Chapter 15
Derivatives

Exercise 15.1

1. Find the limit of the following function at given points.

a. $f(x) = \dfrac{\log(1+x)}{x}$ at $x = 0$

Solution: Since, we have,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Now, $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log(1+x)}{x}$$

$$= \lim_{x\to 0} \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]$$

$$= \lim_{x\to 0}\left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right]$$

$$= \left[1 - \frac{0}{2} + \frac{0^2}{3} - \frac{0^3}{4} + \ldots\right]$$

$$= 1$$

$$\therefore \lim_{x\to 0} f(x) = 1$$

b. $f(x) = \left(\dfrac{1-x}{1+x}\right)^{1/x}$ at $x = 0$

Solution: Here, $y = \left(\dfrac{1-x}{1+x}\right)^{1/x}$

Taking log on both sides,

$$\log y = \frac{1}{x}\log\left(\frac{1-x}{1+x}\right) \qquad [\because \log m^n = n\log m]$$

or, $$\log y = \frac{1}{x}\log\left[1 - \frac{2x}{1+x}\right]$$

$$= \frac{1}{x}\,\frac{\log\left[1 + \left(\frac{-2x}{1+x}\right)\right]}{\left(\frac{-2x}{1+x}\right)} \times \frac{-2x}{1+x}$$

$$= \frac{\log\left[1 + \left(\frac{-2x}{1+x}\right)\right]}{\frac{-2x}{1+x}} \times \frac{-2}{1+x}$$

Taking $\lim\limits_{x\to 0}$ on both sides, we have,

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left[1 + \left(\frac{-2x}{1+x}\right)\right]}{\frac{-2x}{1+x}} \times \frac{-2}{1+x}$$

$$= 1 \times \lim_{x\to 0} -\frac{2}{1+x} \qquad \left[\because \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$$

$$= \frac{-2}{1+0}$$

$$\therefore \lim_{x\to 0} \log y = -2$$

i.e, $\lim_{x\to 0} y = e^{-2}$ [$\because \log_x x = y \Leftrightarrow x = e^y$]

i.e, $\lim_{x\to 0} \left(\frac{1-x}{1+x}\right)^{1/x} = e^{-2}$

c. $f(x) = \frac{(1+x)^n - 1}{x}$ at x = 0

Solution: $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(1+x)^n - 1}{x}$

$= \lim_{x\to 0} \frac{(1+x-1)\,[(1+x)^{n-1} + (1+x)^{n-2} + (1+x)^{n-3} + \ldots + (1+x) + 1]}{x}$

$= \lim_{x\to 0} (1+x)^{n-1} + (1+x)^{x-2} + (1+x)^{n-3} + \ldots (1+x) + 1$

$= 1^{n-1} + 1^{n-2} + 1^{n-3} + \ldots 1 + 1$

$= n-1 + 1$

$= n$

2.a. A function f(x) is defined as follows:

$$f(x) = \begin{cases} -x-3 & \text{for } x \le -2 \\ \frac{2}{3}x + \frac{1}{3} & \text{for } -2 < x < 1 \\ x^2 & \text{for } x \ge 1 \end{cases}$$

Test the continuity of f(x) of x = –2 and x = 1.

Solution:

Note: A function f(x) is said to be continuous at a point x = a if and only if,

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

Here, To test the continuity of f(x) at x = –2, we proceed as follows:

Here, f(x) at x = –2 is

$f(-2) = -x - 3$ [$\because f(x) = -x - 3$ for $x \le -2$]

$= -2 - 3 = -5$ (a finite value)

Now, left hand limit of f(x) at x = –2 is,

$\lim_{x\to -2^-} f(x) = \lim_{x\to 2^-} (-x-3)$ [$\because f(x) = -x - 3$ for $x \le -2$]

$= -2 - 3 = -5$

Finally,

Right hand limit of f(1) at x = –2 is,

$\lim_{x\to -2^+} f(x) = \lim_{x\to 2^+} \left(\frac{2}{3}x + \frac{1}{3}\right)$ [$\because f(x) = \frac{2}{3}x + \frac{1}{3}$ for $-2 < x < 1$]

$= \frac{2}{3} \times (-2) + \frac{1}{3} = -\frac{4}{3} + \frac{1}{3} = \frac{-4+1}{3} = -\frac{3}{3} = -1$

$\therefore \lim_{x\to -2^-} f(x) = f(-2) \ne \lim_{x\to -2^+} f(x)$

So, f(x) is discontinuous at a point x = –2.

2nd Part;

Again testing the continuity of f(x) at x = 1

For the functional value, $f(1) = x^2$ [$\because f(x) = x^2$ for $x \ge 1$]

$= 1^2 = 1$

LHL at x = 1 is,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(\frac{2}{3}x + \frac{1}{3}\right)$ [$\because f(x) = \frac{2}{3}x + \frac{1}{3}$ for $-2 < x < 1$]

$= \frac{2}{3} \times 1 + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1$

RHL at x = 1 is,

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x^2)$ [$\because f(x) = x^2$ for $x \geq 1$]

$= 1^2 = 1$

Here, we have,

$\lim_{x\to1^-} f(x) = f(1) = \lim_{x\to1^+} f(x)$

Thus, the given function is continuous at x = 1

b. Show that the following function is continuous at x = 4

$$f(x) = \begin{cases} \frac{x^2 - 16}{x - 4} & \text{for } x \neq 4 \\ 8 & \text{for } x = 4 \end{cases}$$

Solution: Testing functional value at x = 4;

f(4) = 8 [$\because$ f(x) = 8 when x = 4]

Again, testing the limiting value at x = 4, we have,

$\lim_{x\to4} f(x) = \lim_{x\to4} \left(\frac{x^2 - 16}{x-4}\right)$ $\left[\because f(x) = \frac{x^2-16}{x-4} \text{ for } x \neq 4\right]$

$= \lim_{x\to4} \left(\frac{x^2-4^2}{x-4}\right)$ $\left[\because \text{at } x = 4, \frac{4^3-16}{4-4} = \% \text{ form}\right]$

$= \lim_{x\to4} \frac{(x+4)(x-4)}{(x-4)} = \lim_{x\to4} (x+4)$ [$\because x \neq 4$]

$= 4 + 4 = 8$

Here, we see, $\lim_{x\to4} f(x) = f(4)$

Therefore, f(x) is continuous at x = 4

c. A function f(x) is defined as follows:

$$f(x) = \begin{cases} 3 + 2x & \text{for } -3/2 \leq x < 0 \\ 3 - 2x & \text{for } 0 \leq x < 3/2 \\ -3 - 2x & \text{for } x \geq 3/2 \end{cases}$$

Test the continuity of f(x) at x = 0 and x =- 3

Solution: Testing the continuity of f(x) at x = 0,

For the functional value, f(0) = 3–2x

or, $f(0) = 3 - 2\times0$

$= 3$ (a finite value)

Again, LHL of f(x) at x = 0,

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} (3 + 2x)$ [$\because f(x) = 3+2x$ for $-\frac{3}{2} \leq x < 0$]

$= 3 + 2\times0$

$= 3$

Finally, RHL off f(x) at x = 0

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} (3 - 2x)$ [$\because f(x) = 3 - 2x$ for $0 \leq x < 3/2$]

$= 3 - 2\times0$

$= 3$

Here, we see, $\lim_{x\to0^-} f(x) = f(0) = \lim_{x\to0^+} f(x)$

$\therefore$ The given function is continuous at point x = 0.

