0$

or, $\cos x\,(2\sin x + 1) = 0$

Either $\cos x = 0$

or, $\sin x = -\frac{1}{2}$

$\therefore$ $x = (2x + 1)\frac{\pi}{2}$

$\sin x = \sin\left(-\frac{\pi}{6}\right)$

$\therefore$ $x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

$\therefore$ $x = (2n + 1)\frac{\pi}{2}, n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

b. $\tan^3 x = 3\tan x = 0$

or, $\tan x\,(\tan^2 x - 3) = 0$

Either $\tan x = 0$ or, $\tan^2 x - 3 = 0$

$\therefore$ $x = n\pi$ $\tan^2 x = (\sqrt{3})^2$

$\tan^2 x = \tan^2\frac{\pi}{3}$

$\therefore \quad x = n\pi \pm \frac{\pi}{3}$

$\therefore \quad x = n\pi,\ n\pi \pm \frac{\pi}{3}$

c. Sinax + cosbx = 0

or, –sinax = cosbx

$\cos bx = \cos\left(\frac{\pi}{2} + ax\right)$

$\therefore \quad bx = 2ns\pi \pm \left(\frac{\pi}{2} + ax\right)$ ($\because \cos\theta = \cos\propto \Rightarrow \theta = 2n\pi \pm \propto \ \forall n \in z$)

Taking positive sign	Taking negative sign
$bx = 2n\pi + \frac{\pi}{2} + ax$	$bx = 2n\pi - \frac{\pi}{2} - ax$
$(b-a)\,x = 2n\pi + \frac{\pi}{2}$	$(b + a)x = 2n\pi - \frac{\pi}{2}$
$\therefore \quad x = 1(4n + 1)\,\frac{\pi}{2}$	$x = \frac{1}{(b+a)}\,(4n - 1)\frac{\pi}{2}$

Hence, $x = \frac{(4n + 1)}{b - a}\,\frac{\pi}{2},\ \frac{(4n - 1)}{b+a}\,\frac{\pi}{2}$

d. tanx + cotx = 2

or, $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 2$

or, $\sin^2x + \cos^2x = 2\sin x \cdot \cos x$

$1 = \sin 2x$

$\therefore \quad \sin 2x = \sin\frac{\pi}{2}$

$\therefore \quad 2x = n\pi \pm (-1)^n\,\frac{\pi}{2}$ ($\because \sin\theta = \sin\propto\ \theta = n\pi \pm (-1)^n \propto,\ \forall n$)

$x = \frac{n\pi}{2} + (-1)^n\,\frac{\pi}{4}$

4. Solution:

a. $4\cos^2x = 6\sin^2x = 5$

$4 - 4\sin^2x + 6\sin^2x = 5$

$2\sin^2x = 1$

$\sin^2x = \left(\frac{1}{\sqrt{2}}\right)^2$

$\sin^2x = \sin^2\frac{\pi}{4}$

$\therefore \quad x = n\pi \pm \frac{\pi}{4}$ ($\because \sin^2\theta = \sin^2\propto \Rightarrow \theta = n\pi \pm \propto\ \forall n \in z$)

b. $\cos^2x - \sin^2x + \cos x = 0$

$\cos^2x - 1 + \cos^2x + \cos x = 0$

or, $2\cos^2x + \cos x - 1 = 0$

or, $2\cos^2x + 2\cos x - \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

either $2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$\therefore$ $\cos x = \cos\frac{\pi}{3}$

$\therefore$ $x = 2n\pi \pm \frac{\pi}{3}$

or, $\cos x + 1 = 0$

$\cos x = -1$

$\cos x = \cos\pi$

$\therefore$ $x = 2n\pi \pm \pi$

c. $3\cos^2x + 5\sin^2x = 4$

or, $3 - 3\sin^2x + \sin^2x = 4$

$2\sin^2x = 1$

$\therefore$ $\sin^2x = \left(\frac{1}{\sqrt{2}}\right)^2$

$\sin^2x = \sin^2\frac{\pi}{4}$

$\therefore$ $x = n\pi \pm \frac{\pi}{4}$

d. $4\sin^2x - 8\cos x + 1 = 0$

$4 - 4\cos^2x - 8\cos x + 1 = 0$

or, $4\cos^2x + 8\cos x - 5 = 0$

or, $4\cos^2x + 10\cos x - 2\cos x - 5 = 0$

or, $2\cos x(2\cos x + 5) - 1(2\cos x + 5) = 0$

$(2\cos x - 1)(2\cos x + 5) = 0$

Either $2\cos x - 1 = 0$ or, $\cos x = -\frac{5}{2}$

$\cos x = \frac{1}{2}$

$\cos x = \cos\frac{\pi}{3}$

$\therefore$ $x = 2n\pi \pm \frac{\pi}{3}$

5. Solution:

a. $\cos x + \cos 2x + \cos 3x = 0$

$(\cos x + \cos 3x) + \cos 2x = 0$

or, $2\cos\left(\frac{x + 3x}{2}\right) . \cos\left(\frac{3x - x}{2}\right) + \cos 2x = 0$

$2\cos 2x . \cos x + \cos 2x = 0$

or, $\cos 2x(2\cos x + 1) = 0$

Either $\cos 2x = 0$ or, $2\cos x + 1 = 0$

$\cos 2x = \cos\frac{\pi}{2}$ $\qquad$ $\cos x = -\frac{1}{2}$

$\therefore$ $2x = (2n + 1)\frac{\pi}{2}$ $\qquad$ $\cos x = \cos\frac{2\pi}{3}$

$\therefore$ $x = (2n+1)\frac{\pi}{4}$ $\qquad$ $\therefore$ $x = 2n\pi \pm \frac{2\pi}{3}$

b. $\text{Sin}3x + \sin x = \sin^2x$

$2\sin\left(\frac{3x + x}{2}\right) . \cos\left(\frac{3x - 3}{2}\right) = \sin^2x$

or, $2\sin 2x . \cos x - \sin^2x = 0$

sin2x (2cosx – 1) = 0

Either sin2x = 0 or, 2 cosx – 1 = 0

$\sin 2x = 0$

$2x = n\pi$

$\therefore \quad x = \frac{n\pi}{3}$

$\cos x = \frac{1}{2}$

$\cos x = \cos\frac{\pi}{3}$

$\therefore \quad x = 2n\pi \pm \frac{\pi}{3}$

$\therefore \quad x = \frac{n\pi}{2}, 2n\pi \pm \frac{\pi}{3}$

c. cos3x + cosx = cos2x

or, 2cos2x . cosx = cos2x

cos2x (2 cosx – 1) = 0

Either cos3x = 0 or, 2cosx – 1 = 0

$\therefore \quad 2x = (2n+1)\frac{\pi}{2}$

$\therefore \quad x = (2n+1)\frac{\pi}{4}$

$\cos x = \frac{1}{2}$

$\cos x = \cos\frac{\pi}{3}$

$\therefore \quad x = 2n\pi \pm \frac{\pi}{3}$

Hence, the general solution

$x = (2n+1)\frac{\pi}{4}, (6n \pm 1)\frac{\pi}{3}$

d. 2tanx – cotx = –1

$2\tan x - \frac{1}{\tan x} = -1$

or, $2\tan^2 x - 1 = -\tan x$

or, $2\tan^2 x + \tan x - 1 = 0$

$2\tan^2 x + 2\tan x - \tan x - 1 = 0$

or, 2tanx (tanx + 1) – 1(tanx + 1) = 0

(tanx + 1) (2tanx – 1) = 0

Either tanx + 1 = 0

tanx = –1

$\tan x = \tan\frac{3\pi}{4}$ or, $\tan\left(-\frac{\pi}{4}\right)$

$\therefore \quad x = n\pi + \frac{3\pi}{4}$ or, $n\pi - \frac{\pi}{4}$

or, 2tanx – 1 = 0

$\tan x = \frac{1}{2}$

$\therefore \quad x = \tan^{-1}\frac{1}{2}$

$\therefore \quad x = n\pi + \tan^{-1}\frac{1}{2}$

Hence, the general solution are

$x = n\pi - \frac{\pi}{4}, n\pi + \tan^{-1}\frac{1}{2}$

6. Solution:

a. $\sqrt{3}\sin x - \cos x = \sqrt{2}$

Dividing each term by 2

$$\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \frac{1}{\sqrt{2}}$$

$$\sin\frac{\pi}{3}\sin x - \cos\frac{\pi}{3}\cos x = \frac{1}{\sqrt{2}}$$

$$-\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$$

or, $\cos\left(x + \frac{\pi}{3}\right) = -\frac{1}{\sqrt{2}}$

$$\cos\left(x + \frac{\pi}{3}\right) = \cos\left(\frac{3\pi}{4}\right)$$

$\therefore$ $x + \frac{\pi}{3} = 2n\pi \pm \frac{3\pi}{4}$

$$x = 2n\pi - \frac{\pi}{3} \pm \frac{3\pi}{4}$$

or, $\cos\frac{\pi}{6}\sin x - \sin\frac{\pi}{6}\cos x = \frac{1}{\sqrt{2}}$

$$\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$$

$$\sin\left(x - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}$$

$\therefore$ $x - \frac{\pi}{6} = n\pi \pm (-1)^n\frac{\pi}{4}$

$\therefore$ $x = n\pi + \frac{\pi}{6} \pm (-1)^n\frac{\pi}{4}$

$$\sin x = \sin\frac{3\pi}{2}$$

$\therefore$ $x = n\pi \pm (-1)^n\frac{3\pi}{2}$

or, $2\sin x = 1$

$$\sin x = \sin\frac{\pi}{6}$$

$\therefore$ $x = n\pi \pm (-1)^n\frac{\pi}{6}$

b. $\tan x + \sec x = \sqrt{3}$

$$\frac{\sin x}{\cos x} + \frac{1}{\cos x} = \sqrt{3}$$

or, $\sin x + 1 = \sqrt{3}\cos x$

or, $\sqrt{3}\cos x - \sin x = 1$ ... ... (i)

Dividing (i) by $\sqrt{\sqrt{3}^2 + (-1)^2} = 2$

$\therefore$ $\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$

or, $\cos\left(x + \frac{\pi}{6}\right) = \cos\left(2x\ \pi \pm \frac{\pi}{3}\right)$

$\Rightarrow$ $x = 2n\pi + \frac{\pi}{6}, 2n\pi - \frac{\pi}{2}$

c. $\text{Sinx} + \sqrt{3}\cos x = \sqrt{2}$

Dividing both sides by

$\sqrt{(\text{coeff. of sim})^2 + (\text{coeff. of cosx})^2} = \sqrt{1 + + (\sqrt{3})^2 = 2}$

$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{1}{\sqrt{2}}$

or, $\sin x . \sin\frac{\pi}{6} + \cos x . \cos\frac{\pi}{6} = \cos\frac{\pi}{4}$

$\cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}$

$\therefore \quad x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$

$\therefore \quad x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$

d. $\sqrt{2}\sec x + \tan x = 1$

$\sqrt{2} + \sin x = \cos x$

or, $\cos x - \sin x = \sqrt{2}$

Dividing both sides by $\sqrt{2}$

$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = 1$

$\cos\frac{\pi}{4} . \cos x - \sin\frac{\pi}{4} . \sin x = \cos 0$

$\therefore \quad \cos\left(x + \frac{\pi}{4}\right) = \cos 0^o$

$\therefore \quad x + \frac{\pi}{4} = 2n\pi \pm 0$

$x = 2n\pi - \frac{\pi}{4}$

e. $\text{Sinx} + \cos x = -\frac{1}{\sqrt{2}}$

Dividing both sides by $\sqrt{2}$

$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2}$

$\cos\left(x - \frac{\pi}{4}\right) = \cos\left(\frac{2\pi}{3}\right)$

$\therefore \quad x - \frac{\pi}{4} = 2n\pi \pm \frac{2\pi}{3}$

$\therefore \quad x = 2n\pi \pm 2n\pi \pm \frac{2\pi}{3} + \frac{\pi}{4}$

7. Solution:

a. $\sin 2x + \sin 4x + \sin 6x = 0$

or, $(\sin 2x + \sin 6x) + \sin 6x = 0$

or, $2\sin 4x . \cos 2x + \sin 4x = 0$

$\sin 4x\,(2\cos 2x + 1) = 0$

Either, $\sin 4x = 0$ or, $2\cos 2\,x + 1 = 0$

$\therefore \quad 4x = n\pi$ $\quad \cos 2x = -\frac{1}{2}$

$\therefore \quad x = \frac{n\pi}{4}$

$\cos 2x = \cos\left(\frac{2\pi}{3}\right)$

$2x = 2n\pi \pm \frac{2\pi}{3}$

$\therefore \quad x = n\pi \pm \frac{\pi}{3}$

$\therefore \quad x = \frac{n\pi}{4}, n\pi \pm \frac{\pi}{3}$

b. $(\sin x + \sin 5x) + \sin 3x = 0$

or, $1\sin 3x \,.\, \cos 2x + \sin 3x = 0$

or, $\sin 3x\,(2\cos 2x + 1) = 0$

Either $\sin 3x = 0$

$\therefore \quad 3x = n\pi$

$\therefore \quad x = \frac{n\pi}{3}$

or, $2\cos 2x + 1 = 0$

$\cos 2x = -\frac{1}{2}$

$\cos 2x = -\frac{1}{2}$

$\cos 2x = \cos\left(\frac{2\pi}{3}\right)$

$\therefore \quad 2x = 2n\pi \pm \frac{2\pi}{3}$

$x = n\pi \pm \frac{\pi}{3}$

Hence, $x = \frac{n\pi}{3}, n\pi \pm \frac{\pi}{3}$

c. $\cos 3x + \cos x - \cos 2x = 0$

or, $2\cos 2x \,.\, \cos x - \cos 2x = 0$

or, $\cos 2x\,(2\cos x - 1) = 0$

Either $\cos 2x = 0$

$\therefore \quad 2x = (2n + 1)\frac{\pi}{2}$

$\therefore \quad x = (2n + 1)\frac{\pi}{4}$

or, $2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$\cos x = \cos\frac{\pi}{3}$

$\therefore \quad x = 2n\pi \pm \frac{\pi}{3}$

Hence, $x = (2n + 1)\frac{\pi}{4}, 2n\pi \pm \frac{\pi}{3}$

d. $\cos x + \sin x = \cos 2x + \sin 2x$

or, $\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}}\cos 2x + \frac{1}{\sqrt{2}}\sin 2x$

or, $\cos\frac{\pi}{4} \,.\, \cos x + \sin\frac{\pi}{4}\sin x = \cos\frac{\pi}{4}\cos 2x + \sin\frac{\pi}{4} \,.\sin 2x$

or, $\cos\left(x - \frac{\pi}{4}\right) = \cos\left(2x - \frac{\pi}{4}\right)$

$\therefore \quad 2x - \frac{\pi}{4} = 2n\pi \pm \left(x - \frac{\pi}{4}\right)$

$$2x - \frac{\pi}{4} = \begin{cases} 2n\pi + x - \frac{\pi}{4} \\ 2n\pi - x + \frac{\pi}{4} \end{cases}$$

Either $x = 2n\pi$

or, $3x = 2n\pi + \frac{\pi}{2}$

i.e. $x = \frac{2n\pi}{3} + \frac{\pi}{6} = (4n + 1)\frac{\pi}{6}$

Hence, $x = 2n\pi, (4nx + 1)\frac{\pi}{6}$

e. $\tan x + \tan 2x = 1 - \tan x . \tan 2x$

or, $\frac{\tan 2x + \tan x}{1 - \tan 2x . \tan x} = 1$

or, $\tan(2x + x) = 1$

$\tan 3x = \tan\frac{\pi}{4}$

$\therefore$ $3x = n\pi + \frac{\pi}{4}$ $(\because \tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha)$

$\therefore$ $x = \frac{n\pi}{3} + \frac{\pi}{12}$

f. $\tan x + \tan 2x + \sqrt{3}\tan x . \tan 2x = \sqrt{3}$

or, $\tan x + \tan 2x = \sqrt{3}(1 - \tan x . \tan 2x)$

or, $\frac{\tan 2x + \tan x}{1 - \tan 2x. \tan x} = \sqrt{3}$

or, $\tan(2x + x) = \sqrt{3}$

$\tan 3x = \tan\left(\frac{\pi}{3}\right)$

$\therefore$ $3x = n\pi + \frac{\pi}{3}$

$\therefore$ $x = \frac{n\pi}{3} + \frac{\pi}{9}$

g. $\tan 3x + \tan 4x + \tan 7x = \tan 3x . \tan 4x . \tan 7x$

$\tan 3x + \tan 4x = -\tan 7x(1 - \tan 3x . \tan 4)$

or, $\frac{\tan 3x + \tan 4x}{1 - \tan 3x . \tan 4x} = -\tan 7x$

$\tan 7x = -\tan 7x$

$2\tan 7x = 0$ $\tan 7x = 0$

$7x = n\pi$

$\therefore$ $x = \frac{n\pi}{7}$

h. $\tan^2 x + (1 - \sqrt{3})\tan x - \sqrt{3} = 0$

or, $\tan^2 x - \sqrt{3}\tan x + \tan x - \sqrt{3} = 0$

$\tan x(\tan x - \sqrt{3}) + 1(\tan x - \sqrt{3}) = 0$

$(\tan x - \sqrt{3})(\tan x + 1) = 0$

Either $\tan x - \sqrt{3} = 0$

$\tan x = \tan\frac{\pi}{3}$

$\therefore$ $x = n\pi + \frac{\pi}{3}$

or, $\tan x = -1$

$\tan x = \tan\left(-\frac{\pi}{4}\right)$

$\therefore \quad x = n\pi - \frac{\pi}{4}$

Hence, $x = n\pi + \frac{\pi}{3}$, $n\pi - \frac{\pi}{4}$ $n \in z$

i. $\tan(\theta + \alpha) . \tan(\theta - \alpha) = 1$

or, $\left(\frac{\tan\theta + \tan\alpha}{1 - \tan\theta . \tan\alpha}\right)\left(\frac{\tan\theta - \tan\alpha}{1 + \tan\theta . \tan\alpha}\right) = 1$

$\frac{\tan^2\theta - \tan^2\alpha}{1 - \tan^2\theta . \tan^2\alpha} = 1$

or, $\tan^2\theta - \tan^2\alpha = 1 - \tan^2\theta . \tan^2\alpha$

or, $\tan^2\theta - \tan^2\theta + \tan^2\theta . \tan^2\alpha = 1$

$\tan^2\theta + \tan^2\theta . \tan^2\theta = 1 + \tan^2\alpha$

$\tan^2\theta(1 + \tan^2\alpha) = (1 + \tan^2\alpha)$

$\tan^2\theta = 1$

$\tan^2\theta \ \ \tan^2\frac{\pi}{4}$

$\therefore \quad \theta = n\pi \pm \frac{\pi}{4}$

8. Solution:

a. $2\sin^2x + 6 - 6\sin^2x = 5$

$2\sin^2x + 6 - 6\sin^2x = 5$

$-4\sin^2x = -1$

$\sin^2x = \left(\frac{1}{2}\right)^2$

$\sin^2x = \left(\sin\frac{\pi}{6}\right)^2$

$\therefore \quad x = n\pi \pm \frac{\pi}{6}$

b. $\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right) = 4$

or, $\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4} . \tan\theta} + \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4} . \tan\theta} = 4$

or, $\frac{1 + \tan\theta}{1 - \tan\theta} + \frac{1 - \tan\theta}{1 + \tan\theta} = 4$

or, $(1 + \tan\theta)^2 + (1 - \tan\theta)^2 = 4(1 + \tan\theta)(1 - \tan\theta)$

or, $1 + 2\tan\theta + \tan^2\theta + 1 - 2\tan\theta + \tan^2\theta = 4 - 4\tan^2\theta$

or, $6\tan^2\theta = 2$

$\tan^2\theta = \left(\frac{1}{\sqrt{3}}\right)^2$

$\therefore \quad \tan^2\theta = \tan2\left(\frac{\pi}{6}\right)$

$\therefore \quad \theta = n\pi \pm \frac{\pi}{6}$, $n \in z$

c. $2\sin^2x + \sin^2 2x = 2$

or, $2\sin^2x + 4\sin^2x \cdot \cos^2x - 2 = 0$

or, $2\sin^2x + 4\sin^2x\,(1-\sin^2x) - 2 = 0$

or, $2\sin^2x + 4\sin^2x\ 4\sin^4x - 2 = 0$

or, $-4\sin^4x - 6\sin^2x - 2 = 0$

or, $2\sin^4x - 3\sin^2x + 1 = 0$

$(\sin^2x - 1)(2\sin^2x - 1) = 0$

Either

$\sin^2x - 1 = 0$ or, $2\sin^2x - 1 = 0$

$\sin^2x = 1$ $\sin^2x = \left(\frac{1}{\sqrt{2}}\right)^2$

$\sin^2x = \sin^2\frac{\pi}{2}$ $\sin^2x = \sin^2\left(\frac{\pi}{4}\right)$

$\therefore\ x = n\pi \pm \frac{\pi}{2}$ $\therefore\ x = n\pi \pm \frac{\pi}{4}$

Hence, $x = n\pi \pm \frac{\pi}{2}, n\pi \pm \frac{\pi}{4}$

d. $\tan px = \cot qx$

$\frac{\sin px}{\cos px} = \frac{\cos qx}{\sin qx}$

or, $\cos px.\cos qx - \sin px \cdot \sin qx = 0$

$\cos(px + qx) = 0$

or, $\cos(p + q)x = 0$

$\therefore\ (p + q)x = (2n + 1)\frac{\pi}{2}$

$\therefore\ x = \frac{(2n + 1)}{p + q} \cdot \frac{\pi}{2}$

9. Solution:

a. $\tan^2x = \tan x\ (-\pi \le x \le \pi)$

or, $\frac{2\tan x}{1 - \tan^2x} = \tan x$

$2\tan x - \tan x\,(1 - \tan^2x) = 0$

$\tan x\,(2 - 1 + \tan^2x) = 0$

$\tan x\,(1 + \tan^2x) = 0$

Either $\tan x = 0$

$\tan x = \tan 0^o, \tan\pi, \tan(-\pi)$

$\therefore\ x = 0^o, \pi, -\pi$

b. $\sin x = \frac{1}{2}$ and $\cos x = -\frac{\sqrt{3}}{2}$ $(0 \le x \le 2\pi)$

Since, sine of an angle is positive and cosine of the same angle is negative, so the angle must lie in the second quadrant.

$\therefore\ x = \pi - \frac{\pi}{6}$ satisfies both equations

$\therefore\ x = \frac{5\pi}{6}$

c. $\tan x - 3\cot x = 2\tan^3x$ $(0 \le x \le 360^o)$

$\tan x - \frac{3}{\tan x} = 2\left(\frac{3\tan x - \tan^3x}{1 - 3\tan^2x}\right)$

or, $\dfrac{\tan^2x - 3}{\tan x} = \dfrac{6\tan x - 2\tan^3x}{1 - 3\tan^2x}$

or, $\tan^2x - 3\tan^4x - 3 + 9\tan^2x = 6\tan^2x - 2\tan^4x$

or, $\tan^3x - 4\tan^2x + 3 = 0$

$\tan^4x - 3\tan^2x - \tan^2x + 3 = 0$

$\tan^2x(\tan^2x - 3) - 1(\tan^2x - 3) = 0$

$(\tan^2x - 1)(\tan^2x - 3) = 0$

Either, $\tan^2x - 1 = 0$

$\Rightarrow$ $\tan^2x = 1$

$\tan x = \pm 1$

$\tan x = \tan\dfrac{\pi}{4}, \tan\dfrac{3\pi}{4}, \tan\dfrac{5\pi}{4}, \tan\dfrac{7\pi}{4}$

$\therefore$ $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

or, $\tan^2x = 3$

$\therefore$ $\tan x = \pm\sqrt{3}$

$\tan x = \tan\dfrac{\pi}{3}, \tan\dfrac{2\pi}{3}, \tan\dfrac{4\pi}{4}, \tan\dfrac{5\pi}{3}$

$\therefore$ $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

Hence, $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

10. Given equations

$\cot x + \cot y = 2$ ... ... (i) and $2\sin x.\sin y = 1$ ... ... (ii)

$\Rightarrow$ $\dfrac{\cos x.\sin y + \sin x\cos y}{\sin x.\sin y} = 2$

or, $\sin x\cos y + \cos x.\sin y = 2\sin x.\sin y$

$\sin(x + y) = 1$ using (ii)

$\sin(x + y) = \sin 90^o$

$\therefore$ $x + y = 90^o$ ... ... (iii)

Also, $2\sin x.\sin y = 1$

or, $\cos(x - y) = \cos(x + y) = 1$

$\cos(x - y) = \cos 90^o = 1$

$\cos(x - y) = 0 = 1$

$\cos(x - y) = 1$

$\cos(x - y) = \cos 0^o$

$\therefore$ $x - y = 0^o$ ... ... (iv)

Solving (iii) and (iv) we get

$x = 45^o = \dfrac{\pi}{4}$ and $y = 45^o = \dfrac{\pi}{4}$

$\therefore$ $x = \dfrac{\pi}{4}, y = \dfrac{\pi}{4}$

Chapter 10: Conic Section

Exercise 10.1

1.

a. $\frac{x^2}{16} + \frac{y^2}{4} = 1 \dots \dots \dots (1)$

Comparing (1) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 16, b^2 = 4$

$\therefore \quad a = 4, b = 2$

Now, eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{16}} = \frac{\sqrt{3}}{2}$

Co-ordinate of vertices = $(\pm a, 0) = (\pm 4, 0)$

Co-ordinate of foci = $(\pm ae, 0) = \left(14 . \sqrt{3}2 , 0\right)$

$= (\pm 2\sqrt{3}, 0)$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 4}{4} = 2$

Major axis = $2a = 2\times4 = 8$

Minor axis = $2b = 2\times2 = 4$

b. $\frac{x^2}{9} + \frac{y^2}{25} = 1 \dots \dots \dots (1)$

Compare (1) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 9, b^2 = 25$

$\therefore \quad a = 3, b = 5$

Now, eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

Co–ordinate of vertices = $(0, \pm b) = (0, \pm 5)$

Co-ordinate of foci = $(0, \pm be)$

$= \left(0, \pm \times \frac{4}{5}\right) = (0, \pm 4)$

Length of latus rectum $= \frac{2a^2}{b} = \frac{2\times9}{5} = \frac{18}{5}$

Major axis = $2b = 2\times5 = 10$

Minor axis = $2a = 2\times3 = 6$

c. $3x^2 + 4y^2 = 36$

or, $\frac{3x^2}{36} + \frac{4y^2}{36} = 1$

or, $\frac{x^2}{12} + \frac{y^2}{9} = 1 \dots \dots \dots (1)$

Comparing (1) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 12, b^2 = 9$

$\therefore \quad a = 2\sqrt{3}, b = 3$

Now, eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{12}} = \frac{\sqrt{3}}{\sqrt{12}} = \frac{1}{2}$

Co–ordinate of vertices = $(\pm a, 0) = (\pm 2\sqrt{3}, 0)$

Co–ordinate of foci = $(\pm ae, 0) = (\pm 2\sqrt{3}\ \frac{1}{2}, 0)$

$= (\pm\sqrt{3}, 0)$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{2\sqrt{3}} = 3\sqrt{3}$

Major axis = $2a = 2\times2\sqrt{3} = 4\sqrt{3}$

Minor axis = $2b = 2\times3 = 6$

d, e are similar to a, b, c

2.

a. Focus = (–2, 0), vertex = (5, 0)

Solution:

Here, a = 5, ae = 2

$\therefore \quad 5e = 2 \Rightarrow e = \frac{2}{5}$

Now, using $b^2 = a^2 (1 - e^2)$

$$= 25\left(1 - \frac{4}{25}\right) = 21$$

So, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{25} + \frac{y^2}{21} = 1$

b. Vertex = (0, 10), eccentricity = $\frac{4}{5}$

Solution: Here, major axis is along the y–axis.

So, $b = 10 \Rightarrow b^2 = 100$ and $e = \frac{4}{5}$

Now, using $a^2 = b^2 (1 - e^2) = 100\left(1 - \frac{16}{25}\right) = 36$

So, the equation of ellipse is $\frac{x^2}{36} + \frac{y^2}{100} = 1$

c. Foci = (± 2, 0), eccentricity = $\frac{1}{2}$

Solution: Here, foci = (± 2, 0) = (± ae, 0)

$\Rightarrow \quad ae = 2$ and $e = \frac{1}{2}$

$\Rightarrow \quad a = \frac{2}{1/2} = 4$

and $e^2 = 1 - \frac{b^2}{16} \Rightarrow \frac{1}{4} = \frac{16 - b^2}{16}$

$\Rightarrow 4 = 16 - b^2$

$\Rightarrow b^2 = 12$

Using equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

or, $3x^2 + 4y^2 = 48$

d. Vertex = (0, 8) and passing through $\left(3, \frac{32}{5}\right)$

Solution: Here, major axis is along the y–axis.

So, b = 8

The equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{64} = 1$

which passes through $\left(3, \frac{32}{5}\right)$, so

$$\frac{9}{a^2} + \frac{(32/5)^5}{64} = 1$$

or, $\frac{9}{a^2} + \frac{1025}{25 \times 64} = 1$

or, $\frac{9}{a^2} + \frac{16}{25} = 1$

or, $\frac{9}{a^2} = \frac{9}{25} \Rightarrow a^2 = 25$

$\therefore$ The equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{64} = 1$

e. Passing through the points (1, 4) and (–3, 2)

Solution: Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ … … … (1)

Since, (1) passes through (1, 4) and (–3, 2), so

$\frac{1}{a^2} + \frac{16}{b^2} = 1$ and $\frac{9}{a^2} + \frac{4}{b^2} = 1$

Solving these two equations, we get

$a^2 = \frac{140}{12} = \frac{35}{3}$

and $b^2 = \frac{140}{8} = \frac{35}{2}$

From equation (1), equation of ellipse is $\frac{x^2}{35/3} + \frac{y^2}{35/2} = 1$

or, $\frac{3x^2}{35} + \frac{2y^2}{35} = 1$

or, $3x^2 + 2y^2 = 35$

3.

a. $\frac{(x+2)^2}{16} + \frac{(y-5)^2}{9} = 1$ … … … (1)

Solution: Comparing (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

We get, $h = -2$, $k = 5$, $a^2 = 16$, $b^2 = 9$

$\therefore$ $a = 4$, $b = 3$

Now, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

The co–ordinate of vertices = $(h \pm a, k)$

= $(-2 \pm 4, 5)$ = (–6, 5) and (2, 5)

So, the co–ordinate of centre = $\left(-\frac{6+2}{2}, \frac{5+5}{2}\right) = (-2, 5)$

And co–ordinate of foci = $(h \pm ae, k)$

$$= \left(-2 \pm 4 \cdot \frac{\sqrt{7}}{4}, 5\right) = (-2 \pm \sqrt{7}, 5)$$

b. $\frac{(x-3)^2}{9} + \frac{(y-5)^2}{25} = 1$

We have,

$\frac{(x-3)^2}{9} + \frac{(y-5)^2}{25} = 1$ which is in the form of $\frac{(x-h)^2}{9} + \frac{(y-k)^2}{25} = 1$

where $h = 3$, $k = 5$, $a^2 = 9$ and $b^2 = 25$

$\therefore$ $a = 3$ and $b = 5$

Since (b) (a) 0. So, the ellipse in along y–axis.

eccentricity (e) = $\sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

Co–ordinate of the center = (h, k) = (3, 5)

Foci of the ellipse = $(h, k \pm be) = \left(3, 5 \pm 5 \times \frac{4}{5}\right)$ = (3, 1) and (3, 9)

c. $x^2 + 4y^2 - 4x + 24y + 24 = 0$

or, $(x-2)^2 + 4(y+3)^2 = 4 + 36 - 24 = 16$

or, $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$ … … … (1)

Comparing (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$a^2 = 16$, $b^2 = 4$, $h = 2$, $k = -3$

Now, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{16}} = \frac{\sqrt{3}}{2}$

Foci = (h ± ae, k) = $\left(2 \pm 4.\frac{\sqrt{3}}{2}, -3\right) = (2 \pm \sqrt{3}, -3)$

and centre (h, k) = (2, –3)

d. We have,

$9x^2 + 5y^2 - 30y = 0$

$\Rightarrow\ 9x^2 + 5(y^2 - 6y) = 0$

$\Rightarrow\ 9x^2 + 5(y^2 - 2.y.3 + 3^2 - 3^2) = 0$

$\Rightarrow\ 9x^2 + 5[(y - 3)^2 - 9] = 0$

$\Rightarrow\ 9x^2 + 5(y - 3)^2 = 45$

Dividing by 45 on both sides, we get

$\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$ which is in the form of $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; where h = 0, k = 3, $a^2 = 5$ and $b^2 = 9$

Since b > a > 0. So, the ellipse is along y–axis.

Hence,

Eccentricity (e) = $\sqrt{1 - a^2/b^2} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

Co–ordinate of the center = (h, k) = (0, 3)

Foci of the ellipse = (h, k ± be) = $\left(0, 3 \pm 3 \times \frac{2}{3}\right)$ = (0,5) and (0, 1)

e. We have,

$9x^2 + 4y^2 + 40y + 18x + 73 = 0$

$\Rightarrow\ (9x^2 + 18x) + (4y^2 + 40y) + 73 = 0$

$\Rightarrow\ 9[x^2 + 2.x.1 + 1^2 - 1^2] + 4[y^2 + 2.5.y + 5^2 - 5^2] + 73 = 0$

$\Rightarrow\ 9[(x + 1)^2 - 1] + 4[(y + 5)^2 - 25] + 73 = 0$

$\Rightarrow\ 9(x + 1)^2 - 9 + 4(y + 5)^2 - 100 + 73 = 0$

$\Rightarrow\ 9(x + 1)^2 + 4(y + 5)^2 = 36$

$\Rightarrow\ \frac{(x+1)^2}{4} + \frac{(y+5)^2}{9} = 1$; which is in the form of $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; where h = –1, k = –5, $a^2 = 4$ and $b^2 = 9$

$\therefore$ a = 2 and b = 3

Since b > a > 0. So, the ellipse is along y–axis

Hence,

eccentricity (e) = $\sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

Co–ordinate of the center (h, k) = (–1, –5)

Foci of the ellipse = (h, k ± be) = $\left(-1, -5 \pm 3 \times \frac{\sqrt{5}}{3}\right) = (-1, -5 \pm \sqrt{5})$

4.

a. Major axis is twice its minor axis and which passes through the point (0, 1)

Solution:

The equation of the ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ … … … (1)

It is given that a = 2b and ellipse passes through (0, 1)

So, $\frac{0}{a^2} + \frac{y^2}{b^2} = 1$

or, $b^2 = 1$ $\therefore$ b = 1

and a = 2b = 2

from (1), $\frac{x^2}{4} + \frac{y^2}{1} = 1$

or, $x^2 + 4y^2 = 4$ is the required equation of an ellipse.

b. Latus rectum 3 and eccentricity is $\frac{1}{\sqrt{2}}$

Solution: Here, equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

It is given, length of latus rectum = 3

or, $\frac{2b^2}{a} = 3 \quad \therefore b^2 = \frac{3a}{2}$

Using $e^2 = 1 - \frac{b^2}{a^2}$ or, $\frac{1}{2} = 1 - \frac{3a}{2a^2}$

or, $a = 2a - 3$ or, $a = 3$

and $b^2 = \frac{3.3}{2} = \frac{9}{2}$

So, the equation of ellipse is $\frac{x^2}{a} + \frac{y^2}{9/2} = 1$

or, $x^2 + 2y^2 = 9$

c. Distance between the two foci is 8 and the semi-latus rectum is 6.

Solution:

Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let a > b distance between foci = 8

i.e, $2ae = 8$

$\therefore \quad ae = 4$

and semi latus rectum = $\frac{b^2}{a} = 6$

or, $b^2 = 6a$

Using $b^2 = a^2 (1-e^2)$

or, $6a = a^2\left(1 - \frac{16}{a^2}\right) \qquad (\therefore e = 4/a)$

or, $6 = a\left(\frac{a^2 - 16}{a^2}\right)$

or, $a^2 - 6a - 16 = 0$

or, $a = 8, -2$ (but $a \neq -2$)

So, $b^2 = 6 \times 8 = 48$

$\therefore$ The equation of ellipse is $\frac{x^2}{64} + \frac{y^3}{48} = 1$

or, $3x^2 + 4y^2 = 192$

d. Latus rectum is equal to the half its major axis and which passes through the point (4, 3).

Solution:

Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ … … … (1)

which passes through (4, 3)

So, $\frac{16}{a^2} + \frac{9}{b^2} = 1$ … … … (ii)

Also, $\frac{2b^2}{a} = \frac{1}{2}2a$

or, $ab^2 = a^2$

Put $a^2 = 2b^2$ in (ii), then

$\frac{16}{2b^2} + \frac{9}{b^2} = 1$

or, $8 + 9 = b^2 \quad \therefore b^2 = 17$

and, $a^2 = 2 \times b^2 = 34$

So, from (1), equation of ellipse is $\frac{x^2}{34} + \frac{y^2}{17} = 1$

or, $x^2 + 2y^2 = 34$

e. Foci are at $(\pm 2, 0)$ and length of latus rectum is 6.

Solution: Here, foci = $(\pm ae, 0) = (\pm 2, 0)$

$\Rightarrow \quad ae = 2 \qquad \therefore e = \frac{2}{a}$

and length of latus rectum $\frac{2b^2}{a} = 6$

or, $b^2 = 3a$

Also, $e = \sqrt{1 - \frac{b^2}{a^2}}$

or, $\frac{2}{a} = \sqrt{1 - \frac{3a}{a^2}}$

or, $\frac{4}{a^2} = 1 - \frac{3}{a}$

or, $\frac{4}{a} = a - 3$

or, $4 = a^2 - 3a$

or, $a^2 - 3a - 4 = 0$

or, $a^2 - 4a + a - 4 = 0$

or, $a(a - 4) + 1(a - 4) = 0$

$\therefore$ $a = -1, 4$ (but $a \neq -1$)

and $b^2 = 3\times4 = 12$

Hence, the equation of ellipse is

$\frac{x^2}{16} + \frac{y^2}{12} = 1$

Exercise 10.2

1.

a. $\frac{x^2}{25} - \frac{y^2}{16} = 1$ … … … (1)

Compare (1) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 25, b^2 = 16$ $\therefore$ $a = 5, b = 4$

Now, eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{25}} = \frac{\sqrt{41}}{5}$

Co–ordinate of vertices $(\pm a, 0) = (\pm 5, 0)$

Co–ordinate of foci $(\pm ae, 0) = (\pm 5. \frac{\sqrt{41}}{5}, 0)$

$= (\pm \sqrt{41}, 0)$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2\times16}{5} = \frac{32}{5}$

Length of transverse axis = 2a = 2×5 = 10

Length of conjugate axis = 2b = 2×4 = 8

b. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

We have,

$\frac{x^2}{9} - \frac{y^2}{25} = 1$ which is in the form of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; where $a^2 = 9$ and $b^2 = 25$

$\therefore$ a = 3 and b = 5

Since the hyperbola is along y–axis

Hence,

eccentricity (e) = $\sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{9}{25}} = \frac{\sqrt{34}}{5}$

Co–ordinate of the vertices = $(0, \pm b) = (0, \pm 5)$

Foci of the hyperbola = $(0, \pm be) = \left(0, \pm \times \frac{\sqrt{34}}{5}\right) = (0, \pm \sqrt{34})$

Lenth of the latus rectum = $\frac{2a^2}{b} = \frac{2 \times 9}{5} = \frac{18}{5}$

Length of transverse axis = 2b = 2×b = 2×5 = 10

Length of conjugate axis = 2a = 2×3 = 6

c. $3x^2 - 4y^2 = 36$

or, $\frac{x^2}{12} - \frac{y^2}{9} = 1$ which is in the form of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; where $a^2 = 12$, $b^2 = 9$

$\therefore$ a = $2\sqrt{3}$ and b = 3

Since the hyperbola is along x–axis

Hence, eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{12}} = \frac{\sqrt{21}}{2\sqrt{3}} = \frac{\sqrt{7}}{2}$

Co–ordinate of the vertices = $(\pm a, 0) = (\pm 2\sqrt{3}, 0)$

Foci of the hyperbola = $(\pm ae, 0) = \left(\pm 2\sqrt{3} . \frac{\sqrt{7}}{2}, 0\right) = (\pm\sqrt{21}, 0)$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \times 9}{2\sqrt{3}} = 3\sqrt{3}$

Length of the transverse axis = 2a = $2\times 2\sqrt{3} = 4\sqrt{3}$

Length of the conjugate axis = 2b = $2\times 3 = 6$

2.

a. $\frac{(x+1)^2}{144} - \frac{(y-1)^2}{25} = 1$ ……… (1)

Compare (1) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get

$h = -1, k = 1, a^2 = 144, b^2 = 25$

Now, eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{25}{144}} = \frac{13}{12}$

Cor–ordinate of vertices = $(h \pm a, k)$

= $(-1 \pm 12, 1) = (-13, 1)$ and $(11, 1)$

Co–ordinate of foci = $(h \pm ae, k) = \left(-1 \pm 12 \times \frac{13}{12}, 1\right)$

= $(-14, 1)$ and $(12, 1)$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2 \times 25}{12} = \frac{25}{6}$

Length of conjugate axis = 2b = 2 × 5 = 10

Length of transverse axis = 2a = 2 × 12 = 24

b. $5x^2 - 20y^2 - 20x = 0$

or, $x^2 - 4y^2 - 4x = 0$

or, $(x-2)^2 - 4y^2 = 4$

or, $\frac{(x-2)^2}{4} - \frac{y^2}{1} = 1$ ……… (1)

Compare (1) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, then (h, k) = (2, 0), a = 2, b = 1

Now, eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$

Co–ordinate of vertices = $(h \pm a, k) = (2 \pm 2, 0)$

= (4, 0)| and (0, 0)

Co–ordinate of foci = $(h \pm ae, k)$

= $\left(2 \pm 2. \frac{\sqrt{5}}{2}, 0\right) = (2 \pm \sqrt{5}, 0)$

Length of latus rectum = $\frac{2b^2}{a} = 2 . \frac{1}{2} = 1$

Length of transverse axis = 2a = 2.2 = 4

Length of conjugate axis = 2b = 2.1 = 2

c. $\mathbf{16x^2 - 9y^2 + 96x - 72y + 144 = 0}$

or, $16(x^2 + 6x) - 9(y^2 + 8y) + 144 = 0$

or, $16(x+3)^2 - 9(y+4)^2 + 144 - 144 + 144 = 0$

or, $16(x+3)^2 - 9(y+4)^2 = -144$

or, $\frac{(x+3)^2}{9} - \frac{(y+4)^2}{16} = -1$ ……… (1)

Compare (1) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$, we get

$(h, k) = (-3, -4)$, $a^2 = 9$, $b^2 = 16$. $(b > a)$

Now, eccentricity $(e) = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$

Co–ordinate of vertices $= (h, k \pm b) = (-3, -4 \pm 4)$
$= (-3, 0)$ and $(-3, -8)$

Co–ordinate of foci $= (h, k \pm be)$

$= (-3, -4 \pm 4.\frac{5}{4}) = (-3, 1)$ and $(-3, -9)$

Length of latus rectum $= \frac{2a^2}{b} = 2 . \frac{9}{4} = \frac{9}{2}$

Length of transverse axis $= 2b = 2\times4 = 8$

Length of conjugate axis $= 2a = 2\times3 = 6$

3.

a. Transverse and conjugate axis are respectively 4 and 5.

Solution:

Let the equation of hyperbola be

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ … … … (1)

Where, $2a = 4$ and $2b = 5 \Rightarrow a = 2, b = \frac{5}{2}$

$\therefore$ from (1), equation of hyperbola is

$\frac{x^2}{4} - \frac{y^2}{25/4} = 1$

or, $\frac{x^2}{4} - \frac{4y^2}{25} = 1$

b. Foci $= (\pm 3, 0)$, eccentricity $(e) = \frac{3}{2}$

Here, e = 3 and ae = 3

$\Rightarrow$ $a = \frac{3x^2}{3} = 2$

Using, $b^2 = a^2(e^2 - 1)$

or, $b^2 = 4\left(\frac{9}{4} - 1\right) = 5$

$\therefore$ The equation of hyperbola is

$\frac{x^2}{4} - \frac{y^2}{5} = 1$

c. Latus rectum is 4 and eccentricity is 3

Solution: Here, e = 3 and $\frac{2b^2}{a} = 4$

Now, $b^2 = \frac{4a}{2}$

Using $b^2 = a^2(e^2 - 1)$

$\frac{4a}{2} = a^2(9 - 1)$

or, $2 = 8a \Rightarrow a = \frac{1}{4}$ $\qquad \therefore a^2 = \frac{1}{16}$

and $b^2 = \frac{4a}{2} = \frac{4\,\frac{1}{4}}{2} = \frac{1}{2}$

So, the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

or, $\frac{x^2}{1/16} - \frac{y^2}{1/2} = 1$

or, $16x^2 - 2y^2 = 1$

d. Vertex at (0, 8) and passing through (4, 8$\sqrt{2}$)

Solution: Here, vertex = (0, ± b) = (0, 8)

$\Rightarrow$ b = 8

Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ … … … (1)

Which passes through (4, 8$\sqrt{2}$), then

$$\frac{4^2}{a^2} - \frac{(8\sqrt{2})^2}{64} = -1$$

or, $\frac{16}{a^2} - \frac{128}{64} = -1$

or, $\frac{16}{a^2} = -1 + 2 \Rightarrow a^2 = 16$

Hence, from (1), $\frac{x^2}{16} - \frac{y^2}{64} = -1$

e. Vertices at (0, ± 7), e = $\frac{4}{3}$

Solution: Here, b = 7, e = $\frac{4}{3}$

Using $a^2 = b^2(e^2 - 1)$

$$= 49\left(\frac{16}{9} - 1\right) = 49 \times \frac{7}{9} = \frac{343}{9}$$

Hence, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

or, $\frac{x^2}{343/9} - \frac{y^2}{49} = -1$

or, $9x^2 - 7y^2 = -343$

or, $9x^2 - 7y^2 + 343 = 0$

f. Focus at (6, 0) and a vertex at (4, 0)

Solution: Here, ae = 6 and a = 4

Then, e = $\frac{6}{4} = \frac{3}{2}$

Using $b^2 = a^2(e^2 - 1)$

$b^2 = 16\left(\frac{9}{4} - 1\right) = 16 \times \frac{5}{4} = 20$

Now, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{16} - \frac{y^2}{20} = 1$

Chapter 11
Co-ordinate in Space

Exercise 11.1

1.

a. A(–2, 1, 0) and B(3, 5, –2)

Here,

$x_1 = -2$ $\qquad$ $x_2 = 3$

$y_1 = 1$ $\qquad$ $y_2 = 5$

$z_1 = 0$ $\qquad$ $z_2 = -2$

Using distance formulae,

$$
\begin{aligned}
AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\\
&= \sqrt{(3 - (-2))^2 + (5 - 1)^2 + (-2 - 0)^2}\\
&= \sqrt{5^2 + 4^2 + (-2)^2}\\
&= \sqrt{25 + 16 + 4}\\
&= \sqrt{45}\\
&= 3\sqrt{5} \text{ units}
\end{aligned}
$$

b. P(–4, 7, –7) and Q(–2, 1, –10)

Here,

$x_1 = -4$ $\qquad$ $x_2 = -2$

$y_1 = 7$ $\qquad$ $y_2 = 1$

$z_1 = -7$ $\qquad$ $z_2 = -10$

Using distance formula,

$$
\begin{aligned}
PQ &= \sqrt{(x_2 - x_1)_2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\\
&= \sqrt{(-2 - (-4))^2 + (1 - 7)^2 + (-10 - (-7))^2}\\
&= \sqrt{(-2 + 4)^2 + (-6)^2 + (-10 + 7)^2}\\
&= \sqrt{2^2 + (-6)^2 + (-3)^2}\\
&= \sqrt{4 + 36 + 9}\\
&= \sqrt{49}\\
&= 7 \text{ units}
\end{aligned}
$$

2. Show that the following points are collinear.

a. A(3, –2, 4), B(1, 1, 1) and C(–1, 4, –2)

Solution:

Using distance formula

$$
\begin{aligned}
AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\\
&= \sqrt{(1 - 3)^2 + (1 + 2)^2 + (1 - 4)^2}\\
&= \sqrt{(-2)^2 + 3^2 + (-3)^2}\\
&= \sqrt{4 + 9 + 9}\\
&= \sqrt{22} \text{ units}
\end{aligned}
$$

Again,

$$
\begin{aligned}
BC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\\
&= \sqrt{(-1 - 1)^2 + (4 - 1)^2 + (-2 - 1)^2}\\
&= \sqrt{(-2)^2 + (3)^2 + (-3)^2}\\
&= \sqrt{4 + 9 + 9}\\
&= \sqrt{22} \text{ units}
\end{aligned}
$$

Finally,

$$AC = \sqrt{(-1-3)^2 + (4+2)^2 + (-2-4)^2}$$
$$= \sqrt{(-4)^2 + (6)^2 + (-6)^2}$$
$$= \sqrt{16 + 36 + 36}$$
$$= \sqrt{88}$$
$$= 2\sqrt{22} \text{ units}$$

Now,

$AB + BC = 2\sqrt{22}$

Since, AB + BC = AC, the given points are collinear.

b. P(1, –2, 3), Q(2, 3, –4) and R(0, –7, 10)

Solution:

Using distance formula,

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(2-1)^2 + (3+2)^2 + (-4-3)^2}$$
$$= \sqrt{1^2 + 5^2 + (-7)^2}$$
$$= \sqrt{1 + 25 + 49}$$
$$= \sqrt{75}$$
$$= 5\sqrt{3} \text{ units}$$

Again,

$$QR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(0-2)^2 + (-7-3)^2 + (10+4)^2}$$
$$= \sqrt{(-2)^2 + (-10)^2 + (14)^2}$$
$$= \sqrt{4 + 100 + 196}$$
$$= \sqrt{300}$$
$$= 10\sqrt{3} \text{ units}$$

Finally,

$$PR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(0-1)^2 + (-7+2)^2 + (10-3)^2}$$
$$= \sqrt{(-1)^2 + (-5)^2 + 7^2}$$
$$= \sqrt{1 + 25 + 49}$$
$$= \sqrt{75}$$
$$= 5\sqrt{3} \text{ units}$$

Now, $PQ + PR = 10\sqrt{3}$

Since PQ + PR = QR, the given points are collinear.

c. x(1, 2, 3) y(4, 0, 4) and z(–2, 4, 2)

Solution:

Using distance formula,

$$xy = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(4-1)^2 + (0-2)^2 + (4-3)^2}$$
$$= \sqrt{(4-1)^2 + (0-2)^2 + (4-3)^2}$$
$$= \sqrt{3^2 + (-2)^2 + 1^2}$$
$$= \sqrt{9 + 4 + 1}$$
$$= \sqrt{14} \text{ units}$$

Again,

$$\begin{aligned}
yz &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 (z_2 - z_1)^2} \\
&= \sqrt{(-2-4)^2 + (4-0)^2 + (z-4)^2} \\
&= \sqrt{(-6)^2 + 4^2 + (-2)^2} \\
&= \sqrt{36 + 16 + 4} \\
&= \sqrt{56} \\
&= 2\sqrt{14} \text{ units}
\end{aligned}$$

Finally,

$$\begin{aligned}
xz &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 (z_2 - z_1)^2} \\
&= \sqrt{(-2-1)^2 + (4-2)^2 + (2-3)^2} \\
&= \sqrt{(-3)^2 + 2^2 + (-1)^2} \\
&= \sqrt{9 + 4 + 1} \\
&= \sqrt{14} \text{ units}
\end{aligned}$$

Since, yz = xz + xy

So, the points are collinear.

3. Find the mid-points

a. (–2, 6, –4) and (4, 0, 8)

Here,

$x_1 = -2$ $\quad$ $x_2 = 4$

$y_1 = 6$ $\quad$ $y_2 = 0$

$z_1 = -4$ $\quad$ $z_2 = 8$

Now,

Using mid-point formula,

$$\begin{aligned}
M(x, y, z) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) \\
&= \left(\frac{-2+4}{2}, \frac{6+0}{2}, \frac{-4+8}{2}\right) \\
&= \left(\frac{2}{2}, \frac{6}{2}, \frac{4}{2}\right)
\end{aligned}$$

$\therefore$ Mid–points = (1, 3, 2)

b. (–1, –2, –1) and (4, 7, 6)

Here,

$x_1 = -1$ $\quad$ $x_2 = 4$

$y_1 = -2$ $\quad$ $y_2 = 7$

$z_1 = -1$ $\quad$ $z_2 = 6$

Now,

Using mid-point formula,

$$\begin{aligned}
M(x, y, z) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) \\
&= \left(\frac{-1+4}{2}, \frac{-2+7}{2}, \frac{-1+6}{2}\right) \\
&= \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)
\end{aligned}$$

$\therefore$ Mid–point $= \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$

4. Here,

Let the points of line be A(3, 3, 1) and B(3, –6, 4). And the ratio that divides the line is 2:1. So,

$x_1 = 3$ $\quad$ $x_2 = 3$

$y_1 = 3$ $\quad$ $y_2 = -6$

$z_1 = 1$ $\quad$ $z_2 = 4$

Also,

m : n = 2 : 1

Using section formula we get,

$$P(x, y, z) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

$$= \left(\frac{2\times3 + 1\times3}{2+1}, \frac{2\times(-6) + 1\times3}{2+1}, \frac{2\times4 + 1\times1}{2+1}\right)$$

$$= \left(\frac{9}{3}, \frac{-9}{3}, \frac{9}{3}\right)$$

$\therefore$ P(x, y, z) = (3, – 3, 3)

5. Here,

Let the point of the line be M(3, 4, –5) and N(1, 3, –2) and the ratio that divides the time is 5 : 4. So,

$x_1 = 3$ $\quad$ $x_2 = 1$

$y_1 = 4$ $\quad$ $y_2 = 3$ $\quad$ m : n = 5 : 4

$z_1 = -5$ $\quad$ $z_2 = -2$

Using section formula we get

$$P(x, y, z) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

$$= \left(\frac{5\times1 - 4\times3}{5-4}, \frac{5\times3 - 4\times4}{5-4}, \frac{5\times(-2) - 4\times(-5)}{5-4}\right)$$

$$= (5 - 12, 15 - 16, -10 + 20)$$

$\therefore$ P(x, y, z) = (–7, –1, 10)

6. Here,

The internal section of two points P(2, –4, 3) and Q(x, y, z) in the ratio 2 : 1 is A(–2, 2, –3). So,

$x_1 = 2$ $\quad$ $x_2 = x$ $\quad$ and internal point m : n

$y_1 = -4$ $\quad$ $y_2 = y$ $\quad$ p = –2 = 2 : 1

$z_1 = 3$ $\quad$ $z_2 = z$ $\quad$ q = 2

r = –3

Now,

Using section formula we get,

$$p = \frac{-mx_2 + nx_1}{m+n}$$

$$\text{or,}\quad -2 = \frac{2\times x + 1\times2}{2+1}$$

or, $-2 \times 3 = 2x + 2$

or, $-6 = 2x + 2$

or, $2x = -8$

$\therefore$ $x = -4$

Similarly,

$$q = \frac{my_2 + ny_1}{m+n}$$

or, $2 = \dfrac{2 \times y + 1 \times (-4)}{2 + 1}$

or, $6 = 2y - 4$

or, $2y = 6 + 4$

or, $y = \dfrac{10}{2}$

$\therefore$ $y = 5$

Again,

$$r = \frac{mz_2 + nz_1}{m + n}$$

or, $-3 = \dfrac{2 \times z + 1 \times 3}{2 + 1}$

or, $-3 \times 3 = 2z + 3$

or, $-9 = 2z + 3$

or, $2z = -9 - 3$

or, $z = \dfrac{-12}{2}$

$\therefore$ $z = -6$

Hence, $Q(x, y, z) = (-4, 5, -6)$

7. **Solution:**

Given, $A(x_1, y_1, z_1) = (3, 2, -4)$

$B(x, y, z) = (5, 4, -6)$

and $C(x_2, y_2, z_2) = (9, 8, -10)$ be three collinear points.

So, let B divide AC in the ratio x : 1

So,

$$x = \frac{kx_2 + x_1}{k + 1}$$

or, $5 = \dfrac{k \times 9 + 3}{k + 1}$

or, $5k + 5 = 9k + 3$

or, $5k - 9k = 3 - 5$

or, $-4k = -2$

$\therefore$ $k = \dfrac{1}{2}$ i.e. $k : 1 = 1 : 2$

$\therefore$ B divides AC in the ratio of 1 : 2

b. Solution:

Let, xy – plane divides the line joining the points (–2, 4, 7) and (3, –5, –8) in the ratio of k : 1.

At the xy – plane, $z = 0$

Now,

Using the section formula,

$$z = \frac{mz_2 + nz_1}{m + n}$$

or, $z = \dfrac{kz_2 + z_1}{k + 1}$

or, $0 = \dfrac{k(-8) + 7}{k + 1}$

or, $k8 + 7 = 0$

$\therefore$ $k = \dfrac{7}{8}$ or, $k : 1 = 7 : 8$

$\therefore$ Required ratio is 7:8

c. Solution:

Let xz plane divides the line jointing the points A(1, 2, 3) and B(4, –4, 9) in the ratio k : 1.

At the xz – plane, y = 0

Now,

Using the section formula,

$$y = \frac{my_2 + ny_1}{m + n}$$

or, $y = \frac{k\,x_2 + y_1}{k + 1}$

or, $0 = \frac{k(-4) + 2}{k + 1}$

or, $-4k + 2 = 0$

or, $4k = 2$

$\therefore$ $k = \frac{1}{2}$ or, k : 1 = 1 : 2

$\therefore$ Required ratio is 1:2

Then using $x = \frac{mx_2 + nx_1}{m + n} = 2$ and $z = \frac{mz_2 + nz_1}{m + n} = 5$

$\therefore$ The required point is (2, 0, 5).

8. Solution

a. Let P(x, y, z) be any point on the locus. Let

Let A(1, 2, 1) and B(3, –4, 2) be two points.

By the given condition

$PA = PB$

or, $PA^2 = PB^2$

or, $(x - 1)^2 + (y - 2)^2 + (2 - 1)^2 = (x - 3)^2 + (y + 4)^2 + (z - 2)^2$

or, $x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 2z + 1 = x^2 - 6x + 9 + y^2 + 8y + 16 + z^2 - 4z + 4$

or, $x^2 - x^2 + y^2 - y^2 + z^2 - z^2 - 2x + 6x - 4y - 8y - 2z + 4z + 1 + 4 + 1 - 9 - 16 - 4 = 0$

or, $4x - 12y + 2z - 23 = 0$ is the required equation of locus.

b. Solution:

Here, $PA^2 = (x + 1)^2 + (y - 2)^2 + (z - 1)^2$

$= x^2 + y^2 + z^2 + 2x - 4y + 2z + 6$

and $PB^2 = x^2 + (y - 3)^2 + (z + 2)^2$

$= x^2 + y^2 + z^2 - 6y + 4z + 113$

Since $PA^2 + PB^2 = 6$

So, we have

$x^2 + y^2 + z^2 + 2x - 4y + 2z + 6 + x^2 + y^2 + z^2 - 6y + 4z + 13 = 6$

$\Rightarrow$ $2x^2 + 2y^2 + 2z^2 + 2x - 10y + 6z + 13 = 0$

which is the required equation of the locus of a point.

9. Solution

Let (2, –3, 1) and (3, –4, 5) divides the plane 2x + y + z = 7 in the ratio k : 1.

So,

$$(x, y, z) = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

$$= \left(\frac{k.3 + 2}{k + 1}, \frac{-4k - 3}{k + 1}, \frac{-5k + 1}{k + 1}\right) \quad \text{... (i)}$$

Also, (x, y, z) satisfies the plane 2x + y + z = 7

So, $2\left(\frac{3k + 2}{k + 1}\right) + \left(\frac{-4k - 3}{k + 1}\right) + \left(\frac{-5k + 1}{k + 1}\right) = 7$

or, $6k + 4 - 4k - 3 - 5k + 1 = 7k + 7$

or, $2k - 5k - 7k = 7 - 4 + 3 - 1$

$\therefore$ $k = \frac{-1}{2}$

Put the value of $k = \frac{-1}{2}$ in (i), we get the required point $(x, y, z) = (1, -2, 7)$.

10. Let P(x, y, z) be the centre of the sphere passing through the points A(3, 2, 2), B(–1, 1, 3), C(2, 1, 2) and D(1, 0, 4).

Here,

According to the question,

$P(x, y, z) = (1, 3, 4)$

Now,

Using distance formula,

$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ we have,

or, $AP^2 = (x - 3)^2 + (y - 2)^2 + (z - 2)^2$

$= x^2 - 6x + 9 + y^2 - 4y + 4 + z^2 - 4z + 4$ .... (i)

$BP^2 = (x + 1)^2 + (y - 1)^2 + (z - 3)^2$

$= x^2 + 2x + 1 + y^2 - 2y + 1 + z^2 - 6z + 9$ .... (ii)

$CP^2 = (x - 2)^2 + (y - 1)^2 + (z - 2)^2$

$= x^2 - 4x + 4 + y^2 - 2y + 1 + z^2 - 4z + 4$ .... (iii)

$DP^2 = (x - 1)^2 + (y - 0)^2 + (z - 4)^2$

$= x^2 - 2x + 1 + y^2 + z^2 - 8z + 16$ .... (iv)

$\therefore$ $AP = BP$

$\Rightarrow$ $AP^2 = BP^2$

$\Rightarrow$ $x^2 + y^2 + z^2 - 6x - 4y - 4z + 17 = x^2 + y^2 + z^2 + 2x - 2y - 6z + 11$

$\Rightarrow$ $2x + 6x - 2y + 4y - 6z + 4z = 17 - 11$

$\Rightarrow$ $8x + 2y - 2z = 6$

$\Rightarrow$ $4x + y - z = 3$

Similarly making (i) equal to (iii) and (i) equal to (iv)

We have,

$x^2 - 6x + 17 + y^2 - 4y + z2 - 4z = x^2 + y^2 + z^2 - 4x - 2y - 4z + 9$

or, $-2x - 2y = -8$

or, $x + y = 4$ .... (B)

Also, solving equation (iii) and (iv) we get

$-x - y + 2z = 4$ .... (C)

Hence, solving equation (A), (B) and (C) we get

$P(x, y, z) = (1, 3, 4)$

Hence proved.

Also, radius (r) = AP

$= \sqrt{x^2 + y^2 + z^2 - 6x - 4y - 4z + 17}$

$= \sqrt{1 + 9 + 16 - 6 - 12 - 16 + 17}$

$= \sqrt{9}$

= 3 units

11. Solution:

Let A(0, 7, 10), B(–1, 6, 6) and C(–4, 9, 6) be three points.

For AB

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z^2 - z_1)^2$

$= (-1 - 0)^2 + (6 - 7)^2 + (6 - 70)^2$

$= 1 + 1 + 16$

$= 18$

$AB = \sqrt{18} = 3\sqrt{2}$ units

For BC

$BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$= (-4 + 1)2 + (9 - 6)2 + (6 - 6)2$

$= 9 + 9 + 0$

$= 18$

$BC = \sqrt{18} = 3\sqrt{2}$ units

For AC

$CA^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$= (0 + 4)2 + (7 - 9)^2 + (10 - 6)^2$

$= 16 + 4 + 16$

$= 36$

$CA = \sqrt{36}$

$= 6$ units

Now,

By Pythagoras theorem

We have,

$CA^2 = AB^2 + BC^2$

Also AB = AC

$\therefore$ A, B and C are the vertices of right angled isosceles triangle.

11. Solution:

b. Let A(2, 0, –4) B(4, 2, 4) and C(10, 2, –2) be three points.

$AB^2 = 14 - 2)^2 + (2 - 0)^2 + (4 + 4)^2 = 4 + 4 + 64 = 72$

Again,

$BC^2 = (10 - 4)^2 + (2 - 2)^2 + (-2 - 4)^2 = 36 + 0 + 36 = 72$

Similarly,

$CA^2 = (2 - 10)^2 + (0 - 2)^2 + (-4 + 2)^2 = 64 + 4 + 4 = 72$

So, AB = BC = CA

$\therefore$ A, B and C are the vertices of an equilateral triangle.

12. Solution

a. Let A(1, 2, 3), B(–1, –2, –1), C(2, 3, 2)and D(4, 7, 6) be four points

$AB^2 = (-1 - 1)^2 + (-2 - 2)^2 + (-1 - 3)^2$

$= 36$

$\therefore AB = 6$

$BC^2 = (2 + 1)^2 + (3 + 2)^2 + (2 + 1)^2$

$= 43$

$\therefore BC = \sqrt{43}$

$CD^2 = (4 - 2)^2 + (7 - 3)^2 + (6 - 2)^2$

$= 36$

$\therefore CD = 6$

$DA^2 = (1 - 4)^2 + (2 - 7)^2 + (3 - 6)^2$

$= 43$

$\therefore DA = \sqrt{43}$

Hence, AB = CD and BC = DA so, A, B, C and D are the vertices of a parallelogram.

Here,

$AC = \sqrt{(2 - 1)^2 + (3 - 2)^2 + (2 - 3)^2}$

$= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ units

Again,

BD $= \sqrt{(4+2)^2 + (7+2)^2 + (6+1)^2}$

$= \sqrt{36 + 81 + 49} = \sqrt{166}$ units

Since, the two diagonals AC and BD are not equal.

$\therefore$ The points A, B, C and D do not represent a rectangle.

b. Solution

Let, $D(\overline{x}, \overline{y}, \overline{z})$ be the point of intersection of the diagonals AC and BD.

For AC : A (–5, 5, 2) and C(–3, –3, 0)

The coordinates of the mid-point AC

$$= \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2}\right) = \left(\frac{-3-5}{2}, \frac{5-3}{2}, \frac{2+0}{2}\right) = (-4, 1, 1)$$

For BD : B(–9, –1, 2) and $D(\overline{x}, \overline{y}, \overline{z})$

Since mid point of AC = midpoint of BD

$x = \frac{x_1 + x_2}{2}$

or, $-4 = \frac{-9 + x}{2}$ or, $-8 = -9 + \overline{x}$ $\therefore \overline{x} = -8 + 9 = 1$

$y = \frac{y_1 + y_2}{2}$

or, $1 = \frac{-1 + \overline{y}}{2}$ or, $2 = -1 + \overline{y}$ $\therefore \overline{y} = 2 + 1 = 3$

$z = \frac{z_1 + z_2}{2}$

or, $1 = \frac{2 + z}{2}$ or, $2 = 2 + \overline{z}$ $\therefore \overline{z} = 2 - 2 = 0$

$\therefore$ The coordinates of $D = (\overline{x}, \overline{y}, \overline{z}) = (1, 3, 0)$

13. Solution:

Let $(x_1, y_1, z_1) = (2, 6, -4)$

$(x_2, y_2, z_2) = (15, -10, 16)$

$(x, y, z) = (7, -2, 5)$ and $(x_3, y_3, z_3) = ?$

By the centroid formula,

$x = \frac{x_1 + x_2 + x_3}{3}$

or, $7 = \frac{2 + 15 + x_3}{3}$

or, $21 = 17 + x_3$

$\therefore$ $x_3 = 21 - 17 = 4$

$y = \frac{y_1 + y_2 + y_3}{3}$

or, $-6 = -4 + y_3$

$\therefore$ $y_3 = -6 + 4 = -2$

$z = \frac{z_1 + z_2 + z_3}{3}$

or, $5 = \frac{-4 + 16 + z_3}{3}$

or, $15 = 12 + z_3$

or, $z_3 = 15 - 12$

$\therefore$ $z_3 = 3$

$\therefore$ $(x_3, y_3, z_3) = (4, -2, 3)$

Exercise 11.2

1. Solution:

$\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{4}$

But $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or, $\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} + \cos^2\gamma = 1$

or, $\frac{1}{2} + \frac{1}{2} + \cos^2\gamma = 1$

or, $1 + \cos^2\gamma = 1$

or, $\cos^2\gamma = 0$

or, $\cos\gamma = 0$

$\therefore$ $\gamma = \cos^{-1}(0)$

$\therefore$ $\gamma = \frac{\pi}{2}$

$\therefore$ The angle is $\frac{\pi}{2}$.

2. Solution:

Let the angle made by a line with 3 axes be α, α, α.

Now, we know

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or, $\cos^2\alpha + \cos^2\alpha \cos^2\alpha = 1$

or, $3\cos^2\alpha = 1$

or, $\cos\alpha = \pm\frac{1}{\sqrt{3}}$

Similarly, for $\cos^2\alpha$ and $\cos^2\alpha$

$\cos\alpha = \pm\frac{1}{\sqrt{3}}$

and $\cos\alpha = \pm\frac{1}{\sqrt{3}}$

$\therefore$ The direction cosines $= \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

3. Solution:

Given,f α, β and γ be the angles made by the line with the co-ordinates axis.

$\text{Cos}^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Multiplying by 2 on both sides we get,

$2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma = 2$

or, $1 + \cos^2\alpha + 1 + \cos^2\beta + 1 + \cos^2\gamma = 2$

or, $3 + \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2$

$\therefore$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 1 = 0$

4. Solution:

a. Here, a = 6, b = 2 and c = –3

The direction cosines are

$l = \frac{1}{\sqrt{a^2 + b^2 + c^2}} = \frac{6}{\sqrt{6^2 + 2^2 + (-3)^2}} = \frac{6}{\sqrt{49}} = \frac{6}{7}$

$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{2}{\sqrt{49}} = \frac{2}{7}$

and, $n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{-3}{\sqrt{49}} = \frac{-3}{7}$

$\therefore$ The direction cosines (l, m, n) = $\left(\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}\right)$

b. Solution:

Here, a = –1, b = –2, and c = –3

Now,

The direction cosines are:

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}},\quad m = \frac{b}{\sqrt{a^2+b^2+c^2}},\quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{-1}{\sqrt{(-1)^2+(-2)^2+(-3)^2}},\quad = \frac{-2}{\sqrt{(-1)^2+(-2)^2+(-3)^2}},\quad = \frac{-3}{\sqrt{(-1)^2+(-2)^2+(-3)^2}}$$

$$= \frac{-1}{\sqrt{14}},\quad = \frac{-2}{\sqrt{14}},\quad = \frac{-3}{\sqrt{14}}$$

$\therefore$ The direction cosines are (l, m, n) = $\left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$

5. Solution:

a. $(x_1, y_1, z_1) = (-2, 1, -8)$ and $(x_2, y_2, z_2) = (4, 3, -5)$

$$PQ = r = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$= \sqrt{(4+2)^2+(3-1)^2\,(-5+8)^2}$$
$$= \sqrt{6^2+2^2+3^2}$$
$$= \sqrt{36+4+9}$$
$$= \sqrt{49} = 7$$

$\therefore$ The direction cosines of PQ are

$$l = \frac{x_2-x_1}{r},\quad m = \frac{y_2-y_1}{r},\quad n = \frac{z_2-z_1}{r}$$

$$= \frac{4+2}{7},\quad = \frac{3-1}{7},\quad = \frac{-5+8}{7}$$

$$= \frac{6}{7},\quad = \frac{2}{7},\quad = \frac{3}{7}$$

$\therefore$ The direction cosines are (l, m, n) = $\left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

b. Solution:

$(x_1, y_1, z_1) = (5, 2, 8)$ and $(x_2, y_2, z_2) = (7, -1, 9)$

$$AB = r = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$= \sqrt{(7-5)^2+(-1-2)^2+(9-8)^2}$$
$$= \sqrt{2^2+(-3)^2+1^2}$$
$$= \sqrt{4+9+1}$$
$$= \sqrt{14}$$

$\therefore$ The direction cosines of AB are:

$$l = \frac{x_2-x_1}{r},\ m = \frac{y_2-y_1}{r},\ n = \frac{z_2-z_1}{r}$$

$$= \frac{7-5}{\sqrt{14}} = \frac{-1-3}{\sqrt{14}} = \frac{9-8}{\sqrt{14}}$$

$= \frac{2}{\sqrt{14}} = \frac{-3}{\sqrt{14}} = \frac{1}{\sqrt{14}}$

$\therefore$ The direction cosines (l, m, n) = $\left(\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$

6. Solution:

a. $a_1 = 1$ $b_1 = 2$, $c_1 = 2$

$a_2 = 2$, $b_2 = 3$, $c_2 = 6$

We have,

$$\text{Cos}\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{1\times2 + 2\times3 + 2\times6}{\sqrt{1^2 + 2^2 + 2^2}\sqrt{2^2 + 3^2 + 6^2}} = \frac{20}{3\times7}$$

or, $\theta = \cos^{-1}\left(\frac{20}{21}\right)$

$\therefore$ $\theta = \cos^{-1}\left(\frac{20}{21}\right)$

b. For AB :

$AB = \sqrt{(1 + 2)^2 + (4 - 1)^2 + (2 - 2)^2}$

$= \sqrt{9 + 9 + 0} = 3\sqrt{2}$ units

and $(l_1, m_1, n_1) = \left(\frac{-2 - 1}{3\sqrt{2}}, \frac{1 - 4}{3\sqrt{2}}, \frac{2 - 2}{3\sqrt{2}}\right) = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

For BC

$BC = \sqrt{(2 + 2)^2 + (-3 - 1)^2 + (4 - 2)^2}$

$= \sqrt{16 + 16 + 4} = \sqrt{36} = 6$ units

and $(l_2, m_2, n_2) = \left(\frac{2 + 2}{6}, \frac{-3 - 1}{6}, \frac{4 - 2}{6}\right) = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

We know,

$\cos B = l, l_2 + m_1m_2 + n_1n_2$

$= \frac{-1}{\sqrt{2}} \times \frac{2}{3} + -\frac{1}{\sqrt{2}} \times \frac{-2}{3} + 0 \times \frac{1}{3}$

$= \frac{-\sqrt{2}}{3} + \frac{\sqrt{2}}{3} + 0 = 0$

or, $\cos B = 0$

$\therefore$ $B = \frac{\pi}{2}$

$\therefore$ $B = \frac{\pi}{2} = 90^o$

Hence, the lines are perpendicular.

For AC

$AC = \sqrt{(2 - 1)^2 + (-3 - 4)^2 + (4 - 2)^2}$

$= \sqrt{1 + 49 + 4}$

$= \sqrt{54} = 3\sqrt{6}$ units.

Again, $(l_2, m_2, n_2) = \frac{2 - 1}{3\sqrt{6}}, \frac{-3 - 4}{3\sqrt{6}}, \frac{4 - 2}{3\sqrt{6}} = \left(\frac{1}{3\sqrt{6}}, \frac{-7}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}\right)$

Similarly,

$\text{Cos } A = l_1 l_2 + m_1m_2 + n_1n_2$

$= \frac{-1}{\sqrt{2}} \times \frac{1}{3\sqrt{6}} + \frac{-1}{\sqrt{2}} \times \frac{-7}{3\sqrt{6}} + 0 \times \frac{2}{3\sqrt{6}}$

$= \frac{-\sqrt{3}}{18} + \frac{7\sqrt{3}}{18} + 0$

$= \frac{1}{\sqrt{3}}$

$\therefore$ $\text{CosA} = \frac{1}{\sqrt{3}}$

7. Solution:

a. For the live joining the points (1, 2, 3) and (–1, –2, –3) $(x_1, y_1, z_1) = (1, 2, 3)$ and $(x_2, y_2, z_2) = (-1, -2, -3)$

$a_1 = x_2 - x_1 = -1 - 1 = -2$

$b_1 = y_2 - y_1 = -2 - 2 = -4$

$c_1 = z_2 - z_1 = -3 - 3 = -6$

For the live joining the points (2, 3, 4) and (5, 9, 13) $(x_1, y_1, z_1) = (2, 3, 4)$ and $(x_2, y_2, z_2) = (5, 9, 13)$

$a_2 = x_2 - x_1 = 5 - 2 = 3$

$b_2 = y_2 - y_1 = 9 - 3 = 6$

$c_2 = z_2 - z_1 = 13 - 4 = 9$

Now,

$\frac{a_1}{a_2} = \frac{-2}{3}, \frac{b_1}{b_2} = \frac{-4}{6} = \frac{-2}{3}, \frac{c_1}{c_2} = \frac{-6}{9} = \frac{-2}{3}$

$\therefore$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Hence, the two lines are parallel.

