Chapter 2 Binomial Theorem

Exercise 2.1

1.a. $(3x + 2y)^5$

We know that, $(a + x)^n = c(n, 0)\, a^n + c(n, 1)\, a^{n-1} x + c(n, 2)\, a^{n-2} x^2 + ... + c(n, r)\, a^{n-r} x^r + ... + c(n, n)\, x^n$.

$\therefore$ $(3x + 2y)^5 = c(5, 0)\,(3x)^5 + c(5, 1)\,(3x)4\,(2y) + c(5, 2)\,(3x)^3\,(2y)^2 + c(5, 3)\,(3x)^2\,(2y)^3 + c(5, 4)\,(3x)^1\,(2y)^4 + c(5, 5)\,(2y)^5$

$= 243x^5 + 810x^4y + 1080x^3y^2 + 270x^2y^3 + 240xy^4 + 32y^5$

b. $(2x - 3y)^6$

$= 6c_0(2x)^6 + 6c_1(2x)^5\,(-3y) + 6c_2(2x)^4\,(-3y)^2 + 6c_3(2x)^3\,(-3y)^3 + 6c_4(2x)^2\,(-3y)^4 + 6c_5(2x)\,(-3y)^5 + 6c_6(-3y)^6$

$= 64x^6 - 576x^5y + 2160x^4y^2 - 4320x^3y^3 + 4860x^2y^4 - 2916xy^5 + 729y^6$.

c. $\left(x + \frac{1}{y}\right)^{11} = x^{11} + 11x^{10}\frac{1}{y} + 55\frac{x^9}{y^2} + 165\frac{x^8}{y^3} + 330\frac{x^7}{y^4} + 462\frac{x^6}{y^5} + 462\frac{x^5}{y^6} + 330\frac{x^4}{y^7} + 165\frac{x^8}{y^8} + 55\frac{x^2}{y^9} + 11\frac{x}{y^{10}} + \frac{1}{y^{11}}$

d. $\left(x + \frac{1}{x}\right)^6 = 6c_0\, x^6 + 6c_1\, x^5\frac{1}{x} + 6c_2\, x^4.\frac{1}{x^2} + 6c_3\, x^3\frac{1}{x^3} + 6c_4\, x^2\frac{1}{x^4} + 6c_5\, x\frac{1}{x^5} + 6c_6\frac{1}{x^6}$

$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

e. $\left(x - \frac{1}{x}\right)^7$

$= x^7 + {}^7c_1\, x^6\left(-\frac{1}{x}\right) + {}^7c_2\, x^5\left(-\frac{1}{x}\right)^2 + {}^7c_3\, x^4\left(-\frac{1}{x}\right)^3 + {}^7c_4\, x^3\left(-\frac{1}{x}\right)^4 + {}^7c_5\, x^2\left(-\frac{1}{x}\right)^5 + {}^7c_6\, x\left(-\frac{1}{x}\right)^6 + \left(-\frac{1}{x}\right)^7$

$= x^7 + 7x^5 + 21x^3 - 35x + 35.\frac{1}{x} - 21.\frac{1}{x^3} + 7.\frac{1}{x^5} - \frac{1}{x^7}$

f. $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6$

$= \left(\frac{2x}{3}\right)^6 + {}^6c_1\left(\frac{2x}{3}\right)^5.\left(\frac{-3}{2x}\right) + {}^6c_2\left(\frac{2x}{3}\right)^4 {}^6c_3\left(\frac{2x}{3}\right)^3.\left(\frac{-3}{2x}\right)^4 + {}^6c_4\left(\frac{2x}{3}\right)^2.\left(\frac{-3}{2x}\right)^4 + {}^6c_5\left(\frac{2x}{3}\right).\left(\frac{-3}{2x}\right)^5 + \left(\frac{-3}{2x}\right)^6$

$= \frac{64x^6}{729} - \frac{96}{81}x^4 + \frac{20}{3}x^2 - 21 + \frac{135}{4}.\frac{1}{x^2} - \frac{243}{8}.\frac{1}{x^4} + \frac{729}{64}.\frac{1}{x^6}$

g. $(1 + 2x - 3x^2)^5 = (1 + 2x)^5 + {}^5c_1\,(1 + 2x)^4.(-3x^2) + {}^5c_2\,(1 + 2x)^3\,(-3x^2)^2 + {}^5c_3\,(1 + 2x)^2.(-3x^2)^3 + {}^5c_4\,(1 + 2x).(-3x^2)^4 + (-3x^2)^5$

$= 1 + 10x + 25x^2 - 40x^3 - 190x^5 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$

2.a. We know that the general term t_{r+1} of expansion of $(a + x)^n$ is given by $t_{r+1} = n_{C_r}\, a^{n-r}\, x^r$

Here,

$(a + x)^n \Rightarrow \left(\frac{2x}{3} + \frac{3}{2x}\right)^6$

$\therefore$ $a \Rightarrow \frac{2x}{3}$, $x \Rightarrow \frac{3}{2x}$ and $n \Rightarrow 6$

For 7th term, put r = 6

$t_{6+1} = {}^6C_6\left(\frac{2x}{3}\right)^{6-6}\left(\frac{3}{2x}\right)^6$

$\therefore$ $t_7 = 1.\ 1.\ \frac{729}{64x^6}$

$\therefore$ $t_7 = \frac{729}{64x^6}$

b. The total number of terms of the expansion of $\left(\frac{x}{y} - \frac{2y}{x^2}\right)^6$ is 7.

So, there is no 10^{th} term.

c. For 5^{th} term, put r = 4.

$t_{r+1} = t_{4+1} = 12_{C_4}\ (2x)^{2-4}\ y^4 = 495 \times 2^8x^8y^4 = 126720\ x^8y^4$

d. Given,

$\left(2x^2 + \frac{1}{x}\right)^8$

Here, $n \rightarrow 8$

$a \rightarrow 2x^2$ and $x \rightarrow \frac{1}{x}$

We know that, $t_{r+1} = n_{C_r}\ a^{n-r}\ x^r$

$t_5 = t_{4+1} = 8_{C_4}\ (2x^2)^4 \left(\frac{1}{x}\right)^4 = 1120x^4$

e. $\left(x - \frac{1}{x}\right)^7$

$t_6 = t_{5+1} = {}^7C_5\ x^{7-5} \left(-\frac{1}{x}\right)^5 = 21x^2 \left(-\frac{1}{x^5}\right) = -\frac{21}{x^3}$

3.a. $(x^2 - y)^6$

Here, n = 6

The general term $(t_{r+1}) = {}^6C_r\ (x^2)^{6-r}\ (-y)^r$

$= (-1)^r\ {}^6C_r\ x^{12-2r}\ y^r$

b. Given, $\left(x^2 - \frac{1}{x}\right)^{12}$

Here, n = 12

The general term $(t_{r+1}) = {}^{12}C_r\ (x^2) \left(-\frac{1}{x}\right)^r = (-1)^r\ {}^{12}C_r\ x^{24-3r}$

c. Here, n = 10

The general term $(t_{r+1}) = {}^{10}C_r \left(\frac{x}{b}\right)^{10-r} \left(-\frac{b}{x}\right)^r = (-1)^r\ {}^{10}C_r \left(\frac{x}{b}\right)^{10-2r}$

d. Given, $\left(x - \frac{1}{x}\right)^{12}$

The general term $(t_{r+1}) = {}^{12}C_r\ (x)^{12-r} \left(-\frac{1}{x}\right)^r = (-1)^r\ {}^{12}C_r\ x^{12-2r}$

4.a. The general term $(t_{r+1}) = {}^{11}C_r\ (x^2)^{11-r} \left(-\frac{1}{x}\right)^r = (-1)^r\ {}^{12}C_r\ x^{12-2r}$

For x^7, 22 – 3r = 7

15 = 3r

$\therefore$ r = 5

$\therefore$ the coeff. of x^7 is 11_{C_r} i.e. ${}^{11}C_5 = 462$

b. The general term $(t_{r+1}) = {}^7C_r\ (x)^{7-r} \left(\frac{1}{2x}\right)^r = {}^7C_r \cdot \frac{1}{2^r}\ x^{7-2r}$

For x^5, we must have

7 – 2r = 5

r = 1

$\therefore$ Coeff. $x^5 = 7_{C_1}.\ \frac{1}{2} = \frac{7}{2}$

c. We have,

$t_{r+1} = 9c_r\,(3x^2)^{9-r}\left(\frac{-1}{3x}\right)^r$

$= 9c_r\,.\,3^{9-2r}\,.\,x^{18-2r-r}.\,(-1)^r$

$= (-1)^r\,3^{9-2r}\,.\,9c_r\,x^{18-3r}$

Here, $18 - 3r = 6$

$\therefore$ $r = 4$

$\therefore$ The coeff. of x^6 is $(-1)^4\,3^{9-8}\,.\,9c_4$

$= 3 \times 9c_4 = 378$

d. The general term $(t_{r+1}) = {}^9C_r\,(ax^4)^{9-r}\,(-bx)^r$

$= {}^9C_r\,a^{9-r}\,(-b)^r\,x^{36-3r}$]