2nd part:

Testing the continuity of f(x) at x = 3/2

Functional value, $f\left(\frac{3}{2}\right) = -3-2x$ [$\because$ f(x) = –3–2x for $x \geq 3/2$]

$$= -3 \times -2 \times \frac{3}{2} = -6$$

LHL at $x = \frac{3}{2}$, $\lim_{x\to 3/2^-} f(x) = \lim_{x\to 3/2^-} (3-2x)$ [$\because$ f(x) = 3–2x for $0 \leq x < 3/2$]

$$= 3 - 2\times\frac{3}{2}$$

Here, we see,

$$\lim_{x\to 3/2^-} f(x) \neq x\left(\frac{3}{2}\right)$$

Therefore, the given function is discontinuous at $x = \frac{3}{2}$

3.a. Show that the function f(x) defined by,

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} \text{ forr} \neq 0 \\ 0 \text{ for } x \to 0 \end{cases}$$ is continuous at x = 0

Proof: Functional values, f(x) = 0 [$\because$ f(x) = 0 for x = 0]

Limiting value, $\lim_{x\to 0} f(x)$

$$= \lim_{x\to 0} x^2 \sin\frac{1}{x} \qquad [\because f(x) = x^2 \sin\frac{1}{x} \text{ for } x \neq 0]$$

$= 0$

$\therefore$ $\lim_{x\to 0} f(x) = f(0)$. Thus the function is continuous at x = 0

Hence proved.

b. Examine for continuity at x = 0 for the function of f(x) defined by,

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Functional values at x = 0;

f(0) = 1 (finite values) [$\because$ f(x) = 1 for x = 0]

Limiting values at x = 0,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos x}{x^2} \qquad \left[f(x) = \frac{1-\cos x}{x^2} \text{ for } x \neq 0\right]$$

$$= \lim_{x\to 0} \frac{(1-\cos x)}{x^2} \times \frac{1+\cos x}{(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \times \frac{1}{1+\cos x}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \times \lim_{x\to 0} \frac{1}{1+\cos x}$$

$$= (1)^2 \times \frac{1}{2} = \frac{1}{2} \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

Here, $\lim_{x\to 0} f(x) \neq f(0)$

$\therefore$ The given function is discontinuous at x = 0.

Exercise 15.2

1. Find from first principles the derivative of (1–5).

a. (i) $e^{\sin x}$.

Solution: Let, $f(x) = e^{\sin x}$

We know by the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h\to 0} \frac{e^{\sin(x+h)} - e^{\sin x}}{h} \; \ldots\ldots\ldots \text{(i)}$$

Put, $y=\sin x \Rightarrow y+k = \sin(x+h)$

$\Rightarrow$ $k = \sin(x+h) - y$

where $k \to 0$ as $h \to 0$

$\Rightarrow$ $\sin(x+h) = y + k$

Now, from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^y}{h}$$

$$= \lim_{h\to 0} \frac{e^y (e^k - 1)}{k} \times \frac{k}{h}$$

$$= e^y . \lim_{h\to 0} \frac{e^k - 1}{k} \times \frac{k}{h}$$

$$= e^y . \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \left[\because \lim_{h\to 0} \frac{e^k - 1}{k} = 1\right]$$

$$= e^y . \lim_{h\to 0} \frac{2\cos\frac{x+h+x}{2} \times \sin\frac{x+h-x}{2}}{h}$$

$$= e^y . \lim_{h\to 0} 2\cos\left(x + \frac{h}{2}\right) . \frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right) \times 2}$$

$$= e^y \times \frac{1}{2} \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right)} \times 2\cos\left(x + \frac{h}{2}\right)$$

$$= \frac{1}{2} e^y \times \lim_{h\to 0} 2\cos\left(x + \frac{h}{2}\right)$$

$$= e^{\sin x} \cos\left(x + \frac{h}{2}\right)$$

$$= e^{\sin x} . \cos x$$

ii. $e^{\tan x}$

Solution: Let, $f(x) = e^{\tan x}$

By the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{\tan(x+h)} - e^{\tan x}}{h} \quad \ldots\ldots\ldots \text{(i)}$$

Put, $y = \tan x \Rightarrow y+k = \tan(x+h)$, where $k \to 0$ when $h \to 0$

$\Rightarrow k = \tan(x+h) - e^{\tan x}$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^{y}}{h}$$

$$= \lim_{h\to 0} \frac{e^{y}(e^{k}-1)}{h}$$

$$= e^{y} . \lim_{h\to 0} \frac{e^{k}-1}{k} \times \frac{k}{h}$$

$$= e^{y} . \frac{\tan(x+h) - \tan x}{h}$$

$$= \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{h \cos x \cos(x+h)}$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos x . \cos(x+h)} \qquad [\because \sin A \cos B - \sin B \cos A = \sin(A-B)]$$

$$= e^{y} . \lim_{h\to 0} \frac{\sin h}{h} . \lim_{h\to 0} \frac{1}{\cos x . \cos(x+h)}$$

$$= e^{\tan x} . 1 \times \frac{1}{\cos x \cos x}$$

$$= e^{\tan x} . \sec^{2}x$$

iii. e^{x^2}

Solution: Let, $f(x) = e^{x^2}$

Since, by the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e(x+h)^{2} - e^{x^2}}{h} \quad \ldots\ldots\ldots \text{(i)}$$

Put, $y = x^2 \Rightarrow y+k = (x+h)^2$ where $k\to 0$ when $h\to 0$

$\Rightarrow \quad k = (x+h)^2 - y = (x+h)^2 - x^2$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^{y}}{h}$$

$$= \lim_{h\to 0} \frac{e^{y}(e^{k}-1)}{k} \times \frac{k}{h}$$

$$= e^{y} \lim_{h\to 0} \frac{e^{k}-1}{k} \times \lim_{h\to 0} \frac{k}{h}$$

$$= e^{y} \times 1 \times \lim_{h\to 0} \frac{(x+h)^{2} - x^{2}}{h}$$

$$= e^{y} \times \lim_{h\to 0} \frac{x^{2} + 2xh + h^{2} - x^{2}}{h}$$

$$= e^{y} \times \lim_{h\to 0} \frac{2xh}{h} + \lim_{h\to 0} \frac{h^{2}}{h}$$

$$= e^{x^2} \times 2x + 0$$

$$= 2x^{0x^2}$$

b. i. $a\sin\frac{x}{a}$

Solution: Let, $f(x) = a\sin\frac{x}{a}$

Since by the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{a\sin(x+h) - a\sin\frac{x}{a}}{a}}{h} \quad \ldots\ldots\ldots \text{(i)}$$

Put, $y = \frac{x}{0} \Rightarrow y+k = \frac{x+h}{a}$, where $k \to 0$ when $h \to 0$

$\Rightarrow \quad k = \frac{x+h}{a} - y \Rightarrow k = \frac{x+h}{a} - \frac{x}{a}$

from (i) we have,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{a\sin(4+k) - a\sin y}{h}$$

$$= a\lim_{h\to 0}\frac{\left[2\cos\frac{y+k+y}{2}\,.\,\sin\frac{y+k-y}{2}\right]}{h}$$

$$= a\lim_{h\to 0}\left[\frac{2\cos(y+k/2)\,.\,\sin\frac{k}{2}}{h}\right]$$

$$= 2a\left[\lim_{h\to 0}\cos\left(y+\frac{k}{2}\right)\,.\,\lim_{h\to 0}\frac{\sin\frac{k}{2}}{\frac{k}{2}}\times\frac{\frac{k}{2}}{h}\right]$$

$$= 2a\left[\cos y\,.\,\lim_{h\to 0}\frac{k}{2h}\right] \qquad [\because k \to 0 \text{ when } h \to 0]$$

$$= 2a\cos y\,.\,\lim_{h\to 0}\frac{\left(\frac{x+h}{a}-\frac{x}{a}\right)}{2h}$$

$$= a\cos y \times \lim_{h\to 0}\frac{\left(\frac{x+h-x}{a}\right)}{h}$$

$$= a\cos y \times \frac{1}{x}\lim_{h\to 0}\frac{h}{h}$$

$$= \cos y$$

$$= \cos\frac{x}{a}$$

ii. $\sin x^2$

Solution: Let, $f(x) = \sin x^2$

Since by the definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)^2 - \sin x^2}{h} \quad \ldots\ldots\ldots \text{(i)}$$