7. Solution:

b. For the line joining the points (0, 4, 1) and (2, 6, 2) $(x_1, y_1, z_1) = (0, 4, 1)$ and $(x_2, y_2, z_2) = (2, 6, 2)$

$a_1 = x_2 - x_1 = 2 - 0 = 2$

$b_1 = y_2 - y = 6 - 4 = 2$

$c_1 = 2_2 - 2_1 = 2 - 1 = 1$

For the line joining the points (4, 5, 0) and (2, 6, 2) $(x_1, y_1, z_1) = (4, 5, 0)$ and $(x_2, y_2, z_2) = (2, 6, 2)$

$a_2 = x_2 - x_1 = 2 - 4 = -2$

$b_2 = y_2 - y_1 = 6 - 5 = 1$

$c_2 = z_2 - z_1 = 2 - 0 = 2$

Now,

$a_1\ a_2 = -4 \quad b_1\ b_2 = 2 \qquad c_1\ c_2 = 2$

$\therefore$ $a_1a_2 + b_1b_2 + c_1c_2 = 0$

or, $-4 + 2 + 2 = 0$

$-4 + 4 = 0$

$\therefore$ $0 = 0$

8.a. Here,

The two points of a line (1, 2, 3) and (4, 5, k) is parallel to two points of a line (–4, 3, –6) and (2, 9, 2).

So,

$a_1 = x_2 - x_1, \qquad b_1 = y_2 - y_1, \qquad c_1 = k - 3$

$= 4 - 1 \qquad = 5 - 2 \qquad = k - 3$

$= 3 \qquad = 3$

Again,

$a_2 = x_2 - x_1, \qquad b_2 = 9 - 3, \qquad c_2 = z_2 - z_1$

$= 2 + 4 \qquad = 6 \qquad = 2 + 6$

$= 6 \qquad = 8$

Since the two lines are parallel to each other we know,

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

or, $\frac{3}{6} = \frac{3}{6} = \frac{k-3}{8}$

or, $\frac{3}{6} = \frac{k-3}{8}$

or, $24 = 64 - 18$

or, $24 + 18 = 64$

or, $k = \frac{42}{6}$

$\therefore$ $k = 7$

b. Solution

Here, the line joining the points (k, 2, 3) and (–1, –2, –3) is perpendicular to the line joining the points (–2, 1, 5) and (3, 3, 2)

Now,

The direction ratios are:

$a_1 = x_2 - x_1$, $b_1 = y_2 - y_1$, $c_1 = z_2 - z_1$

$= -1 - k$ $= -2 - 2$ $= -3 - 3$

$= -4$ $= -6$

Again,

$a_2 = x_2 - x_1$, $b_2 = y_2 - y_1$, $c_2 = z_2 - z_1$

$= 3 + 2$ $= 3 - 1$ $= 2 - 5$

$= 5$ $= 2$ $= -3$

Since the two lines are perpendicular to each other. We know,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

or, $(-1 - k) \times 5 + (-4) \times 2 + (-6) \times (-3) = 0$

or, $18 - 13 = 5k + 0$

or, $5 - 0 = 5k$

$\therefore$ $k = 1$

9. Here,

Let l, m and n be the direction cosines of the line perpendicular to the given line.

$l + 2m + 3n = 0$ .... (i)

$-l + 3m + 5n = 0$ .... (ii)

By cross = multiplication method

$\frac{l}{10-9} = \frac{m}{-5-3} = \frac{n}{3+2}$

or, $\frac{l}{1} = \frac{m}{-8} = \frac{n}{5}$

$= \frac{1}{\sqrt{1^2 + 8^2 + 5^2}} = \frac{1}{\sqrt{90}}$

$\therefore$ $l = \frac{1}{\sqrt{90}}$, $m = \frac{-8}{\sqrt{90}}$, $n = \frac{5}{\sqrt{90}}$

10. Solution

Suppose l, m, n are the direction cosines of a line and its length

Then, $lr = 1$

$mr = 2$

$nr = 2$

Squaring and adding there relations

$r^2(l^2 + m^2 + n^2) = 12 + 22 + 22 = 9$

or, $r^2 = 9$

$\therefore$ $r = 3$ units

So length of the line = 3

Hence, $l = \frac{1}{r} = \frac{1}{3}$

$m = \frac{2}{r} = \frac{2}{3}$

$n = \frac{2}{r} = \frac{2}{3}$

11. Solution

$(x_1, y_1, z_1) = (3, -1, 2)$ and $(x_2, y_2, z_2) = (5, -7, 4)$

Here,

a. $a = 1, b = -1, c = 2$

$\therefore \quad l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{1 + 1 + 4}} = \frac{1}{\sqrt{6}}$

$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{-1}{\sqrt{1 + 1 + 4}} = \frac{-1}{\sqrt{6}}$

$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{2}{\sqrt{1 + 1 = 4}} = \frac{2}{\sqrt{6}}$

The required projection on the line

$= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1) n$

$= (5 - 3) \times \frac{1}{\sqrt{6}} + (-7 + 1) \times -\frac{1}{\sqrt{6}} + (4-2) \times \frac{2}{\sqrt{6}} = 2\sqrt{6}$

b. Here,

$r = \sqrt{(1 - 0)^2 + (3-1)^2 + (7-0)^2}$

$= \sqrt{54}$

$= 3\sqrt{6}$

Now,

$l = \frac{x_2 - x}{r} = \frac{1 - 0}{3\sqrt{6}} = \frac{1}{3\sqrt{6}}$

$m = \frac{y_2 - y_1}{r} = \frac{3 - 1}{3\sqrt{6}} = \frac{2}{3\sqrt{6}}$

$n = \frac{z_2 - z_1}{r} = \frac{7 - 0}{3\sqrt{6}} = \frac{7}{3\sqrt{6}}$

The projection $= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$

$= (5-3) \times \frac{1}{3\sqrt{6}} + (-7 + 1) \times \frac{2}{3\sqrt{6}} + (4-2) \times \frac{7}{3\sqrt{6}}$

$= \frac{4}{3\sqrt{6}}$

12. Solution

a. Here, $4l + 3m - 2n = 0$ .... (i)

$lm + mn + nl + 0$ .... (ii)

From the equation (i), $n = \frac{4l + 3m}{2}$

From the equation (ii)

$lm - m\frac{(4l + 3m)}{2} + l\frac{(4l + 3m)}{2} = 0$

or, $lm - \frac{4ml}{2} - \frac{3m^2}{2} + \frac{4l^2}{2} + \frac{3ml}{2} = 0$

or, $lm - 2ml - \frac{3}{2}m^2 + 2l^2 + \frac{3ml}{2} = 0$

or, $2l^2 + \frac{ml}{2} - \frac{3}{2}m^2 = 0$

or, $4l^2 + ml - 3m^2 = 0$

or, $4l^2 + 4ml - 3ml - 3m^2 = 0$

or, $4l\,(4l + m) - 3m\,(l + m) = 0$

or, $(l + m)(4l - 3m) = 0$

$\therefore$ $l + m = 0$ .... (iii)

$4l - 3m = 0$ .... (iv)

From equation (i) and (iii)

$4l + 3m - 2n = 0$ and $l + m + 0n = 0$

$\therefore \frac{l}{0+2} = \frac{m}{-2-0} = \frac{n}{4-3}$

or, $\frac{l}{2} = \frac{m}{-2} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{4 + k + 1}} = \frac{1}{3}$

$\therefore l = \frac{2}{3}, m = -\frac{2}{3}, n = \frac{1}{3}$

From equation (i) and (iv)

$4l + 3m - 2n = 0$ and $4l - 3m + 0n = 0$

$\therefore \frac{l}{0-6} = \frac{m}{-8-0} = \frac{n}{-12-12}$

or, $\frac{l}{6} = \frac{m}{8} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{36 + 64 + 576}} = \frac{1}{\sqrt{676}} = \frac{1}{26}$

$\therefore l = \frac{6}{26} = \frac{3}{13}$

$m = \frac{8}{26} = \frac{4}{13}$

$n = \frac{24}{26} = \frac{12}{13}$

b. Solution

Here,

$2l + 2m - n = 0$ .... (i)

$mn + nl + lm = 0$ .... (ii)

Using equation (i) in equation (ii) we have

$m(2l + 2m) + l(2l + 2m) + lm = 0$

or, $2lm + 2m^2 + 2l^2 + 2lm + lm = 0$

or, $2m^2 + 5lm + 2l^2 = 0$

or, $2m^2 + (4 + 1)\, lm + 2l^2 = 0$

or, $2m^2 + 4lm + lm + 2l^2 = 0$

or, $2m(m + 2l) + l(m + 2l) = 0$

or, $(m + 2l)(2m + l) = 0$

$\therefore$ $m + 2l = 0$ .... (iii)

$2m + l = 0$ .... (iv)

from (i) and (iii)

$2l + 2m - n = 0$ and $2l + m + 0.n = 0$

$\therefore \frac{l}{0+1} = \frac{m}{0+2} = \frac{n}{2-4}$

or, $\frac{l}{1} = \frac{m}{2} = \frac{n}{-2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{1 + k + 4}} = \frac{1}{3}$

$\therefore l = \frac{1}{3}, m = \frac{2}{3}, n = -\frac{2}{3}$

from (i) and (iv)

$2l + 2m - n = 0$ and $l + 2m + 0.n = 0$

$\therefore \frac{l}{0+2} = \frac{m}{0+1} = \frac{n}{4-2}$

or, $\frac{l}{2} = \frac{m}{1} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{4 + 1 + 4}} = \frac{1}{3}$

$\therefore \quad l = \frac{2}{3}, m = \frac{1}{3}, n = \frac{2}{3}$

13. Solution

Here, $l + m + n = 0$ .... (i)

or, $n = -1 - m$

Putting the value of n in

$l^2 + m^2 - n^2 = 0$ .... (ii)

or, $l^2 + m^2 - (-1 - m)^2 = 0$

or, $2lm = 0$

or, $lm = 0$

$\therefore \quad l = 0$ .... (iii)

and $m = 0$ .... (iv)

from (i) and (iii)

$l + m + n = 0$ and $l + 0.m + 0.n = 0$

or, $\frac{l}{0-0} = \frac{m}{1-0} = \frac{n}{0-1}$

or, $\frac{l}{0} = \frac{m}{1} = \frac{n}{-1} = \frac{1}{\sqrt{0+1+1}} = \frac{1}{\sqrt{2}}$

$\therefore \quad l = 0, m = \frac{1}{\sqrt{2}}, n = -\frac{1}{\sqrt{2}}$

from (i) and (iv)

$l + m + n = 0$ and $0.l + m + 0.n = 0$

$\therefore \quad \frac{l}{0-1} = \frac{m}{0-0} = \frac{n}{1-0}$

or, $\frac{l}{-1} = \frac{m}{0} = \frac{n}{1} = \frac{1}{\sqrt{1+0+1}} = \frac{1}{\sqrt{2}}$

$\therefore \quad l = -\frac{1}{\sqrt{2}}, m = 0, n = \frac{1}{\sqrt{2}}$

$\therefore \quad \cos\theta = 0\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right).0 + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

$\therefore \quad \theta = \frac{2\pi}{3} = 120^o$

14. Solution

Given, relations are

$al + bm + cn = 0$ .... (i)

$fmn + gnl + klm = 0$ .... (ii)

Eliminating n between (i) and (ii) we have

$fm\left(-\frac{al + bm}{c}\right) + g\left(-\frac{al + bm}{c}\right) l + hlm = 0$

$\Rightarrow \quad agl2 + (af + bg - ch)lm + bfm2 = 0$

$\Rightarrow \quad ag\left(\frac{l}{m}\right)^2 + (ch - af + bg)\left(\frac{l}{m}\right) + bf = 0$

which is quadratic in $\frac{l}{m}$. Let the two roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$

$\frac{l_1}{m_1}\ \frac{l_2}{m_2} = \frac{bf}{ag}$

$\Rightarrow \quad \frac{l_1\,l_2}{bf} = \frac{m_1m_2}{ag}$

$\Rightarrow \quad \frac{l_1l_2}{f/a} = \frac{m_1m_2}{g/b}$ .... (iii)

Similarly, if we eliminate l between (i) and (ii) have,

$\frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c}$ .... (iv)

From equation (iii) and (iv)

$\frac{l_1 l_2}{f/a} = \frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c}$

If each ratio be equal to k, then

$l_1l_2 = k\frac{f}{a}$, $m_1m_2 = k\frac{g}{b}$, $n_1n_2 = k\frac{h}{c}$

The two lines will be perpendicular if

$l_1l_2 + m_1m_2 + n_1n_2 + 0$

$k\frac{f}{a} + k\frac{g}{b} + k\frac{h}{c} = 0$

i.e. $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$ proved.

Exercise 11.3

1. Solution:

a. The equation of the plane is $2x + 3y + 4z = 24$

Dividing both sides by 24, $\frac{2x}{24} + \frac{3y}{24} + \frac{4z}{24} = \frac{24}{24}$

or, $\frac{x}{12} + \frac{4}{8} + \frac{z}{6} = 1$

The intercepts on the x–axis, y–axis and z–axis are 12, 8 and d 6 respectively.

b. To reduce the equation of the plane $2x – y + 2z = 4$ into normal form,

Divide each term by $\sqrt{2^2 + (–1)^2 + 2^2} = 3$

$\therefore \frac{24}{3} – \frac{y}{3} + \frac{2z}{3} = \frac{4}{3}$ is in normal form where length of perpendicular from origin is $\frac{4}{3}$ units.

The dc's are

$$\frac{2}{\sqrt{2^2 + (–1)^2 + 2^2}}, \frac{–1}{\sqrt{2^2 + (–1)^2 + 2^2}}, \frac{2}{\sqrt{2^2 + (–1)^2 + 2^2}}$$

i.e. $\frac{2}{3}, –\frac{1}{3}, \frac{2}{3}$

2. Solution:

a. The equation of plane which cuts intercepts 2, 3, 4 on the coordinate axes is

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

or, $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$

$\therefore$ $6x + 4y + 3z = 12$

b. Here, $a = b = c$

The equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$x + y + z = a$ .... (i)

If the plane (i) passes through (2, 3, 4) then $2 + 3 + 4 = a$

$\therefore$ $a = 9$

From equation (i) $x + y + z = 9$ which is the equation of the plane.

3. Solution:

a.(i) The equation of the plane through (2, 3, –3) $a(x – 2) + b(y –3) + c(z + 3) = 0$ .... (i)

If the plane passes through (1, 1, –2) and (–1, 1, 2) then,

$a(1 – 2) + b(1 – 3) + c(–2 + 3) = 0$

or, $–a –2b + c = 0$ .... (ii)

Again,

$a(–1 – 2) + b(1 – 3) + c(2 + 3) = 0$

or, $-3a - 2b + 5c = 0$ .... (iii)

From (ii) and (iii) and cross multiplication gives

$$\frac{a}{-10+2} = \frac{b}{-3+5} = \frac{c}{2-6}$$

$$\frac{a}{-8} = \frac{b}{2} = \frac{c}{-4} = k \text{ (say)}$$

$\therefore$ $a = -8k$, $b = 2k$ $c = -4k$

Substituting the values of a, b and c in equation (i) we get,

$-8h(x-2) + 2k(y-3) - 4k(2+3) = 0$

or, $-8x + 16 + 2y - 6 - 4 - 12 = 0$

or, $-8x + 2y - 4z - 2 = 0$

or, $-4x + y + 4z - 2 = 0$

or, $4x - y - 4z + 2 = 0$

(ii). The equation of the plane through (2, 2, 2) is

$a(x-2) + b(y-2) + c(z-2) = 0$ .... (i)

If the plane passes through (3, 1, 1) and (6, –4, –6) then,

$a(3-2) + b(1-2) + c(1-2) = 0$

or, $a - b - c = 0$ .... (ii)

Again,

$a(6-2) + b(-4-2) + c(-6-2) = 0$

or, $4a - 6b - 8c = 0$ .... (iii)

From (ii) and (iii) and cross multiplication gives

$$\frac{a}{8-6} = \frac{b}{-4+8} = \frac{c}{-6+4}$$

$$\frac{a}{2} = \frac{b}{4} = \frac{c}{-2} = k \text{ (say)}$$

$\therefore$ $a = 2k$, $b = 4k$, $c = -2k$

Substituting the values of a, b and c in equation (i)

$2k(x-2) + (4k)\, y-2) + (-2)(z-2) = 0$

or, $2k - 4 + 4y - 8 - 2z + 4 = 0$

or, $2x + 4y - 2z = 8$

or, $2x + 4y - 2z = 8$

or, $x + 2y - z = 4$

b. The points (1, 3, –1), (3, 5, 1), (0, 2, –2) and (2, 1, –2) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-1 & 5-3 & 1+1 \\ 0-1 & 2-3 & -2+1 \\ 2-1 & 1-3 & -2+1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 1 & -2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow -2\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = 0$$

$\Rightarrow$ $-2 \times 0 = 0 \Rightarrow 0 = 0$ (true)

Hence, the given four points are coplanar.

4. Solution:

The equation of the plane parallel to the plane $ax + by + cz = 0$ is

$ax + by = cz + k = 0$ .... (i)

If the plane (i) passes through (α, β, γ) then
$a\alpha = b\beta = c\gamma + k = 0$
$\therefore \quad k = -a\alpha - b\beta - c\gamma$
Substituting the value of k in (i) we have
$ax + by + cz = a\alpha + b\beta + c\gamma$
which is the required equation of the plane.

5. Solution:

a. $x + 2y + 3z = 6$ and $3x - 3y + z = 1$
Here,
$x + 2y + 3z - 6 = 0$ and $3x - 3y + 2 - 1 = 0$

$a_1 = 1$ $\quad a_2 = 3$
$b_1 = 2$ $\quad b_2 = -3$
$c_1 = 3$ $\quad c_2 = 1$

Now,
Angle between the two planes

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$
$$= \frac{1 \times 3 + 2 \times (-3) + 3 \times 1}{\sqrt{1^2 + 2^2 + 3^2}\sqrt{3^2 + (-3)^2 + 1^2}}$$
$$= \frac{0}{\sqrt{1 + 4 + 9}\sqrt{9 + 9 + 1}}$$

$\therefore \quad \cos\theta = 0$

$\therefore \quad \theta = \frac{\pi}{2}$

b. $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$
Here,
Repeating the same procedure as No. 5a

$\therefore \quad \theta = \frac{\pi}{2}$

6. Solution:

a. The equation of the two planes is given:
$x - 6y + 8z = 4$ .... (i)
$4x + ky + z = 7$ .... (ii)
Now,

$a_1 = 1$ $\quad a_2 = 4$
$b_1 = -6$ $\quad b_2 = k$
$c_1 = 8$ $\quad c_2 = 1$

We know, the equation for two planes being perpendicular is given by,
$a_1a_2 = b_1b_2 + c_1c_2 = 0$
or, $1.4 + (-6).\ k + 8.1 = 0$
or, $4 - 6k + 8 = 0$
or, $-6k = -12$
$\therefore \quad k = 2$

b. The two equations of the plane is given
$2x - ky + 5z = 9$ .... (i)
$2x + 3y + 4z = 7$ .... (ii)
We know,
The equation of two planes being perpendicular is given by
$a_1a_2 + b_1b_2 + c_1c_2 = 0$
or, $2 \times 2 + (-4) \times 3 + 5 \times 4 = 0$
or, $-3x = -24$
$\therefore \quad k = 8$

7. Solution:

For parallel

Two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel, if

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Here,

$a_1 = 2, b_1 = 3\ c_1 = -4$

$a_2 = 10, b_2 = 15, c_2 = -20$

$\frac{a_1}{a_2} = \frac{2}{10} = \frac{1}{5}, \frac{b_1}{b_2} = \frac{3}{15} = \frac{1}{5}, \frac{c_1}{c_2} = \frac{-4}{-20} = \frac{-1}{-5} = \frac{1}{5}$

$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\therefore$ The planes $2x + 3y - 4z = 3$ and $10x + 15y - 202 = 12$ are parallel.

For perpendicular

Two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$

are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Here, $a_1 = 2, b_1 = 3, c_1 = -4$

$a_2 = 3, b_2 = 2, c_2 = 3$

And

$a_1a_2 + b_1b_2 + c_1c_2$

$= 2 \times 3 + 3 \times 2 - 4 \times 3$

$= 6 + 6 - 12$

$= 12 - 12$

$= 0$

$\therefore$ The two planes $2x + 3y - 4z = 3$ and $3x + 2y + 3z = 5$ are perpendicular.

8. Solution:

a. An equation of a plane passing through $(-2, 3, 4)$ so, the equation is

$a(x + 2) + b(y - 3) + c(z - 4) = 0$ .... (i)

Now,

It is perpendicular to the equation $2x + 3y + 4z = 6$ then,

$a \times 2 + b \times 3 + c \times 4 = 0$

or, $2a + 3b + 4c = 0$ .... (ii)

Again,

It is perpendicular to the equation $3x + 2y + 2 = 9$ then,

$a \times 3 + 2 \times b + 2 \times c = 0$

or, $3a + 2b + 2c = 0$ .... (iii)

By cross multiplication

$2a + 3b + 4c = 0$

$3a + 2b + 2c = 0$

$\therefore \quad \frac{a}{6 - 8} = \frac{b}{12 - 4} = \frac{c}{4 - 9} = k \text{ (say)}$

$\frac{a}{-2} = \frac{b}{8} = \frac{c}{-5} = k \text{ (say)}$

$\Rightarrow a = -2k, b = 8k, c = -5k$

Substituting the values of a, b, c in equal (i) we have,

$-2k(x + 2) + 8k(y - 3) + (-5k)(z - 4) = 0$

or, $-2x - 4 + 8y - 24 - 5z + 20 = 0$

or, $-2x + 8y - 5z - 28 + 20 = 0$

or, $-2x + 8y - 5z - 8 = 0$

or, $2x - 8y + 5z + 8 = 0$ is the required equation of the plane.

b. Solution:

Here,
Repeating the same procedure as in No. 8a
So, the required equation of the plane is $2x - y + 3z = 9$

9. Solution:

a. The equation of the plane through P(a, b, c) is
$A(x - a) + B(y - b) + C(z - c) = 0$ .... (i)
The direction cosines of OP are proportional to $a - 0, b - 0, c - 0$
i.e., a, b, c
Since the plane (i) is perpendicular to OP,
$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = k$ (let)
$A = ak, B = bk, C = ck$
Substituting the values of A, B, C in (i)
$ak(x - a) + bk(y - b) + ck(z - c) = 0$
or, $a(x - a) + b(y - b) + c(z - c) = 0$
$\therefore$ $ax + by + cz = a^2 + b^2 + c^2$

b. Solution:
The equation of the plane through P(3, 2, 1) is
$a(x - 3) + b(y - 2) + c(z - 1) = 0$ .... (i)
The direction cosines of MN are proportional to $2 + 5, -4 - 3, 5 - 7$
i.e, 7, –7, –2
Since the plane (i) is perpendicular to MN,
$\frac{a}{7} = \frac{b}{-7} = \frac{c}{-2} = k$ (say)
$\therefore$ $a = 7k, b = -7k, c = -2k$
Now,
Substituting the value of a, b, c in equation (i) we have,
$7k(x - 3) - 7k(y - 2) - 2k(z - 1) = 0$
or, $7x - 21 - 7y + 14 - 2z + 2 = 0$
or, $7x - 7y - 2z - 5 = 0$ is the required equation of the plane.

10. Solution:

a. Any plane passing through (–1, 1, 2) is
$a(x + 1) + b(y - 1) + c(z - 2) = 0$ .... (i)
But, it passes through (1, –1, 1) so
$a(1 + 1) + b(-1 - 1) + c(1 - 2) = 0$
or, $2a - 2b - c = 0$ .... (ii)
The plane (i) is perpendicular to the given plane $x + 2y + 2z = 5$.
i.e. if $a + 2b + 2c = 0$ ... ... (iii)
From equation (ii) and (iii) we have
$\frac{a}{-4+2} = \frac{b}{-1-4} = \frac{c}{4+2}$
$\therefore$ $\frac{a}{-2} = \frac{b}{-5} = \frac{c}{6} = k$ (say)
$\Rightarrow$ $a = -2k, b = -5k, c = 6k$
From equation (i) we get
$-2k(x + 1) - 5k(y - 1) + 6k(z - 2) = 0$
or, $-2x - 2 - 5y + 5 + 6z - 12 = 0$
or, $-2x - 5y + 6z - 9 = 0$
or, $2x + 5y - 6z + 9 = 0$

b. Here, two planes are
$x + y + z = 5$ .... (i)
and $2x + 3y + 4z - 5 = 0$ .... (ii)
Then the equation of plane through intersection of (i) and (ii) is
$x + y + z - 5 + \lambda(2x + 3y + 4z - 5) = 0$

or, $x + y + z - 5 + 2\lambda x + 3\lambda y + 4\lambda z - 5\lambda = 0$
or, $(1 + z\lambda), c + (1 + 3\lambda) y + (1 + 4\lambda) z - 5 - 5\lambda = 0$ .... (iii)
and the plane (iii) passes through (0, 0, 0) so
$(1 + 2\lambda) 0 + (1 + 3\lambda) 0 + (1 + 4\lambda) 0 - 5 - 5\lambda = 0$
or, $-5 - 5\lambda = 0$
or, $-5 = 5\lambda$
$\Rightarrow$ $\lambda = -1$
So, the required equation of plane is
$x + y + z - 5 - 1(2x + 3y + 4z - 5) = 0$
or, $x + y + z - 5 - 2x - 3y - 4z + 5 = 0$
or, $-x - 2y - 3z = 0$
$\therefore$ $x + 2y + 3z = 0$

c. Here, two planes are:
$x + 2y + 3z + 4 = 0$ ... ... (i)
$4x + 3y + 4z + 1 = 0$ ... ... (ii)
The equation of the plane through the intersection is given by,
$x + 2y + 3z + 4 + \lambda(4x + 3y + 4z + 1) = 0$
or, $x + 2y + 3z + 4 + 4\lambda x + 3\lambda y + 4\lambda z + \lambda = 0$
or, $x(1 + 4\lambda) + y(2 + 3\lambda) + z(3 + 4\lambda) + 4 + \lambda = 0$ ... ... (ii)
And, the plane (iii) passes through the point (1, –3, –1)
$1(1 + 4\lambda) + (-3)(2 + 3\lambda) + (-1)(3 + 4\lambda) + 4 + \lambda = 0$
or, $1 + 4\lambda - 6 - 9\lambda - 3 - 4\lambda + 4 + \lambda = 0$
or, $1 - 6 - 3 + 4 - 9\lambda + \lambda = 0$
or, $-9 + 5 - 8\lambda = 0$
or, $-4 = 8\lambda$
$\therefore$ $\lambda = -\frac{4}{8}$
$= -\frac{1}{2}$

So, the required equation of the plane is $x + 2y + 32 + 4 + \left(-\frac{1}{2}\right)(4x + 3y + 4z + 1) = 0$

or, $x + 2y + 3z + 4 - \frac{4x}{2} - \frac{3y}{2} - \frac{4z}{2} - \frac{1}{2} = 0$

or, $x + 2y + 3z + 4 - 2x - \frac{3y}{2} - 2z = \frac{1}{2}$

or, $-x - \frac{3y}{2} + 2y + z + 4 - \frac{1}{2} = 0$

or, $-x - \frac{3y + 4y}{2} + z + \frac{8 - 7}{2} = 0$

or, $-2x + y + 2z + 7 = 0$
or, $2x - y - 2z = 7$ is the required equation of the plane.

11. Solution:
The equation of the plane through the intersection of the given planes is
$x + 2y + 3z - 4 + \lambda(2x + y - z) = 0$ .... (i)
or, $x + 2y + 3z - 4 + 2\lambda x + \lambda y - \lambda z = 0$
or, $(1 + 2\lambda) x + (2 + \lambda) y + (3 - \lambda) z - 4 = 0$
Since the plane (i) is perpendicular to the plane: $5x + 3y + 6z + 8 = 0$
So, $(1 + 2\lambda) 5 + (2 + \lambda) 3 + (3 - \lambda). 6 = 0$ $[\because a_1a_2 + b_1b_2 + c_1c_2 = 0]$
or, $5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$
or, $29 + 7\lambda = 0$
or, $\lambda = -\frac{29}{7}$

Substituting the value of λ in equation (i) we get

or, $x + 2y + 3z - 4 - \frac{29}{7}(2x + y - z) = 0$

or, $x + 2y + 3z - 4 - \frac{58x}{7} - \frac{29y}{7} + \frac{29z}{7} = 0$

or, $7x - 58x + 14y - 29y + 21z + 29z - 28 = 0$

or, $-51x - 15y + 50z - 28 = 0$

$\therefore$ $51x + 15y - 50z + 28 = 0$ is the required equation of the planes.

12.a. Solution:

i. The given plane is,

$2x - 3y + 3z + 27 = 0$

The distance from the point (3, 4, –5) to the plane $2x - 3y + 3z + 27 = 0$ is,

$$\pm \frac{2 \times 3 - 3 \times 4 + 3 \times -5 + 27}{\sqrt{2^2 + (-3)^2 + 3^2}}$$

$$= \pm \frac{(6 - 12 - 15 + 27)}{\sqrt{4 + 9 + 9}}$$

$$= \pm \frac{6}{\sqrt{22}} = \frac{6}{\sqrt{22}} \text{ (in magnitude)}$$

ii. Similar to No. 12a(i)

b. Here, the two points (1, –1, 3) and (3, 3, 3) are equidistance from the equation of the plane $5x + 2y - 7z + 9 = 0$.

Firstly,

$x_1 = 1, y_1 = -1, z_1 = 3$

and the equation of the plane is given by $ax_1 + by_1 + cz_1 + d = 0$

So,

$$\text{Distance} = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{5 \times 1 + 2 \times (-1) + (-7) \times 3 + 9}{\sqrt{5^2 + 2^2 + (-7)^2}}\right|$$

$$= \left|\frac{5 - 2 - 21 + 9}{\sqrt{25 + 4 + 49}}\right|$$

$$= \left|-\frac{9}{\sqrt{78}}\right|$$

$$\text{Distance} = \frac{9}{\sqrt{78}} \text{ units}$$

Similarly, $x_2 = 3, y_2 = 3, z_2 = 3$

$$\text{Distance} = \left|\frac{ax_2 + by_2 + cz_2 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{5 \times 3 + 2 \times 3 + (-7) \times 3 + 9}{\sqrt{5^2 + 2^2 + (-7)^2}}\right|$$

$$= \left|\frac{15 + 6 - 21 + 9}{\sqrt{25 + 4 + 49}}\right|$$

$$= \left|\frac{9}{\sqrt{78}}\right|$$

$\therefore$ $\text{Distance} = \frac{9}{\sqrt{78}}$ units.

Since the two point (1, –1, 3) and (3, 3, 3) are in at the same distance from the given plane.

$\therefore$ They are at equidistance from the plane.

c. Solution:

The equation of two parallel planes is given by,

3x + 3y – 6z + 1 = 0 .... (i)

6x + 4y – 12z + 9 = 0 .... (ii)

Now,

In equation (i) let y = z = 0 then, $x = -\frac{1}{3}$

i.e, $\left(-\frac{1}{3}, 0, 0\right)$ is the point which lies in equation (i) plane.

i.e, $(x_1, y_1, z_1) = \left(-\frac{1}{3}, 0, 0\right)$

and

$$\text{Distance} = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$= \left|\frac{6 \times \left(-\frac{1}{3}\right) + 4 \times 0 - 12 \times 0 + 9}{\sqrt{6^2 + 4^2 + (-12)^2}}\right| = \left|\frac{7}{\sqrt{36 + 16 = 144}}\right| = \left|\frac{7}{\sqrt{196}}\right| = \left|\frac{7}{14}\right|$$

$\therefore$ Distance $= \frac{1}{2}$

Hence, according to the qn, the distance between the two planes is $\frac{1}{2}$ units

13. Solution:

Let the plane (ΔABC) whose vertices are

A(a, 0, 0), B (0, b, 0), and (0, 0, c) at a distance of 3p units from the origin.

So, the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

or, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$ .... (i)

Since, 1^v distance of the equation (i) from (0, 0, 0) is 3p,

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{\frac{1}{a} . 0 + \frac{1}{b} . 0 + \frac{1}{c} . 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right|$$

or, $3p = \frac{1}{\sqrt{\frac{1}{q^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$

or, $\frac{1}{(3p)^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ .... (ii)

Let (α, β, α) be the centroid of ΔABC

So,

$\alpha = \frac{a + 0 + 0}{3}$

$\therefore$ $\alpha = \frac{a}{3} \Rightarrow a = 3\alpha$

Similarly, $b = 3\beta$ and $c = 3\nu$

Now,

From equation (ii)

$\frac{1}{9p^2} = \frac{1}{(3\alpha)^2} + \frac{1}{(3\beta)^2} + \frac{1}{(3\nu)^2}$

or, $\frac{1}{9p^2} = \frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\nu^2}$

or, $\frac{1}{p^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\nu^2}$

$\therefore$ $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\nu^2} = \frac{1}{p^2}$ proved.

Hence, the locus of the centroid of ΔABC is

$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$

Chapter 12
Product of Vectors

Exercise 12.1

1. Solution

a. Given, $\vec{a} = 2\vec{i} - 3\vec{k} = (2, 0, -3)$

$\vec{b} = 2\vec{j} + 4\vec{k} = (0, 2, 4)$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -3 \\ 0 & 2 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & -3 \\ 2 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -3 \\ 0 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$

$= 6\vec{i} - 8\vec{j} + 4\vec{k}$

b. Given vectors

$\vec{a} = 2\vec{i} + 4\vec{k} = (2, 0, 4)$

$\vec{b} = 3\vec{j} - 2\vec{k} = (0, 3 -2)$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 4 \\ 0 & 3 & -2 \end{vmatrix} = -12\vec{i} + 4\vec{j} + 6\vec{k}$

c. Given,

$\vec{a} = 20\vec{i} + 3\vec{j} + \vec{k}$

$\vec{b} = -\vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ -1 & -2 & 3 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix}$

$= 11\vec{i} - 7\vec{j} - \vec{k}$

2. Given $\vec{a} = 3\vec{i} + 4\vec{j} - 5\vec{k}$

$\vec{b} = 7\vec{i} - 3\vec{j} + 6\vec{k}$

$\vec{a}+\vec{b} = (3\vec{i} + 4\vec{j} - 5\vec{k}) + (7\vec{i} - 3\vec{j} + 6\vec{k}) = 10\vec{i} + \vec{j} + \vec{k}$

$\vec{a} - \vec{b} = (3i + 4j - 5k) - (7\vec{i} - 3\vec{j} + 6\vec{k}) = -4\vec{i} + 7\vec{j} - 11\vec{k}$

$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 10 & 1 & 1 \\ -4 & 7 & -11 \end{vmatrix}$

$= -18\vec{i} + 106\vec{j} + 74\vec{k}$

$|(\vec{a}+\vec{b}) \times (\vec{a} - \vec{b})| = \sqrt{(-18)^2 + (106)^2 + (74)+} = \sqrt{17036}$

3. Here, $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}\vec{i}$

$(\vec{a} + \vec{b}) = (1 + 2)\vec{i} + (1 + 3)\vec{j} + (1 + 1)\vec{k} = 3\vec{i} + 4\vec{j} + 2\vec{k}$

$(\vec{a} - \vec{b}) = (1 - 2)\vec{i} + (1 - 3)\vec{j} + (1 - 1)\vec{k} = -\vec{i} - 2\vec{j} + 0\vec{k}$

$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 2 \\ -1 & -2 & 0 \end{vmatrix}$

$= (0 + 4)\vec{i} - \vec{j}(0 + 2) + \vec{k}(-6 + 4)$

$= 4\vec{i} - 2\vec{j} - 2\vec{k}$

$|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| = \sqrt{4^2 + (-2)^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$

4. Solution:

a. Given vectors $\vec{a} = 4\vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{b} = 5\vec{i} + \vec{j} - 4\vec{k}$

The vector orthogonal to each of given vectors is given by

$\vec{a}\times\vec{b}$

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 3 \\ 5 & 1 & -4 \end{vmatrix} = (8 - 3)\ \vec{i} - (-16 - 15)\vec{j} + (4 + 10)\vec{k}$

$= 5\vec{i} + 31\vec{j} + 14\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{5^2 + 31^2 + 14^2} = \sqrt{1182}$

Unit vector is given as $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \dfrac{5\vec{i} + 31\vec{j} + 14\vec{k}}{\sqrt{1182}}$

b. Here,

$\vec{a} = (6, 3, -5)$ and $\vec{b} = (1, -4, 2)$

$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 3 & -5 \\ 1 & -4 & 2 \end{vmatrix}$

$(6 - 20)\ \vec{i} - (12 + 5)\vec{j} + (-24 - 4)\vec{k}$

$= -14\vec{i} - 17\vec{j} - 27\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-14)^2 + (-17)^2 + (-27)^2} = \sqrt{1214}$

$\therefore$ Unit vector is $\dfrac{-14\vec{i} - 17\vec{j} - 27\vec{k}}{\sqrt{1214}}$

5. Solution:

a. Given, $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$

$\vec{b} = 4\vec{i} - 7\vec{k}$

$a = |\vec{a}| = \sqrt{4 + 9 + 25} = \sqrt{38}$

$b = |\vec{b}| = \sqrt{4^2 + (-7)^2} = \sqrt{65}$

$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ 4 & 0 & -7 \end{vmatrix} = 21\vec{i} + 34\vec{j} + 12\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{21^2 + 34^2 + 12^2} = \sqrt{1741}$

We know that $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|} = \frac{\sqrt{1741}}{\sqrt{38}\,.\,\sqrt{65}} = \sqrt{\frac{1741}{2470}}$

b. Given, $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 9)$

$a = |\vec{a}| = \sqrt{9 + 1 + 4} = \sqrt{14}$

$b = |\vec{b}| = \sqrt{4 + 4 + 81} = \sqrt{89}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 9 \end{vmatrix} = 13\vec{i} - 23\vec{j} - 8\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{13^2 + (-23)^2 + (-8)^2} = \sqrt{762}$

$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{762}}{\sqrt{14 \times 89}} = \sqrt{\frac{762}{14 \times 89}} = \sqrt{\frac{381}{623}}$

c. Given rectors are $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$

$a = |\vec{a}| = \sqrt{9 + 1 + 4} = \sqrt{14}$

$b = |\vec{b}| = \sqrt{4 + 4 + 16} = \sqrt{24}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 8\vec{i} - 8\vec{j} - \vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{64 + 64 + 64} = \sqrt{192}$

$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{ab} = \sqrt{\frac{192}{14 \times 24}} = \frac{2}{\sqrt{7}}$

6. Let O be the origin

Given $\vec{OP} = \vec{i} + \vec{j} + 2\vec{k}$

$\vec{OQ} = 2\vec{i} + 3\vec{j} + \vec{k}$

$\vec{OP} = 3\vec{i} - \vec{j} + 4\vec{k}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \vec{OQ} = (2, 3, 1) - (1, 1, 2) = (1, 2, -1) = \vec{i} + 2\vec{j} - \vec{k}$

$\vec{QR} = \vec{OR} - \vec{OQ} = (3, -1, 4) - (2, 3, 1) = (1, -4, 3) = \vec{i} - 4\vec{j} + 3\vec{k}$

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 1 & -4 & 3 \end{vmatrix} = 2\vec{i} - 4\vec{j} - 6\vec{k}$$

Hence, $2\vec{i} - 4\vec{j} - 6\vec{k}$ is a vector perpendicular to both $\vec{PQ}$ and $\vec{QR}$ and hence perpendicular to the plane PQR.

7. Solution:

a. $\vec{a} = 3\vec{i} + \vec{j} + \vec{k}$

$\vec{b} = \vec{i} - 2\vec{j} - \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \vec{i} + 4\vec{j} - 7\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{1 + 16 + 49} = \sqrt{66}$

$\therefore$ Area of triangle determined by $\vec{a}$ and $\vec{b}$ is given by

$\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{66}$ sq. units

b. Given vectors $\vec{a} = (3, 4, 0)$ and $\vec{b} = (-5, 7, 0)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ -5 & 7 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 41\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{0^2 + 0^2 + 41^2} = 41$

Area of triangle determined by the vectors $\vec{a}$ and $\vec{b}$ is given by

$\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2} \times 41 = 20\frac{1}{2}$ sq. unit

8. Let 0 be the origin. Let A, B and C be vertices of triangle

Then $\vec{OA} = 3\vec{i} - \vec{j} + 2\vec{k}$

$\vec{OB} = \vec{i} - \vec{j} - 3\vec{k}$

$\vec{OC} = 4\vec{i} - 3\vec{j} + 2\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = -2\vec{i} + 0\vec{j} - 5\vec{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = 3\vec{i} - 2\vec{j} + 5\vec{k}$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & -5 \\ 3 & -2 & 5 \end{vmatrix} = -10\vec{i} - 5\vec{j} + 4\vec{k}$$

$|\vec{AB} \times \vec{BC}| = \sqrt{100 + 25 + 16} = \sqrt{141}$

$\therefore$ Area of triangle ABC $= \frac{1}{2}|\vec{AB} \times \vec{BC}| = \frac{1}{2}\sqrt{141}$ sq. units

9. Solution:

a. Given,

$\vec{a} = 7\vec{i} + 8\vec{j} - \vec{k}$

$\vec{b} = 10\vec{i} - 11\vec{j} + 12\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 8 & -1 \\ 10 & -11 & 12 \end{vmatrix} = (96 - 11)\vec{i} - (84 + 10)\vec{j} + (-77 - 80)\vec{k}$$

$= 85\vec{i} - 94\vec{j} - 157\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{85^2 + 94^2 + 157^2} = \sqrt{40710}$

$\therefore$ Area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\sqrt{40710}$ sq. units.

b. $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$

$\vec{b} = \vec{i} - 2\vec{j} + 4\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -2 & 4 \end{vmatrix} = (8+6)\vec{i} - (4 - 3)\vec{j} + (-2 - 2)\vec{k}$$

$= 14\vec{i} - \vec{j} - 4\vec{k}$

$|\vec{a}\times\vec{b}| = \sqrt{14^2 + 1^2 + 4^2} = \sqrt{196 + 1 + 16} = \sqrt{2131}$

$\therefore$ Area of parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{213}$ sq units.

c. Given, $\vec{a} = (1, -2, 3)$ and $\vec{b} = (3, 2, 2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 3 & 2 & 2 \end{vmatrix} = (-4-6)\vec{i} - (2-9)\vec{j} + (2+6)\vec{k}\vec{k}$$

$= -10\vec{i} + 7\vec{j} + 8\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{100 + 49 + 64} = \sqrt{213}$

$\therefore$ Area of parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{213}$ sq units

d. Given,

$\vec{a} = (1, -2, 3)$ and $\vec{b} = (3, 2, 2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 3 & 2 & 2 \end{vmatrix} = (-4-6)\vec{i} - (2-9)\vec{j} + (2+6)\vec{k}$$

$= -10\vec{i} + 7\vec{j} + 8\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{100 + 49 + 64} = \sqrt{213}$

Area of parallelogram whose adjacent sides $\vec{a}$ and $\vec{b}$ is given by

$|\vec{a} \times \vec{b}| = \sqrt{213}$ square units.

10. Let $\vec{d_1} = \vec{i} + \vec{j} - \vec{k}$ and $\vec{d_2} = \vec{i} - \vec{j} + \vec{k}$ be two diagonals of a parallelogram.

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 0\vec{i} - 2\vec{j} - 2\vec{k}$$

$|\vec{d_1} \times \vec{d_2}| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Area of parallelogram whose diagonals $\vec{d_1}$ and $\vec{d_2}$ is given by

$\frac{1}{2}|\vec{d_1} \times \vec{d_2}|$ sq. units

$= \frac{1}{2} . 2\sqrt{2}$ sq. units

$= \sqrt{2}$ sq. units

11. Solution:

a. Given $|\vec{a}| = 15$, $|\vec{b}| = 4$ and $|\vec{a} \times \vec{b}| = 36$

If θ be the angle between two vectors $\vec{a}$ and $\vec{b}$ then

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{36}{15 \times 4} = \frac{9}{15} = \frac{3}{5}$$

$$\therefore \quad \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

Also, we know that $\cos\theta = \dfrac{\vec{a} . \vec{b}}{|\vec{a}||\vec{b}|}$

$\therefore \quad \vec{a} . \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$= 15 \times 4 \times \frac{4}{5}$

$= 48$

$\therefore \quad \vec{a}.\vec{b} = 48$

b. Given, $|\vec{a}| = 9$, $|\vec{b}| = 5$ and $\vec{a}.\vec{b} = 36$

If θ be the angle between $\vec{a}$ and $\vec{b}$

Then, $\cos\theta = \dfrac{\vec{a}.\vec{b}}{ab} = \dfrac{36}{9\times5} = \dfrac{4}{5}$

$\therefore \quad \sin\theta = \sqrt{1\ \cos^2\theta} = \dfrac{3}{5}$

Also, $|\vec{a}\times\vec{b}| = |\vec{a}|\ |\vec{b}| \sin\theta$

$= 9 \times 5 \times \dfrac{3}{5}$

$= 27$

c. LHS

$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a} + \vec{c}\times\vec{b}$

$= \vec{a}\times\vec{b} - \vec{c}\times\vec{a} + \vec{b}\times\vec{c} - \vec{a}\times\vec{b} + \vec{c}\times\vec{a} - \vec{b}\times\vec{c}$

$= 0$ RHS

d. Suppose

$\vec{a} + \vec{b} + \vec{c} = 0$

$\vec{a} + \vec{b} = -\vec{c}$ ... ... (i)

Taking cross product with $\vec{a}$ on both sides

$\vec{a} \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{c})$

or, $\vec{a}\times\vec{a} + \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$

or, $0 + \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$

or, $\vec{a}\times\vec{b} = \vec{c}\times\vec{a}$ ... ... (ii)

Again, taking cross product with $\vec{b}$ on equation (i) both sides

$\vec{b} \times (\vec{a} + \vec{b}) = \vec{b} \times (-\vec{c})$

$\vec{b} \times \vec{a} + \vec{b}\times\vec{b} = -\vec{b}\times\vec{c}$

or, $-\vec{a}\times\vec{b} = 0 = -\vec{b}\times\vec{c}$

$\therefore \quad \vec{a}\times\vec{b} = \vec{b}\times\vec{c}$ ... ... (iii)

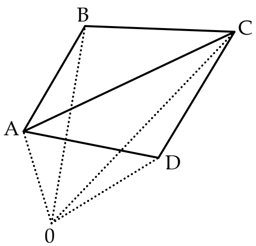
Combining (ii) and (iii) we get,

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ Proved.

12. Solution:

Let 0 be the origin suppose A, B, C, D are vertices of a quadrilateral ABCD.

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$ and $\vec{OD} = \vec{d}$



Now, $\vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

$\vec{BC} = \vec{OC} - \vec{OB} = \vec{c} - \vec{b}$

$\vec{AC} = \vec{OC} - \vec{OA} = \vec{c} - \vec{a}$

$\vec{CD} = \vec{OD} - \vec{OC} = \vec{d} - \vec{c}$

Vector area of $\Delta ABC = \dfrac{1}{2} \vec{AB} \times \vec{BC}$

$= \dfrac{1}{2} (\vec{b} - \vec{a}) \times (\vec{c} - \vec{b})$

$= \dfrac{1}{2} [\vec{b}\times\vec{c} - \vec{b}\times\vec{b} - \vec{a}\times\vec{c} + \vec{a}\times\vec{b}]$

$= \dfrac{1}{2} [\vec{b}\times\vec{c} - 0 + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}]$

$$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}]$$

Again, vector area of $\Delta ACD = \frac{1}{2}(\vec{AC}\times\vec{CD})$

$$= \frac{1}{2}[(\vec{c} - \vec{a}) \times (\vec{d} - \vec{c})]$$

$$= \frac{1}{2}[\vec{c}\times\vec{d} - \vec{c}\times\vec{c} - \vec{a}\times\vec{d} + \vec{a}\times\vec{c}]$$

$$= \frac{1}{2}[\vec{c}\times\vec{d} + \vec{d}\times\vec{a} + \vec{a}\times\vec{c}]$$

$\therefore$ Vector of quadrilateral ABCD = vector area of ΔABC + vector area of ΔACD

$$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{c}+\vec{d} + \vec{d}\times\vec{a} + \vec{a}\times\vec{b}\times\vec{c}]$$

$$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{c}+\vec{d} + \vec{d}\times\vec{a} - \vec{c}\times\vec{a}]$$

$$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}]$$

13. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be the position vectors of the vertices of a quadrilateral ABCD, then the vector area of this quadrilateral is given by

$\frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}]$ ... ... (i)

Now, $\vec{AC} = \vec{OC} - \vec{OA} = \vec{C} - \vec{a}$

$\vec{BD} = \vec{OD} - \vec{OB} = \vec{d} - \vec{b}$

$\overrightarrow{AC} \times \overrightarrow{BD} = (\vec{c} - \vec{a}) \times (\vec{d}-\vec{b})$

$= \vec{c}\times\vec{d} - \vec{c}\times\vec{b} - \vec{a}\times\vec{d} + \vec{a}\times\vec{b}$

$= \vec{c}\times\vec{d} + \vec{b}\times\vec{c} + \vec{d}\times\vec{a} + \vec{a}\times\vec{d}$

$= \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}$

$\therefore \frac{1}{2}\vec{AC} \times \vec{BD} = \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}]$ ... ... (ii)

$\therefore$ Combining the result of (i) and (ii),

Vector area of quadrilateral ABCD = $\frac{1}{2}\vec{AC} \times \vec{BD}$ proved.

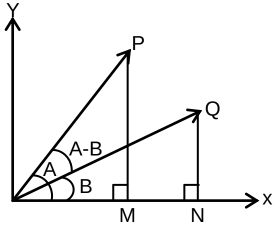
14. Solution:

Let OX and OY be two co-ordinate axes. Let P and Q be two points even that $\sphericalangle XOP = A$ and $XOQ = B$ so that $\sphericalangle POQ = A–B$. Let $OP = r_1$ and $OQ = r_2$.
Draw 1_r PM and QN on x–axis.

$\vec{OP}$ = (OM, MP) = (OP cosA, OP sinA) = (r_1 cosA, r_1 sinA)

$\vec{OQ}$ = (ON, NQ) = (OQ cosB, OQ sinB) = (r_2 cosB, r_2 sinB)

Since, angle between OQ and OP is A–B.



$\therefore |\vec{OP}\times \vec{OQ}| = |\vec{OP}||\vec{OQ}| \sin(A - B)$ ... ... (i)

Now, $\vec{OP}\times \vec{OQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_1\cos A & r_1\sin A & 0 \\ r_2\cos B & r_2\sin B & 0 \end{vmatrix}$

$= 0\vec{i} - 0\vec{j} + (r_1r_2 \cos A \sin B - r_1r_2 \sin A . \cos B)\vec{k}$

$= (0, 0, -r_1r_2 (\sin A \cos B - \cos A \sin B)$

$|\vec{OP}\times\vec{OQ}| = \sqrt{0^2 + 0^2 + (-r_1r_2)^2 (\sin A \cos B - \cos A.\sin B)^2}$

$= r_1r_2 (\sin A . \cos B - \cos A . \sin B)$

from (i)

$\frac{|\vec{OP}\times\vec{OQ}|}{|\vec{OP}||\vec{0Q}|} = \sin(A - B)$

$\therefore \quad \sin(A - B) = \frac{r_1r_2\ (\sin A \cos B - \cos A - \sin B)}{r_1r_2}$

$\therefore \quad \sin(A–B) = \sin A.\ \cos B – \cos A\ .\ \sin B$

15. Suppose ABC be a triangle in which $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$

Now, by vector addition,

$\vec{AB} + \vec{Bc} = \vec{Ac}$

$\Rightarrow \quad \vec{c} + \vec{d} = -\vec{b}$

$\therefore \quad \vec{a} + \vec{b} + \vec{c} = \vec{o}$ ... ... (i)

Multiplying (i) vectorically by $\vec{a}$, we get

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{o}$

$\vec{a}\times\vec{a} + \vec{a}\times\vec{b} + \vec{a}\times\vec{c} = 0$

$\therefore \quad \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$ ... ... (ii) $\quad (\because\ \vec{a}\times\vec{a} = 0 \text{ and } \vec{a}\times\vec{c} = -\vec{c}\times\vec{a})$

Similarly, multiplying (i) vectorially by $\vec{b}$, we get

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c}$ ... ... (iii)

Combining (ii) and (iii) we get

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

or, $\quad |\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$

ab $\sin(\pi–c)$ = bc $\sin(\pi–A)$ = ca $\sin(\pi–B)$

or, $\quad$ absinC = bcsinA = casinB

Dividing by abc, we get

$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ proved.

16. Given,

$\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$

$\vec{b} = 2\vec{i} + \vec{j} - 3\vec{k}$

$\vec{c} = \vec{i} - 2\vec{j} + 2\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \vec{i} + 13\vec{j} + 5\vec{k}$$

$$(\vec{a}\times\vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 13 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 36\vec{i} + 3\vec{j} - 15\vec{k} \text{ ... ... (i)}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & -2 & 2 \end{vmatrix} = -4\vec{i} - 7\vec{j} - 5\vec{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ -4 & -7 & -5 \end{vmatrix} = 19\vec{i} + 7\vec{j} - 25\vec{k} \text{ ... ... (ii)}$$

From (i) and (ii),

Hence $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

17. Given,

$\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{c} = \vec{i} - \vec{j}$

Let $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{a}.\vec{b} = (\vec{i} + \vec{j} + \vec{k}) . (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$

$3 = b_1 + b_2 + b_3$ ... ... (i)

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} = (b_3 - b_2)\vec{i} + (b_1 - b_3)\vec{j} + (b_2 - b_1)\vec{k}$$

or, $\vec{c} = (b_3 - b_2)\vec{i} + (b_1 - b_3)\vec{j} + (b_2 - b_1)\vec{k}$

$\vec{i} - \vec{j} = (b_3 - b_2)\vec{i} + (b_1 - b_3)\vec{j} + (b_2 - b_1)\vec{k}$

Equating corresponding vectors

$b_3 - b_2 = 1$, $b_1 - b_3 = -1$ and $b_2 - b_1 = 0$

i.e. $b_2 - b_1 = 0$

$\therefore$ $b_1 = b_2$ ... ... (ii)

$b_3 = 1 + b_2$ ... ... (iii)

$b_3 = 1 + b_1$ ... ... (iv)

$b_1 + b_2 + b_3 = 3$

$b_1 + b_1 + 1 + b_1 = 3$

$3b_1 = 2$

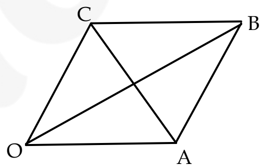
$\therefore$ $b_1 = \frac{2}{3}$

$\therefore$ from (ii) $b_2 = \frac{2}{3}$

from (iv) $b_3 = 1 + \frac{2}{3} = \frac{5}{3}$

$\therefore$ $\vec{b} = (b_1, b_2, b_3) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{5}{3}\vec{k} = \frac{1}{3}(2\vec{i} + 2\vec{j} + 5\vec{k})$

18. Let $\vec{OA} = \vec{a}$ and $\vec{OC} = \vec{b}$



Then, $\vec{OB} = \vec{OA} + \vec{AB}$

$= \vec{A} = O\vec{c} = \vec{a} + \vec{b}$

and $\vec{AC} = \vec{AO} + \vec{OC}$

$= -\vec{a} + \vec{b} = \vec{b} - \vec{a}$

Now, the vector equation of the line OB is

$\vec{r} = t(\vec{a} + \vec{b})$ ... ... (i)

Where t is a scalar.

Again, the vector equation of the straight line AC is

$\vec{r} = (1 - 5)\,\vec{a} + 5\vec{b}$ ... ... (ii) where s is a scalar.

If two diagonals OB and AC meet at M, then for M, we have

$t(\vec{a} + \vec{b}) = (1 - 5)\,\vec{a} + 5\vec{b}$

Equating the coeff. of like vectors,

t = 1–5 and t = 5

Solving $t = 5 = \frac{1}{2}$

$\therefore$ The position vector of M i.e. $\overrightarrow{OM} = \frac{1}{2}(\vec{a}+\vec{b}) = \frac{1}{2}\ \vec{OB}$

Also, $\vec{AM} = \vec{AO} + \vec{OM}$

$= -\vec{a} + \frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{2}(\vec{b} - \vec{a}) = \frac{1}{2}\vec{AC}$

$\therefore$ Hence the diagonals bisect each other.

Again, $\vec{OB}.\vec{AC} = (\vec{a} + \vec{b}) . (\vec{b} - \vec{a})$

$= (\vec{b})^2 - (\vec{a})^2$

$= b^2 - a^2$

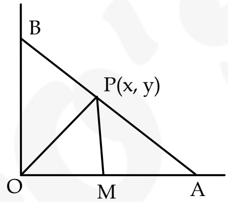
$= OC^2 - OA^2$

$= 0$ ($\because$ OC = OA)

So, the diagonals of a rhombus are right angles.

$\therefore$ The diagonals of a rhombus bisect each other at right angles.

19. Let $\vec{i}$ and $\vec{j}$ be the unit vectors along two mutually perpendicular straight lines OX and OY respectively. Let OA = a and OB = b.



Then $\vec{OA} = a\vec{i}$

$\vec{OB} = b\vec{j}$

Let P(x, y) be a point on the line AB.

From P, draw PM $\perp_r$ to OA.

Then $\vec{OM} = x\vec{i}$ and $\vec{MP} = y\vec{j}$

Join OP

By vector addition,

$\vec{OP} = \vec{OM} + \vec{MP}$

$x\vec{i} + y\vec{j}$ ... ... (i)

Again, the vector equation of the straight line AB is

$\vec{r} = (l - t)\, a\vec{i} + tb\vec{j}$ ... ... (ii)

For P, the point of intersection of OP and AB, we have

$x\vec{i} + y\vec{j} = (1-t)a\vec{i} + tb\vec{j}$

Equating the coeff. of like vectors,

x = (1 – t)a and y = tb

$\frac{x}{a} = 1 - t \qquad \frac{y}{b} = t$

$\therefore \frac{x}{a} = 1 - \frac{y}{b}$

$\therefore \boxed{\frac{x}{a} + \frac{y}{b} = 1}$

Exercise 12.2

1. Solution:

a. Here, $\vec{a} = (1, 2, 0)$, $\vec{b}$ (2, 0, 3) and $\vec{c} = (2, -1, 2)$

Then, $\vec{b}\times\vec{c} = (2, 0, 3) \times (1, -1, 2)$

$$\begin{matrix} 2 & 0 & 3 & 2 & 0 \\ 1 & -1 & 2 & 1 & -1 \end{matrix}$$

$= (0\times2 - (-1) \times 3, 3\times1 - 2\times2, 2\times-1 - 1\times0$

$= (3, -1, -2)$

Now, $\vec{a}(\vec{b} \times \vec{c}) = (1, 2, 0) . (3, -1, -2)$

$= 1 \times 3 + 2 \times -1 + 0 \times -2$

$= 1$

b. Here, $\vec{a} = (-1, 2, 3)$, $\vec{b} = (0, 1, -2)$ and $\vec{c} = (3, 0, -1)$

Then $\vec{b}\times\vec{c} = (0, 1, -2) \times (3, 0, -1)$

$$= \begin{array}{cccccc} 0 & 1 & -2 & 0 & 1 \\ 3 & 0 & -1 & 3 & 0 \end{array}$$

$= (-1 + 0, -6 + 0, 0 - 3)$

$= (-1, -6, -3)$

and $\vec{a} \times \vec{b} = (-1, 2, 3) \times (0, 1, -2)$

$$= \begin{array}{cccccc} -1 & 2 & 3 & -1 & 2 \\ 0 & 1 & -2 & 0 & 1 \end{array}$$

$= (-4, -3, 0, -2, -1, -0)$

$= (-7, -2, -1)$

Now, $\vec{a} \,.\, (\vec{b} \times \vec{c}) = (-1, 2, 3) \,.\, (-1, -6, -3)$

$= (-1\times-1 + 2\times-6 + 3\times-3)$

$= -20$

and $(\vec{a}\times\vec{b}) \,.\, \vec{c} = (-7, -2, -1) \,.\, (3, 0, -1)$

$= -7 \times 3 + (-2) \times 0 + (-1) \times -1$

$= -21 + 0 + 1$

$= -20$

2. We have,

$\vec{a} = \vec{i} - \vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j}$ and $c = 3\vec{k}$

Now, $\vec{a}\,(\vec{b} \times \vec{c}) = (\vec{i} - \vec{j} - \vec{k}) \,.\, [(2\vec{i} + \vec{j}) \times (3\vec{k})]$

$= (\vec{i} - \vec{j} - \vec{k}) \,.\, (-6\vec{j} + 3\vec{i})$

$= (\vec{i} - \vec{j} - \vec{k}) \,.\, (3\vec{i} - 6\vec{j})$

$= 3 + 6 - 0 = 9$

3. Solution:

a. Here, the adjacent edges of a parallelepiped are represented by the vectors $\vec{i}+\vec{j}$, $2\vec{i} - \vec{j} + \vec{k}$ and $\vec{i}+2\vec{j}-3\vec{k}$

$= (1, 1, 0), (2, -1, 1)$ and $(1, 2, -3)$

i.e., $\vec{a}.(\vec{b} \times \vec{c})$ ... ... (i)

Where, $\vec{b}\times\vec{c} = (2, -1, 1) \times (1, 2, -3)$

$$= \begin{array}{cccccc} 2 & -1 & 1 & 2 & -1 \\ 1 & 2 & -3 & 1 & 2 \end{array}$$

$= (3 - 2, 1 + 6, 4 + 1)$

$= (1, 7, 5)$

$\therefore$ from (i) volume $= (1, 1, 0) \,.\, (1, 7, 5)$

$= (1.1 + 1.7 + 0.5)$

$= (1 + 7 + 0)$

$= 8$

b. Let $\vec{a} = 5\vec{i} + 2\vec{j} - 3\vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} - 4\vec{k}$ and $\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k}$

The volume of the parallelepiped represented by the given three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by

$$\vec{a} \,.\, (\vec{b} \times \vec{c}) = (5\vec{i} + 2\vec{j} - 3\vec{k}) \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -4 \\ 3 & -1 & 2 \end{vmatrix}$$

$= (5\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{i} - 16\vec{j} - 11\vec{k})$
$= 10 - 32 + 33 = 11$ cu units

4. Here, the three concurrent edges of a parallelogpiped are given by $2\vec{i} + 3\vec{j} + m\vec{k}$, $\vec{i} - 2\vec{j}$ and $3\vec{i} + \vec{j} - 2\vec{k}$ and volume = 26 cu.unit.

Let the three edges be denoted by

$\vec{a} = (2, 3, -m)$, $\vec{b} = (1, -2, 0)$ and $\vec{c} = (3, 1, -2)$

Now, $\vec{b}\times\vec{c} = (1, -2, 0) \times$

$$= \begin{array}{cccccc} 1 & -2 & 0 & 1 & -2 \\ 3 & 1 & -2 & 3 & 1 \end{array}$$

$= (4 - 1, 0 + 2, 1 + 6) = (3, 2, 7)$

Using the volume of parallelepiped $= \vec{a} \cdot (\vec{b}\times\vec{c})$

or, $26 = (2, 3, -m) \cdot (3, 2, 7)$
or, $26 = (6 + 6 - 7m)$
or, $26 = 12 - 7m$
or, $7m = 12 = 26 = -14$
$\therefore$ $m = -2$

Chapter 13
Correlation and Regression

Exercise 13.1

1. Solution:

a. Here, cov (x, y) = –16.5

var (x) = 2.89

var (y) = 100

Coefficient off correlation (r) = $\dfrac{\text{cov(x, y)}}{\sqrt{\text{val(x)}}\,.\,\sqrt{\text{var(y)}}}$

$= \dfrac{-16.5}{\sqrt{2.89}\,.\,\sqrt{100}}$

$= \dfrac{-16.5}{1.7\times10}$

$= \dfrac{-16.5}{17}$

$= -0.97$

b. Here,

Given, N = 13

$\Sigma x = 117$

$\Sigma x^3 = 1{,}313$

$\Sigma y = 260$

$\Sigma y^2 = 6580$

$\Sigma xy = 2827$

Coefficient of correlation (r) = $\dfrac{n\Sigma xy - \Sigma x.\,\Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\,.\,\sqrt{n\Sigma y - (\Sigma y)^2}}$

$= \dfrac{3 \times 2827 - 117\times260}{\sqrt{13\times1313} - \sqrt{13\times6580 - \ldots.}}$

$= \dfrac{36751 - 30420}{\sqrt{3380} - \sqrt{17940}}$

$= \dfrac{6331}{58.13\times133.94}$

$= 0.81$

c. Solution:

Here, n = 15

$\sigma x = 3.2$

$\sigma y = 3.y$

$\Sigma(x - \overline{x})\,(y - \overline{y}) = 122$

Coefficient of correlation (r) = $\dfrac{\Sigma(x - \overline{x})\,(y - \overline{y})}{n\sigma x\sigma y}$

$= \dfrac{122}{15\times3.2\times3.y}$

$= \dfrac{122}{163.2}$

$= 0.75$

2. a. Solution:

Karl Pearson's coefficient of correlation between 'x' and 'y' (i) = 0.28

Cor (x, y) = 0.76

Val (x) = 9

σy = ?

Now, We have,

Coefficient of correlation (r) = $\dfrac{\text{cov}(x, y)}{\sqrt{\text{var}(x)}.\sqrt{\text{var}(y)}}$

or, $0.26 = \dfrac{0.76}{\sqrt{9}\,.\sqrt{\text{var}(y)}}$

or, $0.36 = \dfrac{1}{3.\sqrt{\text{var}(y)}}$

or, $1.105\,.\sqrt{\text{var}(y)} = 1$

or, $\sqrt{\text{var}\ (y)} = 0.904$

$\therefore\ \sigma_y = 0.904$

b. Solution:

Correlation coefficient between x and y (r) = 0.85

Cov (x,y) = 6.5

Var (x) = 6.1

Standard derivation of y (σ_y) = ?

Now, we have,

Coefficient of correlation (r) = $\dfrac{\text{cov}(x,y)}{\sqrt{\text{var}(x)}\,.\sqrt{\text{var}(y)}}$

or, $0.85 = \dfrac{6.5}{\sqrt{6.1}\,.\sqrt{\text{var}(y)}}$

or, $0.13076 \times 24698\,.\sqrt{\text{var}(y)} = 1$

or, $\sqrt{\text{var}(y)} = \dfrac{1}{0.32295}$

or, $\sqrt{\text{var}(y)} = 3.096$

or, $\sigma_y = 3.1$

Hence, the required σ_y is 3.1

3. Solution:

a. Here,

Maths(x)	Biology (y)	$x = (x-\bar{x})$	$y=(y-\bar{y})$	xy	x^2	y^2
48	45	14	10	140	196	100
35	20	1	–15	–15	1	225
17	40	–17	5	–85	289	25
23	25	–11	–10	110	121	100
47	45	13	10	130	169	100
$\Sigma x = 170$	$\Sigma y = 175$			$\Sigma xy=280$	$\Sigma x^2=776$	$\Sigma y^2=550$

$\bar{x} = \dfrac{\Sigma x}{n} = \dfrac{170}{5} = 34,\qquad \bar{y} = \dfrac{\Sigma y}{N} = \dfrac{175}{5} = 35$

Karl Pearn's coefficient of correlation (r) = $\dfrac{\Sigma xy}{\sqrt{\Sigma x^2}\,.\sqrt{\Sigma y^2}}$

$= \dfrac{280}{\sqrt{776}\,.\sqrt{550}}$

$= \dfrac{280}{\sqrt{776}\,.\sqrt{550}}$

$$= \frac{280}{27.85 \times 23.45}$$

$$= \frac{280}{653.0325}$$

$$= 0.42$$

b. Calculation of correlation coefficient

Data of price (Rs.) (x)	Demand (Rs.) (y)	u = x – 19	$v = v_1 - 20$	u^2	v^2	uv
14	24	–5	–4	25	16	20
16	22	–3	–2	9	4	6
19	20	0	0	0	0	0
22	24	3	4	9	16	12
24	23	5	3	25	9	15
30	28	11	8	121	64	88
		11	9	189	109	141

∴ Required correlation coefficient is given by

$$r = \frac{n\Sigma uv - \Sigma u\Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}}$$

$$= \frac{6 \times 141 - 11 \times 9}{\sqrt{6 \times 189 - (11)^2}\sqrt{6 \times 109 - (9)^2}}$$

$$= \frac{846 - 99}{\sqrt{1013}\sqrt{573}} = \frac{747}{31.83 \times 23.94} = \frac{747}{762.01} = 0.98$$

c. Calculation of co-varience and correlation coefficient

x	y	$x-\bar{x}$	$(x-\bar{x})^2$	$y-\bar{y}$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
20	7	5	25	–5	25	–25
10	15	–5	25	3	9	–15
20	12	5	25	0	0	0
10	16	–5	25	4	16	–20
17	17	2	4	5	25	10
12	10	–3	9	–2	4	6
15	11	0	0	–1	1	0
16	8	1	1	–4	16	–4
$\Sigma x = 120$	$\Sigma y = 96$		114		94	$\Sigma(x-\bar{x})(y-\bar{y}) = -48$

Here, $\bar{x} = \frac{\Sigma x}{n}$ and $\bar{y} = \frac{\Sigma y}{n}$

$\Rightarrow$ $\bar{x} = \frac{120}{8}$ and $\bar{y} = \frac{96}{8}$

∴ $\bar{x} = 15$ and $\bar{y} = 12$

∴ Co–varience of (x, y) $= \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n} = \frac{-48}{8} = -6$

And, the correlation coefficient is

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2}\sqrt{(y-\bar{y})^2}} = \frac{-48}{\sqrt{114}\sqrt{94}} = \frac{-48}{10.67 \times 9.70} = \frac{-48}{103.50} = -0.463$$

4. Solution:

a. Here,

x	y	$x=(x-\bar{x})$	$y=(y-\bar{y})$	xy	x^2	y^2
6	9	–0	1	0	0	1

2	11	–4	3	–12	16	9
10	5	4	–3	–12	16	9
4	8	–2	0	0	4	0
8	7	2	–1	–2	4	1

$\Sigma xyh = -26, \Sigma x^2 = 40, \Sigma y^2 = 20$

Hence,

$\bar{y} = \frac{\Sigma y}{n}, \bar{x} = 6$

or, $8 = \frac{35 + a}{5}$

or, $40 - 35 \pm a$

$\therefore$ $a = 5$

Coefficient of correlation (r) $= \frac{\Sigma xy}{\sqrt{\Sigma x^2} . \sqrt{\Sigma y^2}} = \frac{-26}{\sqrt{40} . \sqrt{20}} = \frac{-26}{\sqrt{800}} = \frac{-26}{28.28} = -0.92$

b. Solution:

x	y	$x=(x-\bar{x})$	$y=(y-\bar{y})$	x^2	y^2	xy
10	9	–3	–5	9	25	15
12	12	–1	–2	1	4	2
20	15	7	1	49	1	7
$x_1(6)$	18	–7	4	49	16	–28
16	14	3	0	9	0	0
14	16	1	2	1	4	2
$\Sigma x = 72 + x$	$\Sigma y = 84$	0	0	118	50	–2

It is given that,

$\bar{x} = 13$ and $\bar{y} = \frac{\Sigma y}{n}$

$\Rightarrow \frac{\Sigma x}{n} = 13$ and $\bar{y} = \frac{84}{6}$

$\Rightarrow \frac{72 + x_1}{6} = 13$ and $\bar{y} = 14$

$\therefore$ $x_1 = 6$

$\therefore$ The correlation coefficient is given by

$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2} \sqrt{\Sigma y^2}} = \frac{-2}{\sqrt{118} \sqrt{50}}$

$= \frac{-2}{10.86 \times 7.07} = \frac{-2}{76.78}$

$= -0.026$

5. Solution:

x	y	x^2	y^2	xy
41	22	1618	484	902
44	24	1936	576	1051
45	25	2025	625	1125
48	27	2304	729	1296
40	21	1600	441	840
42	22	1764	484	924
44	23	1936	529	1012
$\Sigma x=304$	$\Sigma y=164$	$\Sigma x^2 = 13266$	$\Sigma y^2=3868$	$\Sigma xy=7155$

No. of items (n) = 7

Coefficient of collection (r) $= \frac{n\Sigma xy - \Sigma x. \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2} . \sqrt{n\Sigma y^2 - (\Sigma y)^2}}$

$$= \frac{7\times7155 - 304\times164}{\sqrt{7\times13246 - (304)^2}.\sqrt{7\times3868.(64)^2}}$$

$$= \frac{223}{\sqrt{306.\sqrt{180}}}$$

$= 0.976$

b. Solution

Here,

x	y	xy	x^2	y^2
10	9	96	100	81
12	12	144	144	144
20	16	300	400	225
6	18	108	36	324
16	14	224	256	190
14	16	224	196	256
$\Sigma x=78$	$\Sigma y=8y$	$\Sigma xy=1090$	$\Sigma x^2=1132$	$\Sigma y^2=1226$

Here,

$\overline{x} = 13$

We have, $\overline{x} = \frac{\Sigma x}{n}$

or, $13 = \frac{72 + 9}{6}$

or, $78 = 72+9$

$\therefore$ $a = 6$

Now, Coefficient of correlation (r) $= \frac{n\Sigma xy - \Sigma x.\ \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\ .\ \sqrt{n\Sigma y^2 - (\Sigma y)^2}}$

$$= \frac{6\times1090 - 78\times84}{\sqrt{6\times1132 - (78)^2}\sqrt{6\times1226 - (34)^2}}$$

$$= \frac{-12}{\sqrt{708}\sqrt{300}}$$

$$= \frac{-12}{26.60\times1732} = -0.01$$

6. Solution:

Math (x)	Eng(y)	xy	x^2	y^2
45	35	1575	2025	1225
70	90	6300	4900	8100
65	70	4550	4225	4900
30	40	1200	900	1600
90	95	8550	8100	9025
40	40	1600	1600	1600
50	60	3000	2500	3600
75	80	6000	5625	6400
85	80	6800	7225	6400
60	50	3000	3600	2500
$\Sigma x=610$	$\Sigma y=640$	$\Sigma xy=42575$	$\Sigma x^2=40700$	$\Sigma y^2=45350$

Coefficient of correlation (r) $= \frac{n\Sigma xy - \Sigma x.\ \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\ .\ \sqrt{n\Sigma y^2 - (\Sigma y)^2}}$

$$= \frac{10\times42575 - 610\times640}{\sqrt{10\times40700 - (610)^2}\ .\ \sqrt{10\times45360 - ...}}$$

$$= \frac{425750 - 390400}{\sqrt{34900}\,.\,\sqrt{43900}}$$
$$= \frac{35350}{186.81 \times 209.52}$$
$$= \frac{35350}{39140.4312}$$
$$= 0.9033$$

7. Solution

Here, No. of observations (n) = 9

$\Sigma xy = 731$

$\Sigma x^2 = 285$

$\Sigma xy^2 = 2085$

$\bar{x} = 5$

$\bar{y} = 15$

Coefficient of co-relation (r) $= \dfrac{\Sigma xy - n\bar{x}\,\bar{y}}{\sqrt{\Sigma x^2 - n\bar{x}^2}\,.\,\sqrt{\Sigma y^2 - n\bar{y}^2}}$

$$= \frac{731 - 9\times5\times15}{\sqrt{285 - 9\times25}\,.\,\sqrt{2085 - 9\times225}}$$
$$= \frac{56}{7.74 \times 7.74}$$
$$= \frac{56}{7.74 \times 7.74}$$
$$= 0.934$$

Yes, it is positive.