For x^{12}, we must have

$36 - 3r = 12$

$\therefore$ $r = 8$

The required coeff. of x^{12} is ${}^9C_8\,a^{9-8}\,(-b)^8$

$= 9ab^8$

e. We have, $t_{r+1} = {}^9C_r\,(2x)^{9-r}\left(-\frac{1}{3x^2}\right)^r = (-1)^r\,{}^9C_r\,\frac{2^{9-r}}{3^r}\,x^{9-3}$

For x^{-6}, $9-3r = -6$

$9 + 6 = 3r$

$\therefore$ $r = 5$

Coeff. of $x^{-6} = (-1)^5\,{}^9C_5\,\frac{2^{9-5}}{3^5} = -\frac{2016}{243} = -\frac{224}{27}$

5.a. The general term $(t_{r+1}) = {}^8C_r\,(2x)^{8-r}\left(-\frac{1}{3x^2}\right)^r$

$= {}^8C_r\left(-\frac{1}{3}\right)^r 2^{8-r}\,x^{8-3r}$

For the independent of x,
we must have $8 - 3r = 0$

$\therefore$ $r = \frac{8}{3}$ (not possible)

$\therefore$ There is no term which is free from x.

b. $\left(x + \frac{1}{x}\right)^{10}$

Here, $t_{r+1} = 10c_r\,x^{10-r}\left(\frac{1}{x}\right)^r = 10c_r\,x^{10-2r}$

For free from x, $10-2r = 0$

$\therefore$ $r = 5$

$\therefore$ $t_{r+1} = t_{5+1} = t_6$ is the required term.

c. $t_{r+1} = 15c_r\,(x^2)^{15-r}\left(-\frac{1}{x^3}\right)^r$

$= 15c_r\,(-1)^r\,x^{30-4r}$

$\therefore$ $30 - 4r = 0$

$r = \frac{15}{2}$ (not possible)

$\therefore$ no term has free from x

d. $t_{r+1} = 10c_r\left(\frac{3x^2}{2}\right)^{10-r}\left(-\frac{1}{3x}\right)^r$

$= (-1)r\,10c_r\,\frac{3^{10-2r}}{2^{10-r}}\,x^{20-3r}$

For x^0, $20-3r = 0$

$r = \frac{20}{3}$ (not possible)

$\therefore$ No term is free from x.

e. The general term $(t_{r+1}) = 14c_r\,(x^2)^{14-r}\left(-\frac{1}{x^2}\right)^r$

$= (-1)^r\,14c_r\,x^{24-4r}$

$\therefore$ For free of x, we have $24 - 4r = 0$

$\therefore$ 7^{th} term is required term.

6.a. $(3 + x)^6$

Here n = 6, there is a single middle term.

$\therefore$ Middle term is

$t_{\frac{n}{2}+1} + t_{3+1} = t_4$

Using $t_{r+1} = nc_r\,a^{n-r}\,n^r$

$t_{3+1} = 6c_3\,3^{6-3}\,x^3 = 6c_3\,3^3\,x^3 = 540x^3$

b. $\left(x - \frac{1}{2y}\right)^{10}$

Since n = 10 (even), there is a single middle term

$\therefore$ Middle term is

$t_{\frac{n}{2}+1} = t_{5+1} = 10c_5\,(x)^{10-5}\left(-\frac{1}{2y}\right)^5 = 10c_5\,x^5\left(-\frac{1}{2}\right)^5\frac{1}{y^5} = -\frac{3}{8}\left(\frac{x}{y}\right)^5$

c. $\left(1 - \frac{x^2}{2}\right)^{14}$

Here, n = 14 (even), there is a single middle term

$t_{\frac{n}{2}+1} = t_{7+1}$

$\therefore$ $t_{7+1} = 14c_7\,(1)^{14-7}\left(-\frac{x^2}{2}\right)^7$ $\quad (\because t_{r+1} = nc_r\,a^{n-r}\,x^2)$

$= -14c_7\,\frac{x^{14}}{2^7}$

$= -\frac{429}{16}\,x^{14}$

d. Since n = 10, there is a middle term.

$t_{5+1} = 10c_5\,(x^2)^5\left(-\frac{2}{x}\right)^5 = -2^5\,.\,10c_3\,x^5 = -8064x^5$

e. Since 2n is even, there is single middle term

$t_{\frac{2n}{2}+1}$

i.e. t_{n+1}

$= 2nc_n\,(an)^n\left(-\frac{1}{an}\right)^n$

$= 2nc_n\,(-1)^n$

$= (-1)^n\,\frac{(2n)\,!}{n!\,n!})$

$= (-1)^n\,\frac{1.2.3....\,(2n-2)\,(2n-1).\,2n}{n!\,n!}$

$= \frac{(-1)^n\,2^n(1.2.3....\,n)\,(1.3.5....\,(2n-1))}{n!\,n!} = \frac{(-2)^n\,\{1.3.5....\,(2n-1)}{n!}$

f. There is a single middle term

$t_{\frac{n}{2}+1} = t_{4+1} = 8c_4\,(2x^2)^4\left(\frac{1}{x}\right)^4 = 8c_4\,2^4\,.\,x^4 = 1120x^4$

7.a. Here, n = 5 (odd), there are two middle terms.

i.e. $t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e. t_{2+1} and t_{3+1}

$t_{2+1} = 5c_2\ (x^2)^3\ (a^2)^2 = 10x^6a^4$

$t_{3+1} = 5c_3\ (x^2)^2\ (a^2)^3 = 10x^4a^6$

b. Here, n = 11 (odd), there are two middle terms.

i.e. $t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e t_{5+1} and t_{6+1}

Now, $t_{5+1} = 11c_5\ (x^4)^{11-5} \left(-\frac{1}{x^3}\right)^5 = (-1)^5\ 11c_5\ x^9 = -462x^9$

$t_{6+1} = 11c_6\ (x^4)^{11-6} \left(-\frac{1}{x^3}\right)^6 = 11c_6\ x^2 = -462x^2$

c. Here, n = 13 (odd), there are two middle terms.

$t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e. t_{6+1} and t_{7+1}

Now, $t_{6+1} = 13c_6\ (x)^7 \left(-\frac{1}{x}\right)^6 = 13c_6\ x = 1716x$

$t_{7+1} = 13c_7\ x^6 \left(-\frac{1}{x}\right)^7 = -\frac{1716}{x}$

d. Given, $\left(2x + \frac{1}{x}\right)^{17}$

Since n = 17, there are two middle terms.

$t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e. t_{8+1} and t_{9+1}

$t_{8+1} = 17c_8\ (2x)^9 \left(\frac{1}{x}\right)^8 = 2^9.c(17,8)x = 144446720x$

$t_{9+1} = 17c_9\ (2x)^8 \left(\frac{1}{x}\right)^9 = 2^8\ c(17, 9)\ .\ \frac{1}{x} = \frac{6223360}{x}$

e. Here, $\left(x + \frac{1}{x}\right)^{2n+1}$

Since (2n+1) is odd, there are two middle terms.

$t_{\frac{2n+1-1}{2}+1}$ and $t_{\frac{2n+1+1}{2}+1}$

i.e. t_{n+1} and $t_{(n+1)+1)}$

Now, $t_{n+1} = 2n+1c_n\ (x)^{2n+1-n} \left(\frac{1}{x}\right)^n = 2n+1c_n\ .\ x$

$t_{(n+1)+1} = 2n+1c_{n+1}\ x^{2n+1-n-1} \left(\frac{1}{x}\right)^{n+1} = 2n+1c_{n+1}\ .\ x^{-1}$

8.a. $(1+x)^{2n}$

Since 2n is even for any n, there is a single middle term.

$t_{\frac{n}{2}+1}$

i.e. $t_{\frac{2n}{2}+1}$ t_{n+1}

$\therefore\ t_{n+1} = 2nc_n\ x^n = n(2n, n)\ x^n.$

$= (2x)^n\ .\ \frac{1.3.5.\ ...\ (2n-1)}{n!}$

b. Since (2n+1) is an odd number, there is only one middle term given by

$t_{\frac{2n}{2}+1}$ i.e. t_{n+1}

We know $t_{r+1} = n_{c_r}\, a^{n-r}\, n^r$ where $a \Rightarrow x$ $x - \frac{1}{x}$

$n \Rightarrow 2n$

$\therefore$ Using r = n is above formula, we get

$$t_{n+1} = 2n_{c_n}\, a^{2n-n}\left(-\frac{1}{x}\right)^n = \frac{2n!}{n!\,n!}(-1)^n$$

$$= \frac{1\times2\times3\times4\times5\times6\times\ldots\times(2n-3)\times(2n-2)\times(2n-1)\times2n}{n!\,n!}(-1)^n$$

$$= \frac{\{1\times3\times5\times\ldots\times(2n-1)\}\,[2\times4\times6\times\ldots\times(2n-2)\times2n]}{n!\times n!}(-1)^n$$

$$= \frac{\{1\times3\times5\times\ldots\times(2n-1)\}\times2^n\{1\times2\times3\times\ldots(n-1)\times n\}}{n!\,n!}(-1)^n$$

$$= \frac{1.3.5.\ldots(2n-1)}{n!}(-2)^n$$

c. $\left(\frac{x}{y} - \frac{y}{x}\right)^{2n+1}$

Since (2n+1) is odd for any n, the number of terms of the expansion is (2n+2), which is even, so there are two middle terms, given by