Put, $y = x^2 \Rightarrow y+k = (x+h)^2$ where $k\to 0$ as $h\to 0$

$\Rightarrow k = (x+h)^2 - x^2$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\sin(y+k) - \sin y}{h}$$

$$= \frac{\lim_{h\to 0} 2\cos\frac{y+k+y}{2} . \sin\frac{y+k-y}{2}}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(y+\frac{k}{2}\right) . \sin\frac{k}{2}}{h}$$

$$= 2\lim_{h\to 0} \cos\left(y+\frac{0}{2}\right) . \frac{\sin\frac{k}{2}}{\frac{k}{2}} = x\,\frac{\frac{k}{2}}{h}$$

$$= 2\cos y \lim_{h\to 0} \frac{k}{2h}$$

$$= \cos y \times \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \cos y \lim_{h\to 0} \frac{x^2 + h^2 + 2xh - x^2}{h}$$

$$= \cos y \lim_{h\to 0} \frac{h(h+2x)}{h}$$

$$= \cos y \lim_{h\to 0} (h+2x)$$

$$= 2x \cos y$$

$$= 2x \cos x^2$$

iii. $\sqrt{\tan x}$

Solution: Let, $f(x) = \sqrt{\tan x}$

Since by the definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h} \times \frac{\sqrt{\tan(x+h)} + \sqrt{\tan x}}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$= \lim_{h\to 0} (\tan(x+h) - \tan x) \times \frac{1}{h\left[\sqrt{\text{ten}(x+h)} + \sqrt{\tan x}\right]}$$

$$= \lim_{h\to 0} \left(\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right) \times \frac{1}{h\left[\sqrt{\text{ten}(x+h)} + \sqrt{\tan x}\right]}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos x(\cos x(x+h))} \times \frac{1}{h\left[\sqrt{\tan(x+h)} + \sqrt{\tan}\right]}$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{\cos x \cos(x+h)} \times \frac{1}{h\left[\sqrt{\tan(x+h)} + \sqrt{\tan}\right]}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \times \frac{1}{\cos(x+h)\cos x} \times \lim_{h\to 0} \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan}}$$

$$= 1 \times \frac{1}{\cos^2 x} \times \frac{1}{\sqrt{\tan x} + \sqrt{\tan x}}$$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}$$

c. i. Log tanx

Solution: Let, f(x) = log (tanx)

Since by the definition of derivative,

$$\frac{d(f(x))}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log \tan(x+h) - \log(\tan x)}{h} \quad \text{... .... (i)}$$

Put, y = tanx ⇒ y+k = tan(x + h) where k → 0 as h → 0

k = tan (x + n) – tanx

$$\text{from (i), } \lim_{h\to 0} \frac{\log (y + k) - \log y}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(1 + \frac{k}{y}\right)}{\frac{k}{y}} \times \frac{k/y}{h}$$

$$= \frac{1}{y} \times \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\frac{\sin(X+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos x \,.\, \cos(x+h)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos x \,.\, \cos(x+h)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\sin h}{n\cos x \,.\, \cos(x+h)}$$

$$= \frac{1}{y} \times 1 \times \frac{1}{\cos^2 x}$$

$$= \frac{1}{\tan x} \times \frac{1}{\cos^2 x}$$

$= \frac{1}{\tan x} . \sec^2 x$**Error! Bookmark not defined.**

ii. Log $\sec x^2$

Solution: Let, f(x) = f(x) = log $\sec x^2$

Since by the definition of derivatives,

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log \sec(x+h)^2 - \log \sec x^2}{h} \quad \text{... ... ... (i)}$$

Put, y = $\sec x^2$ ⇒ y+k = $\sec(x + h)^2$ where k→0, as h→0

⇒ k = $\sec(x + h)^2 - \sec x^2$

from (i)

$$= \lim_{h\to 0} \frac{\log (y + k) - \log y}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log(1+k/y)}{k/y} \times \frac{k/y}{h}$$

$$= 1 \times \lim_{h\to 0} \frac{\sec(x+n)^2 - \sec x^2}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\frac{1}{\cos(x+n)^2} - \frac{1}{\cos x^2}}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\frac{1}{\cos(x+h)^2} - \frac{1}{\cos x^2}}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\cos x^2 - \cos(x+h)^2}{h \,.\, \cos(x+h)^2 \,.\, \cos x^2}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{-2\sin\frac{x^2+(x+h)^2}{2} \sin\frac{x^2-(x+1)^2}{2}}{h \,.\, \cos(x+h)^2 \,.\, \cos x^2}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{-2\sin\frac{x^2+(x+h)^2}{2} \,.\, \sin\frac{x^2-(x+h)^2}{2}}{h.\, \cos(x+h)^2 \,.\, \cos x^2}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{-2\sin\left(\frac{x^2+2xh+h^2}{2}\right) \sin\left(\frac{-2xh-h^2}{2}\right)}{h.\, \cos(x+h)^2 \,.\, \cos x^2}$$

$$= \frac{1}{y}\, x(-2) \frac{\sin x^2}{\cos x^2 \cos x^2} \times \lim_{h\to 0} (-1) \times \frac{\sin\left(\frac{2xh+h^2}{2}\right)}{h}$$

$$= \frac{2}{y} \tan x^2 \,.\, \sec x^2 \times \lim_{h\to 0} \frac{\sin(2x+h)/2}{(2x+h)/2} \times \frac{(2x+h)}{2}$$

$$= \frac{2}{\sec x^2} \tan x^2 \,.\, \sec x^2 \times 1 \times \frac{2x}{2}$$

$$= 2x \tan x^2$$

iii. log (cosecx)

Solution: Let, f(x) = log(cosecx)

Since by definition of derivatives,

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log \operatorname{cosec}(x+h) - \log \operatorname{cosec} x}{h} \quad \ldots\ldots\ldots \text{(i)}$$

Put y = cosecx $\Rightarrow$ y+k = cosec(x + h) where $k\to 0$ as $h\to 0$

$\Rightarrow$ k = cosecx (x + h) – cosecx

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log(y+k) - \log y}{h}$$

$$= \lim_{h\to0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to0} \frac{\log(1+k/y)}{k/y} \times \frac{k/y}{k}$$

$$= \lim_{h\to0} \frac{k}{yh}$$

$$= \frac{1}{y} \lim_{h\to0} \frac{\frac{1}{\sin(x+h)} - \frac{1}{\sin x}}{h}$$

$$= \frac{1}{y} \lim_{h\to0} \frac{\sin x - \sin(x+h)}{\sin x \sin(x+h) \times h}$$

$$= \frac{1}{y} \lim_{h\to0} \frac{2\cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{h \times \sin x \sin(x+h)}$$

$$= \frac{1}{y} \lim_{h\to0} \frac{2\cos(x+h/2)\sin(-h/2)}{h \times \sin x \sin(x+h)}$$

$$= \frac{2}{y} \frac{\cos x}{\sin x \,.\, \sin x} \times \lim_{h\to0} (-1) \frac{\sin h/2}{h/2 \times 2}$$

$$= \frac{2}{\text{cosecl}} \,.\, \text{colx} \,.\, \text{cosec}x \times \left(\frac{-1}{x}\right)$$

$$= \frac{-\text{cosec}x \,.\, \text{colx}}{\text{cosec}x}$$

$$= -\cot x$$

d. i. $\cot^{-1}x$

Solution: Let, $f(x) = \cot^{-1}x$

$\therefore \quad f(x+h) = \cot^{-1}(x+h)$

We know, $f'(x) = \lim_{h\to0} \frac{\cot^{-1}(x+h) - \cot^{-1}x}{h}$

Let $\cot^{-1}x = y$ and $\cot^{-1}(x+h) = y + k$

$\therefore \quad y + k - y = k = \cot^{-1}(x+h) - \cot^{-1}x$

When $h \to 0$ then $k \to 0$

Also, $x + h = \cot(y+k)$ and $x = \cot y$

$x + h - x = h = \cot(y+k) - \cot y$

Now, $f'(x) = \lim_{k\to0} \frac{y+k-y}{h} = \lim_{k\to0} \frac{k}{\cot(y+k) - \cot y}$

$$= \lim_{k\to0} \frac{k}{\frac{\cot(y+k)}{\sin(y+k)} - \frac{\cos y}{\sin y}}$$

$$= \lim_{k\to0} \frac{k}{\frac{\sin y(\text{con } y+k) - \text{con } y \sin(y+k)}{\sin(y+k) \,.\, \sin y}}$$

$$= \lim_{k\to0} \frac{k\sin(y+k) \,.\, \sin y}{\sin(y+k+y)} = \lim_{k\to0} \frac{-k}{\sin k} \times \frac{\sin(y+k)}{\sin y}$$

$$= -1 \times \sin(y+0) \,.\, \sin y = -\sin^2 y = \frac{-1}{\text{cosec}^2 y}$$

$$= \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$$

ii. $\log \tan^{-1} x$

Solution: Let, $f(x) = \log\tan^{-1} x$

Since by definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log \tan^{-1}(r+h) - \log \tan{-1}^{x}}{h} \quad \ldots \ldots \ldots \text{(i)}$$