8. Solution:

Here,

Collected $\Sigma x = 120 - 8 - 12 + 8 + 10 = 118$

Collected $\Sigma y = 90 - 10 - 7 + 12 + 8 = 93$

Collected $\Sigma x^2 = 600 - 8^2 - 12^2 + 8^2 + 10^2 = 556$

Collected $\Sigma y^2 = 250 - 10^2 - 7^2 + 12^2 + 8^2 = 309$

Collected $\Sigma xy = 356 - 8\times10 - 12\times7 + 8\times12 + 10\times8 = 368$

Now,

Corrected value of f = 8

$$r_e = \frac{n\Sigma xy - \Sigma x.\ \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\,.\,\sqrt{n\Sigma y^2 - (\Sigma y^2)}}$$
$$= \frac{30\times368 - 118\times93}{\sqrt{30\times556 - (118)^2}\sqrt{30\times309 - (93)^2}}$$
$$= \frac{11040 - 10974}{\sqrt{16680 - 13924}.\,\sqrt{9270 - 8649}}$$
$$= \frac{66}{52.49 \times 24.91}$$
$$= \frac{66}{1307.5259}$$
$$= 0.05$$

9. Here,

x	Rank (R_x)	y	Rank (R_y)	d = $R_x - R_y$	d^2
39 65	8 6	47 53	10 8	–2 –2	4 4

62	7	58	7	0	0
90	2	86	2	0	0
82	3	62	5	–2	4
75	5	68	4	1	1
25	10	60	6	4	16
98	1	91	1	0	0
36	9	51	9	0	0
78	4	84	3	1	1
					$\Sigma d^2 = 30$

$$\text{Rank } (\rho) = 1 - \frac{6 - \Sigma d^2}{n(n^2 - 1)}$$

$$= 1 - \frac{6 \times 30}{10 \times 99}$$

$$= 0.818$$

10. Here,

Officer (x)	Officer (y)	xy	x^2	y^2
1	1	1	1	1
7	6	42	49	36
4	5	20	16	25
2	2	4	4	4
3	3	9	9	9
6	4	24	36	16
5	7	35	25	49
9	11	99	81	121
10	8	80	100	64
8	10	80	64	100
11	9	99	121	81
$\Sigma x = 66$	$\Sigma y = 66$	$\Sigma xy = 493$	$\Sigma x^2 = 506$	$\Sigma y^2 = 506$

$$\text{Correlation coefficient (r)} = \frac{n\Sigma xy - \Sigma x.\ \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\ .\ \sqrt{n\Sigma y^2 - (\Sigma y^2)}}$$

$$= \frac{11 \times 493 - 66 \times 66}{\sqrt{11 \times 506 - (66)^2}\sqrt{11 \times 506 - (66)^2}}$$

$$= \frac{5423 - 4356}{\sqrt{1210}\ .\ \sqrt{1210}}$$

$$= \frac{1067}{1210}$$

$$= 0.882$$

11. Calculation for spearman's rank correlation

Teaching method	Rank of st A (R_A)	Rank of std. B (R_B)	$d = R_A - R_B$	d^2
I	2	1	1	1
II	1	3	–2	4
III	3	2	1	1
IV	5	4	1	1
V	4	7	–3	9
VI	6	5	1	1
VII	7	6	1	1
				$\Sigma d^2 = 18$

Rank (ρ) $= 1 - \frac{6 \sum d^2}{n(n^2 - 1)}$

$= 1 - \frac{6 \times 18}{7 \times 48} = 1 - \frac{108}{336} = \frac{228}{336} = 0.68$

Exercise 13.2

1. Solution:

a. Here, $\sigma_x = 20$, $\sigma_y = 15$, $r = 0.48$

The regression coefficients are $b_{yx} = r\frac{\sigma_y}{\sigma_x} = 0.48 \times \frac{15}{20} = 0.36$

and $b_{xy} = r\frac{\sigma_x}{\sigma_y} = 0.48 \times \frac{20}{15} = 0.64$

b. $\sigma_x = 8$, $\sigma_y = 10$, $r = -0.6$

The regression coefficients are

$b_{yx} = r\frac{\sigma_y}{\sigma_x} = -0.6 \times \frac{10}{8} = -0.75$

and $b_{xy} = r\frac{\sigma_x}{\sigma_y} = -0.6 \times \frac{8}{10} = -0.48$

2. Solution:

a. Here, $b_{yx} = 0.35$, $b_{xy} = 1.8$

Now, correlation coefficient (r) $= \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{0.35 \times 1.8} = \sqrt{0.63} = 0.7937$

b. We have, $\Sigma x = 60$, $\Sigma y = 40$, $\Sigma xy = 1150$

$\Sigma x^2 = 4160$, $\Sigma y^2 = 1720$, $N = 10$

Here, $b_{yx} = \frac{N\Sigma xy - \Sigma x\Sigma y}{N\Sigma x^2 - (\Sigma x)^2} = \frac{10 \times 1150 - 60 \times 40}{10 \times 4160 - (60)^2}$

$= \frac{11500 - 2400}{4160 - 3600} = \frac{9100}{38000} = 0.2394$

$b_{xy} = \frac{N\Sigma xy - \Sigma x\Sigma y}{N\Sigma y^2 - (\Sigma y)^2} = \frac{10 \times 1150 - 60 \times 40}{10 \times 1720 - (40)^2} = \frac{9100}{15600} = 0.5833$

And, $\overline{x} = \frac{\Sigma x}{N} = \frac{60}{10} = 6$, $\overline{y} = \frac{\Sigma y}{N} = \frac{40}{10} = 4$

Now, Regression equation of y on x is,

$y - \overline{y} = b_{yx}(x - \overline{x})$

$\Rightarrow$ $y - 4 = 0.2394(x - 6)$

$\Rightarrow$ $y - 4 = 0.2394x - 1.4364$

$\Rightarrow$ $yh = 2.5636 + 0.2394x$

Regression equation of x on y is,

$x - \overline{x} = b_{xy}(y - \overline{y})$

$\Rightarrow$ $x - 6 = 0.5833(Y - 4)$

$\Rightarrow$ $x - 6 = 0.5833y - 2.3332$

$\Rightarrow$ $x = 3.6668 + 0.5833y$

And, the correlation coefficient is

$r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{0.2394 \times 0.5833} = \sqrt{0.1396} = 0.3736$

c. We have,

$b_{yx} = 2.002$, $b_{xy} = -0.461$, $\overline{x} = 87.2$,

$\overline{y} = 127.2$

$\therefore$ Correlation coefficient is,

$r = \sqrt{-2.002 \times -0.461} = 0.9606 > 0$

Since $b_{yx} < 0$, $b_{xy} < 0$ and $r > 0$

So, it is not possible.

3. Solution:

a. Here, $s\sum xy = 750$, $\sum x^2 = 2085$, $\sum y^2 = 285$, $\sum x = 135$, $\sum y = 45$, $N = 9$

Now, The regression coefficient of x only is

$$b_{xy} = \frac{n\sum xy - \sum x.\sum y}{n\sum y^2 - (\sum y)^2}$$

$$= \frac{9 \times 750 - 135 \times 45}{9 \times 285 - (45)^2} = \frac{6750 - 6075}{2565 - 2025} = \frac{675}{540} = 1.25$$

b. Here, $\sum x = 60$, $\sum y = 40$, $\sum xy = 1150$, $\sum x^2 = 4160$, $\sum y^2 = 1720$, $n = 10$

Now, the point through which the regression lines intersect to each other is

$$(\overline{x}, \overline{y}) = \left(\frac{\sum x}{n}, \frac{\sum y}{n}\right) = \left(\frac{60}{10}, \frac{40}{10}\right) = (6, 4)$$

Since the equation of regression line of y on x is

$y - \overline{y} = b_{yx}(x - \overline{x})$ ... ... (i) and x on y is

$x - \overline{x} = b_{xy}(y - \overline{y})$ ... ... (ii) where

$$b_{yx} = \frac{n\sum xy - \sum x . \sum y}{n\sum x^2 - (\sum x^2)} = \frac{10 \times 1150 - 60 \times 40}{10 \times 4160 - 60^2}$$

$$= \frac{11500 - 2400}{41600 - 3600}$$

$$= \frac{9100}{38000} = 0.239$$

and $$b_{xy} = \frac{n\sum xy - \sum x. \sum y}{n\sum y^2 - (\sum y)^2} = \frac{10 \times 1150 - 60\times40}{10\times1720 - 40^2}$$

$$= \frac{9100}{15600} = 0.583$$

Hence, from (i) and (ii), the required equations are

$y - 4 = 0.239 (x - 6)$

or, $y = 2.566 + 0.239x$

and $x - 6 = 0.583 (y - 4)$

or, $x = 3.668 + 0.583y$

Also, correlation coefficient (r) $= \sqrt{b_{yx} . b_{xy}}$

$= \sqrt{0.239 \times 0.583}$

$= \sqrt{0.139337} = 0.373$

c. Here, $b_{yx} = -2.002$ and $b_{xy} = -0.461$

Now, $r = \sqrt{b_{yx} . b_{xy}} = \sqrt{-2.002 \times -0.461} = \sqrt{0.922922} = -0.961$

Also, **Error! Bookmark not defined.** $\overline{x} = 87.2$, $\overline{y} = 127.2$, $y = 133$, $x = ?$

Using, $x - \overline{x} = b_{xy}(y - \overline{y})$,

or, $x - 87.2 = -0.461 (y - 127.2)$

or, $x - 87.2 = -461 (133 - 127.2)$

or, $x - 87.2 = -2.6738$

or, $x = 84.5262$

d. Let average price in Birgunj $(\overline{x})$ = Rs. 65

Average price in Kathmandu $(\overline{y})$ = Rs. 67

$\sigma_x = 2.5$, $\sigma_y = 3.5$, $r = 0.08$

Now, $b_{yx} = r . \frac{\sigma_y}{\sigma_x} = 0.08 \times \frac{3.5}{2.5} = 0.112$

The equation of regression line of y on x is

$y - \overline{y} = b_{yx}(x - \overline{x})$

or, $y - 67 = 0.112 (x - 65)$

or, $y - 67 = 0.112 \times -7.28$

or, $y = 59.72 + 0.112x$

If x = Rs. 70, then $y = 59.72 + 0.112 \times 70$ = Rs. 67.56

4. Solution:

a. Here, the regression equations ae $3x + 2y - 26 = 0$; $6x + y - 31 = 0$

or, $2y = -3x + 26$ and $6x = -y + 31$

or, $y = \frac{-3}{2}x + 13$, $x = -\frac{1}{6}y + \frac{31}{6}$

Implies the regression coefficient as $b_{yx} = -\frac{3}{2}$ and $b_{xy} = -\frac{1}{6}$

Now, correlation coefficient (r) = $\sqrt{byx \cdot bxy}$

$$= \sqrt{\frac{-3}{2} \times -\frac{1}{6}} = \sqrt{\frac{1}{4}} = -0.5$$

After solving the given equations, we get the intersection point (x, y) = (4, 7)

i.e, means of x = $\overline{x}$ = 4 and means of y = $\overline{y}$ = 7

b. We have, the given two regression equations are

$3x + 4y = 65$ and $3x + y = 31$

Since $(\overline{x}, \overline{y})$ lies on the given regression lines.

$3\overline{x} + 4\overline{y} = 65$ ... ... (i)

$3\overline{x} + \overline{y} = 32$ ... ... (ii)

Subtracting (ii) from (i), we get

$3\overline{y} = 33$

$\therefore \quad \overline{y} = 11$

from (i), $3\overline{x} + 4\times 11 = 65 \Rightarrow 3\overline{x} = 65 - 44 = 21$

$\therefore \quad \overline{x} = 7\ 33$

$\therefore \quad \overline{y} = 11$

For the regression coefficients, we have

$3x + 4y = 65$

$\Rightarrow \quad 4y = 65 - 3x$

$\therefore \quad y = \frac{65}{4} - \frac{3}{4}x$ which is in the form of $y = a + bx$; where

$b = -\frac{3}{4} \therefore b = b_{yx} = -\frac{3}{4}$

Again, we have,

$3x + y = 32$

$\Rightarrow 3x = 32 - y$

$\therefore \quad x = \frac{32}{3} - \frac{1}{3}y$ which is in the form of $x = a + by$; where $b = -\frac{1}{3}$

$\therefore \quad b = b_{xy} = -\frac{1}{3}$

$\therefore \quad$ Correlation coefficient (r) = $\sqrt{b_{xy} + b_{xy}} = \pm\sqrt{-\frac{3}{4} \times -\frac{1}{3}}$

$\therefore \quad r = -\frac{1}{2}$

5. Solution,

Here, $b_{xy} = 1.5$, $b_{yx} = 0.65$

$\overline{x} = 36$ and $\overline{y} = 52$

Regression equation of y on x is

$y - \overline{y} = b_{yx}(x - \overline{x})$

$\Rightarrow \quad y - 52 = 0.65(x - 36)$

$\Rightarrow \quad y = 0.65x - 23.4 + 52$

$\therefore \quad y = 0.65x + 28.6$ ... ... (i)

And, the regression equation of x on y is

$x - \overline{x} = b_{xy}(y - \overline{y})$

$\Rightarrow \quad x - 36 = 1.5(y - 52)$

$\Rightarrow \quad x = 1.5y - 78 + 36$

$\therefore \quad x = 1.5y - 42$ ... ... (ii)

When x = 60 then from (i),

$y = 0.65 \times 60 + 28.6 = 67.6$

6. Solution:

a. Given, $\overline{x} = 20$, $\overline{y} = 120$, $cov_x = 25$, $cov_y = 28.83$, $r = 0.8$, $x = ?$ if $y = 150$

$cov_x = 25$, then $25 = \dfrac{\sigma_x}{\overline{x}} \times 100$

or, $\dfrac{25 \times 20}{100} = \sigma_x$

or, $\sigma_x = 5$

$Cov_y = 28.83 \Rightarrow 28.83 = \dfrac{\sigma_y}{\overline{y}} \times 100$

or, $\dfrac{28.83 \times 120}{100} = \sigma_y$

or, $\sigma_y = 34.596$

$\therefore \quad b_{xy} = r\dfrac{\sigma_x}{\sigma_y} = 0.8 \times \dfrac{5}{34.596} = 0.1156$

Now, the equation of regression line of x on y is

$x - \overline{x} = b_{xy}(y - \overline{y})$

or, $x - 20 = 0.1156(y - 120)$

or, $x = 0.1156y - 13.872 + 20$

$\therefore \quad x = 0.1156y + 6.128$

when $y = 150$, $x = 0.1156 \times 150 + 6.128 = 23.5$

b. Given,

$n = 50$

$\dfrac{\sigma_y}{\sigma_x} = \dfrac{5}{2}$

$b_{yx} = r\dfrac{\sigma_y}{\sigma_x} = \dfrac{5r}{2}$ ... ... (i)

and $b_{xy} = \dfrac{2r}{5}$

Also, $4y - 5x - 8 = 0$

$\Rightarrow \quad y = \dfrac{5x}{4} + 2$

i.e. $b_{yx} = \dfrac{5}{4}$

from (i), $\dfrac{5r}{2} = \dfrac{5}{4}$

or, $r = \dfrac{1}{2}$

Now, $\overline{x} = 40$. Let the average marks for mathematics by $\overline{y}$.

So, $4\overline{y} - 5\overline{x} = 8$

$\Rightarrow \quad 4\overline{y} = 8 + 200$

$\Rightarrow \quad \overline{y} = \frac{208}{4} = 52$

7. Here, average of rainfall $(\overline{x}) = 26.7$cm

   Average of yield $(\overline{y}) = 508.4$kg
   $\sigma_x = 4.6$, $\sigma_y = 36.8$, $r = 0.52$

   For x on y:
   $b_{xy} = r\frac{\sigma_x}{\sigma_y} = 0.52 \times \frac{4.6}{36.8} = 0.065$

   So, the equation is $x - \overline{x} = b_{xy}(y - \overline{y})$
   or, $x - 26.7 = 0.065(y - 508.4)$
   or, $x - 26.7 = 0.065y - 33.046$
   or, $x = 0.065y - 6.346$
   When yield (y) = 600kg, $x = 0.065 \times 600 - 6.346$
   $= 32.654$cm
   For y on x
   $b_{yx} = r\frac{\sigma_y}{\sigma_x} = 0.52 \times \frac{36.8}{4.6} = 4.16$

   So, the equation is $y - \overline{y} = b_{yx}(x - \overline{x})$
   or, $y - 5084 = 4.16(x - 26.7)$
   or, $y - 5084 = 4.16x - 111.072$
   or, $y = 4.16x + 397.328$
   When rainfall (x) = 29cm
   $y = 4.16 \times 29 + 397.328 = 517.968$kg

8. Given, Mean no. of workers on strike $(\overline{x}) = 800$

   Mean loss of daily production $(\overline{y}) = 35000$
   Standard deviation of no. of workers on strike $(\sigma_x) = 100$
   Standard deviation of daily production $(\sigma_y) = 2,000$
   Coefficient of correlation between x and y (r) = 0.8
   When x = 1800, y = ?
   Now, $b_{yx} = r\frac{\sigma_y}{\sigma_x} = 0.8 \times \frac{2000}{100} = 16$

   The equation of regression of y on x is $y - \overline{y} = b_{yx}(x - \overline{x})$
   or, $y - 35000 = 16(x - 800)$
   or, $y - 35000 = 16x - 12800$
   or, $y = 16x - 22200$
   When x = 1800, $y = 16 \times 1800 - 22200 = 6600$

9. Solution:

x	y	xy	x^2
1	50	50	1
5	60	300	25
6	80	480	36
8	100	800	64
10	110	1100	100
$\sum x = 30$	$\sum y = 400$	$\sum xy = 2730$	$\sum x = 226$

Here, $\overline{x} = \frac{\sum x}{n} = \frac{30}{5} = 6$

$\overline{y} = \frac{\sum y}{n} = \frac{400}{5} = 80$

$b_{yx} = \frac{n\sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2} = \frac{5 \times 273 - 30 \times 400}{5 \times 226 - (30)^2} = \frac{13650 - 12000}{1130 - 900} = \frac{1650}{230} = 7.174$

The regression equation of y on x is

$y - \overline{y} = b_{yx} (x - \overline{x})$

or, $y - 80 = 7.174 (x - 6)$

or, $y - 80 = 7.174x - 43.04$

or, $y = 7.174x + 36.96$

When x = 15000,

$y = 7.174 \times 15000 + 36.96$

$= \text{Rs. } 107645.66$

10. Calculation regression equation of x on y

x	y	x^2	y^2	xy
2	18	4	324	36
4	12	16	144	48
5	10	25	100	50
6	8	36	64	48
8	7	64	49	56
11	5	121	25	55
36	**60**	**266**	**706**	**293**

Here, $\overline{x} = \frac{\Sigma x}{n}$ and $\overline{y} = \frac{\Sigma y}{n}$

$\Rightarrow \overline{x} = \frac{36}{6}$ and $\overline{y} = \frac{60}{6}$

$\therefore \overline{x} = 6 \qquad \therefore \overline{y} = 10$

Again, $b_{xy} = \frac{n\Sigma xy - \Sigma x\Sigma y}{n\Sigma y^2 - (\Sigma y)^2} = \frac{6 \times 293 - 36 \times 60}{6 \times 706 - (60)^2}$

$= \frac{1758 - 2160}{4236 - 3600} = \frac{-402}{636} = -0.6320$

$\therefore$ Regression equation of x on y is,

$x - \overline{x} = b_{xy} (y - \overline{y})$

$\Rightarrow x - 6 = -0.6320 (y - 10)$

$\Rightarrow x - 6 = -0.6320y + 6.32$

$\therefore x = 12.32 - 0.6320y$

When y = 12 then $x = 12.32 - 0.6320 \times 12$

$\therefore x = 4.736$

11. Here, n = 25, $\sum x = 125$, $\sum y = 100$, $\sum x^2 = 650$, $\sum y^2 = 460$, $\sum xy = 508$

But (8, 12) and (6, 8) were copied wrong as (6, 14) and (8, 6) respectively.

So, correct values are

$n = 25, \sum x = 125 + 8 - 6 + 6 - 8 = 125$

$\sum y = 100 + 12 - 14 + 8 - 6 = 100$

$\sum x^2 = 650 + 8^2 - 6^2 + 8^2 = 650$

$\sum y^2 = 460 + 12^2 - 14^2 + 8^2 - 6^2 = 436$

$\sum xy = 508 + 8\times12 + 6\times8 - 6\times14 - 8\times6 = 520$

Now, $b_{xy} = \frac{n\sum xy - \sum x.\sum y}{n\sum y^2 - (\sum y)^2} = \frac{25 \times 520 - 125 \times 100}{25 \times 436 - (100)^2} = \frac{13000 - 12500}{10900 - 100000} = \frac{5}{9}$

and $b_{yx} = \frac{n\sum xy - \sum x.\sum y}{n\sum y^2 - (\sum y)^2} = \frac{25 \times 520 - 12500}{25 \times 650 - (125)^2} = \frac{500}{625} = \frac{4}{5}$

Now, coefficient of correlation (r) $= \sqrt{b_{yx} \times b_{xy}} = \sqrt{\frac{5}{9} \times \frac{4}{5}} = \frac{2}{3}$

Now, $\overline{x} = \frac{\sum x}{n} = \frac{125}{25} = 5$

$\overline{y} = \frac{\sum y}{n} = \frac{100}{25} = 4$

The equation of regression line of x on y is

$x - \overline{x} = b_{xy} (y - \overline{y})$

or, $x - 5 = \frac{5}{9} (y - 4)$

or, $9x - 45 = 5y - 20$

or, $9x - 5y = 25$

and the equation of regression line of y on x is

$y - \overline{y} = b_{yx} (x - \overline{x})$

or, $y - 4 = \frac{4}{5} (x - 5)$

or, $5y - 20 = 4x - 20$

or, $4x - 5y = 0$

Chapter 14
Probability

Exercise 14.1

4. n = Total no. of cards = 52

a. No. of club = 13

No. of diamond = 13

m = No. of possible cases = 13 + 13 = 26

P(Either a club or diamond) = $\frac{m}{n} = \frac{26}{52} = \frac{1}{2}$

b. There are four kings

$\therefore$ No. of possible cases = 52 – 4 = 48

$\therefore$ P(Not of king) = $\frac{48}{52} = \frac{12}{13}$

c. There are 12 face cards and 13 club cards.

$\therefore$ m = no. of cases = 12 + 13 – 3 = 22

$\therefore$ P(Either a face or a club) = $\frac{m}{n} = \frac{22}{52} = \frac{11}{26}$

5. From 20 tickets marked from 1 to 20, one is drawn at random. Find the probability that

a. It is an odd number

b. A multiple of 4 or 5

Solution:

a. P(Odd number) = ?

Among 20 tickets, there are 10 tickets marked with odd number.

$\therefore$ P(Odd number) = $\frac{m}{n} = \frac{10}{20} = \frac{1}{2}$

b. P(A multiple of 4 or 5) = ?

There are 5 tickets marked with multiple of 4 and 4 tickets marked with multiple of 5.

= P(Multiple of 4) + P(multiple of 5) – P(Multiple of 4) $\times$ P(Multiple of 5)

$= \frac{5}{20} + \frac{4}{20} - \frac{5}{20} \times \frac{4}{20} = \frac{2}{5}$

6. Given that,

P(A) = Probability that A solves the problem = $\frac{1}{2}$

P(B) = Probability that B solves the problem = $\frac{1}{3}$

P(C) = Probability that C solves the problem = $\frac{1}{4}$

P(D) = Probability that D solves the problem = $\frac{1}{5}$

$P(\overline{A})$ = Probability that A not solve the problem = $1 - \frac{1}{2} = \frac{1}{2}$

$P(\overline{B})$ = Probability that B not solve the problem $1 - \frac{1}{3} = \frac{2}{3}$

$P(\overline{C})$ = Probability that C not solve the problem = $1 - \frac{1}{4} = \frac{3}{4}$

$P(\overline{D})$ = Probability that D not solve the problem = $1 - \frac{1}{5} = \frac{4}{5}$

b. Probability that A, B, C, D not solve the problem = $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5}$

a. Probability that A, B, C, D solve the problem = $1 - \frac{1}{5} = \frac{4}{5}$

7. Suppose 4 cards are drawn from a well-shuffled deck of 52 cards.
   a. What is the probability that all 4 are spade?
   b. What is the probability that all 4 are black?

Solution:

a. There are 13 spades

Now, n = Total no. of possible cases

= No. of selection of 4 cards out of 52

= $52c_4$

= No. of favourable cases

= No. of selection of 4 spades out of 13 = $13c_4$

P(4 are spades) = ?

Now, P(4 are spades) = $\frac{m}{n} = \frac{13c_4}{52c_4} = \frac{13!}{9!\ 4!} \times \frac{48! \times 41}{52!} = \frac{11}{4165}$

b. There are 26 black. So, we have to choose 4 black among 26 blacks.

Now, n = Total no. of possible cases

= No. of selection of 4 cards out of 52 = $52c_4$

m = No. of favourable cases

= No. of selection of 4 black out of 26 black

= $26c_4$

P(4 are black) = ?

Now, P(4 are black) = $\frac{m}{n} = \frac{26c_4}{52c_4} = \frac{46}{833}$

8. Solution:

a. Two cards can be drawn from a pack of 52 playing cards in $^{52}C_2$ ways

i.e. $\frac{52 \times 51}{2}$ = 1326 ways

The event that two kings appear in a single draw can appear in $^{4}C_2$ ways

∴ The probability that the two cards drawn from a pack of 52 cards are kings

$= \frac{6}{1326} = \frac{1}{221}$

b. One king and one queen can be selected as $\frac{4}{52} \times \frac{4}{51}$ ways.

One queen and one king can be selected as = $\frac{4}{52} \times \frac{4}{51}$ ways

Total no. of ways = $\frac{4}{52} \times \frac{4}{51} + \frac{4}{52} \times \frac{4}{51} = \frac{8}{663}$

9. Here,

Total no. of candidates = 9

Total no. of men = 6

Total no. of women = 3

Total no. of vacancy = 2

∴ Out of 2, one man and one woman can be selected in the following ways.

∴ $m = bc_1 \times 9c_1$

∴ Total no. of vacancy can be chosen from total no. of candidates as

$n = 9c_2$

∴ P(1 man and 1 woman) = $\frac{6c_1 \times 9c_1}{9c_2} = \frac{18}{36}$ 0.5

10. Since the bag consists of 7 white and 9 black balls.

∴ Total balls = 7 + 9 = 16

Total number of possible cases means the number of selection of 2 balls out of 16.

Since, the selection of 1 white and 1 black. So, the number of favourable cases is the selection of balls with 1 white and 1 black

∴ m = No. of favourable cases

= No. of selection of 1 white out of 7 and 1 black out of 9

= $7c_1 \times 9c_1$

n = Total no. of possible cases

= No. of selection of 2 balls out of 16.

= $16c_2$

∴ P(1 white and 1 black) = $\frac{m}{n} = \frac{7c_1 \times 9c_1}{16c_2} = \frac{63}{120}$

11. There are 6 + 8 = 14 balls (Total)

a. P(both white) = ?

P(First white) = $\frac{6}{14}$ and P(second white) = $\frac{5}{13}$

P(Both white) = $\frac{6}{14} \times \frac{5}{13} = \frac{15}{91}$

b. P(Both red) = ?

P(First red) = $\frac{8}{14}$, P(Second red) = $\frac{7}{13}$

∴ P(Both red) = $\frac{8}{14} \times \frac{7}{13} = \frac{4}{13}$

c. Since balls are drawn one after another without replacement.

P(One red and one white) = ?

∴ P(First red) = $\frac{8}{14}$, P(Second white) = $\frac{6}{13}$

P(First white) = $\frac{6}{14}$ P(Second red) = $\frac{8}{13}$

∴ P(One red and one white) = $\frac{6}{14} \times \frac{8}{13} + \frac{8}{13} \times \frac{6}{13} = \frac{48}{91}$

12. Given, P(A) = 0.40, P(B) = 0.80, P(B/A) = 0.60, P(A/B) = ? P(A∪B) = ?

∴ $P(B/A) = \frac{P(A\cap B)}{P(A)}$

⇒ $P(A\cap B) = P(A) . P(B/A) = 0.40 \times 0.60 = 0.24$

∴ $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{0.24}{0.80} = 0.3$

$P(A\cup B) = P(A) + P(B) – P(A\cap B)$

$= 0.40 + 0.80 – 0.3$

= 0.9

13. A box contains six red and four black balls. Two balls are drawn one at a time without replacing the first ball compute $P(R_2/131)$. Also find the probability that both are red balls.

Solution:

$P(R_2/B_1)$ = Probability of getting a red ball given that the first ball is black.

First black ball

n = Total no. of possible cases

Total no. of balls = 6 + 4 = 10

m = No. of favourable cases

= No. of black balls

= 4

$P(B_1) = \frac{m}{n} = \frac{4}{10}$

Second Red Ball

One black ball which is drawn is not replaced.

n = Total no. of possible cases

= No. of remaining balls

= 6 + 3 = 9

m = No. of favourable cases

= No. of red balls = 6

$P(R_2/B_1)\ \frac{m}{n} = \frac{6}{9} = \frac{2}{3}$

$P(R_1)$ = Probability of getting a red ball

$= \frac{m}{n} = \frac{6}{10}$

$P(R_2/R_1)$ Probability that second ball is red when first also red $= \frac{m}{n} = \frac{5}{9}$

$\therefore\ P(R_1 \cap R_2) = P(R_1) \cdot P(R_2/R_1)$

$= \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$

14. A lot contains 12 items of which 5 are defective. If 5 items are chosen from the lot at random. One after another without replacement. Find the probability that all the five are defective.

Solution: We have,

No. of total items = 12

No. of defective items = 5

$\therefore$ Probability of getting first item defective, $P(A) = \frac{5}{12}$

Since, second item is drawn without replacement of first items.

So, probability of getting second item defective $P(B) = \frac{4}{11}$

Similarly,

Probability of getting 3rd item defective, $P(C) = \frac{3}{10}$

Probability of getting 4th item defective, $P(D) = \frac{2}{9}$

Probability of getting 5th item defective, $P(E) = \frac{1}{8}$

Probability of getting all items defective

$= P(A) \times P(B) \times P(C) \times P(D) \times P(E)$

$= \frac{5}{12} \times \frac{4}{11} \times \frac{3}{11} \times \frac{1}{9} \times \frac{1}{8}$

$= \frac{1}{792}$

15. A bag contains 3 white, 2 black and 4 red balls. Two balls are drawn, the first replaced before the second is drawn, what is the probability that
    i. They will be of same colour?
    ii. They will be of different colour?

Solution: We have,

No. of white balls = 3
No. of black balls = 2
No. of red balls = 4
Total no. of balls = 9

Let P(W) = Probability of getting a white ball $= \frac{3}{9} = \frac{1}{3}$

P(B) = Probability of getting black ball $= \frac{2}{9}$

P(R) = Probability of getting red ball $= \frac{4}{9}$

a. P(They will be of same colour) = P(WW or BB or RR)

$= P(WW) + P(BB) + P(RR)$

$= P(W) \times P(W) + P(B) \times P(B) + P(R) \times P(R)$

$= \frac{1}{3} \times \frac{1}{3} + \frac{2}{9} \times \frac{2}{9} + \frac{4}{9} \times \frac{4}{9}$

$= \frac{29}{81}$

b. Probability of getting different colours, there should be either WB or BW or BR or RB or WR or RW

$\therefore$ P(That they are of different colour)

= P(WB or BW or BR or RB or WR or RW)

$= P(W) \times P(B) + P(B) \times P(W) + P(B) \times P(R) + P(W) \times P(R) + P(R) \times P(W)$

$= \frac{1}{3} \times \frac{2}{9} + \frac{2}{9} \times \frac{1}{3} + \frac{2}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{2}{9} + \frac{1}{3} \times \frac{4}{9} + \frac{4}{9} \times \frac{1}{3}$

$= \frac{52}{81}$

Exercise 14.2

1. We have, mean = np = 25 ... ... (i)
   Variance = npq = 5 ... ... (ii)
   from (i) and (ii)
   25q = 5

   or, $q = \frac{5}{25} = \frac{1}{5}$

   $\therefore$ $p = 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$

   $\therefore$ from (ii), $n \cdot \frac{4}{5} \cdot \frac{1}{5} = 5$

   $\therefore$ $p = \frac{4}{5}, q = \frac{1}{5}$

2. We have,

$p = \frac{3}{5}$, $n = 50$

$\therefore \quad q = 1 - p = 1 - \frac{3}{5} = \frac{2}{5}$

$\therefore \quad \text{Mean} = np = 50 \times \frac{3}{5} = 30$

$\text{S.D} = \sqrt{npq} = \sqrt{50 \times \frac{3}{5} \times \frac{2}{5}} = 2\sqrt{3}$

3. We have, mean = np = 4 ... ... (i)

S.D. = $\sqrt{npq} = \sqrt{3}$

or, npq = 3 ... ... (ii)

$\therefore$ from (i) and (ii)

$4q = 3$

$\Rightarrow \quad q = \frac{3}{4}$

$\therefore \quad p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

$\therefore$ from (i), $n \times \frac{1}{4} = 4 \Rightarrow n = 16$

$\therefore$ Binomial distribution = $(q + p)^n = \left(\frac{3}{4} + \frac{1}{4}\right)^{16}$

4. We have,

Mean = np = 7 ... ... (i)

Variance = npq = 11 ... ... (ii)

$\therefore$ from (i) and (ii)

$7 \times q = 11$

or, $q = \frac{11}{7} = 1.57 > 1$

Since, q is probability of failture, which cannot be greater than 1. So, the given statement is not correct.

5. Here,

p = probability of getting ahead = $\frac{1}{2}$

q = probability of getting a tail = $\frac{1}{2}$

n = no. of trials = 4

p(r) = probability of r success in n trials

$= n_{c_r} p^r q^{n-r}$

a. p(2) = probability of 2 heads in 4 trials

$= 4_{c_2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{4 \times 3}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{8}$

b. p(at least two heads) = p(2) + p(3) + p(4)

$= 4_{c_2}\, p^2q^2 + 4_{c_3}p^3q + 4_{c_4}\, p^4$

$= \frac{4 \times 3}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 \frac{1}{2} + 1\left(\frac{1}{2}\right)^4$

$= \frac{11}{16}$

c. Plat least one head = p(1) + p(2) + p(3) + p(4)

$= 4_{c_1}\, p1q^3 + 4_{c_2}\, p^2q^2 + 4_{c_3}\, p^3q + 4_{c_4}\, p^4$

$= 4_{c_2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 + 4_{c_2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + 4_{c_3} \left(\frac{1}{2}\right)^3 \frac{1}{2} + 1. \left(\frac{1}{2}\right)^4$

$= \frac{1}{4} + \frac{11}{16}$

$= \frac{15}{16}$

6. Let p be the probability of getting 3 or 6.

$\therefore \quad p = \frac{2}{6} = \frac{1}{3}$, $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

n = no. of trials = 4

Now, probability of r success out of n trials is given by

$p(r) = n_{c_r} p^r q^{n-r} = 4_{c_r} \left(\frac{1}{3}\right)^r . \left(\frac{1}{3}\right)^{4-r} = r \frac{1}{81} 4_{c_r}$

a. Probability of getting at least one success = $p(\geq 1) = p(1) + p(2) + p(3) + p(4)$

$= 4_{c_1} p^1q^{4-1} + 4_{c_2} p^2q^{4-2} + 4_{c_3} p^3p^{4-3} + 4_{c_4} p^4q^{4-4}$

$= \frac{4!}{3!\ 1!} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 + \frac{4!}{2!2!} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 + \frac{4!}{3!\ 1!} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 + \frac{4!}{0!\ 4!} \left(\frac{1}{3}\right)^4$

$= \frac{4\times8}{81} + \frac{6\times4}{81} + \frac{4\times2}{81} + \frac{1}{81}$

$= \frac{65}{81}$

b. Probability of exactly two success = $p(2)$

$= 4_{c_2} p^2q^2$

$= \frac{4!}{2!\ 2!} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2$

$= \frac{4 \times 3}{2} \times \frac{4}{81} = \frac{8}{27}$

7. Let X represents the number of diamond cards among the five cards drawn. Since the drawing off cards is with replacement, the trials are Bornouli trial.

In a well-shuffled deck of 52 cards, there are 13 diamond cards.

$\Rightarrow \quad p = \frac{13}{52} = \frac{1}{4}$, $q = 1 - \frac{1}{4} = \frac{3}{4}$

X has a Binomial distribution with $n = 5$ and $p = \frac{1}{4}$

$\therefore \quad p(x = x) = n_{c_x} p^xq^{n-x}$, where $x = 0, 1, 2, \dots n$

$= 5_{c_x} \left(\frac{3}{4}\right)^{5-x} \left(\frac{1}{4}\right)^x$

a. p(all 5 cards ae diamond) = $p(x = 5)$

$= 5_{c_5} \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^5 = \frac{1}{1024}$

b. p(only 3 cards ae diamond) = $p(x = 3)$

$= 5_{c_3} \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^3 = \frac{45}{512}$

c. p(none is spade) = $p(x = 0)$

$= 5_{c_0} \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^0 = \frac{243}{1024}$

8. Let p = the event of getting a head 10 coins being tossed simultaneously is the same as one coin being tossed 10 times.

$p(x = r) = 10_{c_r} p^r.q^{n-r} = 10_{c_r} \left(\frac{1}{2}\right)^{10}$

a. p(exactly 6 heads) = $10_{c_6} \left(\frac{1}{2}\right)^{10}$

$= \frac{10!}{6!\ 4!} \times \frac{1}{1024} = \frac{105}{512}$

b. p(at least 7 heads) = p(7 heads or 8 heads or 9 heads or 10 heads)

$= 1 - [p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3) + p(x = 4) + p(x = 5) + p(x = 6)] \left(\frac{1}{2}\right)^{10}$

$= 1 - (1 + 10 + 45 + 120 + 210 + 252 + 210) \frac{1}{1024}$

$= \frac{176}{1024}$

c. p(not more than 3 heads) = $p(x \le 34)$
$= p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3)$
$= (10_{c_0} + 10_{c_1} + 10_{c_2} + 10_{c_3}).\left(\frac{1}{2}\right)^{10}$
$= (1 + 10 + 45 + 120) . \frac{1}{1024} = \frac{11}{64}$

9. Given,
Probability of getting a six in one thrown (p) = $\frac{1}{6}$
$\therefore \quad q = 1 - p = \frac{5}{6}$
No. of trials (n) = 4
Now, probability of r success in 4 trials is given by
$p(r) = n_{c_r}\, p^r.q^{n-r} = 4_{c_r} \left(\frac{1}{6}\right)^r . \left(\frac{5}{6}\right)^{4-r}$ ... ... (i)

a. $p(\text{no six}) = p(0) = 4_{c_0} \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^{4-0} = \frac{625}{1296}$

b. $p(\text{exactly 1 six}) = p(1) = 4_{c_x} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{4-1} = \frac{125}{324}$

c. $p(\text{exactly two sixes}) = p(2) = 4_{c_2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{4-2} = \frac{25}{216}$

10. Probability of fail = $\frac{40}{100} = \frac{2}{5} = q$
Probability of pass = $1 - \frac{2}{5} = \frac{3}{5} = p$
$n = 6,\ q = \frac{2}{5}$
$x \to$ R.V.
We have Binomial condition,
$P(x = r) = n_{c_r}\, p^r.q^{n-r}$
$p(x \ge 4) = ?$
$\therefore \quad p(x \ge 4) = p(x = 4) + p(x = 5) + p(x = 6)$
$= 6_{c_4} \left(\frac{3}{4}\right)^4 \left(\frac{2}{5}\right)^4 + 6_{c_5} \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right) + 6_{c_6} \left(\frac{3}{5}\right)^6 \left(\frac{2}{5}\right)^0$
$= \frac{6!}{4!\,2!} \times \frac{3^4 \times 2^2}{5^6} + \frac{6!}{5!1!} \times \frac{3^5 \times 2}{5^6} + \frac{6!}{6!} \times \frac{3^6}{5^6}$
$= \frac{1701}{3125}$

11. Given,
$p = 60\% = \frac{60}{100} = \frac{3}{5}$
$\therefore \quad q = -1 - P = 1 - \frac{3}{5} = \frac{2}{5}$
n = number of trials = 10
Now, probability of r successes in 10 trials is given by
$p(r) = 10_{c_r}\, p^r.q^{10-r} = 10_{c_r} \left(\frac{3}{5}\right)^r \left(\frac{2}{5}\right)^{10-r}$

a. P(None of them male) = p(0)
$= 10_{c_0} \left(\frac{3}{5}\right)^0 \left(\frac{2}{5}\right)^{10-r}$
$= \frac{10!}{0!\,10!} \times 1 \times \frac{2^{10}}{5^{10}} = 0.0001049$

b. $P(\text{Exactly three male}) = p(3) = 10_{c_3} \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^{10-3}$

$= \frac{10!}{7!\,3!} \times \frac{3^3 \times 2^7}{5^3 \times 5^7} = 0.04246$

c. P(More than 4 are male) = P(r > 4)

$= p(5) + p(6) + p(7) + p(8) + p(9) + p(10)$

$= 10_{c5}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^{10-5} + 10_{c6}\left(\frac{3}{5}\right)^6\left(\frac{2}{5}\right)^{10-6} + 10_{c7}\left(\frac{3}{5}\right)^7\left(\frac{2}{5}\right)^{10-7}$

$+ 10_{c8}\left(\frac{3}{5}\right)^8\left(\frac{2}{5}\right)^{10-8} + 10_{c9}\left(\frac{3}{5}\right)^9\left(\frac{2}{5}\right)^{10-9} + 10_{c10}\left(\frac{3}{5}\right)^{10}\left(\frac{2}{5}\right)^0$

$= \frac{10!}{5!\,5!}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^5 + \frac{10!}{6!\,4!}\left(\frac{3}{5}\right)^9 + \frac{10!}{3!\,7!}\left(\frac{3}{5}\right)^7\left(\frac{2}{5}\right)^3 + \frac{10!}{8!\,2!}\left(\frac{3}{5}\right)^8\left(\frac{2}{5}\right)^2$

$+ \frac{10!}{9!\,1!}\left(\frac{3}{5}\right)^9\left(\frac{2}{5}\right) + \frac{10!}{10!}\left(\frac{3}{5}\right)^{10} = 0.9447$

12. Here,

p = Probability off hitting a target $= \frac{1}{5}$

$q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

n = No. of hitting = 6

p(r) = Probability of r successful hitting $= n_{c_r}\, p^r q^{n-r}$

a. p(Exactly once) = p(1) = ?

$= 6_{c1}\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^{6-1} = \frac{6!}{5!\,1!} \times \frac{1}{5} \times \left(\frac{4}{5}\right)^5 = 0.3932$

b. p(Exactly twice) = p(2)

$= 6_{c2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^{6-2} = \frac{6!}{4!\,2!} \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^4 = 0.24576$

13. Given,

p = Probability that a bomb dropped $= \frac{1}{4}$

$q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

n = no. of dropped = 5

a. P(None will strike target) = p(0) $= n_{c_r}\, p^r q^{n-r}$

$= 5_{c0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{5-3}$

$= \frac{5!}{2!\,3!} \times \left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = 0.879$

c. p(At least three will strike target) = $p(x \le 3)$

$= p(3) + p(4) + p(5)$

$= 5_{c3}\left(\frac{1}{4}\right)^3 . \left(\frac{3}{4}\right)^{5-3} + 5_{c4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^{5-4} \times 5_{c5}\left(\frac{1}{4}\right)^5$

$= 5_{c3}\left(\frac{1}{4}\right)^3 . \left(\frac{3}{4}\right)^2 + \frac{5!}{4!\,1!} \times \left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1 + \frac{5!}{5!}\left(\frac{1}{4}\right)^5$

$= 0.1035$

14. Given,

p = detective products = 20% $= \frac{20}{100} = \frac{!}{5}$

$q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$, n = 4

Now, the probability of r defective in 4 trials is given by

$p(r) = 4_{c_r}\, p^r . q^{4-r} = 4_{c_r}\left(\frac{1}{5}\right)^r\left(\frac{4}{5}\right)^{4-r}$

a. p(No chip is defective) = p(0)

$= 4_{c0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{4-0}$

$$= \frac{4!}{4!\ 0!} \times 1 \times \left(\frac{4}{5}\right)^4 = 0.4096$$

b. p(One chip is defective) = p(1)

$$= 4_{c_1} \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{4-1}$$

$$= \frac{4!}{1!\ 3!} \times \frac{1}{5} \times \left(\frac{4}{5}\right)^3 = \frac{4 \times 4 \times 4 \times 4}{5 \times 5 \times 5 \times 5} = 0.4096$$

c. p(more than one chip care defective)

$$= p(2) + p(3) + p(4)$$

$$= 4_{c_2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{4-2} + 4_{c_3} \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^{4-3} + 4_{c_4} \left(\frac{1}{5}\right)^4$$

$$= \frac{4!}{2!\ 2!} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^2 + \frac{4!}{3!\ 1!} \times \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right) + \frac{4!}{4!} \left(\frac{1}{5}\right)^4 = 0.1808$$

Chapter 15
Derivatives

Exercise 15.1

1. Find the limit of the following function at given points.

a. $f(x) = \dfrac{\log(1+x)}{x}$ at $x = 0$

Solution: Since, we have,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$\text{Now, } \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log(1+x)}{x}$$

$$= \lim_{x\to 0} \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]$$

$$= \lim_{x\to 0}\left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right]$$

$$= \left[1 - \frac{0}{2} + \frac{0^2}{3} - \frac{0^3}{4} + \ldots\right]$$

$$= 1$$

$$\therefore \lim_{x\to 0} f(x) = 1$$

b. $f(x) = \left(\dfrac{1-x}{1+x}\right)^{1/x}$ at $x = 0$

Solution: Here, $y = \left(\dfrac{1-x}{1+x}\right)^{1/x}$

Taking log on both sides,

$$\log y = \frac{1}{x}\log\left(\frac{1-x}{1+x}\right) \qquad [\because \log m^n = n\log m]$$

$$\text{or, } \log y = \frac{1}{x}\log\left[1 - \frac{2x}{1+x}\right]$$

$$= \frac{1}{x}\,\frac{\log\left[1 + \left(\frac{-2x}{1+x}\right)\right]}{\left(\frac{-2x}{1+x}\right)} \times \frac{-2x}{1+x}$$

$$= \frac{\log\left[1 + \left(\frac{-2x}{1+x}\right)\right]}{\frac{-2x}{1+x}} \times \frac{-2}{1+x}$$

Taking $\lim\limits_{x\to 0}$ on both sides, we have,

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left[1 + \left(\frac{-2x}{1+x}\right)\right]}{\frac{-2x}{1+x}} \times \frac{-2}{1+x}$$

$$= 1 \times \lim_{x\to 0} -\frac{2}{1+x} \qquad \left[\because \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$$

$$= \frac{-2}{1+0}$$

$$\therefore \lim_{x\to 0} \log y = -2$$

i.e, $\lim_{x\to 0} y = e^{-2}$ [$\because \log_x x = y \Leftrightarrow x = e^y$]

i.e, $\lim_{x\to 0} \left(\frac{1-x}{1+x}\right)^{1/x} = e^{-2}$

c. $f(x) = \frac{(1+x)^n - 1}{x}$ at x = 0

Solution: $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(1+x)^n - 1}{x}$

$$= \lim_{x\to 0} \frac{(1+x-1)\,[(1+x)^{n-1} + (1+x)^{n-2} + (1+x)^{n-3} + \ldots + (1+x) + 1]}{x}$$

$$= \lim_{x\to 0} (1+x)^{n-1} + (1+x)^{x-2} + (1+x)^{n-3} + \ldots (1+x) + 1$$

$$= 1^{n-1} + 1^{n-2} + 1^{n-3} + \ldots 1 + 1$$

$$= n-1+1$$

$$= n$$

2.a. A function f(x) is defined as follows:

$$f(x) = \begin{cases} -x-3 & \text{for } x \le -2 \\ \frac{2}{3}x + \frac{1}{3} & \text{for } -2 < x < 1 \\ x^2 & \text{for } x \ge 1 \end{cases}$$

Test the continuity of f(x) of x = –2 and x = 1.

Solution:

Note: A function f(x) is said to be continuous at a point x = a if and only if,

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

Here, To test the continuity of f(x) at x = –2, we proceed as follows:

Here, f(x) at x = –2 is

$f(-2) = -x - 3$ [$\because$ f(x) = –x – 3 for x ≤ –2]

$= -2 - 3 = -5$ (a finite value)

Now, left hand limit of f(x) at x = –2 is,

$\lim_{x\to -2^-} f(x) = \lim_{x\to 2^-} (-x-3)$ [$\because$ f(x) = –x – 3 for x ≤ –2]

$= -2 - 3 = -5$

Finally,

Right hand limit of f(1) at x = –2 is,

$\lim_{x\to -2^+} f(x) = \lim_{x\to 2^+} \left(\frac{2}{3}x + \frac{1}{3}\right)$ [$\because f(x) = \frac{2}{3}x + \frac{1}{3}$ for –2 < x < 1]

$= \frac{2}{3} \times (-2) + \frac{1}{3} = -\frac{4}{3} + \frac{1}{3} = \frac{-4+1}{3} = -\frac{3}{3} = -1$

$\therefore \lim_{x\to -2^-} f(x) = f(-2) \ne \lim_{x\to -2^+} f(x)$

So, f(x) is discontinuous at a point x = –2.

2nd Part;

Again testing the continuity of f(x) at x = 1

For the functional value, $f(1) = x^2$ [$\because f(x) = x^2$ for x ≥ 1]

$= 1^2 = 1$

LHL at x = 1 is,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(\frac{2}{3}x + \frac{1}{3}\right)$ [$\because f(x) = \frac{2}{3}x + \frac{1}{3}$ for –2 < x < 1]

$= \frac{2}{3} \times 1 + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1$

RHL at x = 1 is,

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x^2)$ [$\because$ $f(x) = x^2$ for $x \geq 1$]

$= 1^2 = 1$

Here, we have,

$\lim_{x\to1^-} f(x) = f(1) = \lim_{x\to1^+} f(x)$

Thus, the given function is continuous at x = 1

b. Show that the following function is continuous at x = 4

$$f(x) = \begin{cases} \frac{x^2 - 16}{x - 4} \text{ for } x \neq 4 \\ 8 \text{ for } x = 4 \end{cases}$$

Solution: Testing functional value at x = 4;

f(4) = 8 [$\because$ f(x) = 8 when x = 4]

Again, testing the limiting value at x = 4, we have,

$\lim_{x\to4} f(x) = \lim_{x\to4} \left(\frac{x^2 - 16}{x-4}\right)$ $\left[\because f(x) = \frac{x^2-16}{x-4} \text{ for } x \neq 4\right]$

$= \lim_{x\to4} \left(\frac{x^2-4^2}{x-4}\right)$ $\left[\because \text{at } x = 4, \frac{4^3-16}{4-4} = \% \text{ form}\right]$

$= \lim_{x\to4} \frac{(x + 4)(x - 4)}{(x - 4)} = \lim_{x\to4} (x + 4)$ [$\because$ $x \neq 4$]

$= 4 + 4 = 8$

Here, we see, $\lim_{x\to4} f(x) = f(4)$

Therefore, f(x) is continuous at x = 4

c. A function f(x) is defined as follows:

$$f(x) = \begin{cases} 3 + 2x \text{ for } -3/2 \leq x < 0 \\ 3 - 2x \text{ for } 0 \leq x < 3/2 \\ -3 - 2x \text{ for } x \geq 3/2 \end{cases}$$

Test the continuity of f(x) at x = 0 and x =- 3

Solution: Testing the continuity of f(x) at x = 0,

For the functional value, f(0) = 3–2x

or, $f(0) = 3-2\times0$

= 3 (a finite value)

Again, LHL of f(x) at x = 0,

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} (3 + 2x)$ [$\because$ $f(x) = 3+2x$ for $-\frac{3}{2} \leq x < 0$]

$= 3 + 2\times0$

$= 3$

Finally, RHL off f(x) at x = 0

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} (3 - 2x)$ [$\because$ $f(x) = 3 - 2x$ for $0 \leq x < 3/2$]

$= 3 - 2\times0$

$= 3$

Here, we see, $\lim_{x\to0^-} f(x) = f(0) = \lim_{x\to0^+} f(x)$

$\therefore$ The given function is continuous at point x = 0.

2nd part:

Testing the continuity of f(x) at x = 3/2

Functional value, $f\left(\frac{3}{2}\right) = -3-2x$ $\qquad [\because f(x) = -3-2x \text{ for } x \geq 3/2]$

$$= -3 \times -2 \times \frac{3}{2} = -6$$

LHL at $x = \frac{3}{2}$, $\lim\limits_{x \to 3/2^-} f(x) = \lim\limits_{x \to 3/2^-} (3-2x)$ $\qquad [\because f(x) = 3-2x \text{ for } 0 \leq x < 3/2]$

$$= 3 - 2 \times \frac{3}{2}$$

Here, we see,

$$\lim_{x \to 3/2^-} f(x) \neq x\left(\frac{3}{2}\right)$$

Therefore, the given function is discontinuous at $x = \frac{3}{2}$

3.a. Show that the function f(x) defined by,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} \text{ for } r \neq 0 \\ 0 \text{ for } x \to 0 \end{cases} \text{ is continuous at } x = 0$$

Proof: Functional values, f(x) = 0 $\qquad [\because f(x) = 0 \text{ for } x = 0]$

Limiting value, $\lim\limits_{x \to 0} f(x)$

$$= \lim_{x \to 0} x^2 \sin \frac{1}{x} \qquad [\because f(x) = x^2 \sin \frac{1}{x} \text{ for } x \neq 0]$$

$$= 0$$

$\therefore$ $\lim\limits_{x \to 0} f(x) = f(0)$. Thus the function is continuous at x = 0

Hence proved.

b. Examine for continuity at x = 0 for the function of f(x) defined by,

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Functional values at x = 0;

f(0) = 1 (finite values) $\qquad [\because f(x) = 1 \text{ for } x = 0]$

Limiting values at x = 0,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1 - \cos x}{x^2} \qquad \left[f(x) = \frac{1 - \cos x}{x^2} \text{ for } x \neq 0\right]$$

$$= \lim_{x \to 0} \frac{(1 - \cos x)}{x^2} \times \frac{1 + \cos x}{(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2 (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \times \frac{1}{1 + \cos x}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \times \lim_{x \to 0} \frac{1}{1 + \cos x}$$

$$= (1)^2 \times \frac{1}{2} = \frac{1}{2} \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

Here, $\lim_{x\to 0} f(x) \neq f(0)$

$\therefore$ The given function is discontinuous at x = 0.

Exercise 15.2

1. Find from first principles the derivative of (1–5).

a. (i) $e^{\sin x}$.

Solution: Let, $f(x) = e^{\sin x}$

We know by the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h\to 0} \frac{e^{\sin(x+h)} - e^{\sin x}}{h} \ldots\ldots\ldots \text{(i)}$$

Put, $y = \sin x \Rightarrow y + k = \sin(x+h)$

$\Rightarrow$ $k = \sin(x+h) - y$

where $k \to 0$ as $h \to 0$

$\Rightarrow$ $\sin(x+h) = y + k$

Now, from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^y}{h}$$

$$= \lim_{h\to 0} \frac{e^y(e^k - 1)}{k} \times \frac{k}{h}$$

$$= e^y . \lim_{h\to 0} \frac{e^k - 1}{k} \times \frac{k}{h}$$

$$= e^y . \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \left[\because \lim_{h\to 0} \frac{e^k - 1}{k} = 1\right]$$

$$= e^y . \lim_{h\to 0} \frac{2\cos\frac{x+h+x}{2} \times \sin\frac{x+h-x}{2}}{h}$$

$$= e^y . \lim_{h\to 0} 2\cos\left(x + \frac{h}{2}\right) . \frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right) \times 2}$$

$$= e^y \times \frac{1}{2} \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right)} \times 2\cos\left(x + \frac{h}{2}\right)$$

$$= \frac{1}{2} e^y \times \lim_{h\to 0} 2\cos\left(x + \frac{h}{2}\right)$$

$$= e^{\sin x} \cos\left(x + \frac{h}{2}\right)$$

$$= e^{\sin x} . \cos x$$

ii. $e^{\tan x}$

Solution: Let, $f(x) = e^{\tan x}$

By the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{\tan(x+h)} - e^{\tan x}}{h} \quad \ldots\ldots\ldots \text{(i)}$$

Put, $y = \tan x \Rightarrow y+k = \tan(x+h)$, where $k \to 0$ when $h \to 0$

$\Rightarrow k = \tan(x+h) - e^{\tan x}$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^{y}}{h}$$

$$= \lim_{h\to 0} \frac{e^{y}(e^{k}-1)}{h}$$

$$= e^{y} \cdot \lim_{h\to 0} \frac{e^{k}-1}{k} \times \frac{k}{h}$$

$$= e^{y} \cdot \frac{\tan(x+h) - \tan x}{h}$$

$$= \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{h \cos x \cos(x+h)}$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos x \cdot \cos(x+h)} \qquad [\because \sin A \cos B - \sin B \cos A = \sin(A-B)]$$

$$= e^{y} \cdot \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{1}{\cos x \cdot \cos(x+h)}$$

$$= e^{\tan x} \cdot 1 \times \frac{1}{\cos x \cos x}$$

$$= e^{\tan x} \cdot \sec^2 x$$

iii. e^{x^2}

Solution: Let, $f(x) = e^{x^2}$

Since, by the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e(x+h)^2 - e^{x^2}}{h} \quad \ldots\ldots\ldots \text{(i)}$$

Put, $y = x^2 \Rightarrow y+k = (x+h)^2$ where $k\to 0$ when $h\to 0$

$\Rightarrow \quad k = (x+h)^2 - y = (x+h)^2 - x^2$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^{y}}{h}$$

$$= \lim_{h\to 0} \frac{e^{y}(e^{k}-1)}{k} \times \frac{k}{h}$$

$$= e^{y} \lim_{h\to 0} \frac{e^{k}-1}{k} \times \lim_{h\to 0} \frac{k}{h}$$

$$= e^{y} \times 1 \times \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= e^{y} \times \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= e^{y} \times \lim_{h\to 0} \frac{2xh}{h} + \lim_{h\to 0} \frac{h^2}{h}$$

$$= e^{x^2} \times 2x + 0$$

$$= 2x^{0x^2}$$

b. i. $a\sin\frac{x}{a}$

Solution: Let, $f(x) = a\sin\frac{x}{a}$

Since by the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to0}\frac{\frac{a\sin(x+h) - a\sin\frac{x}{a}}{a}}{h} \quad \dots\dots\dots (i)$$

Put, $y = \frac{x}{0} \Rightarrow y+k = \frac{x+h}{a}$, where $k \to 0$ when $h \to 0$

$\Rightarrow \quad k = \frac{x+h}{a} - y \Rightarrow k = \frac{x+h}{a} - \frac{x}{a}$

from (i) we have,

$$\frac{d}{dx}(f(x)) = \lim_{h\to0}\frac{a\sin(4+k) - a\sin y}{h}$$

$$= a\lim_{h\to0}\frac{\left[2\cos\frac{y+k+y}{2}\cdot\sin\frac{y+k-y}{2}\right]}{h}$$

$$= a\lim_{h\to0}\left[\frac{2\cos(y+k/2)\cdot\sin\frac{k}{2}}{h}\right]$$

$$= 2a\left[\lim_{h\to0}\cos\left(y+\frac{k}{2}\right)\cdot\lim_{h\to0}\frac{\sin\frac{k}{2}}{\frac{k}{2}}\times\frac{\frac{k}{2}}{h}\right]$$

$$= 2a\left[\cos y\cdot\lim_{h\to0}\frac{k}{2h}\right] \qquad [\because k\to0 \text{ when } h\to0]$$

$$= 2a\cos y\cdot\lim_{h\to0}\frac{\left(\frac{x+h}{a}-\frac{x}{a}\right)}{2h}$$

$$= a\cos y\times\lim_{h\to0}\frac{\left(\frac{x+h-x}{a}\right)}{h}$$

$$= a\cos y\times\frac{1}{x}\lim_{h\to0}\frac{h}{h}$$

$$= \cos y$$

$$= \cos\frac{x}{a}$$

ii. $\sin x^2$

Solution: Let, $f(x) = \sin x^2$

Since by the definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to0}\frac{\sin(x+h)^2 - \sin x^2}{h} \quad \dots\dots\dots (i)$$

Put, $y = x^2 \Rightarrow y+k = (x+h)^2$ where $k\to0$ as $h\to0$

$\Rightarrow k = (x+h)^2 - x^2$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\sin(y+k) - \sin y}{h}$$

$$= \frac{\lim_{h\to 0} 2\cos\frac{y+k+y}{2} . \sin\frac{y+k-y}{2}}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(y+\frac{k}{2}\right) . \sin\frac{k}{2}}{h}$$

$$= 2\lim_{h\to 0} \cos\left(y+\frac{0}{2}\right) . \frac{\sin\frac{k}{2}}{\frac{k}{2}} = x\frac{\frac{k}{2}}{h}$$

$$= 2\cos y \lim_{h\to 0} \frac{k}{2h}$$

$$= \cos y \times \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \cos y \lim_{h\to 0} \frac{x^2 + h^2 + 2xh - x^2}{h}$$

$$= \cos y \lim_{h\to 0} \frac{h(h+2x)}{h}$$

$$= \cos y \lim_{h\to 0} (h+2x)$$

$$= 2x \cos y$$

$$= 2x \cos x^2$$

iii. $\sqrt{\tan x}$

Solution: Let, $f(x) = \sqrt{\tan x}$

Since by the definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h} \times \frac{\sqrt{\tan(x+h)} + \sqrt{\tan x}}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$= \lim_{h\to 0} (\tan(x+h) - \tan x) \times \frac{1}{h\left[\sqrt{\text{ten}(x+h)} + \sqrt{\tan x}\right]}$$

$$= \lim_{h\to 0} \left(\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right) \times \frac{1}{h\left[\sqrt{\text{ten}(x+h)} + \sqrt{\tan x}\right]}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos x(\cos x(x+h))} \times \frac{1}{h\left[\sqrt{\tan(x+h)} + \sqrt{\tan}\right]}$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{\cos x \cos(x+h)} \times \frac{1}{h\left[\sqrt{\tan(x+h)} + \sqrt{\tan}\right]}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \times \frac{1}{\cos(x+h)\cos x} \times \lim_{h\to 0} \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan}}$$

$$= 1 \times \frac{1}{\cos^2 x} \times \frac{1}{\sqrt{\tan x} + \sqrt{\tan x}}$$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}$$

c. i. Log tanx

Solution: Let, f(x) = log (tanx)

Since by the definition of derivative,

$$\frac{d(f(x))}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log \tan(x+h) - \log(\tanx)}{h} \quad \text{... .... (i)}$$

Put, y = tanx $\Rightarrow$ y+k = tan(x + h) where k $\to$ 0 as h $\to$ 0

k = tan (x + n) – tanx

from (i), $\lim_{h\to 0} \frac{\log (y + k) - \log y}{h}$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(1 + \frac{k}{y}\right)}{\frac{k}{y}} \times \frac{k/y}{h}$$

$$= \frac{1}{y} \times \lim_{h\to 0} \frac{\tan (x + h) - \tan x}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\frac{\sin(X + h)}{\cos(x + h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\sin(x + h) \cos x - \cos(x + h) \sin x}{h\cos x \cdot \cos(x + h)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\sin(x + h - x)}{h \cos x \cdot \cos (x + h)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\sin h}{n \cos x \cdot \cos(x + h)}$$

$$= \frac{1}{y} \times 1 \times \frac{1}{\cos^2 x}$$

$$= \frac{1}{\tan x} \times \frac{1}{\cos^2 x}$$

$= \frac{1}{\tan x} \cdot \sec^2 x$**Error! Bookmark not defined.**

ii. Log $\sec x^2$

Solution: Let, f(x) = f(x) = log $\sec x^2$

Since by the definition of derivatives,

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log \sec(x+h)^2 - \log \sec x^2}{h} \quad \text{... ... ... (i)}$$

Put, y = $\sec x^2 \Rightarrow$ y+k = $\sec(x + h)^2$ where k$\to$0, as h$\to$0

$\Rightarrow$ k = $\sec (x + h)^2 - \sec x^2$

from (i)

$$= \lim_{h\to 0} \frac{\log (y + k) - \log y}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log(1+k/y)}{k/y} \times \frac{k/y}{h}$$

$$= 1 \times \lim_{h\to 0} \frac{\sec(x+n)^2 - \sec x^2}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\frac{1}{\cos(x+n)^2} - \frac{1}{\cos x^2}}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\frac{1}{\cos(x+h)^2} - \frac{1}{\cos x^2}}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\cos x^2 - \cos(x+h)^2}{h\,.\cos(x+h)^2\,.\cos x^2}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{-2\sin\frac{x^2+(x+h)^2}{2}\sin\frac{x^2-(x+1)^2}{2}}{h\,.\cos(x+h)^2\,.\cos x^2}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{-2\sin\frac{x^2+(x+h)^2}{2}\,.\sin\frac{x^2-(x+h)^2}{2}}{h.\cos(x+h)^2\,.\cos x^2}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{-2\sin\left(\frac{x^2+2xh+h^2}{2}\right)\sin\left(\frac{-2xh-h^2}{2}\right)}{h.\cos(x+h)^2\,.\cos x^2}$$

$$= \frac{1}{y}x(-2)\frac{\sin x^2}{\cos x^2\cos x^2} \times \lim_{h\to 0}(-1) \times \frac{\sin\left(\frac{2xh+h^2}{2}\right)}{h}$$

$$= \frac{2}{y}\tan x^2\,.\sec x^2 \times \lim_{h\to 0}\frac{\sin(2x+h)/2}{(2x+h)/2} \times \frac{(2x+h)}{2}$$

$$= \frac{2}{\sec x^2}\tan x^2\,.\sec x^2 \times 1 \times \frac{2x}{2}$$

$$= 2x\tan x^2$$

iii. log (cosecx)

Solution: Let, $f(x) = \log(\operatorname{cosec}x)$

Since by definition of derivatives,

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\log\operatorname{cosec}(x+h) - \log\operatorname{cosec}x}{h} \quad \dots\dots\dots \text{(i)}$$

Put $y = \operatorname{cosec}x \Rightarrow y+k = \operatorname{cosec}(x+h)$ where $k\to 0$ as $h\to 0$

$\Rightarrow k = \operatorname{cosec}x\,(x+h) - \operatorname{cosec}x$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\log(y+k) - \log y}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log(1 + k/y)}{k/y} \times \frac{k/y}{k}$$

$$= \lim_{h\to 0} \frac{k}{yh}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\frac{1}{\sin(x+h)} - \frac{1}{\sin x}}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\sin x - \sin(x+h)}{\sin x \sin(x+h) \times h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{2\cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{h \times \sin x \sin(x+h)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{2\cos(x+h/2)\sin(-h/2)}{h \times \sin x \sin(x+h)}$$

$$= \frac{2}{y} \frac{\cos x}{\sin x \,.\, \sin x} \times \lim_{h\to 0} (-1) \frac{\sin h/2}{h/2 \times 2}$$

$$= \frac{2}{\text{cosecl}} \,.\, \text{colx} \,.\, \text{cosec}x \times \left(\frac{-1}{x}\right)$$

$$= \frac{-\text{cosec}x \,.\, \text{colx}}{\text{cosec}x}$$

$$= -\cot x$$

d. i. $\cot^{-1}x$

Solution: Let, $f(x) = \cot^{-1}x$

$\therefore \quad f(x+h) = \cot^{-1}(x+h)$

We know, $f'(x) = \lim_{h\to 0} \frac{\cot^{-1}(x+h) - \cot^{-1}x}{h}$

Let $\cot^{-1}x = y$ and $\cot^{-1}(x+h) = y + k$

$\therefore \quad y + k - y = k = \cot^{-1}(x+h) - \cot^{-1}x$

When $h \to 0$ then $k \to 0$

Also, $x + h = \cot(y+k)$ and $x = \cot y$

$x + h - x = h = \cot(y+k) - \cot y$

Now, $f'(x) = \lim_{k\to 0} \frac{y+k-y}{h} = \lim_{k\to 0} \frac{k}{\cot(y+k) - \cot y}$

$$= \lim_{k\to 0} \frac{k}{\frac{\cot(y+k)}{\sin(y+k)} - \frac{\cos y}{\sin y}}$$

$$= \lim_{k\to 0} \frac{k}{\frac{\sin y(\text{con}\, y+k) - \text{con}\, y \sin(y+k)}{\sin(y+k) \,.\, \sin y}}$$

$$= \lim_{k\to 0} \frac{k\sin(y+k) \,.\, \sin y}{\sin(y+k+y)} = \lim_{k\to 0} \frac{-k}{\sin k} \times \frac{\sin(y+k)}{\sin y}$$

$$= -1 \times \sin(y+0) \,.\, \sin y = -\sin^2 y = \frac{-1}{\text{cosec}^2 y}$$

$$= \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$$

ii. $\log \tan^{-1} x$

Solution: Let, $f(x) = \log\tan^{-1} x$

Since by definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log \tan^{-1}(r+h) - \log \tan{-1}^{x}}{h} \quad \ldots\ldots\ldots \text{(i)}$$

Put, $y = \tan^{-1} x \Rightarrow y + k = \tan^{-1}(x+h)$ where $k \to 0$ as $h \to 0$

or, $\tan y = x$ or, $\tan(y+k) - \tan y = h$

Now, from (i)

$$\frac{d}{dx}(f(r) = \lim_{h\to 0} \frac{\log(y+k) - \log y}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log(1+k/y)}{k/y} \times \frac{k}{yh}$$

$$= \frac{1}{y} \times \lim_{h\to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{k \times \cos(y+k) \,.\, \cos y}{\sin(y+k) \,.\, \cos y - \sin y \,.\, \cos(y+k)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{k \cos(y+k) \cos y}{\sin(y+k-y)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{k\cos y \,.\, \cos(y+k)}{\sin k}$$

$$= \frac{1}{y}\left(\lim_{h\to 0} \frac{k}{\frac{\sin k}{k} \times k}\right) \lim_{h\to 0} \cos y \,.\, \cos(y+k)$$

$$= \frac{1}{\tan^{-1}x} \times 1 \times \cos^2 y$$

$$= \frac{\cos^2 y}{\tan^{-1}x} = \frac{1}{\tan^{-1}x \sec^2 y} = \frac{1}{\tan^{-1}x\,(1+\tan^2 y)} = \frac{1}{\tan^{-1}x\,(1+x^2)}$$

iii. $e^{\tan^{-1}x}$

Solution: Let, $f(x) = e\tan^{-1}x$

Since by definition of derivatives

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c^{\tan-1\,(x+h)} - e^{\tan-1}\,x}{h}$$

Put, $y = \tan^{-1} x \Rightarrow y+k = \tan^{-1}(x+h)$ where $k \to 0$ as $h \to 0$

or, $\tan y = x \Rightarrow \tan(y+k) = x+h$

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^{y}}{h}$$

$$= \left(\lim_{h\to 0} \frac{e^{y}(ek-1)}{k}\right) \times \lim_{h\to 0} \frac{k}{h}$$

$$= e^{y} \,.\, \lim_{h\to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$= e^{y} \,.\, \lim_{h\to 0} \frac{ky\cos(y+k)\cos y}{\sin(y+k)\cos y - \sin y \,.\, \cos(y+k)}$$

$$= e^y . \lim_{h\to 0} \frac{k \cos y \cos (y + k)}{\sin(y + k - y)}$$

$$= e^y . \lim_{h\to 0} \frac{ky \cos y \cos (y + k)}{\frac{\sin k}{k} \times k}$$

$$= e^y . \lim_{h\to 0} \frac{\cos y \cos (y + k)}{\frac{\sin k}{k}}$$

$$= e^y . \cos^2 y$$

$$= e^{\tan^{-1}}x . \frac{1}{\sec^2 y}$$

$$= e^{\tan^{-1}}x . \frac{1}{1 + \tan^2 y}$$

$$= e^{\tan^{-1}x} . \frac{1}{1 + x^2}$$

e.i. 3^{x2}

Solution: Let, $f(x) = 3^{x2} = e\log 3x2 = e^{x2 \log 3}$.

Since by definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h\to 0} \; e^{(x+b)^2 \log 3} - e^{x^2 \log 3} \text{ ... ... (i)}$$

Put, $y = x^2\log 3 \Rightarrow y + k = (x + h)^2 \log 3$ where $k \to 0$ as $h\to 0$

$\Rightarrow k = (x + h)^2 \log 3 - x^2 \log 3$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^y}{h}$$

$$= \lim_{h\to 0} \frac{e^y (e^k - 1)}{k} \times \frac{k}{h}$$

$$= e^y \lim_{h\to 0} \left(\frac{e^k - 1}{k}\right) \times \lim_{h\to 0} \frac{(x + h)^2 \log 3 - x^2\log 3}{h}$$

$$= e^y . \lim_{h\to 0} \frac{x^2\log 3 + h^2\log 3 + 2xh\log 3 - x^2\log^3}{h}$$

$$= e^y . \lim_{h\to 0} \frac{h(h\log 3 + 2x\log 3)}{h}$$

$$= e^{x^2\log 3} \times 2x\log 3$$

$$= e^{x^2\log 3}\, 2x\log 3 - e^{\log 3^{x^2}}\, 2\times\log 3 = 3^{x^2}\, 2x\log 3$$

ii. x^x

Solution: Let, $f(x) = x^x = e^{\log x^x} = e^{x\log x}$

Since by definition of derivatives

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{(x+h) \log (x + h)} - e^{x\log x}}{h} \text{ ... ... (i)}$$

Put, $y = x\log x \Rightarrow y+k = (x+h) \log (x + h)$ where $x\to 0$ cos $h\to 0$

From (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^y}{h}$$

$$= \lim_{h\to 0} \frac{e^y(e^k - 1)}{k} \times \frac{k}{h}$$

$$= e^y \left(\lim_{h\to 0} \frac{e^k - 1}{k}\right) \times \lim_{h\to 0} \frac{(x+h)\log(x+h) - x\log x}{h}$$

$$= e^y . \lim_{h\to 0} \frac{x\log(x+h) + h\log(x+h) - x\log x}{h}$$

$$= e^y . \lim_{h\to 0} \left[\frac{x[\log(x+h) - \log x]}{h} + \frac{h\log(x+h)}{h}\right]$$

$$= e^y \left[\lim_{h\to 0} \left\{\frac{x\log(1+h/x)}{\frac{h}{x}x^x}\right\} + \lim_{h\to 0} \frac{h\log(x+h)}{h}\right]$$

$$= e^y [1 + \log(x+0)]$$
$$= e^{x\log x}[1+\log x]$$
$$= x^x[1+\log x]$$

iii. a^{2x}

Solution: Let $f(x) = a^{2x} = e^{\log a^{2x}} = e^{2x\log a}$

Since by definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{2(x+h)\log a} - e^{2x\log a}}{h}$$

$$= \lim_{h\to 0} \frac{e^{2x\log a} . e^{2h\log a} - e^{2x\log a}}{h}$$

$$= \lim_{h\to 0} e^{2x\log a} \frac{[e^{2h\log a} - 1]}{h}$$

$$= e^{2x\log a} \times \lim_{h\to 0} \frac{e^{h\log a} - 1}{\frac{2h\log a}{2\log a}}$$

$$= e^{\log a^{2x}} \times 2\log a \left[\lim_{h\to 0} \frac{e^{2h\log a} - 1}{2h\log a}\right]$$

$$= 2a^{2x}\log a$$

2.a. A function f(x) defined as follows:

$$f(x) = \begin{cases} 1 + \sin x \text{ for } 0 \le x < \pi/2 \\ 2 + (x - \pi/2)^2 \text{ for } \pi/2 \le x < \infty \end{cases}$$

Does $f'(\pi/2)$ exists?

Solution: To show whether $f'(\pi/2)$ exists or not, we show,

Left hand derivative = Right level derivative at $\frac{\pi}{2}$

Now, For right hand derivative,

$$Rf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{(\pi/2 + h) - f(\pi/2)}{h} = \lim_{h\to 0} \frac{2 + (\pi/2 + h - \pi/2)2 - 2 - (\pi/2 - \pi/2)^2}{h}$$

$$= \lim_{h\to 0} \frac{2 + h^2 - 2}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$$

For, left hand derivative at $x = \frac{\pi}{2}$

$$Lf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{f(\pi/2 - h) - f(\pi/2)}{-h}$$

$$= \lim_{h\to 0} \frac{1 + \sin(\pi/2 - h) - 2 - (\pi/2 - \pi/2)^2}{-h}$$

$$= \lim_{h\to 0} \frac{1 + \sin(\pi/2 - h) - 2}{-h} = \lim_{h\to 0} \frac{1 + \cos h - 2}{-h}$$

$$= \lim_{h\to 0} \frac{1 - \cos h}{h} = \ldots\ldots = \lim_{h\to 0} \frac{1 + \cos h - 2}{-h}$$

$\therefore$ $Rf'\left(\frac{\pi}{2}\right) = Lf'\left(\frac{\pi}{2}\right) = 0$. Therefore, $f'\left(\frac{\pi}{2}\right)$ exists.

b. Show that $f(x) = |x-2|$ is continuous but not differentiate at $x = 2$.