$t_{\frac{2n+1-1}{2}+1}$ and $t_{\frac{2n+1+1}{2}+1}$

i.e. t_{n+1} and $t_{(n+1)+1}$

Now, $t_{n+1} = 2n+1_{c_n}\left(\frac{x}{y}\right)^{2n+1-n}\left(-\frac{-y}{x}\right)$

$= 2n+1_{c_n}\left(\frac{x}{y}\right)$

$= c(2n+1, n)\,\frac{x}{y}$

$t_{(n+1)+1} = c(2n+1, n+1)\,\frac{y}{x}$

9. Let ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ be the three consecutive coefficients in the expansion of $(1+x)^n$. Then,

$n_{c_{r-1}} = 165$ .... (i)

$n_{c_r} = 330$ .... (ii)

$n_{c_{r+1}}$ 462 .... (iii)

Dividing (i) by (ii), we get,

$\frac{n_{c_{r-1}}}{n_{c_r}} = \frac{165}{330}$

$\Rightarrow \frac{n!}{(n-r+1)!\,(r-1)!} \times \frac{(n-r)!\,r!}{n!} = \frac{1}{2}$

or, $\frac{(n-r)!\,r!}{(r-1)!\,(n-r+1)!} = \frac{1}{2}$

or, $\frac{(n-r)!\,(r-1)!\,r}{(r-1)!\,(n-r+1)\,(n-r)!} = \frac{1}{2}$

$\therefore$ $\frac{r}{n-r+1} = \frac{1}{2}$

or, $2r = n-r+1$

$3r = n+1$ .... (iv)

Again, dividing (ii) by (iii) we get

$\frac{n_{c_r}}{n_{c_{r+1}}} = \frac{330}{462} \Rightarrow \frac{n!}{(n-r)!\,r!} \times \frac{(n-r-1)!\,(r+1)!}{n!} = \frac{5}{7}$

or, $\frac{(n-r-1)!\,(r+1)!}{(n-r)!\,r!} = \frac{5}{7}$

or, $\frac{(n-r-1)!\ r!\ (r+1)}{(n-r)\ (n-r-1)!\ r!} = \frac{5}{7}$

or, $\frac{r+1}{n-r} = \frac{5}{7}$

or, $7r + 7 = 5n - 5r$

$12r = 5n - 7$ .... (v)

from (iv) and (v), we get

$4(n+1) = 5n - 7$

$4n + 4 = 5n - 7$

$\therefore$ $n = 11$ and $r = 4$

10. Let $n_{c_{r-1}}$, n_{c_r} and $n_{c_{r+1}}$ be three consecutive coefficients of the expansion of $(1+n)^n$.

Acc^r to question $n_{c_{r-1}} : n_{c_r} : n_{c_{r+1}} = 1 : 7 : 42$

Let $n_{c_{r-1}}$ k .... (i) $n_{c_r} = 7k$ .... (ii) and $n_{c_{r+1}} = 42k$ .... (iii)

Dividing (ii) by (i), $\frac{n_{c_r}}{n_{c_{r-1}}} = \frac{7k}{k}$

$\frac{n!}{(n-r)!\ r!} \times \frac{(n-r+1)\ !\ (r-1)!}{n!} = 7$

or, $\frac{(n-r+1)\ (n-r)!}{(n-r)!\ (r-1)!} = 7$

or, $n-r+1 = 7r$

$n+1 = 8r$ .... (iv)

Again, dividing (iii) by (ii)

$\frac{n_{c_{r+1}}}{n_{c_r}} = \frac{42k}{7k}$

$\frac{n!}{(n-r-1)!\ (r+1)!} \times \frac{(n-r)!\ r!}{n!} = 6$

or, $\frac{(n-r)\ (n-r-1)!\ r!}{(r+1)\ r!\ (n-r-1)!} = 6$

or, $n-r = 6r + 6$

$n = 7r+6$ .... (v)

From (iv) and (v)

$7r + 6 + 1 = 8r$

$7 = r$

$\therefore$ $r = 7$

$\therefore$ $n = 55$

11. Here, $(x+y)^n$

$\therefore$ $t_{r+1} = n_{c_r}\ x^{n-r}\ y^r$

4^{th} term $(t_4) = t_{3+1} = n_{c_3}\ x^{n-3}\ y^3$

13^{th} term $(t_{13}) = t_{12+1} = n_{c_{12}}\ x^{n-12}\ y^{12}$

Acc^r to question,

Coeff. of 4^{th} term = coeff. of 13^{th} terms

$n_{c_3} = n_{c_{12}}$

$\frac{n!}{(n-3)!\ 3!} = \frac{n!}{(n-12)!\ 12!}$

or, $(n-12)!\ 12! = (n-3)!\ 3!$

or, $(15-12)!\ 12! = (15-3)!\ 3!$

or, $3!\ 12! = 12!\ 3!$

$\therefore$ $n = 15$

12. We have $(1+x)^{20}$.

r^{th} term $(t_r) = t_{(r-1)+1} = {}^{20}c_{r-1}\ x^{r-1}$

$\therefore$ Coeff. of r^{th} term is ${}^{20}c_{r-1}$

$(r+4)^{th}$ term $(t_{r+4}) = t_{(r+3)+1} = {}^{20}c_{r+3}\ x^{r+3}$

$\therefore$ Coeff. of $(r+4)^{th}$ term is $^{20}c_{r+3}$

Acc^r to question,

$^{20}c_{r-1} = {}^{20}c_{r+3} \Rightarrow r-1+r+3 = 20$

$r = 9$

13. We have $(1+x)^{m+n}$

Comparing it with $(a+x)^n$, we get $a \Rightarrow 1$, $x \Rightarrow x$ $n \to m+n$.

We know that the general term of expansion of $(a+n)^n$ is $t_{r+1} = nc_r\, a^{n-r}\, x^r$.

Here, $t_{r+1} = m+nc_r\, (1)^{m+n-r} \,.\, x^r$

$\therefore$ $t_{r+1} = m+nc_r\, x^r$ .... (i)

for x^m, put $r = m$, in (i)

Then $t_{m+1} = m+nc_m\, x^m$

for x^n, put $r = n$ in (i)

Then $t_{n+1} = m+nc_n\, x^n$

Now, $m+nc_m = \dfrac{(m+n)!}{(m+n-m)!\, m!} = \dfrac{(m+n)!}{n!\, m!} = \dfrac{(m+n)!}{n!\, m!}$

$= \dfrac{(m+n)!}{(m+n-n)!\, n!} = m+nc_n$

Hence, $m+nc_m = m+nc_n$

This proves that the coeff. of x^m and x^n are equal.

14. Here, $(1+x)^{2n}$

The general term $t_{r+1} = 2nc_r\, x^r$ .... (i)

for 2^{nd} term, put r=1 in (i)

$\therefore$ $t_2 = 2nc_1\, x^1$

When r = 2 in (i) $\qquad t_3 = 2nc_2\, x^2$

When r = 3 in (i) $\qquad t_4 = 2nc_3\, x^3$

Acc^r to question $2nc_1$, $2nc_2$ and $2nc_3$ are in AP.

Then $2nc_2 = \dfrac{2nc_1 + 2nc_3}{2}$

or, $2.\dfrac{(2n)!}{(2n-2)!\, 2!} = \dfrac{(2n)!}{(2n-1)!} + \dfrac{(2n)!}{(2n-3)!\, 3!}$

or, $\dfrac{(2n)!}{(2n-2)!} = 2n!\left[\dfrac{1}{(2n-1)!} + \dfrac{1}{(2n-3)!\, 3!}\right]$

or, $\dfrac{1}{(2n-2)!} = \dfrac{3! + (2n-2)\,(2n-3)}{(2n-1)!\, 3!}$

or, $\dfrac{1}{(2n-2)!} = \dfrac{6 + (2n-2)\,(2n-3)}{(2n-1)\,(2n-2)!\, 6}$

or, $6(2n-1) = 6 + (2n-2)\,(2n-3)$

or, $12n-6 = 6 + 4n^2 - 6n - 4n + 6$

or, $4n^2 - 10n - 12n + 18 = 0$

$4n^2 - 22n + 18 = 0$

$2(2n^2 - 11n + 9) = 0$

$2n^2 - 11n + 9 = 0$

15. The general term of the expansion of $(1+x)^n$ is $t_{r+1} = {}^nC_r\, x^r$

The 2^{nd}, 3^{rd} and 4^{th} terms are respectively,

$nc_1\, x^1$, $^nC_2\, x^2$ and $nc_3\, x^3$

Acc^r to question, $^nC_1$, $^nC_2$ and $^nC_3$ are in AP.