Put, $y = \tan^{-1} x \Rightarrow y + k = \tan^{-1}(x + h)$ where $k \to 0$ as $h \to 0$

or, $\tan y = x$ or, $\tan(y + k) - \tan y = h$

Now, from (i)

$$\frac{d}{dx}(f(r) = \lim_{h\to 0} \frac{\log(y+k) - \log y}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log(1 + k/y)}{k/y} \times \frac{k}{yh}$$

$$= \frac{1}{y} \times \lim_{h\to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{k \times \cos(y+k) \cdot \cos y}{\sin(y+k) \cdot \cos y - \sin y \cdot \cos(y+k)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{k \cos(y+k) \cos y}{\sin(y+k-y)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{k\cos y \cdot \cos(y+k)}{\sin k}$$

$$= \frac{1}{y}\left(\lim_{h\to 0} \frac{k}{\frac{\sin k}{k} \times k}\right) \lim_{h\to 0} \cos y \cdot \cos(y+k)$$

$$= \frac{1}{\tan^{-1}x} \times 1 \times \cos^2 y$$

$$= \frac{\cos^2 y}{\tan^{-1}x} = \frac{1}{\tan^{-1}x \sec^2 y} = \frac{1}{\tan^{-1}x(1 + \tan^2 y)} = \frac{1}{\tan^{-1} x(1 + x^2)}$$

iii. $e^{\tan^{-1}x}$

Solution: Let, $f(x) = e\tan^{-1}x$

Since by definition of derivatives

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c^{\tan-1(x+h)} - e^{\tan-1} x}{h}$$

Put, $y = \tan^{-1} x \Rightarrow y+k = \tan^{-1}(x + h)$ where $k \to 0$ as $h \to 0$

or, $\tan y = x \Rightarrow \tan(y + k) = x+h$

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^{y}}{h}$$

$$= \left(\lim_{h\to 0} \frac{e^{y}(ek-1)}{k}\right) \times \lim_{h\to 0} \frac{k}{h}$$

$$= e^{y} \cdot \lim_{h\to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$= e^{y} \cdot \lim_{h\to 0} \frac{ky\cos(y+k)\cos y}{\sin(y+k)\cos y - \sin y \cdot \cos(y+k)}$$

$$= e^y . \lim_{h\to 0} \frac{k \cos y \cos (y + k)}{\sin(y + k - y)}$$

$$= e^y . \lim_{h\to 0} \frac{ky \cos y \cos (y + k)}{\frac{\sin k}{k} \times k}$$

$$= e^y . \lim_{h\to 0} \frac{\cos y \cos (y + k)}{\frac{\sin k}{k}}$$

$$= e^y . \cos^2 y$$

$$= e^{\tan^{-1}}x . \frac{1}{\sec^2 y}$$

$$= e^{\tan^{-1}}x . \frac{1}{1 + \tan^2 y}$$

$$= e^{\tan^{-1}x} . \frac{1}{1 + x^2}$$

e.i. 3^{x2}

Solution: Let, $f(x) = 3^{x^2} = e\log 3x2 = e^{x2 \log 3}$.

Since by definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h\to 0} \; e^{(x+b)^2 \log 3} - e^{x^2 \log 3} \; \ldots \ldots \ldots (i)$$

Put, $y = x^2 \log 3 \Rightarrow y + k = (x + h)^2 \log 3$ where $k \to 0$ as $h\to 0$

$\Rightarrow k = (x + h)^2 \log 3 - x^2 \log 3$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^y}{h}$$

$$= \lim_{h\to 0} \frac{e^y (e^k - 1)}{k} \times \frac{k}{h}$$

$$= e^y \lim_{h\to 0} \left(\frac{e^k - 1}{k}\right) \times \lim_{h\to 0} \frac{(x + h)^2 \log 3 - x^2 \log 3}{h}$$

$$= e^y . \lim_{h\to 0} \frac{x^2 \log 3 + h^2 \log 3 + 2xh\log 3 - x^2 \log^3}{h}$$

$$= e^y . \lim_{h\to 0} \frac{h(h\log 3 + 2x\log 3)}{h}$$

$$= e^{x^2 \log 3} \times 2x\log 3$$

$$= e^{x^2 \log 3}\, 2x\log 3 - e^{\log 3^{x^2}}\, 2\times \log 3 = 3^{x^2}\, 2x\log 3$$

ii. x^x

Solution: Let, $f(x) = x^x = e^{\log x^x} = e^{x\log x}$

Since by definition of derivatives

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{(x+h) \log (x + h)} - e^{x\log x}}{h} \ldots \ldots \ldots (i)$$

Put, $y = x\log x \Rightarrow y+k = (x+h) \log (x + h)$ where $x\to 0$ cos $h\to 0$

From (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^y}{h}$$

$$= \lim_{h\to 0} \frac{e^y(e^k - 1)}{k} \times \frac{k}{h}$$

$$= e^y \left(\lim_{h\to 0} \frac{e^k - 1}{k}\right) \times \lim_{h\to 0} \frac{(x+h)\log(x+h) - x\log x}{h}$$

$$= e^y . \lim_{h\to 0} \frac{x\log(x+h) + h\log(x+h) - x\log x}{h}$$

$$= e^y . \lim_{h\to 0} \left[\frac{x[\log(x+h) - \log x]}{h} + \frac{h\log(x+h)}{h}\right]$$

$$= e^y \left[\lim_{h\to 0} \left\{\frac{x\log(1+h/x)}{\frac{h}{x}\, x^x}\right\} + \lim_{h\to 0} \frac{h\log(x+h)}{h}\right]$$

$$= e^y [1 + \log(x + 0)]$$
$$= e^{x\log x} [1+\log x]$$
$$= x^x [1 + \log x]$$

iii. a^{2x}

Solution: Let $f(x) = a^{2x} = e^{\log a^{2x}} = e^{2x\log a}$

Since by definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{2(x+h)\log a} - e^{2x\log a}}{h}$$

$$= \lim_{h\to 0} \frac{e^{2x\log a} . e^{2h\log a} - e^{2x\log a}}{h}$$

$$= \lim_{h\to 0} e^{2x\log a} \frac{[e^{2h\log a} - 1]}{h}$$

$$= e^{2x\log a} \times \lim_{h\to 0} \frac{e^{h\log a} - 1}{\frac{2h\log a}{2\log a}}$$

$$= e^{\log a^{2x}} \times 2\log a \left[\lim_{h\to 0} \frac{e^{2h\log a} - 1}{2h\log a}\right]$$

$$= 2a^{2x} \log a$$

2.a. A function f(x) defined as follows:

$$f(x) = \begin{cases} 1 + \sin x \text{ for } 0 \le x < \pi/2 \\ 2 + (x - \pi/2)^2 \text{ for } \pi/2 \le x < \infty \end{cases}$$

Does $f'(\pi/2)$ exists?

Solution: To show whether $f'(\pi/2)$ exists or not, we show,

Left hand derivative = Right level derivative at $\frac{\pi}{2}$

Now, For right hand derivative,

$$Rf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{(\pi/2 + h) - f(\pi/2)}{h} = \lim_{h\to 0} \frac{2 + (\pi/2 + h - \pi/2)2 - 2 - (\pi/2 - \pi/2)^2}{h}$$

$$= \lim_{h\to 0} \frac{2 + h^2 - 2}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$$

For, left hand derivative at $x = \frac{\pi}{2}$

$$Lf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{f(\pi/2 - h) - f(\pi/2)}{-h}$$

$$= \lim_{h\to 0} \frac{1 + \sin(\pi/2 - h) - 2 - (\pi/2 - \pi/2)^2}{-h}$$

$$= \lim_{h\to0} \frac{1 + \sin(\pi/2 - h) - 2}{-h} = \lim_{h\to0} \frac{1 + \cos h - 2}{-h}$$

$$= \lim_{h\to0} \frac{1 - \cos h}{h} = ... = \lim_{h\to0} \frac{1 + \cos h - 2}{-h}$$

$\therefore$ $Rf'\left(\frac{\pi}{2}\right) = Lf'\left(\frac{\pi}{2}\right) = 0$. Therefore, $f'\left(\frac{\pi}{2}\right)$ exists.

b. Show that $f(x) = |x-2|$ is continuous but not differentiate at $x = 2$.