Solution: Testing the continuous at $x = 2$

R.H.L at $x = 2$ is,

$$= \lim_{h\to 2^+} f(x) = \lim_{h\to 2^+} |x - 2| = \lim_{h\to 2^+} (x - 2) = 0$$

L.H.L. at $x = 2$ is,

$$= \lim_{h\to 2^-} f(x) = \lim_{h\to 2^-} |x - 2| = \lim_{h\to 2^-} - (x - 2) = 0$$

Functional value at $x = 2$ is,

$f(2) = |2 - 2| = 0$

$\therefore$ L.H.L = R.H.L = functional value

$\therefore$ The given function is continuous at $x = 2$

2^{nd} part,

$$Rf'(2) = \lim_{h\to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h\to 0} \frac{|2 + h - 2| - 0}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h} = 1$$

$$Lf'(2) = \lim_{h\to 0} \frac{f(2 - h) - f(2)}{h} = \lim_{h\to 0} \frac{|2 - h - 2| - 0}{-h}$$

$$= \lim_{h\to 0} \frac{|-h|}{-h}$$

$$= \lim_{h\to 0} \frac{h}{-h} = -1$$

Here, $Rf'(2) \neq Lf'(2)$

Thus, the given function is not differentiable at $x = 2$, though it is continuous at $x = 2$

3. Find the derivative of;

$$y = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log\left(x + \sqrt{x^2 + a^2}\right)$$

Solution: Differentiate both sides w.r.to x we get,

$$\frac{dy}{dx} = \left[\frac{x.\ 2x}{2\sqrt{x^2 + a^2}\times 2} + \frac{\sqrt{x^2 + a^2}}{2}\right] + \frac{a^2}{2} \frac{\left(1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right)}{(x + \sqrt{x^2 + a^2})}$$

$$= \frac{x^2}{2\sqrt{x^2 + a^2}} + \frac{\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \frac{\left(\sqrt{x^2 + a^2} + x\right)}{\sqrt{x^2 + a^2}\left(x + \sqrt{x^2 + a^2}\right)}$$

$$= \frac{1}{2}\left[\frac{x^2 + x^2 + a^2 + a^2}{\sqrt{x^2 + a^2}}\right] = \frac{(x^2 + a^2)}{\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2}$$

4.a. Let $y = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$

Put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

Now, $y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$= \tan^{-1}(\tan 2\theta)$

$= 2\theta$

$y = 2\tan^{-1}x$

Now, differentiating on both sides, we get

$$\frac{dy}{dx} = 2\frac{d\tan^{-1}x}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

b. Let $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$

Differentiating on both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d\cos^{-1}\frac{2x}{1+x^2}}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \times \left[\frac{(1+x^2)^2 - 2x \times 2x}{(1+x^2)^2}\right]$$

$$= -\frac{(1+x^2)}{\sqrt{(1+x^2)^2 - 4x^2}}\left[\frac{2+2x^2-4x^2}{(1+x^2)^2}\right]$$

$$= -\frac{(2-2x^2)}{\sqrt{1+2x^2+x^4-4x^2}} \times \frac{1}{(1+x^2)}$$

$$= -\frac{(1-x^2)}{(1-x^2)} \times \frac{1}{(1+x^2)}$$

$$\frac{dy}{dx} = -\frac{2}{(1+x^2)}$$

c. Let $y = \sin^{-1}(3x - 4x^3)$

Put $x = \sin\theta$

$y = \sin^{-1}(3\sin\theta - 4\sin3\theta)$

$y = \sin^{-1}(\sin3\theta)$

$y = 3\theta = 3\sin^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

d. Let $y = \sin^{-1}\sqrt{1-x^2}$

Put $x = \cos\theta$

$\therefore \quad y = \sin^{-1}\sqrt{1-\cos^2\theta}$

$= \sin^{-1}\sin\theta$

$= \theta$

$y = \cos^{-1}x$

Now, differentiating on both sides, we get

$$\frac{dy}{dx} = \frac{d(\cos^{-1}x)}{dx}$$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

e. Let $y = \sin^{-1} 2x\sqrt{1-x^2}$

Put $x = \cos\theta$ then

$y = \sin^{-1} 2\cos\theta\sqrt{1-\cos^2\theta}$

$= \sin^{-1}(2\cos\theta \times \sin\theta)$

$= \sin^{-1}\sin2\theta$

$y = 2\theta = 2\cos^{-1}x$

Now, differentiating on both sides, we get

$$\frac{dy}{dx} = 2 \times -\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

f. Let $y = \sec^{-1}(\tan x)$

Differentiating on both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d\sec^{-1}(\tan x)}{dx}$$

$$= \frac{1}{\tan x\sqrt{\tan^2 x - 1}} \times x\sec^2 x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sec^2 x}{\tan x\sqrt{\tan^2 x - 1}}$$

g. $y = \tan^{-1}\frac{\sqrt{1+x^2} - 1}{x}$

Solution: Put, $x = \tan\theta$

Now, $y = \tan^{-1}\frac{\sqrt{1+\tan^2 x} - 1}{x}$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}$$

$$= \tan^{-1}\tan\theta/2 = \theta/2 = \frac{1}{2}\tan a^{-1}x$$

Differentiate both sides w.r.to x,

h. $\text{Sin}^{-1}\frac{2x}{1+x^2} + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ prove that $\frac{dy}{dX} = \frac{4}{1+x^2}$

Solution: Put, $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

L.H.S. $\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$$= \sin^{-1}\sin 2\theta + \sec^{-1}\left(\frac{\sec^2\theta}{1-\frac{\sin^2\theta}{\cos^2\theta}}\right)$$

$$= 2\theta + \sec^{-1}\left(\frac{1}{\cos^2\theta - \sin^2\theta}\right)$$

$$= 2\theta + \sec^{-1}\sec 2\theta$$

$$= 40 = 4\tan^{-1}x$$

Differentiate both sides by x, we get,

$\frac{dy}{dx} = \frac{4}{1+x^2}$ R.H.S. Proved.

5.a. To prove $\frac{dy}{dx} = \frac{y^2}{x(1-y\text{Ln}x)}$

We have,

$y = x^y$

Taking Ln on both sides, we get

Lny = yLnx

Now, differentiating on both sides we get

$\frac{1}{y}\frac{dy}{dx} = Lnx\frac{dy}{dx} + y.\frac{1}{x}$

$\Rightarrow \left(\frac{1}{y} - Lnx\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow (1 - yLnx)\frac{dy}{dx} = \frac{y^2}{x}$

$\therefore \frac{dy}{dx} = \frac{y^2}{x(1 - y\,Lnx)}$

b. To prove $\frac{dy}{dx} = \frac{y}{x}$

We have, $x^p.y^q\ \ (x + y)^{P+q}$

Taking Ln on both sides, we get

$Ln(x^p.y^q) = Ln(x + y)^{p+q}$

$\Rightarrow$ pLnx + qLny = (p + q) Ln (x + y)

Now, differentiating on both sides, we get

$p.\frac{1}{x} + q\frac{1}{y}\frac{dy}{dx} = (p + q).\frac{1}{(x + y)}\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow \frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p + q}{x + y} + \frac{p + q}{x + y}\frac{dy}{dx}$

$\Rightarrow \left(\frac{q}{y} - \frac{p + q}{x + y}\right)\frac{dy}{dx} = \frac{p + q}{x + y} - \frac{p}{x}$

$\Rightarrow \frac{(qx + qy - py - qy)}{y(x + y)}.\frac{dy}{dx} = \frac{px + qx - px - py}{x(x + y)}$

$\Rightarrow \frac{(qx - py)}{y(x + y)}\frac{dy}{dx} = \frac{(qx - py)}{x(x + y)}$

$\therefore \frac{dy}{dx} = \frac{y}{x}$

c. To prove $\frac{dy}{dx} = -\frac{y(y + x\,Lny)}{x(yLnx - x)}$

We have,

$x^y . y^x = 1$

Taking Ln on both sides we get

yLnx + xLny = 0

Now, differentiating on both sides w.r.t. x, we get

$Lnx\frac{dy}{dx} + y.\frac{1}{x} + Lny.1 + x.\frac{1}{y}\frac{dy}{dx} = 0$

$\Rightarrow \left(Lnx + \frac{x}{y}\right)\frac{dy}{dx} = -\frac{y}{x} - Lny$

$\Rightarrow \frac{(y\,Lnx + x)}{y}\frac{dy}{dx} = \frac{-y - xLny}{x}$

$\therefore \frac{dy}{dx} = \frac{-y(y + xLny)}{x(yLnx - x)}$

d. If siny = x cos (a+y) show that $\frac{dy}{dx} = \frac{\cos^2(a + y)}{\cos a}$

Solution: Differentiate both sides w.r. to x we ge,t

$\cos y\frac{dy}{dx} = -x\sin(a + y)\frac{dy}{dx} + 1.\cos(a + y)$

or, $(\cos y + x\sin(a + y))\frac{dy}{dx} = \cos(a + y)$

or, $\frac{dy}{dx} = \frac{\cos(a + y)}{\cos y + x\sin(a + y)}$

or, $\frac{dy}{dx} = \frac{\cos(a+y)}{\cos y + \frac{\sin y}{\cos(a+y)\sin(a+y)}} = \frac{\cos^2(a+y)}{\cos(a+y-y)} = \frac{\cos^2(a+y)}{\cos a}$ proved.

e. If $y = \frac{\cos x - \sin x}{\cos x + \sin x}$, show that $\frac{dy}{dx} + y^2 + 1 = 0$

Solution: Here, $y = \frac{\cos x - \sin x}{\cos x + \sin x} \times \frac{\cos x - \sin x}{\cos x - \sin x}$

$y = \frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x} = \frac{\cos 2x + \sin 2x - 2\sin x \cos x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \frac{\sin 2x}{\cos 2x} = \sec 2x - \tan 2x$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 2\sec 2x \tan 2x - 2\sec^2 x.$

Again, $y^2 = \sec^2 2x + \tan^2 2x - 2\sec 2x \, . \tan 2x$

Now, L.H.S. $\frac{dy}{dx} + y^2 + 1$

$= 2\sec 2x \tan 2x - 2\sec^2 2x + \sec^2 2x + \tan^2 2x - 2\sec 2x \tan 2x + 1$

$= -\sec^2 2x + \tan^2 2x + 1$

$= -1 + 1$

$= 0$ R.H.S.

6. Find the derivative with respect to x of following.

a. Let $y = x^{\sin x}$

$\text{Ln}y = \sin x \ln x$

Differentiating on both sides, we get

$\frac{1}{y} . \frac{dy}{dx} = \ln x \, . \cos x + \sin x \, \frac{1}{x}$

$\Rightarrow \frac{dy}{dX} = y \left[\ln x \cos x + \frac{\sin x}{x}\right]$

$\therefore \frac{dy}{dx} = x^{\sin x} \left[\frac{\sin x}{x} + \cos x \, . \ln x\right]$

b. Let $y = (\sin x)^{\cos x}$

$\text{Ln}y = \cos x \, . \text{Ln}(\sin x)$

$\frac{1}{y}\frac{dy}{dx} = -\text{Ln}(\sin x) \, . \sin x + \cos x \times \cot x$

$\therefore \frac{dy}{dx} = (\sin x)^{\cos x} [\cos x \, . \cot x - \sin x \, , \text{Ln}(\sin x)]$

c. $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

Solution: Let, $y = u + v$ where, $u = (\sin x)^{\cos x}$ and $v = (\cos x)^{\sin x}$

Now, $u = (\sin x)^{\cos x}$

Taking log on both sides,

or, $\log u = \cos x \log \sin x$

Differentiate both sides w.r. to x, we get,

$\frac{1}{u}\frac{dy}{dx} = \frac{\cos x \, . \cos x}{\sin x} + \log \sin x \, . (-\sin x)$

$= \frac{\cos^2 x}{\sin x} - \sin x \log \sin x$

or, $\frac{dy}{dx} = (\sin x)^{\cos x} [\cos x \, . \cot x - \sin x \, . \log \sin x]$

Similarly, $\frac{1}{v}\frac{dv}{dx} = \sin x \, . \frac{-\sin x}{\cos x} + \cos x \text{ vg} \cos x$

$\frac{dv}{dx} = (\cos x)^{\sin x}$ [cosx log cosx – sinx . tanx]

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dx} + \frac{dv}{dx}$

$= (\sin x)^{\cos x}$ [cosx . cotx – sinx . log sinx] + $(\cos x)^{\sin x}$ [cosx. logcosx – sinx . tanx]

d. $x^{\tan x} + (\tan)^x$

Solution: Let, y = u + v, where, $u = x^{\tan x}$ and $v = (\tan x)^x$.

if, $u = (x)^{\tan x}$

Taking log on both sides,

logu = tanx logx

Differentiate both sides w.r. to x, we get,

$\frac{1}{u}\frac{dy}{dx} = \tan x \frac{1}{x} + \log x . \sec^2 x$

$\frac{dy}{dx} = (x)^{\tan x}\left[\frac{\tan x}{x} + \log x . \sec^2 x\right]$

Again similarly,

$\frac{1}{v}\frac{dv}{dx} = x.\frac{\sec^2 x}{\tan x} + \log \tan x.1$

$\frac{dv}{dx} = (\tan x)^x \left[\log \tan x + \frac{x \sec^2 x}{\tan x}\right]$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dx} + \frac{dv}{dx} = (x)^{\tan x}\left[\frac{\tan x}{x} + \log x \sec^2 x\right] + (\tan x)^x \log \tan x + \frac{x\sec^2 x}{\tan x}$

e. $(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution: Let, y = u+v where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

Now, $u = (\sin x)^x$

Taking log on both sides,

logu = x log sinx

Differentiate both sides w.r.to x, we get,

$\frac{1}{u}\frac{dy}{dx} = x\frac{\cos x}{\sin x} + \log \sin x.1$

$\frac{dy}{dx} = (\sin x)^x$ [xcotx + log sinx]

and, $v = \sin^{-1}\sqrt{x}$

Differentiate both sides w.r. to x, we get,

$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} . \frac{d(\sqrt{x})}{dx} \qquad \left[\because \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}\right]$

$= \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dx} + \frac{dv}{dx}$

$= (\sin x)^x$. [xcotx + log sinx] + $\frac{1}{2\sqrt{x}\sqrt{1-x}}$

7.a. If $y = e^{x+e^{x+e^{x+...}}}$. Show that $\frac{dy}{dx} = \frac{y}{1-y}$

Solution: Let, $y = e^{x+e^{x+e^{x+...}}}$. Then we have,

$y = e^{x+y}$

Taking log on both sides we get,

logy = (x + y) loge

Differentiate both sides w.r. to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

or, $\left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$

or, $\left(\frac{1 - y}{y}\right)\frac{dy}{dx} = 1$

or, $\frac{dy}{dx} = \frac{y}{1 - y}$

b. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots}}}$

Show that, $\frac{dy}{dx} = \frac{\cos x}{2y - 1}$

Proof: Here, we can rewrite y as,

$y = \sqrt{\sin x + y}$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = \frac{1}{2\sqrt{\sin x + y}} \times \frac{d}{dx}(\sin x + y)$

$\left(2\sqrt{\sin x + y}\right)\frac{dy}{dx} = \cos x + \frac{dy}{dx}$

or, $2y\frac{dy}{dx} = \frac{dy}{dx} = \cos x$

or, $\frac{dy}{dx} = \frac{\cos x}{2y - 1}$ proved.

c. If $y = (\cos x)\cos x^{\cos x\ldots}$ prove that $\frac{dy}{dx} = \frac{y^2 \tan x}{y\log \cos x^{-1}}$

Proof: Here, we can rewrite the problem as in the form $y = (\cos x)^y$

Taking log on both sides we have,

$\log y = y \log \cos x$

Differentiate both sides w.r. to x, we get,

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = y\frac{-\sin x}{\cos x} + \log\cos x\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y} - \log \cos x\right)\frac{dy}{dx} = -y\tan x$

$\Rightarrow \frac{dy}{dx} = \frac{-y^2\ \mathrm{tnax}}{1 - y\log \cos x} = \frac{y^2 \tan x}{y \log \cos x - 1}$ proved.

Exercise 15.3

Find the derivative with respect to x of the following:

1. $e^{\cosh x/a}$

Solution: Let, $y = e^{\cosh x/a}$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = \frac{d}{dx}(e^{\cosh x/a})$

$= e^{\cosh x/a} . s^{\mathrm{inh}x/a}\frac{d(x/a)}{dx}$

$= \frac{1}{a}\sinh\frac{x}{a}\ e^{\cosh x/a}$

2. Log tanhx

Solution: Let, $y = \log \tanh x$

Differentiate both sides w.r.to x, we get,

$\frac{dy}{dx} = \frac{d}{dx}(\log \tanh x)$

$$= \frac{1}{\tanh x} \frac{d}{dx} (\tanh x)$$

$$= \frac{1}{\tanh x} . \operatorname{sech}^2 x = \frac{\cosh x}{\sinh x . \cosh^2 x}$$

$$= \frac{1}{\sinh x \cosh x} = \frac{2}{\sinh 2x} = 2\operatorname{cosech} 2x$$

3. Tanh (arcsinx)

Solution: Let, $y = \tanh (\sin^{-1}x)$

Differentiate both sides w.r.to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx} (\tanh \sin^{-1}x) = \operatorname{sech}^2 \sin^{-1}x \frac{d}{dx} (\sin^{-1}x)$$

$$= \operatorname{sech}^2 \sin^{-1} \frac{x}{\sqrt{1-x^2}}$$

4. $\operatorname{Sech}^{-1}x = \operatorname{cosech}^{-1}x$

Solution: Let, $y = \operatorname{sech}^{-1}x = - \operatorname{cosch}^{-1}x$

Differentiate both sides w.r. to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx} [\operatorname{sech}^{-1}x - \operatorname{cosech}^{-1}x]$$

$$= \frac{-1 - -1}{x\sqrt{1-x^2}\ x\sqrt{x^2+1}}$$

$$= \frac{1}{x} \left[\frac{1}{\sqrt{x^2+1}} - \frac{1}{\sqrt{1-x^2}}\right]$$

5. $x^{\cosh x}$

Solution: Let, $y = x^{\cosh x}$

Taking log on both sides we have,

$\log y = \cosh x \log x$

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y} \frac{dy}{dx} = \cosh x \ \frac{1}{x} + \log x . \sinh x$$

$$\frac{dy}{dx} = x^{\cosh x} \left[\frac{\cosh x + x \sinh x \log x}{x}\right]$$

6. $x^{\sinh x^2/a}$

Solution: Let, $y = x^{\sinh x^2/a}$

Taking log on both sides we get,

$$\log y = \sinh \frac{x^2}{a} \log x$$

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y} \frac{dy}{dx} = \sinh x^2/a \ \frac{1}{x} + \log x . \cos x \frac{x^2}{a} = \frac{2x}{a}$$

$$\frac{dy}{dx} = x \sinh x^2/a \left[\frac{\sin x^2/a}{x} + \frac{2x \log x}{a} \cosh x^2/a\right]$$

7. $x^{\cosh^{-1}x/a}$

Solution: Let, $y = x^{\cosh^{-1}x/a}$

Taking log on both sides we get,

$\log y = x^{\cosh^{-1}x/a} \log x$

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y} \frac{dy}{dx} = \cosh^{-1}x/a \frac{1}{x} + \log x \frac{1}{\sqrt{\frac{x^2}{a^2} - 1}} \times \frac{1}{a}$$

$$\frac{dy}{dx} = x^{\cosh^{-1}x/a} \left[\frac{1}{x} \cosh^{-1}x/a + \frac{\log x}{\sqrt{x^2 - a^2}}\right]$$

8. $(\log x)^{\sinh x}$

Solution: Let, $y = (\log x)^{\sinh x}$

Taking log on both sides we get,

$\log y = \sinh x \log(\log x)$

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \sinh x \frac{1}{\log x} . \frac{1}{x} + \log(\log x) . \cosh x$$

$$\text{or, } \frac{dy}{dx} = (\log x)^{\sinh x}\left[\frac{\sinh x}{x\log x} + \cosh x \log(\log x)\right]$$

9. $(\sinh x)^{\cosh^{-1}x}$

Solution: Let, $y = (\sinh x)^{\cosh^{-1}x}$

Taking log on both sides w.r. to x, we get,

$\log y = \cosh^{-1}x \log(\sinh x)$

Differentiate both sides w.r. to x, we get,

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \cosh^{-1}x . \frac{1}{\sinh x} . \cosh x + \log(\sinh x)\frac{1}{\sqrt{x^2-1}}$$

$$\text{or, } \frac{dy}{dx} = (\sinh x)^{\cosh^{-1}x}\left[\cosh^{-1}x \coth x + \frac{\log \sinh x}{\sqrt{x^2-1}}\right]$$

10. $(\cosh x)^{\cosh x}$

Solution: Let, $y = (\cosh x)^{\cosh x}$

Taking log on both sides we get,

$\log y = \cosh x \log(\cosh x)$

Differentiate both sides w.r. to x, we get,

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \cosh x \frac{\sinh x}{\cosh x} + \log(\cosh x) . \sinh x$$

$$\text{or, } \frac{dy}{dx} = (\cosh x)^{\cosh x}[\cosh x \tanh x + \sinh x \log(\cosh x)]$$

11. $(\tanh \frac{x}{a})^{\log x}$

Solution: Let, $y = (\tanh \frac{x}{a})^{\log x}$

Taking log on both sides w.r. to x, we get,

$\log y = \log x \log(\tanh\frac{x}{a})$

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \log x \frac{\text{sech}^2 x/a}{\tanh x/a} \times \frac{1}{a} + \log\tanh\frac{x}{a} . \frac{1}{x}$$

$$\text{or, } \frac{dy}{dx} = (\tanh\frac{x}{a})^{\log x}\left[\frac{\cos x . \cosh x/a}{\cosh^2 x/a . \sinh x/a}\frac{1}{a} + \log\tanh\frac{-}{a} . \frac{1}{x}\right]$$

$$\text{or, } \frac{dy}{dx} = (\tanh\frac{x}{a})^{\log x}\left[\frac{2\log x}{2\sinh x/a \cosh x/a} \times \frac{1}{a} + \log\tanh x/a . \frac{1}{x}\right]$$

$$\text{or, } \frac{dy}{dx} = (\tanh\frac{x}{a})^{\log x}\left[\frac{2}{a}\text{cosech}2 x/a . \log x + \log\tanh x/a . \frac{1}{x}\right]$$

$$\text{or, } \frac{dy}{dx} = (\tanh\frac{x}{a})^{\log x}\left[\frac{2}{a}\log x \text{ cosech } 2x/a + \frac{1}{x}\log\tanh x/a\right]$$

12. $(\sinh^{-1}x + \cosh^{-1}x)^x$

Solution: Let, $y = (\sinh^{-1}x + \cosh^{-1}x)^x$

Taking log on both sides w.r. to x, we get,

$\log y = x\log(\sinh^{-1}x + \cosh^{-1}x)$

Differentiate both sides w.r. to x, we get,

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \frac{x}{(\sinh^{-1}x + \cosh^{-1}x)}\left[\frac{d}{dx}(\sinh^{-1}x + \cosh^{-1}x)\right] + \log(\sinh^{-1}x + \cosh^{-1}x)$$

or, $\frac{1}{y}\frac{dy}{dx} = \frac{x}{(\sinh^{-1}x + \cosh^{-1}x)}\left(\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{x^2-1}}\right) + \log(\sinh^{-1}x + \cosh^{-1}x)$

or, $\frac{dy}{dx} = (\sinh^{-1}x + \cosh^{-1}x)^x \left[\frac{x}{(\sinh^{-1}x + \cosh^{-1}x)}\left(\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{x^2-1}}\right)\right]$
$+ \log(\sinh^{-1}x + \cosh^{-1}x)$

13. $(\sinh\frac{x}{a} + \cosh\frac{x}{a})^{nx}$

Solution: Let, $y = (\sinh\frac{x}{a} + \cosh\frac{x}{a})^{nx}$

Taking log on both sides we get,

$\log y = nx \log(\sinh\frac{x}{a} + \cosh\frac{x}{a})$

Differentiate both sides w.r. to x, we get,

$\frac{1}{y}\frac{dy}{dx} = nx \cdot \frac{(\cosh\frac{x}{a} + \sinh\frac{x}{a})}{(\sinh\frac{x}{a} + \cosh\frac{x}{a})}\frac{2}{a} \text{inlog}\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)$

$\frac{dy}{dx} = ny\left[\frac{2x}{a} + \log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right]$

$= n\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}\left[\frac{2x}{a} + \log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right]$

Exercise 15.4

1. Find the slope and inclination with the x–axis of the tangent of the following:
   a. $y = y = x^3 + 2x + 7$ at $x = 1$

Solution: Given, $y = x^3 + 2x + 7$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 3x^2 + 2$

Slope at x = 1 is, $\left.\frac{dy}{dx}\right|_{x=1}$

$\therefore \quad \frac{dy}{dx} = 3\times1^2 + 2 = 5$

Since slope $m = \tan\theta$, θ is angle from x–axis.

$\therefore \quad \text{Tan}\theta = 5$

$\Rightarrow \quad \theta = \tan^{-1}5$

b. $x^2 - y^2 = 9$ at (3, 0)

Solution: Given, $y = x^2 - y^2 = 9$

Differentiate both sides w.r. to x, we get,

$3x - 2y\frac{dy}{dx} = 2x$

or, $\frac{dy}{dx} = \frac{x}{y}$

Slope at (3, 0) is, $\left.\frac{dy}{dx}\right|_{(3,0)} = \frac{3}{0} = \infty$

Again, $\tan\theta = \infty$

$\theta = \tan^{-1}\infty = \tan{-1}\tan\frac{\pi}{2} = \frac{\pi}{2}$

c. $y = -3x - x^4$ at $x = -1$

Solution: Given, $y = -3x - x^4$

Differentiate both sides w.r. to x,

$\frac{dy}{dx} = -3 - 4x^3$

Slope at x = –1 is, $\frac{dy}{dx}_{x=-1}$

$= -3-4(-1)^3 = -3+4 = 1$

Again, $\tan\theta = 1$

$\theta = \tan^{-1}1$

$= \tan{-1} \tan\frac{\pi}{4} = \frac{\pi}{4}$

2. Obtain the equation to the tangent to the parabola $y^2 - 8x$ at (2, –4)

Solution: Since we know the equation of tangent be the parabola $y^2 = 4ax$ at (x_1, y_1) is,

$yy_1 = 2a(x + x_1)$

Now, $y^2 = 8x = 4 \times 2x$ and $(x_1, y_1) = (2, -4)$

$\therefore$ Required equation of tangent is

$y(-4) = 4(x + 2)$

or, $-4y = 4x + 8$

or, $-y = x + 2$

or, $x + y + 2 = 0$

Given, $y^2 = 8x$

$3y\frac{dy}{dx} = 8$

$\frac{dy}{dx} = \frac{4}{y}$

$\left.\frac{dy}{dx}\right|_{(2,-4)} = \frac{4}{-4} = -1$

Now, required equation of tangent,

$y - y_1 = \ln(x - x_1)$

or, $y + 4 = -1(x - 2)$

or, $y + 4 = -x + 2$

or, $x + y + 2 = 0$

3. Find the equation of tangent and normal to the curve.

a. $y = 2x^2 - 3x - 1$ at (1, –2)

Solution: Given $y = 2x^2 - 3x - 1$

Differentiate both sides w.r. to x, we get

$\frac{dy}{dx} = 4x - 3$

Slope of tangent say (m_1) at (1, –2) is $m_1 = \frac{dy}{dx} = 4 - 3 = 1$

Then slope of normal is say m_2 is given by

$m_1 \times m_2 = -1$

$m_2 = -\frac{1}{1} = -1$

Now, equation of tangent is,

$y - (-2) = 1\,(x - 1)$

$\Rightarrow$ $y + 2 = x - 1$

$\Rightarrow$ $x - y - 3 = 0$

Again equation of normal at (1, –2) is,

$y - (-2) = -1\,(x - 1) \Rightarrow y + 2 = -x + 1$

$\Rightarrow x + y + 1 = 0$

b. $y = x^3$ at (2, 8)

Solution: Given, $y = x^3$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 3x^2$

Slope (m_1) $\left.\frac{dy}{dx}\right|_{(2, 8)} = 3\times2^2 = 12$

Equation of tangent at (2, 8) is

$y - 8 = 12(x - 2)$

$\Rightarrow$ $y - 8 = 12x - 24$

$\Rightarrow$ $12x - y - 16 = 0$

Again, Slope of normal is $= -\frac{1}{12}$

Now, equation of normal is,

$y - 8 = -\frac{1}{12}(x - 2)$

$\Rightarrow$ $12y - 96 = -x + 2$

$\Rightarrow$ $x + 12y - 98 = 0$

c. $x^2 - y^2 = 16$ at (6, 3)

Solution: Given, $x^2 - y^2 = 16$

Differentiate both sides w.r. to x, we get

$2x - 2y\frac{dy}{dx} = 0$

or, $2y\frac{dy}{dx} = 2x$

or, $\frac{dy}{dx} = \frac{x}{y}$

$\left.\frac{dy}{dx}\right|_{(6, 3)} = \frac{6}{3} = 2$

Now, equation of tangent at (6, 3) is,

$y - 3 = 2(x - 6)$

or, $y - 3 = 2x - 12$

or, $2x - y - 9 = 0$

Again, equation of normal is,

$y - 3 = -\frac{1}{2}(x - 6)$

or, $2y - 6 = -x + 6$

or, $x + 2y - 12 = 0$

d. $x^2 + 3xy + y^2 = 11$ at (2, 1)

Solution: Given, $x^2 + 3xy + y^2 = 11$

Differentiate both sides w.r. to x, we get,

$2x + 3\left[x\frac{dy}{dx} + y\right] + y^2 = 11$

or, $2x + 3x\frac{dy}{dx} + 3y + y^2 = 11$

or, $3x\frac{dy}{dx} = 11 - 2x - 3y - y^2$

or, $\frac{dy}{dx} = \frac{11 - 2x - 3y - y^2}{3x}$

Now, $\left.\frac{dy}{dx}\right|_{(2, 1)} = \frac{11 - 2\times2 - 3\times1 - 1}{3\times2} = \frac{11 - 8}{6} = \frac{3}{6} = \frac{1}{2}$

$\therefore$ Equation of tangent at (2, 1) is,

$y-1 = \frac{1}{2}(x - 2) \Rightarrow 2y - 2 = x - 2 \Rightarrow x - 2y = 0$

Again, equation of normal is,

$y - 1 = -2(x - 2)$

or, $y - 1 = -2x + 4$

or, $2x + y - 5 = 0$

e. $x^{2/3} + y^{2/3} = 2$ at (1, 1)

Solution: Differentiate w.r.t. to x, we get,

$$\frac{2}{3}x^{(2/3-1)} + \frac{2}{3}y^{(2/3-1)}\frac{dy}{dx} = 0$$

or, $\frac{2}{3}x^{-1/2} + \frac{2}{3}y^{-1/2}\frac{dy}{dx} = 0$

or, $y^{-1/2}\frac{dy}{dx} = x^{-1/2}$

or, $\frac{dy}{dx} = \frac{x^{1/2}}{y^{-1/2}} = \frac{y^{1/2}}{x^{1/2}} = \frac{\sqrt{y}}{\sqrt{x}}$

$\left.\frac{dy}{dx}\right|_{(1,\,1)}$ is 1

Now, equation of tangent at (1, 1) is,

$y - 1 = 1(x - 1)$

$\Rightarrow$ $y-1 = x-1 \Rightarrow x-y = 0$

Again, equation of normal is,

$y - 1 = -1(x - 1)$

$\Rightarrow$ $y - 1 = -x + 1$

$\Rightarrow$ $x + y - 2 = 0$

4. Find the points on the curve where the tangents are parallel to x–axis.

a. $y = 2x - x^2$

Solution: Differentiate both sides w.r.t. to x, we get,

$\frac{dy}{dx} = 2 - 2x$

If the tangent are parallel to x–axis, then the slope v must be zero

i.e. $\frac{dy}{dx}$

$\therefore$ $\frac{dy}{dx} = 2 - 2x = 0$

$\Rightarrow$ $2 = 2x$

$\Rightarrow$ $x = 1$

If $x = 1$, then $y = 2 - 1^2 = 1$

$\therefore$ The required point is (1, 1)

b. $y = 2x^2 - 6x + 9$

Solution: Differentiate both sides w.r.t. to x, we get,

$\frac{dy}{dx} = 4x - 6]$

Now, $\frac{dy}{dx} = 0 \Rightarrow 4x = 6$

$\Rightarrow x = \frac{3}{2}$

If $x = \frac{3}{2}$, then $y = 2 \times \frac{9}{4} - 6 \times \frac{3}{2} + 9$

$= \frac{9}{2} - 9 + 9 = \frac{9}{2}$

$\therefore$ Required point is $\left(\frac{3}{2}, \frac{9}{2}\right)$

c. $x^2 + y^2 = 16$

Solution: Differentiate both sides w.r.t. to x, we get,

$2x + 2y\dfrac{dy}{dx} = 0$

or, $2y\dfrac{dy}{dx} = -2x$

or, $\dfrac{dy}{dx} = -\dfrac{x}{y}$

Now, $\dfrac{dy}{dx} = 0 \Rightarrow -\dfrac{x}{y} = 0 \Rightarrow -x = 0$

$\Rightarrow x = 0$

If x = 0 then y = ±4

Therefore the required point (0, ±4)

5. Find the points are the curve $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ at which the tangent are 0.

   a. Parallel to x–axis b. Parallel to γ–axis.

Solution: Given, $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

Differentiate both sides w.r. to x, we get,

$\dfrac{2x}{9} + \dfrac{9y}{16}\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{y}{16}\dfrac{dy}{dx} = -\dfrac{x}{9}$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{16x}{9y}$

a. For, parallel to x–axis

$\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{-16x}{9y} = 0$

$\Rightarrow x = 0$

If x = 0 then, $y^2 = 16$

$y = \pm 4$

Therefore required points (0, ± 4)

b. For parallel to γ-axis,

$\dfrac{dy}{dx} = \infty$

$\Rightarrow \dfrac{-16x}{9y} = \infty$ which is possible only when y = 0

Then we have,

$\dfrac{x^2}{9} = 0$

$\therefore x = \pm 3$

Therefore required points (± 3, 0)

6. Show that the tangents to the curve $y = 2x^3 - 3$ at the point where x = 2 and x = –2 are parallel.

Solution: Given, $y = 2x^3 - 3$

Differentiate both sides w.r. to x, we get,

$\dfrac{dy}{dx} = 6x^2$

Slop at x = 2 i.e, $\dfrac{dy}{dx}_{x=2}$ is $\dfrac{dy}{dx} = 6\times 2^2 = 24.$

Again slope at x = –2 is, $\dfrac{dy}{dx}_{x=-2} = 6\times(-2)^2 = 24$

Hence, if x = 2 and x = –2 slope are equal it means the tangent are parallel.

7.a. Find the equation of tangent line to the curves $y = x^2 - 2x + 7$ which is parallel to the line $2x - y + 9 = 0$

Solution: Given, curve, $y = x^2 - 2x + 7$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 2x - 2$(slop of tangent)

Again slop of the line $2x - y + 9 = 0$, obtained by comparing to $y = mx + c$ i.e. $y = 2x + 9$. Therefore the slope of given line?

If the required tangent is parallel to the given line then slope must be equal

$\therefore \quad 2x - 2 = 2$

or, $x = 2$

Put, $x = 2$ in $y = x^2 - 2x + 7$ we get,

$y = 2^2 - 2\times2 + 7 = 7$

$\therefore$ Required point is, (2, 7)

Now, the equation at tangent is,

$y - 7 = 2(x - 2)$

or, $y-7 = 2x - 4$

or, $2x - y + 3 = 0$

b. Find the point on the curve $y^2 = 4x + 1$ at which the tangent is perpendiculars to the line $7x + 2y = 1$.

Solution: Given, curve, $y^2 = 4x + 1$,

$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

Again slope of $7x + 2y = 1$

or, $2y = -7x + 1$

$y = \frac{-7}{2}x + \frac{1}{2}$ is $\frac{-7}{2}$

If the tangent to the curve $y^2 = 4x+1$ is perpendicular to the line $7x + 2y = 1$, product of slope must be -1.

$\therefore \quad \frac{2}{y} \times -\frac{7}{2} = -1 \Rightarrow 14 = 2y \Rightarrow y = 7$

Putting $y = 7$ in $y^2 = 4x + 1$ we get,

$49 = 4x + 1$

$48 = 4x \Rightarrow x = 12$

$\therefore$ The required point is (12, 7)

8. Show that equation of tangent to the curve $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 - 2$ at (a, b) is $\frac{x}{a} + \frac{y}{b} = 2$

Solution: Given, curve is, $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 2$

Differentiate both sides w.r. to x, we get,

$\frac{3x^2}{a^3} + \frac{3y^2}{b^3}\frac{dy}{dx} = 0$

or, $\frac{dy}{dx} = \frac{x^2}{a^3} \times \frac{b^3}{y^2}$

Now, $\left.\frac{dy}{dx}\right|_{(a,\, b)} = -\frac{a^2b^3}{a^3b^2} = -\frac{b}{a}$

Now, equation of tangent at (a, b) is,

$y - b = -\frac{b}{a}(x - a)$

$\Rightarrow \quad ay - ab = -bx + ab$

$\Rightarrow \quad ay + bx = 2ab$. Dividing both sides by ab

We get, $\frac{x}{a} + \frac{y}{b} = 1$ proved.

9. Find the angle of intersection of the following curves:
   a. $y = x^3$ and $6y = 7-x^2$ at (1, 1)

Solution: Solving we get,

$6x^3 + x^2 - 7 = 0$

or, $6x^3 - 6x^2 + 7x2 - 7x + 7x - 7 = 0$

or, $6x^2 (x - 1) + 7x(x - 1) + 7 = (x - 1) = 0$

or, $(x - 1) (6x^2 + 7x + 7) = 0$

$\therefore$ $x = 1$ as $6x^2 + 7x + 7 = 0$ does not have any real values.

If x = 1 then y = 1

Now, from $y = x^3$

$\Rightarrow \dfrac{dy}{dx} = 3x^2$ $\qquad \therefore \dfrac{dy}{dx\,(1,1)}$ (6ay m_1) = 3

Again, from, $6y - 7+x^2 = 0$

$$6\frac{dy}{dx} + 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{6} = -\frac{x}{3}$$

$\therefore \dfrac{dy}{dx}$ at (1, 1) (say m_2) $= \dfrac{-1}{3}$

If θ be the angle between two curves, then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{3 + \frac{1}{3}}{1 + 3 \times \left(-\frac{1}{3}\right)}\right| = \left|\frac{\frac{10}{3}}{0}\right| = \infty$$

$\therefore$ $\theta = \tan^{-1}\infty$

$\therefore$ $\theta = \dfrac{\pi}{2}$

b. $y = x^3$ and $y = 2x$ at (4, 8)

Solution: Solving we get,

$x^3 = 2x$ $\qquad$ [$\because x \neq 0$ otherwise it doesn't remains curves]

$\Rightarrow$ $x^2 = 2$ $\quad \therefore x = \pm\sqrt{2}$

Now, if $x = \sqrt{2}$. Then $y = \pm 2\sqrt{2}$

From $y = x^3$

$\dfrac{dy}{dx} = 3x^2 = 48$

$\left.\dfrac{dy}{dx}\right|_{(\sqrt{2} + 2\sqrt{2})}$ (m_1 say) $= 3(\sqrt{2})^2 = 6$

From, $y = 2x$

$\dfrac{dy}{dx} = 2$ $\qquad \therefore \left.\dfrac{dy}{dx}\right|_{(\sqrt{2} + 2\sqrt{2})}$ (m_2 say) = 2

Now, by using the formula,

$$\tan\theta \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{6 - 2}{1 + 6\times2}\right| = \left|\frac{4}{1 + 12}\right| = \left|\frac{4}{13}\right|$$

$\therefore$ $\theta = \tan^{-1}\dfrac{4}{13}$

c. $y = 6 -x^2$ and $x^3 = 4y$ at (2, 4)

Solution: Here, $y = 6 - x^2$

$\dfrac{dy}{dx} = -2x$

Say $m_1 = \dfrac{dy}{dx\,(2,4)} = -2\times2 = -4$

and, $x^3 = 4y$

$3x^2 = \dfrac{4dy}{dx}$ $\qquad \therefore \dfrac{dy}{dx} = \dfrac{3}{4}x^2$

say $m_2 = \frac{dy}{dx}(2, 4) = \frac{3}{4} \times 4 = 3$

$$\frac{d}{dx}(f(x)) = \lim_{n\to 0} \frac{\log(y+k) - \log y}{h}$$

$$= \lim_{n\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{n\to 0} \frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}} \times \frac{\frac{k}{y}}{h}$$

$$= \lim_{n\to 0} \times \frac{k}{yh} \qquad \left[\because \lim_{n\to 0} \frac{\log(1+x)}{x} = 1\right]$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\sin(x+h).\cos x - \sin x.\cos(x+h)}{h.\cos x.\cos(x+h)}$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\sin(x+h-x)}{h\cos x\cos(x+h)}$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\sinh}{x} \times \lim_{n\to 0} \frac{1}{\cos x\cos(x+h)}$$

$$= \frac{1}{\tan x} \times 1 \times \frac{1}{\cos^2 x}$$

$$= \frac{\sec^2 x}{\tan x}$$

If θ be the angle then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-4-3}{1+(-4)(3)}\right| = \left|\frac{-7}{11}\right|$$

$$\therefore \quad \theta = \tan^{-1}\frac{7}{11}$$

d. $x^2 + y^2 = 5$ and $y^2 = 4x$

Solution: Solving we ge,t

$x^2 + 4x - 5 = 0$

or, $x^2 + 5x - x - 5 = 0$

or, $x(x + 5) -1(x + 5) = 0$

$\therefore$ $x = 1, -5$

If $x = 1$ then $y^2 = 4 \Rightarrow y = 2$

If $x = -5$ then $y^2 = -20$ (does not give real values)

Now from $x^2 + y^2 = 5$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$m_1 \text{ (say)} = \frac{dy}{dx}(1, 2) = -\frac{1}{2}$$

Again, $y^2 = 4x$

$$dy\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

$$\text{or,} \quad m_2\text{(say)} = \frac{dy}{dx}_{(1,2)} = \frac{2}{2} = 1$$

$\therefore$ $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\dfrac{\frac{-1}{2} - 1}{1 + \left(-1\frac{1}{2}\right)(1)}\right| = \left|\dfrac{\frac{-3}{2}}{\frac{1}{2}}\right| = |3|$

$\therefore$ $\theta = \tan^{-1}(3)$

Exercise 15.5

1. Verify the Rolle's theorem for each of the following functions.

a. $f(x) = x^2 + 2$ in $[-2, 2]$

Solution: Here, $f(x) = x^2 + 2$

i. Since the polynomial function is continuous. Hence given function is continuous in [–2, 2].

ii. Again, $f(x) = x^2 + 2$

$f'(x) = 2x$ which gives real values for all values of x in (2, 2)

Hence f(x) is differentiable in (–2, 2)

iii. Since $f(a) = f(-2) = (-2)^2 + 2 = 6$

and $f(b) = f(2) = 2^2 + 2 = 6$

$\therefore$ $f(a) = f(b)$.

Here f(x) satisfies all the condition of Rolle's theorem so these exists $C\in(a, b)$ such that $f'(c) = 0$

Now, $f(c) = 2 = 0$

$\Rightarrow$ $c = 0 \in (-2, 2)$

b. $f(x) = x^3 - 4x$ in $[0, 2]$

Solution: Here $f(x) = x^3 - 4x$

Here, $f'(x) = 3x^2 - 4$. Which is defined for all values in (0, 2). Hence the given function is differentiable in (0, 2). Again since differentiable function is continuous. So the given function is continuous on [0, 2].

Now, $f(a) = f(0) = o3 - 4\times0 = 0$

and $f(b) = f(2) = 2^3 - 4\times2 = 8 - 8 = 0$

$\therefore$ $f(a) = f(b)$

So by Rolle's theorem these exits $c\in(a, b)$ s.t. $f'(c) = 0$

Now, $f'(x) = 3x^2 - 4$

$f'(c) = 3x^2 - 4$

$f'(c) = 0 \Rightarrow 3c^2 - 4 = 0 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm\sqrt{\frac{4}{3}}$

$\therefore$ $c = \sqrt{\frac{4}{3}} \in (0, 2)$.

c. $f(x) = \sin 2x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Solution: Since we know sine function is continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and differentiable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $f(a) = f\left(\frac{\pi}{2}\right) = \sin 2 \times \frac{\pi}{2} = \sin(-\pi) = -\sin\pi = 0$

and $f(b) = f\left(\frac{\pi}{2}\right) = \sin 2\frac{\pi}{2} = \sin\pi = 0$

Thus, by Rolle's theorem there exists $c\in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in which $f'(c) = 0$

$\Rightarrow 2\cos 2c = 0$

$\Rightarrow \cos 2c = 0 = \cos\frac{\pi}{2}$

$\Rightarrow 2c = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

d. $f(x) = \cos 2x$ in $[-\pi,\pi]$

Solution: Since cosine function is continuous in $[-\pi,\pi]$ and is also differentiable $(-\pi,\pi)$

Now, $f(a) = f(-\pi) = \cos 2(-\pi) = \cos 2\pi = 1$

$f(b) = f(\pi) = \cos 2\pi = 1$

Hence by Rolle's Theorem $c \in (-\pi, \pi)$ such that f'(c) = *l*

$\Rightarrow$ $f'(c) = -\sin 2c = 0$

$\Rightarrow$ $\sin 2c = \sin 0$

$\Rightarrow$ $2c = 0$

$\Rightarrow$ $c = 0 \in (-\pi, \pi)$

e. $f(x) = \sqrt{25 - x^2}$ in $[-5, 5]$

Solution: Since given function is continuous in $[-5, 5]$ and also gives real values for all values in $(-5, 5)$ and hence differentiable.

Now, $f(-5) = \sqrt{25 - 25} = 0$, $f(b) = f(5) = \sqrt{25 - 25} = 0$

$\therefore$ $f(a) = f(b)$

Now, by Rolle's theorem, $\ni c \in (-5, 5)$ s.t. $f'(c) = 0$

$\Rightarrow$ $f'(c) = \frac{-2x}{2\sqrt{25 - x^2}} = 0$

$\Rightarrow$ $-2x = 0 \Rightarrow x = 0 \in (-5, 5)$

f. $f(x) = (x - 1)(x - 2)(x - 3) = (x - 1)(x^2 - 5x + 6) = x^3 - x^2 - 5x^2 + 5x + 6x - 6 = x^3 - 6x^2 + 6x$

Solution: Since polynomial function is continuous. So given function is continuous in [1, 3] and is also defined for all values in (1, 3) so is differentiable.

Now, f(1) = 0 f(b)

Now by Rolle's theorem,

$f'(c) = 3c^2 - 12c + 11 = 0$

$\Rightarrow$ $c = \frac{6 \pm \sqrt{3}}{3}$

$\therefore$ $c = \frac{6 + \sqrt{3}}{3} \in (1, 3)$

g. $f(x) = \sin x + \cos x$ in $[0, 2\pi]$

Solution: Since the sum of two continuous function is continuous. So given function is continuous in $[0, 2\pi]$.

The given function defined all values in $[0, 2\pi]$. Hence differentiable in $(0, 2\pi)$

Again, $f(0) = \sin 0 + \cos 0$

$= 0 + 1 = 1$

and $f(2\pi) = \sin 2\pi + \cos 2\pi = 0+1 = 1$

Now by Rolle's theorem $\ni c \in (0, 2\pi)$ s.l. $f'(c) = 0$

$\Rightarrow$ $f'(c) = \cos c - \sin c = 0$

$\Rightarrow$ $\cos c = \sin c$

Which is positive for $c = \frac{\pi}{4} \in (0, 2\pi)$

2. By using Rolle's theorem find a point on each of the curves given by the following. Where the tangent is parallel to x–axis.

a. $f(x) = 6x - x^2$ in (0, 6)

Solution: Being a polynomial function continuous in [0,6]. lso gives real values in (0, 6). Therefore the given function is differentiable in (0, 6)

$f(a) = 6\times 0 - 0^2 = 0$

and $f(b) = 6\times 6 - 6^2 = 0$

$\therefore$ $f(a) = f(b)$

So by Rolle's theorem, $\exists c \in (0, 6)$ s.t. $f'(c) = 6 - 2c$

$\Rightarrow$ f'(c) = 6 – 2c = 0

$\Rightarrow$ 2c = 6 $\Rightarrow$ c = 3

Thus the tngent to the given curve is parallel to x–axis t x = 3.

$\therefore$ If x = 3 . Then y = 6×3 – 3^2 = 18 – 9 = 9

Therefore the required points is (3, 9)

b. f(x) = $2x^2$ – 4x in [0, 2]

Solution: Given f(x) = $2x^2$ – 4x

Since the polynomial function is continuous in [0, 2]

Also the given function gives definite values for all values in (0, 2). Hence the function is also differentiable in (0, 2)

Now, f(0) = 2×0^2 = – 4×0 = 0

f(2) = 2×2^2 – 4×2 = 8 – 8 = 0

$\therefore$ f(0) = f(2)

Here, all the conditioned of Rolle's theorem is satisfied os $\ni$ c$\in$(0, 2) s.t. f'(c) = 0

$\Rightarrow$ f'(c) = 4x – 4 = 0

$\Rightarrow$ 4x = 4 $\Rightarrow$ x = 1$\in$(0, 2)

Thus, the tangent to the curve $2x^2$ – 4x is parallel to x–axis at the point x=1.

$\therefore$ When x=1, y = $2x^2$ = 4x

= 2×1 – 4×1

= –2

So the required point is (1, –2)

3. Verify the mean value theorem for each of the following function on the given interval.

a. f(x) = $3x^2$ – 2 in [2, 3]

Solution: Since, being the polynomial function f(x) = $3x^2$ – 2 is continuous in [2, 3]

Also f(x) = $3x^2$ – 2 is defined for all values in (2, 3). Hence is differentiable in (2, 3).

Again, f(2) = 3×2^2 – 2 = 12 – 2 = 10

and f(3) = 3×3^2 – 2 = 27 – 2 = 25

$\therefore$ f(2) $\neq$ f(3)

Hence all the condition of mean value theorem satisfied. So by the theorem $\exists$ c$\in$(a, b) s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(2)}{3–2} = \frac{25 - 10}{1} = 15 \ldots \ldots \ldots (*)$$

We have, f(x) = $3x^2$ – 2 ... ... (**)

f'(c) = 6c

$\therefore$ from (*) and (**)

$$6c = 15 \Rightarrow c = \frac{15}{6} \in (2, 3)$$

b. f(x) = x^2 in [1, 2]

Solution: Since quadratic function is continuous for all values of x so the given function is continuous in [1, 2]

f(x) = x^2 have a definite values in [1,2] so is differentiable in (1, 2)

Now, f(1) = 1 and f(2) = 4

$\therefore$ f(1) $\neq$ f(2)

All condition of M.V.T satisfied so $\exists$ c$\in$(a, b)

$$\text{Such that, } f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(2) - f(1)}{2–1} = \frac{4–1}{1} = 3 \ldots \ldots \ldots (*)$$

Again, f'(x) = 2x $\therefore$ f'(c) = 2c ... ... (**)

from (*) and (**)

$$2c = 3 \Rightarrow c = \frac{3}{2} = 1.5 \in (1, 2)$$

c. f(x) = x(x – 1) (x – 2) in [0, ½]

Solution: f(x) = x(x^2 – 3x + 2) = x^3 – $3x^2$ + 2x

Since being polynomial function is continuous so the given function is continuous in [0, ½]. All gives definite values in (0, ½). So is differentiable in (0, ½)

Now, $f(0) = 0$ and $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2\times\frac{1}{2}$

$= \frac{1}{8} - \frac{3}{4} + 1 = \frac{1-6+8}{8} = \frac{3}{8}$

$\therefore \quad f(0) \neq f\left(\frac{1}{2}\right)$

Therefore M.V. theorem applicable, so $\exists$ c$\in$ (a, b) such that f'(c)

$$= \frac{f(b)-f(a)}{b-a} = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} = \frac{\frac{3}{8} - 0}{\frac{1}{2}} = \frac{3}{8}\times 2 = \frac{3}{4}$$

Also, $f'(x) = 3x^2 - 6xc + 2$

$f'(c) = 3c^2 - 6c + 2$ ... ... (**)

from (*) and (**) we have,

$3c^2 - 6c + 2 = \frac{3}{4}$

$\Rightarrow \quad 12c^2 - 24c + 8 = 3$

$\Rightarrow \quad 12c^2 - 24c + 5 = 0$

$\Rightarrow \quad c = \frac{24 \pm \sqrt{576-240}}{24} = \frac{24 \pm \sqrt{16\times 21}}{24}$

$= \frac{4(6 \pm \sqrt{21})}{24} = \frac{6 \pm 4.58}{6}$ (Appro.)

Taking positive sing,

$c = \frac{1.42}{6} = 0.23 \in (0, ½)$

d. $f(x) = e^x$ in [0, 1]

Solution: Since exponential function is continuous

$\therefore$ The given function is continuous in [0, 1]

Also differential in (0, 1)

Now, $f(0) = e^0 = 1$

$f(1) = e^1 = 2.718$ (Approx)

$\therefore \quad f(0) \neq f(1)$

Now by M.V. theorem $\ni$ c$\in$(0,b) s.t.

$f'(c) = \frac{f(b)-f(a)}{b-} = \frac{f(1)-f(0)}{1-0} = \frac{2.718-1}{1} = 1.718$ ... ... (*)

and $f'(x) = e^x$

$f'(c) = e^c$ ... ... (**)

$\therefore$ from (*) and (**)

$e^c = 1.718$

Taking log on both sides,

c loge = log (1.718)

$\therefore \quad c = 0.236 \in (0, 1)$

e. $f(x) = \sqrt{a^2 - 4}$ in [2, a]

Solution: Given function is continuous in [2, a]

Also differentiable in (2, a)

Now, $f(2) = 0$

$f(a) = \sqrt{a^2 - 4}$

$\therefore \quad f(2) \neq f(a)$ for all $a > 2$

So by M.V.T $\exists$ c$\in$(2, a) for which $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\sqrt{a^2-4} - 0}{a-2}$ ... ... (*)

Also, $f'(x) = \frac{2x}{2\sqrt{x^2-4}}$

$\therefore \quad f'(c) = \frac{c}{\sqrt{c^2-4}}$ ... ... (**)

from (*) and (**)

$\frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{a^2-4}}{a-2}$

or, $\frac{c^2}{c^2-4} = \frac{a^2-4}{(a-2)(a-2)}$

or, $\frac{c^2}{c^2-4} = \frac{a+2}{a-2}$

or, $\frac{c^2-4}{c^2} = \frac{a+2}{a-2}$

or, $1 - \frac{4}{c^2} = \frac{a-2}{a+2}$

or, $\frac{4}{c^2} = 1 - \frac{a-2}{a+2}$

or, $\frac{4}{c^2} = \frac{a+2-a+2}{a+2}$

or, $\frac{4}{c^2} = \frac{4}{a+2}$

or, $c^2 = + 2$

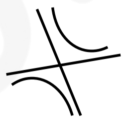
$c = \pm\sqrt{a+2}$

If $c = \sqrt{a+2}$ for all $a > 2$ then, $c = \sqrt{a+2} \in (2, a)$

4. Show that the mean value theorem is not applicable to the function $f(x) = \frac{1}{x}$ in (–1, 1).

Solution: Given function $f(x) = \frac{1}{x}$ in (–1, 1) since the given function is not defined at $x = 0 \in$ (–1, 1). Hence the function is not differentiable at $x = 0 \in (-1, 1)$. To satisfy the M.V.T, f(x) should be differentiable for all $x \in (-1, 1)$

Moreover the graph of $f(x) = \frac{1}{x}$ is,



Here, we cannot draw a tangent at x = 0. So, the function is not differentiable. Hence M.V. theorem for the underlying function in the defined interval is not applicable.

5. Find the points on the curve $f(x) = (x-3)^2$ where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Solution: Let, the chord joining the ending points be, (a, f(a)) = (3, 0) and (b, f(b)) = (4, 1)

Since $f(x) = (x-2)^2$ is continuous in [3, 4]

Also exist for all values in (3, 4) and hence differentiable.

Also, $f(a) \neq f(b)$

By M.V. theorem $\ni c \in (3, 4)$ s.l.

$f'(c) = \frac{f(b)-f(a)}{b-a}$

Now the slope of the chord joining (3, 0), (4, 1) is, $\frac{f(b)-f(a)}{b-a} = \frac{1-0}{4-3} = 1$ ... ... (*)

Since, $f(x) = (x-3)^2$

$f'(x) = \frac{dy}{dx} = 2(x-3)$

$f'(c) = 2(c-3)$ ... ... (**)

from (*) and (**)

$2c - 6 = 1 \Rightarrow 2c = 7 \Rightarrow c = \frac{7}{2} \in (3, 4)$

If $x = \frac{7}{2}$ then $y = \left(\frac{7}{2} - 3\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\therefore$ The tangent at $\left(\frac{7}{2}, \frac{1}{4}\right)$ is parallel to be chord joining (3, 0) and (4, 1).

6. Find the point on the curve $f(x) = x^3 - x^2 + 2$ where the tangent is parallel to the line joining the points (1, 2) and (3, 20).

Solution: Since the given function is continuous on [1, 3] being polynomial and $f'(x) = 3x^2 - 2x$ exist for all (1, 3) and f(a) = 2 and f(b) = 20

$\therefore$ $f(a) \neq f(b)$

So by M.V. theorem $\ni c \in (1, 3)$ s.t.

$f'(c) = \frac{f(b) - f(a)}{b - a}$

Now the slope of chord joining (1, 2) and (3, 20)

i.e, $\frac{f(b) - f(a)}{b - a}$ is, $\frac{20 - 2}{3 - 1} = \frac{18}{2} = 9$ ... ... (*)

And, $f'(c) = 3c^2 - 2c$ ... ... (**)

From (*) and (**)

$3c^2 - 2c = 9 \Rightarrow 3c^2 - 2c - 9 = 0$

Solving $c = \frac{1 \pm 2\sqrt{7}}{3} = \frac{1 + 2 \times 2.64}{3}$ (Appro.)

$= \frac{1+5.29}{3}$ (Appro.) (Taking positive sign)

$= \frac{6.29}{3} \in (1, 3)$

If x = 2.1 then, $y = (2.1)^3 - (2.1)^2 + 2$

$= 9.26 - 4.41 + 2$

$= 6.85$

$\therefore$ The required point is (2.1, 6.85)

Exercise - 15.6

1. By using L Hospital's rule, evaluate:

a. $\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$

Solution: Here, $\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$ (% form)

$= \lim_{x \to 3} \frac{3x^2}{2x}$ (Differentiate w.r.to x) $= \frac{3 \times 3^2}{2 \times 3} = \frac{9}{2}$

b. $\lim_{x \to a} \frac{x+- a^n}{x - a}$

$= \lim_{x \to a} \frac{nx^{n-1}}{1}$ [Differentiate w.r.to x]

$= na^{n-1}$

c. $\lim_{x \to 0} \frac{3x - \sin x}{x}$

Solution: $\lim_{x \to 0} \frac{3x - \sin x}{x} =$ (% form)

$= \frac{3 - \cos x}{1}$ [Differentiate w.r. to x]

$= \frac{3 - \cos 0}{1} = \frac{3-1}{1} = 2$

d. $\lim_{x\to\infty} \frac{5x++ 4x - 3}{2x^2 - 3x + 5}$

Solution: $\lim_{x\to\infty} \frac{5x^2 + 4x - 3}{2x^2 - 3x + 5} = \left(\frac{\infty}{\infty} \text{ form}\right)$

$= \lim_{x\to\infty} \frac{10x + 4}{4x - 3}$ (Differentiate w.r.to x] $\left(\frac{\infty}{\infty} \text{ form}\right)$

$= \frac{10}{-4} = \frac{5}{2}$

e. $\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{\sin^2 x}$

Solution: $\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{\sin^3 c}$ = (% form)

$= \lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x}{2\sin x \cos x}$ (% form)

$= \lim_{x\to 0} \frac{e^x + e^{-x} + 2\cos x}{2\cos 2x} = \frac{1+1+2}{2} = \frac{4}{2} = 2$

f. $\lim_{x\to 0} \frac{x - \sin x}{\tan^3 x}$

Solution: Since, $\lim_{x\to 0} \frac{x - \sin x}{\tan^3 x}$ = (% form)

$= \lim_{x\to 0} \frac{x - \sin x}{x^3 \times \left(\frac{\tan x}{x}\right)^3}$

$= \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^3 \times \lim_{x\to 0} \frac{x - \sin x}{x^3}$

$= 1 \times \lim_{x\to 0} \frac{x - \sin x}{x^3}$ (% form)

$= \lim_{x\to 0} \frac{1 \cos x}{3x^2}$ (% form) [Differentiate w.r. to x]

$= \lim_{x\to 0} \frac{0 + \sin x}{6x}$ [Differentiate w.r.to x]

$= \lim_{x\to 0} \left[\frac{\sin x}{x}\right] \times \frac{1}{6}$

$= 1 \times \frac{1}{6} = \frac{1}{6}$

g. $\lim_{x\to 0} \frac{x - \tan x}{x^3}$

Solution: Here, $\lim_{x\to 0} \frac{x - \tan x}{x^3}$ = (% form)

By L–Hospital rule, differentiate numerator and denominator w.r.to x, we get,

$\lim_{x\to 0} \frac{1 - \sec^2 x}{3x^2}$ (% form) [Differentiate w.r. to x]

$= \lim_{x\to 0} \frac{\tan^2 x}{3x^2}$

$= -\frac{1}{3} \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^2$

$= -\frac{1}{3} \times 1 = -\frac{1}{3}$

h. $\lim\limits_{x\to\pi/2} \frac{\tan^5 x}{\tan x}$

Solution: Here, $\lim\limits_{x\to\pi/2} \frac{\tan^5 x}{\tan x} \left(\frac{\infty}{\infty} \text{ form}\right)$

Using L-Hospital rule,

$\lim\limits_{x\to\pi/2} \frac{5\sec^2 5x}{\sec^2 x}$ [Differentiate w.r. to x]

$= \lim\limits_{x\to\pi/2} \frac{5 \cos^2 x}{\cos^2 5x}$ (% form)

$= \lim\limits_{x\to\pi/2} \frac{-5 \times 2 \cos x \sin x}{-5\ 2\cos 5x \sin 5x}$ [Differentiate w.r. to x]

$= \lim\limits_{x\to\pi/2} \frac{\sin 2x}{\sin 10x}$

$= \lim\limits_{x\to\pi/2} \frac{2\cos 2x}{10 \cos 10x}$

$= \frac{1}{5}\left(\frac{\cos\pi}{\cos 5\pi}\right) = \frac{1}{5}\left(\frac{-1}{-1}\right) = \frac{1}{5}$

i. $\lim\limits_{x\to 0} \frac{(e^x - 1)\tan x}{x^2}$

Solution: Here, $\lim\limits_{x\to 0} \frac{(e^x - 1)\tan x}{x^2}$ (% form)

$= \lim\limits_{x\to 0} \frac{(e^x - 1)}{x} \times \lim\limits_{x\to 0} \frac{\tan x}{x}$

$= \lim\limits_{x\to 0} \frac{e^x - 1}{x} \times 1$ [% form]

$= \frac{e^x}{1}$ [Differentiate w.r. to x]

$= e^0 = 1$

j. $\lim\limits_{x\to 0} \frac{\log \tan x}{\log x}$

Solution: Since, $\lim\limits_{x\to 0} \frac{\log \tan x}{\log x} \left(\frac{-\infty}{\infty} \text{ form}\right)$

By using L–Hospital rule

$= \lim\limits_{x\to 0} \frac{x - x \sec^2 x}{\tan x}$ [Differentiate w.r. to x]

$= \lim\limits_{x\to 0} \sec^2 x \times \lim\limits_{x\to 0} \frac{x}{\tan x}$

$= 1\times 1 = 1$

2. Solution

a. $\lim\limits_{x\to\infty} \frac{x^4}{e^x}$

Solution: Since, $\lim\limits_{x\to\infty} \frac{x^4}{e^\infty} \left(\frac{\infty}{\infty} \text{ form}\right)$

Using L–Hospital rule,

$= \lim\limits_{x\to\infty} \frac{4x^3}{e^x}$ [Differentiate num and dere. w.r.to x]

Again $\frac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{12x^2}{e^x} \text{ [Differentiate w.r. to x]}$$

Again $\frac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{24x}{e^x}$$

Again $\frac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{24}{e^x} = \frac{24}{e^\infty} = \frac{24}{\infty} = 0$$

b. $\lim_{x\to\infty} \frac{\log (x^2 + 1)}{\log (x^3 + 1)}$

Solution: Since, $\lim_{x\to\infty} \frac{\log (x^2 + 1)}{\log (x^3 + 1)} \left(\frac{\infty}{\infty} \text{ form}\right)$

Using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{2x(x^3 + 1)}{3x^2(x^2 + 1)} \text{ [Differentiate w.r. to x]}$$

$$= \frac{2}{3} \lim_{x\to\infty} \frac{x^3 + 1}{x(x^2 + 1)}$$

Again $\frac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \frac{2}{3} \frac{3x^2}{3x^2 + 1}$$

$$= \frac{2}{3} \lim_{x\to\infty} \left(\frac{3x^2 + 1}{3x^2 + 1} - \frac{1}{3x^2 + 1}\right)$$

$$= \frac{2}{3} \lim_{x\to\infty} \left(1 - \frac{1}{3x^2 + 1}\right)$$

$$= \frac{2}{3} (1 - 0) = \frac{2}{3}$$

c. $\lim_{x\to 0} x^x$

Solution: Since, $\lim_{x\to 0} x^x$ (0^0 forms)

Using L–Hospital rule, for this let,

$y = x^x \Rightarrow \log y = x \log x$

Taking limit as x tends to 0

$$\lim_{x\to 0} \log y = \lim_{x\to 0} x \log x$$

$$\text{or, } \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log x}{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{1/x}{-1/x^2} \text{ [Differentiate w.r. to x]}$$

$$= \lim_{x\to 0} -x$$

$$\therefore \lim_{x\to 0} \log y = 0$$

$$\therefore \lim_{x\to 0} y = e^0$$

$\therefore \lim_{x\to 0} x^x = 1$

d. $\lim_{x\to 0} \sin x \log x^2$

Solution: Since, $\lim_{x\to 0} \sin x \log x^2$ ($0.\infty$ forms)

$= \lim_{x\to 0} \frac{\log x^2}{\text{cosec}x}$

$= \lim_{x\to 0} \frac{2x/x^2}{-\text{coxcx.cot}x}$ [Differentiate w.r. to x]

$= \lim_{x\to 0} -\frac{2}{x} \times \frac{\tan x}{\text{cosec}x}$

$= -2 \times \lim_{x\to 0} \left(\frac{\tan x}{x}\right) \times \lim_{x\to 0} \sin x$

$= -2\times1\times0 = 0$

e. $\lim_{x\to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$ ($\infty - \infty$ forms)

$= \lim_{x\to 0} \left[\frac{\sin^2 x - x^2}{x^2 \sin^2 x}\right]$

$= \lim_{x\to 0} \frac{\left(\frac{\sin x}{x}\right)^2 - 1}{1 - \cos^2 x}$

$= \lim_{x\to 0} \frac{\left(\frac{\sin x}{x} + 1\right)\left(\frac{\sin x}{x} - 1\right)}{(1 + \cos x)(1 - \cos x)}$

$= \lim_{x\to 0} \frac{\sin x - x}{x(1 - \cos x)}$

$= \lim_{x\to 0} \frac{\cos x - 1}{x\sin x + (1 - \cos x)}$ [Differentiate w.r. to x]

$= \lim_{x\to 0} \frac{-\sin x}{x\cos x + \sin x + \sin x} = \lim_{x\to 0} \frac{-\sin x}{x\cos x + 2.\sin x}$

$= \lim_{x\to 0} \frac{-\cos x}{-x\sin x + \cos x + 2\cos x}$

$= \frac{-\cos 0^o}{-0 + 1+2\times1} = -\frac{1}{3}$

Exercise – 15.7

1. Find Δy and dy of the following.

a. $y = x^3 + 3$ for $x = 2$ and $\Delta x = 0.1$

Solution: Since we know, $\Delta y = f(x + \Delta x) - f(x)$ and $dy = f'(x)\, dx$.

$\therefore$ $dy = 3x^2\, dx$
$= 3\times2^2 \times 0.1$
$= 12\times0.1$
$= 1.2$

Again, $\Delta y = f(x + \Delta x) - f(x)$
$= f(2 + 0.1) - f(2)$
$= f(2.1) - f(2)$
$= (2.1)^3 + 3 - (2^3 + 3)$
$= 9.261 + 3 - 11$
$= 12.261 - 11$

$= 1.261$

b. $y = \sqrt{x}$ for $x = 4$ and $\Delta x = 0.41$

Solution: Now, $dy = f'(x)\,dx$

$= \frac{1}{2\sqrt{x}} dx = \frac{1}{2\sqrt{4}} \times 0.41 = \frac{1}{4} \times 0.41 = 0.1025$

and, $\Delta y = f(x|\Delta x) - f(x)$

$= f(4+0\,.41) - f(4)$

$= \sqrt{4.41} - \sqrt{4}$

$= 2.1 - 2$

$= 0.1$

c. $y = x^3 + 3x$ when $x = 2$ and $\Delta x = 0.2$

Solution: Since, $dy = f'(x)\,dx$

$= (2x + 3)dx$

$= (2\times2+3) \times 0.2$

$= -7 \times 0.2$

$= 1.4$

and, $\Delta y = f(x + \Delta x) - f(x)$

$= f(2 + 0.2) - f(2)$

$= f(2.2) - f(2)$

$= (2.2)^2 + 3\times2.2 - (2^2 + 3\times2)$

$= 4.84 + 6.6 - 10$

$= 11.44 - 10$

$= 1.44$

2. Find the approximate change in the volume of a cube of side xm caused by increasing the side by 2%.

Solution: Here, side of cube = xm

$\therefore \quad dx = 2\%$ of x

$= \frac{2}{100} x = 0.02x$

Now, the volume of cube having side x,

$v = x^3$

Now the change in volume,

$dv = 3x^2dx = 3x^2\,(0.02x) = 0.06x^3$

3. If $y = x^4 - 10$ and if x changes form 2 v 1.99. What is the exact and approximates change in y?

Solution: Since, $x = 2$ and $x + \Delta x = 1.99 \Rightarrow \Delta x = 1.99 - 2 = -0.01$

Now, $\frac{dy}{dx} = 4x^3$

$\Rightarrow dy = 4x^3dx$

At, $x = 2$, $dy = 4\times(2)^3 \times (-0.01)$

$= -0.32$

Again, if $x = 2$, then $y = x^4 - 10$

$= 2^4 - 10 = 6$

Since, $y + \Delta y = 6 + (-0.32) = 5.68$

4. If the radious of a sphere changes from 3cm to 3.01cm. Find the approximate increase in its volume.

Solution: Let, $x = 3$cm then $x + \Delta x = 3.01$

$\Rightarrow \quad \Delta x = 3.01 - 3$

$\Rightarrow \quad \Delta x = 0.01$

Since volume of sphere,

$v = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi \times 3(3)^2 \times 0.01$

$= 0.04\pi \times 9$

$= 0.36\pi$

5. Find the approximate increase in the surface area of a cube of the edge from 10 to 10.01. Calculate percent error in the surface area.

Solution: Let, $a = 10$ then $a + \Delta a = 10.01$

$\Delta a = 10.01 - 10 = 0.01$

Since surface area of cube is

$A = 6a^2$

$= 12a\ da$

$= 12 \times 10 \times 0.01$

$= 120 \times 0.01$

$= 1.2$

Again for percent error

Since we know that percentage error $= \frac{\text{Change}}{\text{orginal}} \times 100$

$= \frac{6(10.01 - 10)^2}{6 \times 10^2} \times 100$

$= (0.01)^2$

$= 0.0001\%$

6. A circular cupper plate is heated so that its radious increases from 5cm to 5.06 cm. Find the approximate increase in area and also the actual increase in area.

Solution: Let, $r = 5$. Then $r + \Delta r = 5.06$

$\Delta r = 5.06 - 5 = 0.06$

Now, $A = \pi r^2$

$dA = 2\pi r\ dr$

$= 2\pi \times 5 \times 0.06$

$= 0.6\pi$

Again, actual increase in area,

$= \pi(5.06)^2 - \pi(5)^2$

$= \pi(25.603 - 25)$

$= \pi \times 0.603$

$= 0.603\pi$

7. The radious of sphere is found by measurement to be 209cm with possible error of 0.02 of a centimeter. Find the consequent error in the surface.