Then $2.\,{}^nC_2 = {}^nC_1 + {}^nC_3$

$2.\dfrac{n!}{(n-2)!\, 2!} = \dfrac{n!}{(n-1)!} + \dfrac{n!}{(n-3)!\, 3!}$

or, $\frac{n!}{(n-2)!} = n!\left(\frac{1}{(n-1)!} + \frac{1}{6(n-3)!}\right)$

or, $\frac{1}{(n-2)!} = \frac{6 + (n-2)(n-1)}{6(n-1)!}$

or, $\frac{1}{(n-2)!} = \frac{6 + n^2-n-2n+2}{6(n-1)(n-2)!}$

or, $6n-6 = 6+n^2-3n+2$

or, $n^2-9n+14 = 0$

or, $n^2-7n-2n+14 = 0$

$(n-7)(n-2) = 0$

Either $n = 7$ or $n = 2$ (not possible)

$\therefore$ $n = 7$

16. The coeff. of $(2r+1)^{th}$ term is $^{21}c_{2r}$ .... (i)

The coeff. of $(3r+2)^{th}$ term is $21c_{3r+1}$ .... (ii)

Acc^r to question,

$21c_{3r+1} = 21c_{2r}$

$\therefore$ Then $3r + 1 + 2r = 21$

$5r = 20$

$\therefore$ $r = 4$

17. Let us suppose that x^r and x^{r+1} occurs in the $(r+1)^{th}$ and $(r+2)^{th}$ terms in the expansion of $(1+x)^{2n+1}$

Then,

$t_{r+1} = nc_r\, a^{n-r}\, x^r$ and $t_{r+2} = nc_{r+1}\, a^{n-r-1}\, x^{r+1}$

where $a \Rightarrow 1$, $x \Rightarrow x$ $n \Rightarrow 2n+1$

$\therefore$ $t_{r+1} = {}^{2n+1}c_r\, 1^{2n+1-r}\, x^r$ and $t_{r+2} = {}^{2n+1}c_{r+1}\, (1)^{2n+1-r-1}\, x^{r+1}$

$\Rightarrow$ $t_{r+1} = {}^{2n+1}c_r\, x^r$ and $t_{r+2} = {}^{2n+1}c_{r+1}\, x^{r+1}$ .... (i)

Now, by question, coefficient x^r = coefficient of x^{r+1}

$\Rightarrow$ ${}^{2n+1}c_r = {}^{2n+1}c_{r+1}$

or, $\frac{(2n+1)!}{r!\,(2n+1-r)!} = \frac{(2n+1)!}{(r+1)!\,(2n+1-r-1)!}$

or, $r!(2n-r+1)! = (r+1)!\,(2n-r)!$

or, $r!\,(2n-r)!\,(2n-r+1) = r!(r+1)\,(2n-r)!$

or, $2n-r+1 = r+1$

$2n = 2r$

$\therefore$ $r = n$

18. Since the number of terms in the expansion of $(1+x)^{2n}$ is $2n+1$, odd number. So there is only one middle term given by $t_{\frac{2n}{2}+1}$ i.e. t_{n+1}.

Now, coefficient of $(n+1)^{th}$ term = $^{2n}c_3$ m

Again, the number of terms in the expansion of $(1+x)^{2n-1}$ is $2n-1+1 = 2n$, even number. So, there are two middle terms given by $t_{\frac{2n-1+1}{2}}$, $t_{\frac{2n-1+1}{2}+1}$ i.e. t_n, t_{n+1}

Now, the coefficients of two middle terms are $^{2n-1}c_{n-1}$ and $^{2n-1}c_n$

$\therefore$ $^{2n-1}c_{n-1} + {}^{2n-1}c_n = \frac{(2n-1)!}{(n-1)!\,n!} + \frac{(2n-1)!}{n!(n-1)!}$

$= \frac{2(2n-1)!}{n!(n-1)!}$

$= \frac{2n(2n-1)!}{n!\,n(n-1)!}$

$= \frac{(2n)!}{n!n!}$

$= 2nc_n$

Hence proved

19. Since $(1+x)^n = nc_0 + nc_1\, x + nc_2\, x^2 + ... + nc_n\, x^n$
Using x = 1 and –1, we get
$(1+1)^n = nc_0 + nc_1 + nc_2 + ... + nc_n$ .... (*)
$(1–n)^n = nc_0 – nc_1 + nc_2 – .... + nc_n$ .... (**)
Again, $(1+x)^{n–1} = {}^{n–1}c_0 + {}^{n–1}c_1\, x^1 + {}^{n–1}c_2\, x^2 + ... + {}^{n–1}c_{n–1}\, x^{n–1}$
Using x=1 and –1, we get
$(1+1)^{n–1} = {}^{n–1}c_0 + {}^{n–1}c_1 + {}^{n–1}c_2 + ... + {}^{n–1}c_{n–1}$ .... (***)
$(1–1)^{n–1} = {}^{n–1}c_0 – {}^{n–1}c_1 + {}^{n–1}c_2 + ... + (–1)^{n–1}\, {}^{n–1}c_{n–1}$ .... (****)

a. $C_1 – 2.C_2 + 3.C_3 – ..... + n(–1)^{n–1}\, C_n$

$= n – 2\dfrac{n(n–1)}{2!} + 3.\dfrac{n(n–1)\,(n–2)}{3!} – ..... + n.\,(–1)^{n–1}.\,1$

$= n\left[1 – \dfrac{(n–1)}{1!} + \dfrac{(n–1)\,(n–2)}{2!} – ...... + (–1)n–1\right]$

$= n[{}^{n–1}c_0 – {}^{n–1}c_1 + {}^{n–1}c_2 – ...... + (–1)^{n–1}\, {}^{n–1}c_{n–1}]$
$= n(1–1)^{n–1}$ (By using formula (****) above)
$= n×0$
= 0 Hence, proved

b. $C_1 + 2.C_2 + 3.C_3 + ... + n.\, C_n$

$n + 2.\dfrac{n(n–1)}{2!} + 3.\dfrac{n(n–1)\,(n–2)}{3!} + ... + n.1$

$= n[\,{}^{n–1}c_0 + {}^{n–1}c_1 + {}^{n–1}c_2 + ... + {}^{n–1}c_{n–1}]$
$= n(1+1)^{n–1}$ (Using formula (***) above)
$= n.2^{n–1}$ Hence proved

c. $C_0 + 2.C_1 + 3.C_2 + ... + (n+1)\,.\,C_n$
$= (C_0 + C_1 + C_2 + ... + C_n) + (C_1 + 2C_2 + 3C_3 + ... + n.\, C_n)$

$= (1+1)^n + \left[n + \dfrac{n(n–1)}{1!} + \dfrac{n(n–1)\,(n–2)}{2!} + ... + n\right]$

$= 2^n + n\left[1 + \dfrac{(n–1)}{1!} + \dfrac{(n–1)(n–2)}{2!} + ... + 1\right]$

$= 2^n + n[\,{}^{n–1}c_0 + {}^{n–1}c_1 + {}^{n–1}c_2 + ... + {}^{n–1}c_{n–1}]$
$= 2^n + n.\,(1+1)^{n–1}$ (By using formula *** above)
$= 2^n + n\,.\,2^{n–1}$
$= 2^{n–1}.\,2 + n.2^{n–1}$
$= (n+2)2^{n–1}$ Hence proved

d. $C_0 + 2.C_1 + 4.C_2 + ... + 2n.\, C_n\, \boxed{1+n.2^n}$

Similar as above

e. Since $(1+x)^n = C_0 + C_1\, x + C_2\, x^2 + ... + C_n\, x^n$
$\therefore\ (1+x)^{2n} = (C_0 + C_1x + C_2x^2 + ...\, C_nx^n)\,(C_0 + C_1x + C^2x^2 + ... + C_nx^n)$
$\therefore\ {}^{2n}c_0 + {}^{2n}c_1\, x + {}^{2n}c_2\, x^2 + ... + {}^{2n}c_n\, x^n + ... + {}^{2n}c_{2n}\, x^{2n}$ = (") (")
Equating the coefficient of x^n in both sides.

Coefficient of x^n in LHS $= {}^{2n}c_n = \dfrac{(2n)!}{n!\,n!}$ .... (i)

Coefficient of x^n in RHS $= C_0C_n + C_1C_{n–1} + C_2C_{n–2} + ... + ... + C_nC_0$ ...... (ii)
Equating (i) and (ii), we get

$C_0C_n + C_1C_{n–1} + ... + C_nC_0 = \dfrac{(2n)!}{n!\,n!}$ Hence proved

f. Since $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ .... (i)
$(x+1)^n = C_0x^n + C_1x^{n–1} + C_2x^{n–2} + ... + C_n$ .... (ii)
Multiplying (i) and (ii), we get
$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + ... + C_nx^n)\,(C_0x^n + C_1x^{n–1} + ... + C_n)$
Equating the coefficient of x^n both sides, we get

Coefficient of x^n in LHS = $^{2n}c_n = \frac{(2n)!}{n!\,n!}$ .... (iii)

Coefficient of x^n in RHS = $C^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2$ .... (iv)

Equating (iii) and (iv), we get,

$C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \frac{(2n)!}{n!\,n!}$ Hence proved

g. Since $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ .... (i)

$(x+1)^n = C_0x^n = C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n$ ..... (ii)

Multiplying (i) and (ii), we get

$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n)\,(C_0x^n + C_1x^{n-1} + \ldots\ldots + C_n)$

Equating the coefficient of x^{n-1} both sides,

Coeff. of x^{n-1} in LHS = $^{2n}c_{n-1} = \frac{(2n)!}{(2n-n+1)!\,(n-1)!} = \frac{(2n)!}{(n+1)!\,(n-1)!}$ ...... (iii)

Coefficient of x^{n-1} in RHS = $C_0C_1 + C_1C_2 + \ldots. + C_{n-2}\,C_{n-1} + C_{n-1}\,C_n$ .... (iv)

Equating (iii) and (iv), we get

$C_0C_1 + C_1C_2 + \ldots\ldots + C_{n-2}\,C_{n-1} + C_{n-1}\,C_n = \frac{(2n)!}{(n+1)!\,(n-1)!}$ Proved.

h. Since, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ ...... (i)

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n$ .... (ii)

Multiplying (i) and (ii)

$(1+x)^{2n} = (C_0+C_1x + C_2x^2 + \ldots\ldots + C_nx^n)\,(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n)$

This is identify, so coeff. of any power of x in LHS and coeff. of same power of x in RHS must be equal.