Solution: Testing the continuous at $x = 2$

R.H.L at $x = 2$ is,

$$= \lim_{h\to2^+} f(x) = \lim_{h\to2^+} |x - 2| = \lim_{h\to2^+} (x - 2) = 0$$

L.H.L. at $x = 2$ is,

$$= \lim_{h\to2^-} f(x) = \lim_{h\to2^-} |x - 2| = \lim_{h\to2^-} -(x - 2) = 0$$

Functional value at $x = 2$ is,

$f(2) = |2 - 2| = 0$

$\therefore$ L.H.L = R.H.L = functional value

$\therefore$ The given function is continuous at $x = 2$

2nd part,

$$Rf'(2) = \lim_{h\to0} \frac{f(2 + h) - f(2)}{h} = \lim_{h\to0} \frac{|2 + h - 2| - 0}{h}$$

$$= \lim_{h\to0} \frac{|h|}{h} = 1$$

$$Lf'(2) = \lim_{h\to0} \frac{f(2 - h) - f(2)}{h} = \lim_{h\to0} \frac{|2 - h - 2| - 0}{-h}$$

$$= \lim_{h\to0} \frac{|-h|}{-h}$$

$$= \lim_{h\to0} \frac{h}{-h} = -1$$

Here, $Rf'(2) \neq Lf'(2)$

Thus, the given function is not differentiable at $x = 2$, though it is continuous at $x = 2$

3. Find the derivative of;

$$y = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log\left(x + \sqrt{x^2 + a^2}\right)$$

Solution: Differentiate both sides w.r.to x we get,

$$\frac{dy}{dx} = \left[\frac{x.\ 2x}{2\sqrt{x^2 + a^2 \times 2} + \frac{\sqrt{x^2 + a^2}}{2}}\right] + \frac{a^2}{2} \frac{\left(1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right)}{(x + \sqrt{x^2 + a^2})}$$

$$= \frac{x^2}{2\sqrt{x^2 + a^2}} + \frac{\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \frac{\left(\sqrt{x^2 + a^2} + x\right)}{\sqrt{x^2 + a^2}\left(x + \sqrt{x^2 + a^2}\right)}$$

$$= \frac{1}{2}\left[\frac{x^2 + x^2 + a^2 + a^2}{\sqrt{x^2 + a^2}}\right] = \frac{(x^2 + a^2)}{\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2}$$

4.a. Let $y = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$

Put $x = \tan\theta$

$dx = \sec^2 \theta\, d\theta$

Now, $y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$= \tan^{-1}(\tan 2\theta)$

$= 2\theta$

$y = 2\tan^{-1}x$

Now, differentiating on both sides, we get

$$\frac{dy}{dx} = 2\frac{d\tan^{-1}x}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

b. Let $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$

Differentiating on both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d\cos^{-1}\frac{2x}{1+x^2}}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \times \left[\frac{(1+x^2)^2 - 2x \times 2x}{(1+x^2)^2}\right]$$

$$= -\frac{(1+x^2)}{\sqrt{(1+x^2)^2-4x^2}}\left[\frac{2+2x^2-4x^2}{(1+x^2)^2}\right]$$

$$= -\frac{(2-2x^2)}{\sqrt{1+2x^2+x^4-4x^2}} \times \frac{1}{(1+x^2)}$$

$$= -\frac{(1-x^2)}{(1-x^2)} \times \frac{1}{(1+x^2)}$$

$$\frac{dy}{dx} = -\frac{2}{(1+x^2)}$$

c. Let $y = \sin^{-1}(3x - 4x^3)$

Put $x = \sin\theta$

$y = \sin^{-1}(3\sin\theta - 4\sin3\theta)$

$y = \sin^{-1}(\sin3\theta)$

$y = 3\theta = 3\sin^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

d. Let $y = \sin^{-1}\sqrt{1-x^2}$

Put $x = \cos\theta$

$\therefore \quad y = \sin^{-1}\sqrt{1-\cos^2\theta}$

$= \sin^{-1}\sin\theta$

$= \theta$

$y = \cos^{-1}x$

Now, differentiating on both sides, we get

$$\frac{dy}{dx} = \frac{d(\cos^{-1}x)}{dx}$$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

e. Let $y = \sin^{-1}2x\sqrt{1-x^2}$

Put $x = \cos\theta$ then

$y = \sin^{-1}2\cos\theta\sqrt{1-\cos^2\theta}$

$= \sin^{-1}(2\cos\theta \times \sin\theta)$

$= \sin^{-1}\sin2\theta$

$y = 2\theta = 2\cos^{-1}x$

Now, differentiating on both sides, we get

$\frac{dy}{dx} = 2 \times -\frac{1}{\sqrt{1-x^2}}$

$\therefore \quad \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$

f. Let $y = \sec^{-1}(\tan x)$

Differentiating on both sides w.r.t. x, we get

$\frac{dy}{dx} = \frac{d\sec^{-1}(\tan x)}{dx}$

$= \frac{1}{\tan x\sqrt{\tan^2 x - 1}} \times x \sec^2 x$

$\therefore \quad \frac{dy}{dx} = \frac{\sec^2 x}{\tan x\sqrt{\tan^2 x - 1}}$

g. $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Solution: Put, $x = \tan\theta$

Now, $y = \tan^{-1}\frac{\sqrt{1+\tan^2 x}-1}{x}$

$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}$

$= \tan^{-1}\tan\theta/2 = \theta/2 = \frac{1}{2}\tan a^{-1}x$

Differentiate both sides w.r.to x,

h. $\sin^{-1}\frac{2x}{1+x^2} + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ prove that $\frac{dy}{dX} = \frac{4}{1+x^2}$

Solution: Put, $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

L.H.S. $\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$= \sin^{-1}\sin 2\theta + \sec^{-1}\left(\frac{\sec^2\theta}{1-\frac{\sin^2\theta}{\cos^2\theta}}\right)$

$= 2\theta + \sec^{-1}\left(\frac{1}{\cos^2\theta - \sin^2\theta}\right)$

$= 2\theta + \sec^{-1}\sec 2\theta$

$= 40 = 4\tan^{-1}x$

Differentiate both sides by x, we get,

$\frac{dy}{dx} = \frac{4}{1+x^2}$ R.H.S. Proved.

5.a. To prove $\frac{dy}{dx} = \frac{y^2}{x(1-y\mathrm{Ln}x)}$

We have,

$y = x^y$

Taking Ln on both sides, we get

Lny = yLnx

Now, differentiating on both sides we get

$$\frac{1}{y}\frac{dy}{dx} = \text{Lnx}\frac{dy}{dx} + y.\frac{1}{x}$$

$$\Rightarrow \left(\frac{1}{y} - \text{Lnx}\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow (1 - y\text{Lnx})\frac{dy}{dx} = \frac{y^2}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x(1 - y\,\text{Lnx})}$$

b. To prove $\frac{dy}{dx} = \frac{y}{x}$

We have, $x^p.y^q \ (x + y)^{P+q}$

Taking Ln on both sides, we get

$\text{Ln}(x^p.y^q) = \text{Ln}(x + y)^{p+q}$

$\Rightarrow$ pLnx + qLny = (p + q) Ln (x + y)

Now, differentiating on both sides, we get

$$p.\frac{1}{x} + q\frac{1}{y}\frac{dy}{dx} = (p + q)\,.\,\frac{1}{(x + y)}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p + q}{x + y} + \frac{p + q}{x + y}\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{q}{y} - \frac{p + q}{x + y}\right)\frac{dy}{dx} = \frac{p + q}{x + y} - \frac{p}{x}$$

$$\Rightarrow \frac{(qx + qy - py - qy)}{y(x + y)}\,.\,\frac{dy}{dx} = \frac{px + qx - px - py}{x(x + y)}$$

$$\Rightarrow \frac{(qx - py)}{y(x + y)}\frac{dy}{dx} = \frac{(qx - py)}{x(x + y)}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}$$

c. To prove $\frac{dy}{dx} = -\frac{y(y + x\,\text{Lny})}{x(y\text{Lnx} - x)}$

We have,

$x^y \,.\, y^x = 1$

Taking Ln on both sides we get

yLnx + xLny = 0

Now, differentiating on both sides w.r.t. x, we get

$$\text{Lnx}\frac{dy}{dx} + y\,.\,\frac{1}{x} + \text{Lny}\,.\,1 + x\,.\,\frac{1}{y}\frac{dy}{dx} = 0$$

$$\Rightarrow \left(\text{Lnx} + \frac{x}{y}\right)\frac{dy}{dx} = -\frac{y}{x} - \text{Lny}$$

$$\Rightarrow \frac{(y\,\text{Lnx} + x)}{y}\frac{dy}{dx} = \frac{-y - x\text{Lny}}{x}$$

$$\therefore \frac{dy}{dx} = \frac{-y(y + x\text{Lny})}{x(y\text{Lnx} - x)}$$

d. If siny = x cos (a+y) show that $\frac{dy}{dx} = \frac{\cos^2 (a + y)}{\cos a}$