Solution: Here, $r = 20$cm and $\Delta r = 0.02$

Then, $A = 4\pi r^2$

$= 4 \times \frac{22}{7} \times (20)^2 = \frac{3500}{7} = 5028.58$

Now since, $\frac{\Delta A}{A} = 2\frac{\Delta r}{r}$

$\Delta A = 2 \times \frac{0.02}{20} \times 5028.58 = 10.05$ sqcm

Chapter 16: Anti-derivatives

Exercise 16.1

1. Solution

a. $\int \frac{dx}{4x^2 + 9}$

$\int \frac{dx}{(2x)^2 + 3^2}$

Put y = 2x

$\frac{dy}{2} = dx$

Now, $= \frac{1}{2} \int \frac{dy}{y^2 + 3^2}$

$= \frac{1}{2} \frac{1}{3} \tan^{-1} \frac{y}{3} + c$

$= \frac{1}{6} \tan^{-1} \frac{2x}{3} = c$

b. $\int \frac{xdx}{x^4 + 3}$

$= \int \frac{xdx}{(x^2)^2 \ (\sqrt{3})^2}$

Put $y = x^2$

$\frac{dy}{2} = xdx$

$= \frac{1}{2} \int \frac{dy}{y^2 + (\sqrt{3})^2}$

$= \frac{1}{2} . \frac{1}{\sqrt{3}} \tan^{-1} \frac{4}{\sqrt{3}} + c$

$= \frac{1}{2\sqrt{3}} \tan^{-1} \frac{x^2}{\sqrt{3}} + c$

c. $\int \frac{2x + 3}{4x^2 + 1} dx$

$I = I_1 + I_2$

$I_1 = \int \frac{2x}{4x^2 + 1} dx \qquad I_2 = 3 \int \frac{dx}{4x^2 + 1}$

where,

$I_1 = \int \frac{2x}{4x^2 + 1} dx$

Put $y = 4x^2 + 1$

$\frac{dy}{4} = 2x . dx$

Now, $= \frac{1}{4} \int \frac{dy}{y} = \frac{1}{4} \log y + c$

$= \frac{1}{4} \log (4x^2 + 1) + c$

$I_2 = 3\int \frac{dx}{4x^2 + 1} = 3\int \frac{dx}{(2x)^2 + 1^2}$

Put $2x = y = dx = \frac{dy}{2}$

Then, $\frac{3}{2}\int \frac{dy}{y^2 + 1^2}$

$= \frac{3}{2} \tan^{-1} y + c_2 = \frac{3}{2} \tan^{-1} 2x + c_2$

Hence, $J = J_1 + J_2$

$= \frac{1}{4} \log (4x^2 + 1) + \frac{3}{2} \tan^{-1} 2x + c$

d. $\int \frac{x^2 dx}{x^6 - 8} = \int \frac{x^2 dx}{(x^3)^2 - 3^2}$

Put $y = x^3$

$\frac{dy}{3} = x^2\, dx$

Now, $\frac{1}{3}\int \frac{dy}{y^2 - 3^2} = \frac{1}{3}\,\frac{1}{2.3} \log \frac{y-3}{y+3} + c = \frac{1}{18} \log \frac{x^3 - 3}{x^3 + 3} + c$

e. $\int \frac{2x}{1 - x^4}\, dx = \int \frac{2x}{1 - (x^2)^2}\, dx$

Let $x^2 = y$

$2x.dx = dy$

$= \int \frac{dy}{1 - y^2} = \frac{1}{2.1} \log \frac{1+y}{1-y} + c = \frac{1}{2} \log \left(\frac{1 + x^2}{1 - x^4}\right) + C$

f. $\int \frac{dx}{x^2 + 6x + 8} = \int \frac{dx}{x^2 + 2.3x + 9 - 9 + 8}$

$= \int \frac{dx}{(x+3)^2 - 1^2}$

Put $y = x + 3$

$dy = dx$

Now, $\int \frac{dy}{y^2 - 1^2} = \frac{1}{2.1} \log \frac{y-1}{y+1} + c$

$= \frac{1}{2} \log \frac{x + 3 - 1}{x + 3 + 1} + c = \frac{1}{2} \log \frac{x+2}{x+4} + c$

g. $\int \frac{\cos x dx}{\sin^2 x + 4\sin x + 5}$

Put $y = \sin x$, $dy = \cos x.dx$

$= \int \frac{dy}{y^2 + 4y + 5}$

$= \int \frac{dy}{y^2 + 2.2y + 4 - 4 + 5}$

$$= \int \frac{dy}{(y+2)^2 + 1^2}$$

$$= \frac{1}{2} \tan^{-1} \frac{y+2}{1} + c$$

$\text{Tan}^{-1} (\sin x + 2) + c$

h. $$\int \frac{xdx}{x^4 - x^2 - 1}$$

$$\int \frac{xdx}{(x^2)^2 - x^2 - 1}$$

Put $y = x^2$
$dy = 2x.dx$
$\frac{dy}{2} = xdx$

Now, $= \frac{1}{2} \int \frac{dy}{y^2 - y - 1}$

$$= \frac{1}{2} \int \frac{dy}{y^2 - 2.y.\frac{1}{2} + \frac{1}{4} - \frac{1}{4} - 1}$$

$$= \frac{1}{2} \int \frac{dy}{\left(y - \frac{1}{2}\right)^2 - \frac{5}{4}}$$

$$= \frac{1}{2} \int \frac{dy}{\left(y - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \frac{1}{2} \frac{1}{2.\frac{\sqrt{5}}{2}} \log \frac{y - \frac{1}{2} - \frac{\sqrt{5}}{2}}{y - \frac{1}{2} + \frac{\sqrt{5}}{2}} + c$$

$$= \frac{1}{2\sqrt{5}} \log \frac{2x^2 - 1 - \sqrt{5}}{2y - 1 + \sqrt{5}} + c$$

$$= \frac{1}{2\sqrt{5}} \log \frac{2x^2 - 1 - \sqrt{5}}{2x^2 - 1 + \sqrt{5}} + c$$

i. $$\int \frac{e^x \, dx}{e^{2x} + 2e^x + 5}$$

Put $e^x = y$
$dy = e^x . dx$

Now, $= \int \frac{dy}{y^2 + 2y + 5}$

$$= \int \frac{dy}{y^2 + 2.y.1 + 1 - 1 + 5}$$

$$= \int \frac{dy}{(y+1)^2 + (2)^2}$$

$= \int \frac{dy}{(y + 1)^2 + (2)^2}$

$= \frac{1}{2} \tan^{-1} \frac{y + 1}{2} + c$

$= \frac{1}{2} \tan^{-1} \frac{e^x + 1}{2} + c$

j. $\int \frac{dx}{9x^2 + 12x + 13}$

$= \int \frac{dx}{(3x)^2 + 2.3x \,.\, 2 + 4 - 4 + 13}$

$= \int \frac{dx}{(3x + 2)^2 + 3^2}$

Put y = 3x +2

$\frac{dy}{3} = dx$

$= \frac{1}{3} \int \frac{dy}{y^2 + 3^2}$

$= \frac{1}{3} \cdot \frac{1}{3} \tan^{-1} \frac{y}{3} + c$

$= \frac{1}{9} \tan^{-1} \frac{3x + 2}{3} + c$

k. $\int \frac{dx}{1 - 6x - 9x^2} = -\int \frac{dx}{(3x)^2 + 6x - 1}$

$= -\int \frac{dx}{(3x)^2 + 2.3x \,.\, 1 + 1 - 1 - 1}$

$= -\int \frac{dx}{(3x)^2 + 2.3x \,.\, 1 + 1 - 1 - 1}$

$= -\int \frac{dx}{(3x + 1)^2 - (\sqrt{2})^2}$

Put y = 3x +1

$\frac{dy}{3} = dx$

Now, $-\frac{1}{3} \int \frac{dy}{y^2 - (\sqrt{2})^2} = -\frac{1}{3} \cdot \frac{1}{2.\sqrt{2}} \log \frac{y - \sqrt{2}}{y + \sqrt{2}} + c$

$= -\frac{1}{6\sqrt{2}} \log \frac{3x + 1 - \sqrt{2}}{3x + 1 + \sqrt{2}} = c$

l. $\int \frac{3x + 5}{x^2 + 4x + 20} dx$

$p \int \frac{2x + 4}{x^2 + 4x + 20} dx + q \int \frac{dx}{x^2 + 4x + 20}$

$\therefore$ 2p = 3 and 4p + q = 5

or, $p = \frac{3}{2}$ and q = –1

$$\text{Now, } \int \frac{3x+5}{x^2+4x+20}\,dx = \frac{3}{2}\int \frac{(2x+4)\,dx}{x^2+4x+20} - \int \frac{dx}{x^2+4x+20}$$

$$= \frac{3}{2}\log(x^2+4x+20) - \int \frac{dx}{(x+2)^2+4^2}$$

$$= \frac{3}{2}\log(x^2+4x+20) - \frac{1}{4}\tan^{-1}\frac{x+2}{4} + c$$

m. $\displaystyle\int \frac{(2x+2)}{(3+2x-x^2)}\,dx$

$$I = \int \frac{(2x+2)}{(3+2x-x^2)}\,dx = -\int \frac{-2x+2-4}{3+2x-x^2}\,dx$$

$$= -\int \frac{2-2x}{3+2x-x^2}\,dx + 4\int \frac{1}{3+2x-x^2}\,dx$$

$$= -\log(3+2x-x^2) + 4\int \frac{1}{(2)^2-(x-1)}\,dx$$

$$= -\log(3+2x-x^2) + \frac{4}{2.2}\log\frac{2+x-1}{2-x-1} + c$$

$$= -\log(3+2x-x^2) + \log\frac{x+1}{3-x} + c$$

$$= \text{Ln}\left(\frac{x+1}{3-x}\right) - \text{Ln}(3+2x-x^2) + c$$

n. $\displaystyle\int \frac{6x+2}{9x^2+6x+26}\,dx$

$$= \frac{1}{3}\int \frac{18x+6}{9x^2+6x+26}\,dx$$

$$= \frac{1}{3}\log(9x^2+6x+26) + c$$

2. Solution:

a. $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$

$$= \int \frac{dx}{\sqrt{x^2-2^2}}$$

$$= \log\left(x+\sqrt{x^2-4}\right) + c$$

b. $$\int \frac{dx}{\sqrt{x^2+x-2}} = \int \frac{dx}{\sqrt{x^2+2x\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}-2}}$$

$$= \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}} = \log\left[\left(\frac{2x+1}{2}\right)+\sqrt{x^2+x-2}\right] + c$$

c. $$\int \frac{dx}{\sqrt{2x^2+3x+4}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+\frac{3}{2}x+2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+2\cdot\frac{3}{4}\cdot x+\frac{9}{16}-\frac{9}{16}+2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{\sqrt{23}}{4}\right)}}^{2} = \frac{1}{\sqrt{2}} \log \left(x + \frac{3}{4} + \sqrt{x^2 + \frac{3}{2}x + 2}\right) + c$$

d. $\displaystyle\int \frac{dx}{\sqrt{2ax + x^2}} = \int \frac{dx}{\sqrt{a^2 + 2ax + n^2 - a^2}}$

$$= \int \frac{dx}{(a + x)^2 - a^2} = \log\left(a + x + \sqrt{2ax + x^2}\right) + c$$

e. $\displaystyle\int \frac{dx}{\sqrt{5 - x + x^2}} = \int \frac{dx}{\sqrt{x^2 - x + 5}}$

$$= \int \frac{dx}{\sqrt{x^2 - 2.x \,.\, \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + 5}}$$

$$= \int \frac{dx}{\sqrt{x^2 - 2.x \,.\, \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + 5}}$$

$$= \int \frac{dx}{\sqrt{\left(x \;\frac{1}{2}\right)^2 + \left(\frac{\sqrt{19}}{2}\right)^2}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{19}}{2}\right)^2}}$$

$$= \log\left(\frac{2x - 1}{2} + \sqrt{x^2 - x + 5}\right) + c$$

f. $\displaystyle\int \frac{dx}{\sqrt{6 + x - x^2}} = -\int \frac{dx}{\sqrt{x^2 - x - 6}}$

$$= -\int \frac{dx}{\sqrt{x^2 - 2.x \,.\, \frac{1}{2} + \frac{1}{4} - \frac{1}{4} - 6}}$$

$$= -\int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{5}{2}\right)^2}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}}$$

$$= \sin^{-1} \frac{\frac{2x - 1}{2}}{\frac{5}{2}} + c$$

$$= \sin^{-1} \frac{2x - 1}{5} + c$$

g. $\displaystyle\int \frac{dx}{\sqrt{2 - 2x - x^2}}$

$$= -\int \frac{dx}{\sqrt{x^2 + 2x - 2}}$$

$$= -\int \frac{dx}{\sqrt{x^2 + 2x - 2}}$$

$$= -\int \frac{dx}{\sqrt{x^2 + 2x + 1 - 1 - 2}}$$

$$= -\int \frac{dx}{\sqrt{(x + 1)^2 - (\sqrt{3})^2}}$$

$$= \int \frac{dx}{\sqrt{(\sqrt{3})^2 - (x + 1)^2}}$$

$$= \sin^{-1} \frac{x + 1}{\sqrt{3}} + c$$

h. $\displaystyle\int \frac{dx}{\sqrt{9x^2 + 6x + 10}}$

$$= \int \frac{dx}{\sqrt{(3x)^2 + 6x + 10}}$$

$$= \int \frac{dx}{\sqrt{(3x)^2 + 2.3x.1 + 1 - 1 + 10}}$$

$$= \int \frac{dx}{\sqrt{(3x + 1)^2 + 3^2}}$$

Put y = 3x + 1

$\frac{dy}{3} = dx$

Now, $\displaystyle\frac{1}{3} \int \frac{dx}{\sqrt{y^2 + 3^2}}$

$$= \frac{1}{3} . \frac{1}{3} \tan^{-1} \frac{3x + 1}{3} + c$$

$$= \frac{1}{9} \tan^{-1} \frac{3x + 1}{3} + c$$

i. $\displaystyle\int \frac{xdx}{\sqrt{a^4 + x^4}}$

$$\int \frac{xdx}{\sqrt{(a^2)^2 + (x^2)^2}}$$

Put $x^2 = y$

$xdx = \frac{dy}{2}$

Now, $\displaystyle\frac{1}{2} \int \frac{dy}{\sqrt{a^2 + y^2}}$

$= \frac{1}{2} \log \left(y + \sqrt{a^2 + y^2}\right) + c$

$= \frac{1}{2} \log \left(x^2 + \sqrt{a^4 + x^4}\right) + c$

j. $\int \frac{xdx}{\sqrt{x^4 + 2x^2 + 10}}$

$= \int \frac{dx}{\sqrt{(x^2)^2 + 2x^2 + 10}}$

Put $x^2 = y$

$x.dx = \frac{dy}{2}$

Now, $\frac{1}{2} \int \frac{dy}{\sqrt{y^2 + 2y + 10}}$

$= \frac{1}{2} \int \frac{dy}{\sqrt{y^2 + 2.y.1 + 1 - 1 + 10}}$

$= \frac{1}{2} \int \frac{dy}{\sqrt{(y + 1)^2 + 3^2}}$

$= \frac{1}{2} \log \left(y + 1 + \sqrt{x^4 + 2x^2 + 10}\right) + c$

$= \frac{1}{2} \log \left(x^2 + 1 + \sqrt{x^4 + 2x^2 + 10}\right) + c$

k. $\int \frac{2x + 3}{\sqrt{x^2 + 4x + 20}} dx$

$= \int \frac{2x + 3 + 1 - 1}{\sqrt{x^2 + 4x + 20}} dx$

$= \int \frac{2x + 4}{\sqrt{x^2 + 4x + 20}} dx - \int \frac{dx}{\sqrt{x^2 + 4x + 20}}$

$=2\sqrt{x^2 + 4x + 20} - \int \frac{dx}{\sqrt{x^2 + 2.2x + 4 - 4 + 20}}$

$= 2\sqrt{x^2 + 4x + 20} - \int \frac{dx}{\sqrt{(x + 2)^2 + 4^2}}$

$= 2\sqrt{x^2 + 4x + 20} - \log \left(x + 2 + \sqrt{x^2 + 4x + 20}\right) + c$

l. $\int \frac{x - 2}{\sqrt{2x^2 - 8x + 5}} dx$

$= \frac{1}{2} \int \frac{2x - 8}{\sqrt{2x^2 - 8x + 5}} dx$

$= \frac{1}{2} 2\sqrt{2x^2 - 8x + 5} + c$

$= \frac{1}{2} \sqrt{2x^2 - 8x + 5} + c$

m. $\int \frac{x\,dx}{\sqrt{7 + 6x - x^2}}$

This equation can be written as

$$p\int \frac{6 - 2x}{\sqrt{7 + 6x - x^2}}\,dx + q\int \frac{1}{\sqrt{7 + 6x - x^2}}\,dx \quad \text{... .... (i)}$$

By comparing

$-2p = 1$ $6p+q = 0$

$\therefore \quad p = \frac{-1}{2}$ $6 \times \left(\frac{-1}{2}\right) + q = 0$

$-3 + q = 0$

$\therefore \quad q = 3$

Put the value of p and q in equation (i)

$$= \frac{-1}{2} \int \frac{6 - 2x}{\sqrt{7 + 6x - x^2}}\,dx + 3 \int \frac{1}{\sqrt{7 + 6x - x^2}}\,dx$$

$$= \frac{-1}{2} \int (6 - 2x)\,(7 + 6x - x^2)^{-1/2}\,dx + 3 \int \frac{1}{\sqrt{-(x^2 - 6x - 7)}}\,dx$$

$$= \frac{-1}{2} \times \frac{2}{1}\,(7 + 6x - x^2)^{1/2} + 3 \int \frac{1}{\sqrt{\{x^2 - 2.3x + 9 - 9 - 7}}\,dx$$

$$= -\sqrt{7 + 6x - x^2} + 3 \int \frac{dx}{\sqrt{-\{(x - 3)^2 - (4)^2\}}}$$

$$= -\sqrt{7 + 6x - x^2} + 3 \times \sin^{-1}\left(\frac{x - 3}{4}\right) + c$$

n. $I = \int \frac{dx}{\sqrt{(x + a)\,(x + b)}}$

$$= \int \frac{1}{\sqrt{x^2 + bx + ax + ab}}\,dx$$

$$= \int \frac{1}{\sqrt{x^2 + (a + b)\,.\,x + ab}}\,dx$$

$$= \int \frac{1}{\sqrt{(x)2 + 2.x\left(\frac{a + b}{2}\right) + \left(\frac{a + b}{2}\right)^2 - \left(\frac{a + b}{2}\right)^2 + ab}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{a + b}{2}\right)^2 - \left(\frac{a^2 + 2ab + b^2 - 4ab}{4}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{a + b}{2}\right)^2 - \left(\frac{a^2 - 2ab + b^2}{4}\right)}}$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{a + b}{2}\right)^2 - \left(\frac{a - b}{2}\right)^2}}\,dx$$

$$= \log \left(x + \frac{a + b}{2} + \sqrt{(x + a)(x + b)} \right) + c$$

o. $$\int \frac{dx}{(11 + x)\sqrt{2 + x}}$$

Put $z^2 = 2 + x$

$2z \, . \, dz = dx$

Then $I = \int \frac{2z \, . \, dz}{(9 + z^2)^2}$

$$= 2\int \frac{dz}{z^2 + 3^2}$$

$$= \frac{2}{3} \tan^{-1} \frac{2}{3} + c$$

$$= \frac{2}{3} \tan^{-1} \frac{2 + x}{3} + c$$

p. $$J = \int \frac{dx}{(4x + 3)\sqrt{x + 3}}$$

Put $x + 3 = y^2$

$\therefore \quad dx = 2y.dy$

$\therefore \quad x = y^2 - 3$

$$J = \int \frac{2ydy}{[4(y^2 - 3) + 3]y}$$

$$= 2\int \frac{dy}{4y^2 - 9}$$

$$= 2\int \frac{dy}{(2y)^2 - 3^2}$$

$$= 2 \, . \, \frac{1}{2} \, . \, \frac{1}{2.3} \log \frac{2y - 3}{2y + 3} + c$$

$$= \frac{1}{6} \log \frac{2\sqrt{x + 3} - 3}{2\sqrt{x + 3} + 3} + c$$

q. $$I = \int \sqrt{\frac{1 + x}{1 - x}} \, dx$$

$$= \int \sqrt{\frac{1 + x}{1 - x} \times \frac{1 + x}{1 + x}} \, dx$$

$$= \int \sqrt{\frac{(1 + x)^2}{1 - x^2}} \, dx$$

$$= \int \frac{1 + x}{\sqrt{1 - x^2}} \, dx$$

$$= \int \frac{1}{\sqrt{1 - x^2}} \, dx + \int \frac{x}{\sqrt{1 - x^2}} \, dx$$

$= \int \frac{1}{\sqrt{1-x^2}} dx - \frac{1}{2} \int \frac{-2x\, dx}{\sqrt{1-x^2}}$

$= \sin^{-1} x - \frac{1}{2} 2\sqrt{1-x^2} + c$

$= \sin^{-1} x - \sqrt{1-x^2} + c$

Exercise 16.2

1. $\int \sqrt{x^2 - 36}\, dx$

$I = \int \sqrt{x^2 - 36}\, dx$

$= \sqrt{(x) - (6)^2}\, dx$

$= \frac{x\sqrt{x^2-36}}{2} + \frac{(6)^2}{2} \log \left(x + \sqrt{x^2 - 36}\right) + c$

$= \frac{1}{2} x\sqrt{x^2 - 36} - 18 \log \left(x + \sqrt{x^2 - 36}\right) + c$

2. $\int \sqrt{1 - 4x^2}\, dx$

$I = \int \sqrt{1 - 4x^2\, dx} = \int \sqrt{(1)^2 - (2x)^2\, dx}$

Put y = 2x

$\frac{dy}{2} - dx$

Now, $I = \frac{1}{2} \int \sqrt{1^2 - y^2}\, dx$

$= \frac{1}{2} \int \sqrt{1^2 - y^2}\, dx$

$= \frac{1}{2} \cdot \frac{1}{2} \left[y\sqrt{1^2 - y^2 + \frac{1}{2}\, 12 \sin^{-1} \frac{y}{1}} \right] + c$

$= \frac{1}{4} y\sqrt{1^2 - y^2} + \frac{1}{4} \sin^{-1} \frac{y}{1} + c$

$= \frac{2x}{4} \sqrt{1^2 - 4x^2} + \frac{1}{4} \sin^{-1} 2x + c$

$= \frac{x\sqrt{1-4x^2}}{2} + \frac{1}{4} \sin^{-1} 2x + c$

3. $I = \int \sqrt{3x^2 + 5}\, dx = \int \sqrt{(\sqrt{3}x)^2 + (\sqrt{5})^2}\, dx$

Put $y = \sqrt{3}\, x$

$\therefore \quad \frac{dy}{\sqrt{3}} = dx$

Now, $I = \frac{1}{\sqrt{3}} \int \sqrt{y^2 + (\sqrt{5})^2}\, dx$

$I = \frac{1}{\sqrt{3}} \int \sqrt{y^2 + (\sqrt{5})^2\, dx}$

$$= \frac{1}{\sqrt{3}} \left[\frac{y\sqrt{y^2 + (\sqrt{5})^2}}{2} + \frac{\sqrt{(5)^2}}{2} \log \left(y + \sqrt{y^2 + (\sqrt{5})^2}\right) \right]$$

$$= \frac{\sqrt{3}x}{\sqrt{3}2} \sqrt{(\sqrt{3}x)^2 + (\sqrt{5^2})} + \frac{5}{2\sqrt{3}} \log \left(\sqrt{3}x + \sqrt{(\sqrt{3}x^2) + \sqrt{(5)^2}}\right) + c$$

$$= \frac{x\sqrt{3x^2 + 5}}{2} + \frac{5}{2\sqrt{3}} \log \left(\sqrt{3}\, x + \sqrt{3x+ 5}\right) + c$$

4. $I = \int \sqrt{3 - 2x - x^2}\, dx$

$$= -\int \sqrt{x^2 + 2x - 3}\, dx$$

$$= -\int \sqrt{x^2 + 2.x.1 + 1 - 1 - 3}\, dx$$

$$= \int \sqrt{-(x + 1)^2 + (2)^2}\, dx$$

$$= \frac{1}{2} (n+1) ((x + 1)^2 - 2^2) + \frac{1}{2} (2)^2 \log (x + 1 + \sqrt{(x + 1)^2 - 2^2}) + c$$

$$= \int \sqrt{(2)^2 - (x + 1)^2}$$

$$= \frac{1}{2} (x + 1) \left(\sqrt{3 - 2x - x^2}\right) + \frac{4}{2} \sin^{-1} \left(\frac{x + 1}{2}\right) + c$$

$$= \frac{(x + 1) \sqrt{3 - 2x - x^2}}{2} + 2\sin^{-1} \left(\frac{x + 1}{2}\right) + c$$

5. $I = \int \sqrt{5 - 2x + x^2}\, dx$

$$= \int \sqrt{4 + 1 - 2x + x^2}\, dx$$

$$= \int \sqrt{(2)^3 + (x - 1)^2}\, dx$$

$$= \frac{(x - 1) \sqrt{(2)2 + (x - 1)^2}}{2} + \frac{(2)^2}{2} \log \left(x - 1 + \sqrt{(2)^2 + (x - 1)^2}\right) + c$$

$$= \frac{1}{2} (x - 1) \sqrt{5 - 2x + x^2} + 2 \log \left(x - 1 + \sqrt{5 - 2x + x^2}\right) + c$$

6. $\int \sqrt{18x - x^2 - 65}\, dx$

$$= \int \sqrt{81 - 81 + 2.9x - x^2 - 65}\, dx$$

$$= \int \sqrt{81 - 65 - (x^2 - 18x + 9^2)}\, dx$$

$$= \int \sqrt{(4)^2 - (x - 9)^2)}\, dx$$

$$= \frac{1}{2} (x - 9) \sqrt{16 - (x - 9)^2} + \frac{1}{2}\, 16 \sin^{-1} \frac{x - 9}{4} + c$$

$$= \frac{1}{2} (x - 9) \sqrt{18x - x^2 - 65} + 8 \sin^{-1} \frac{x - 9}{4} + c$$

7. $\int \sqrt{5x^2 + 8x + 4}\, dx$

$= \int \sqrt{5\left(x^2 + \frac{8x}{5} + \frac{4}{5}\right)}\, dx$

$= \sqrt{5} \int \sqrt{\left(x2 + 2.x.\frac{4}{5} + \frac{16}{25} - \frac{16}{25} + \frac{4}{5}\right)}\, dx$

$= \sqrt{5} \int \sqrt{(x + 4/5)^2 + (2/5)^2}\, dx$

$= \sqrt{5}\left[\frac{\left(x + \frac{4}{5}\right)}{2}\sqrt{(x + 4/5)^2 + \left(\frac{2}{5}\right)^2} + \frac{\left(\frac{2}{5}\right)^2}{2} \text{Ln}\left\{\left(x + \frac{4}{5}\right) + \sqrt{(x + 4/5)^2 + (2/5)^2}\right\}\right]$

$= \frac{(5x + 4)\sqrt{5x^2 + 8x + 4}}{10} + \frac{2}{5\sqrt{2}} \text{Ln}\left[(5x + 4) + \sqrt{5x^2 + 8x + 4}\right] + c$

8. $I = \int \sqrt{(x - \alpha)(\beta - x)}\, dx$

Put $x - \alpha = y$

$\therefore \quad dx = dy$

$\therefore \quad x = y + \alpha$

$I = \int \sqrt{y(\beta - y\ \alpha)}\, dy$

$= \int \sqrt{(\beta - \alpha)\, y - y^2}\, dy$

$= \int \sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - \left(y - \frac{\beta - \alpha}{2}\right)^2} dy$

$= \left(y - \frac{\beta - \alpha}{2}\right)\sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - \left(y - \frac{\beta - \alpha}{2}\right)^2} + \frac{1}{2}\frac{(\beta - \alpha)^2}{4} \sin^{-1}\frac{y - \left(\frac{\beta - \alpha}{2}\right)}{\left(\frac{\beta - \alpha}{2}\right)} + c$

$= \left(x - \alpha - \frac{\beta - \alpha}{2}\right)\sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - \left(x - \alpha - \frac{\beta - \alpha}{2}\right)^2} + \frac{1}{8}(\beta - \alpha)^2 \sin^{-1}\frac{x - \alpha - \frac{\beta-\alpha}{2}}{\frac{\beta-\alpha}{2}} + c$

$= \frac{1}{2}(2x - 2\alpha - \beta + \alpha)\sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - \left(x - \frac{\alpha + \beta}{2}\right)^2} + \frac{1}{8}(\beta - \alpha)^2 \sin^{-1}\frac{2x - \alpha - \beta}{\beta - \alpha} + c$

$= \frac{1}{4}(2x - \alpha - \beta)\sqrt{(x - \alpha)(\beta - x)} + \frac{1}{8}(\beta - \alpha)^2 \sin^{-1}\frac{2x - \alpha - \beta}{\beta - \alpha} + c$

9. $I = \int \sqrt{2ax - x^2}\, dx$

$= \int \sqrt{a^2 - (a^2 - 2ax + x^2)}\, dx$

$= \int \sqrt{a^2 - (x - a)^2}\, dx$

$$= \frac{1}{2}(x-a)\sqrt{(a)^2-(x-a)^2} + \frac{(a)^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$$

$$= \frac{1}{2}(x-a)\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$$

10. $I = \int (2x-5)\sqrt{x^2-4x+5}\,dx$

$$= \int (2x-4-1)\sqrt{x^2-4x+s}\,dx$$

$$I_1 = \int (2x-4)\sqrt{x^2-4x+5}\,dx - I_2 = \int \sqrt{x^2-4x+5}\,dx$$

Put $y = x^2 - 4x + 5$

$dy = (2x-4)\,dx \quad \therefore\ I = I_1 + I_2$

$$= \int \sqrt{y}\,.\,dy - \int \sqrt{x^2-2.2x+4-4+5}\,dx$$

$$= \frac{2}{3}(y)^{3/2} - \int \sqrt{(x-2)^2+(1)^2}\,dx$$

$$= \frac{2}{3}(y)^{3/2} - \left[\frac{(x-2)\sqrt{x^2-4x+5}}{2} + \frac{1}{2}\log\left(x-2+\sqrt{x^2-4x+5}\right)\right] + c$$

$$= \frac{2}{3}(x^2-4x+5)^{3/2} - \frac{(x-2)\sqrt{x^2-4x+5}}{2} - \frac{1}{2}\log[(x-2)+\sqrt{x^2-4x+5}) + c$$

11. $\int (2-x)\sqrt{16-6x-x^2}\,dx$

$$I = \int (2-x)\sqrt{16-6x-x^2}\,dx$$

$$= \frac{1}{2}\int (4-2x)\sqrt{16-6x-x^2}\,dx$$

$$= \frac{1}{2}\int \{10+(-6-2x)\}\sqrt{16-6x-x^2}\,dx$$

$$5\int \sqrt{16-6x-x^2}\,dx + \frac{1}{2}\int (-6-2x)\sqrt{16-6x-x^2}$$

$I = I_1 + I_2$

$$I_1 = 5\int \sqrt{16-6x-x^2}\,dx$$

$$= 5\int \sqrt{25-(9+6x+x^2)}\,dx$$

$$= 5\int \sqrt{(5)^2-(x+3)^2}\,dx$$

$$= 5\left\{\frac{1}{2}(x+3)\sqrt{(5)^2-(x+3)^2} + \frac{52}{2}\sin^{-1}\left(\frac{x+3}{5}\right)\right\} + c_1$$

$$= \frac{5}{2}(X+3)\sqrt{16-6X-X^2} + \frac{125}{2}\,\text{Sin}^{-1}\left(\frac{x+3}{5}\right) C_1$$

$$I_2 = \frac{1}{2}\int (-6 - 2x)\sqrt{16 - 6x - x^2}\, dx$$

Put $16 - 6x - x^2 = y$

$\therefore$ $dy = (-6 - 2x)\, dx$

$$I_2 = \frac{1}{2}\int y^{1/2}\, dy = \frac{1}{3} y^{3/2} + C_2 \; z \; \frac{1}{3}(16 - 6x - x^2)^{3/2} + C_2$$

$$I = I_1 + I_2 = \frac{5}{2}(x + 3)\sqrt{16 - 6x - x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x + 3}{2}\right) + \frac{1}{3}(16 - 6x - x^2)^{3/2} + C$$

12. $\int (2x + 1)\sqrt{4x^2 + 20x + 21}\, dx$

$$I = \int (2x + 1)\sqrt{4x^2 + 20x + 21}\, dx$$

$$= \frac{1}{4}\int (8x + 4)\sqrt{4x^2 + 20x + 21}\, dx$$

$$= \frac{1}{4}\int \{(8x + 20) - 16\}\sqrt{4x^2 + 20x + 21}\, dx$$

$$= \frac{1}{4}\int (8x + 20)\sqrt{4x^2 + 20x + 21}\, dx - 4\int \sqrt{4x^2 + 20x + 21}\, dx$$

$= I_1 - I_2$

$$I_1 = \frac{1}{4}\int (8x + 20)\sqrt{4x^2 + 20x + 21}\, dx$$

Put $4x^2 + 20x + 21 = y$

$(8x + 20)\, dx = dy$

$$I_1 = \frac{1}{4}\int y^{1/2}\, dy = \frac{1}{4}\frac{y^{3/2}}{3/2} + C_1$$

$$= \frac{1}{6}(4x^2 + 20x + 21)^{3/2} + C_1$$

$$I_2 = 4\int \sqrt{4x^2 + 20x + 21}\, dx$$

$$= 4\int \sqrt{(4x^2 + 20x + 25) - 4}\, dx$$

$$= 4\int \sqrt{(2x + 5)^2 - (2)^2}\, dx$$

Put $2x + 5 = y$

$\therefore$ $dx = \frac{1}{2} y$

$$I_2 = 2\int \sqrt{(y)^2 - (2)^2}\, dy$$

$$= 2\left\{\frac{1}{2} y\sqrt{(y)^2 - (2)^2} - \frac{(2)^2}{2}\log\left(y + \sqrt{y^2 - 4}\right)\right\} + C_2$$

$$= y\sqrt{y^2 - 4} - 4\log\left(y + \sqrt{y^2 - 4}\right) + C_2$$

$$= (2x + 5)\sqrt{(2x + 5)^2 - 4} - 4\log\left(2x + 5 + \sqrt{(2x + 5)^2 - 4}\right)$$

$$= (2x + 5)\sqrt{2x^2 + 20x + 21} - 4\log\left(2x + 5 + \sqrt{4x^2 + 20x + 2}\right)$$

$I = I_1 - I_2$

$= \frac{1}{6}(4x^2 + 20x + 21)^{3/2} - (2x + 5)\sqrt{4x^2 + 20x + 21} + 4\log(2x + 5 + \sqrt{4x^2 + 20x + 21} + C$

13. $I = \int (2x + 3)\sqrt{x^2 - 2x - 3}\, dx$

$= \int (2x + 3 - 5 + 5)\sqrt{x^2 - 2x - 3}\, dx$

$= \int (2x - 2)\sqrt{x^2 - 2x - 3}\, dx + 5\int \sqrt{x^2 - 2x - 3}\, dx$

$I = I_1 + I_2$

$I_1 = \int (2x - 2)\sqrt{x^2 - 2x - 3}\, dx$

Put $y = x^2 - 2x - 3$

$\therefore \quad dy = (2x - 2)\, dx$

$= \int \sqrt{y}\, dy = \frac{y^{3/2}}{3/2} + C = \frac{2}{3}\sqrt{x^2 - 2x - 3} + C$

$I_2 = 5\int \sqrt{x^2 - 2x - 3}\, dx$

$= 5\int \sqrt{x^2 - 2x + 1 - 1 - 3}\, dx$

$= 5\int \sqrt{(x - 1)^2 - (2)^2}\, dx$

$= 5\left[\frac{1}{2}(x - 1)\sqrt{x^2 - 2x - 3} - \frac{4}{2}\log(x - 1 + \sqrt{x^2 - 2x - 3})\right] + C$

$= \frac{5}{2}(x - 1)\sqrt{x^2 - 2x - 3} - \frac{20}{2}\log(x - 1 + \sqrt{x^2 - 2x - 3}) + C$

$= \frac{5}{2}(x - 1)\sqrt{x^2 - 2x - 3} - 10\log(x - 1 + \sqrt{x^2 - 2x - 3}) + C$

$\therefore I = I_1 + I_2$

$= \frac{2}{3}(x^2 - 2x - 3)^{3/2} + \frac{5}{2}(x - 1)\sqrt{x^2 - 2x - 3} - 10\log(x - 1 + (x^2 - 2x - 3) + C$

14. $I = \int e^{3x} . \sin^5 x\, dx$

We suppose $u = \sin 5x$ $v = e^{3x}$

By using formula $\int e^{ax} \sin bx dx = \int \frac{e^{ax}(a \sin bx - b\cos bx)}{a^2 + b^2}$

We have,

$I = \frac{1}{3^2 + 5^2} . e^{3x}(3\sin 5x - 5\cos 5x) + C$

$\therefore \quad I = \frac{1}{34} e^{3x}(3 \sin 5x - 5\cos 5x) + C$

15. $I = \int e^x \cos 3x\, dx$

By using formula $\int e^{ax} . \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

We have,

$$I = \frac{1}{1^2 + 3^2} e^x (1.\cos 3x + 3\sin 3x) + C$$

$$= \frac{1}{10} e^x (\cos 3x + 3\sin 3x) + C$$

16. $I = \int e^{2x} \sin (x + 1)\, dx$

By using formula $\int e^{ax} . \sin bx = e^{ax} \frac{(a\sin bx - b\cos bx)}{a^2 + b^2}$

We have,

$$I = \frac{1}{(2)^2 + 1^2} e^{2x} [2 \sin (x + 1) - \cos (x + 1)] + C$$

$$= \frac{1}{5} e^{2x} [2\sin(x + 1) - \cos(x + 1)] + C$$

Exercise 16.3

1. $I = \int \frac{dx}{1 + 2\sin^2 x}$

Dividing number and de no. by $\cos^2 x$

$$\int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{2\sin^2 x}{\cos^2 x}} dx$$

$$= \int \frac{\sec^2 x}{\sec^2 x + 2\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 + \tan^2 x + 2\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 + (\sqrt{3} \tan x)^2} dx$$

Put $y = \sqrt{3} \tan x$

$$\therefore \quad I = \frac{1}{\sqrt{3}} \int \frac{dy}{1^2 + y^2}$$

$$= \frac{1}{\sqrt{3}} \frac{1}{1} \tan^{-1} \frac{y}{1} + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} (\sqrt{3} \tan x) + C$$

2. $I = \int \frac{dx}{5 + 4\cos x}$

$$\int \frac{dx}{5\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}\right) + 4 \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)} dx$$

$$= \int \frac{1}{5 \sin^2 \frac{x}{2} + 5\cos^2 \frac{x}{2} + 4\cos^2 \frac{x}{2} - 4\sin^2 \frac{x}{2}} dx$$

$$\int \frac{1}{9\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}\, dx$$

Dividing deno. and nu no. by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{9 + \tan^2\frac{x}{2}}\, dx$$

Put $\tan\frac{x}{2} = y$

or, $\sec^2\frac{x}{2}\frac{1}{2} = dy$

$\therefore$ $2dy = \sec^2 x$

Now, $I = 2\int \frac{dy}{(3)^2 + y^2}$

$$= \frac{2}{3}\tan^{-1}\frac{\left(\tan\frac{x}{2}\right)}{3} + C$$

$$= \frac{2}{3}\tan^{-1}\frac{1}{3}\left(\tan\frac{x}{2}\right) + C$$

3. $I = \int \frac{dx}{1 - 3\sin x}$

$$= \int \frac{dx}{\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) - 2.3\sin\frac{x}{2}\cos\frac{x}{2}}$$

Dividing by $\cos^2\frac{x}{2}$ in deno-and num.

$$\int \frac{\sec^2\frac{x}{2}}{ran^2\frac{x}{2} + - 6\tan\frac{x}{2}}\, dx$$

Put $y = \tan\frac{x}{2}$

$2dy = \sec^2\frac{x}{2}\, dx$

Now,

$\therefore$ $I = 2\int \frac{dy}{y^2 - 6y + 1}$

$$= 2\int \frac{dy}{y^2 - 2.3y + 9 - 9 + 1}$$

$$= 2\int \frac{dy}{(y-3)^2 - (2\sqrt{2^2})}$$

$$= 2\frac{1}{2.2\sqrt{2}}\log\frac{y - 3 - 2\sqrt{2}}{y - 3 + 2\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}} \log \frac{\tan\frac{x}{2} - 3 - 2\sqrt{2}}{\text{Tan}\frac{x}{2} - 3 + 2\sqrt{2}} + C$$

4. $I = \int \frac{dx}{a^2 \sin^2 x - b^2 \cos^2 x}$

Dividing number and deno. by $\cos^2 x$ then

$$I = \int \frac{\sec^2 x}{a^2 \text{Tan}^2 x - b^2} dx$$

Put $y = a\text{Tan}x$

$\frac{dy}{a} = \sec^3 x \,.\, dx$

Now,

$$\therefore \quad I = \frac{1}{a} \int \frac{dy}{y^2 - b^2}$$

$$= \frac{1}{2.ba} \log \frac{y-b}{y+b} + C$$

$$= \frac{1}{2ba} \log \left(\frac{a \tan x - b}{a \tan x + b}\right) + C$$

5. $I = \int \frac{dx}{4 \cos x - 1}$

$$= \int \frac{dx}{4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) - \left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{4\cos^2\frac{x}{2} - 4\sin^2\frac{x}{2} - \sin^2\frac{x}{2} - \cos^2\frac{x}{2}}$$

$$= \int \frac{dx}{3\cos^2\frac{x}{2} - 5\sin^2\frac{x}{2}}$$

Dividing deno. and num. by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{3 - (\sqrt{5})^2 \text{Tan}^2\frac{x}{2}}$$

Put $y = \sqrt{5}\,\text{Tan}\frac{x}{2}$

$$\therefore \quad \frac{2dy}{\sqrt{5}} = \sec^2\frac{x}{2}$$

Now, $I = \frac{2}{\sqrt{5}} \int \frac{dy}{(\sqrt{3})^2 - \left(\sqrt{5}\,\text{Tan}\frac{x}{2}\right)^2}$

$$= \frac{2}{\sqrt{5}} \int \frac{dy}{(\sqrt{3})^2 - y^2}$$

$$= \frac{2}{\sqrt{5}} \frac{1}{2\sqrt{3}} \log \frac{\sqrt{3}+y}{\sqrt{3}-y} + C$$

$$= \frac{1}{\sqrt{15}} \log \left(\frac{\sqrt{3} + \sqrt{5}\ \text{Tan}\,\frac{x}{2}}{\sqrt{3} - \sqrt{5}\ \text{Tan}\,\frac{x}{2}} \right) + C$$

6. $I = \int \frac{dx}{2 + 3\cos x}$

$$= \int \frac{dx}{2\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + 3\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{5\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

Dividing deno. and num. by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{(\sqrt{5})^2 - \text{Tan}^2\frac{x}{2}}\, dx$$

Put $y = \text{Tan}\,\frac{x}{2}$

$2dy = \sec^2\frac{x}{2}\, dx$

Now,

$$\therefore \quad I = 2\int \frac{dy}{(\sqrt{5})^2 - y^2} = 2\frac{1}{2\sqrt{5}} \log \frac{\sqrt{5}+y}{\sqrt{5}-y} + C = \frac{1}{\sqrt{5}} \log \left(\frac{\sqrt{5} + \text{Tan}\,\frac{x}{2}}{\sqrt{5} - \text{Tan}\,\frac{x}{2}} \right) + C$$

7. $I = \int \frac{\sin x \cos x}{(\sin x + \cos x)^2}\, dx$

$$= \frac{1}{2} \int \frac{2\sin x - \cos x}{(\sin x + \cos x)^2}\, dx$$

$$= \frac{1}{2} \int \frac{\sin^2 x\, dx}{(\sin x + \cos x)^2}$$

$$= \frac{1}{2} \int \frac{(1 + \sin 2x) - 1}{(\sin x + \cos x)^2}\, dx$$

$$= \frac{1}{2} \int \frac{(\sin x + \cos x)^2 - 1}{(\sin x + \cos x)^2}\, dx$$

$$= \frac{1}{2} \int \frac{(\sin x + \cos x)^2}{(\sin x + \cos x)^2}\, dx - \int \frac{dx}{(\sin x + \cos x)^2}$$

$$I_1 = \frac{1}{2} x - I_2 = \int \frac{dx}{(\sin x + \cos x)^2}$$

Dividing deno. and num. by $\cos^2 x$

$I = I_1 + I_2$

$= \frac{x}{2} - \int \frac{\sec^2 dx}{(1 + \text{Tan}x)^2}$ Put Tan x + 1 = y, dy = $\sec^2$ x.dx

$= \frac{x}{2} - \int \frac{dy}{y^2} = \frac{x}{2} + \frac{1}{y} + C = \frac{x}{2} + \frac{1}{\text{Tan } x + 1} + C$

8. $I = \int \frac{dx}{\sin x + \cos x}$

Put $1 = r\cos\theta$ $\quad\quad$ $1 = r\sin\theta$

So that $r^2 = 2$

$\therefore \quad r = \sqrt{2}$

$\text{Tan}\theta = 1$

$\therefore \quad \theta = \frac{\pi}{4}$

$I = \int \frac{dx}{r\cos\theta \cdot \sin x + r\sin\theta \cdot \cos x} = \frac{1}{r} \int \frac{dx}{\sin(x + \theta)} = \frac{1}{r} \int \text{cosec}(x + \theta)\, dx$

$= \frac{1}{r} \log \text{Tan} \frac{1}{2}(x + \theta) + C = \frac{1}{\sqrt{2}} \log \text{Tan} \frac{1}{2}\left(x + \frac{\pi}{4}\right) + C$

$= \frac{1}{\sqrt{2}} \log \text{Tan}\left(\frac{x}{2} + \frac{\pi}{8}\right) + C$

9. $I = \int \frac{dx}{1 + \sin x + \cos x}$

$= \int \frac{dx}{(1 + \cos x) + \sin x}$

$= \int \frac{dx}{2\cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}$

Dividing by $\cos^2 \frac{x}{2}$ in deno. and num.

$= \frac{1}{2} \int \frac{\sec^2 \frac{x}{2}}{1 + \text{Tan} \frac{x}{2}} dx$

Put $1 + \text{Tan} \frac{x}{2} = y$

$\sec^2 \frac{x}{2} \frac{1}{2} dx = dy$

$\therefore \quad \sec^2 \frac{x}{2} dx = 2dy$

$I = \int \frac{dy}{y}$

$= \log y + C$

$= \log\left(1 + \text{Tan} \frac{x}{2}\right) + C$

10. $I = \int \frac{dx}{3 + 2\sin x + \cos x}$

$$= \int \frac{1}{3\cos^2\frac{x}{2} + 3\sin^2\frac{x}{2} + 4\sin\frac{x}{2}\,.\,\cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\,dx$$

$$= \int \frac{1}{4\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2} + 4\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

Dividing denominator and numerator by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{4 + 2\mathrm{Tan}^2\frac{x}{2} + 4\mathrm{Tan}\frac{x}{2}}\,dx$$

$$= \frac{1}{4}\int \frac{\sec^2\frac{x}{2}}{1 + \frac{1}{2}\tan^2\frac{x}{2} + \tan\frac{x}{2}}$$

Put $\tan\frac{x}{2} = y$

$\Rightarrow \sec^2\frac{x}{2}\,dx = 2dy$

from (i)

$$I = \frac{1}{4}\int \frac{2dy}{1 + \frac{1}{2}y^2 + y} = \int \frac{dy}{y^2 + 2y + z} = \int \frac{dy}{(y+1)^2 + 1^2}$$

$$= \frac{1}{1}\tan^{-1}(y+1) + C = \tan^{-1}\left(\tan\frac{x}{2} + 1\right) + C$$

11. $I = \int \frac{1}{1 - \sin x + \cos x}$

$$= \int \frac{1}{(1 + \cos x) - \sin x}$$

$$= \int \frac{1}{2\cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2} - \sin\frac{x}{2}\cos\frac{x}{2}}$$

Dividing by deno. and num $\cos^2\frac{x}{2}$

$$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{1 - \mathrm{Tan}\frac{x}{2}}\,dx$$

Put $1 - \mathrm{Tan}\frac{x}{2} = y$

$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dy$

$\sec^2\frac{x}{2}\,dx = 2dy$

$$= -\frac{2}{2}\int\frac{dy}{y}$$

$$= -\log y + C$$

$$= -\log\left(1 - \text{Tan}\,\frac{x}{2}\right) + C$$

12. $\displaystyle\int\frac{dx}{2 + \cos x - \sin x}$

$$= \int\frac{dx}{2\sin^2\frac{x}{2} + 2\cos^2\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int\frac{1}{\sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2}}\,dx$$

$$= \int\frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} - 2\tan\frac{x}{2} + 1}dx$$

Put $\tan\frac{x}{2} = y$ $\frac{1}{2}\sec^2\frac{x}{2}\;dx = dy$ $\sec^2\frac{x}{2}\,dx = 2dy$

$$= \int\frac{2dy}{y^2 - 2y + 1}$$

$$= 2\int\frac{1}{(y-1)^2}\,dy = -\frac{2}{(y-1)} + c$$

$$= -\frac{2}{\left(\tan\frac{x}{2} - 1\right)}$$

13. $\displaystyle\int\frac{dx}{\cos x - \sqrt{3}\sin x}$

Put $1 = r\sin\theta$, $\sqrt{3} = r\cos\theta$
So that $r^2 = 3 + 1 = 4$
$\therefore\quad r = 2$

Also, $\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\therefore\quad \theta = \frac{\pi}{6}$

Now, $\displaystyle\int\frac{dx}{r\sin\theta\,.\,\cos x - r\cos\theta\,\sin x}$

$$= \int\frac{dx}{r[\sin(\theta - x)]}$$

$$= \frac{1}{2}\int\cos(\theta - x)\,dx$$

$$= \frac{1}{2}\left|\text{Ln}\tan\left(\frac{\theta - x}{2}\right)\right| + c$$

$$= \frac{1}{2} \text{Ln} \tan\left(\frac{\pi}{12} - \frac{x}{2}\right) + c$$

14. $I = \int \frac{1}{1 + 2\sin x} dx$

$$= \int \frac{dx}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2.2 \sin\frac{x}{2}\cos\frac{x}{2}}$$

Dividing by $\cos^2\frac{x}{2}$ in deno and num

$$= \int \frac{\sec^2\frac{x}{2}}{1 + \tan^2\frac{x}{2} + 4 \text{Tan}\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}}{\left(\tan\frac{x}{2} + 2\right)^2 - 3}$$

Put $\tan\frac{x}{2} + 2 = y$

$\text{Sec}^2\frac{x}{2} dx = 2dy$

$$\therefore \quad I = 2\int \frac{dy}{(y)^2 - (\sqrt{3})^2}$$

$$= 2\frac{1}{2\sqrt{3}} \log\frac{y - \sqrt{3}}{y + \sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}} \log\frac{\text{Tan}\frac{x}{2} + 2 - \sqrt{3}}{\text{Tan}\frac{x}{2} + 2 + \sqrt{3}} + C$$

15. $I = \int \frac{dx}{2 + \sin x}$

$$= \int \frac{dx}{2\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

Dividing by deno and num by $\cos^2\frac{x}{2}$

$$= \frac{1}{2} \int \frac{\sec^2\frac{x}{2}}{1 + \text{Tan}^2\frac{x}{2} + \text{Tan}\frac{x}{2}} dx$$

Put $\tan\frac{x}{2} = y$

$\text{Sec}^2\frac{x}{2} dx = 2dy$

$$I = \frac{2}{2}\int \frac{dy}{1 + y^2 + y}$$

$$= \int \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2y+1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2\text{Tan}\frac{x}{2}+1}{\sqrt{3}} + C$$

16. $I = \int \frac{dx}{4+3\sin x}$

or, $I = \int \frac{dx}{4\left(\cos x^2\frac{x}{2} - \sin x^2\frac{x}{2}\right) + 3.2\sin x\frac{x}{2}\cos x\frac{x}{2}}$

$$= \int \frac{\sec^2\frac{x}{2}}{4\left(1-\text{Tanx}^2\frac{x}{2}\right) + 6\,\text{Tanx}\frac{x}{2}}\,dx$$

Let $\tan x\frac{x}{2} = y$

$\text{Secx}^2\frac{x}{2} = 2dy$

Now,

$\therefore \quad I = 2\int \frac{dy}{4(1-y^2)+6y}$

$$= 2\int \frac{dy}{4 - \left(2y - \frac{3}{2}\right)^2 + \frac{9}{4}}$$

$$= 2\int \frac{dy}{\left(\frac{5}{2}\right)^2 - \left(2y+\frac{3}{2}\right)^2}$$

$$= 2.\frac{1}{2}\,\frac{1}{2.\frac{5}{2}} \log\left\{\frac{\frac{5}{2}+2y-\frac{3}{2}}{\frac{5}{2}-\left(2y-\frac{3}{2}\right)}\right\} + C$$

$$= \frac{1}{5}\log\left(\frac{2y+1}{4-2y}\right) + C$$

$$= \frac{1}{5}\log\left[\frac{2\,\text{Tanx}\frac{x}{2}+1}{4-2\,\text{Tanx}\frac{x}{2}}\right] + C$$

17. $I = \int \frac{dx}{4+3\cosh x}$

$$I = \int \frac{dx}{4\left(\cos h^2 \frac{x}{2} + \sin h^2 \frac{x}{2}\right) + 3\left(\cos h^2 \frac{x}{2} - \sin h^2 \frac{x}{2}\right)}$$

$$= \int \frac{dx}{7\cos h^2 \frac{x}{2} - \sin h^2 \frac{x}{2}}$$

Dividing by deno and num by $\cos^2 \frac{x}{2}$

$$= \int \frac{\sec h^2 \frac{x}{2}}{7 - \tan h^2 \frac{x}{2}}$$

Let $\tan h \frac{x}{2} = y$

$\sec h^2 \frac{x}{2} = 2dy$

$$\therefore\ I = 2\int \frac{dy}{7 - y^2}$$

$$= 2\int \frac{dy}{((\sqrt{7})^2 - (y)^2)}$$

$$= 2.\frac{1}{2.\sqrt{7}} \log \left(\frac{\sqrt{7} + y}{\sqrt{7} - y}\right) + C$$

$$= \frac{1}{\sqrt{7}} \log \left[\frac{\sqrt{7} + \tan h \frac{x}{2}}{\sqrt{7} - \tan h \frac{x}{2}}\right] + C$$

18. $I = \int \frac{\tan hx}{36 \sec hx + \cos hx}\, dx$

Multiplying by cosx deno. and num.

$$= \int \frac{\cos hx. \frac{\sin hx}{\cos hx}}{36 \frac{\cos hx}{\cos hx} + \cos hx \,.\, \cos hx}$$

$$= \int \frac{\sin hx}{(6)^2 + \cos h^2 x}\, dx$$

Put $\cos hx = y$

$\therefore \quad \sin hx \,.\, dx = dy$

$$\therefore \quad \int \frac{dy}{(6)^2 + (y)^2}$$

$$= \frac{1}{6} \tan^{-1} \frac{y}{6} + C$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{\cos h\, x}{6}\right) + C$$

19. $I = \int \frac{\tan hx\, dx}{\cos hx + \sec hx\, 64}$

Multiplying cosx x in deno. and num

$$= \int \frac{\cos hx \frac{\sin hx}{\cos hx}}{\cos hx \,.\, \cos hx + 64 \,.\, \cos hx \frac{1}{\cos hx}}$$

$$= \int \frac{\sin hx}{\cos h^2 x + 64} dx$$

Let cos hx = y

$$\sin h\, x = \frac{dy}{dx}$$

sin h x . dx = dy

$$\therefore \quad I = \int \frac{dy}{y^2 + 64}$$

$$= \int \frac{dy}{y^2 + (8)^2}$$

$$= \frac{1}{8} \text{Tan}^{-1} \frac{y}{8} + C$$

$$= \frac{1}{8} \text{Tan}^{-1} \left(\frac{\cosh x}{8}\right) + C$$

20. $I = \int \frac{\sin h\, x}{4\tan hx - \text{cosec } hx \,.\, \text{sech}} dx$

$$= \int \frac{\sin x\, x}{4 \frac{\sin hx}{\cos hx} - \frac{1}{\text{cosec } hx \,.\, \sec hx}} dx$$

$$= \int \frac{\sin h^2 x \cos hx}{4\sin h^2 x - 1}$$

Let sin hx = y

cos hx.dx = dy

$$\therefore \quad I = \int \frac{y^2\, dy}{4y^2 - 1} = \frac{1}{4} \int \frac{y^2\, dy}{4y^2 - 1} = \frac{1}{4} \int \frac{(14y^2 - 1) + 1}{(4y^2 - 1)} dy$$

$$= \frac{1}{4} \int \left(1 - \frac{1}{4y^2 - y}\right) dy = \frac{1}{4} \left\{ y + \int \frac{1}{(2y)^2 - (1)^2} dy \right\}$$

$$= \frac{1}{4} \left\{ y + \frac{1}{2} \frac{1}{2.1} \log \left(\frac{2y - 1}{2y + 1}\right) \right\} + C$$

$$= \frac{1}{4} \left\{ \sin hx + \frac{1}{4} \log \left(\frac{2\sin hx - 1}{2\sin hx + 1}\right) \right\} + C$$

Exercise 16.4

1. $\int \frac{2x}{(2x + 3)(3x + 5)} dx$

Let $\frac{2x}{(2x + 3)(3x + 5)} = \frac{A}{2x + 3} + \frac{B}{3x + 5}$

$2x = A(3x + 5) + B(2x + 3)$

Equating the coefficient of x and constant terms we to get

$3A + 2B = 2$

$5A + 3B = 0$

A = –6 and B = 10

$$\therefore \quad \frac{2x}{(2x+3)(3x+5)} = \frac{-6}{2x+3} + \frac{10}{3x+5}$$

So, we have by integration

$$\int \frac{dx}{(2x+3)(3x+5)} = -3 \log (2x+3) + \frac{10}{3} \log (3x+5) + C$$

2. $\displaystyle\int \frac{3x}{(x-a)(x-b)} dx$

Let $\displaystyle\frac{3x}{(x-a)(x-b)} = \frac{A}{(2x+3)} + \frac{B}{x-b}$

Put x = a

3a = A(x – b)

$\therefore \quad A = \dfrac{3a}{a-b}$

x = b, 3 b = B(b – a)

$B = \dfrac{-3b}{a-b}$

$$\therefore \quad \frac{3x}{(x-a)(x-b)} = \frac{3a}{(x-a)(a-b)} - \frac{3b}{(a-b)(x-b)}$$

$$\text{or,} \quad \int \frac{3x}{(x-a)(x-b)} dx = \frac{3}{a-b} \int \left\{ \frac{a}{x-a} - \frac{b}{x-b} \right\} dx$$

$$= \frac{3}{a-b} \; [a \log (x-a) - b \log (x \;\; b)] + C$$

3. $\displaystyle\int \frac{1}{(x+2)(x+3)^2} dx$

Let $\displaystyle\frac{1}{(x+x2)(x+3)^2} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

$1 = A(x+3)^2 + B(x+2)(x+3) + C(x+2)$

Put x = –2

1 = A.1

$\therefore$ A = 1

x = –3

I = C(–1)

$\therefore$ C = –1

x = 0, 1 – 9A + B.2.3 + C.2

6B = –6

$\therefore$ B = –1

$$\therefore \quad \frac{1}{(x+2)(x+3)^2} = \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{(x+3)^2}$$

$$\int \frac{1}{(x+2)(x+3)^2} dx = \int \left\{ \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{(x+3)^2} \right\} dx$$

$$\text{Log } (x+2) - \log (x+3) + \frac{1}{x+3} + C = \log \frac{x+2}{x+3} + \frac{1}{x+3} + C$$

4. $\displaystyle\int \frac{x^2 \, dx}{(x-a)(x-b)(x-c)}$

Let $\displaystyle\frac{x^2}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

$x^2 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$

$\therefore$ Put x = b

$b^2 = B(b-a)(b-c) + c(b-c)(b-b)$

Similarly,

Again put x = a

$$\frac{a^2}{(a-b)(a-c)} = A$$

$$\frac{c^2}{(c-b)(c-a)} = C$$

Then,

$$= \frac{a^2}{(a-b)(a-c)} \cdot \int \frac{1}{x-a}dx + \frac{b^2}{(b-a)(b-c)} \int \frac{1}{x-b}dx + \frac{c^2}{(c-b)(c-a)} \int \frac{1}{x-c}dx$$

$$= \frac{a^2}{(a-b)(a-c)} \log(x-a) + \frac{b^2}{(b-a)(b-c)} \log(x-c) + \frac{c^2}{(c-b)(c-a)} \log(x-c) + c$$

5. $\int \frac{x^2+1}{x-1}dx$

$$\int \frac{x^2}{x-1}dx + \int \frac{1}{x-1}dx$$

$$= \int \frac{x^2-1+1}{x-1}dx + \int \frac{1}{x-1}dx$$

$$= \int x + 1dx + \int \frac{1}{x-1} + \int \frac{1}{x+1}dx$$

$$= \frac{x^2}{2} + x + 2\log(x-1) + C$$

6. $\int \frac{dx}{1+x+x^2+x^3}$

$$\int \frac{dx}{x^3+x^2+x+1}$$

$$\int \frac{1}{x^2(x+1)+(x+1)}dx$$

$$\int \frac{1}{(x^2+1)(x+1)}dx$$

Let, $\frac{1}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+c}{x^2+1}$

$1 = A(x^2+1) + (Bx+c)(x+1)$

Put x = 0

$1 = A + C$ ... ... (i)

Put the value of $A = \frac{1}{2}$

$x = -1$

$1 = 2A$

$\therefore$ $\frac{1}{2} = A \Rightarrow C = \frac{1}{2}$

Put x + 1

$1 = 2A + (B + C)\,2$

$1 = 2 \times \frac{1}{2} + \left(B + \frac{1}{2}\right) 2$

$1 = 1\frac{1}{2}\left(B + \frac{1}{2}\right)$

$0 = 2\left(B + \frac{1}{2}\right)$

$\therefore \quad -\frac{1}{2} = B$

$= \frac{1}{2}\int \frac{1}{(x+1)} + \int \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1} dx$

$= \frac{1}{2}\log(x+1) - \frac{1}{2}\times\frac{1}{2}\int \frac{2x+2}{x^2-1} dx$

$= \frac{1}{2}\log(x+1) - \frac{1}{4}\log(x^2+1) + 2\int \frac{1}{x^2+1} dx$

$= \frac{1}{2}\log(x+1) - \frac{1}{4}\log(x^2+1) + 2\,\mathrm{Tan}^{-1}x + c$

7. $\int \frac{7x^2 - 18x + 13}{(x-3)(x^2+2)} dx$

Let, $\frac{7x^2 - 18x + 13}{(x-3)(x^2+2)} = \frac{A}{(x-3)} + \frac{Bx + C}{(x^2+2)}$

$7x^2 - 18x + 13 = (x^2+2)A + (Bx + C)(x-3)$

Put $x = 0$

$13 = 2A - 3C$ ... ... (i)

Again put $x = 3$

$7\times3^2 = -18\times3 + 13 = 11A$

$63 - 54 + 13 = 11A$

or, $22 = 11A$

$\therefore \quad A = 2$

Put the value of A in equation (i)

$13 = 2\times2 - 3c$

or, $\frac{13-2}{3} = -C$

$\therefore \quad C = -3$

Put $x = 1$

$2 = 3A - 2(B + C)$

$2 = 6 - 2(B - 3)$

$-4 = -2(B - 3)$

or, $2 = B - 3$

$\therefore \quad B = 5$

$= \int \frac{7x^2 - 18x + 13}{(x-3)(x^2+2)} dx = 2\int \frac{1}{x-3} dx + \int \frac{5x - 3dx}{x^2+2}$

$= 2\log(x-3) + \frac{5}{2}\int \frac{2x - \frac{3}{5}\times 2}{x^2+2} dx$

$= 2\log(x-3) + \frac{5}{2}\int \frac{1}{x^2+2} dx - \frac{6}{5}\times\frac{5}{2}\int \frac{1}{x^2+2} dx$

$= 2\log(x-3) + \frac{5}{2}(x^2+2) - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

8. $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

Let I = $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

$$= \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 1} dx$$

Put $x + \frac{1}{x} = y$

$\therefore \quad \left(1 - \frac{1}{x^2}\right) dx = dy$

$I = \int \frac{dy}{y+- 1}$

$$= \frac{1}{2.1} \log \frac{y - 1}{y + 1} + C$$

$$= \frac{1}{2} \log \frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} + C$$

$$= \frac{1}{2} \log \frac{x^2 - x + 1}{x^2 + x + 1} + C$$

9. $\int \frac{1}{x^4 - 1} dx$

Let $\frac{1}{x^4 - 1} dx = \frac{1}{(x - 1)(x + 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{x + D}{x^2 + 1}$

Put

$x = 1 \qquad 1 = A.2.2 \qquad \therefore A = \frac{1}{4}$

$x = -1 \qquad 1 = B.(-2).2 \qquad \therefore B = \frac{-1}{4}$

$x = 0 \qquad 1 = A + B(-1) + (-1) D$

$1 = \frac{1}{4} + \frac{1}{4} - D$

$\therefore \quad D = -\frac{1}{2}$

Equating the coefficients of x^3

$0 = A + B + C$

or, $\frac{1}{4} - \frac{1}{4} + C$

$\therefore \quad C = 0$

$\therefore \quad \frac{1}{x^4 - 1} = \frac{1}{4}\frac{1}{x - 1} - \frac{1}{4}\frac{1}{x + 1} - \frac{1}{2}\frac{1}{x^2 + 1}$

$$\int \frac{1}{x^4-1}\,dx = \int \left\{\frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x^2+1}\right\} dx$$

$$= \frac{1}{4}\log(x-1) - \frac{1}{4}\log(x+1) - \frac{1}{2}\text{Tan}^{-1} x + c$$

$$= \frac{1}{4}\log\frac{x-1}{x+1} - \frac{1}{2}\tan^{-1} x + c$$

10. $\displaystyle\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$

$$\frac{x^2}{(x^2+a^2)(x^2+b^2)} = \frac{y}{(y+a^2)(y+b^2)} = \frac{A}{y+a^2} + \frac{B}{y+b^2}$$

where $y = x^2$

$y = A(y + b^2) + B(y + a^2)$

When $y = -a^2$

$-a^2 = A(-a^2 + b^2)$

$\therefore \quad A = \dfrac{a^2}{a^2-b^2}$

When $y = -b^2$

$-b^2 = B(-b^2 + a^2)$

$\therefore \quad B = -\dfrac{b^2}{a^2-b^2}$

$$\frac{y}{(y+a^2)(y+b^2)} = \frac{a^2}{a^2-b^2}\frac{1}{y+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{y+b^2}$$

or, $\displaystyle \frac{x^2}{(x^2+a^2)(x^2+b^2)} = \frac{a^2}{a^2-b^2}\frac{1}{x^2+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{x^2+b^2}$

or, $\displaystyle \int \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \int \left[\frac{-a^2}{a^2-b^2}\frac{1}{x^2+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{x^2+b^2}\right] dx$

$$= \frac{a^2}{a^2-b^2}\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{b^2}{a^2-b^2}\frac{1}{b}\text{Tan}^{-1}\frac{x}{b} + c$$

$$= \frac{a}{a^2-b^2}\tan^{-1}\frac{x}{a} - \frac{b}{a^2-b^2}\tan^{-1}\frac{x}{b} + C$$

11. $\displaystyle\int \frac{x^2+4}{x^4+16}\,dx$

$$= \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}\,dx = \int \frac{1+\frac{4}{x^2}}{\left(x-\frac{4}{x^2}\right)^2+8}$$

Put $x - \dfrac{4}{x} = y$

$\Rightarrow 1 + \dfrac{4}{x^2}\,dx = dy$

$$I = \int \frac{dy}{y^2+(2\sqrt{2})^2} = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{y}{2\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x^2-4}{2\sqrt{2}} + C$$

12. $\displaystyle\int \frac{x^3}{(x-a)(x-b)(x-c)}$

Let $\dfrac{x^3}{(x-a)(x-b)(x-c)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{c}{(x-c)}$

$\Rightarrow \quad x^3 = A(x-b)(x-c) + B(x-a)(x-c) + c(x-a)(x-b)$ ... (i)

Putting x = a, x = b, x = c turn by turn, we get,

$A = \dfrac{a^3}{(a-b)(a-c)}, \; B = \dfrac{b^3}{(b-a)(b-c)}, \; c = \dfrac{c^3}{(c-a)(c-b)}$

Now, $\displaystyle\int \frac{x^3}{(x-a)(x-b)(x-c)}$

$$= \frac{a^3}{(a-b)(a-c)} \int \frac{1}{x-a}\,dx + \frac{b^3}{(b-a)(b-c)} \int \frac{1}{x-b}\,dx + \frac{c^3}{(c-a)(c-a)} \int \frac{1}{x-c}\,dx$$

$$= \frac{a^3}{(a-b)(a-c)} \ln(x-a) + \frac{b^3}{(b-a)(b-c)} \ln(x-b) + \frac{c^3}{(c-a)(c-b)} \ln(x-c) + c$$

13. $\displaystyle\int \frac{x^3\,dx}{2x^4 - x^2 - 10}$

Put $x^2 = y$

$2xdx = dy$

$\therefore \quad dx = \dfrac{dy}{2x}$

Now, $\displaystyle\int \frac{x^3dx}{2x^4 - x^2 - 10} = \int \frac{x^2\,xdx}{2(x^2)^2 = x^2 - 10} = \int \frac{y \,.\, x}{2y^2 - y - 10} \cdot \frac{dy}{2x}$

$$= \frac{1}{2} \int \frac{y\,dy}{2y^2 - 5y + 4y - 10} = \frac{1}{2} \int \frac{y\,dy}{y(2y-5) + 2(2y-5)}$$

$$= \frac{1}{2} \int \frac{y\,dy}{(y+2)(2y-5)} \qquad \text{... (i)}$$

Let $\dfrac{y}{(y+2)(2y-5)} = \dfrac{A}{(y+2)} + \dfrac{B}{(2y-5)}$

$\Rightarrow \quad y = A(2y-5) + B(y+2)$ ... (ii)

Putting y = –2 in (ii), we get,

$-2 = A(-4-5) + B \times 0 \Rightarrow A = \dfrac{2}{9}$

Again, putting $y = \dfrac{5}{2}$ in (ii), we get

$\dfrac{5}{2} = A \times 0 + B\left(\dfrac{5}{2} + 2\right) \Rightarrow B = \dfrac{5}{9}$

$\therefore \quad \dfrac{y}{(y+2)(2y-5)} = \dfrac{2}{9(y+2)} + \dfrac{5}{9(2y-5)}$

from (i) $\displaystyle\frac{1}{2} \int \frac{y\,dy}{(y+2)(2y-5)} = \frac{1}{9} \int \frac{1}{y+2}\,dy + \frac{5}{18} \int \frac{1}{2y-5}\,dy$

$$= \frac{1}{9} \ln(y+2) + \frac{5}{36} \ln(2y-5) + c$$

$$= \frac{5}{36} \ln(2x^2 - 5) + \frac{1}{9} \ln(x^2 + 2) + c$$

14. $\displaystyle\int \frac{dx}{(x-1)^2 (x-3)^2}$

Put $x - 1 = z(x-3)$

$\Rightarrow \quad x - zx = 1 - 3z \Rightarrow x = \dfrac{1-3z}{1-z}$

$$dx = \frac{(1-z) - 3 - (1-3z)\times(-1)}{(1-z)^2}\,dz$$

$$\Rightarrow dx = \frac{-3 + 3z + 1 - 3z}{(1-z)^2}\,dz = \frac{-2}{(1-z)^2}\,dz$$

Here, $$\frac{1}{(x-1)^2\,(x-3)^2} = \frac{1}{z^2(x-3)^2\,(x-3)^2} = \frac{1}{z^2\left[\frac{1-3z}{1-z} - 3\right]^4}$$

$$= \frac{1}{z^2\left[\frac{1-3z-3+3z}{1-z}\right]^4} = \frac{(1-z)^4}{-2z^2}$$

Now, $$\int \frac{dx}{(x-1)^2\,(x-3)^2} = \int \frac{-2}{(1-z)^2} \times \frac{(1-z)^4}{-2z^2}$$

$$= \int \frac{(1-z)^2}{z^2}\,dz = \int \frac{1-2z+z^2}{z^2}\,dz$$

$$= \int \left(\frac{1}{z^2} - \frac{2}{z} + 1\right) dz = -\frac{1}{z} - 2\ln z + z + c$$

$$= -\left(\frac{x-3}{x-1}\right) - 2\ln\left(\frac{x-1}{x-3}\right) + \left(\frac{x-1}{x-3}\right) + c$$

15. $I = \int \frac{dx}{(x-1)^2\,(x-4)^3}$

Put $x - 1 = z(x-3)$

$\therefore \quad x = \frac{3z-1}{z-1}$

or, $dx = \frac{3(z-1)-(3z-1)}{(z-1)^2}\,dz = \frac{3z-3-3z+1}{(z-1)^2}dz = \frac{-2}{(z-1)^2}\,dz$

Also, $$\frac{1}{(x-1)^2\,(x-3)^3} = \frac{1}{z^2(x-3)^5} = \frac{1}{z^2\left(\frac{3z-1}{z-1} - 3\right)^5} = \frac{(z-1)^5}{32z^2}$$

So, $$I = \int \frac{(z-1)^5}{32z^2}\cdot\frac{-2}{(z-1)^2}\,dz$$

$$= \frac{-1}{16}\int \frac{(z-1)^3}{z^2}\,dz$$

$$= \frac{-1}{16}\int \frac{z^3 - 3z^2 + 3z - 1}{z^2}dz$$

$$= \frac{-1}{16}\int \left(z - 3 + \frac{3}{z} - \frac{1}{z^2}\right) dz$$

$$= \frac{-1}{16}\left(\frac{z^2}{2} - 3z + 3\ln z + \frac{1}{z}\right) + C$$

$$= \frac{-1}{32}z^2 + \frac{3}{16}z - \frac{3}{16}\ln z - \frac{1}{16}z + C$$

$$= \frac{-1}{32}\left(\frac{x-1}{x-3}\right)^2 + \frac{3}{16}\,\frac{x-1}{x-3} - \frac{3}{16}\ln\left|\frac{x-1}{x-3}\right| - \frac{1}{16}\left(\frac{x-1}{(x-3)}\right) + C$$

Chapter 17: Differential Equations

Exercise 17.1

1. Solution

a. Given, $\frac{dy}{dx} = 4x$

Here, $\frac{dy}{dx}$ is the first order derivative, so its order is 1

Here, the power of $\frac{dy}{dx}$ is 1. So its degree is 1

b. Given, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$

Here, $\frac{d^2y}{dx^2}$ is the second order derivative, so it's order is 2

Here, the power of y is 1. So, it's degree is 1.

c. Given, $\frac{d^2y}{dx^2} = xe^x$

Here, $\frac{d^2y}{dx^2}$ is the second order derivative. So, its' order is 2

Here, the power of x is 1, so its degree is 1.

d. Given, $x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2$

Here, $\frac{dy}{dx}$ is the first order derivative, so it's order is 1.

Here, the power of y^2 is 2. So it's degree is 2.

e. $\frac{dy}{dx} = \sqrt{\frac{1-x^2}{1-y^2}}$

Here, $\frac{dy}{dx}$ is the first order derivative. So its order is 1.

Here, the power of $\frac{dy}{dx}$ is 2. So, its degree is 2.

f. Given, $\frac{dy}{dx} + 3y\left(\frac{d^2y}{dx^2}\right)^3 = 0$

Here, $\frac{d^2y}{dx^2}$ is the second order derivative. So, its order is 2.

Here, the power of $\frac{d^2y}{dx^2}$ is 3. So, its' degree is 3.