Coeff. of x^{n-2} in LHS = $^{2n}c_{n-2} = \frac{(2n)!}{(n+2)!\,(n-2)!}$ .... (iii)

Coeff. of x^{n-2} in RHS = $C_0C_2 + C_1C_3 + \ldots\ldots + C_{n-2}\,C_n$ .... (iv)

Equating (iii) and (iv) we get

$C_0C_2 + C_1C_3 + \ldots\ldots + C_{n-2}\,C_n = \frac{(2n)!}{(n-2)!\,(n+2)!}$ Hence proved

i. Same as (h)

Equating the coeff. of x^{n-r} in both sides.

Coeff. of x^{n-r} in LHS = $^{2n}c_{n-r} = \frac{(2n)!}{(n+r)!\,(n-r)!}$ .... (iii)

Coeff. of x^{n-r} in RHS = $C_0C_r + C_1C_{r+1} + \ldots\ldots + C_{n-r}\,C_n$ ...... (iv)

Equating (iii) and (iv), we get

$C_0C_r + C_1C_{r+1} + \ldots\ldots + C_{n-r}\,C_n = \frac{(2n)!}{(n-r)!\,(n+r)!}$ Proved.

20. Let $^nc_{r-1}$, ^nc_r, $^nc_{r+1}$ and $^nc_{r+2}$ be four consecutive coefficients in the expansion of $(1+x)^n$ such that $^nc_{r-1} = a$, $^nc_r = b$ $^nc_{r+1} = c$ and $^nc_{r+2} = d$.

Now, $\frac{a}{a+b} = \frac{1}{1+\frac{b}{a}} = \frac{1}{1+\frac{^nc_r}{^nc_{r-1}}} = \frac{1}{1+\frac{n!}{(n-r)!\,r!}\times\frac{(n-r+1)!\,(r-1)}{n!}} = \frac{1}{1+\frac{(n-r+1)}{r}} = \frac{r}{n+1}$

$\therefore\quad \frac{a}{a+b} = \frac{r}{n+1}$ .... (i)

Similarly, we can prove

$\frac{b}{b+c} = \frac{r+1}{n+1}$ .... (ii) and $\frac{c}{c+d} = \frac{r+2}{n+1}$ .... (iii)

$\therefore\quad \frac{a}{a+b} + \frac{c}{c+d} = \frac{r}{n+1} + \frac{r+2}{n+1} = \frac{2(r+1)}{n+1} = \frac{2b}{b+c}$

$\therefore\quad \frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}$ proved.

Exercise 2.2

1.a. $(2 - 3x)^{-3}$

$= 2^{-3}\left(1 - \frac{3}{2}x\right)^{-3}$

The expansion is valid when $\left|\frac{3}{2}x\right| < 1$ i.e. $|x| < \frac{2}{3}$

Now, $(2 - 3x)^{-3} = 2^{-3}\left(1 - \frac{3}{2}x\right)^{-3}$

$$= \frac{1}{2^3}\left[1 + (-3)\left(-\frac{3}{2}x\right) + \frac{(-3)(-3-1)}{2!}\left(-\frac{3}{2}x\right)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}\left(-\frac{3}{2}x\right)^3 + \ldots \text{ to } \infty\right]$$

$$= \frac{1}{2^3}\left[1 + \frac{9}{2}x + \frac{27}{2}x^2 + \frac{135}{4}x^3 + \ldots \text{ to } \infty\right]$$

$$= \frac{1}{8}\left(1 + \frac{9x}{2} + \frac{27}{2}x^2 + \frac{135}{4}x^3 + \ldots \text{ to } \infty\right)$$

b. Here $(2 + 3x)^{5/2}$

$e^{5/2}\left(1 + \frac{3}{2}x\right)^{5/2}$

The expansion is valid when $\left|\frac{3x}{2}\right| < 1$ i.e. $|x| < \frac{2}{3}$

We know that,

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}n^2 + \frac{n(n-1)(n-2)}{3!}n^3 + \ldots$

Now, $2^{5/2}\left(1 + \frac{3}{2}x\right)^{5/2}$

$$= 2^{5/2}\left[1 + \left(\frac{5}{2}\right)\left(\frac{3}{2}x\right) + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\left(\frac{3x}{2}\right)^2 + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)}{3!}\left(\frac{3}{2}x\right)^3 + \ldots\right]$$

$$= 2^{5/2}\left[1 + \frac{15}{4}x + \frac{135x^2}{64} + \frac{135x^3}{128} + \ldots \text{ to } \infty\right]$$

c. $(5 + 4x)^{-1/2}$

$5^{-1/2}\left[1 + \frac{4}{5}x\right]^{-1/2}$

This expansion is valid when $\left|\frac{4x}{5}\right| < 1$ i.e. $|x| < \frac{5}{4}$

$5^{-1/2}$

$$\left[1 + \left(-\frac{1}{2}\right)\frac{4}{5}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{4x}{5}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}\left(\frac{4x}{5}\right)^3 + \ldots\right]$$

$$= \frac{1}{\sqrt{5}}\left[1 - \frac{2x}{5} + \frac{6x^2}{25} - \frac{4x^3}{25} + \ldots \text{ to } \infty\right]$$

d. $(3 - 2x^2)^{-2/3}$

$3^{-2/3}\left(1 - \frac{2}{3}x^2\right)^{-2/3}$

The expansion is valid when $\left|\frac{2x^2}{3}\right| < 1$ i.e. $|x^2| < \frac{3}{2}$

Now, $e^{-2/3}\left[1 - \frac{2}{3}x^2\right]^{-2/3}$

$$= e^{-2/3}\left[1 + \left(\frac{-2}{3}\right)\left(-\frac{2}{3}x^2\right)\right] + \frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)}{2!}\left(\frac{-2}{3}x^2\right)^2 + \ldots \text{ to } \infty)$$

$$= e^{-2/3}\left[1 + \frac{4x^2}{9} + \frac{20x^4}{81} + \ldots \text{ to } \infty\right]$$

2.a. $(1+x)^{1/2}$

$$1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)x^2}{2!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \ldots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}x^3 \ldots \text{ to } \infty$$

b. $(1 + x^2)^{-1/2}$

$$1 + \left(\frac{-1}{2}\right)x^2 + \frac{\left(\frac{-1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(x^2)^3 + \ldots$$

$$= 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \ldots \text{ to } \infty$$

c. $(1 + x)^{1/4}$

$$1 + \left(+\frac{1}{4}\right)x + \frac{\left(+\frac{1}{4}\right)\left(+\frac{1}{4}-1\right)}{2!}x^2 + \frac{\left(\frac{+1}{4}\right)\left(+\frac{1}{4}-2\right)\left(\frac{1}{4}-2\right)}{3!}x^3 + \ldots$$

$$= 1 + \frac{x}{4} - \frac{3x^2}{32} + \frac{7x^3}{128} - \ldots \text{ to } \infty$$

d. $(1 - x^2)^{-1/3}$

$$1 + \left(-\frac{1}{3}\right)(-x^2) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(-x^2)^2 + \ldots \text{ to } \infty$$

$$= 1 + \frac{1}{3}x^2 + \frac{2x^4}{9} + \ldots \text{ to } \infty$$

3.a. $(1.03)^{-5}$

$(1 + 0.03)^{-5}$

$$= 1 + (-5)(0.03) + \frac{(-5)(-5-1)(0.03)^2}{2!} + \frac{(-5)(-5-1)(-5-2)}{3!}(0.03)^3 + \ldots$$

$= 0.915$

b. $(0.01)^{1/2}$

$(1 - 0.99)^{1/2}$

$$= 1 + \frac{1}{2}(-0.99) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-0.99)^2 + \ldots$$

$= 0.1$

c. $(28)^{1/3}$

$(27 + 1)^{1/3}$

$$= 3\left(1 + \frac{1}{27}\right)^{1/3}$$

$$= 3\left[1 + \frac{1}{3}\cdot\frac{1}{27} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{27}\right)^2 + \ldots\right]$$

$= 3.0.37$

d. $\sqrt{17} = (16 + 1)^{1/2} = 4\left(1 + \frac{1}{16}\right)^{1/2}$

$$4\left[1 + \frac{1}{2}\frac{1}{16} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(\frac{1}{16}\right)^2 + \ldots\right]$$