Solution: Differentiate both sides w.r. to x we ge,t

$$\cos y\frac{dy}{dx} = -x\sin(a + y)\frac{dy}{dx} + 1.\cos(a + y)$$

or, $(\cos y + x\sin(a + y))\frac{dy}{dx} = \cos(a + y)$

or, $\frac{dy}{dx} = \frac{\cos(a + y)}{\cos y + x\sin(a + y)}$

or, $\frac{dy}{dx} = \frac{\cos(a+y)}{\cos y + \frac{\sin y}{\cos(a+y)\sin(a+y)}} = \frac{\cos^2(a+y)}{\cos(a+y-y)} = \frac{\cos^2(a+y)}{\cos a}$ proved.

e. If $y = \frac{\cos x - \sin x}{\cos x + \sin x}$, show that $\frac{dy}{dx} + y^2 + 1 = 0$

Solution: Here, $y = \frac{\cos x - \sin x}{\cos x + \sin x} \times \frac{\cos x - \sin x}{\cos x - \sin x}$

$y = \frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x} = \frac{\cos 2x + \sin 2x - 2\sin x \cos x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \frac{\sin 2x}{\cos 2x} = \sec 2x - \tan 2x$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 2\sec 2x \tan 2x - 2\sec^2 x.$

Again, $y^2 = \sec^2 2x + \tan^2 2x - 2\sec 2x . \tan 2x$

Now, L.H.S. $\frac{dy}{dx} + y^2 + 1$

$= 2\sec 2x \tan 2x - 2\sec^2 2x + \sec^2 2x + \tan^2 2x - 2\sec 2x \tan 2x + 1$

$= -\sec^2 2x + \tan^2 2x + 1$

$= -1 + 1$

$= 0$ R.H.S.

6. Find the derivative with respect to x of following.

a. Let $y = x^{\sin x}$

$\text{Ln}y = \sin x \ln x$

Differentiating on both sides, we get

$\frac{1}{y} . \frac{dy}{dx} = \ln x . \cos x + \sin x \frac{1}{x}$

$\Rightarrow \frac{dy}{dX} = y\left[\ln x \cos x + \frac{\sin x}{x}\right]$

$\therefore \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x . \ln x\right]$

b. Let $y = (\sin x)^{\cos x}$

$\text{Ln}y = \cos x . \text{Ln}(\sin x)$

$\frac{1}{y}\frac{dy}{dx} = -\text{Ln}(\sin x) . \sin x + \cos x \times \cot x$

$\therefore \frac{dy}{dx} = (\sin x)^{\cos x} [\cos x . \cot x - \sin x , \text{Ln}(\sin x)]$

c. $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

Solution: Let, $y = u + v$ where, $u = (\sin x)^{\cos x}$ and $v = (\cos x)^{\sin x}$

Now, $u = (\sin x)^{\cos x}$

Taking log on both sides,

or, $\log u = \cos x \log \sin x$

Differentiate both sides w.r. to x, we get,

$\frac{1}{u}\frac{dy}{dx} = \frac{\cos x . \cos x}{\sin x} + \log \sin x . (-\sin x)$

$= \frac{\cos^2 x}{\sin x} - \sin x \log \sin x$

or, $\frac{dy}{dx} = (\sin x)^{\cos x} [\cos x . \cot x - \sin x . \log \sin x]$

Similarly, $\frac{1}{v}\frac{dv}{dx} = \sin x . \frac{-\sin x}{\cos x} + \cos x \text{ vg } \cos x$

$\frac{dv}{dx} = (\cos x)^{\sin x}$ [cosx log cosx – sinx . tanx]

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dx} + \frac{dv}{dx}$

$= (\sin x)^{\cos x}$ [cosx . cotx – sinx . log sinx] + $(\cos x)^{\sin x}$ [cosx. logcosx – sinx . tanx]

d. $x^{\tan x} + (\tan)^x$

Solution: Let, y = u + v, where, $u = x^{\tan x}$ and $v = (\tan x)^x$.

if, $u = (x)^{\tan x}$

Taking log on both sides,

logu = tanx logx

Differentiate both sides w.r. to x, we get,

$\frac{1}{u}\frac{dy}{dx} = \tan x \frac{1}{x} + \log x . \sec^2 x$

$\frac{dy}{dx} = (x)^{\tan x}\left[\frac{\tan x}{x} + \log x . \sec^2 x\right]$

Again similarly,

$\frac{1}{v}\frac{dv}{dx} = x.\frac{\sec^2 x}{\tan x} + \log \tan x.1$

$\frac{dv}{dx} = (\tan x)^x\left[\log \tan x + \frac{x \sec^2 x}{\tan x}\right]$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dx} + \frac{dv}{dx} = (x)^{\tan x}\left[\frac{\tan x}{x} + \log x \sec^2 x\right] + (\tan x)^x \log \tan x + \frac{x\sec^2 x}{\tan x}$

e. $(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution: Let, y = u+v where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

Now, $u = (\sin x)^x$

Taking log on both sides,

logu = x log sinx

Differentiate both sides w.r.to x, we get,

$\frac{1}{u}\frac{dy}{dx} = x\frac{\cos x}{\sin x} + \log \sin x.1$

$\frac{dy}{dx} = (\sin x)^x [x\cot x + \log \sin x]$

and, $v = \sin^{-1}\sqrt{x}$

Differentiate both sides w.r. to x, we get,

$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} . \frac{d(\sqrt{x})}{dx} \qquad \left[\because \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}\right]$

$= \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dx} + \frac{dv}{dx}$

$= (\sin x)^x . [x\cot x + \log \sin x] + \frac{1}{2\sqrt{x}\sqrt{1-x}}$

7.a. If $y = e^{x+e^{x+e^{x+\dots}}}$. Show that $\frac{dy}{dx} = \frac{y}{1-y}$

Solution: Let, $y = e^{x+e^{x+e^{x+\dots}}}$. Then we have,

$y = e^{x+y}$

Taking log on both sides we get,

logy = (x + y) loge

Differentiate both sides w.r. to x, we get

$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$

or, $\left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$

or, $\left(\frac{1-y}{y}\right)\frac{dy}{dx} = 1$

or, $\frac{dy}{dx} = \frac{y}{1-y}$

b. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots}}}$

Show that, $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

Proof: Here, we can rewrite y as,

$y = \sqrt{\sin x + y}$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = \frac{1}{2\sqrt{\sin x + y}} \times \frac{d}{dx}(\sin x + y)$

$\left(2\sqrt{\sin x + y}\right)\frac{dy}{dx} = \cos x + \frac{dy}{dx}$

or, $2y\frac{dy}{dx} = \frac{dy}{dx} = \cos x$

or, $\frac{dy}{dx} = \frac{\cos x}{2y-1}$ proved.

c. If $y = (\cos x)\cos x^{\cos x\ldots}$ prove that $\frac{dy}{dx} = \frac{y^2\tan x}{y\log\cos x^{-1}}$

Proof: Here, we can rewrite the problem as in the form $y = (\cos x)^y$

Taking log on both sides we have,

$\log y = y \log \cos x$

Differentiate both sides w.r. to x, we get,

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = y\frac{-\sin x}{\cos x} + \log\cos x\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y} - \log\cos x\right)\frac{dy}{dx} = -y\tan x$

$\Rightarrow \frac{dy}{dx} = \frac{-y^2\,\text{tnax}}{1 - y\log\cos x} = \frac{y^2\tan x}{y\log\cos x - 1}$ proved.