Exercise 17.2

1.

a. $\frac{dy}{dx} = \frac{x+4}{y+2}$

$(y + 2)\, dy = (x + 4)\, dx$

Integrating both sides

$\int (y + 2)\, dy = \int (x + 4)\, dx$

or, $\frac{y^2}{2} + 2y = \frac{x^2}{2} + 4x + c$

or, $y^2 + 4y = x^2 + 8x + c$

b. $x^2dx + y^2dy = 0$

Integrating

$\int x^2dx + \int y^2dy = \int 0$

$\frac{x^3}{3} + \frac{y^3}{3} = \frac{c}{3}$

$\therefore \quad x^3 + y^3 = c$

c. $y\frac{dy}{dx} = \cos x$

$ydy = \cos x\, dx$

Integrating both sides

$\int ydy = \int \cos x\, dx$

or, $\frac{y^2}{2} = \sin x + c'$ where c' is constant

$\therefore \quad y^2 = 2\sin x + 2c'$

$\therefore \quad y^2 = 2\sin x + c$

d. $\frac{dy}{dx} = e^{x+y}$

$\frac{dy}{dx} = e^x . e^y$

$e^{-y}\, dy = e^x\, dx$

Integrating

$\int e^{-y}\, dy = \int e^x\, dx$

$-e^{-y} = e^x + c$

or, $\frac{-1}{e^y} = c^x + c$

or, $-1 = e^y\,(e^x + c)$

e. $e^{x-y}\, dx + e^{y-x}\, dy = 0$

or, $e^x .e^{-y}\, dx + e^y . e^{-x}\, dy = 0$

or, $\frac{e^x}{d^y}\, dx + \frac{e^y}{e^x}\, dy = 0$

$e^{2x}\, dx + e^{2y}\, dy = 0$

Integrating

$\int e^{2x}\, dx + \int e^{2y}\, dy = \int 0$

$\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{c}{2}$

$\therefore \quad e^{2x} + e^{2y} = c$

f. $\frac{dy}{dx} = e^{x-y} + x^3e^{-y}$

$\frac{dy}{dx} = \frac{(e^x + x^3)}{e^y}$

$e^y\, dy = e^xdx + x^3dx$

Integrating

$$\int e^y dy = \int e^x dx + \int x^3 dx$$

$$e^y = e^x + \frac{x^4}{4} + c$$

2. Solution:

a. $(x + 2)\frac{dy}{dx} = y + 2$

or, $\left(\frac{1}{y+2}\right) dy = \frac{dy}{x+2}$

Integrating both sides

$$\int \frac{1}{x+2} dy = \int \frac{1}{x+2} dx$$

$\ln (y + 2) = \ln(x + 2) + \ln c$

$\ln (y + 2) = \ln\{c(x+2)\}$

$\therefore \quad y + 2 = c(x + 2)$

b. $x\frac{dy}{dx} + y - 1 = 0$

$x\frac{dy}{dx} = (1 - y)$

or, $\frac{1}{1-y} dy = \frac{dx}{x}$

Integrating

$-\ln (l - y) = \ln x + \ln c$

$\ln(1 - y)^{-1} = \ln cx$

$\therefore \quad cx = \frac{1}{1-y}$

$x(1 - y) = \frac{1}{c}$

$\therefore \quad x(1 - y) = c$

c. $\text{Cos}x \,.\, \cos y \frac{dy}{dx} = -\sin x \,.\, \sin y$

$\frac{\cos y}{\sin y} dy = -\frac{\sin x}{\cos x} dx$

$\cot y \, dy = -\tan x \, dx$

Integrating both sides

$$\int \cot y \, dy = -\int \tan x \, dx$$

$\log \sin y = \log \cos x + \log c$

$\sin y = c \cos x$

d. $\text{Sec}^2 x \,.\, \tan y dx \;\; \sec^2 y \tan x \, dy = 0$

or, $\frac{\sec^2 x}{\tan x} = -\frac{\sec^2 y}{\tan y} dy$

Integrating both sides we get

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\text{tax}y} dy = \int 0$$

$\ln(\tan x) + \ln(\tan y) = \ln c$

$\ln(\tan x \,.\, \tan y) = \ln c$

$\therefore \quad \tan x \,.\, \tan y = c$

e. $(e^y + 1) \cos 2dx + e^y \,.\, \sin x \, dy = 0$

$$\frac{\cos x\,dx}{\sin x} + \frac{e^y\,dx}{e^y + 1} = 0$$

Integrating both sides

$$\int \frac{\cos x}{\sin x}\,dx + \int \frac{e^y}{1+ey}\,dy = \int 0$$

$\ln \sin x + \ln (e^y + 1) = \ln c$

$\ln \sin x(1 + e^y) = \ln c$

$\therefore \quad \sin x\,(1 + e^y) = c$

f. $(xy^2 + x)\,dx + (x^2y + y)\,dy = 0$

or, $x(1 + y^2)\,dx + y(1 + x^2)\,dy = 0$

or, $\frac{x\,dx}{1+x^2} + \frac{y}{1+y^2}\,dy = 0$

or, $\left(\frac{2x}{1+x^2}\right) dx + \left(\frac{2y}{1+y^2}\right) dy = 0$

Integrating both sides

$$\int \frac{dx}{1+x^2} + \int \frac{2y}{1+y^2}\,dy = \int 0$$

$\ln(1 + x^2) + \ln(1 + y^2) = \ln c$

$\therefore \quad \ln(1 + x^2)\,(1 + y^2) = \ln c$

$\therefore \quad (1 + x^2)\,(1 + y^2) = c$

$\ln(1+y) = x - \frac{x^2}{2} + c$

g. $\sqrt{1+x^2}\,dy + \sqrt{1 + y^2}\,dx = 0$

or, $\frac{dy}{\sqrt{1 + y^2}} + \frac{dx}{\sqrt{1 + x^2}} = 0$

Integrating

$$\int \frac{1}{\sqrt{1 + y^2}}\,dy + \int \frac{1}{\sqrt{1 + x^2}}\,dx = \int 0$$

or, $\ln (y + \sqrt{1+y^2}) + \ln(x + \sqrt{1+x^2}) = \ln c$

or, $\ln \{(x + \sqrt{1+x^2})\,(y + \sqrt{1+y^2})\} = \ln c$

or, $(x + \sqrt{1+x^2})\,(y + \sqrt{1+y^2}) = c$

h. $x\sqrt{1 - y^2}\,dx + y\sqrt{1 - x^2}\,dy = 0$

$\frac{x}{\sqrt{1 - x^2}}\,dx + \frac{y}{\sqrt{1 - y^2}}\,dy = 0$

Put $l–x^1 = u$ and $l–y^2 = v$

Then, $–2xcdx = du$, $–2ydy = dv$

So, $\frac{(-2x)dx}{\sqrt{1 - x^2}} + \frac{(-2y)\,dy}{\sqrt{1 - y^2}} = 0$

$\frac{dy}{\sqrt{u}} + \frac{dv}{\sqrt{v}} = 0$

$u^{-1/2}\,du + v^{-1/2}\,dv = 0$

Integrating

$2u^{1/2} + 2v^{1/2} = 2c$

$\therefore \quad \sqrt{u} + \sqrt{v} = c$

$\therefore \quad \sqrt{1 - x^2} + \sqrt{1 - y^2} = c$

i. $(1 - x^2)dy + xydx = x^2ydx$

or, $(1 - x^2)dy = xy\,(1 + y)dx$

or, $\frac{dy}{y(1+y)} = \frac{x\,dx}{1-x^2}$

or, $\left(\frac{1}{y} - \frac{1}{y+1}\right) dy = \frac{1}{-2}\left(\frac{-2x}{1-x^2}\right) dx$

Integrating

$\ln y - \ln(y+1) = -\frac{1}{2}\ln(1-x^2) + \ln c$

$\ln\left(\frac{y}{1+y}\right) = \ln(1-x^2)^{-1/2} + \ln c \Rightarrow \frac{y}{1+y} = c(1-x^2)^{-1/2}$

3. Solution:

a. $\sec^2 y\,(1+x^2)dy + 2x.\tan y\,dx = 0$

or, $\frac{\sec^2 y}{\tan y}\,dy + \frac{2x}{1+x^2}\,dy = 0$

Integrating, we get,

$\ln(\tan y) + \ln(1+x^2) = \ln c$

$(1+x^2)\tan y = c$

When x = 1, there $y = \frac{\pi}{4}$

$\therefore$ $(1+1)\tan\frac{\pi}{4} = c$

$2.1 = c$

$\therefore$ $c = 2$

b. $\cos y\,dy + \cos x \sin y\,dx = 0$

$\cos y\,dy = -\cos x.\sin y\,dx$

$\frac{\cos y}{\sin y}\,dy = -\cos x\,dx$

Integrating, we get

$\ln \sin y = -\sin x + c$

$\sin x + \ln \sin y = c$

When $x = \frac{\pi}{2}$ then $y = \frac{\pi}{2}$

then c = 1

$\therefore$ $\sin x + \ln \sin y = 1$

Exercise 17.3

1. Solution:

a. $x\frac{dy}{dx} = y + x$

$\frac{dy}{dx} = \frac{y}{x} + 1$ ... ... (i) It is a homogenous

Thus, differential equation

Put y = vx

Then $\frac{dy}{dx} = v + x\frac{dy}{dx}$

(i) becomes

$v + x\frac{dv}{dx} = v + 1$

$\therefore$ $x\frac{dv}{dx} = 1$

$dv = \frac{dx}{x}$

Integrating both sides

s $\int dv = \int \frac{1}{x} dx$

$v = \ln x + c$

$\frac{y}{x} = \ln x + c$

b. $\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$

It is homogeneous differential equation.

Put $y = vx$

$\frac{dy}{dx} = v + x.\frac{dv}{dx}$

Then $v + x.\frac{dv}{dx} = v^2 + v$

$x\frac{dv}{dx} = v^2$

$v^{-2}\, dv = \frac{1}{x} dx$

Integrating both sides

$\int v^{-2} dv = \int \frac{1}{x} dx$

$-v^{-1} = \ln(cx)$

$-\frac{1}{v} \ln cx$

$-\frac{x}{y} = \ln(cx)$

c. $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ ... ... (i)

It is homogenous differential equation,

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(i) becomes, $v + x\frac{dv}{dx} = v + \tan v$

$x\frac{dv}{dx} = \tan v$

$\cot v\, dv = \frac{1}{x} dx$

Integrating $\int \cot v\, dv = \int \frac{1}{x} dx$

$\log \sin v = \log cx$

$\sin\left(\frac{y}{x}\right) = cx$

d. $x\left(\frac{dy}{dx} + \tan\frac{y}{x}\right) = y$

or, $\frac{dy}{dx} + \tan\frac{y}{x} = \frac{y}{x}$ ... ... (i)

Put $y = vx$ then $\frac{dy}{dx} = v + x \,.\, \frac{dv}{dx}$

$v + x\frac{dv}{dx} + \tan v = v$

$x\dfrac{dv}{dx} = -\tan v$

$-\cot v\, dv = \dfrac{dx}{x}$

Integrating

$\int -\cot v\, dv = \int \dfrac{1}{x}\, dx$

$-\log \sin v = \log cx$

$\sin^{-1} v = cx$

$\sin^{-1}\left(\dfrac{y}{x}\right) = cx$

$\dfrac{1}{c} = x \sin\left(\dfrac{y}{x}\right)$

$\therefore \quad x \sin\left(\dfrac{y}{x}\right) = c$

e. $\dfrac{dy}{dx} - \dfrac{y}{x} - \sin\dfrac{y}{x} = 0$ ... ... (i)

Put $y = vx$ then $\dfrac{dy}{dx}\, v + x\dfrac{dv}{dx}$

(i) becomes,

$v + x\,.\,\dfrac{dv}{dx} - v - \sin v = 0$

$x\dfrac{dv}{dx} = \sin v$

$\dfrac{dv}{\sin v} = \dfrac{dx}{x}$

$\text{cosec}\, v\, dv = \dfrac{1}{x}\, dx$

Integrating

$\int \text{cosec}\, v dv = \int \dfrac{1}{x}\, dx$

$\log (\text{cosec}\, v - \cot v) = \log x + \log c$

$\text{cosec}\, v - \cot v = cx$

$\text{cosec}\left(\dfrac{y}{x}\right) = \cot\left(\dfrac{y}{x}\right) = cx$

f. $\dfrac{dy}{dx} = \dfrac{y}{x} + \cos^2\left(\dfrac{y}{x}\right)$

Put $y = vx$

$\dfrac{dy}{dx} = v + x\,.\,\dfrac{dv}{dx}$

$v + x\,.\,\dfrac{dv}{dx} = v + \cos^2 v$

$x\dfrac{dv}{dx} = \cos^2 v$

$\sec^2 v\, dv = \dfrac{1}{x}\, dx$

Integrating

$\int \sec^2 v\, dx = \int \dfrac{1}{x}\, dx$

$\tan v = \ln(cx)$

$\tan\left(\frac{y}{x}\right) = \ln(cx)$

$\tan\left(\frac{y}{x}\right) = \ln(cx)$

2. Solution:

a. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$y^2 = (xy - x^2)\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{y^2}{xy - x^2}$ ... ... (i)

Put $y = vx$

Then, $v + x\frac{dv}{dx} = \frac{v^2x^2}{x^2v - x^2}$

$v + x\frac{dv}{dx} = \frac{v^2}{v-1}$

$x\frac{dv}{dx} = \frac{v^2}{v-1} - v$

$x\frac{dv}{dx} = \frac{v^2 - v^2 + v}{v-1}$

$x\frac{dv}{dx} = \frac{v}{v-1}$

$\frac{(v-1)}{v}dv = \frac{1}{x}dx$

Integrating

$v - \ln v = \ln x + \ln c$

$v = \ln(cxv)$

$\frac{y}{x} = \ln(cy)$

b. $x^2ydx - (x^3 + y^3)dy = c$

$\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$ ... ... (i)

Put $y = vx$

$\frac{dy}{dx} = v + x.\frac{dv}{dx}$

(i) becomes

$v + x.\frac{dy}{dx} = \frac{v}{1 + v^3}$

$x\frac{dv}{dx} = \frac{v - v - v^4}{1 + v^3}$

$x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$

$\frac{(1+v^3)}{v^4}dv = -\frac{dx}{x}$

$v^{-4}dv + \frac{1}{v}dv = -\frac{1}{x}dx$

Integrating

$\frac{v^{-3}}{-3} + \ln v = -\ln x - \ln c$

$\frac{1}{3v^3} = \ln(cxv)$

$\frac{x^3}{3y^3} = \ln(cy)$

c. $(x^2 + y^2)\,dx - 2xy\,dy = 0$

$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ ... ... (i)

$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$

$x\frac{dv}{dx} = \frac{1 + v^2 - 2v^2}{2v}$

$x\frac{dv}{dx} = \frac{1 - v^2}{2v}$

$\left(\frac{2v}{1-v^2}\right) dv = \frac{dx}{x}$

$-\ln(1-v^2) = \ln x + \ln c$

$\ln(1-v^2)^{-1} = \ln(cx)$

$(1 - v^2)^{-1} = cx$

$\frac{1}{1 - v^2} = cx$

$\frac{x^2}{x^2 - y^2} = cx$

$\frac{x}{x^2 - y^2} = c$

$\therefore\ x^2 - y^2 = \frac{x}{c}$

$\therefore\ x^2 - y^2 = cx$

d. $(x + y)\,dx + (y - x)\,dy = 0$

$(y - x)\,dy = -(x + y)\,dx$

$\therefore\ \frac{dy}{dx} = -\frac{(x + y)}{y - x}$

$\therefore\ \frac{dy}{dx} = \frac{x + y}{x - y}$ ... ... (i)

Put $y = vx$, then $\frac{dy}{dx} = v + x.\frac{dv}{dx}$

Then (i) reduces to

$v + x.\frac{dv}{dx} = \frac{x + vx}{x - vx}$

$v + x\frac{dv}{dx} = \frac{1 + v}{1 - v}$

or, $x\frac{dv}{dx} = \frac{1 + v}{1 - v} - v$

$x\frac{dv}{dx} = \frac{1 + v - v + v^2}{1 - v}$

$x\frac{dv}{dx} = \frac{1 + v^2}{1-v}$

$\left(\frac{1 - v}{1 + v^2}\right) dv = \frac{dx}{x}$

$\frac{1}{1+v^2}dv - \frac{1}{2}\left(\frac{2v}{1 + v^2}\right) dv = \frac{1}{x}dx$

Integrating

$\tan^{-1} v = \frac{1}{2}\ln(1 + v^2) = \ln x + c$

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln x + \ln\sqrt{1 + v^2} + c$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln \sqrt{x^2 + y^2} + c$$

e. $(x + y)^2\, dx = xy\, dy$

$$\frac{dy}{dx} = \frac{(x + y)^2}{xy}$$

$$v + x\frac{dv}{dx} = \frac{(x + vx)^2}{x^2 v}$$

$$v + x\frac{dv}{dx} = \frac{(1 + v2)^2}{v}$$

$$x\frac{dv}{dx} = \frac{(1 + v)^2 - v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1 + 2v}{v}$$

$$\frac{v}{1+2v}\, dv = \frac{dx}{x}$$

$$\frac{1}{2}\frac{(1 + 2v - 1)}{1 + 2v}\, dv = \frac{1}{x}\, dx$$

$$\frac{1}{2}\left(1 - \frac{1}{1+2v}\right) dv = \frac{1}{x}\, dx$$

$$\frac{1}{2}\, dv - \frac{1}{4}\left(\frac{2}{1+2v}\right) dv = \frac{1}{x}\, dx$$

Integrating

$$\frac{1}{2}v - \frac{1}{4}\log(1 + 2v) = \log(cx)$$

$$\frac{1}{2}\left(\frac{y}{x}\right) - \frac{1}{4}\log\left(1 + \frac{2y}{x}\right) = \log(cx)$$

f. $x^2\frac{dy}{dx} = \frac{y(x + y)}{2}$

$$\frac{dy}{dx} = \frac{y(x + y)}{2x^2} \quad \dots\dots\dots \text{(i)}$$

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(i) becomes,

or, $v + x \cdot \frac{dv}{dx} = \frac{vx(x + vx)}{2x^2}$

or, $v + x \cdot \frac{dv}{dx} = \frac{v(1 + v)}{2}$

or, $x\frac{dv}{dx} = \frac{v + v^2}{2} - v$

or, $x\frac{dv}{dx} = \frac{v + v^2 - 2v}{2}$

or, $x\frac{dv}{dx} = \frac{v^2 - v}{2}$

$$\frac{1}{v(v - 1)}\, dv = \frac{dx}{2x}$$

$$\left(\frac{1}{v-1} - \frac{1}{v}\right) dv = \frac{1}{2}\frac{1}{x}\, dx$$

Integrating

$$\ln(v-1) - \ln(v) = \frac{1}{2}\ln x + \ln c$$

$$\ln\left(\frac{v-1}{v}\right) \ \ln(c\sqrt{x})$$

$$\therefore \quad \frac{v-1}{v} = c\sqrt{x}$$

$$1 - \frac{x}{y} = c\sqrt{x}$$

$$y - x = cy\sqrt{x}$$

3. Solution:

a. $\frac{dy}{dx} = \frac{y+1}{x+y+1}$

Put $y + 1 = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\,.\,\frac{dv}{dx} = \frac{vx}{x+vx}$$

$$v + x\frac{dv}{dx} = \frac{v}{1+v}$$

$$x\frac{dv}{dx} = \frac{v}{1+v} - v$$

$$x\frac{dv}{dx} = \frac{v - v - v^2}{1+v}$$

$$\frac{(1+v)}{v^2}\,dv = -\frac{dx}{x}$$

$$\left(v^{-2} + \frac{1}{v}\right) dv = -\frac{1}{x}\,dx$$

Integrating

$$-v^{-1} + \ln v = -\ln x - \ln c$$

$$\ln(cvx) = \frac{1}{v}$$

$$\ln(cx\,(y+1)) = \frac{x}{y+1}$$

b. $\frac{dy}{dx} = \frac{y+x+1}{x+1}$

Put $y = v(x+1)$

$$\frac{dy}{dx} = v + (x+1)\frac{dv}{dx}$$

So, $v + (x+1)\frac{dv}{dx} = v + 1$

$$(x+1)\frac{dv}{dx} = 1$$

$$dv = \frac{dx}{x+1}$$

Integrating

$$\int dv = \int \frac{1}{x+1}\,dx$$

$$v = \ln(x+1) + c$$

$$\frac{y}{x+1} = \ln(x+1) + c$$

$$\therefore \quad y = (x+1)\{\ln(x+1) + c\}$$

Exercise 17.4

1. Solution:

a. $xdx - ydy = 0$

Integrating both sides

$$\int xdx - \int ydy = \int 0$$

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{c}{2}$$

$$x^2 - y^2 = c$$

b. $xdy - ydx = 0$

$$\frac{xdy - ydx}{x^2} = 0$$

$$d\left(\frac{y}{x}\right) = 0$$

Integrating

$$\int d\left(\frac{y}{x}\right) = \int 0$$

$$\frac{y}{x} = c$$

$\therefore \quad y = cx$

c. $(x + y^2)dx = 2xydy$

$$\frac{xdy}{x^2} = \frac{2xy\,dy - y^2dx}{x^2}$$

$$\frac{1}{x}\,dx = \frac{2xy\,dy - y^2dx}{x^2}$$

$$\frac{1}{x}\,dx = d\left(\frac{y^2}{x}\right)$$

Integrating we get

$$\ln x = \frac{y^2}{x} + c$$

$$x\ln x = y^2 + cx$$

d. $ydx - \frac{x}{2}\,dy = 0$

$$ydx = \frac{x}{2}\,dy$$

or, $\frac{dx}{x} = \frac{dy}{y}$

$$\int \frac{dy}{y} = 2\int \frac{1}{x}\,dx$$

$$\ln y = 2\ln x + \ln c$$

$$\ln y = \ln x^2 c$$

$$y = cx^2$$

e. $\frac{1}{x+1}\,dx + \frac{1}{y+1}\,dy = 0$

Integrating

$$\int \frac{1}{x+1}\,dx \neq \int \frac{1}{y+1}\,dy = \int 0$$

or, $\ln(x + 1) + \ln(y + 1) = \ln c$

or, $\ln(x + 1)\,(y + 1) = \ln c$

$\therefore \quad (x + 1)\,(y + 1) = c$

f. $\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0$

Integrating

$\int \frac{dy}{1+x^2} + \int \frac{dy}{1+y^2} = \int 0$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}c$

or, $\tan^{-1}\left(\frac{x+y}{1+xy}\right) = \tan^{-1}c$

$\therefore \quad \frac{x+y}{1-xy} = c$

$\therefore \quad x + y = c(1 - xy)$

g. $\frac{x}{1+x^2}dx + \frac{y}{1+y^2}dy = 0$

or, $\frac{2x}{1+x^2}dx + \frac{2y}{1+y^2}dy = 0$

Integrating

$\ln(1 + x^2) + \ln(1 + y^2) = \ln c$

$\therefore \quad \ln(1 + x^2)(1 + y^2) = \ln c$

$\therefore \quad (1 + x^2)(1 + y^2) = c$

h. $(x + y)\,dy + (y - 2)\,dx = 0$

$xdy + ydy + ydx - xdx = 0$

$(xdy + ydx) + ydy - xdx = 0$

$d(xy) + ydy - xdx = 0$

Integrating

$\int d(xy) + \int ydy - \int xdx = \int 0$

$xy + \frac{y^2}{2} - \frac{x^2}{2} = \frac{c}{2}$

$2xy + y^2 - x^2 = c$

i. $2xy\,dx + (x^2 - y^2)\,dy = 0$

$2xydx + x^2dy - y^2dy = 0$

$d(x^2y) - y^2dy = 0$

Integrating, we get

$x^2y - \frac{y^3}{3} = c$

$3x^2y - y^3 = c$

j. $(x^2 + xy^2)\,dx + (x^2y + y^2)\,dy = 0$

or, $x^2dx + xy^2dx + x^2y\,dy + y^2dy$

or, $x^2dx + (xy^2dx + x^2ydy) + y^2dy$

or, $x^2dx + \frac{1}{2}d(x^2y^2) + y^2dy = 0$

Integrating

$\frac{x^3}{3} + \frac{1}{2}(x^2y^2) + \frac{y^3}{3} = c$

$2x^3 + 3x^2y^2 + 2y^3 = c$

2. Solution:

a. $\cos x.\cos y\,dy = \sin x.\sin y\,dx = 0$

$\cos x.\cos y dy = \sin x.\sin y\,dx$

$\frac{\cos y}{\sin y}dy = \frac{\sin x}{\cos x}dx$

$\frac{\cos y}{\sin y}dy - \frac{\sin x}{\cos x}dx = 0$

Integrating

$$\int \frac{\cos y}{\sin y}\,dy - \int \frac{\sin x}{\cos x}\,dx = \int 0$$

$\ln(\sin y) + \ln(\cos x) = \ln c$

$\ln(\sin y.\cos x) = \ln c$

$\sin y.\cos x = c$

b. $\sin x \cos x\,dx - \sin y.\cos y\,dy = 0$

$\frac{1}{2}\sin 2x\,dx - \frac{1}{2}\sin 2y\,dy = 0$

or, $\sin 2x\,dx - \sin 2y dy = 0$

Integrating

$$\int \sin 2x\,dx - \int \sin 2y dy = \int 0$$

$\frac{-\cos 2x}{2} + \frac{\cos 2x}{2} = \frac{c}{2}$

$\therefore\quad \cos 2y - \cos 2x = c$

c. $\frac{dy}{dx} = \frac{1-\cos y}{1+\cos x}$

$\frac{dy}{dx} = \frac{2\sin^2 y/2}{2\cos^2 x/2}$

or, $\operatorname{cosec}^2 \frac{y}{2}\,dy = \sec^2 \frac{x}{2}\,dx$

Integrating

$$\int \operatorname{cosec}^2 \frac{y}{2}\,dy = \int \sec^2 \frac{x}{2}\,dx$$

or, $\frac{-\cot \frac{y}{2}}{\frac{1}{2}} = \frac{\tan \frac{x}{2}}{\frac{1}{2}} + \frac{c}{\frac{1}{2}}$

$\therefore\quad -\cot \frac{y}{2} = \tan \frac{x}{2}\, c$

d. $\frac{dy}{dx} = \frac{4x - y + 5}{x + 5y + 4}$

or, $4xdx - ydx + 5dx = xdy + 5ydy + 4dy$

or, $4xdx + 5dx = (xdy + ydx) + 5ydy + 4dy$

$4xdx + 5dx = d(xy) + 5ydy + 4dy$

Integrating both sides

$$\int 4xdx + 5\int dx = \int d(xy) + 5\int ydy + 4\int dy$$

$\frac{4x^2}{2} + 5x = xy + \frac{5y^2}{2} + 4y + c$

$4x^2 + 10x = 2xy + 5y^2 + 8y + c$

$4x^2 - 5y^2 + 10x - 8y - 2xy = c$

e. $\frac{dy}{dx} = \frac{x + y + 1}{2y - x + 2}$

or, $2ydy - xdy + 2dy = xdx + ydx + dy$

$2ydy + 2dy = xdx + ydx + xdy + dx$

$2ydy + 2dy = xdx + d(xy) + dx$

Integrating

$\int 2ydy + \int 2dy = \int xdx + \int d(xy) + \int dx$

$y^2 + 2y = \frac{x^2}{2} + xy + x + c$

$2y^2 + 4y = x^2 + 2xy + 2x + c$

$2y^2 - x^2 - 2xy + 4y - 2x = c$

f. $\frac{dy}{dx} = \frac{x - 3(y+1)}{y + 3(x+1)}$

or, $ydy + 3(x + 1)\,dy = xdx - 3)(y + 1)\,dx$

or, $ydy + 3xdy + 3dy = xdx - 3ydx - 3dx$

or, $ydy + 3(xdy + ydx) + 3dy = xdx - 3dy$

$ydy + 3d(xy) + 3dy = (x - 3)\,dx$

Integrating we get

$\frac{y^2}{2} + 3xy + 3y = \frac{x^2}{2} - 3x + c$

$\therefore\ y^2 + 6xy + 6y = x^2 - 6x + 6$

g. $(\sin x.\tan y - 1)\,dx = \cos x.\sec^2 y\,dy = 0$

$\sin x.\tan y\,dx - \cos x.\sec^2 y\,dy = dx$

$-d(\cos x.\tan y) = dx$

$d(\cos x.\tan y) + dx = 0$

Integrating both sides

$\int d(\cos x.\tan y) + \int dx = \int 0$

$\cos x.\tan y + x = c$

Exercise 17.5

1. Solution:

a. Given, differential equation is $\frac{dy}{dx} + y = 1$ ... ... (i)

It is a linear differentiate equation of the type $\frac{dy}{dx} + py = Q$

Here, $p = 1$, $Q = 1$

$\int pdx = \int 1dx = x$

Integrating factor (I.F.) = $e^{spdx} = e^x$

Integrating equation (i) both sides by I.F.

$\left[\frac{dy}{dx} + y\right] e^x = 1 \times e^x$

Comparing both sides, we get $\int d(y.e^x) = \int e^x\,dx$

$\therefore\ y.e^x = e^x + c$

$\therefore\ y = 1 + ce^{-x}$

b. Given,

$\frac{dy}{dx} + y = e^x$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + py = Q$, we get

$p = 1$, $Q = e^x$

$\int pdx = \int 1dx = x$

Integrating factor (I.F.) is given by $e^{\int pdx}$

I.F. = e^x

Multiplying (i) both sides by I.F., we get

$$\left(\frac{dy}{dx} + y\right) e^x = e^x . e^x$$

or, $d(y.e^x) = e^{2x}$

Integrating both sides

$$\int d(y.e^x) = \int e^{2x}\, dx$$

$$y.e^x = \frac{e^{2x}}{2} + c$$

$$\therefore \quad y = \frac{e^x}{2} + ce^{-x}$$

c. $\dfrac{dy}{dx} + 2y = \dfrac{1}{2}(x^2 - x)$ ... ... (i)

Here, p = 2 and Q = $\frac{1}{2}(x^2 - x)$

$\int pdx = 2x$

I.F. = $e^{\int pdx} = e^{2x}$

Multiplying (i) both sides by I.F.

$$\left(\frac{dy}{dx} + 2y\right) e^{2x} = \frac{1}{2}(x^2 - x)e^{2x}$$

$$d(y.e^{2x})\, e^{2x} = \frac{1}{2}(x^2 - x)\, e^{2x}$$

Integrating both sides

$$\int d(y.e^{2x}) = \frac{1}{2}\int (x^2 - x)\, e^{2x}$$

$$y.e^{2x} = \frac{1}{2}\left[(x{+}{-}\, x)\frac{e^{2x}}{2} - \int (2x - 1)\frac{e^{2x}}{2}\, dx\right]$$

$$y.e^{2x} = \frac{1}{2}(x^2 - x)\, e^{2x} - \frac{1}{4}\int (2x-1)\, e^{2x}dx$$

$$y.e^{2x} = \frac{1}{4}(x^2 - x)\, e^{2x} - \frac{1}{4}\left[(2x - 1)\frac{e^{2x}}{2} - \int \frac{2e^{2x}}{2}\, dx\right]$$

$$y.e^{2x} = \frac{1}{4}(x^2 - x)\, e^{2x} - \frac{1}{8}(2x - 1)\, e^{2x} + \frac{1}{8}e^{2x} + c$$

$$y.e^{2x} = \frac{1}{8}e^{2x}(2x^2 - 2x - 2x + 1 + 1) + c$$

$$\Rightarrow y.e^{2x} = \frac{1}{4}(x - 1)^2\, e^{2x} + c$$

d. $\text{Sec}3x\dfrac{dy}{dx} - 2y\sec 3x = 1$

$\therefore \quad \dfrac{dy}{dx} - 2y = \cos 3x$ ... ... (i) It is a linear differential equation of the form $\dfrac{dy}{dx} + py = Q$

Here, p = –2 and Q = cos3x

Now, $\int pdx = -\int 2dx = -2x$

I.F. = $e^{\int pdx} = e^{-2x}$

Multiplying (i) by I.F.

$\left(\frac{dy}{dx} - 2y\right) e^{-2x} = \cos3x \, . \, e^{-2x}$

or, $d(y.e^{-2x}) = co3x \, . \, e^{-2x}$

Integrating both sides,

$\int d(y.e^{-2x}) = \int \cos3x \, . \, e^{-2x} \, dx$

$y.e^{-2x} = \int \cos3x.e^{-2x} \, dx$ ... ... (ii)

Let $I = \int \cos3x.e^{-2x} \, dx$

or, $I = \cos3x \int e^{-2x} \, dx - \int \left\{\frac{d}{dx}(\cos3x) \, . \int e^{-2x} \, dx\right\} dx$

$= \frac{\cos3x. \, e^{-2x}}{-2} - \int \frac{-3}{-2} \sin3x \, . \, e^{-2x} \, dx)$

$I = -\frac{1}{2}\cos3x \, . \, e^{-2x} - \frac{3}{5}\int \sin3x.e^{-2x} \, dx$

or, $I = -\frac{1}{2}\cos3x \, . \, e^{-2x} - \frac{3}{2}\left[\sin3x \int e^{-2x} \, dx - \int\left\{\frac{d}{dx}(\sin3x) \, . \int e^{-3x} \, dx\right\}\right]$

$I = -\frac{1}{2}\cos3x \, . \, e^{-2x} - \frac{3}{2}\left[\sin3x \, . \, \frac{e^{-2x}}{-2} - \int \frac{3}{-2}\cos3x \, . \, e^{-2x} \, dx\right]$

$I = -\frac{1}{2}\cos3x \, . \, e^{-2x} + \frac{3}{4}\sin3x \, e^{-2x} - \frac{9}{4} I$

or, $\frac{13}{4} I = -\frac{1}{2}\cos3x \, . \, e^{-2x} + \frac{3}{4}\sin3x \, e^{-2x}$

$I = \frac{1}{13}(3\sin3x - 2\cos3x) \, e^{-2x}$

from (ii),

$y \, . \, e^{-2x} = \frac{1}{13}(3\sin3x - 2\cos3x) \, e^{-2x} + c$

$\therefore \quad y = \frac{1}{13}(3\sin3x - 2\cos3x) + ce^{2x}$

e. $\cos^2x \frac{dy}{dx} + y = \tan x \Rightarrow \frac{dy}{dx} + \sec^2x.y = \tan x \, . \, \sec^2x$ ... ... (i)

$p = \sec^2x$ and $Q = \sec^2x.\tan x$

$\int p dx = \int \sec^2x dx = \tan x$

$I.F. = e^{\int p dx} = e^{\tan x}$

Multiplying (i) by I.F. we get,

$d(y. \, e^{\tan x}) = \tan x. \, \sec^2x.e^{\tan x}$

Integrating

$\int d(Iy.etanx) = \int e^{\tan x} \tan x \, . \, \sec^2x \, dx$

$y.e^{\tan x} = \int e^u.udu$ where $u = \tan x$

$y.e^{\tan x} = u.e^u - e^u$

Integrating by parts

$y.e^{\tan x} = (u - 1)\, e^{u}$

$y.e^{\tan x} = (\tan x - 1)\, e^{\tan x} + c$

$\therefore \quad y = (\tan x - 1) + c.e^{-\tan x}$

f. $x \dfrac{dy}{dx} + 2y = x^2 \log x$

$\dfrac{dy}{dx} + \dfrac{2}{x} y = x \log x \text{ ... ... (i)}$

Comparing (i) with $\dfrac{dy}{dx} + py = Q$, we get

$p = \dfrac{2}{x}$ and $Q = x \log x$

$\int p dx = \int \dfrac{2}{x} dx = 2 \ln x = \log x^2$

I.F. $= e^{\int pdx} = e^{\log x^2} = x^2$

Multiplying (i) by I.F.

$\left(\dfrac{dy}{dx} + \dfrac{2}{x} y\right) x^2 = x^3 \log x$

$d(y.x^2) = x^3.\log x$

Integrating both sites, we get

$\int d(y.x^2) = \int x^3 \log x \, dx$

$y.x^2 = \log x. \int x^3 dx - \int \left\{\dfrac{d}{dx} (\log x) . \int x^3 \, dx\right\} dx$

$y.x^2 = \log x . \dfrac{x^4}{4} - \int \dfrac{1}{x} . \dfrac{x^4}{4} dx$

$y.x^2 = \dfrac{x^4}{4} \log x - \dfrac{1}{4} s \int x^3 \, dx$

$y.x^2 = \dfrac{x^4}{4} \log x - \dfrac{1}{16} x^4 + c$

$\therefore \quad y = \dfrac{x^2}{4} \log x - \dfrac{x^2}{16} + cx^{-2}$

g. $x \dfrac{dy}{dx} - x = 1 + y$

$x \dfrac{dy}{dx} - y = 1 + x$

$\dfrac{dy}{dx} - \dfrac{1}{x} y = \left(\dfrac{1+x}{x}\right) \text{ ... ... (i)}$

Comparing (i) with $\dfrac{dy}{dx} + py = Q$, we get

$p = -\dfrac{1}{x}$ and $Q = \dfrac{1 + x}{x}$

$\int p dx = \int -\dfrac{1}{x} dx = -\ln x = \ln x^{-1}$

I.F. $= e^{\int pdx} = e^{\int pdx} = e^{\ln x^{-1}} = x^{-1}$

Multiplying (i) both sides by I.F.

$\left(\dfrac{dy}{dx} - \dfrac{x}{1} y\right) x^{-1} = \left(\dfrac{1+x}{x}\right) x^{-1}$

$d(y.x^{-1}) = \frac{1+x}{x^2}$

$d(y.x^{-1}) = x^{-2} + \frac{1}{x}$

Integrating both sides,

$\int(y.x^{-1}) = \int x^{-2}\,dx + \int \frac{1}{x}\,dx$

$y.x^{-1} = -x^{-1} + \ln x + c$

$y = -1 + x\ln x + cx$

h. $\frac{dy}{dx} + \frac{y}{x} = e^x$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + p.y = Q$ we get

$p = \frac{1}{x}$ and $Q = e^x$

$\int pdx = \ln x$

$I.F. = e^{\int pdx} = e^{\ln x} = x$

Multiplying (i) both sides by I.F.

$\left(\frac{dy}{dx} + \frac{y}{x}\right) x = x.e^x$

$d(y.x) = x.e^x$

Integrating both sides,

$\int d(y.x) = \int x.e^x\,dx$

$y.x = xe^x - e^x + c$

$y = \frac{(x-1)}{x} e^x + \frac{c}{x}$

i. $(1 + x^2)\frac{dy}{dx} + y = \tan^{-1}x$

$\frac{dy}{dx} + \left(\frac{1}{1+x^2}\right) y = \frac{\tan^{-1}x}{(1+x^2)}$ ... . (i)

Comparing (i) with $\frac{dy}{dx} + py = Q$, we get

$p = \frac{1}{1+x^2}$, $Q = \frac{\tan^{-1}x}{1+x^2}$

$\int pdx = \int \frac{1}{1+x^2}\,dx = \tan^{-1}x$

$I.F. = e^{\int pdx} = e^{\tan^{-1}x}$

Multiplying (i) by I.F. both sides

We get,

$d(y.e^{\tan^{-1}x}) = \frac{\tan^{-1}x}{1+x^2} . e^{\tan^{-1}x}$

Integrating both sides

$\int d(y.\, e^{\tan^{-1}x}) = \int e^{\tan^{-1}x} \frac{\tan^{-1}x}{1+x^2}\,dx$

$y.e^{\tan^{-1}x} = \int e^{\tan^{-1}x} \frac{\tan^{-1}x}{1+x^2}\,dx$

Put $e^{\tan^{-1}x} = u$ in RHS

Then $\frac{1}{1+x^2}\,dx = du$

Then,

$y.e^{\tan^{-1}x} = \int e^u.u\,du$

$y.e^{\tan^{-1}x} = \int u.e^u\,dx$

$y.\,e^{\tan^{-1}x} = u\int e^u\,du - \int\left\{\frac{du}{du}.\int e^u\,du\right\}du$

$y.\,e^{\tan^{-1}x} = u.e^u - \int 1.e^u\,du$

$y.\,e^{\tan^{-1}x} = ue^u = e^u + c$

$y.e^{\tan^{-1}x} = \tan^{-1}x\,e^{\tan^{-1}x} - e^{\tan^{-1}x} + c$

$y = \tan^{-1}x - 1 + \frac{c}{e^{\tan^{-1}x}}$

2. Solution:

a. $(1-x^2)\frac{dy}{dx} - x.y = 1$

$\therefore\ \frac{dy}{dX} + \left(\frac{-x}{1-x^2}\right)y = \frac{1}{1-x^2}$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + py = Q$, we get

$p = \frac{-x}{1-x^2}$ and $Q = \frac{1}{1-x^2}$

$\int pdx = \frac{1}{2}\int\frac{-2x}{1-x^2}\,dx = \frac{1}{2}\ln(1-x^2) = \ln\sqrt{1-x^2}$

I.F. $= e^{\int pdx} = e^{\ln\sqrt{(1-x^2)}} = \sqrt{1-x^2}$

Multiplying (i) both sides by I.F.

$\left[\frac{dy}{dx} + \left(\frac{-x}{1-x^2}\right)y\right]\sqrt{1-x^2} = \frac{1}{(1-x^2)}\sqrt{1-x^2}$

$d(y\,.\sqrt{1-x^2}) = \frac{1}{\sqrt{1-x^2}}$

Integrating

$\int d(y.\sqrt{1-x^2}) = \int\frac{1}{\sqrt{1-x^2}}\,dx$

$y.\sqrt{1-x^2} = \sin^{-1}x + c$

b. $\sec x\,.\frac{dy}{dx} - y = \sin x$

$\frac{dy}{dx} - \cos x\,.\,y = \sin x\,.\cos x$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + p.y = Q$, we get

$p = -\cos x$ and $Q = \sin x.\cos x$

$\int pdx = -\int\cos x dx = -\sin x$

I.F. $= e^{\int pdx} = e^{-\sin x}$

Multiplying (i) both sides by I.F.

$d(y.e^{\sin x}) = e^{-\sin x} . \sin x . \cos x \, dx$

Integrating

$\int d(y.e^{-\sin x}) = \int e^{-\sin x} . \sin x.\cos x \, dx$

$y.e^{-\sin x} = \int e^{-u} . u \, du$ where $\sin x = u$

$y.e^{-\sin x} = -u \, e^{-u} - e^{-u} + c$

$y.e^{-\sin x} = (-1-u) \, e^{-u} = + c$

$y.e^{-\sin x} = (-1 - \sin x) \, e^{-\sin x} + c$

$\therefore \quad y = (1 - \sin x) + c \, e^{\sin x}$

$\therefore \quad y + 1 + \sin x = c \, e^{\sin x}$

c. $\cos^2 x \frac{dy}{dx} + y = 1$

$\frac{dy}{dx} + \sec^2 x.y = \sec^2 x$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + p.y = Q$, we get

$\sec^2 x$ and $Q = \sec^2 x$

$\int pdx = \int \sec^2 x \, dx = \tan x$

Integrating factor (I.F.) $= e^{\int pdx} = e^{\tan x}$

Multiplying (i) both sides by I.F.

$\left(\frac{dy}{dx} + \sec^2 x.y\right) e^{\tan x} = \sec^2 x \, e^{\tan x}$

$d(ye^{\tan x}) = e^{\tan x} \sec^2 x$

Integrating both sides,

$y.e^{\tan x} = \int e^{\tan x} . \sec^2 x \, dx$

$y.e^{\tan x} = e^{\tan x} + c$

$y = 1 + c \, e^{-\tan x}$

$\text{Sin}x \frac{dy}{dx} + y.\cos x = \sin 2x$

$\frac{dy}{dx} + \cot x \; y = 2\cos x$ ... ... (i)

d. Here, $p = \cot x$ and $q = 2\cos x$

$\int pdx = \int \cot x \, dx = \log \sin x$

I.F. $= e^{\int pdx} = e^{\log \sin x} = \sin x$

Multiplying (i) by I.F. we get

$d(y.\sin x) = 2\cos x . \sin x$

Integrating

$\int d(y.\sin x) = \int \sin 2x \, dx$

$y.\sin x = -\frac{\cos 2x}{2} + c$

$y = -\frac{1}{2} \cos 2x . \text{cosec}x + c.\text{cosec}x$

3. Solution:

a. $(1+x)\dfrac{dy}{dx} - xy = 1 - x$

$\dfrac{dy}{dx} - \dfrac{x}{1+x}.Y = \dfrac{1-X}{1+X}$ ... ... (i)

$p = -\dfrac{x}{1+x}$ and $Q = \dfrac{1-x}{1+x}$

$\int pdx = -\int \dfrac{x}{x+1}dx = -\int \dfrac{x+1-1}{x+1}dx = -\int 1dx + \int \dfrac{1}{x+1} = -x + \ln(x+1)$

I.F. = $e^{\int pdx} = e^{-x + \ln(x+1)} = e^{-x}(x+1)$

Multiplying (i) by by I.F. we get

$d(y.e^{x}(x+1)) = (1-x)e^{-x}$

Integrating $\int d(y(x+1)e^{-x}) = \int e^{-x}(1-x)dx$

or, $y(x+1)e^{-x} = \dfrac{(1-x)e^{-x}}{-1} + e^{-x} + c$

or, $ye^{-x}(x+1) = -(1-x)e^{-x} + e^{-x} + c$

or, $ye^{-x}(x+1) = e^{-x}(1-1+x) + c$

or, $y(x+1)e^{-x} = ex^{-x} + c$

$\therefore$ $y(x+1) = x + ce^{x}$

b. $\dfrac{dy}{dx} + \dfrac{4x}{1+x^2}.y = -\dfrac{1}{(x^2+1)^2}$ ... ... (i)

Here, $p = \dfrac{4x}{x^2+1}$ and $Q = \dfrac{-1}{(x^2+1)^2}$

$\int pdx = 2\int \dfrac{2x}{x^2+1}dx = 2\ln(x^2+1) = \ln(x^2+1)^2$

I.F. $= e^{\int pdx} = (x^2+1)^2$

Multiplying (i) by I.F. we get

$d\{y.(x^2+1)^2\} = -1$

Integrating $\int d(y(x^2+1)^2) = -\int 1dx$

$y(x^2+1)^2 = -x + c$

c. $(x^2-1)\dfrac{dy}{dx} + 2xy = \dfrac{2}{x^2-1}$

$\therefore$ $\dfrac{dy}{dx} + \dfrac{2x}{2^2-1}y = \dfrac{2}{(x^2-1)}$ ... ... (i)

Comparing (i) with $\dfrac{dy}{dx} + p.y = Q$, we get

$p = \dfrac{2x}{x^2-1}$ and $Q = \dfrac{2}{(x^2-1)^2}$

Now, $\int pdx = \int \dfrac{2x}{x^2-1}dx = \ln(x^2-1)$

I.F. $= e^{\int pdx} = e^{\ln(x^2-1)} = (x^2-1)$

Multiplying (i) both sides by I.F.

$\left(\dfrac{dy}{dx} + \dfrac{2x}{x^2-1}y\right)(x^2-1) = \dfrac{2}{(x^2-1)^2}(x^2-1)$

$d\{y(x^2-1)\} = \dfrac{2}{x^2-1}$

Integrating

$$\int d(y.(x^2-1)) = 2\int \frac{1}{x^2-1}\,dx$$

$$y.(x^2-1) = 2.\log\left(\frac{x-1}{x+1}\right) + c$$

d. $\dfrac{dy}{dx} + \dfrac{y}{x.\ln x} = \dfrac{1}{x}$ ... ... (i)

Comparing (i) with $\dfrac{dy}{dx} + p.y = Q$

$p = \dfrac{1}{x.\ln x}$ and $Q = \dfrac{1}{x}$

$$\int pdx = \int \frac{1}{x.\ln x}\,dx$$

Put $\ln x = u$ then $\dfrac{1}{x}\,dx = du$

$$s\int pdx = \int \frac{du}{u} = \ln u$$

$I.F = e^{\int pdx} = e^{\ln u} = u = \ln x$

Multiplying (i) by I.F. we get

$d(y.\ln x) = \dfrac{1}{x}\ln x$

Integrating both sides

$$\int d(y.\ln x) = \int (1,x)\ln x\,dx$$

$$y.\ln x = \int vdv \text{ where } \ln x = v$$

$$y.\ln x = \frac{v^2}{2} + c$$

$$y.\ln x = \frac{(\ln x)^2}{2} + c$$

$$\therefore\quad y = \frac{1}{2}\ln x + \frac{c}{\ln x}$$

e. $\dfrac{dy}{dx} + \dfrac{y}{x} + y^2$

$\dfrac{1}{y^2}\dfrac{dy}{dx} + \dfrac{1}{x.y} = 1$ ... ... (i)

Put $\dfrac{1}{y} = z$ then $-\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

(i) becomes

$-\dfrac{dz}{dx} + \dfrac{1}{x}.z = 1$

$\dfrac{dz}{dx} - \dfrac{1}{x}.z = -1$ ... ... (ii)

Comparing (ii) with $\dfrac{dy}{dx} + p.y = Q$ we get

$p = -\dfrac{1}{x}$ and $Q = -1$

$$\int pdx = -\int \frac{1}{x}\,dx = -\ln x = \ln x^{-1}$$

I.F. $= e^{\int pdx} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$

Multiplying (ii) by I.F.

$$\left(\frac{dz}{dx} - \frac{1}{x} . z\right)\frac{1}{x} = -1 . \frac{1}{x}$$

$$d\left(z . \frac{1}{x}\right) = -\frac{1}{x}$$

Integrating both sides, we get

$$\int d\left(z . \frac{1}{x}\right) = -\frac{1}{x}$$

Integrating both sides, we get

$$\int d\left(z . \frac{1}{x}\right) = -\int \frac{1}{x} dx$$

$$z.\frac{1}{x} = -\ln x + c$$

$$\frac{1}{y} = -x\ln x + cx$$

f. $\frac{dy}{dx} + xy = xy^3$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x \text{ ... ... (i)}$$

Put $\frac{1}{y^2} = z$

Then $-\frac{1}{y^3}\frac{dy}{dx} = \frac{dz}{dx}$

(i) becomes

$$\frac{dz}{dx} - xz = -x \text{ ... ... (ii)}$$

Comparing (ii) with $\frac{dy}{dx} + p.y = Q$, we get

$p = -x$ and $Q = -x$

$$\int pdx = -\int xdx = e- \frac{x^2}{2}$$

I.F. $= e^{\int pdx} = e^{-\frac{x^2}{2}}$

Multiplying (ii) by I.F.

$$d(z,e^{-x^2/2}) = -x.e^{-\frac{x^2}{2}}$$

Integrating

$$z.e^{-\frac{x^2}{2}} = \int -x . e^{-\frac{x^2}{2}} dx$$

$$z.e^{-\frac{x^2}{2}} = \int e^u \, du \text{ where } u = -\frac{x^2}{2}$$

$$\frac{1}{y^2}\ e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} + c$$

$$\frac{1}{y^2} = 1 + c\, e^{\frac{x^2}{2}}$$

g. $(1 - x^2)\frac{dy}{dx} + x.y = xy^2$

$$\frac{1}{y^2}\frac{dy}{dx} + \left(\frac{x}{1-x^2}\right)\frac{1}{y} = \frac{x}{1-x^2} \text{ ... ... (i)}$$

Put $\frac{1}{y} = z$ then $\frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$

Then, $-\frac{dz}{dx} + \left(\frac{x}{1-x+}\right) . z = \frac{x}{1-x^2}$

$$\frac{dz}{dx} - \left(\frac{x}{1-x^2}\right) z = -\frac{x}{1-x^2} \text{ ... ... (ii)}$$

Here, $p = \frac{-x}{1-x^2}$ and $Q = \frac{-x}{1-x^2}$

$$\int pdx = \frac{1}{2}\int -\frac{2x}{1-x^2} = \frac{1}{2}\ln(1-x^2) = \ln\sqrt{1-x^2}$$

I.F. $= e^{\int pdx} = e^{\ln\sqrt{1-x^2}} = \sqrt{1-x^2}$

Multiplying (ii) by I.F. we get

$d(z.\sqrt{1-x^2} = -\frac{x}{1-x^2}\sqrt{1-x^2}$

Integrating

$$z\sqrt{1-x^2} = \int \frac{x}{\sqrt{1-x^2}} dx$$

$$z.\sqrt{1-x^2} = \frac{1}{2}\int \frac{du}{\sqrt{u}} \text{ where } 1-x^2 = u$$

$$\frac{1}{y}.\sqrt{1-x^2} = \sqrt{1-x^2} + c$$

Chapter 18: Linear Programming

Exercise 18

1. Solution:

a. Here, Max. $z = 3x + 5y$

Subject to the constraints

$x + 2y \leq 5$

$2x - 3y \leq 7$

$x, y \geq 0$

Let, r, s be any two non-negative slack variables. Then,

$x + 2y + r = 5$

$2x - 3y + 5 = 7$ and

$z = 3x + 5y$

The reformulation of LP into standard form as

$x + 2y + r + 0.s + 0.z = 5$

$2x - 3y + 0.r + s + 0.z = 7$

$-3x - 5y + 0.r + 0.s + z = 0$

$x, y, r, z \geq 0$

b. Here, Max $z = 10x + 15y$

Subject to

$x + 2y = \leq 20$

$x + y \leq 16$

$x, y \geq 0$

Let, r, s be any two non-negative slack variables.

$x + 2y + r = 20$

$x + y + s = 16$

The reformulation of LP into standard form as;

$x + 2y + r + 0.s + 0.z = 20$

$x + y + 0.r + s + 0.z = 16$

$-10x - 15y + 0.r + 0.5 + z = 0$

$x, y, r, s \geq 0$

c. Here, Min. $z = x_1 + x_2$

Subject to $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

Let, r, s be any two non–negative slack variables. Then,

$2x_1 + x_2 + r = 4$

$x_1 + 7x_2 + s = 7$

The reformulation of LP into standard form as;

$2x_1 + x_2 + r + 0.s + 0.z = 4$

$x_1 + 7x_2 + 0.r + s + 0.z = 7$

$-x_1 - x_2 + 0.r + 0.s + z = 0$

$x_1, x_2, r, s \geq 0$

d. Here, Min. $z = 7x + 5y$

Subject to $4x + 3y \leq 12 \Rightarrow -4x - 3y \geq -12$

$x + 2y \leq 6 \Rightarrow -x - 2y \geq -6$

Let, r, s be any two non–negative slack variables.

$-4x - 3y + r = -12$

$-x - 2y + s = -6$

The reformulation of LP problem into standard forms

$-4x - 3y + r + 0.s + 0.z = -12$

$-x - 2y + 0.r + s + 0.z = -6$

$-7x - 5y + 0.r + 0.s + z = 0$
$x, y, r, s \geq 0$

2. Solution:

a. Here, given equations are $x + y + z = 6$
$4x + 3y + z = 12$
There are 3 variables and 2 equations so, there are two basic solution and one non-basic.

Solution:

Case – I: if $z = 0$, then,
$x + y = 6$ ... ... (i)
$4x + 3y = 12$ ... ... (ii)
Solving equation (i) and (ii)
$\therefore \quad y = \frac{12}{5}$ (basic)
$\therefore \quad x = \frac{6}{5}$ (basic)
$\therefore \quad z = 0$ (non–basic)

Case – II: if $y = 0$
$x + z = 6$ ... ... (iii)
$4x + z = 12$ ... ... (iv)
Solving (iii) and (iv0
$\therefore \quad x = 2$ (basic)
$\therefore \quad z = 4$ (basic)
$\therefore \quad y = 0$ (non–basic)

Case–III: if $x = 0$
$y + z = 6$ ... ... (v)
$3y + z = 12$ ... ... (vi)
Solving (v) and (vi)
$\therefore \quad y = 6$ (basic)
$\therefore \quad z = -6$ (basic)
$\therefore \quad x = 0$ (non basic)

b. Here,
Given equations are
$x + 2y + z = 4$
$2x + y + 5z = 5$
There are 3 variables in 2 equations among them 2 are basic and 1 is non-basic.

Case–I: if $z = 0$
$x + 2y = 4$ ... ... (i)
$2x + y = 5$ ... ... (ii)
Solving equation (i) and equation (ii)
$\therefore \quad y = 1$ (basic)
$\therefore \quad x = 2$ (basic)
$\therefore \quad z = 0$ (non–basic)

Case–II: if $y = 0$
$x + z = 4$ ... ... (iii)
$2x + 5z = 5$ ... ... (v)
Solving (iii) and (iv)
$\therefore \quad z = -1$ (basic)
$\therefore \quad y = 0$ (non–basic)
$\therefore \quad x = 3$ (basic)

Case–III: if $x = 0$
$2y + z = 4$ ... ... (v)
$y + 5z = 5$ ... ... (vi)

Solving (v) and (vi)

$\therefore$ $z = \frac{2}{3}$ (basic)

$\therefore$ $y = \frac{5}{3}$ (basic)

$\therefore$ $x = 0$ (non–basic)

3. Solution:

a. Given equations are

$x + 2y - z = 3$

$x - y + z = 5$

There are3 variables and 2 equations. So, among them 2 are basic and 1 is non-basic.

Case–I: if $z = 0$

$x + 2y = 3$ ... ... (i)

$x - y = 5$ ... ... (ii)

Solving (i) and (ii)

$\therefore$ $x = \frac{13}{3}$ (basic)

$\therefore$ $y = \frac{-2}{3}$ (basic)

$\therefore$ $z = 0$ (non–basic)

Case–II: if $y = 0$

$x - z = 3$ ... ... (iii)

$x + z = 5$ ... ... (iv)

Solving (ii) and (i)

$\therefore$ $x = 4$ (basic)

$\therefore$ $z = 1$ (basic)

$\therefore$ $y = 0$ (non–basic)

Case–III: if $x = 0$

$2y - z = 3$ ... ... (v)

$\underline{-y + z = 5 \text{ ... ... (vi)}}$

Solving (v) and (vi)

$\therefore$ $y = 8$ (basic)

$\therefore$ $z = 13$ (basic)

$\therefore$ $x = 0$ (non-basic)

Since, the case II and III are non-negative, so they give basic feasible solution.

$\therefore$ The basic feasible solution are (4, 0, 1) and (0, 8, 13)

b. Here, the given equations are

$2x + 3y + z = 12$

$x + 2y - 3z = 5$

There are 3 variables and 2 equations. Among them 2 are basic and 1 is non-basic.

Case–I: if $z = 0$

$2x + 3y = 12$ ... ... (i)

$x + 2y = 5$ ... ... (ii)

Solving (i) and (ii)

$\therefore$ $y = -2$ (basic)

$\therefore$ $x = 9$ (basic)

$\therefore$ $z = 0$ (non–basic)

Case–II: if $y = 0$

$2x + z = 12$ ... ... (iii)

$x - 3z = 5$ ... ... (iv)

Solving (iii) and (iv)

$\therefore$ $z = \frac{2}{7}$ (basic)

$\therefore \quad x = \frac{41}{7}$ (basic)

$\therefore \quad y = 0$ (non–basic)

Case III: If x = 0,

$3y + z = 12$ .... (v)

$2y - 3z = 5$ .... (vi)

solving (v) and (vi), we get

$y = \frac{41}{11}$ and $z = \frac{9}{11}$

Since, the cases II and III are non-negative, so the basic feasible solution are

$\left(\frac{41}{7}, 0, \frac{3}{7}\right)$ and $\left(0, \frac{41}{11}, \frac{9}{11}\right)$

4. Solution:

a. Here, max. $z = 2x + 3y$

Subject to $x + 2y \le 10$

$2x + y = \le 14$

$x, y \ge 0$

Let, r, s be any two non–negative slack variables.

$x + 2y + r = 10$

$2x + y + s = 14$ and

$z = 2x + 3y$

$\Rightarrow \quad x + 2y + r + 0.s + 0.z = 10$

$2x + y + 0.r + s + 0.z = 14$

$-2x - 3y + 0.r + 0.s + z = 0$

The simplex tableau;

Basic variables	x	y	r	s	z	RHS
r s	1 2	② 1	1 0	0 1	0 0	10 14
	–2	–3	0	0	1	0

The most negativity entry is –3 so, y column is pivot column. Then, $\frac{10}{2} = 5$, $\frac{14}{1} = 14$

Here, 5 < 14 so 2 is pivot element.

$R_1 \rightarrow \frac{1}{2} R_1$

Basic variables	x	y	r	s	z	RHS
y x	$\frac{1}{2}$ 2	1 1	$\frac{1}{2}$ 0	0 1	0 0	5 14
	–2	–3	0	0	1	0

$R_2 \rightarrow R_1$, $R_3 \rightarrow 3R_1$

Basic variables	x	y	r	s	z	RHS
y s	$\frac{1}{2}$ $\frac{3}{2}$	1 0	$\frac{1}{2}$ $\frac{-1}{2}$	0 1	0 0	5 9
	$-\frac{1}{2}$	0	$\frac{3}{2}$	0	1	15

All the values in last row is not positive. So, it is not optimal solution.

Here, the most negativity entry is $\frac{-1}{2}$ so x column is pivot column. Then,

$\frac{5}{\frac{1}{2}} = \frac{5 \times 2}{10}, \frac{9}{\frac{3}{2}} = \frac{9 \times 2}{3} = 6$

Here, 6 < 10 so, $\frac{3}{2}$ is pivot element.

$R_2 \to \frac{2}{3} R_2$

Basic variables	x	y	r	s	z	RHS
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	5
x	1	0	$\frac{-1}{3}$	$\frac{2}{3}$	0	6
	$\frac{-1}{2}$	0	$\frac{3}{2}$	0	1	15

$R_1 \to R_1 \frac{-1}{2} R_2$, $R_3 \to \frac{1}{2} R_2 + R_3$

Basic variables	x	y	r	s	z	RHS
y	0	1	$\frac{5}{6}$	$\frac{-1}{3}$	0	2
x	1	0	$\frac{-1}{3}$	$\frac{2}{3}$	0	6
	0	0	$\frac{4}{3}$	$\frac{1}{3}$	1	18

Here, all the elements in R_3 are positive so, it is optimal solution. The maximum value is 18 at x = 6 and y = 2.

b. Here, max. z = 9x + y

Subject to $2x + y \leq 8$

$4x + 3y \leq 18$

$x, y \geq 0$

Let, r, s be any two non-negative slack variables. Then,

2x + y = r = 8

4x + 3y + 5 = 18

z = 9x + y

$\Rightarrow 2x + y + r + 0.s + 0.z = 8$

$4x + 3y + 0.r + s + 0.z = 18$

$-9x - y + 0.r + 0.s + z = 0$

The simplex tableau is

Basic variables	x	y	r	s	z	RHS
r	2	1	1	0	0	8
s	4	3	0	1	0	18
	–9	–1	0	0	1	0

The most negativity entry is –9 so, x column is pivot column. Then, $\frac{8}{2} = 4$, $\frac{18}{4} = 4.5$

Here, 2 is pivot element.

$R_1 \to \frac{1}{2} R_1$

Basic variables	x	y	r	s	z	RHS
x	1	2	$\frac{1}{2}$	0	0	4
s	4	3	0	1	0	18
	–9	–1	0	0	1	0

$R_2 \to R_2 - 4R_1$

Basic variables	x	y	r	s	z	RHS
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	4
s	0	1	–2	1	0	2

	–9	–1	0	0	1	0

$R_3 \to R_3 + 9R_1$

Basic variables	x	y	r	s	z	RHS
x s	1 0	$\frac{1}{2}$ 1	$\frac{1}{2}$ –2	0 1	0 0	4 2
	0	$\frac{7}{2}$	$\frac{9}{2}$	0	1	36

Here, all the element in R_3 are positive so, it is optimal solution.

The maximum value is 36 at x = 4 is y = 0.

c. Here, max. $f = 6x_1 - 9x_2$

Subject to $2x_1 - 3x_2 \le 6$

$x_1 + x_2 \le 20$

$x_1, x_2 \ge 0$

Let, r, s be any two non–negative slack variables then,

$2x_1 - 3x_2 + r = 6$

$x_1 + x_2 + s = 20$ and

$f = 6x_1 - 9x_2$

$\Rightarrow$ $2x_1 - 3x_2 + r + 0.s + 0.f = 6$

$x_1 + x_2 + 0.r + s + 0.f = 20$

$-6x_1 + 9x_2 + 0.r + 0.s + f = 0$

The initial simplex tableau is;

Basic variables	x	y	r	s	z	RHS
r s	② 1	–3 1	1 0	0 1	0 0	6 20
	–6	9	0	0	1	0

The most negativity entry is –6 so, x column is pivot column. Then, $\frac{6}{2} = 3$, $\frac{20}{1} = 20$.

Here, 3 < 20 so, 2 is pivot column.

$R_1 \to \frac{1}{2} R_1$

Basic variables	x	y	r	s	z	RHS
x s	1 1	$\frac{-3}{2}$ 1	$\frac{1}{2}$ 0	0 1	0 0	3 20
	–6	9	0	0	1	0

Now, $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 6R_1$

Basic variables	x	y	r	s	z	RHS
x s	1 0	$\frac{-3}{2}$ $\frac{5}{2}$	$\frac{1}{2}$ $\frac{-1}{2}$	0 1	0 0	3 17
	0	0	3	0	0	18

Here, all the elements in last row are positive so, the maximum value is 18 at x = 3 and y = 0.

d. Here, max. $z = 7x_1 + 5x_2$

Subject to $x_1 + 2x_2 = 6$

$4x_1 + 3x_2 \le 12$

$x_1, x_2 \ge 0$

Let, r, s be any two non–negative slack variables. Then,

$x_1 + 2x_2 + r = 6$

$4x_1 + 3x_2 + 5 = 12$ and

$z = 7x_1 + 5x_2$

$\Rightarrow$ $x_1 + 2x_2 + r + 0.s + 0.z = 6$

$4x_1 + 3x_2 + 0.r + s + 0.z = 12$

$-7x_1 - 5x_2 + 0.r + 0.s + z = 0$

The simplex tableau is;

Basic variables	x_1	x_2	r	s	z	RHS
r s	1 ④	2 3	1 0	0 1	0 0	6 12
	–7	–5	0	0	1	0

The most negativity entry is –7 so, x_1 column is pivot column. Then,

$\frac{6}{1} = 6, \frac{12}{4} = 3$

Here, 3 < 6 so, 4 is pivot element.

$R_2 \to \frac{1}{4} R_2$

Basic variables	x_1	x_2	r	s	z	RHS
r x_1	1 1	2 $\frac{3}{4}$	1 0	0 $\frac{1}{4}$	0 0	6 3
	–7	–5	0	0	1	0

$R_1 \to R_1 - R_2, R_3 \to R_3 + 7R_2$

Basic variables	x_1	x_2	r	s	z	RHS
r x_1	0 1	$\frac{5}{4}$ $\frac{3}{4}$	1 0	$\frac{-1}{4}$ $\frac{1}{4}$	0 0	3 3
	0	$\frac{1}{4}$	0	$\frac{7}{4}$	1	21

Here, all the elements in last row are positive so, it is optimal solution.

The maximum value is 21 at $x_1 = 3$ and $x_2 = 0$.

5. Solution:

a. Here, mix. $W = 3x + 2y$

Subject to $2x + y \geq 6$

$x + y \geq 4$

$x, y \geq 0$

The augmented matrix is A =

$$\left(\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 1 & 4 \\ \hline 3 & 2 & 0 \end{array}\right)$$

Then, the augmented dual problem is

$A^T =$

$$\left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 2 \\ \hline 6 & 4 & 0 \end{array}\right)$$

Hence, max. $z = 6x_1 + 4y_1$ s.t.

$2x_1 + y_1 \leq 3$

$x_1 + y_1 = \leq 2$

$x_1, y_1 \geq 0$

Let, x and y be two non-negative slack variables,

$2x + y + x = 3$

$x_1 + y_1 + y = 2$ and

$z = 6x_1 + 4y_1$

$\Rightarrow \quad 2x_1 + y_1 + x + 0.y + 0.z = 3$

$\quad x_1 + y_1 + 0.x + y + 0.z = 2$

$\quad -6x_1 - 4y_1 + 0.x + 0.y + z = 0$

The simplex tableau is

Basic variables	x_1	x_2	r	s	z	RHS
x	②	1	1	0	0	3
y	1	1	0	1	0	2
	–6	–4	0	0	1	0

The most negativity entry is –6 so, x_1 column is pivot column. Then,

$\frac{3}{2} = 1.5 < \frac{2}{1} = 2$

So, 2 is pivot element.

$R_1 \rightarrow \frac{1}{2} R_1$

Basic variables	x	y	r	s	z	RHS
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
y	1	1	0	1	0	2
	–6	–4	0	0	1	0

$R_2 \rightarrow R_2 - R_1,\ R_3 \rightarrow R_3 + 6R_2$

Basic variables	x_1	y_1	r	s	z	RHS
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
y	0	$\frac{1}{2}$	$\frac{-1}{2}$	1	0	$\frac{1}{2}$
	0	–1	3	0	1	9

Here, all the elements in last raw are not positive so it is not optimal solution.

Here, the most negativity entry is –1 so, y_1 column is pivot column. Then,

$\frac{\frac{3}{2}}{\frac{1}{2}} = 3 > \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

so, $\frac{1}{2}$ is pivot element.

$R_2 \rightarrow 2R_2$

Basic variables	x_1	y_1	r	s	z	RHS
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
y_1	0	1	–1	2	0	1
	0	–1	3	0	1	9

$R_1 \rightarrow R_1 - \frac{1}{2} R_2,\ R_3 \rightarrow R_3 + R_2$

Basic variables	x	y	r	s	z	RHS
x_1	1	0	1	–1	0	1
y_1	0	1	–1	2	0	1
	0	0	2	2	1	10

Here, all the elements in last row are positive so, it is optimal solution.

The maximum value is 10 at $x_1 = 1$ and $y_1 = 1$.

Hence, the corresponding min. W = 10 at x = 2 and y = 2

b. Here, min. $W = 18x + 12y$

Subject to $2x + y \geq 8$

$6x + 6y \geq 36$

$\Rightarrow \quad x + y \geq 6$

$x, y \geq 0$

The augmented matrix A =

$$\left(\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & 1 & 6 \\ \hline 18 & 12 & 0 \end{array}\right)$$

The augmented dual problem is

$A^T =$

$$\left(\begin{array}{cc|c} 2 & 1 & 18 \\ 1 & 1 & 12 \\ \hline 8 & 6 & 0 \end{array}\right)$$

Hence, max $z = 8x_1 + 6y_1$, $2x_1 + y_1 \leq 18$ s.t.

$x_1 + y_1 \leq 12$

$x_1, y_1 \geq 0$

Let, x, y be any two non–negative slack variables then,

$2x_1 + y_1 + x = 18$

$x_1 + y_1 + y = 12$ and

$z = 8x_1 + 6y_1$

$\Rightarrow \quad 2x_1 + y_1 + x + 0.y + 0.z = 18$

$x_1 + y_1 + 0.x + y + 0.z = 12$

$-8x_1 - 6y_1 + 0.x + 0.y + z = 0$

Basic variables	x_1	y_1	x	y	z	RHS
x	2	1	1	0	0	18
y	1	1	0	1	0	12
	–8	–6	0	0	1	0

The most negatively entry is –8 so x_1, column is pivot column. Then,

$\frac{18}{2} = 9 < \frac{12}{1} = 12$

Here, 2 is pivot element.

$R_1 \to \frac{1}{2} R_1$

Basic variables	x_1	y_1	x	y	z	RHS
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	9
y	1	1	0	1	0	12
	–8	–6	0	0	1	0

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + 8R_1$

Basic variables	x_1	y_1	x	y	z	RHS
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	9
y	0	$\frac{1}{2}$	$\frac{-1}{2}$	1	0	3
	0	–2	4	0	1	72

All the elements in last row are not positive. So, it is not optimal solution.

The most negative entry is –2 so y_1 column is pivot column. Then,

$\frac{9}{\frac{1}{2}} = 18 > \frac{3}{\frac{1}{2}} = 6$

Here, $\frac{1}{2}$ is pivot element.

$R_2 \to 2R_2$

Basic variables	x_1	y_1	x	y	z	RHS
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	9
y_1	0	1	–1	2	0	–6
	0	–2	4	0	1	72

$R_1 \to R_1 - \frac{1}{2}R_2,\ R_3 \to R_3 + R_2$

Basic variables	x_1	y_1	x	y	z	RHS
x_1	1	0	1	–1	0	6
y_1	0	1	–1	2	0	6
	0	0	2	4	1	84

Hence, the min. W = 84 at (2, 4)

c. Here, min. $f = x + 4y$

Subject to $x + 2y \geq 8$

$3x + 2y \geq 12$

The augmented matrix is A =

$$\left|\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 2 & 12 \\ \hline 1 & 4 & 0 \end{array}\right|$$

The augmented dual problem is

$$A^T = \left|\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 2 & 4 \\ \hline 8 & 12 & 0 \end{array}\right|$$

Hence, max. $G = 8x_1 + 12y_1,\ x_1 + 3y_1 \leq 1$

$x_1 + y_1 \leq 2$

$x_1\ y_1 \leq 0$

Let x, y are two non–negative slack variables then,

$\Rightarrow\quad x_1 + 3y_1 + x = 1$

$\quad x_1 + y_1 + y = 2$

$\quad -8x_1 - 12y_1 + G = 0$

Then,

$x_1 + 3y_1 + x + 0.y + 0.G = 1$

$x_1 + y_1 + 0.x + 0.y + G = 0$

The simplex tableau is

Basic variables	x_1	y_1	x	y	G	RHS
x	1	3	1	0	0	1
y	1	1	0	1	0	2
	–8	–12	0	0	1	0

The most negativity entry is –12 so, y_1 column is pivot column. Then,

$\frac{1}{3} = 0.33 < \frac{2}{1} = 2$

So, 3 is pivot element.

$R_1 \to \frac{1}{3}R_1$

Basic variables	x_1	y_1	x	y	G	RHS
y_1	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$
y	1	1	0	1	0	2
	–8	–12	0	0	1	0

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + 12R_1$

Basic variables	x_1	y_1	x	y	G	RHS
y_1	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$
y	$\frac{2}{3}$	0	$\frac{-1}{3}$	0	0	$\frac{5}{3}$
	–4	0	4	0	1	4

All the elements in last row are not positive. So it is not optimal solution. The most negativity entry is –4 so, x_1 column is pivot column. Then,

$$\frac{\frac{1}{3}}{\frac{1}{3}} = 1 < \frac{\frac{5}{3}}{\frac{2}{3}} = 2.5$$

So $\frac{1}{3}$ is pivot element.

$R_1 \to 3R_1$

Basic variables	x_1	y_1	x	y	G	RHS
x_1	1	3	1	0	0	1
y	$\frac{2}{3}$	0	$\frac{-1}{3}$	0	0	$\frac{5}{3}$
	–4	0	4	0	1	4

$R_2 \to R_2 - \frac{2}{3}R_1$, $R_3 \to R_3 + 4R_1$

Basic variables	x_1	y_1	x	y	G	RHS
x_1	1	3	1	0	0	1
y	0	–2	$\frac{-11}{9}$	0	0	$\frac{1}{9}$
	0	12	8	0	1	8

The maximum value is 8 at $x_1 = 1$ and $y_1 = 0$.

Hence, the corresponding min. F = 8 at x = 8, and y = 0

d. Here, min. W = 14x + 20y

Subject to $7x + 6y \geq 20$

$x + 2y \geq 4$

$x, y \geq 0$

The augmented matrix is A =

$$\left|\begin{array}{cc|c} 7 & 6 & 20 \\ 1 & 2 & 4 \\ \hline 14 & 20 & 0 \end{array}\right|$$

The augmented dual problem is A^T =

$$\left|\begin{array}{cc|c} 7 & 1 & 14 \\ 6 & 2 & 20 \\ \hline 20 & 4 & 0 \end{array}\right|$$

Max $z = 20x_1 + 4y_1$, $7x_1 + y_1 \leq 14$

$6x_1 + 2y_1 \leq 20$

$x, y \geq 0$

Let, x, y be any two non–negative slack variables.

$7x_1 + y_1 + x = 14$

$6x_1 + 2y_1 + 0.x + y + 0.w = 20$

$-20x_1 - 4y_1 + 0.x + 0.y + w = 0$

The simplex tableau is

Basic variables	x_1	y_1	r	s	z	RHS
x	7	1	1	0	0	14
y	6	2	0	1	0	20

	–20	–4	0	0	1	0

The most negativity entry is –20 so, x_1 column is pivot column. Then,

$\frac{14}{7} = 2 < \frac{20}{6} = 3.3$

$R_1 \to \frac{1}{7} R_1$

Basic variables	x_1	y_1	r	s	z	RHS
x_1	1	$\frac{1}{7}$	$\frac{1}{7}$	0	0	2
y	6	2	0	1	0	20
	–20	–4	0	0	1	0

$R_2 \to R_2 - 6R_1$

Basic variables	x_1	y_1	r	s	z	RHS
x_1	1	$\frac{1}{7}$	$\frac{1}{7}$	0	0	2
y	0	$\frac{8}{7}$	$\frac{-6}{7}$	1	0	8
	–20	–4	0	0	1	0

$R_3 \to R_3 + 20R_1$

Basic variables	x_1	y_1	x	y	z	RHS
x_1	1	$\frac{1}{7}$	$\frac{1}{7}$	0	0	2
y	0	$\frac{8}{7}$	$\frac{-6}{7}$	1	0	8
	0	$\frac{-8}{7}$	$\frac{20}{7}$	0	1	40

All the elements in last row are not positive so, it is not optimal solution. The most negativity element is $\frac{-8}{7}$. So, y_1 column is pivot column. Then,

$\frac{2}{\frac{1}{7}} = 14, \frac{8}{\frac{8}{7}} = 7$

Here, $\frac{8}{7}$ is pivot element.

$R_2 \to \frac{7}{8} R_2$

Basic variables	x_1	y_1	r	s	z	RHS
x_1	1	$\frac{1}{7}$	$\frac{1}{7}$	0	0	2
y_1	0	1	$\frac{-3}{4}$	$\frac{7}{8}$	0	7
	0	$\frac{-8}{7}$	$\frac{20}{7}$	0	1	40

$R_1 \to R_1 - \frac{1}{7} R_2$

Basic variables	x_1	y_1	r	s	z	RHS
x_1	1	0	$\frac{1}{4}$	$\frac{-1}{8}$	0	1
y_1	0	1	$\frac{-3}{4}$	$\frac{7}{8}$	0	7
	0	$\frac{-8}{7}$	$\frac{20}{7}$	0	1	40

$R_3 \to R_3 + \frac{8}{7} R_2$

Basic variables	x_1	y_1	r	s	z	RHS
x_1	1	0	$\frac{1}{4}$	$\frac{-1}{8}$	0	1
y_1	0	1	$\frac{-3}{4}$	$\frac{7}{8}$	0	7
	0	0	2	1	1	48

All the elements in last row are positive so it is optimal solution.
The maximum value is 48 at $x_1 = 1$ and $y_1 = 7$.
Hence, the corresponding min. w = 48 at x = 2 and y = 1

Chapter 19: System of Linear Equation

Exercise 19.1

1. **a. Solution:**

Given equations are

$4x + 5y = 12$ ... (i)

$3x + 2y = 9$ ... (ii)

Multiplying by 3 in (i) & 4 in eq. (ii) and subtracting eq. (ii) from eq. (i)

Forward elimination

$12x + 15y = 36$

$12x + 8y = 36$

$- \quad - \quad\quad -$

$7y = 0$

$\therefore y = 0$

Backward substitution

Put the value of y in eq. (i), we get

$4x + 5 \times 0 = 12$

or, $4x = 12$

$x = 3$

$\therefore \quad x = 3, y = 0$

b. Solution:

Given equation are

$5x + 2y = 4$ ... (i)

$7x + 3y = 5$ ... (ii)

Multiplying by 7 in eq. (i) & 5 in eq. (ii) and subtracting eq. (ii) from eq. (i)

Forward elimination

$35x + 14y = 28$

$35x + 15y = 25$

$- \quad - \quad\quad -$

$-y = 3$

$\therefore y = -3$

Backward substitution

$5x + 2y = 4$

or, $5x + 2 \times (-3) = 4$

or, $5x = 4 + 6$

or, $5x = 10$

$\therefore \quad x = 2$

Hence, the value of x & y are 2 and –3 respectively.

c. Solution:

$5x - 3y = 8$ ... (i)

$2x + 5y = 59$ ... (ii)

Forward elimination

Multiplying by 2 in eq. (ii) & 5 in eq. (ii) and subtracting eq. (ii) from eq. (i)

$10x - 6y = 16$

$10x + 25y = 295$

$- \quad - \quad\quad -$

$-31y = -279$

$\therefore y = 9$

Backward substitution

Put the value of y in eq. (i)

$5x - 3 \times 9 = 8$

or, $5x = 8 + 27$

or, $5x = 35$

$\therefore$ $x = 7$

Hence, $x = 7$ & $y = 9$

d. Solution

$2x - 3y = 7$ ... (i)

$3x + y = 5$ ... (ii)

Forward elimination

Multiplying by 3 in eq. (i) & 2 in eq. (ii) and subtracting eq. (ii) from eq. (i)

$6x - 9y = 21$

$6x + 2y = 10$

$-\quad -\quad -$

$-11y = 11$

$\therefore$ $y = -1$

Backward substitution,

Put the value of y in eq. (ii)

or, $2x - 3y = 7$

or, $2x - 3 \times (-1) = 7$

or, $2x = 7 - 3$

$\therefore$ $x = 2$

Hence, the required value of x & y are 2 & –1 respectively.