$= 4.123$

e. $\left(\frac{96}{101}\right)^{1/3}$

$$\left(1 - \frac{5}{101}\right)^{1/3}$$

$$1 + \frac{1}{3}\left(\frac{-5}{101}\right) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{-5}{101}\right)^2 + \ldots = 0.983$$

4.a. Let $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.2} + \ldots$ to $\infty = (1 + x)^n$

Then, $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots$ to $\infty = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$

Equating corresponding term, we get,

$nx = \frac{1}{4}$

$\therefore \quad x = \frac{1}{4n}$ ... ... (i) and $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{4.8}$

or, $\frac{n(n-1)}{2}\frac{1}{(4n)^2} = \frac{1.3}{4.8}$

$\frac{n(n-1)}{2.16\,n^2} = \frac{3}{32}$

$n - 1 = 3n$

$-2n = 1$

$\therefore \quad n = -\frac{1}{2}$

from (i) $x = \frac{1}{4(-1/2)} = -\frac{1}{2}$

Hence, $(1 + x)^n = \left(1 - \frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2} = 2^{1/2} = \sqrt{2}$

$\therefore \quad 1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots$ to $\infty = \sqrt{2}$

b. Let $(1+x)^n$ be equal to $1 + \frac{1.2}{2.3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \ldots$ to ∞

i.e. $1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$ to $\infty = 1 + \frac{1.2}{2.3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \ldots$ to ∞

Equating corresponding term, we get

$nx = \frac{1.2}{2.3} \Rightarrow x = \frac{1}{3n}$ ... ... (i) and $\frac{n(n-1)}{2}x^2 = \frac{1.3}{3.6}$

$\frac{n(n-1)}{2}\frac{1}{n^2} = \frac{1}{6}$

$n - 1 = 3n$

$-2n = 1$

$\therefore \quad n = -\frac{1}{2}$

from (i) $x = \frac{1}{3\left(\frac{-1}{2}\right)} = -\frac{2}{3}$

$\therefore \quad (1 + x)^n = \left(1 - \frac{2}{3}\right)^{-1/2} = \left(\frac{1}{3}\right)^{-1/2} = 3^{1/2} = \sqrt{3}$

c. Let $1 - \frac{1}{6} + \frac{1.3}{6.12} + \frac{1.3.5}{6.12.18} + \ldots$ to $\infty = (1 + x)^n$

$1 - \frac{1}{6} + \frac{1.3}{6.12} + \frac{1.3.5}{6.12.18} + \ldots$ to $\infty = (1 + x)^n$

Equating corresponding term

$nx = -\frac{1}{6}$

$\therefore \quad x = -\frac{1}{6n} \ldots \ldots \ldots$ (i)

$\frac{n(n-1)}{2} x^2 = \frac{1.3}{6.12}$

or, $\frac{n(n-1)}{2} \left(\frac{-1}{6n}\right)^2 = \frac{1}{24}$

$\frac{n(n-1)}{2} \frac{1}{36.n^2} = \frac{1}{24}$

or, $n - 1 = 3n, n = -\frac{1}{2}$

from (i) $x = -\frac{1}{6\left(-\frac{1}{2}\right)} = \frac{1}{3}$

$\therefore \quad (1 + x)^n = \left(1 + \frac{1}{3}\right)^{-1/2} = \left(\frac{4}{3}\right)^{-1/2} = \left(\frac{3}{4}\right)^{1/2} = \frac{\sqrt{3}}{2}$

d. Let $1 + \frac{1}{4} + \frac{1.4}{4.8} + \ldots$ to $\infty = (1 + x)^n$

$1 + \frac{1}{4} + \frac{1.4}{4.8} + \ldots$ to $\infty = 1 + nx + \frac{n(n-1)}{2} x^2 + \ldots$

$\therefore \quad nx = \frac{1}{4}$ and $\frac{n(n-1)}{2} x^2 = \frac{1.4}{4.8}$

$x = \frac{1}{4n} \ldots \ldots \ldots$ (i)

or, $\frac{n(n-1)}{2} . \frac{1}{16n^2} = \frac{1}{8}$

or, $n - 1 = 4n$

$3n = -1$

$\therefore \quad n = -\frac{1}{3}$

$\therefore \quad x = -\frac{3}{4}$

$\therefore \quad (1 + x)^n = \left(1 - \frac{3}{4}\right)^{-1/3} = \left(\frac{1}{4}\right)^{-1/3} = 4^{1/3} = 2^{2/3}$ proved.

e. Let $1 + \frac{1}{4} - \frac{1.1}{4.8} + \frac{1.1.3}{4.8.12} - \ldots = 1 + nx + \frac{n(n-1)}{2} x^2 + \ldots$

Equating, $nx = \frac{1}{4}$

$\therefore \quad x = \frac{1}{4n} \ldots \ldots \ldots$ (i)

$\frac{n(n-1)}{2} x^2 = -\frac{1}{32}$

$\frac{n(n-1)}{2} . \frac{1}{16n^2} = -\frac{1}{32}$

$n - 1 = -n$

$2 n = 1$

$\therefore \quad n = \frac{1}{2}$

$\therefore \quad x = \frac{1}{2}$

$\therefore \quad (1 + n)^n = \left(1 + \frac{1}{2}\right)^{1/2} = \left(\frac{3}{2}\right)^{1/2} = \sqrt{\frac{3}{2}}$

Hence, $1 + \frac{1}{4} - \frac{1.1}{4.8} + \frac{1.1.3}{4.8.12} - \frac{1.1.3.5}{4.8.12.16} + ... = \sqrt{\frac{3}{2}}$

5. Given, $y = 2x + 3x^2 + 4x^3 + ...$ to ∞

$1 + y = 1 + 2x + 3x^2 + 4x^3 + ...$ to ∞

or, $1 + y = (1 - x)^{-2}$

or, $(1 - x) = (1 + y)^{-1/2}$

or, $1 - x = 1 - \frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{10}y^3 + ...$

$\therefore$ $x = \frac{1}{2}y - \frac{3}{8}y^2 + \frac{5}{16}y^3 - ...$ to ∞

6. Here, $(1 - x + x^2 - x^3 + ...$ to $\infty)(1 + x + x^2 + x^3 + ...$ to $\infty)$

$= (1 + x)^{-1} . (1 - x)^{-1}$

$= (1 - x^2)^{-1}$

$= 1 + (-1)(-x^2) + \frac{(-1)(-1-1)}{2}(-x^2)^2 + ...$ to ∞

$= 1 + x^2 + x^4 + ...$ to ∞ proved.

7. Here,

$(1 + x + x^2 + x^3 + ...$ to $\infty)(1 + 2x + 3x^2 + ...$ to $\infty)$

$= (1 - x)^{-1}(1 - x)^{-2}$

$= (1 - x)^{-3}$

$= 1 + (-3)(-x) + \frac{(-3)(-3-1)}{2!}(-x)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}(-x)^3 + ...$

$= 1 + 3x + 6x^2 + ...$ to ∞

Exercise 2.3

1.a. $\frac{e^{5x} + e^x}{e^{3x}} = e^{2x} + e^{-2x}$

We know that, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ to ∞

$\therefore$ $e^{2x} + e^{-2x} = \left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(3x)^3}{3!} + ... \text{ to } \infty\right) + \left(1 - \frac{2x}{1!} + \frac{(2x)^2}{2!}\right) - \frac{(2x)^3}{3!} + ... \text{ to } \infty)$

$= 2 + 2\frac{(2x)^2}{2!} + 2\frac{(2x)^4}{4!} + ...$ to ∞

$= 2\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + ...\right) = 2\left(1 + \frac{2^2.x^2}{2!} + \frac{2^4.x^4}{4!} + \frac{2^6.x^6}{6!} ... + \infty\right)$

b. $\frac{e^{7x} + e^x}{2e^{4x}}$

$= \frac{1}{2}[e^{3x} + e^{-3x}]$

$= \frac{1}{2}\left[\left(1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + ...\right)\right] + \left(1 - \frac{3x}{1!} + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + ...\right)$

$= \frac{1}{2}\left[2 + \frac{2(3x)^2}{2!} + 2((3x)^4,4!) + ...\right]$

$= 1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!}$

$= 1 + \frac{3^2.x^2}{2!} + \frac{3^4.x^4}{4!} + \frac{3^6\,x^6}{6!} + ... \infty$

c. $\frac{e^{4x} - 1}{e^{2x}} = e^{2x} - e^{-2x} = \left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + ..\right) - \left(1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + ...\right)$

$= 2\left(\frac{2x}{1!} + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + ...\right)$

2.a. $(e^1) . (e^{-1})$

$e^0 = 1$ proved.

b. $\frac{(1+1)}{1!} + \left(\frac{3+1}{3!}\right) + \frac{(5+1)}{5!} + ...$ to ∞

$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + ... = e$

c. n^{th} term of given series $(t_n) = \frac{1+2+3+...+n}{(n-1)!} = \frac{\frac{n}{2}(n+1)}{(n-1)!} = \frac{n(n+1)}{2(n-1)!}$

d. $t_n = \frac{1+3+5+...(2n-1)}{n!} = \frac{n^2}{n!} = \frac{n.n}{n!} = \frac{n.n}{n(n-1)!}$

$t_n = \frac{n}{(n-1)!} = \frac{n-1+1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$

$S_n = \Sigma t_n = \Sigma \frac{1}{(n-2)!} + \Sigma \frac{1}{(n-1)!}$

$= \left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...\right)$

$= \left(\frac{1}{0} + 1 + \frac{1}{1!} + \frac{1}{2!} + ...\right) + \left(1 + \frac{1}{1!} + \frac{1}{2!} + ...\right)$

$= e + e$

$= 2e$

e. $t_n = \frac{2+4+6+...+2n}{n!} = \frac{n(n+1)}{n!}$

$= \frac{n(n+1)}{n(n-1)!} = \frac{n-1+2}{(n-1)!} = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$

$t_n = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$

$s_n = \Sigma t_n = \sum_{n=1}^{\infty} \frac{1}{(n-2)!} + 2\sum_{n=1}^{\infty} \frac{1}{(n-1)!}$

$= e + 2e$

$= 3e$

f. $3 + \frac{5}{1!} + \frac{7}{2!} + \frac{9}{3!} + ...$ to ∞

Let t_n be the n^{th} term of above series.