Exercise 15.3

Find the derivative with respect to x of the following:

1. $e^{\cosh x/a}$

Solution: Let, $y = e^{\cosh x/a}$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = \frac{d}{dx}(e^{\cosh x/a})$

$= e^{\cosh x/a} . s^{\text{inh} x/a}\frac{d(x/a)}{dx}$

$= \frac{1}{a}\sinh\frac{x}{a}\ e^{\cosh x/a}$

2. Log tanhx

Solution: Let, y = log tanhx

Differentiate both sides w.r.to x, we get,

$\frac{dy}{dx} = \frac{d}{dx}(\log\tanh x)$

$= \frac{1}{\tanh x}\frac{d}{dx}(\tanh x)$

$= \frac{1}{\tanh x} \cdot \operatorname{sech}^2 x = \frac{\cosh x}{\sinh x \cdot \cosh^2 x}$

$= \frac{1}{\sinh x \cosh x} = \frac{2}{\sinh 2x} = 2\operatorname{cosech} 2x$

3. Tanh (arcsinx)

Solution: Let, $y = \tanh(\sin^{-1}x)$

Differentiate both sides w.r.to x, we get,

$\frac{dy}{dx} = \frac{d}{dx}(\tanh \sin^{-1}x) = \operatorname{sech}^2 \sin^{-1}x \frac{d}{dx}(\sin^{-1}x)$

$= \operatorname{sech}^2 \sin^{-1} \frac{x}{\sqrt{1-x^2}}$

4. $\operatorname{Sech}^{-1}x = \operatorname{cosech}^{-1}x$

Solution: Let, $y = \operatorname{sech}^{-1}x = -\operatorname{cosch}^{-1}x$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = \frac{d}{dx}[\operatorname{sech}^{-1}x - \operatorname{cosech}^{-1}x]$

$= \frac{-1 - -1}{x\sqrt{1-x^2}\ x\sqrt{x^2+1}}$

$= \frac{1}{x}\left[\frac{1}{\sqrt{x^2+1}} - \frac{1}{\sqrt{1-x^2}}\right]$

5. $x^{\cosh x}$

Solution: Let, $y = x^{\cosh x}$

Taking log on both sides we have,

$\log y = \cosh x \log x$

Differentiate both sides w.r. to x, we get,

$\frac{1}{y}\frac{dy}{dx} = \cosh x \ \frac{1}{x} + \log x \cdot \sinh x$

$\frac{dy}{dx} = x^{\cosh x}\left[\frac{\cosh x + x \sinh x \log x}{x}\right]$

6. $x^{\sinh x^2/a}$

Solution: Let, $y = x^{\sinh x^2/a}$

Taking log on both sides we get,

$\log y = \sinh \frac{x^2}{a} \log x$

Differentiate both sides w.r. to x, we get,

$\frac{1}{y}\frac{dy}{dx} = \sinh x^2/a \ \frac{1}{x} + \log x \cdot \cos x \frac{x^2}{a} = \frac{2x}{a}$

$\frac{dy}{dx} = x \sinh x^2/a \left[\frac{\sin x^2/a}{x} + \frac{2x \log x}{a} \cosh x^2/a\right]$

7. $x^{\cosh^{-1}x/a}$

Solution: Let, $y = x^{\cosh^{-1}x/a}$

Taking log on both sides we get,

$\log y = x^{\cosh^{-1}x/a} \log x$

Differentiate both sides w.r. to x, we get,

$\frac{1}{y}\frac{dy}{dx} = \cosh^{-1}x/a \frac{1}{x} + \log x \frac{1}{\sqrt{\frac{x^2}{a^2} - 1}} \times \frac{1}{a}$

$\frac{dy}{dx} = x^{\cosh^{-1}x/a}\left[\frac{1}{x}\cosh^{-1}x/a + \frac{\log x}{\sqrt{x^2 - a^2}}\right]$

8. $(\log x)^{\sinh x}$

Solution: Let, $y = (\log x)^{\sinh x}$

Taking log on both sides we get,

$\log y = \sinh x \log(\log x)$

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \sinh x \frac{1}{\log x} . \frac{1}{x} + \log(\log x) . \cosh x$$

$$\text{or, } \frac{dy}{dx} = (\log x)^{\sinh x}\left[\frac{\sinh x}{x\log x} + \cosh x \log(\log x)\right]$$

9. $(\sinh x)^{\cosh^{-1}x}$

Solution: Let, $y = (\sinh x)^{\cosh^{-1}x}$

Taking log on both sides w.r. to x, we get,

$\log y = \cosh^{-1}x \log(\sinh x)$

Differentiate both sides w.r. to x, we get,

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \cosh^{-1}x . \frac{1}{\sinh x} . \cosh x + \log(\sinh x)\frac{1}{\sqrt{x^2-1}}$$

$$\text{or, } \frac{dy}{dx} = (\sinh x)^{\cosh^{-1}x}\left[\cosh^{-1}x \coth x + \frac{\log \sinh x}{\sqrt{x^2-1}}\right]$$

10. $(\cosh x)^{\cosh x}$

Solution: Let, $y = (\cosh x)^{\cosh x}$

Taking log on both sides we get,

$\log y = \cosh x \log(\cosh x)$

Differentiate both sides w.r. to x, we get,

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \cosh x \frac{\sinh x}{\cosh x} + \log(\cosh x) . \sinh x$$

$$\text{or, } \frac{dy}{dx} = (\cosh x)^{\cosh x}[\cosh x \tanh x + \sinh x \log(\cosh x)]$$

11. $(\tanh \frac{x}{a})^{\log x}$

Solution: Let, $y = (\tanh \frac{x}{a})^{\log x}$

Taking log on both sides w.r. to x, we get,

$\log y = \log x \log(\tanh\frac{x}{a})$

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \log x \frac{\text{sech}^2 x/a}{\tanh x/a} \times \frac{1}{a} + \log \tanh \frac{x}{a} . \frac{1}{x}$$

$$\text{or, } \frac{dy}{dx} = (\tanh \frac{x}{a})^{\log x}\left[\frac{\cos x . \cosh x/a}{\cosh^2 x/a . \sinh x/a}\frac{1}{a} + \log \tanh \frac{\;}{a} . \frac{1}{x}\right]$$

$$\text{or, } \frac{dy}{dx} = (\tanh \frac{x}{a})^{\log x}\left[\frac{2 \log x}{2 \sinh x/a \cosh x/a} \times \frac{1}{a} + \log \tanh x/a . \frac{1}{x}\right]$$

$$\text{or, } \frac{dy}{dx} = (\tanh \frac{x}{a})^{\log x}\left[\frac{2}{a} \text{cosech} 2\, x/a . \log x + \log \tanh x/a . \frac{1}{x}\right]$$

$$\text{or, } \frac{dy}{dx} = (\tanh \frac{x}{a})^{\log x}\left[\frac{2}{a} \log x \,\text{cosech}\, 2x/a + \frac{1}{x} \log \tanh x/a\right]$$

12. $(\sinh^{-1}x + \cosh^{-1}x)^x$

Solution: Let, $y = (\sinh^{-1}x + \cosh^{-1}x)^x$

Taking log on both sides w.r. to x, we get,

$\log y = x \log(\sinh^{-1}x + \cosh^{-1}x)$

Differentiate both sides w.r. to x, we get,

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \frac{x}{(\sinh^{-1}x + \cosh^{-1}x)}\left[\frac{d}{dx}(\sinh^{-1}x + \cosh^{-1}x)\right] + \log(\sinh^{-1}x + \cosh^{-1}x)$$

or, $\frac{1}{y}\frac{dy}{dx} = \frac{x}{(\sinh^{-1}x + \cosh^{-1}x)}\left(\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{x^2-1}}\right) + \log(\sinh^{-1}x + \cosh^{-1}x)$

or, $\frac{dy}{dx} = (\sinh^{-1}x + \cosh^{-1}x)^x\left[\frac{x}{(\sinh^{-1}x + \cosh^{-1}x)}\left(\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{x^2-1}}\right)\right.$

$\left. + \log(\sinh^{-1}x + \cosh^{-1}x)\right]$

13. $(\sinh\frac{x}{a} + \cosh\frac{x}{a})^{nx}$

Solution: Let, $y = (\sinh\frac{x}{a} + \cosh\frac{x}{a})^{nx}$

Taking log on both sides we get,

$\log y = nx\log(\sinh\frac{x}{a} + \cosh\frac{x}{a})$

Differentiate both sides w.r. to x, we get,

$\frac{1}{y}\frac{dy}{dx} = nx\cdot\frac{(\cosh\frac{x}{a} + \sinh\frac{x}{a})}{(\sinh\frac{x}{a} + \cosh\frac{x}{a})}\frac{2}{a}\text{inlog}\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)$

$\frac{dy}{dx} = ny\left[\frac{2x}{a} + \log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right]$

$= n\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}\left[\frac{2x}{a} + \log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right]$

Exercise 15.4

1. Find the slope and inclination with the x–axis of the tangent of the following:
   a. $y = y = x^3 + 2x + 7$ at $x = 1$

Solution: Given, $y = x^3 + 2x + 7$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 3x^2 + 2$

Slope at x = 1 is, $\left.\frac{dy}{dx}\right|_{x=1}$

$\therefore \quad \frac{dy}{dx} = 3\times1^2 + 2 = 5$

Since slope $m = \tan\theta$, θ is angle from x–axis.