2. a. Solution:

Here, given equation are

$5x - y + 4z = 5$ ... (i)

$2x + 3y + 5z = 2$ ... (ii)

$5x - 2y + 6z = -1$ ... (iii)

Multiplying by 2 in eq. (i) & 5 in eq. (ii) and subtracting eq. (ii) from eq. (i)

$10x + 15y + 25z = 10$

$10x - 2y + 8z = 10$

$-\quad -\quad -\quad -$

$-17y + 17z = 0$ ... (iv)

Again,

Subtracting eq. (iii) from eq. (i)

$5x - y + 4z = 5$

$5x - 2y + 6z = -1$

$-\quad +\quad -\quad +$

$y - 2z = 6$ ... (v)

Multiplying by 17 in eq. (v) & subtracting eq. (v) from eq. (iv)

$17y + 17z = 0$

$17y - 34z = 102$

$-\quad +\quad -$

$51z = -102$

$\therefore z = -2$

The system of linear equations becomes

$5x - y + yz = 5$

$y - 2z = 6$

$2z = -2$

Put the value of z in eq. (v)

$y - 2 \times (-2) = 6$

or, $y = 6 - y$

$\therefore$ $y = 2$

Again, put the values of x & y in eq. (i)

$5x - y + yz = 5$

or, $5x - 2 + 4 \times (-2) = 5$

or, $5x = 5 + 7 + 8$
or, $5x = 15$
$\therefore$ $x = 3$
Hence, $x = 3$, $y = 2$, $z = -2$

b. Solution:
Here, given equations are
$x - y + 2z = 7$ ... (i)
$3x + 4y - 5z = -5$ ... (ii)
$2x - y + 3z = 12$ ... (iii)
Multiplying by 3 in eq. (i) & subtracting eq. (ii) from eq. (i)
$$\begin{array}{l} 3x - 3y + 6z = 21 \\ 3x + 4y - 5z = -5 \\ \underline{- \quad - \quad + \quad +} \\ -7y + 11z = 26 \text{ ... (iv)} \end{array}$$
Multiplying by 2 in eq. (i) and subtracting eq. (iii) from eq. (i)
$$\begin{array}{l} 2x - 2y + 4z = 14 \\ 2x - y + 3z = 12 \\ \underline{- \quad + \quad - \quad -} \\ -y + 2 = 2 \text{ ... (v)} \end{array}$$
Multiplying by 7 in eq. (v) and subtracting eq. (v) from eq. (iv)
$$\begin{array}{l} -7y + 11z = 26 \\ -7y + 7z = 14 \\ \underline{+ \quad - \quad\quad -} \\ -y + z = 2 \text{ ... (iv)} \end{array}$$
Multiplying by 7 in eq. (v) and subtracting eq. (v) from eq. (iv)
$$\begin{array}{l} -7y + 11z = 26 \\ -7y + 7z = 14 \\ \underline{+ \quad - \quad\quad -} \end{array}$$
or, $4z = 12$
$\therefore$ $z = 3$
The system of linear equations become
$x - y + 2z = 7$
$-y + z = 2$
$z = 3$
Put the value of z in eq. (v)
$-y + 3 = 2$
or, $-y = -3 + 2$
or, $y = 1$
Again,
Put the value of y & z in eq. (i)
$x - 1 + 2 \times 3 = 7$
or, $x = 7 - 6 + 1$
or, $x = 2$
Hence, $x = 2$, $y = 1$, $z = 3$.

c. We have,
$2x + 3y + 3z = 5$ ... (i)
$x - 2y + z = -4$ ...(ii)
$3x - y - 2z = 3$ ...(iii)
from (i) and (ii), we get
$7y + z = 13$ ...(iv)
from (ii) and (iii) we get
$5y - 5z = 15$
$\therefore$ $y - z = 3$ ...(v)

Adding (iv) and (v), we get
$8y = 16$
$\therefore \quad y = 2$
from (v), $2 - z = 3$
$\therefore \quad z = -1$
from (ii), $x - 4 - 1 = -4$
$\Rightarrow \quad x = -4 + 5 = 1$
Hence, $x = 1, y = 2, z = -1$

d. Solution:

Here, given equations are
$x + 2y + 3z = 14$ ... (i)
$3x + 4y + 2z = 17$ ... (ii)
$2x + 3y + z = 11$ ... (iii)
Multiplying by 3 in eq. (i) and subtracting eq. (ii) from eq. (i)

$$\begin{array}{l} 3x + 6y + 9z = 42 \\ 3x + 4y + 2z = 17 \\ - \quad - \quad - \quad - \\ \hline 2y + 7z = 25 \text{ ... (iv)} \end{array}$$

Multiplying by 2 in eq. (i) and subtracting eq. (iii) from (ii)

$$\begin{array}{l} 2x + 4y + 6z = 28 \\ 2x + 3y + z = 11 \\ - \quad - \quad - \quad - \\ \hline y + 5z = 17 \text{ ... (v)} \end{array}$$

Multiplying by 2 in eq. (v) & subtracting eq. (v) from (iv)

$$\begin{array}{l} 2y + 7z = 25 \\ 2y + 10z = 34 \\ - \quad - \quad + \\ \hline +3z = +9 \end{array}$$

$\therefore z = 3$
The system of linear equations becomes
$x + 2y + 3z = 14$
$y + 5z = 17$
$z = 3$
Put the value of z in eq. (v)
or, $y + 5 \times 3 = 17$
or, $y = 17 - 15 = 2$
$\therefore y = 2$
Put the value of x & y in eq. (i)
$x + 2y + 3z = 14$
or, $x + 2 \times 2 + 3 \times 3 = 14$
or, $x = 14 - 9 - 4$
or, $x = 14 - 13$
$\therefore \quad x = 1$
Hence, $x = 1, y = 2$ & $z = 3$

3. (a) Solution:
$x + 3y = 5$ ... (i)
$3x + y = 4$ ... (ii)
Multiplying by 3 in eq. (i) and subtracting eq. (ii) from eq. (i) Forward elimination

$$\begin{array}{l} 3x + 9y = 15 \\ 3x + y = 4 \\ - \quad - \quad - \\ \hline 8y = 11 \end{array}$$

$y = \frac{11}{8}$

Backward substitution

Put the value of y in eq. (i)

$x + \frac{3 \times 11}{8} = 5$

or, $x = 5 - \frac{33}{8}$

or, $x = \frac{7}{8}$

$\therefore \quad x = \frac{7}{8}, y = \frac{11}{8}$

It is consistent and has unique solution.

b. Solution:

Here,

$3x - 2y = 3 \quad \ldots (i)$

$3x - 2y = 6 \quad \ldots (ii)$

Subtracting eq. (ii) from eq. (i)

$$\begin{array}{l} 3x - 2y = 3 \\ 3x - 2y = 6 \\ \underline{- \quad + \quad -} \\ 0 = -3 \end{array}$$

Hence, it is inconsistent and has no solution.

c. **Solution:**

$-2x + 5y = 3 \quad \ldots (i)$

$6x - 15y = -9 \quad \ldots (ii)$

Multiplying by 3 in eq. (i) & adding eq. (i) and eq. (ii)

$$\begin{array}{l} -6x + 15y = 9 \\ 6x - 15y = -9 \\ \underline{- \quad + \qquad +} \\ 0 = 0 \end{array}$$

It is consistent having infinitely many solution.

d. **Solution:**

Given equations are:

$x - 2y - 5z = -12 \quad \ldots (i)$

$2x - y = 7 \quad \ldots (ii)$

$-4x + 5y + 6z = 1 \quad \ldots (iii)$

Multiplying by 4 in eq. (i) and adding eq. (i) & eq. (iii)

$$\begin{array}{l} 4x - 8y - 20z = -48 \\ -4x + 5y + 6z = 1 \\ \underline{- \quad + \qquad +} \\ -3y - 14z = -47 \end{array}$$

or, $3y + 14z = 47 \quad \ldots (iv)$

Multiplying by 2 in eq. (i) and subtracting eq. (ii) from eq. (i)

$$\begin{array}{l} 2x - 4y - 10z = -24 \\ 2x - y = 7 \\ \underline{- \quad + \quad -} \\ -3y - 10z = -31 \ldots (iv) \end{array}$$

Adding eq. (iv) and eq. (v)

$$\begin{array}{l} 3y + 14z = 47 \\ \underline{-3y - 10z = -31} \\ 4z = 16 \end{array}$$

$\therefore z = 4$

The system of linear equations becomes

$x - 2y - 5z = -12$
$3y + 14z = 47$
$z = 4$
Put the value of z in eq. (i)
$3y + 14 \times 4 = 47$
or, $3y = 47 - 56$
or, $3y = -9$
$\therefore$ $y = -3$

e. Here,
$2x - y + 4z = 4$ ... ... (i)
$x + 2y - 3z = 1$ ... ... (ii)
$3x + 3z = 6$ ... ... (iii)
Multiplying by 2 in equation (i) and subtracting equation (ii) from equation (i)
$2x - y + 4z = 4$
$2x + 4y - 6z = 2$
$\underline{- \quad - \quad + \quad -}$
$-5y + 10z = 2$ ... ... (iv)
Multiplying by 3 in equation (i) and subtracting equation (iii) from equation (ii)
$3x + 6y - 92 = 3$
$3x + 3z = 6$
$\underline{- \quad - \quad\quad -}$
$6y - 12z = -3$
or, $2y - 4z = -1$ ... ... (v)
Multiplying by 2 in equation (iv) and adding equation (iv) and equation (v)
$-10y + 20z = 4$
$\underline{10y - 20z = -5}$
$0 -1$
Here, $0 = -1$
It is inconsistent having no solution.

f. Here,
Given equations are
$x + 3y + 4z = 8$ ... ... (i)
$2x + y + 2z = 5$ ... ... (ii)
$5x + 2z = 7$ ... ... (iii)
Multiplying by 2 in equation (i) and subtracting equation (ii) from equation (i)
$2x + 6y + 8z = 16$
$2x + y + 22 = 5$
$\underline{- \quad - \quad - \quad\quad -}$
$5y + 6z = 11$ ... ... (iv)
Multiplying by 5 in equation (i) and subtracting equation (iii) from equation (i)
$5x + 15y + 20z = 40$
$5x + 0.y + 22 = 7$
$\underline{- \quad - \quad\quad - \quad\quad -}$
$15y + 18z = 33$ ... ... (v)
Multiplying by 3 in equation (iv) and subtracting equation (v) from equation (iv)
$15y + 18z = 33$
$15y + 18z = 33$
$\underline{- \quad - \quad\quad\quad -}$
$0 = 0$
It is consistent having infinitely many solution.

Exercise 19.2

1. Solution

a. Here,

$3x_1 + x_2 = 5$ ... ... (i)

$x_1 + 2x_2 = 5$ ... ... (ii)

from equation (i) and equation (ii)

$x_1 = \frac{5 - x_2}{3}$

$x_2 = \frac{5 - x_1}{2}$

Initially, $x_2 = 0$. Iteration – I

$x_1 = \frac{5}{3}$

$= 1.66$

$x_2 = \frac{5 - 1.66}{2}$

$= 1.67$

Iteration – II

$x_1 = \frac{5 - 1.67}{3}$

$= 1.11$

$x_2 = \frac{5 - 1.11}{2}$

$= 1.945$

Iteration – III

$x_1 = \frac{5 - 1.945}{3}$

$= 1.01$

$x_2 = \frac{5 - 1.01}{2}$

$= 1.995$

Iteration – IV

$x_1 = \frac{5 - 1.995}{3}$

$= 1.001$

$x_2 = \frac{5 - 1.001}{2}$

$= 1.9995$

From iteration – III and iteration – IV the value of x_1 and x_2 are nearly equal to 1 and 2.

$\therefore \quad x_1 = 1, x_2 = 2$

b. Here,

$2x_1 - x_2 = 8$ ... ... (i)

$3x_1 + 7x_2 = -5$ ... ... (ii)

from equation (i) and equation (ii)

$x_1 = \frac{8 + x_2}{2}$

$x_2 = \frac{-5 - 3x_1}{7}$

Initially, $x_2 = 0$

Iteration – I

$x_1 = \frac{8 + 0}{2}$

$= 4$

$x_2 = \frac{-5 - 3\times 4}{7}$

$= \frac{-5 - 12}{7}$

$= -2.42$

Iteration – II

$x_1 = \frac{8 - 2.4^2}{2}$

$= 2.79$

$x_2 = \frac{-5 - 3\times 2.79}{7}$

$= \frac{-5 - 8.37}{7}$

$= -1.91$

Iteration – III

$x_1 = \frac{8 - 1.91}{2}$

$= 3.045$

$x_2 = \frac{-5 - 3\times 3.045}{7}$

$= \frac{-5 - 9.135}{7}$

Iteration – IV

$x_1 = \frac{8 - 2.01}{2}$

$= 2.995$

$x_2 = \frac{-5 - 3\times(+2.995)}{7}$

$= \frac{-5 - 8.985}{7}$

$= \frac{-14.135}{7}$

$= -2.01$

$= \frac{-13.985}{7}$

$= -1.99$

Iteration – V

$x_1 = \frac{8 - 1.99}{2}$

$= 3.005$

$x_2 = \frac{-5 - 3\times3.005}{7}$

$= \frac{-5 - 9.015}{7}$

$= -2.002$

Iteration – VI

$x_1 = \frac{8 - 2.002}{2}$

$= 2.999$

$x_2 = \frac{-5 - 3\times2.999}{7}$

$= \frac{-5 - 8.997}{7}$

$= 1.99$

From iteration – V and iteration – VI the value of x_1 and x_2 are nearly equation.

So, $x_1 = 3$, $x_2 = -2$

c. Here,

$3x_1 + x_2 = 5$ ... ... (i)

$x_1 - 3x_2 = 5$ ... ... (ii)

from equation (i) and equation (ii)

$x_1 = \frac{5 - x_2}{3}$

$x_2 = \frac{x_1 - 5}{3}$

Initially, $x_2 = 0$

Iteration – I

$x_1 = \frac{5 - 0}{3}$

$= 1.67$

$x_2 = \frac{1.67 - 5}{3}$

$= -1.11$

Iteration – II

$x_1 = \frac{5 + 1.11}{3}$

$= 2.03$

$x_2 = \frac{2.03 - 5}{3}$

$= -0.99$

Iteration – III

$x_1 = \frac{5 + 0.99}{3}$

$= 1.99$

$x_2 = \frac{1.99 - 5}{3}$

$= -1.00$

Iteration – IV

$x_1 = \frac{5 + 1.00}{3}$

$= 2$

$x_2 = \frac{2 - 5}{3}$

$= -1$

From iteration III and iteration IV the value of x_1 and x_2 are nearly equal so, $x_1 = 2$ and $x_2 = -1$.

d. Here,

$2x - 3y = 7$

$3x + y = 5$

or, $3x + y = 5$ ... (i)

or, $2x - 3y = 7$ ... (ii)

and do in similar way

$\because$The order of given equations are not diagonally dominant, so we should change to order.

2. Solution

a. Given equation are

$2x - y = 1$... ... ... (i)

$-x + 3y - z = 8$ ... ... (ii)

$y - 2z = 5$ ... ... (iii)

From equation (i) (ii) (iii),

$x = \frac{1 + y}{2}$

$y = \frac{8 + x + z}{3}$

$z = \frac{y - 5}{2}$

Initially, y = 0, z = 0

Iteration – I

$x = \frac{1}{2}$

$= 0.5$

$y = \frac{8 + 0.5 + 0}{3}$

$= 2.83$

$z = \frac{2.83 - 5}{2}$

$= -1.085$

Iteration – II

$x = \frac{1 + 2.83}{2}$

$= 1.915$

$y = \frac{8 + 1.915 - 1.085}{3}$

$= 2.943$

$z = \frac{2.943 - 5}{2}$

$= -1.0285$

Iteration – III

$x = \frac{1 + 2.943}{2}$, $y = \frac{8 + 1.975 - 1.0285}{3}$, $z = \frac{2.981.5}{2}$

$= 1.9715$ $= 2.981$ $= -1.0095$

From iteration II and III the value of x, y and z are nearly equal to 2, 3, and –1.

x = 2, y = 3 and z = –1

b. Here,

Given equations are

$3x + y - z = 2$ ... ... (i)

$2x - 5y + z = 20$ ... ... (ii)

$x - 3y - 8z = 3$ ... ... (iii)

From equation(i), (ii) and equation (iii)

$x = \frac{2 - y + z}{3}$

$y = \frac{2x + z - 20}{5}$

$z = \frac{x - 3y - 3}{8}$

Initially, y = 0, z = 0

Iteration – I

$x = \frac{2}{3} = 0.67$

$y = \frac{2\times0.67 + 0.20}{5} = -3.73$

$z = \frac{0.67 + 3\times3.73 - 3}{8} = -2.64$

Iteration – II

$x = \frac{2 + 3.73 - 2.64}{3}$

$= 1.03$

$y = \frac{2\times1.03 - 2.64 - 20}{5}$

$= \frac{2.06 - 22.64}{5}$

$= -4.116$

$z = \frac{1.03 + 3\times4.116 - 3}{8}$

$= 1.29$

Iteration – III

$x = \frac{2 + 4.116 + 1.29}{3}$

$= 2.46$

$y = \frac{2\times2.46 + 1.29 - 20}{5}$

$= -2.75$

$= \frac{2.46 + 3\times2.75 - 3}{8}$

$= 0.96$

Iteration – IV

Iteration – V

$x = \frac{2 + 2.75 + 0.96}{3}$

$= 1.9$

$y = \frac{2\times1.9 + 0.96 - 20}{5}$

$= -3.0$

$z = \frac{1.9 + 3\times3.0 - 3}{8}$

$= 0.98$

$x = \frac{2 + 3.0 + 0.98}{3}$

$= 1.99$

$y = \frac{2\times1.99 + 0.98 - 20}{5}$

$= -3.00$

$z = \frac{1.99 + 3\times3-3}{8}$

$= 0.99$

From iteration IV and V the value of x, y and z are nearly equal to 2, –3 and 1 respectively.

$x = 2, y = -3$ and $z = 1$

c. Here,

Given equations are

$5x_1 + 2x_2 + x_3 = 12$ ... ... (i)

$x_1 = 4x_2 + 2x_3 = 15$ ... ... (ii)

$x_1 = 2x_2 + 5x_3 = 20$ ... ... (iii)

from equation (i), (ii) and (iii)

$x_1 = \frac{12 - x_2 - x_3}{5}$

$x_2 = \frac{15 - 2x_3 - x_1}{4}$

$x_3 = \frac{20 - x_1 - 2x_2}{5}$

Initially, $x_2 = 0$, $x_3 = 0$

Iteration I

$x_1 = \frac{12}{5} = 2.4$

$x_2 = \frac{15 - 2\times0 - 2.4}{4} = 3.15$

$x_3 = \frac{20 - 2.4 - 2\times3.15}{5} = 2.26$

Iteration II

$x_1 = \frac{12 - 3.15 - 2.26}{5} = 1.31$

$x_2 = \frac{15 - 2\times2.26 - 1.31}{4} = 2.29$

$x_3 = \frac{20 - 1.31 - 2\times2.29}{5} = 2.82$

Iteration III

$x_1 = \frac{12 - 2.29 - 2.82}{5} = 1.37$

$x_2 = \frac{15 - 2\times2.82 - 1.37}{4} = 1.9$

$x_3 = \frac{20 - 1.37 - 2\times1.9}{5} = 2.9$

Iteration IV

$x_1 = \frac{12 - 1.9 - 2.9}{5} = 1.44$

$x_2 = \frac{15 - 2\times2.9 - 1.44}{4} = 1.94$

$x_3 = \frac{20 - 1.44 - 2\times1.94}{5} = 2.9$

Iteration V

$x_1 = \frac{12 - 1.94 - 2.9}{5} = 1.4$

$x_2 = \frac{15 - 2\times2.9 - 1.44}{4} = 1.99$

$x_3 = \frac{20 - 1.4 - 2\times199}{5} = 2.9$

From iteration IV and V, the value of x_1, x_3 and x_3 are nearly equal to 1, 2, 3 respectively.

Hence, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$

d. Here,

Given equation are

$x + 10y + z = 6$

$10x + y + z = 6$

$x + y + 10z = 6$

We can write the equations in

$10x + y + z = 6$ ... (i)

$x + 10y + z= 6$ ... (ii)

$x + y + 10z = 6$ ... (iii)

Order as the equations are not in diagonally dominant and do similarly.

e. Here,

$3x + y + z = 13$ ... (i)

$x - 4y - z = -14$ ... (ii)

$2x + 3y + 6z = 37$ ... (iii)

from equation (i), (ii) and (iii)

$x = \frac{13 - y - z}{3}$

$y = \frac{x + 14 - z}{4}$

$z = \frac{37 - 2x - 3y}{6}$

Initially, y = 0, z = 0

Iteration I

$x = 4.33$

$y = \frac{4.33 + 14 - 0}{4} = 4.58$

$z = \frac{37 - 2\times4.33 - 3\times4.58}{6} = 2.43$

Iteration II

$x = \frac{13 - 4.58 - 2.43}{3} = 1.99$

$y = \frac{1.99 + 14 - 2.43}{4} = 3.39$

$z = \frac{37 - 2\times1.99 - 3\times3.39}{6} = 3.80$

Iteration III

$x = \frac{13 - 3.39 - 3.80}{3} = 1.9$

$y = \frac{1.9 + 14 - 3.80}{4} = 3.025$

$z = \frac{37 - 2\times1.9 - 3\times3.025}{6} = 4.0$

Iteration IV

$$x = \frac{13 - 3.025 - 4.0}{3} = 1.99$$

$$y = \frac{1.99 + 14 - 4.0}{4} = 2.9$$

$$z = \frac{37 - 2\times1.99 - 3\times2.9}{6} = 4.0$$

From iteration III and IV the value of x, y and z are nearly equations.

So, x = 2, y = 3, z = 4

3. Solution

a. Given equations are

$3x + 1.52y = 1$ ... ... (i)

$2x + 1.02y = 1$ ... ... (ii)

Multiplying by 2 in equation (i) and 3 in equation (ii) and subtracting equation (ii) from equation (i)

$$\begin{array}{l} 6x + 3.04y = 2 \\ 6x + 3.06y = 3 \\ - \quad - \qquad\quad - \\ \hline -0.02y = -1 \end{array}$$

$y = 50$

Put the value of y in equation (i)

$3x + 1.52\times50 = 1$

or, $3x - 1-76$

or, $3x = -75$

$\therefore$ $x = -25$

If the coefficient of y in equation (ii) is changed to 1.03. Then,

$3x + 1.52y = 1$ ... ... (iii)

$2x + 1.03y = 1$ ... ... (iv)

Multiplying by 2 in equation (iii) and 3 in equation (iv) and subtracting equation (iv) from equation (iii).

$$\begin{array}{l} 6x + 3.04y = 2 \\ 6x + 3.09y = 3 \\ - \quad - \qquad\quad - \\ \hline -0.05y = -1 \end{array}$$

$\therefore$ $y = 20$

Put the value of y in equation (iii)

$3x + 1.52\times20 = 1$

or, $3x = 1 - 30.4$

or, $3x - 29.4$

$\therefore$ $x = -9.8$

It is observed that when a very small change in coefficient of y brings greater change in its solution.

So, $3\times(-9.8) + 1.52\times20 - 1 = 0$

$2\times(-9.8) + 1.02\times20 - 1 = -0.2$

which is very small, so it is ill conditioned.

Exercise 19.3

1. Solution

a. Let, $A = \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}$

We can augment matrix A with unit matrix as,

$$[A : I] = \begin{bmatrix} 3 & 2 & : & 1 & 0 \\ -1 & 6 & : & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2/3 & : & 1/3 & 0 \\ -1 & 6 & : & 0 & 1 \end{bmatrix} \quad [\because R_1 \to \frac{1}{3} R_1]$$

$$\sim \begin{bmatrix} 1 & 2/3 & : & 1/3 & 0 \\ 0 & 20/3 & : & 1/3 & 1 \end{bmatrix} \quad [\because R_2 \to R_2 + R_1]$$

$$\sim \begin{bmatrix} 1 & 2/3 & : & 1/3 & 0 \\ 0 & 1 & : & 1/20 & 3/20 \end{bmatrix} \quad [\because R_2 \to \frac{3}{20} R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & : & 3/10 & -1/10 \\ 0 & 1 & : & 1/20 & 3/20 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{3}{10} & \frac{-1}{10} \\ \frac{1}{20} & \frac{3}{20} \end{bmatrix}$$

b. Here,

Let, $A = \begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix}$

We can augment matrix A with unit matrix as

$$[A : I] = \begin{bmatrix} 2 & -1 & : & 1 & 0 \\ -3 & 3 & : & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{-1}{2} & : & \frac{1}{2} & 0 \\ -3 & 3 & : & 0 & 1 \end{bmatrix} \quad [\because R_1 \to \frac{1}{2} R_1]$$

$$\sim \begin{bmatrix} 1 & \frac{-1}{2} & : & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & : & \frac{3}{2} & 1 \end{bmatrix} \quad [\because R_2 \to R_2 + 3R_1]$$

$$\sim \begin{bmatrix} 1 & \frac{-1}{2} & : & \frac{1}{2} & 0 \\ 0 & 1 & : & 1 & \frac{2}{3} \end{bmatrix} \quad [\because R_2 \to \frac{2}{3} R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & : & 1 & \frac{1}{3} \\ 0 & 1 & : & 1 & \frac{2}{3} \end{bmatrix} \quad [\because R_1 \to R_1 + \frac{1}{2} R_2]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & \frac{1}{3} \\ 1 & \frac{2}{3} \end{bmatrix}$$

c. Here,

Let, $A = \begin{bmatrix} 5 & 3 & -1 \\ 2 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

We can augment matrix with unit matrix as,

$$[A : I] = \begin{bmatrix} 5 & 2 & -1 & : & 1 & 0 & 0 \\ 2 & 4 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 3 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{2}{5} & \frac{-1}{5} & : & \frac{1}{5} & 0 & 0 \\ 2 & 4 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 3 & : & 0 & 0 & 1 \end{bmatrix} \quad [\because R_1 \to \frac{1}{5} R_1]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & \frac{16}{5} & \frac{7}{5} & \frac{-2}{5} & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] [\because R_2 \to R_2 - 2R_1]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & \frac{16}{5} & \frac{7}{5} & \frac{-2}{5} & 1 & 0 \\ 0 & \frac{8}{5} & \frac{16}{5} & \frac{-1}{5} & 0 & 1 \end{array}\right] [\because R_3 \to R_3 - R_1]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{7}{16} & \frac{-1}{8} & \frac{5}{16} & 0 \\ 0 & \frac{8}{5} & \frac{16}{5} & \frac{-1}{5} & 0 & 1 \end{array}\right] [\therefore R_2 \to \frac{6}{16} R_2]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & \frac{-3}{8} & \frac{2}{5} & \frac{-1}{8} & 0 \\ 0 & 1 & \frac{7}{16} & \frac{-1}{8} & \frac{5}{16} & 0 \\ 0 & \frac{8}{5} & \frac{16}{5} & \frac{-1}{5} & 0 & 1 \end{array}\right] [\because R_1 \to R_1 - \frac{2}{5} R_2]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & \frac{-3}{8} & \frac{2}{5} & \frac{-1}{8} & 0 \\ 0 & 1 & \frac{7}{16} & \frac{-1}{8} & \frac{5}{16} & 0 \\ 0 & 0 & \frac{5}{2} & 0 & \frac{-1}{2} & 1 \end{array}\right] [\because R_3 \to R_3 - \frac{8}{5} R_2]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & \frac{-3}{8} & \frac{2}{5} & \frac{-1}{8} & 0 \\ 0 & 1 & \frac{7}{16} & \frac{-1}{8} & \frac{5}{16} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{5} & \frac{2}{5} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & \frac{-3}{8} & \frac{2}{5} & \frac{-1}{8} & 0 \\ 0 & 1 & 0 & \frac{-1}{8} & \frac{2}{5} & \frac{-7}{40} \\ 0 & 0 & 1 & 0 & \frac{-1}{5} & \frac{2}{5} \end{array}\right] [\because R_2 \to R_2 \to \frac{7}{16} R_3]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{2}{5} & \frac{-1}{5} & \frac{3}{20} \\ 0 & 1 & 0 & \frac{-1}{8} & \frac{2}{5} & \frac{-7}{40} \\ 0 & 0 & 1 & 0 & \frac{-1}{5} & \frac{2}{5} \end{array}\right] [\because R_1 \to R_1 + \frac{3}{8} R_3]$$

$$A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{-1}{5} & \frac{3}{20} \\ \frac{-1}{8} & \frac{2}{5} & \frac{-7}{40} \\ 0 & \frac{-1}{5} & \frac{2}{5} \end{bmatrix}$$

d. Here,

Let, $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

We can augment matrix A with unit matrix as,

$$[A : I] = \begin{bmatrix} 1 & 2 & -2 & : & 1 & 0 & 0 \\ -1 & 3 & 0 & : & 0 & 1 & 0 \\ 0 & -2 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & : & 1 & 0 & 0 \\ 0 & 5 & -2 & : & 1 & 1 & 0 \\ 0 & -2 & 1 & : & 0 & 0 & 1 \end{bmatrix} \quad [\because R_2 \to R_2 + R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & : & 1 & 0 & 0 \\ 0 & 1 & \frac{-2}{5} & : & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & -2 & 1 & : & 0 & 0 & 1 \end{bmatrix} \quad [\because R_2 \to \frac{1}{5} R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{-6}{5} & : & \frac{3}{5} & \frac{-2}{5} & 0 \\ 0 & 1 & \frac{-2}{5} & : & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & -2 & 1 & : & 0 & 0 & 1 \end{bmatrix} \quad [\because R_1 \to R_1 - 2R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{-6}{5} & : & \frac{3}{5} & \frac{-2}{5} & 0 \\ 0 & 1 & \frac{-2}{5} & : & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & : & \frac{2}{5} & \frac{2}{5} & 1 \end{bmatrix} \quad [\because R_3 \to R_3 + 2R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{-6}{5} & : & \frac{3}{5} & \frac{-2}{5} & 0 \\ 0 & 1 & \frac{-2}{5} & : & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & : & 2 & 2 & 5 \end{bmatrix} \quad [\because R_3 \to 5R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{-6}{5} & : & \frac{3}{5} & \frac{-2}{5} & 0 \\ 0 & 1 & 0 & : & 1 & 1 & 2 \\ 0 & 0 & 1 & : & 2 & 2 & 5 \end{bmatrix} \quad [\because R_2 \to R_2 + \frac{2}{5} R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 & 2 & 6 \\ 0 & 1 & 0 & : & 1 & 1 & 2 \\ 0 & 0 & 1 & : & 2 & 2 & 5 \end{bmatrix} \quad [\because R_1 \to R_1 + \frac{6}{5} R_3]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

2. Solution:

a. Here, Given equations can be written in matrix form as AX = B.

$$\underset{A}{\begin{bmatrix} 2 & 3 \\ 4 & -9 \end{bmatrix}} \times \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

$X = A^{-1} B$

Matrix A can be augmented with unit matrix a

$$[A : I] = \begin{bmatrix} 2 & 3 & : & 1 & 0 \\ 4 & -9 & : & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{3}{2} & : & \frac{1}{2} & 0 \\ 4 & -9 & : & 0 & 1 \end{bmatrix} \qquad [\because R_1 \to \frac{1}{2} R_1]$$

$$\sim \begin{bmatrix} 1 & \frac{3}{2} & : & \frac{1}{2} & 0 \\ 0 & -15 & : & -2 & 1 \end{bmatrix} \qquad [\because R_2 \to R_2 - 4R_1]$$

$$\sim \begin{bmatrix} 1 & \frac{3}{2} & : & \frac{1}{2} & 0 \\ 0 & 1 & : & \frac{2}{15} & \frac{-1}{15} \end{bmatrix} \qquad [\because R_3 \to \frac{-1}{5} R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & : & \frac{3}{10} & \frac{1}{10} \\ 0 & 1 & : & \frac{2}{15} & \frac{-1}{15} \end{bmatrix} \qquad [\because R_1 \to R]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{2}{15} & \frac{-1}{15} \end{bmatrix}$$

Now, $x = A^{-1} B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{2}{15} & \frac{-1}{15} \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{10} \times (-2) + \frac{1}{10} \times 1 \\ \frac{2}{15} \times (-2) + \left(\frac{-1}{15}\right) \times 1 \end{bmatrix} = \begin{bmatrix} \frac{-6}{10} + \frac{1}{10} \\ \frac{-4}{15} - \frac{1}{15} \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{3} \end{bmatrix}$$

$\therefore x = \frac{-1}{2}, y = \frac{-1}{3}$

b. Here,

$4x + 5y = 9$

$5x - y = 4$

These equations can be written in matrix form as AX = B.

$$\underset{A}{\begin{bmatrix} 4 & 5 \\ 5 & -1 \end{bmatrix}} \times \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 9 \\ 4 \end{bmatrix}}$$

$X = A^{-1} B. \to$ equation (i)

Matrix A can be augmented with unit matrix as,

$$[A : I] = \begin{bmatrix} 4 & 5 & : & 1 & 0 \\ 5 & -1 & : & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{5}{4} & : & \frac{1}{4} & 0 \\ 5 & -1 & : & 0 & 1 \end{bmatrix} \qquad [\because R_1 \to \frac{1}{4} R_1]$$

$$\sim\left[\begin{array}{cc:cc} 1 & \frac{5}{4} & \frac{1}{4} & 0 \\ 0 & \frac{-29}{4} & \frac{-5}{4} & 1 \end{array}\right] \qquad [\because R_2 \to R_2 - 5R_1]$$

$$\sim\left[\begin{array}{cc:cc} 1 & \frac{5}{4} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{5}{29} & \frac{-4}{29} \end{array}\right] \qquad [\because R_3 \to \frac{-4}{29} R_3]$$

$$\sim\left[\begin{array}{cc:cc} 1 & 0 & \frac{1}{29} & \frac{5}{29} \\ 0 & 1 & \frac{5}{29} & \frac{-4}{29} \end{array}\right] \qquad [\because R_1 \to R_1 - \frac{5}{4} R_2]$$

from equation (i)

$X = A^{-1} B$

or, $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{29} & \frac{5}{29} \\ \frac{5}{29} & \frac{-4}{29} \end{bmatrix} = \begin{bmatrix} \frac{9}{29} + \frac{20}{29} \\ \frac{45}{29} - \frac{16}{29} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ $x = 1, y = 1$

c. Here,

The system of equations can be written in matrix form as AX = B.

$$\underset{A}{\begin{bmatrix} 1 & -2 \\ 3 & 7 \end{bmatrix}} \times \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} -7 \\ 5 \end{bmatrix}}$$

$x = A^{-1} B$

Matrix A can be augmented with unit matrix as,

$$[A : I] = \left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right]$$

$$\sim\left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 0 & 13 & -3 & 1 \end{array}\right] \qquad [\because R_2 \to R_2 - 3R_1]$$

$$\sim\left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 0 & 1 & \frac{-3}{13} & \frac{1}{13} \end{array}\right] \qquad [\because R_2 \to \frac{1}{13} R_2]$$

$$\sim\left[\begin{array}{cc:cc} 1 & 0 & \frac{7}{13} & \frac{2}{13} \\ 0 & 1 & \frac{-3}{13} & \frac{1}{13} \end{array}\right] \qquad [\because R_2 \to R_1 + 2R_2]$$

Now,

$x = A^{-1} B$

or, $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{7}{13} & \frac{2}{13} \\ \frac{-3}{13} & \frac{1}{13} \end{bmatrix} \times \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{13} \times (-7) + \frac{2}{13} \times 5 \\ \frac{-3}{13} \times (-7) + \frac{1}{13} \times 5 \end{bmatrix} = \begin{bmatrix} \frac{-49 + 10}{13} \\ \frac{21 + 5}{13} \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$\therefore$ $x = -3, y = 2$

c, d, e & f are similar.

Chapter 20: Parallel Forces

Exercise 20.1

1. Find the resultant of two parallel forces 4 N and 6 N at a distance of 5 m, when they are (a) like parallel, (b) unlike parallel

Solution:

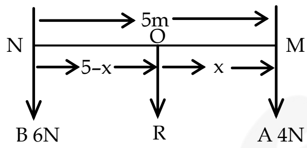
a. Let A and B be two parallel forces acting at points. M and N respectively.

The magnitude of the resultant is given by

R = A + B = 4N + 6N = 10N

The direction of the resultant is same as that of the two forces.

Let the position of the resultant R be at 0, at a at a distance x from M.



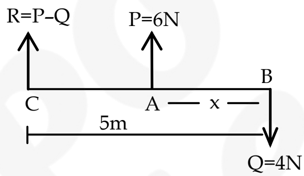
We have, A × Mo = B × No

or, $4 \times x = b\ (45 - x)$

or, $4x + 6x = 30 \Rightarrow x = 3m$

b. When they are unlike parallel

Let the resultant R of unlike parallel force P and Q then R = 6N – 4N = 2N



$\frac{P}{AC} = \frac{Q}{AB} = \frac{R}{BC}$

$\frac{6}{5-x} = \frac{4}{x} = \frac{2}{5}$

$\frac{6}{5-x} = \frac{2}{5}$

$30 = 10 - 2x$

$3x = -10m$

Also, $\frac{4}{x} = \frac{2}{5}$

or, x = 10m

∴ Resultant = 10m

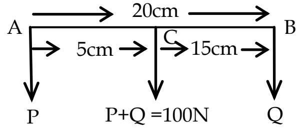
2. Find two like parallel forces at a distance of 20 cm equivalent to 100 N force, the line of action of one of them being at a distance of 5 cm from the given force.

Solution:

Suppose P and Q be two like parallel forces acting at the point A and B such that AB = 20cm. Then the line of action of their resultant is the force P + Q = 100N acting at the point C where AC = 5cm and BC = 15cm. By question, forces are parallel, so we have

$\frac{P}{BC} = \frac{Q}{AC} = \frac{P+Q}{AB}$

or, $\frac{P}{15} = \frac{Q}{5} = \frac{100}{20} = 5$



or, $\frac{P}{15} = 5$ or, $\frac{Q}{5} = 5$

or, $P = 15 \times 5$ or, $Q = 5 \times 5$

= 75N or, Q = 25N

Hence, the required forces are 75N and 25N

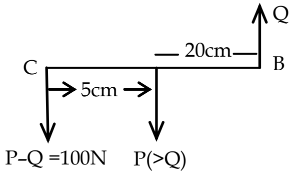
3. Find two unlike parallel forces at a distance of 20 cm equivalent to 100 N force, the line of action of the greater of them being at a distance of 5 cm from the given force.

Solution:

Suppose P and Q be two unlike parallel forces (P > Q) acting at the point A and B such that AB = 20cm. Since forces are unlike, so that their resultant is the force P – Q = 100N acting at the point C, where AC = 5cm. Since forces are parallel. So, we have,

$$\frac{P}{BC} = \frac{Q}{AC} = \frac{P-Q}{BC-AC} = \frac{P-Q}{AB} = \frac{100}{20} = 5$$

$$\frac{P}{25} = \frac{Q}{5} = 5$$



$\therefore \quad \frac{P}{25} = 5$ or, $\frac{Q}{5} = 5$

$\therefore \quad P = 25 \times 5$ or, $Q = 5 \times 5$

$= 125N$ or, $Q = 25N$

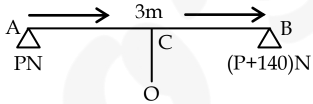
Required forces are 125N and 25N.

4. The extremities of a straight bamboo pole 3 m long rests on two smooth pegs at A and B in the same horizontal line. A heavy load hangs form a point C on the pole. If AC = 3BC and the pressure at B is 140 N more than that at A, find the weight of the load.

Solution:

Let AB be the straight bamboo of 3m long. Let PN and (P + 140)N acting at A and B. Since the heavy load hangs at point C such that AC = 3BC. So, the line of action of the resultant acting at point C. Since forces are parallel,

$$\therefore \quad \frac{P}{BC} = \frac{P+140}{AC} \Rightarrow \frac{P}{BC} = \frac{P+140}{3BC}$$



$\Rightarrow \quad 3P = P + 140$

Hence, P = 70N

Hence, the weight of the load = PV + (P + 140)N

= 70N + (70 + 140)N = 280N

5. A heavy uniform beam 5 m long is supported in a horizontal position by two props, one is at one end and the other is such that the beam projects 1.5 m beyond it. If the weight of the beam is 70 kg wt, find the pressures at the props.

Solution:

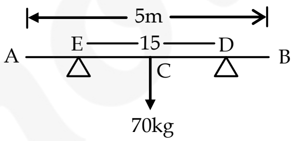
Let AB be the uniform beam AB = 5m

Let C be the midpoint of AB so that AC = CB = 2.5m

Let E and D be two props such that DB = 1.75m, CD = 2.5m – 1.75m = 0.75m, EC = 1.5m – 0.75m = 0.75m

Let R_1 and R_2 be the reactions at E and F.

$$\therefore \quad \frac{R_1}{CD} = \frac{R_2}{EC} = \frac{70}{ED}$$



$$\Rightarrow \quad \frac{R_1}{0.75} = \frac{R_2}{0.75} = \frac{70}{1.5}$$

$$\therefore \quad R_1 = \frac{70 \times 0.75}{1.5} = 35 \text{ kg}$$

$$R_2 = \frac{70 \times 0.75}{1.5} = 35\text{kg}$$

6. P and Q are like parallel forces with the resultant R. If P is moved parallel to itself through a distance x, show that R is displaced by a distance $\frac{Px}{R}$.

Solution:

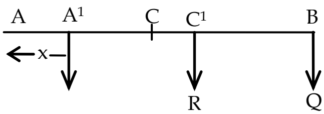
Let R be the resultant of two like parallel forces P and Q acting of A and B respectively. Suppose resultant R acts and P_1 then

$$\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}$$

from $\frac{P}{CB} = \frac{R}{AB} \Rightarrow CB = \frac{P.AB}{R}$ ... ... (i)

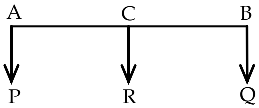
For the second case, let P acts at A^1 such that AA^1 = x then the resultant displace from C to C^1.

So, $\frac{P}{C^1B} = \frac{Q}{A^1C^1} = \frac{R}{A^1B}$



from, $\frac{P}{C^1B} = \frac{R}{A^1B}$

or, $C^1B = \frac{P.\ A^1B}{R}$ ... ... (ii)



Now, $CC^1 = CB - C^1B$

$= \frac{P.AB}{R} - \frac{P.A^1B}{R}$ [$\because$ from (i) and (ii)]

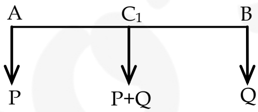
$= \frac{P}{R}(AB - A^1B)$

or, $CC^1 = \frac{PX}{R}$ Hence proved.

7. Two like parallel forces of magnitudes P and Q are acting at the end points A and B of a rod AB of length r. If two opposite forces each of magnitude F are added to P and Q, then prove that the line of action of the new resultant will move a distance $\frac{Fx}{P + Q}$.

Solution:

Suppose the two forces P and Q acting at A and B and let their resultant P+Q acting at C_1.



$\therefore$ $\frac{P}{BC_1} = \frac{Q}{AC_1} = \frac{P + Q}{BC_1 + AC_1}$

or, $\frac{P}{BC_1} = \frac{P + Q}{AB}$

$\therefore$ $BC_1 = P.\frac{AB}{P+Q}$

If the force P is moved parallel to itself through a distance x to D then the resultant act at C_2, where AD = x.

Then, $\frac{P}{BC_2} = \frac{Q}{DC_2} = \frac{P+Q}{BC_2+DC_2}$

or, $\frac{P}{BC_2} = \frac{P + Q}{BD}$

$\therefore$ $BC_2 = \frac{P}{P+Q}.BD$

Now, the required distance which the resultant moves $C_1C_2 = BC_2 - BC_1$

$= \frac{P.BD}{P+Q} - \frac{P.AB}{P+Q}$

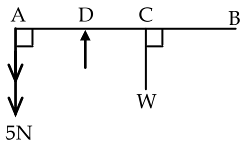
$= \frac{P}{P+Q}(BD - AB) = \frac{P}{P+Q}.AD = \frac{Px}{P+Q}$

8. Straight uniform rod is 3m long when a load of 5N is placed at one end it balances about a point 25cm from that end. Find the weight of the rod.

Solution:

Let W be the weight of the rod AB, acting at centre point C of AB. A load 5N is placed at A, it balances 25cms from that end. i.e. AD = 25cms AB = 3m = 300 cms.

$\therefore$ $AC = BC = \frac{300}{2} = 150$ cms



$\therefore$ DC = AC – AD

= 150 cms – 25cms

= 125cms

Now, using the like parallel forces theorem.

$\frac{5}{DC} = \frac{W}{AD}$

$\therefore$ $W = \frac{5AD}{CD} = \frac{5 \times 25}{125} = 1N$

Exercise 20.2

1. Masses 2kg, 3kg, 5kg and 10kg are suspended from a uniform rod of length 8m, at a distance of 1m, 2m, 3m and 6m respectively from one end. If the mass of rod is 5kg, find the position of the point about which it will balance.

Solution:

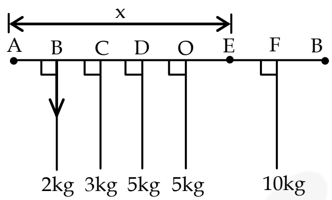
Let AB be the rod whose length is 8m and 0 is the centre of the rod. So OA = OB = 4m.

Given, masses are shown in the figure. Let E be the required point such that AE = xm

So, taking moments about E, $2(x-1) + 3(x-2) + 5(x-3) + 5(x-4) = 10(6-x)$

or, $25x = 103$

$x = 4\frac{3}{25}$ m



i.e. The rod will balance about a point which is at a distance of $4\frac{3}{25}$ m from the end A.

2. Forces equal to 3, 4, 5 and 6N respectively act along the side of a square ABCD taken in order, find the magnitude, direction and line of action of their resultant.

Solution:

Let a be the side of the square ABCD. Let the forces 3, 4, 5 and 6N act along AB, BC, CD and DA respectively. Resolving the forces along and perpendicular to AB, we have,

$x = 3 - 5 = -2$

$y = 4 - 6 = -2$

$\therefore$ The resultant $R = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$ N

If the resultant is inclined at angle θ with AB

$\text{Tan}\theta = \frac{y}{x} = \frac{-2}{-2} = 1 = \text{Tan}\,(180^o + 45^o)$

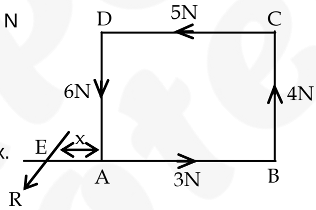
$\therefore$ $\theta = 225^o$

Let the resultant R cut BA produced at E where AE = x.

Taking moment about E,

We have, $4 \times (a + x) + 5a - 6x = 0$

or, $x = \frac{9}{2}a$



3. Two weights of 10 kg and 2 kg hang from the ends of a uniform lever 10 m long and weighing 4 kg. Find the point in the lever about which it will balance.

Solution:

Let A and B be the end points of the lever, 0 be its midpoint and e be the point about which it will balance.

Let AC = x

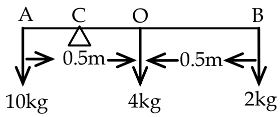
Taking moment about C_1

$10x - 4(0.5 - x) - 2(1 - x) = 0$

or, $10x - 2 + 4x - 2 + 2x = 0$

or, $16x = 4$

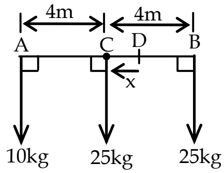
$x = \frac{1}{4} = 0.25$m



This means the lever will balance about the point 25cm from the 10kg end.

4. A uniform rod is of length 8m and weight 25kg and from its extremities are suspended weights 10kg and 25kg respectively. From what point must the rod be suspended so that it may remain in a horizontal position?

Solution: Let AB = 8m be a uniform rod whose weight 25kg act at C, (middle point of AB). Let weight 10kg and 25kg be suspended from A and B 10kg respectively. Also, suppose the rod be suspended at D so that the rod may rest horizontally such that CD = xm.



Taking moment about D,

$10 \times AD + 25 \times CD - 25 \times DB = 0$

or, $10 \times (4+x) + 25x = 5(4 - x)$

or, $8 + 2x + 5x = 20 - 5x \Rightarrow x = 1m$

So the point D must be a distance of (4 + 1) m. i.e. 5m from the end A.

5. ABCD is a square, along AB, CD, AD and DC equal forces, P act. Show that the magnitude of their resultant is equal to double of any components and acts along DC.

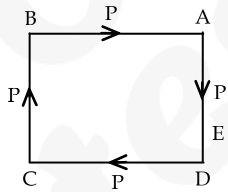
Solution:

Since, equal forces P act along the sides AB, CB, AD and DC of the square ABCD. Resolving the forces along and perpendicular to CD, we have,

$x = -P - P = -2P$, $y = -P + P = 0$

Let R be the resultant, then, $R = \sqrt{x^2 + y^2} = \sqrt{4p^2 + 0} = 2P$

Let θ be the angle made by R with CD. Then



$\text{Tan}\theta = \frac{x}{y} = \frac{0}{-2P} = 0 = \text{Tan}180^o$

$\therefore \quad \theta = 180^o$

Let the resultant cuts AD at E such that DE = x and CD = a

Taking moments about E,

$-p \times DE + p \times EA - p \times CD = 0$

or, $-p \times x + p \times (a - x) - p \times a = 0$

or, $-px + pa - px - pa = 0$

or, $-2px = 0$

$\therefore \quad x = 0$

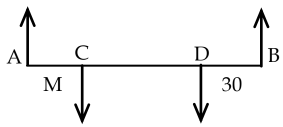
The resultant acts along DC. Hence proved.

6. A light rod of length 72 cm has equal weights attached to it, one at 12 cm from one end and the other at 30 cm from the other end; if it is supported by two vertical strings attached to its ends and if the strings cannot support a tension greater than the weight of 50 kg, what is the greatest magnitude of the equal weight?

Solution:

AB be a light rod of length 72cms. Let W be the equal weight suspended through C and D such that AC = 18cm,

BD =30cm



Now, taking moment about B, we get,

$50 \times AB = W \times CB + W \times DB$

or, $50 \times 72 = W \times 60 + W \times DB$

or, $50 \times 72 = W \times 60 + W \times 30 \quad [\because CB = 72 - 12 = 60]$

or, $W = \frac{50 \times 72}{90} = 40$

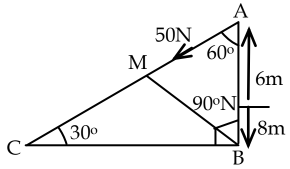
$W = 40$ kgs

7. A pole of length 6 m is placed with its end on a horizontal plane and is pulled by a string attached to its upper end, inclined at an angle of 30° to the horizon. If the tension of the string is equal to 50 N, find the horizontal force which applied at the midpoint of the pole keeps it in a vertical position.

Solution

Suppose AB = 6m, be the pole and its one end is placed in the horizontal line and its upper end situated by the string whose tension is 50N and making angle 30 with the horizontal line BC.

We draw perpendicular BM to the line of action of the tension 50N. Again the angle $\angle BAC = 60^o$. Let F be the horizontal force applied at the point N where BN = 8 metres. Now, taking moment about B, we get,



$50 \times BM = F \times BN$

or, $F = \dfrac{60 \times BM}{BN}$

$F = \dfrac{50 \times B \sin 60^o}{8} = \dfrac{50 \times \sqrt{3/2}}{8} = 5.4125N$

8. Forces equal to P, 2P, 3P act along the sides AB, BC, CA of an equilateral triangle. Find the magnitude and direction of the resultant. Also find where the line of action of the resultant meets BC.

Solution:

Since the force P, 2P, and 3P act along the sides AB, BC, CA of an equilateral triangle ABC. Resolving the forces along and perpendicular to BC, we have,

$x = 2p \cos 0^o + 3p \cos 120^o + p\cos(-120^o)$

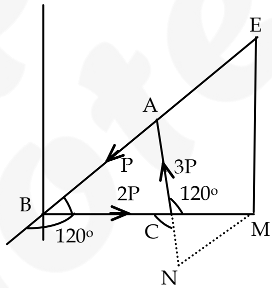
$= 2P - 3P\dfrac{1}{2} - \dfrac{P}{2} = 0$

$y = 2P \sin 0^o + 3P \sin/20^o + P\sin(-120^o)$

$= 0 + 3P\dfrac{\sqrt{3}}{2} - P\dfrac{\sqrt{3}}{2} = \dfrac{2P\sqrt{3}}{2} = P\sqrt{3}$

i. Magnitude of the resultant is given by,

$R = \sqrt{x^2 + y^2} = \sqrt{0 + p^2 . 3} = \sqrt{3}P$



ii. The direction of the resultant $\tan\theta = \dfrac{y}{x} = \dfrac{P\sqrt{3}}{0} = \infty = \text{Tan}90^o$

$\therefore \quad \theta = 90^o$

Direction of resultant is perpendicular to BC.

iii. The position of the line of action of the resultant

$P \times ME - 3P \times MN = 0$

$P(BC + x) \sin 60^o - 3P.CM \sin 60^o = 0$

or, $P(a + x) - 3P.x = 0$

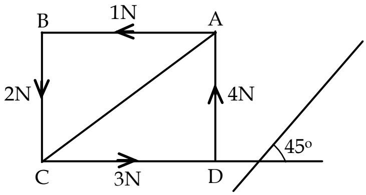
or, $x = \dfrac{a}{2}$ where, a = a side of equilateral triangle.

9. Forces equal to 1 N, 2 N, 3 N, 4 N act along the sides AB, BC, CD, DA, each equal to 2 units, of a square ABCD. Find the magnitude, direction and where the line of action of the resultant intersects C.

Solution:

Let the forces 1N, 2N, 3N and 4N act along the sides AB, BC, CD, DA respectively. Resolving the forces along and perpendicular to CD_1

We have,



$x = 3N - N = 2N$

$y = 4N - 2N = 2N$

Let R be the resultant, then,

$R = \sqrt{x^2 + y^2} = \sqrt{4N^2 + 4N^2} = 2\sqrt{2}N$

Let θ be the angle made by R with AB $\tan\theta$

$= \dfrac{y}{x} = \dfrac{2p}{2p} = 1$

$\theta = 45$

Hence the resultant is parallel to CA. Let the resultant cut CD produced at E, where DE = x.

Let CB = a

$\therefore$ $-4N \times DE + N\times DA + 2N\times CE = 0$

or, $-4x + 1 \times a + 2(a + x) = 0$

or, $-2x = -3a$

or, $x = \frac{3}{2}a$

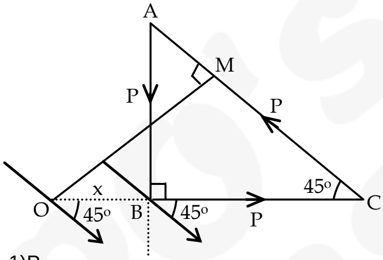
i.e. $DE = \frac{3}{2}CD$

10. Three forces each equal to P act along the sides of an isosceles triangle ABC, right angled at B, taken in order. Find the magnitude, direction and line of action of the resultant.

Solution:

Let the three forces each equal to P act along the sides BCV, CA and AB of an isosceles triangle ABC where $\angle ABC = 90^o$

Resolving the forces along and perpendicular to BC, we have,



$x = P\cos0^o + P\cos135^o + P\cos270^o$

$= P.1 + P\left(-\frac{1}{\sqrt{2}}\right) = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)P$

and, $y = P\sin0^o + P\sin135^o + P\sin270^o$

$= 0 + P.\frac{1}{\sqrt{2}} + P.(-1) = -\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)P$

The resultant $R = \sqrt{x^2 + y^2}$

$= \sqrt{\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2 P^2 + \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2 P^2} = (\sqrt{2}-1)P$

If the resultant is inclined at an angle θ with BC then $\tan\theta = \frac{y}{x} = -1$

$\therefore$ $\theta = 360^o - 45^o = 315^o$

The magnitude of the resultant is $(\sqrt{2}-1)P$ and its direction is parallel to AC.

Let BC = α. Let the resultant cut CB produced at 0 where BO = x

Now taking moment about 0, we have

$P.OM - P.OB = 0$

or, $P.(\alpha + x)\sin45^o - P.x = 0$

or, $\frac{(P\alpha + Px)}{\sqrt{2}} - Px = 0$

or, $\alpha + x - \sqrt{2}x = 0 \Rightarrow x = \frac{\alpha}{\sqrt{2}-1} = \frac{BC}{\sqrt{2}-1}$

$\therefore$ The resultant passes through 0, a point on the line CB produced where $BO = \frac{BC}{\sqrt{2}-1}$.

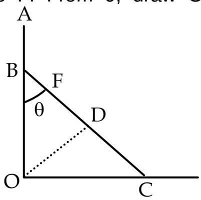
11. At what height from the base of a tree must the end of a rope be fixed so that a man on the ground, pulling at its other end with a given force, may have the greatest tendency to make the tree overturn?

Solution:

Let OB be the tree whose base is 0. Let BC, a rope of length ℓ be fixed on the tree at the point B.

Let a man at C, on the ground pull the rope with the force F. From 0, draw OD perpendicular to BC. Let $\angle OBC = \theta$.

Then the moment of force F about O.



$= F.OD$

$= F.OB\sin\theta$

$= F.BC.\sin\theta.\cos\theta$

$= \frac{1}{2}F.\ell.2\sin\theta.\cos\theta$

$= \frac{1}{2}$ F.ℓ. sin2θ

The tendency to overturn the tree is maximum when the momentum is greatest. i.e. when sin2θ = 1 ⇒ θ = 45°.

Now, OB = BC . sin45° = $\ell \cdot \frac{1}{\sqrt{2}} = \frac{\ell}{\sqrt{2}}$

∴ The rope must be fixed at a distance of $\frac{1}{\sqrt{2}}$ from the base on the tree.

12. Forces proportional to AB, BC and 2CA act along the sides of a triangle ABC taken in order. Show that the resultant is represented in magnitude and direction by CA and that its line of action meets BC at a point x where CX = BC.

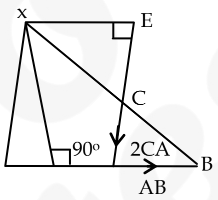
Solution:

Let ABC be a triangle and the forces proportional to AB, BC and 2CA act along the sides of triangle ABC taken in order.

Now, by triangle law of forces, the forces $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ are in equilibrium. Now the resultant of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\overrightarrow{AC}$ along AC.

∴ The resultant of $\overrightarrow{AC}$ along AC and 2 along CA is

$2\overrightarrow{CA} + \overrightarrow{AC} = 2\overrightarrow{CA} - \overrightarrow{CA} = \overrightarrow{CA}$



Hence the resultant of the forces represent in magnitude and direction by CA.

Now, if the line of action of the resultant meets (produced) in x.

Taking moment about x, we get,

BC × O – 2AC × EX + AB × DX = 0

or, 2CA × EX = AB × DX

or, But sinc = $\frac{EX}{CX}$ and sinB = $\frac{DX}{BX}$ and CA sinC = AB sinB ... ... (ii)

Hence, by (ii), equation (i) becomes

2CA × CX sinC = AB × B×sinB

or, 2C × (CA sinC) = B × (AB sinB) [∵ by (ii)]

or, 2CX = BX

Hence the resultant cuts BC in the ratio 2:1. So, we must have CX = BC. Hence proved.

13. The wire passing round a telegraph pole is horizontal and the two portions attached to the pole are inclined at angle 60° to one another. The pole is supported by a wire attached to the middle point of the pole and inclined at 60° to the horizontal. Show that the tension of this wire is $4\sqrt{3}$ times that of the telegraph wire.

Solution:

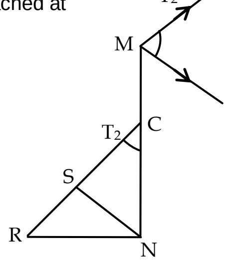
Suppose C is middle point of the telegraph wire MN.

Let the tension of the telegraph wire at M be T_1 and tension of the wire attached at C be T_2.

Since, the two portions attached to the pole are inclined at an angle of 60°.

∴ The resultant tension of the two portions of the wire attached at

M = $2 \times T_1 \cos\frac{60^o}{2}$ ... ... (i)



Again, draw perpendicular from N to RC at S.

Taking moment about N,

We get, $\left(2 \times T_1 \cos\frac{60^o}{2}\right) \times MN = T_2 \times SN$

But SN = CN sin30° = $\frac{MN}{2}$ sin30°

Hence, equation (ii) becomes,

$2T_1 \cos30^o \times MN = T_2 . \frac{MN}{2} \sin30^o$

or, $2T_1 \frac{\sqrt{3}}{2} = T_2 \frac{1}{2} . \frac{1}{2}$

or, $T_1 \sqrt{3} = \frac{T_2}{4} \Rightarrow T_2 = 4\sqrt{3}\, T_1$

i.e. Tension of the wire attached at C is $4\sqrt{3}$ times that of the tension of the telegraph wire.

14. Forces forming a couple are 8N each and the arm of the couple is 5m. Find (a) force of an equivalent couple whose arm is 4m(b) the arm of an equivalent couple each of whose forces is 12N.

Solution: Since length of arm (p) = 5m

Force (f) = 8N

$\therefore$ Couple of moment = P×F = 5×8 = 40Nm.

a. Here moment of couple equivalent = 40Nm.

Arm of couple (P) = 4m

Force of couple (F) = ?

Since, couple of moment = F×P

or, $40 = F \times 4 \Rightarrow F = 10N$

b. Here, couple of moment equivalent = 40 Nm

Force of couple (F) = 12N

Arm of couple (P) = ?

Since, Couple of moment = P×F

or, $40 = P \times 12$

$\therefore \quad P = \frac{40}{12} = \frac{10}{3} = 3\frac{1}{3}$ m

15. Given a couple of moment 20 Nm, (a) find the length of the arm if each force is 5 N, (b) find the magnitude of constituent force if the length of the arm is 2 m.

Solution:

The moment of the given couple = 20Nm.

a. Let x be the arm of the couple each at whose force is 5N. So, its moment = $5 \times x$ Nm.

Given, that, $5x = 20$

$x = 4m$

b. Let f be the one force of couple whose arm is 2m so, its moment is = $F \times 2$Nm

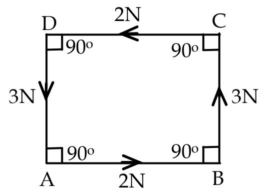
Given that, $F \times 2 = 20$

$F = 10N$

16. Forces of magnitude 2 N, 3 N, 2 N, and 3 N act along the sides of a square of each side 50 cm taken in order. Find the resultant.

Solution:

Let the forces 2, 3, 2, 3 Newton's act along the sides AB, BC, CD and DA respectively of the square ABCD each of length a metre.



Now, the force 2N and 2N Newton's acting along AB and CD forms a couple of moment

= 2×a = 2a Nm.

Again the force 3N and 3N act along BC and DA form a couple of moment

= $3 \times a$ = 3a Nm

Since the both moments are in the same sense. Hence the four forces form a couple of moment

= (2a + 3a)

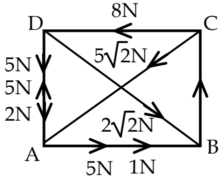
= 5a Nm where a is the size of the square.

17. ABCD is a square whose side is 2m, along AB, BC, CD and DA act forces equal to 1, 2, 8 and 5N and along AC and DB forces equal to $5\sqrt{2}$ and $2\sqrt{2}$ N. Show that they are

equivalent to a couple whose moment is equal to 16Nm.

Solution:

The components of the forces $5\sqrt{2}$N along AC are $5\sqrt{2}\cos45^o$ along DC and $2\sqrt{2}\sin45^o$ along DA. i.e. 2N along DC and 2N along DA.



Total force along AB = (5 + 1)N = 6N = 6N

Total force along CD = (8 – 2)N = 6N

Total force along DA = (5 – 5 + 2)N = 2N

The force 6N and 6N along AB and CD form a couple whose moment = 6×2 = 12 Nm

Again, the forces 2N and 2N along BC and DA form a couple whose moment

= 2 × 2 = 4Nm

The resultant of two couples is a couple. Hence, the resultant of all forces is equivalent to a couple whose moment = (12 + 4) Nm

= 16Nm

18. ABCD is a square. Along the sides AB, BC, CD, DA act forces equal to 3 N, 3 N, 6 N and 2 N. Along the diagonals AC and DB act the forces equal to $\sqrt{2}$ N and $2\sqrt{2}$ N. Prove that the resultant is a couple. If each side of a square is 1m, what is the moment of the couple?

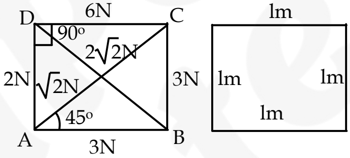
Solution

Here, the resolved part of the force $\sqrt{2}$ along AB = $\sqrt{2}\cos45^o$

AD = $\sqrt{2}\sin45^o$

Hence, total force along AB = $3 + \sqrt{2}\cos45^o$

$= 3 + \sqrt{2} \times \frac{1}{\sqrt{2}} = 4$N



Total force along AD = AD = $2 + \sqrt{2}\sin45^o$

$= 2 + \sqrt{2}\,\frac{1}{\sqrt{2}} = 3$N

Total force along BC = $3 + 2\sqrt{2}\sin45^o = 3 + 2\sqrt{2}\,\frac{1}{\sqrt{2}} = 5$N

Total force along CD = $6 + 2\sqrt{2}\cos45^o = 6 + 2\sqrt{2}\,\frac{1}{\sqrt{2}} = 8$N

Hence the forces 3N, 3N, 6N, 2N act long the sides AB, BC, CD, and DA. Hence, their resultant forces form couple (4, BC), (3, AB), (5, CD) and (8, AD)

Hence, their resultant is a couple.

Now, the moment of couple = 4 × BC + 3×AB

= 4×1 + 3×1 = 7N

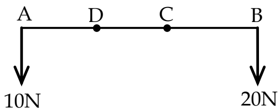
Also, the moment of couple = 5 × CD + 8×AD

= 5×1 + 8×1 = 13N

19. Two like parallel forces of magnitude 10 N and 20 N are acting at two points of a rigid body and 3 m apart. If a couple of moment 12 Nm is combined with them, find the distance by which the resultant is displaced.

Solution:

Let A and B be the points where 10N and 20N forces act. Let C be the point from where the line of action passes. Let C be shifted to D after the couple of moment 12Nm is added.



Now, AC : AB = 20 : 10

or, $\frac{AC}{AC + CB} = \frac{20}{30}$

or, $AC = \frac{20}{30} \times 3 = 2$m

For a couple of moment 12Nm, if the arm is 3m, the constituent forces are of magnitude

4N force. So the force 4N will be added to 10N and subtracted from 20N (because of positive moment). So,

Now, DC = AC – AD = 2m – 2.6m = –0.6m = 0.6

20. ABCD is a rectangle with length AB = 30 cm and breadth BC = 20 cm. Forces of magnitude 4 N act along AB and CD and forces of magnitude 3 N act along AD and CB. Find the perpendicular distance between the resultant of 4 N and 3 N at A and that of those at C.

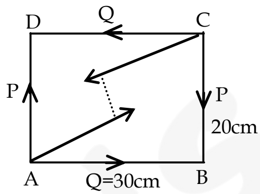
Solution:

Here, AB = CD = 30cms

BC = DA = 20 cms

Force, Q = 4N

P = 3N



The resultant of the forces P and Q at A.

$= \sqrt{P^2 + Q^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + b} = 5N$

Similarly, the resultant force at C,

$= \sqrt{P^2 + Q^2} = 5N$

These two resultant forces form a couple

Moment of these couple $= \sqrt{P^2 + Q^2}\,.d = 5d$

Where d = perpendicular distance between the line of two resultant forces.

Again the given forces form two couple. The moment of these couples.

$= Q.\ AD - P.AB = 4\times20 - 3\times30 = -10$

These two couples are equivalent to a single couple

$\therefore \quad \sqrt{P^2 + Q^2}\,.d = Q.\ AD - P.AB$

or, $5d = -10$

or, $d = -2$cms

$\therefore \quad d = 2$ cms (distance is never negative)

Chapter 21: Dynamics

Exercise 21.1

1. a. An average sized onion has a mass 50g. Find the weight of the apple in Newton? ($g = 9.8 m/s^2$)
   b. A bicycle of mass 20 kg is accelerated at $2 m/sec^2$. Find the force acting on it.
   c. Find the acceleration produced when a force of 5hg wt. acts on a mass of 1 kg.

Solution:

a. Here, mass (m) = 50g = $\frac{50}{100}$ kg = 0.05 kg
   $g = 9.8 \text{ m/s}^2$
   Weight (w) = ?
   $\therefore$ w = mg = 0.05 × 9.8 = 0.49N Ans.

b. Here, mass (m) = 20kg
   Acceleration (a) = 2 m/s^2
   Force (f) = ?
   $\therefore$ f = ma = 20 × 2 = 40N

c. Here, acceleration (a) = ?
   Force (f) = 5kg = 50N
   Mass (m) = 1kg
   $\therefore$ F = ma
   or, $a = \frac{f}{m} = \frac{50}{1}$
   $\therefore$ $a = 50 \text{ m/s}^2$

2. a. A bicycle has mass 50kg.If its velocity increases from 2m/sec to 5m/sec in 6 seconds, find the force exerted on it.
   b. A body of mass 10kg falling from a certain height is brought to rest after striking the ground with a speed of 5m/sec. If the resistance force of the ground is 200N, find the duration of contact.
   c. A car is pushed on a frictional smooth plane with an average force of 50N for 10 sec. If the car with mass 500 kg is at rest in the beginning, find the velocity acquired by the car.

Solution:

a. Mass of bicycle (m) = 50 kg
   Initial velocity (u) = 2 m/s
   Final velocity (v) = 5 m/s
   Time taken (t) = 6 sec
   Force exerted (f) = ?
   $\therefore$ $F = \frac{m(v-u)}{t} = \frac{50(5-2)}{6} = \frac{50 \times 3}{6} = 25N$
   $\therefore$ F = 25N

b. Here, mass of the body (m) = 10kg
   Initial velocity (u) = 5m/s
   Final velocity (v) = 0
   Force on the ground (f) = 200N
   Duration of contact (t) = ?
   Now, applying the formula, $f = \frac{mv - mu}{t}$
   or, $t = \frac{m(v-u)}{f} = \frac{10(0-5)}{200} = -0.25$
   $\therefore$ t = 0.25 sec

c. Here, Average force (f) = 50N
Mass of car (m) = 500kg
Time taken (t) = 10 sec
Initial velocity (u) = 0
Final velocity (v) = ?

Now, we have, $f = \frac{m(v - u)}{t}$

or, $v = \frac{ft}{m} + u$

or, $v = \frac{50 \times 10}{500} + 0$

$\therefore$ v = 1 m/s

3. a. A horse directs a horizontal Jet of water, moving with a velocity of 30 m/sec on a vertical wall. If the mass of water per second striking the wall is 3kg/ sec, find the force on the wall.

b. Sand allowed to fall vertically at a steady rate hits a horizontal floor with a speed $0.04ms^{-1}$. If the force exerted on the floor is 0.004N, find the mass of sand falling per second.

c. Rain drops falling vertically on ground at the rate of 0.3 kgs^{-1} come to rest after hitting the ground. If the resistance force of the ground is 3N, find the velocity of rain drops just before hitting the ground.

Solution:

a. Mass of water per second $\left(\frac{m}{t}\right)$ = 3 kg/sec.

Initial velocity (u) = 30 m/s
Final velocity (v) = 0
Force on the wall (f) = ?

Now, apply, $f = \frac{mv - mu}{t}$

or, $f = \frac{m(v - u)}{t} = \frac{m}{t}(v - u) = 3(0 - 30) = -90$

$\therefore$ f = 90N

b. Here, initial velocity (u) = 0
Fin al velocity (v) = 0.04 m/s
Force exerted on the floor (f) = 0.004N
Mass of sand falling per second (m/t) = ?

Now, applying $f = \frac{mv - mu}{t} = \frac{m}{t}(v - u)$

or, $\frac{m}{t} = \frac{f}{v - u} = \frac{0.004}{(0.04 - 0)} = 0.1$

$\therefore$ $\frac{m}{t} = 0.1$ kg/s

$\therefore$ Mass of sand falling per second = 0.1kg/sec

c. Quantity of rain falling per second $\left(\frac{m}{t}\right)$ = 0.3 kg/s

Force of the ground (f) = 3N
Velocity before hitting the ground (u) = ?
Velocity after hitting the ground (v) = 0

We know, $f = \frac{mv - mu}{t} = \frac{m}{t}(v - u)$

or, 3 = 0.3 (0 – u)
or, u = –10 m/s
$\therefore$ u = 10 m/s

4. a. A force 1 kg wt. acts on a body continuously for seconds and causes it to describe one metre in that time, find the mass of the body.
   b. A body of, mass 25kg is acted upon by a force of 200N. How long will it take to move the body from rest through 64m?
   c. A force of 520N acting on a body for 30 secs increases its velocity from 290 m/sec to 350 m/sec. Find the mass of the body.
   d. A bullet of mass 20g fired into a wall with a velocity of 30m/sec loses its velocity in penetrating into a wall through 3cms. Find the average force exerted by the wall.
   e. How large a force required to bring a motorbike of mass 500 kg moving with a velocity of $50ms^{-1}$ to rest at
      i. a distance of 50m   ii. in 10 secs
   f. A constant force of 20N acting on an object reduces if velocity from $30ms^{-1}$ to $10ms^{-1}$ in 3 secs. Find the mass of the object.
   g. A car of mass 1000 kg travelling at 36 km/hr is brought to rest over a distance of 20m. Find the average braking force.
   h. Find the velocity of a 5kg shot that will just penetrate through a wall 20 cms thick the resistance being 40 tons wt.

Solution:

a. Suppose m be the mass of an object and a be the acceleration. So that

$f = ma$ [$\because f = 1\times9.8 = 9.8N$]

or, $9.8 = ma$

or, $a = \dfrac{9.8}{m}$

Now, $s = \dfrac{1}{2}at^2$

or, $l = \dfrac{1}{2}\times\dfrac{9.8}{m}\times10^2$

or, $l = \dfrac{9.8\times50}{m}$

or, $m = 490$ kg

b. Mass of body (m) = 25kg

Force (f) = 200N

Time taken (t) = ?

Initial velocity (u) = 0

Distance covered (s) = 64m

we have, $f = ma$

or, $a = \dfrac{f}{m} = \dfrac{200}{25} = 8$

$\therefore$ $a = 8\ m/s^2$

Now, $s = ut + \dfrac{1}{2}at^2$

or, $64 = 0 + \dfrac{1}{2}\times8\times t^2$

or, $4t^2 = 64$

or, $t^2 = 16$

$\therefore$ $t = 4$ sec

c. Here, force (f) = 520N

Time (t) = 30 sec

Initial velocity (u) = 290 m/s

Final velocity (v) = 350 m/s

Mass of body (m) = ?