Then, $t_n = \frac{3+(n-1)d}{(n-1)!} = \frac{3+(n-1).2}{(n-1)!} = \frac{2n+1}{(n-1)!}$

$t_n = \frac{2(n-1)+3}{(n-1)!} = \frac{2}{(n-2)!} + \frac{3}{(n-1)!}$

$s_n = \sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} \frac{2}{(n-2)!} + 3\sum_{n=1}^{\infty} \frac{1}{(n-1)!}$

$= 2\left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{!} + \frac{1}{2!} + ...\right) + 3\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...\right)$

$= 2e + 3e$

$= 5e$

g. Here, $\frac{2-1}{2!} + \frac{3-1}{3!} + \frac{4-1}{4!} + ...$

$1 - \frac{1}{2!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{3!} - \frac{1}{4!} + ...$

$= 1$ proved.

h. We know, $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to $\infty = e^x$

Putting x = 1 and x = –1, we get

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = e$ ... ... (i)

and $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots = e^{-1}$ ... ... (ii)

Adding (i) and (ii), we get

$2 + \frac{2}{2!} + \frac{2}{4!} + \ldots = e + e^{-1}$

or, $2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\right) = e + e^{-1}$

or, $1 + \frac{1}{2!} + \frac{1}{4!} + \ldots = \frac{e + e^{-1}}{2}$

$\therefore \quad 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots = \frac{e^2 + 1}{2e}$ ... ... (iii)

Subtracting (ii) from (i), we get

$2 + \frac{2}{3!} + \frac{2}{5!} + \frac{2}{7!} + \ldots = e–e^{-1}$

$1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots = \frac{e^2 - 1}{2e}$ ... ... (iv)

Dividing (iii) by (iv), we get,

$\frac{1 + \frac{1}{2!} + \frac{1}{4!} + \ldots}{1 + \frac{1}{3!} + \frac{1}{5!} + \ldots} = \frac{e^2 + 1}{e^2 - 1}$ Hence proved.

i. $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots$

$\frac{3-1}{3!} + \frac{5-1}{5!} + \frac{7-1}{7!} + \ldots$

$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \ldots$

$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots = e^{-1}$ proved.

j. We know,

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ ... ... (*)

$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$ ... ... (**)

Adding (*) and (**),

$e^x + x^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right)$

or, $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$ ... ... (***)

Subtracting (**) from (*), we get,

$e^x - e^{-x} = 2\left(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$

$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$ ... ... (****)

Now, $\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \text{ to } \infty\right)^2 - \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)^2$

$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right)$

$= e^x \cdot e^{-x} = 1$

k. We know $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$\therefore \quad \frac{1}{2} e^x = \frac{1}{2}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$

$\frac{1}{2} e^2 = \frac{1}{2}\left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + - \frac{2^4}{4!} + \ldots\right)$

$= \frac{1}{2} + 1 + \frac{2}{2!} + \frac{2^2}{3!} + \frac{2^3}{4!} + \ldots$

$\therefore \quad 1 + \frac{2}{2!} + \frac{2^2}{3!} + \frac{2^3}{2!} + \ldots = \frac{1}{2} e^2 - \frac{1}{2} = \frac{1}{2}(e^2 - 1)$

Also, $e - e^{-1} = 1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots$

$\therefore \quad \dfrac{1 + \frac{2}{2!} + \frac{2^2}{3!} + \frac{2^3}{4!} + \ldots}{1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots} = e$ proved.

3.a. We know that,

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to ∞

$\therefore \quad e^{x^2} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots$ to ∞ ... ... (i)

$e^{y^2} = 1 + \frac{y^2}{1!} + \frac{y^4}{2!} + \frac{y^6}{3!} + \ldots$ to ∞ ... ... (ii)

Equation (i) – equation (ii)

$e^{x^2} - e^{y^2} = \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots\right) - \left(1 + \frac{y^2}{1!} + \frac{y^2}{2!} + \frac{y^6}{3!} + \ldots\right)$

$= (x^2 - y^2) + \frac{(x^4 - y^4)}{2!} + \frac{1}{3!}(x^6 - y^6) + \ldots$ Proved.

b. RHS $\dfrac{1}{\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots \text{ to } \infty}$

$= \dfrac{1}{\frac{3-1}{3!} + \frac{5-1}{5!} + \frac{7-1}{7!} + \ldots \text{ to } \infty}$

$= \dfrac{1}{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \ldots \text{ to } \infty}$

$= \dfrac{1}{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots}$

$= \frac{1}{e^{-1}} = e$

LHS $\frac{1+1}{1!} + \frac{3+1}{3!} + \frac{5+1}{5!} + \ldots$ to ∞

$= 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$ to ∞

$= e$ Hence proved

c. We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ to ∞

$\therefore \quad e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \ldots$

$$e^2 - 1 = 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + ...$$

4.a. $\sqrt{e} = e^{1/2}$

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

Put $x = \frac{1}{2}$

$$e^{1/2} = 1 + \frac{\frac{1}{2}}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + ...$$

$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + ...$$

$$= 1 + 0.5 + 0.125 + 0.0208 + ...$$

$$= 1.6458$$

b. $\frac{1}{\sqrt{e}} = e^{-1/2}$

$$= 1 - \frac{\frac{1}{2}}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} - \frac{\left(\frac{1}{2}\right)^3}{3!} + ...$$

$$= 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + ...$$

$$= 1 - 0.5 + 0.125 - 0.0208 + ...$$

$$= 0.6042$$

5.a. Let n^{th} term of above series be t_n

Then $t_n = \frac{1 + 2 + 2^2 + ... + 2^{n-1}}{n!} = \frac{\frac{1(2^n - 1)}{2 - 1}}{n!} = \frac{2^n - 1}{n!}$

$$t_n = \frac{2^n}{n!} - \frac{1}{n!}$$

Let s_∞ be the required sum of the series.

Then, $s_\infty = \Sigma t_n = \Sigma\left(\frac{2^n}{n!} - \frac{1}{n!}\right) = \sum_1^\infty \frac{2^n}{n!} - \sum_1^\infty \frac{1}{n!}$

$$= \left(\frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + ...\right) - \left(\frac{1}{1!} + \frac{1}{2!} + ..\right)$$

$$= (e^2 - 1) - (e - 1)$$

$$= e^2 - e$$

b. $1 + \frac{1}{2!} + \frac{1.3}{4!} + \frac{1.3.5}{6!} + ...$

$$= 1 + \frac{1}{2} + \frac{1.3}{1.2.3.4} + \frac{1.3.5}{1.2.3.4.5.6} + ...$$

$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + ...$$

$$= 1 + \frac{\left(\frac{1}{2}\right)}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + ...$$

$$= e^{1/2}$$

c. Let t_n be the n^{th} term of the given series

Then $t_n = \frac{n(n+1)}{n!} = \frac{n^2 + n}{n!} = \frac{n}{(n-1)!} + \frac{1}{(n-1)!}$

$$= \frac{n+1}{(n-1)!} = \frac{(n-1)+2}{(n-1)!} = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$$

$\therefore$ The given series

$$\Sigma t_n = \Sigma \frac{1}{(n-2)!} + \Sigma \frac{2}{(n-1)!}$$

$$= \left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right) + 2\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

but $(-1)! = \infty$ and $0! = 1$

$$\therefore \quad \Sigma t_n = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) + 2\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

$$= e + 2e$$

$$= 3e$$

d. Let t_n be the n^{th} term of the series then,

$$t_n = \frac{1+(n-1)^2}{(n-1)!} = \frac{2n-1}{(n-1)!} = \frac{2(n-1)+1}{(n-1)!}$$

$$= \frac{2}{(n-2)!} + \frac{1}{(n-1)!}$$

$\therefore$ Sum of the given series

$$\sum_1^{\infty} t_n = 2\sum_1^{\infty} \frac{1}{(n-2)!} + \sum_1^{\infty} \frac{1}{(n-1)!}$$

$$= 2\left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \ldots\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