We have, $f = \frac{m(v-u)}{t}$

or, $m = \frac{ft}{v-u} = \frac{520 \times 30}{350-290} = \frac{520 \times 30}{60} = 260$

$\therefore$ m = 260 kg

d. Mass of bullet (m) = 20gms = 0.02 kg
Initial velocity (u) = 30 m/s
Distance (s) = 3 cms = 0.03m
Average force (f) = ?
Final velocity (v) = 0

$\therefore$ Applying, $v^2 = u^2 + 2as$

or, $0 = 30^2 + 2a \times 0.03$

or, $-900 = 0.06a$

or, $a = -\frac{900}{0.06} = -15000$

$\therefore$ a = —15,000 m/s^2

Now, force, f = ma = 0.02×(–15,000) = –300

$\therefore$ f = 300N

e. Mass of motor bike (m) = 500kg
Initial velocity (u) = 50m/s
Final velocity (v) = 0
Force (f) = ?

(i) A distance of 50m
Since, s = 50m
Now, using, $v^2 = u^2 + 2as$

or, $0^2 = 50^2 + 2a \times 50$

or, 2500 = 100a

or, a = –25 where a is retardation

$\therefore$ a = 25 m/s^2

$\therefore$ f = mass × retardation
= 500 × 25
= 12500N

(ii) In 10 seconds
Now, v = u + at
or, 0 = 50 + a×10
or, 10a = –50
a = –5 m/s^2, where a is retardation
$\therefore$ f = mass × retardation = 500 × 5 = 2500 N

f. Force (f) = 20N
Initial velocity (u) = 30 m/s
Final velocity (v) = 10 m/s
Time taken (t) = 3 sec
Mass (m) = ?

$\therefore$ $F = \frac{m(v-u)}{t}$

or, $m = \frac{Ft}{v-u} = \frac{20 \times 3}{(10-30)} = \frac{20 \times 3}{-20}$

$\therefore$ m = 3kg

g. Final velocity (v) = 0
Distance (s) = 20m
Average force (f) = ?
We have,

$\therefore$ $v^2 = u^2 + 2as$

or, $0^2 = 10^2 + 2.a \times 20$

or, $-100 = 40a$

or, $a = -\frac{100}{40} = -2.5$ m/s^2 where a is retardation.

Now, f = mass × retardation

= 1000 × 2.5

= 2,500N

h. Mass of shot (m) = 5kg

Penetrating space (S) = 20cms = 0.2m

Resistance (f) = 40 tones

= 40×1000×9.8N

If a is the retardation produced by the wall then, f = –ma

or, $a = -\frac{f}{m} = -\frac{40 \times 1000 \times 9.8}{5} = -78400$ m/s^2

Let u is initial velocity and v be final velocity then u = 2, v = 0

Using the formula,

$v^2 = u^2 + 2as$

or, $0^2 = u^2 + 2(-78400) \times 0.2$

or, $u^2 = 31{,}360$

or, $u = \sqrt{31360}$

∴ u = 177.08 m/s

5. Find the velocity of 4 kg shot that will just penetrate through a wall 16 cms thick, the resistance being 4 metric tons weight.

Solution:

Here, mass of the shot (m) = 4kg

Penetrated space (S) = 16cms = 0.16m

Resistance (F) = 4 metric tons

= 4×1,000×9.8 = 392000N

If a is retardation produced by the wall then, F = –ma

or, $a = -\frac{F}{m} = -\frac{39200}{4} = -9800$ m/s^2

Let u is the initial velocity and v is final velocity then, u = 2, v = 0

Using the formula,

$v^2 = u^2 + 2as$

or, $0^2 = u^2 + 2(-9800) \times 0.16$

or, $u^2 = 3136$

or, $u = \sqrt{3136} \Rightarrow u = 56$m/s

6. A resultant force of 25N acts on a mass of 0.5 kg starting from rest. Find.

a. the acceleration

b. the final velocity after 20 secs

c. the distance moved (g = 10 m/ sec^2)

Solution:

Here, force acting (f) = 25N

Mass of body (m) = 0.50 kg

Initial velocity (u) = 0

a. The acceleration in ms^{-2}

Now, f = ma

or, $a = \frac{F}{m} = \frac{25}{0.50} = 50$ m/s^2

b. The final velocity after 20 sec.

⇒ Let v be the velocity after 20 sec.

Then, using v = u + at

or, $v = 0 + 50 \times 20$

$\therefore$ v = 1,000 m/s

Distance of penetration of the target

If a is the retardation of the system, then F = ma

$\Rightarrow$ $a = \frac{F}{m} = \frac{72}{0.006} = 12,000\ m/s^2$

If S is the required distance of penetration of target then, $s = \frac{1}{2}at^2 = \frac{1}{2} \times 12,000 \times (0.01)^2$

$\therefore$ S = 0.6m

c. The distance moved in 20 sec.

If S is required distance moved in 20 sec.

Then, $S = ut + \frac{1}{2}at^2$

or, $S = 0 + \frac{1}{2} \times 50 \times (20)^2$

or, S = 10,000m

S = 10km

7. A body of mass 1 kg is falling under gravity at the rate of 28m/secs. What is the uniform force that will stop it in (a) 0.1 sec (b) 20 cms

Solution:

Here, mass of the body (m) = 1kg

Initial velocity (u) = 28 m/s

Final velocity (v) = 0

Let P be the uniform force applied in upward direction to stop the body. Then force acting on the body are

a. Weight of the body, w = mg = lg in downward direction

b. The force p in upward direction

$\therefore$ Resultant force, F = P – lg ... ... (i)

Case – I:

When time taken t = 0.1 second.

Now, if this resultant force (i) produces the retardation a.

Then, v = u + at

or, $0 = 28 - a \times 0.1 \Rightarrow a = \frac{28}{0.1} = 280\ m/s^2$

Then by second law of motion,

We get,

P – mg = ma

or, $P - 1\times10 = 1\times280 \Rightarrow P = 290N = \frac{290}{10}\ kg = 29kg$

Case – II

When distance is S = 20cm = 0.2m

Now, if the resultant force (i) produce the retardation a,

Then, $v^2 = u^2 + 2aS$

or, $0^2 = (28)^2 + 2\times a\times0.2 \Rightarrow a = \frac{784}{0.4} = 1960 m/s^2$

Then, by second law of motion

P – mg = ma

or, P = 1(10 + 1960)

$P = 1970N = \frac{1970}{10}\ kg = 197kg$

8. a. A body of mass 20kg falls 10m form rest and is then brought to rest penetrating 0.5 m into sand. Find the resistance of the sand on it in kg wt.

b. A mass of 4kg falls 200cms from rest and is then brought to rest by penetrating 20cms into some sand. Find the average thrust of the sand on it.

Solution:

a. Mass of body (m) = 20kg

Distance covered (s) = 10m

Initial velocity (u) = 0

$\therefore$ $v^2 = u^2 + 2gh \Rightarrow v^2 = 20g$ ... ... (i)

The velocity given by (i) is reduced to zero when the body goes to 0.5m into sand. If a is the retardation of the system then,

$(0)^2 = v^2 - 2\times a\times 0.5 \Rightarrow a = v^2$

$\Rightarrow a = 20g\ m/s^2$

Let T be the average thrust of the sand on the body. Now, when the body is penetrating into the sand, then the force acting on the body are

a. A force TN of the sand acting upward
b. The weight 20gN of the body acting downward.

Resultant upward force = (T – 20g)N

Then applying Newton's second law of motion, we have,

$T - mg = ma$

or, $T - 20g = 20\times log$

or, $T = 20g + 200g$

$T = 220$ kgwt

b. Suppose V is the velocity of the body when it falls 200cms from rest under gravity.

Then u = 0, v = v, h = 2m

$\therefore$ $v^2 = u^2 + 2gh$

$v^2 = 0 + 2g \times 2 \Rightarrow v^2 = 4g$ ... ... (i)

The velocity given by (i) is reduced to zero when the body goes to 20cms = 0.2m into sand. If a is the retardation of the system, then

$O^2 = v^2 - 2\times a\times 0.2$

or, $a = \frac{v^2}{0.4} = \frac{4g}{0.4} = 10g\ m/s^2$

Let T be the average thrust of the sand on the body.

Now when the body is penetrating into the sand, then the force acting on the body are

a. A force TN of the sand acting upward
b. The weight 4gN of the body acting downward

$\therefore$ Resultant upward thrust = (T – 4g)N

Then apply Newton's second law of motion,

$T - mg = ma$

or, $T = 4g = 4\times 10g$

or, $T = 40g \Rightarrow T = 40$kg wt

9. A man of mass 70 kg stands on a lift which moves with a uniform acceleration of 2m /sec^2. Find the reaction of the floor when the lift moves (a) up (b) down

Solution:

Here, two forces act on a mass one, the weight mg of the man acting vertically downwards and other, the reaction R of the floor on a man acting vertically upward.

a. Since, the lift is moving upward, so the resultant upward force = R – mg, by Newton's second law.

$R - mg = ma$ (where a is acceleration of lift)

or, $R = m(g + a) = 70(9.8 + 2) = 826N$

b. Since the lift is moving downwards, so the resultant downward force = mg – R

By Newton's second law,

$mg - R = ma$

or, $R = mg - ma$

$= m(g - a)$

$= 70(9.8 - 2)$

$= 546N$

10. A bullet of mass 0.006 kg travelling at 120 m/sec penetrates deeply into a fixed target & is brought to rest in 0.01 secs. Calculate.
The average retarding force exerted on the bullet. ($g = 10ms^{-2}$)

Solution:

Mass of the bullet (m) = 0.006kg
Final velocity of the bullet (v) = 120m/s
Time taken (t) = 0.01 sec.
Initial velocity (u) = 0
If F be the average retarding force on bullet then,

$$F = \frac{\text{Change in momentum}}{\text{Time taken}} = \frac{m(v-u)}{t} = \frac{0.006 \times (120 - 0)}{0.01} = 72N$$

11. A resultant force of 12N acts for 5 sec on a mass of 2kg. What is the change in momentum of the mass? What would be the change in momentum of a mass of 10kg under the same condition?

Solution:

Given, Force act (F) = 12N
Time taken (t) = 5 sec
Change in momentum = ?

Case–I

When mass m is 2kg
Let a is acceleration of the system then, F = ma
or, $12 = 2\times a \Rightarrow a = 6\ m/s^2$
Let v be the velocity then v = u + at
or, $v = 0 + 6 \times 6$
$\Rightarrow$ v = 30 m/s
Change in momentum = mv – mu
$= 2\times30 - 2\times0 = 60\ kgms^{-1}$

Case–II

When mass is m = 10kg
Under the same condition, v = 30 m/s, u = 0 m/s
Change in momentum = 10 (30 – 0) = 300 kgm/s

12. Solution:

a. Mass of bullet (M) = 0.02kg
Mass of rifle (m) = 10kg
Muzzle velocity of bullet (v) = 1,000 m/s
Recoil velocity of rifle (v) = ?
We know,
Mass of the bullet × Muzzle velocity = Mass of rifle × Recoil velocity
or, $0.02 \times 1{,}000 = 10\times v$
or, $v = \frac{0.2 \times 1{,}000}{10} = 20\ m/s$

b. Here, momentum of the bullet = $mv = 330\times0.1$
Momentum of the gun = $Mv = 10\times v$
$\therefore$ Momentum of bullet = Momentum of gun
or, $330 \times 0.1 = 10\times v$
or, $v = \frac{330\times0.1}{10} = 3.3\ m/s$
Since initial velocity of gun and bullet = 0 m/s
Total momentum before firing = mu + Mu
$= 0.1\times0 + 10\times0 = 0$

c. Here, m = Mass of shot = 700kg
v = Velocity of shot = 600 m/s

M = Mass of the gun = 40 metric tons = (40×1000)kg
v = Velocity of the gun = ?
Momentum of the shut = mv = 700×600
Momentum of the gun = Mv = (40×1000) v
By the principal of conservation of linear momentum.
Momentum of shot = momentum of gun (in magnitude)
or, $700 \times 600 = (40 \times 1000)v$
or, $v = \frac{700 \times 600}{40 \times 1000} = 10.5$ m/s
$\therefore$ Velocity of gun = 109.5 m/s

d. Let v be the recoil velocity of the gun
Then moment of the shut = 10×245
Momentum of the gun = $5000 \times v$
But we know momentum of the shot = Momentum of the gun
or, $10 \times 245 = 5000 \times v$
or, v = 0.49 m/s
The gun recoils with velocity 0.49 m/s. Apply a constant force to the gun so that it will stop after recoiling at time $1\frac{1}{4} = \frac{5}{4}$ seconds.
Let a be the retardation then $0 = v - at \Rightarrow a = \frac{v}{t}$
If f is required constant force to be applied then, $f = ma = m \times \frac{v}{t} = \frac{5000 \times 0.49}{5/4}$
$$= \frac{5000 \times 0.49 \times 4}{5} = 1960\text{N}$$

e. Let v be the velocity of the gun then momentum of the shot = 400×400
Momentum of the gun = $80{,}000 \times v$
$\therefore$ Momentum of gun = Momentum of shot
or, $80{,}000 \times v = 400 \times 400$
or, v = 2 m/s
The gun recoils with velocity 2 m/s. Applying a constant force to the gun so that it will stop after recoiling at distance 2 meters.
$\Rightarrow$ Let a be the retardation, then $0^2 = v^2 - 2as \Rightarrow a = \frac{v^2}{25}$
If f is the required constant force to be applied then, $f = ma = m \times \frac{v^2}{25}$
$$= 80{,}000 \times \frac{2^2}{2\times2} = 80{,}000\text{N}$$

f. Let v be the recoil velocity at the gun then, momentum of the shot = 40×140
Momentum of the gun = $7{,}000 \times v$
But, momentum of gun = Momentum of shot $7{,}000 \times v = 40 \times 140$
or, v = 0.8 m/s
The gun recoil with velocity 0.8 m/s. Applying a constant force to the gun so that it wll stop after recoiling at distance 6.4m.
Let a be the retardation, then, $0^2 = v^2 - 2as \Rightarrow a = \frac{v^2}{25}$
If f be the required constant force to be applied then, f = ma
$$= 7{,}000 \times \frac{(0.8)^2}{2 \times 6.4}$$
$$f = 350\text{N}$$

g. Let v be the recoil velocity at gun then momentum of bullet = 2×250
Momentum of gun = $100 \times v$
But, momentum of gun = Momentum of bullet

$100 \times v = 2 \times 250$

$v = 5$ m/s

13. A cricket ball of mass 150g is moving with a velocity of 12m/sec & it hit by a bat so that the ball is turned back with a velocity of 20m/sec. The force of blow acts for 0.01 secs on the ball. Find the average force exerted by the bat on the ball.

Solution:

Here, mass of ball (m) = 150g = 0.15kg

Initial velocity (u) = 12 m/s

Final velocity (v) = 20 m/s

Time duration (t) = 0.01 sec

Let R be the average force exerted on ball by bat then, impulse of force = Change in momentum

$R \times t = m(12 - x - 20)$

$R = \frac{0.15 \times 32}{0.01} = \frac{4.8}{0.01} = 480N$

14. a. A body of mass 2kg moving with a uniform velocity of 40m/sec collides with another at rest. If the two together begin to move with a uniform velocity of 25m/sec. Find the mass of the other.

 b. Two bodies of masses 10 kg and 5 kg move along the opposite directions with velocity $15ms^{-1}$ and $-6ms^{-1}$ respectively collide and stick together. Find their common velocity.

 c. A 20g bullet enters a block of wood of mass 980g with a velocity of $300ms^{-1}$. Find the common velocity the bullet and the wood.

Solution:

a. Mass of first body (m_1) = 2kg

Initial velocity (u_1) = 40 m/s

Mass of second body (m_2) = ?

Initial velocity second body (u_2) = 0 m/s

Le their common velocity of collision be v, then v = 25m/s

Then total momentum before collision = $m_1u_1 + m_2u_2 = 2\times40 + m_2\times0 = 80$

On collision,

$m_1u_1 + m_2u_2 = (m_1 + m_2)\, v$

or, $80 = (2 + m_2)\, 25$

or, $2 + m_2 = \frac{80}{25} = m_2 = 1.2$ kg

b. Let v be the velocity just after collision of two bodies whose combined mass = (10 + 5) kg = 15kg

$\therefore$ Its momentum = (10 + 5)v = 15v

Also, total momentum before collision = $m_1u_1 + m_2u_2$

$= (10\times15 + 5\times-6) = 150 - 30 = 120$

Now, by the principal of conservation of linear momentum, the total momentum of the system remains constant. Hence momentum after collision = momentum before collision

or, $15v = 120$

or, $v = 8$ m/s

c. Mass of bullet (m_1) = 20g = 0.02 kg

Mass of block (m_2) = 980g = 0.98 kg

initial velocity of bullet (u_1) = 300 m/s

Initial velocity of block (u_2) = 0 m/s

Let v be the common velocity of bullet and block, v = ?

Now, by the principal of conservation of linear momentum

We have, $(m_1u_1 + m_2u_2) = (m_1 + m_2)\, v$

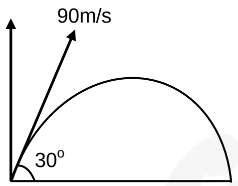
or, $(0.02 \times 300 + 0.98 \times 0) = (0.02 + 0.98)\, v$

or, $(6 + 0) = 1 \times v$

$\Rightarrow$ v = 6 m/s

Exercise 21.2

1. A ball is thrown with a velocity of 98 m/sec at an elevation of 30°, find
   a. the horizontal range,
   b. time of light
   c. magnitude and direction of the velocity after 2 seconds.
   d. position after 2 seconds.



Solution:

Initial velocity (u) = 98m/s

Angle of elevation (θ) = 30^o

a. Horizontal range (R) = $\dfrac{u^2 \sin^2 \theta}{g}$

$= \dfrac{(98)^2 . \sin^2 .30}{10} = \dfrac{9604 \times 50.866}{10} = 831.7\text{m}$

b. Time of flight (T) = $\dfrac{2u \sin\theta}{g} = \dfrac{2 \times 98 \times \sin 30^o}{10} = 9.8$ sec

c. Let v be the striking velocity of the ball making an angle θ with horizontal.

$\therefore$ v_x = horizontal component = $v\cos\theta = 98.\cos 30^o = \dfrac{98\sqrt{3}}{2} = 49\sqrt{3}$ m

v_y = vertical component = $u \sin\alpha - gt = 98 \sin 30^o - 10 \times 2$

$= \dfrac{98}{2} - 20 = 49 - 20 = 29\text{m}$

$\therefore$ Now, $v^2 = v_x^2 + v_y^2 = (49\sqrt{3})^2 - (29)^2$

$= 7202.58 - 841 = 6361.58$

$\therefore$ v = 79.76 m/s

Direction, $\text{Tan}\theta = \dfrac{v_y}{v_x} = \dfrac{29}{49\sqrt{3}} = 0.342$

$\theta = \text{Tan}^{-1}(0.342)$

$\theta = 20^o$

d. If (x, y) be position of projectile after time t = 2 sec

$\therefore$ $x = u\cos\alpha\, t = 98 \times \cos 30^o \times 2$

$= 98 \times \dfrac{\sqrt{3}}{2} \times 2 = 98\sqrt{3}$ m

$y = u\sin\alpha\, t - \dfrac{1}{2} gt^2$

$= 98 \times \sin 30^o \times 2 - \dfrac{1}{2} \times 10 \times 2^2$

$= 98 \times \dfrac{1}{2} \times 2 - \dfrac{40}{2} = 98 - 20 = 78\text{m}$

$\therefore$ Position (x, y) = $(98\sqrt{3}, 78)$

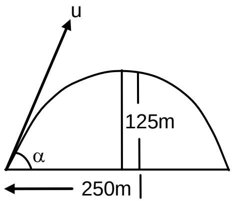
2. a. Find the velocity and the direction of projection of a shot which passes in a horizontal direction just over the top of a wall which is 250 m off and 125m high (g = 9.8ms^{-2}]
   b. A shot is seen to pass horizontally just over a vertical wall 64m high and 96m off. Find the magnitude and direction of the velocity of the shot with which it was fired.

Solution:

a. Let u be the velocity of the projection of a shot making an angle α with the horizon. Since the shot just passes the top of the building, it moves horizontally.

$\therefore$ Max. height (H) = 125m

Horizontal range (R) = 2 × 250m = 500m



$\therefore \quad H = \frac{u^2 \sin^2\alpha}{2g} \Rightarrow 125 = \frac{u^2 \sin^2\alpha}{2g}$ ... ... (i)

and, $R = \frac{u^2 \sin^2\alpha}{g} \Rightarrow 500 = \frac{u^2 \sin^2\alpha}{g}$ ... ... (ii)

Dividing (i) by (ii)

$\frac{1}{4} = \frac{\sin^2\alpha}{2\sin\alpha \cdot \cos\alpha} \Rightarrow \text{Tan}\alpha = 1 = \text{Tan}45^o \Rightarrow \alpha = 45^o$

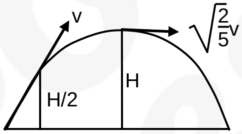
Velocity of projection:

Substituting the value of α in (i) $\Rightarrow 125 = \frac{u^2}{2g} \times \left(\frac{1}{\sqrt{2}}\right)^2$

$\Rightarrow u^2 = 500 \times 9.8 = u = 70\text{m/s}$

b. The velocity of a particle when at its greatest height is $\sqrt{\frac{2}{5}}$ of its velocity when at half its greatest height. Show that the angle of projection is 60°.

Let u be the velocity and α, angle of projection off a particle. Let H be the greatest height. If v be the velocity at $\frac{H}{2}$, then the velocity at H is $\sqrt{\frac{2}{5}}$ v.



$\therefore \quad \sqrt{\frac{2}{5}}\, v = u \cos\alpha$

or, $v^2 = \frac{5}{2} u^2 \cos^2\alpha$ ... ... (i)

Also, $H = \frac{u^2 \sin^2 \alpha}{2g}$ ... ... (ii)

And, $v^2 = u^2 - 2g\frac{H}{2}$

or, $\frac{5}{2}\; u^2 \cos^2\alpha = u^2 - g\frac{u^2 \sin^2 \alpha}{2g}$ (From (i) and (ii)

or, $\frac{5}{2} \cos^2\alpha = 1 - \frac{\sin^2\alpha}{2}$

or, $5\cos^2\alpha = 2 - \sin^2\alpha$

or, $5 - 5\sin^2\alpha = 2 - \sin^2\alpha$

or, $-4\sin^2\alpha = -3$

$\sin^2\alpha = \frac{3}{4}$

$\sin\alpha = \frac{\sqrt{3}}{2}$

$\alpha = 60^o$

3. A projectile thrown from a point in a horizontal plane come back to the plane in 4 secs at a distance of 58.8m from the point of projection, find the velocity of the projectile.

Solution:

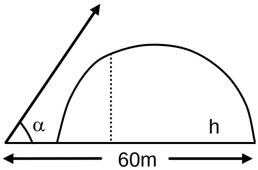
Here, time of projection (t) = 4 sec

Horizontal range (R) = 58.8m

Velocity of projection (u) = ?

Let α be the angle of projection, then

We know,



$t = \frac{2\ u\sin\alpha}{g}$

or, $4 = \frac{2u \sin\alpha}{10}$

or, $u \sin\alpha = 20$ ... ... (i)

and $R = \frac{u^2 \sin^2\alpha}{g}$

$60R = \frac{v^2\ 2\sin\alpha \cos\alpha}{g}$

$u^2\sin\alpha \cdot \cos\alpha = 300$ ... ... (ii)

from (i) and (ii)

$\left(\frac{20}{\sin\alpha}\right)^2 \sin\alpha.\cos\alpha = 300 \Rightarrow \frac{\cos\alpha}{\sin\alpha} = \frac{3}{4}$

or, $\tan\alpha = \frac{4}{3}$, $\therefore$ $\sin\alpha = \frac{4}{5}$

Substituting the value of $\sin\alpha$ in (i)

$u\sin\alpha = 20$

$u \cdot \frac{4}{5} = 20 \Rightarrow u = 25$ m/s

4. Find the angle of projection when the range on a horizontal plane is 4 times the greatest height attained.

Solution:

Angle of projection (α) = ?

Given, Horizontal range = 4 maximum height

or, $\frac{u^2 \sin^2\alpha}{g} = 4 \cdot \frac{u^2 \sin^2\alpha}{2g}$

or, $2\sin\alpha.\cos\alpha = \frac{4\sin^2\alpha}{2}$

or, $1 = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \text{Tan}\alpha = \text{Tan}45^o$

$\therefore$ $\alpha = 45^o$

5. The horizontal range of a projectile is $4\sqrt{3}$ times its maximum height. Find the angle of projection.

Solution:

Angle of projection (α) = ?

Given, Horizontal range = $4\sqrt{3}$ max. height

$\frac{u^2 \sin^2\alpha}{g} = 4\sqrt{3}\, \frac{u^2\, \alpha\sin^2\alpha}{2g}$

or, $2\sin\alpha.\cos\alpha = 2\sqrt{3}\, \sin^2\alpha$

or, $\tan\alpha = \frac{1}{\sqrt{3}} = \tan 30^o$

$\therefore$ $\alpha = 30^o$

6. From the top of a tower 144m high, a particle is projected horizontally with a velocity of 60m/sec. Find its velocity when it reaches the ground.

Solution:

Here, for horizontal projectile, just before hitting the ground,

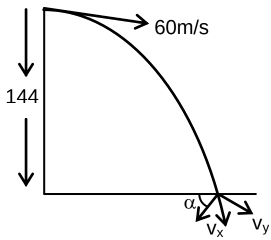
hmax = 144m

$u = 60$m/s, $v = ?$, $g = 10\text{m/s}^2$

Let T = time of flight

v = Velocity with which it hits the ground

α = angle made by $\vec{v}$ with positive x–axis



Now,

$u_x = u = 60$m/s, $u_y = 0$

$v_x = ux = 60$m/s, $v_y = uy + gT = 0 + gT$

We have, $T = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 144}{10}}$

T = 5.37 sec

Again, we have, $v = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{u^2 + (gT)^2}$

$= \sqrt{(60)^2 + (10 \times 5.37)^2} = \sqrt{3600 + 53.7} = 60.45$ m/s

7. A stone is projected from the top of a tower 72.5m high at an angle of 45° which strikes the ground at a distance of 50m from the foot of the tower. Find the velocity of projection.

Solution:

For horizontal projectile, just before hitting the ground hmax = 72.5m, R = 50m, v = ?, g = 10m/s^2, $\alpha = 45^o$.

Let u be the velocity with which body be projected t be the we taken by the body to reach the ground. Now, taking upward direction as positive.

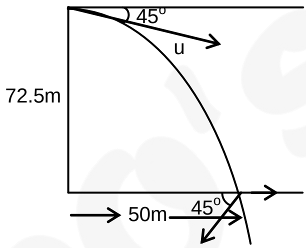
We have,

$-h = u\sin\alpha \cdot t - \frac{1}{2} \times 10 \times t^2$

or, $-72.5 = \frac{ut}{2} - 5t^2$ ... ... (i)

The particle hits at a distance of 50m from the base of the tower, so that

$s = u\cos\alpha.t \Rightarrow \frac{ut}{\sqrt{2}} = 50$ ... ... (ii)



from (i) and (ii)

$-72.5 = \frac{50\sqrt{2}}{2} = 5t^2$

$-72.5 = 35.46 - 5t^2$

$5t^2 = 107.96$

$t^2 = 21.59$

$t = 4.65$

Again, from (ii)

$\frac{u \times 4.65}{\sqrt{2}} = 50$ $\qquad u = \frac{50\sqrt{2}}{4.65} = 15.21$m/s

Hence, required projected velocity = 15.21 m/s

8. A ball is projected from a point with a velocity 64m/sec from the top of a tower 128m high in direction making an angle 30° with the horizon. Find when and at what distance from the foot of the tower it will strike the ground.

Here, initial velocity (u) = 64 m/s

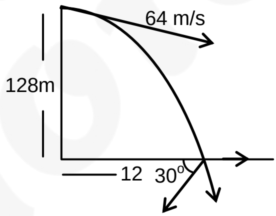
Angle of projection (θ) = 30^o

Height fallen (H) = 128m

Time of flight (T) = ?

Horizontal range (R) = ?

We have,



$H = \frac{1}{2}gT^2 \Rightarrow T = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 128}{10}} = 5.06$ sec

Horizontal range (R) = uT

$= 64 \times 5.06 = 323.8$m

9. A canon ball has the same range R on a horizontal plane for two different angles of projection. If H and H' are the greatest heights and t_1 and t_2 are the time of flights in two paths for which this is possible, prove that

a. $R^2 = 16$ HH' $\qquad$ b. $R = \frac{1}{2}$ gtt'

Solution:

a. Let α and α_1 be two different angle of projections having the same range R.

$R = \frac{u^2\sin^2\alpha}{g} = \frac{u^2\sin^2\alpha_1}{g}$

or, $\frac{\sin^2\alpha}{2\alpha} = \sin 2\alpha_1$

or, $2\alpha = 180 - 2\alpha_1 \Rightarrow \alpha = 90 - \alpha_1$

So, that, $H = \frac{u^2\sin\alpha^2}{2g}$ and, $H^1 = \frac{u^2\sin^2(90-\alpha)}{2g} = \frac{u^2\cos^2\alpha}{2g}$

$t_1 = \frac{2u\sin\alpha}{g}$ and $t_2 = \frac{2u\sin(90-\alpha)}{g} = \frac{2u\cos\alpha}{g}$

Now, $R^2 = \left(\frac{u^2\sin^2\alpha}{g}\right) = \frac{u^2 4\sin^2\alpha.\cos^2\alpha}{g^2}$

$= 4\frac{u^2\sin^2\alpha}{g}.\frac{u^2\cos^2\alpha}{g} = 4.4\frac{u^2\sin^2\alpha}{2g}.\frac{u^2\cos^2\alpha}{2g}$

= 16HH, Hence proved.

b. Again, $R = \frac{u^2\sin^2\alpha}{g} = \frac{u^2 . 2\sin\alpha.\cos\alpha}{g}$

$= \frac{1}{2}g\frac{2u^2\sin\alpha}{g}\times\frac{2u\cos\alpha}{g} = \frac{1}{2}gtt^1$ Hence proved.

10. If 't' be the time in which a projectile reaches a point P of its path and t be the time from P till it strikes the horizontal plane through the point of projection, show that the height of P above the plane is $\frac{1}{2}$ gt t'.

Solution: ***Please see the answer of 9(b)***

11. A particle is projected with a velocity u. If the greatest height attained by the particle be H, prove that the range R on the horizontal plane through the point of projection is

$$R = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$$

Solution:

If α is the angle of projection, then

H = greatest height = $\frac{u^2\sin\alpha}{2g}$

and, R = horizontal range = $\frac{u^2\sin^2\alpha}{g}$

Then, $\frac{u^2}{2g} - H = \frac{u^2}{2g} - \frac{u^2\sin^2\alpha}{2g} = \frac{u^2(1-\sin^2\alpha)}{2g} = \frac{u^2\cos^2\alpha}{2g}$

Now, $4\sqrt{H\left(\frac{u^2}{2g} - H\right)} = 4\sqrt{\frac{u^2\sin^2\alpha}{2g}.\frac{u^2\cos^2\alpha}{2g}}$

$= \frac{4 . u^2\sin\alpha.\cos\alpha}{2g} = \frac{u^2\sin^2\alpha}{g} = R$

$\therefore\quad R = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$

12. If R be the horizontal range and T, the time of flight of a projection, show that $\tan\alpha = \frac{gT^2}{2R}$, where α is the angle of projection.

Solution:

Let u be the velocity of the projection, then

We know,

R = horizontal range = $\frac{u^2\sin^2\alpha}{g}$ ... ... (i)

T = time of flight = $\frac{2u\sin\alpha}{g}$ ... ... (ii)

from (i) and (ii)

$\frac{gT^2}{2R} = g\left(\frac{2u\sin\alpha}{g}\right)^2 = \frac{g.4u^2\sin^2\alpha}{g} = \frac{4u^2\sin^2\alpha}{4u^2\sin\alpha.\cos\alpha} = \tan\alpha$

$\therefore\quad \text{Tan}\alpha = \frac{gT^2}{2R}$ Hence proved.

13. A ball is projected with a velocity of 49m/s, find the two directions along which the ball must be projected so as to have arrange of 122.5m.

Solution: Here, u = 49 m/s, R = 122.5m

Angle of projection (α) = ?

We know that, $R = \frac{u^2 \sin^2\alpha}{g}$

or, $\sin^2\alpha = \frac{Rg}{u^2} = \frac{122.5 \times 9.8}{49 \times 49}$

or, $\sin^2\alpha = \frac{1}{2}$

or, $\sin^2\alpha = \sin30^o$ or $\sin150^o$

$\therefore$ $\alpha = 15^o$ or, 75^o

14. A body is thrown from the top of a tower 96m high with a velocity 80m/sec at an angle of 30° above the horizon. Find the horizontal distance from the foot of the tower to the point where it hits the ground.

Solution:

Here,

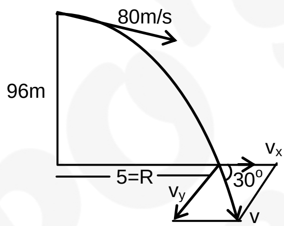
Initial velocity (u) = 80 m/s

Height fallen (H) = 96m

Time of flight (T) = ?

Horizontal range (R) = ?

Angle of projection (Q) = 30^o



We have, $H = \frac{1}{2} gT^2$

$T = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 96}{10}}$

T = 4.38 sec

Horizontal range (R) = u.T

= 80 × 4.38 = 350.54m

15. A ball thrown by a player from a height of 2m at an angle of 30° with the horizon with a velocity of 18m/sec is caught by another player at a height of 0.4m from the ground. How far apart were the two players?

Solution:

Distance between the players (x) = ?

Initial velocity (u) = 18m/s

Angle (θ) = 30^o

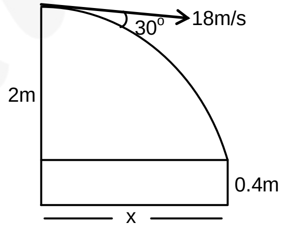
Vertical distance to the traveler (y) = 2 – 0.4 = 1.6m

We have, $y = \frac{1gT^2}{2}$

$t^2 = \frac{2y}{g} = \frac{2\times1.6}{10}$, t = 0.57 sec

$\therefore$ x = u.t

= 18 ×0.57 = 10.18m



Chapter 22
MATHEMATICS FOR ECONOMICS AND FINANCE

Exercise 22.1

1. Find the quadratic supply function Qs = f(P) from the information given.

Price (P)	40	50	80
Quantity supplied (Q)	600	3300	15000

Solution:

Let $Q_s = ap^2 + bp + c$ ... ... (i) be a quadratic supply function.

Then acceleration to question, when p = 40 then $Q_s = 600$

$\therefore$ $a\times40^2 + b\times40 + c = 600 \Rightarrow 1600a - 40b + c = 600$ ... ... (i)

Similarly, other two points are (50, 3300) and (80, 1500)

Then,

$2500a + 50b + c = 3300$ ... ... (ii)

$6400a + 806 + c = 15000$ ... ... (iii)

From (i) and (ii)

$2500a + 50b + c = 3300$

$1600a + 40b + c = 600$

$-\quad -\quad -$

$900a + 10b = 2700$

$90a + b = 270$ ... ... (iv)

from (ii) and (iii)

$6400a + 80b + c = 15000$

$2500a + 50b + c = 3300$

$-\quad -\quad -$

$3900a + 30b = 11700$

$130a + b = 390$ ... ... (v)

from (iv) and (v)

$130a + b = 390$

$90a + b = 270$

$-\quad -\quad -$

$40a = 120$

$\therefore$ $a = 3$

from (iv) b = 0

Substituting the value of a and b in (i) we get c = –4200

Hence, required quadratic supply function is $Q_s = 3p^2 - 4200$

2. The supply and demand functions are given by $P = Q^2 + 12Q + 32$, $P = -Q^2 - 4Q + 200$ respectively. Find equilibrium price and quantity.

Solution:

Given, supply function $P_s = Q^2 + 12Q + 32$

Demand function $P_d = Q^2 - 4Q + 200$

For equilibrium, $P_d = P_s$

i.e. $Q^2 + 12Q + 32 = -Q^2 - 4Q + 200$

$2Q^2 + 16Q - 168 = 0$

$Q^2 + 8Q - 84 = 0$

$Q^2 + 14Q - 6Q - 84 = 0$

$\therefore$ Q = 6, Q = –14 (not possible)

When Q = 6

Then $p = 6^2 + 12\times6 + 32 = 36 + 72 + 32 = 140$

$\therefore$ equilibrium price = 140

3. Given the supply and demand functions

$$Q_S = (P + 5)\sqrt{P + 10}$$

$$Q_d = \frac{210 - 9P - 3P^2}{\sqrt{P + 10}}$$

Calculate the equilibrium price and quantity.

Solution:

Given, $Q_s = (P+5)\sqrt{p + 10}$ and $Q_1 = \dfrac{210 - 9p - 3p^2}{\sqrt{p + 10}}$

For equilibrium condition,

$Qs = Q_d$

i.e. $(P + 5)\sqrt{p + 10} = \dfrac{210 - 9p - 3p^2}{\sqrt{p + 10}}$

or, $(p + 5)(p + 10) = 210 - 9p - 3p^2$

or, $p^2 + 15p + 50 + 3p^2 + 9p - 210 = 0$

or, $4p^2 + 24p - 160 = 0$

or, $p^2 + 6p - 40 = 0$

or, $p^2 + 10p - 4p - 40 = 0$

$\therefore$ p = 4

Then $Q = 9\sqrt{14}$

$\therefore$ equilibrium point $(4, 9\sqrt{14})$

4. The average cost of a product is given as

$$AC = 15Q - 3600 + \frac{486000}{Q}$$

Find the quantity for which the total cost is minimum. Also find the minimum cost.

Solution:

Given, average cost (AC) $= 15Q - 3600 + \dfrac{486,000}{Q}$

Total cost function (TS) = AC $\times$ Q

$\therefore$ $TC = 15Q^2 - 3600Q + 486,000$

Comparing it with $y = ax^2 + bx + c$

a = 15, b = –3600 and c = 486,000

Since a > 0, TC represents a parabola concave upward. Being upward, TC has minimum value at $Q = -\dfrac{b}{2a}$ $\left(x = -\dfrac{b}{2a}\right)$

i.e. $Q = +\dfrac{3600}{30}$

Q = 120

$\therefore$ Total cost is minimum at Q = 120 units

Then the min. total cost is $TC = 15\times120^2 - 3600 \times 120 + 486000$

$= 2,70,000$

5. For the price Rs. P, the quantity demanded is given by Q = 600,000 – 2,500P. Determine the total revenue function R = f(P).

a. What is the concavity of the revenue function?

b. What is the total revenue when price is Rs. 50?
c. Find the price for which the total revenue is maximized.

Solution:

Demand function is given by
$Q = 6,00,000 - 2,500P$
Total revenue function (TR) = $P \times Q$
$\therefore$ $R = 600,000p - 2,500p^2$
Comparing it with $y = ax^2 + bx + c$,
We get $a = -2500$, $b = 600,000$ and $c = 0$
Since $a < 0$, the graph is concave downward parabola.
When p = Rs. 50, then total revenue is $R = -2500 \times 50^2 + 600,000 \times 50$
Rs. 3,62,50,000
$\therefore$ total revenue is maximized at $P = -\frac{b}{2a}$

i.e. $P = \frac{-600,000}{-5,000} = 120$

$\therefore$ Revenue is maximized at P = Rs. 120

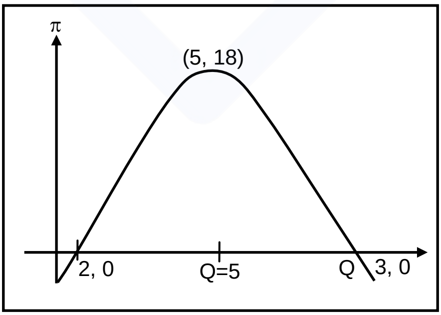
6. Fixed cost = 32
Variable cost = 5
$\therefore$ total cost for producing a units is given by,
$TS = 5Q + 32$
Also, demand function $p = 25 - 2Q$
$\therefore$ total revenue function $TR = 25Q - 2Q^2$

a. For break-even TR = TC
$25Q - 2Q^2 = 5Q + 32$
$20Q - 2Q^2 - 32 = 0$
or, $Q^2 - 10Q + 16 = 0$
$Q^2 - 8Q - 2Q + 16 = 0$
$Q = 2$ or 8

b. Profit function $\pi = TR - TC$
$= 25Q - 2Q^2 - 5Q - 32$
π $-2Q^2 + 20Q - 32$
Since coefficient of Q^2 is negative, parabola is concave downward,
Profit is maximum of $Q = -\frac{20}{-4} = 5$

c. Maximum profit $\pi_{max} = \frac{4ac - b^2}{4a} = \frac{4\times(-2)(-32) - 400}{-8} = 18$

d.



7. Given the fixed cost as 32, variable cost per unit as 5 per unit and the demand function P = 25 – 2Q, express the profit function π in terms of Q.

a. Find the value(s) of Q for break even.

$TR = -2Q^2 + 14Q$

$TC = 2Q + 10$

For break-even, TR = TC

$-2Q^2 + 14Q = 2Q + 10$

$2Q^2 - 12Q + 10 = 0$

$Q^2 - 6Q + 5 = 0$

$Q^2 - 5Q - Q + 5 = 0$

Q = 1 or 5

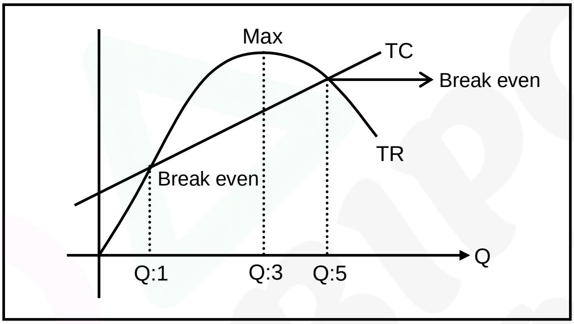
Now, profit function π = TR – TC

$\therefore \quad \pi = Q^2 - 6Q + 5$

It is quadratic function. Since a > 0 concave downward, gives max profit at

$Q = -\frac{b}{2a} = \frac{6}{2} = 3$

b. Find the value of Q for which π is maximum.



c. Max profit $\pi = 3^2 - 6\times3 + 5 = -4$

d. Sketch the graph of π.

Exercise 22.2

1. Solution:

a. Given,

$x_{11} = 200,\ x_{12} = 250,\ d_1 = 450$

$\therefore \quad x_1 = x_{11} + x_{12} + d_1 = 900$

$x_{21} = 125,\ x_{22} = 8,\ d_2 = 225$

$\therefore \quad x_2 = x_{21} + x_{22} + d_2 = 400$

$\therefore$ the consumption matrix or coefficient input matrix is given by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where

$a_{ij} = \frac{x_{ij}}{x_j}$ for all i, j

so, $a_{11} = \frac{x_{11}}{x_1} = \frac{200}{900} = \frac{2}{9}$

$a_{12} = \frac{x_{12}}{x_2} = \frac{125}{400} = \frac{5}{36}$

$a_{21} = \frac{x_{21}}{x_1} = \frac{125}{450} = \frac{5}{16}$

$a_{22} = \frac{x_{22}}{x_2} = \frac{50}{450} = \frac{1}{9}$

$\therefore$ Input coefficient matrix is $\begin{bmatrix} \frac{2}{9} & \frac{5}{16} \\ \frac{5}{36} & \frac{1}{9} \end{bmatrix}$

b. Given,

$x_{11} = 250$, $x_{11} = 140$, $x_{13} = 30$, $d_1 = 80$

Then total output $(x_1) = x_{11} + x_{12} + x_{13} + d_1$

$= 250 + 140 + 30 + 80 = 500$

$x_{21} = 100$, $x_{22} = 105$, $x_{23} = 15$, $d_2 = 130$

$\therefore$ Total output $(x_2) = x_{21} + x_{22} + x_{23} + d_2 = 350$

$x_{31} = 50$, $x_{32} = 35$, $x_{33} = 45$, $d_3 = 20$

$\therefore$ Total output $(x_3) = x_{31} + x_{32} + x_{33} + d_3 = 150$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Matrix where $a_{ij} = \frac{x_{ij}}{x_j}$

$a_{11} = \frac{x_{11}}{x_1} = \frac{250}{500} = 0.5$

$a_{12} = \frac{x_{12}}{x_2} = \frac{140}{350} = 0.4$

$a_{13} = \frac{x_{13}}{x_3} = \frac{30}{150} = 0.2$

$a_{21} = \frac{x_{21}}{x_1} = \frac{100}{500} = 0.2$

$a_{22} = \frac{x_{22}}{x_2} = \frac{105}{350} = 0.3$

$a_{23} = \frac{x_{23}}{x_3} = \frac{15}{150} = 0.1$

$a_{31} = \frac{x_{31}}{x_1} = \frac{50}{500} = 0.1$

$a_{32} = \frac{x_{32}}{x_1} = \frac{35}{200} = 0.1$

$a_{33} = \frac{x_{33}}{x_3} = \frac{45}{150} = 0.3$

Therefore, $A = \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{pmatrix}$

2. Given, consumption input coefficient matrix is sector I, Sector II and Sector III

$$A = \begin{array}{l} \begin{pmatrix} 0.1 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.2 \\ 0.1 & 0.0 & 0.2 \end{pmatrix} \end{array} \begin{array}{l} \text{Sector I} \\ \text{Sector II} \\ \text{Sector III} \end{array}$$

If first sector decides to produce 200 units, then it consumes 0.1×200 units

$= 20$ units of itself

and 0.4×200 units $= 80$ units of sectors 2

and 0.1×200 units $= 20$ units of sector 3

3. Given,

Coefficient matrix A = $\begin{bmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{bmatrix}$ and final demand vector D = $\begin{bmatrix} 18 \\ 11 \end{bmatrix}$

Then technology matrix T = I – A

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{pmatrix}$$

$$T = \begin{bmatrix} 0.9 & -0.6 \\ -0.5 & 0.8 \end{bmatrix}$$

$$|T| = \begin{vmatrix} 0.9 & -0.6 \\ -0.5 & 0.8 \end{vmatrix} = 0.72 - 0.30 = 0.42$$

$$\therefore T^{-1} = \frac{\text{Adj. (T)}}{|T|} = \frac{\begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{bmatrix}}{0.42}$$

Let x = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the gross output to meet the final demand then,

$$x = T^{-1}D = \frac{1}{0.42}\begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{bmatrix}\begin{bmatrix} 18 \\ 11 \end{bmatrix} = \frac{1}{0.42}\begin{bmatrix} 21 \\ 18.9 \end{bmatrix} = \begin{bmatrix} 50 \\ 45 \end{bmatrix}$$

$\therefore$ The production level is 50 units and 45 units respectively.

4. Given, consumption input coefficient matrix be $\begin{vmatrix} 0.2 \\ 0.1 \end{vmatrix}\begin{vmatrix} 0.05 \\ 0.1 \end{vmatrix}$

i.e. A = $\begin{pmatrix} 0.2 & 0.05 \\ 0.1 & 0.1 \end{pmatrix}$

Also given market demand vector D = $\begin{bmatrix} 750 \\ 500 \end{bmatrix}$

Now, technology matrix T = I – A = $\begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{bmatrix}$

$$T^{-1} = \frac{\text{Adj (T)}}{|T|}$$

$$|T| = \begin{vmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{vmatrix} = 0.72 - 0.005 = 0.715$$

$$T^{-1} = \frac{1}{|T|}\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.8 \end{bmatrix}$$

$$= \frac{1}{0.715}\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.8 \end{bmatrix}$$

Using $x = T^{-1}D = \frac{1}{0.715}\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} 750 \\ 500 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 979.02 \\ 664.34 \end{bmatrix}$$

$\therefore$ x_1 = Rs. 979.02, x_2 = Rs. 664.34

5. Given,

$$A = \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.2 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

$$D = \begin{bmatrix} 35 \\ 0 \\ 100 \end{bmatrix}$$

Technology matrix T = I – A

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.2 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.9 & -0.2 & -0.5 \\ -0.2 & 0.9 & -0.2 \\ -0.3 & -0.4 & 0.8 \end{bmatrix}$$

$$|T| = \begin{vmatrix} 0.9 & -0.2 & -0.5 \\ -0.2 & 0.9 & -0.2 \\ -0.3 & -0.4 & 0.8 \end{vmatrix}$$

$$= 0.9\begin{vmatrix} 0.9 & -0.2 \\ -0.4 & 0.8 \end{vmatrix} + 0.2\begin{vmatrix} -0.2 & -0.2 \\ -0.3 & 0.8 \end{vmatrix} - 0.5\begin{vmatrix} -0.2 & 0.9 \\ -0.3 & -0.4 \end{vmatrix}$$

$= 0.9\ (0.72 - 0.08) + 0.2\ (-0.16 - 0.06) - 0.5(0.08 + 0.27)$

$= 0.576 - 0.044 - 0.175$

$= 0.357$

Let $\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$ be a cofactor matrix of T

Then

T_{11} = cofactor of 0.9 = $\begin{vmatrix} 0.9 & -0.2 \\ -0.4 & 0.8 \end{vmatrix} = 0.64$

T_{12} = Cofactor of –0.2 = $-\begin{vmatrix} -0.2 & -0.2 \\ -0.3 & 0.8 \end{vmatrix} = 0.22$

T_{13} = Cofactor of –0.5 = $\begin{vmatrix} -0.2 & 0.9 \\ -0.3 & -0.{.}4 \end{vmatrix} = 0.35$

T_{21}= Cofactor of –0.2 = $-\begin{vmatrix} -0.2 & -0.5 \\ -0.4 & 0.8 \end{vmatrix} = 0.36$

T_{22} = Cofactor 0.9 = $\begin{vmatrix} 0.9 & -0.5 \\ -0.3 & 0.8 \end{vmatrix} = 0.57$

T_{23} = Cofactor of –0.2 = $-\begin{vmatrix} 0.9 & -0.2 \\ -0.3 & -0.4 \end{vmatrix} = 0.42$

T_{31} = Cofactor of –0.3 = $\begin{vmatrix} 0.2 & -0.5 \\ 0.9 & -0.2 \end{vmatrix} = 0.49$

T_{32} = Cofactor of –0.4 = $-\begin{vmatrix} 0.9 & -0.5 \\ -0.2 & -0.2 \end{vmatrix} = 0.28$

T_{33} = Cofactor of 0.8 = $\begin{vmatrix} 0.9 & -0.2 \\ -0.2 & 0.9 \end{vmatrix} = 0.77$

$\therefore$ Cofactor matrix is $\begin{bmatrix} 0.64 & 0.22 & 0.35 \\ 0.36 & 0.57 & 0.42 \\ 0.49 & 0.28 & 0.77 \end{bmatrix}$

$$\text{Adj}\ (T) = \begin{bmatrix} 0.64 & 0.36 & 0.49 \\ 0.22 & 0.57 & 0.28 \\ 0.35 & 0.42 & 0.77 \end{bmatrix}$$

$$T^{-1} = \frac{1}{0.357}\begin{bmatrix} 0.64 & 0.36 & 0.49 \\ 0.22 & 0.57 & 0.28 \\ 0.35 & 0.42 & 0.77 \end{bmatrix}$$

Using $x = T^{-1}D = \frac{1}{0.357}\begin{bmatrix} 35 \\ 0 \\ 100 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \\ 250 \end{bmatrix}$$

$\therefore$ $X_1 = 200, X_2 = 100, X_3 = 250$

6. From, 1(b),

Input coefficient matrix $A = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$ and given,

Demand vector $D = \begin{bmatrix} 400 \\ 110 \\ 250 \end{bmatrix}$

Exercise 22.3

1. Solution:

a. Given, Demand function $p = 100 - Q^2$

at $Q = 8$, $p = 100 - 64 = 36$

Consumer's surplus (C.S.) $= \int_0^8 (100 - Q^2)\, dQ - 36 \times 8$

$= \left[100Q - \frac{Q^3}{3}\right]_0^8 - 288$

$= 341.33$

b. Given demand function

$p = \frac{80}{\sqrt[3]{Q}}$

$p = 80Q^{-1/3}$

When $Q = 64$ then $p = 80(64)^{1/3} = 20$

Consumers' surplus (c.s.) $= \int_0^Q p\,dQ - p\times Q$

C.S. $\int_0^{64} 80\, Q^{-1/3}\, dQ - 20 \times 64$

$= 80\,\frac{3}{2}\left[Q^{\frac{2}{3}}\right]_0^{64} - 1280$

$= 120 \times 16 - 1280$

$= 640$

c. $Q = \frac{10 - p}{2p}$ at $p = 2$

$Q = \frac{5}{p} - \frac{1}{2} \Rightarrow p = \frac{10}{2Q+1}$

When $p = 2$ then $Q = 2$

Consumer's demand (C.S.) = $\int_0^Q pdQ - p\times Q$

$= \int_0^2 \frac{10}{2Q+1} dQ - 2\times2$

$= \frac{10}{2} [\ln(2Q + 1)]_0^Q - 4$

$= 5 \ln 5 - 4$

d. Given, $p = \frac{2Q}{Q^2 + 1}$

When Q = 10, then $p = \frac{20}{101}$

Consumer's surplus (C.S.) = $\int_0^{10} p\, dQ - p \times Q$

$= \int_0^{10} \frac{2Q}{Q^2 + 1} dQ - \frac{20}{101} \times 10$

$[\ln(Q^2 + 1)]_0^{10} - \frac{200}{101}$

$= \ln 101 - \frac{200}{101}$

$= 2.63$

2. Solution:

a. Given, supply sunction p = 12 + 2Q

When Q = 5 then p = 22

Producer surplus (P.S.) = $P\times Q - \int_0^Q pdQ$

$= 22\times5 - \int_0^5 (121 + 2Q)\, dQ$

$= 110 - [12Q + Q^2]_0^5$

$= 110 - 85$

$= 25$

b. Given,

$P = 20\sqrt{Q} + 15$ at Q = 25

When Q = 25 then p = 115

Then P.S. = $P \times Q - \int_0^Q pdQ$

$= 115 \times 25 - \int_0^{25} (20\sqrt{Q} + 15)\, dQ$

$= 2875 - \left[20 \frac{Q^{3/2}}{3/2} + 15Q\right]_0^{25}$

$= 2675 - \left(\frac{40}{3} \times 125 + 375\right)$

$= 2875 - 2041.67$

$= 833.33$

3. Solution:

a. Given, Demand function $p = \dfrac{4000}{Q + 20}$

Supply function p = Q + 50
For equilibrium
Supply = demand
i.e. (Q + 50) (Q + 20) = 4000
Q2 + 70Q – 3000 = 0 $\Rightarrow$ Q = 30
When Q = 30 then p = 80

Now, consumer's surplus = $\int_0^Q$ demand function – P×Q

$$= \int_0^{30} \frac{4000}{Q + 20}\, dQ - 80 \times 30$$

$$= 4000\,[\ln (Q + 20)]_0^{30} - 2400$$

$$= 400 \ln\left(\frac{50}{20}\right) - 2400$$

$$= 1265.16$$

Producer's surplus (P.S.) = P × Q – $\int$supply function

$$= 80 \times 30 - \int_0^{30} (Q + 50)\, dQ$$

$$= 2400 - \left[\frac{Q^2}{2} + 50Q\right]_0^{30}$$

$$= 2400 - (450 + 1500)$$

$$= 450$$

b. Given, $p_d = 74 - Q^2_d$
$p_s = Q^2_s + 2$
For equilibrium, $p_d - p_s = p$
$Q_d = Q_s = Q$
$Q^2 + Q = 74 - Q^2$
$2Q^2 = 72$
Q = 6
When Q = 6 then p = 38

Consumer's surplus (C.S.) = $\int_0^6 p_d\, dQ - p \times Q$

$$\int_0^6 (74 - Q^2_d)\, dQ - 38 \times 6$$

$$\left[74Q - \frac{Q^3}{3}\right]_0^6 - 228 = 144$$

And, P.S. = P × Q – $\int_0^Q p_s\, dQ$

$$= 228 - \int_0^6 (Q^2 + 2)\, dQ$$

$= 228 - \left[\frac{Q^3}{3} + Q\right]_0^6$

$= 144$

c. Given, demand function $p = 100\, e^{-Q/5}$

Supply function $p = 20\, e^{2Q/5}$

For market equilibrium,

Supply function = Demand function

i.e. $20\, e^{2Q/5} = 100\, e^{-Q/5}$

$e^{3Q/5} = 5$

$\frac{3Q}{5} = \ln 5$

$Q = 2.68$

When $Q = 2.68$ then $p = 20\, e^{1.073} = 58.48$

Now, consumer surplus (C.S.) $= \int_0^6 100 e^{-Q/5}\, dQ - 58.48 \times 2.68$

$= 100 \left[\frac{e^{-Q/5}}{-1/5}\right]_0^{2.68} - 156.73$

$= -500\, (e^{-2.68/5} - e^0) - 156.73$

$= 50.73$

Producer surplus (P.S.) $= P \times Q - \int_0^Q 20\, e^{2Q/5}\, dQ$

$= 156.73 - \int_0^{2.68} 20\, e^{2Q/5}\, dQ$

$= 156.73 - 20 \times \frac{5}{2}\, [e^{2Q/5}]_0^{2.68}$

$= 156.73 - 50\, (e^{0.4\times 2.68} - e^0)$

$= 60.7$

4. Given, Supply function $p = 3 + 4Q$

Producer's surplus at $Q = \propto$ is 72

When $Q = \propto$ then $p = 4 \propto 3$

Now, using

P.S. $= p \times Q - \int_0^Q (3 + 4Q)\, dQ$

$72 = (4\alpha + 3)\, \alpha - [3Q + 2Q^2]_0^\alpha$

$72 = (4\alpha^2 + 3\alpha) - (3\alpha + 2\alpha^2)$

$72 = 4\alpha + 3\alpha - 2\alpha\ 2$

$2\alpha^2 = 72$

$\therefore\ \alpha = 6$

5. Given, $Q_d = \gamma - \delta p$, $Q_s = \beta p - \alpha$

At equilibrium, $Q_d = Q_s$

i.e. $\gamma - \delta p = \beta p - \alpha$

$p = \frac{\propto + \gamma}{\beta + \delta}$

When $p = \frac{\propto + \gamma}{\beta + \delta}$ then $Q = \gamma - \delta\left(\frac{\propto + \gamma}{\beta + \delta}\right) = \frac{\beta\gamma + \delta\gamma - \delta\propto - \delta\gamma}{\beta + \delta} = \frac{\beta\gamma - \delta\propto}{\beta + \delta}$

Producer's surplus (P.S.) = $P \times Q - \int_0^Q$ supply function

$$= \left(\frac{\propto + \gamma}{\beta + \delta}\right)\left(\frac{\beta\gamma - \delta\propto}{\beta + \delta}\right) - \int_0^{\beta\gamma - \delta\propto/\beta+\delta}\left(\frac{Q + \propto}{\beta}\right)dQ$$

$$= \frac{(\propto + \gamma)(\beta\gamma - \delta\propto)}{(\beta + \delta)^2} - \frac{1}{-\beta}\int_0^{\beta\gamma - \delta\propto/\beta+\delta}(Q + \alpha)\, dQ$$

$$= \frac{(\propto + \gamma)(\beta\gamma - \delta\propto)}{(\beta + \delta)^2} - \frac{1}{\beta}\left[\frac{Q^2}{2} + \alpha Q\right]_0^{\beta\gamma - \delta\propto/\beta+\delta}$$

$$= \frac{(\propto + \gamma)(\beta\gamma - \delta\propto)}{(\beta + \delta)^2} - \frac{1}{\beta}\left[\frac{\left(\frac{\beta\gamma - \propto\delta}{\beta + \delta}\right)^2}{2} + \frac{\propto(\beta\gamma - \propto\delta)}{\beta + \delta}\right]$$

6. Given, Demand $p = 80 - 6\sqrt{Q}$

when p = 62 then $62 = 80 - 6\sqrt{Q}$

$6\sqrt{Q} = 18$

$\therefore \quad Q = 9$

$$C.S. = \int_0^9 (80 - 6\sqrt{Q})\, dQ - P\times Q$$

$$= \left[80Q - \frac{6Q^{3/2}}{3/2}\right]_0^9 - 62\times 9$$

$= (720 - 4\times 27) - 558$

$= 54$

Again, when p = 56 then $56 = 80 - 6\sqrt{Q}$

$6r\sqrt{Q} = 24$

$\therefore \quad Q = 16$

$$C.S. = \int_0^{16} (80 - 6\sqrt{Q})dQ - 56 \times 16$$

$= [80Q - 4Q^{3/2}]_0^{16} - 896$

$= (1280 - 256) - 896$

$= 128$

Change in C.S. is 128 – 54 = 74

7. Given, supply function $aP - bQ = 1$

$$P = \frac{1 + bQ}{a}$$

Price = 12 and quantity = 6

Producer's surplus (P.S.) = $P \times Q - \int_0^Q$ supply function.

$$18 = 12\times 6 - \int_0^6 \left(\frac{1 + bQ}{a}\right)dQ$$

or, $\frac{1}{a}\left[Q + b\frac{Q^2}{2}\right]_0^6 = 54$

or, $\frac{1}{a}(6 + 18b) = 54$

$18b + 6 = 54a$

$\therefore$ 54a – 18b = 6 ... ... (i)
Since, ap – bQ = 1
a12 – b6 = 1
12a – 6b = 1 ... ... (ii)
Solving (i) and (ii),
54a – 18b = 6
36a – 18b = 3
– + –
18a = 3

$a = \frac{1}{6}$

Substituting the alue of a in (ii) we get,
12a – 6b = 1

$12 \times \frac{1}{6} - 6b = 1$

2 – 6b = 1
6b = 1

$\therefore \quad b = \frac{1}{6}$

Hence, $a = \frac{1}{6}$ and $b = \frac{1}{6}$

Exercise 22.4

1. Form the difference equations from

a. $Y_t = 8(4^t) - 7 \Rightarrow Y^t + 7 = 8(4)^t$
Replacing T by T+1
$Y_{T+1} = 8(4^{t+1}) - 7$
$= 84^t.4 - 7$
$= 4(4_t + 7) - 7$
$= 4Y_t + 28 - 7$
$= 4Y_t + 21$
Hence, the required difference equation is,
$Y_t + = 4Y_t + 21$
Equivalent $Y_t = 4Y_{t+1} + 21$
$\therefore \quad Y_T = 4Y_{t+1} + 21)$

b. $Y_t = A4^t + B.7^t$
We have,
$Y_t = A(4^t) + B(7^t)$
Replacing t by t+1
$Y_{t+1} = A(4^{t+1}) + B(7^{t+1})$
$A4^t.(4) + B7^t.(7)$
From the given equation $(Y_t - A4^t) = B7^t$ so
$Y_{t+1} = 4A4^t + 7(Y_t - A4^t)$
$= 4A4^t + 7Y_t - 7A4^t$
$Y_{t+1} = 7Y_t - 3A4^t$
again,
Replacing t by t+1
$Y_{t+2} = 7Y_{t+1} - 3A(4^{t+1})$
From the given equation $(7Y_t - Y_{t+1}) = 4A4^{t+1})$
$= 7Y_{t+1} - 3(7Y_t - Y_{t+1})$
$= 7Y_{t+1} - 21Y_t + 3Y_{t+1} = 0$

$= 10Y_{t+1} - 21Y_t$

so required difference equation is

$\therefore\ Y_{t+2} - 10Y_{t+1} - 21Y_t = 0$

2. Solve the following difference equation

a. $Y_T = Y_{T-1} + 2,\ Y_o = 2$

Comparing the equation with $Y_T = aY_{T-1} + b$, we have

$a = 1$

$b = 2$

Since $a \neq 1$

Another method,

The complementary function (c.f.) $= Aa^T = A(1)^T$

Let $Y_T = KT$ be a particulars solution

Then $Y_{T-1} = K.(T-1)$

Substituting the value Y_T and Y_{T-1}

$K_T = K(T - 1) + 2$

$K_T - K(T-1) = 2$

$K_T - K_T + K = 2$

$\therefore\quad K = 2$

So, PS = 2

The required general solution is

$Y_T = CF + PS$

$= A(1)^T + 2 \Rightarrow Y_T = A(1)^T + 2$

As given, $Y_o = 2$, then

$2 = A + 2$

$A = 0$

Now, $(Y_T = (1)^T + 2)$

b. $Y_{T+1} = -Y_T + 6,\ Y_o = 4$

$Y_T = -Y_{T+1} + 6$

Comparing the equation with $Y_T = aY_{T+1} + b$, we have

$a = -1$,

$b = 6$

Since $a \neq 1$, the required solution is

$Y_T = Aa^T + \dfrac{b}{1-a}$, where A is constant.

Substituting the value of a, b

$Y_T = A(-1)^T + \dfrac{6}{1+1}$

$Y_T = A(-1)^T + 3$

Putting,

Given $Y_0 = 4$, $T = 0$

Then,

$4 = A(-1)^0 + 3$

$4 - 3 = A$

$A = 1$,

Now,

$Y_T = 1(-1)^T + 3$, is the required solution,

c. $4Y_T = Y_{T-1} + 24$

$Y_T = \dfrac{1}{4}Y_{T-1} + 6$

Comparing the equation with $Y_T = aY_{T+1} + b$, we have

$a = \dfrac{1}{4}$, $b = 6$

Since $a \neq 1$, the required solution is

$Y_T = A4^T + \frac{b}{1-a}$ where A is constant.

Substituting the values of a and b

$$Y_T = A\left(\frac{1}{4}\right)^T + \frac{6}{1-\frac{1}{4}}$$

$$Y_T = A\left(\frac{1}{4}\right)^T + 8 = A(0.25)^t + 8$$

d. $Y_T = -0.5Y_{T-1} + 1$

Comparing with the equation $Y_T = aY_{T+1} + b$, we have

$a = -0.5$

$b = 1$

Since $a \neq 1$ the required

Solution is

$Y_T = Aa^T + \frac{b}{1-a}$, where A is constant.

Substituting the value of a and b

$$Y_T = A(-0.5)^T + \frac{1}{1+0.5}$$

$$Y_T = A(-0.5)^T + 0.66$$

3. Solution

Given, $Y_T = 3Y_{T-1} + 7$, $Y_o = 2$

a. Find the value of Y_1, Y_2, Y_3 without solving d.e. when, Y_1 then, Y_2 then,

$Y_1 = 3Y_{1-1} + 7$	$Y_2 = 3Y_{2-2} + 7$
$= 3Y + 7$	$Y_2 = 3Y_1 + 7$
$3Y = -7$	$= 3\left(\frac{-7}{3}\right) + 7$
$Y = \frac{-7}{3}$	$= -7 + 7 = 0$
$\therefore \quad Y_1 = \frac{-7}{3}$	$\therefore Y_2 = 0$

Y_3, then,

$Y_3 = 3Y_{3-1} + 7$

$= 3Y_2 + 7$

$= 3(0) + 7$

$\therefore \quad Y_3 = 7$

$\therefore \quad Y_1 = \frac{-7}{3}$, $Y_2 = 0$ and $Y_3 = 7$

b. Find Y_1, Y_2, Y_3 using this solution.

Comparing with the equation $Y_T = aY_{T-1} + b$, we have

$a = 3$

$b = 7$

Since, $a \neq 1$, the required solution is,

$$Y_T = Aa^T + \frac{b}{1-a}$$

Substituting the value and a, b

$$Y_T = A(3)^T + \frac{7}{1-3}$$

$$Y_T = A(3)^T - 3.5$$

When, $Y_0 = 2$, then

$2 = A(3)^0 - 3.5$

$2 + 3.5 = A$

$\therefore \quad A = 5.5$

Now, $Y_T = 5.5(3)^T - 3.5$

When, Y_1, Y_2, Y_3, 300

$Y_1 = 5.5(3)^1 - 3.5$	$Y_2 = 5.5(3)^2 - 3.5$	$Y_3 = 5.5(3)^3 - 3.5$
$Y_1 = 13$	$= 46$	$= 145$

$$\therefore \begin{pmatrix} Y_1 = 13 \\ Y_2 = 46 \\ Y_3 = 145 \end{pmatrix}$$

4. Given, $Y_T = 0.3Y_{T-1} + 0.4T + 5$

a. $y_t = 1.008\, y_{t-1} - 4{,}000$

$y_c = m = 1.008$

$y_c = A(1.008)^t$

Particular integral

(y_p) = let $y_t = k$ be

$y_{t-1} = k$

$k - 1.008k = -4{,}000$

$k = 5{,}00{,}000$

$\therefore \quad y_t = A(1.008)^t + 5{,}00{,}000$

$y_0 = 1{,}50{,}000$

$1{,}50{,}000 = A + 5{,}00{,}000$

$A = -3{,}50{,}000$

$y_t = -3{,}50{,}000\,(1.008)^t + 5{,}00{,}000$

$y_{12} = 114881.46$

c. We have, $y_t = A(1.008)^t + 5{,}00{,}000$

To pay the loan, $y_t = 0$

$3{,}50{,}000\,(1.008)^t = 5{,}00{,}000$

$(1.008)^t = 1.43$

$$t = \frac{\ln (1.43)}{\ln(1.008)}$$

$= 45$ months

.

Exercise 22.5

1. Given,

$Q_{ST} = P_{T-1} - 8$

$Q_{dT} = -2P_T + 22$

For equilibrium

$Q_{ST} = Q_{dT}$

So, $P_T - 1 - 8 = -2P_T + 22$

$P_{T-1} - 8 + 2P_T - 22 = 0$

$2P_T = -P_{T-1} + 30$

$$P_T = -\frac{1}{2} P_{T-1} + 15$$

Comparing with $P_T = aP_{T-1} + b$

Now, $a = \dfrac{-1}{2}$, $b = 13$

The general solution Rs. $P_T = AaT + \dfrac{b}{1-a}$

Where, A is constant.

Substituting the values,

$$P_T = A\left(\frac{-1}{2}\right)^T + \frac{15}{1+\frac{1}{2}}$$

$P_T = A\left(\frac{-1}{2}\right)^T + 10$ ... ... (i)

When, $P_o = 11$,

Now, $11 = A + 10$

$11 - 10 = A$

$\therefore \quad A = 1$

Now, $P_T = 1\left(1 - \frac{1}{2}\right)^T + 10$

Putting this expression in $Q_{dT} = -2_{PT} + 22$

$= -2\left[1\left(\frac{-1}{2}\right)^T + 10\right] + 22$

$= -2\left(1 - \frac{1}{2}\right)^T + 2$

Since, $|a| = |-\frac{1}{2}| = | = \frac{1}{2} > 0$. So it is stable.

2. Given,

$Q_{2t} = -5P_T + 35$

$Q_{ST} = 4P_{T-1} - 10$

For equation

$Q_{dT} = Q_{ST}$

$4P_{T-1} - 10 = -5P_T + 35$

$5P_T = -4P_{T-1} + 45$

$P_T = \frac{-4}{5} P_{T-1} + 9$

Comparing with $P_T = aP_{T-1} + b$,

so, $a = \frac{-4}{5}$, $b = 9$

The general solution is $P_T = Aa^T \neq \frac{Aa}{1-a}$ (A is constants)

Substitution the values

$P_T = A\left(\frac{-4}{5}\right)^T + \frac{9}{1 + \frac{4}{5}} = A\left(\frac{-4}{5}\right)^T + 5$

When, $P_o = 6$ then

$6 = A + 5$

$\therefore \quad A = 1$

$\therefore \quad P_T = 1\left(\frac{-4}{5}\right)^T + 5$

Putting this expression is $Q_{dT} = -5P_T + 35$

$= -5\left[1\left(\frac{-4}{5}\right)^T + 3\right] + 33$

$= -5\left(\frac{-4}{5}\right)^T + 10$

$\therefore \quad Q_{dT} = -5\left(\frac{-4}{5}\right)^T + 10$

$P_T = 1\left(\frac{-4}{5}\right)^T + 5 = (-0.8)^t + 5$

3. Given,

$Q_d = -4p + 10$

$Q_s = 6p - 10$

a. For equilibrium, $Q_d \neq Q_s$

$-4p + 10 = 6p - 10$
$-10p = -20$
$p = 2$
Substituting the value of $Q_s = 6p - 10$
$6\times2 - 10$
$\therefore \quad Q = 2$
$\therefore \quad \begin{pmatrix} p = 2 \\ Q = 2 \end{pmatrix}$

4. Given,
$y_t = c_t + I_t$
$= 0.75y_{t-1} + 400 + 200$
$\therefore \quad y_t = 0.75y_{t-1} + 600$
If $t = 1$,
$y_1 = 0.75\, y_0 + 600$
$= 0.75 \times 400 + 600$
$= 900$
So, from $c_t = 0.75\, y_{t-1} + 400$
$c_2 = 0.75y_1 + 400 = 1075$

5. We have,
$y_t = c_t + I_t$
$y_t = 0.7y_{t-1} + 400 + 0.1y_{t-1} + 100$
or, $y_t = 0.8y_{t-1} + 500$
$y_t - 0.8y_{t-1} = 500$ ... ... (i)
Solution of (i) is $y_t = y_c + y_p$ where
y_c = complementary function
y_p = particular integral
For complementary function (y_c) : Reduce (i) into homogeneous form as
$y_t - 0.8y_{t-1} = 0$ ... ... (ii)
Let $y_t = A(m)^t$ be a trial solution.
Then $y_{t-1} = Am^{t-1}$
from (ii)
$Am^t - 0.8\, Am^{t-1} = 0$
$Am^t\, (1 - 0.8m^{-1}) = 0$
$m = 0.8$ since $Am^t \neq 0$
$\therefore \quad y_c = A(0.8)^t$
For particular integral (y_p) :
Let $y_t = k$ be a trial solution of (i).
Then $y_{t-1} = k$
$\therefore$ (i) becomes
$k - 0.8k = 500$
$0.2k = 500$
$k = \dfrac{500}{0.2} = 2500$
$\therefore \quad y_p = 2500$
$\therefore \quad y_t = A(0.8)^t + 2500$ is general solution.
When $t = 0$ then $y_0 = A(0.8)^0 + 2500$
$300 = A + 2500$
$A = 500$
$\therefore \quad y_t = 500\, (0.8)^t + 2500$ is required particular solution for y_t.

6. Given,
$y_t = c_t + I_t$

$y_t = (0.8y_{t-1} + 200) + 1000$
$y_t = 0.8y_{t-1} + 1200$
$y_t - 0.8y_{t-1} = 1200$ ... ... (i)

a. When t = 1 then
$y_1 - 0.8y_0 = 1200$
$y_1 - 0.8 \times 5000 = 1200$
$y_1 = 5200$
When t = 2
Then $y_2 = 0.8y_1 = 1200$
$y_2 \quad = 1200 + 0.8\ 5200 = 5360$
We have,
$c_t \quad = 0.8y_{t-1} + 200$
When t = 2
$c_2 \quad = 0.8 \times y_1 + 200$
$\quad = 0.8 \times 5200 + 200$
$\quad = 4360$

b. The difference equation relating $y_t + y_{t-1}$ is $y_t - 0.8\ y_{t-1} = 1200$ ... ... (i)

c. Its solution is $y_t = y_c + y_p$
For y_c:
$y_t - 0.8y_{t-1} = 0$
$Am^t (1 - 0.8\ m^{-1}) = 0$
$\therefore \quad m = 0.8$ since $Am^t \neq 0$
$y_c = A(0.8)^t$
For y_p:
Let $y_t = k$ be a solution
Then $y_{t-1} = k$
from (i)
$0.2k = 1200$
$k = 6000$
$\therefore \quad y_p = 6000$
Hence $y_t = A(0.8)^t + 6000$
When t = 0
$y_0 = A + 6{,}000$
$\therefore \quad A = -1{,}000$ since $y_0 = 5{,}000$
$\therefore \quad y_t = -1{,}000\ (0.8)^t + 6{,}000$
when t = 2
$y_2 = -1{,}000\ (0.8)^2 + 6{,}000$
5,340 which is in (a).