But $(-1)! = \infty$ and $0! = 1$

$$\therefore \quad \sum_1^{\infty} t_n = 2e + e = 3e$$

e. Let t_n be the n^{th} term

$$t_n = \frac{n^2}{n!} = \frac{n}{(n-1)!} = \frac{n-1+1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$$

$\therefore$ Sum of the series

$$\sum_1^{\infty} t_n = \sum_n^{\infty} \frac{1}{(n-2)!} + \Sigma\frac{1}{(n-1)!} = e+e = 2e$$

6.a. $$t_n = \frac{n^2}{(n+1)!} = \frac{n^2-1+1}{(n+1)!} = \frac{(n^2-1)}{(n+1)!} + \frac{1}{(n+1)!}$$

$$= \frac{(n-1)}{n!} + \frac{1}{(n+1)!}$$

$$= \frac{n}{n!} - \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$= \frac{1}{(n-1)!} - \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$\Sigma t_n = \Sigma\frac{1}{(n-1)!} - \Sigma\frac{1}{n!} + \Sigma\frac{1}{(n+1)!}$$

$$= \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) - \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) + \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\right)$$

$$= e - (e-1) + (e-2)$$

$$= e - e + 1 + e - 2$$

$$= e - 1$$

b. $$t_n = \frac{1}{(n+1)!}$$

$$\Sigma t_n = \sum_1^{\infty} \frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

$$= e - 2$$

c. $t_n = \frac{1}{(n+2)!}$

$$\Sigma t_n = \sum_1^{\infty} \frac{1}{(n+2)!} = \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

$$= \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots\right) - \frac{5}{2}$$

$$= e - \frac{5}{2}$$

d. $t_n = \frac{n^3}{n!} = \frac{n^2}{(n-1)!} = \frac{(n-1)(n+1)}{(n-1)!} + \frac{1}{(n-1)!}$

$$= \frac{n+1}{(n-2)!} + \frac{1}{(n-1)!} = \frac{(n-2)}{(n-2)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$$

$$= \frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$$

$$\Sigma t_n = \Sigma \frac{1}{(n-3)!} + 3\Sigma\frac{1}{(n-2)!} + \Sigma\frac{1}{(n-1)!}$$

$$= \left(\frac{1}{(-2)!} + \frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right) + 3\left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \ldots\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

$$= e + 3e + e$$

$$= 5e$$

e. $t_n = \frac{n(n+1)}{n!} = \frac{n+1}{(n+1)!} = \frac{n-1}{(n-1)!} + \frac{2}{(n-1)!} = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$

$$\Sigma t_n = \Sigma\frac{1}{(n-2)!} + \Sigma \frac{2}{(n-1)!}$$

$$= e + 2e$$

$$= 3e$$

7.a. We know that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ to ∞

$$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

$$\ln 2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots$$

$$= \left(\frac{2-1}{1.2}\right) + \frac{(4-3)}{3.4} + \left(\frac{6-5}{5.6}\right) + \ldots$$

$$= \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$$

b. $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots$ to $\infty = 2 - 2\ln 2$

We have,

$\ln(1 + x) = x - x\frac{2}{2} + x\,\frac{3}{3} - x\frac{4}{4} + x\frac{5}{5} - x\,\frac{6}{6} + x\,\frac{7}{7} - \ldots$

Putting x = 1, we get,

$\ln(1 + 1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots$

$\Rightarrow\ \ln^2 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \ldots$

$\Rightarrow\ \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \ldots = 1 - \ln 2$

$\Rightarrow\ \left(\frac{3-2}{2.3}\right) + \left(\frac{5-4}{4.5}\right) + \left(\frac{7-6}{6.7}\right) + \ldots = 1 - \ln 2$

$\Rightarrow\ \frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots = 1 - \ln 2$

Multiplying by 2 on both sides, we get

$\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots = 2(1 - \ln 2)$

Hence, $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots = 2 - 2\ln 2$

c. We know that,

$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$ to ∞ ... ... (i)

$\log_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$ to ∞ ... ... (ii)

Subtracting (ii) from (i)

$\log_e(1 + x) - \log(1 - x) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + 2\frac{x^7}{7} + \ldots$

or, $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$

$\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots$ to ∞

Put $x = \frac{1}{3}$

$$\frac{1}{2}\log_e\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \frac{\left(\frac{1}{3}\right)^7}{7} + \ldots$$

$$\frac{1}{2}\log_e\left(\frac{\frac{4}{3}}{\frac{2}{3}}\right) = \frac{1}{3} + \frac{1}{3^3.3} + \frac{1}{3^5.5} + \frac{1}{3^7.7} + \ldots$$

$\therefore\ \frac{1}{2}\log_e(2) = \frac{1}{3} + \frac{1}{3^3.3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \ldots$

d. We know,

$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$$

$$\therefore\quad \log(1+x) = \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$$

Put $x = \frac{1}{2}$

$$\log\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) = 2\left(\frac{1}{2} + \frac{1}{3.2^3} + \frac{1}{5.2^5} + \frac{1}{7.2^7} + \ldots\right)$$

$$\log 3 = 1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots$$

e. Given,

$2\ln x - \ln(x+1) - \ln(x-1)$

$= 2\ln x - \{\ln(x+1) + \ln(x-1)\}$

$= \ln x^2 - \ln\{(x+1)(x-1)\}$

$= \ln x^2 - \ln(x^2 - 1)$

$= -[\ln(x^2 - 1) - \ln^2]$

$$= -\ln\left(\frac{x^2-1}{x^2}\right)$$

$$= -\ln\left(1 - \frac{1}{x^2}\right)$$

$$= -\left[-\frac{1/x^2}{1} - \frac{(1/x^2)^2}{2} - \frac{(1/x^2)^3}{3} - \ldots \text{ to } \infty\right]$$

$$= \frac{1}{x^2} + \frac{1}{2.x^2} + \frac{1}{3.x^6} + \ldots \text{ to } \infty$$

f. LHS. $\left(\frac{1}{3} - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{3^2} - \frac{1}{2^2}\right) + \frac{1}{3}\left(\frac{1}{3^3} - \frac{1}{2^3}\right) + \ldots$

$$= \left(\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1}{3}\frac{1}{3^3} + \ldots\right) - \left(\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\cdot\frac{1}{2^3} - \ldots\right)$$

$$= -\left[-\left(\frac{1}{3}\right) - \frac{\left(\frac{1}{3}\right)^2}{2} - \frac{\left(\frac{1}{3}\right)^3}{3} - \ldots\right] - \left[\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \ldots\right]$$

$$= -\log_e\left(1 - \frac{1}{3}\right) - \log_e\left(1 + \frac{1}{2}\right)$$

$$= -\log\frac{2}{3} - \log\frac{3}{2}$$

$$= -\left[\log\frac{2}{3} + \log\frac{3}{2}\right]$$

$$= -\log\left(\frac{2}{3}, \frac{3}{2}\right)$$

$= -\log 1$

$= 0$ Hence proved.

8.a. Sum to infinity the following series:

$$\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots \text{ to } \infty$$

$\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+ \ldots \text{ to } \infty$

$= \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+ \ldots \text{ to } \infty$

$= 1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\ldots\right)$

$= 1 - \text{Ln}(1+1) = 1 - \ln^2$

b. We know that,

$\ln(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}- \ldots \text{ to } \infty$

Putting x = 1 on both sides, we get

$\ln(1+1) = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$

or, $\ln 2 = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots$

$= \left(\frac{2-1}{2}\right)+\left(\frac{4-3}{3.4}\right)+\left(\frac{6-5}{5.6}\right)+\ldots$

$\ln 2 = \frac{1}{2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots \text{ to } \infty$

or, $\ln 2 - \frac{1}{2} = \frac{1}{3.4}+\frac{1}{5.6}+\ldots$

or, $\ln 2 - \frac{1}{2} = \frac{1}{3.4}+\frac{1}{5.6}+\ldots$

$\therefore \quad \frac{1}{3.4}+\frac{1}{5.6}+\ldots \text{ to } \infty = \ln 2 - \frac{1}{2}$

9. Here, $y = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots \text{ to } \infty$

or, $y = \log_e(1+x)$

$\therefore \quad e^y = 1+x$

$1+x = 1+\frac{y}{1!}+\frac{y^2}{2!}+\frac{y^3}{3!}+\ldots \text{ to } \infty$

$\therefore \quad x = y+\frac{y^2}{2!}+\frac{y^3}{3!}+\ldots \text{ to } \infty$

10. Here, $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots \text{ to } \infty$

$y = -\left[-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots \text{ to } \infty\right]$

or, $y = -[\log_e(1-x)]$

or, $-y = \log_e(1-x)$

$\therefore \quad 1-x = e^{-y}$

$1-x = 1-\frac{y}{1!}+\frac{y^2}{2!}-\frac{y^3}{3!}+\frac{y^4}{4!}-\ldots$

$x = y-\frac{y^2}{2!}+\frac{y^3}{3!}-\frac{y^4}{4!}+\ldots \text{ to } \infty$

11. Here, $y = \frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \text{ to } \infty$

$1+y = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \text{ to } \infty$

$1+y = e^x$

Taking 'ln' on both sides

$\ln(1 + y) = x$

$\therefore \quad x = \ln(1 + y)$

$$x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots \text{ to } \infty$$