$\therefore \quad \text{Tan}\theta = 5$

$\Rightarrow \quad \theta = \tan^{-1}5$

b. $x^2 - y^2 = 9$ at (3, 0)

Solution: Given, $y = x^2 - y^2 = 9$

Differentiate both sides w.r. to x, we get,

$3x - 2y\frac{dy}{dx} = 2x$

or, $\frac{dy}{dx} = \frac{x}{y}$

Slope at (3, 0) is, $\left.\frac{dy}{dx}\right|_{(3,0)} = \frac{3}{0} = \infty$

Again, $\tan\theta = \infty$

$\theta = \tan^{-1}\infty = \tan{-1}\tan\frac{\pi}{2} = \frac{\pi}{2}$

c. $y = -3x - x^4$ at $x = -1$

Solution: Given, $y = -3x - x^4$

Differentiate both sides w.r. to x,

$\frac{dy}{dx} = -3 - 4x^3$

Slope at x = –1 is, $\frac{dy}{dx}_{x=-1}$

$= -3-4(-1)^3 = -3+4 = 1$

Again, $\tan\theta = 1$

$\theta = \tan^{-1}1$

$= \tan-1 \tan\frac{\pi}{4} = \frac{\pi}{4}$

2. Obtain the equation to the tangent to the parabola $y^2 - 8x$ at (2, –4)

Solution: Since we know the equation of tangent be the parabola $y^2 = 4ax$ at (x_1, y_1) is,

$yy_1 = 2a(x + x_1)$

Now, $y^2 = 8x = 4 \times 2x$ and $(x_1, y_1) = (2, -4)$

$\therefore$ Required equation of tangent is

$y(-4) = 4(x + 2)$

or, $-4y = 4x + 8$

or, $-y = x + 2$

or, $x + y + 2 = 0$

Given, $y^2 = 8x$

$3y\frac{dy}{dx} = 8$

$\frac{dy}{dx} = \frac{4}{y}$

$\left.\frac{dy}{dx}\right|_{(2,\,-4)} = \frac{4}{-4} = -1$

Now, required equation of tangent,

$y - y_1 = \ln(x - x_1)$

or, $y + 4 = -1(x - 2)$

or, $y + 4 = -x + 2$

or, $x + y + 2 = 0$

3. Find the equation of tangent and normal to the curve.

a. $y = 2x^2 - 3x - 1$ at (1, –2)

Solution: Given $y = 2x^2 - 3x - 1$

Differentiate both sides w.r. to x, we get

$\frac{dy}{dx} = 4x - 3$

Slope of tangent say (m_1) at (1, –2) is $m_1 = \frac{dy}{dx} = 4 - 3 = 1$

Then slope of normal is say m_2 is given by

$m_1 \times m_2 = -1$

$m_2 = -\frac{1}{1} = -1$

Now, equation of tangent is,

$y - (-2) = 1\,(x - 1)$

$\Rightarrow$ $y + 2 = x - 1$

$\Rightarrow$ $x - y - 3 = 0$

Again equation of normal at (1, –2) is,

$y - (-2) = -1\,(x - 1) \Rightarrow y + 2 = -x + 1$

$\Rightarrow x + y + 1 = 0$

b. $y = x^3$ at (2, 8)

Solution: Given, $y = x^3$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 3x^2$

Slope (m_1) $\left.\frac{dy}{dx}\right|_{(2, 8)} = 3\times2^2 = 12$

Equation of tangent at (2, 8) is

$y - 8 = 12 (x - 2)$

$\Rightarrow$ $y - 8 = 12x - 24$

$\Rightarrow$ $12x - y - 16 = 0$

Again, Slope of normal is $= -\frac{1}{12}$

Now, equation of normal is,

$y - 8 = -\frac{1}{12}(x - 2)$

$\Rightarrow$ $12y - 96 = -x + 2$

$\Rightarrow$ $x + 12y - 98 = 0$

c. $x^2 - y^2 = 16$ at (6, 3)

Solution: Given, $x^2 - y^2 = 16$

Differentiate both sides w.r. to x, we get

$2x - 2y\frac{dy}{dx} = 0$

or, $2y\frac{dy}{dx} = 2x$

or, $\frac{dy}{dx} = \frac{x}{y}$

$\left.\frac{dy}{dx}\right|_{(6, 3)} = \frac{6}{3} = 2$

Now, equation of tangent at (6, 3) is,

$y - 3 = 2(x - 6)$

or, $y - 3 = 2x - 12$

or, $2x - y - 9 = 0$

Again, equation of normal is,

$y - 3 = -\frac{1}{2}(x - 6)$

or, $2y - 6 = -x + 6$

or, $x + 2y - 12 = 0$

d. $x^2 + 3xy + y^2 = 11$ at (2, 1)

Solution: Given, $x^2 + 3xy + y^2 = 11$

Differentiate both sides w.r. to x, we get,

$2x + 3\left[x\frac{dy}{dx} + y\right] + y^2 = 11$

or, $2x + 3x\frac{dy}{dx} + 3y + y^2 = 11$

or, $3x\frac{dy}{dx} = 11 - 2x - 3y - y^2$

or, $\frac{dy}{dx} = \frac{11 - 2x - 3y - y^2}{3x}$

Now, $\left.\frac{dy}{dx}\right|_{(2, 1)} = \frac{11 - 2\times2 - 3\times1 - 1}{3\times2} = \frac{11 - 8}{6} = \frac{3}{6} = \frac{1}{2}$

$\therefore$ Equation of tangent at (2, 1) is,

$y - 1 = \frac{1}{2}(x - 2) \Rightarrow 2y - 2 = x - 2 \Rightarrow x - 2y = 0$

Again, equation of normal is,

$y - 1 = -2 (x - 2)$

or, $y - 1 = -2x + 4$

or, $2x + y - 5 = 0$

e. $x^{2/3} + y^{2/3} = 2$ at (1, 1)

Solution: Differentiate w.r.t. to x, we get,

$$\frac{2}{3}x^{(2/3-1)} + \frac{2}{3}y^{(2/3-1)}\frac{dy}{dx} = 0$$

or, $\frac{2}{3}x^{-1/2} + \frac{2}{3}y^{-1/2}\frac{dy}{dx} = 0$

or, $y^{-1/2}\frac{dy}{dx} = x^{-1/2}$

or, $\frac{dy}{dx} = \frac{x^{1/2}}{y^{-1/2}} = \frac{y^{1/2}}{x^{1/2}} = \frac{\sqrt{y}}{\sqrt{x}}$

$\left.\frac{dy}{dx}\right|_{(1,\,1)}$ is 1

Now, equation of tangent at (1, 1) is,

$y - 1 = 1(x - 1)$

$\Rightarrow$ $y-1 = x-1 \Rightarrow x-y = 0$

Again, equation of normal is,

$y - 1 = -1(x - 1)$

$\Rightarrow$ $y - 1 = -x + 1$

$\Rightarrow$ $x + y - 2 = 0$

4. Find the points on the curve where the tangents are parallel to x–axis.

a. $y = 2x - x^2$

Solution: Differentiate both sides w.r.t. to x, we get,

$\frac{dy}{dx} = 2 - 2x$

If the tangent are parallel to x–axis, then the slope v must be zero

i.e. $\frac{dy}{dx}$

$\therefore$ $\frac{dy}{dx} = 2 - 2x = 0$

$\Rightarrow$ $2 = 2x$

$\Rightarrow$ $x = 1$

If x = 1, then $y = 2 - 1^2 = 1$

$\therefore$ The required point is (1, 1)

b. $y = 2x^2 - 6x + 9$

Solution: Differentiate both sides w.r.t. to x, we get,

$\frac{dy}{dx} = 4x - 6]$

Now, $\frac{dy}{dx} = 0 \Rightarrow 4x = 6$

$\Rightarrow x = \frac{3}{2}$

If $x = \frac{3}{2}$, then $y = 2 \times \frac{9}{4} - 6 \times \frac{3}{2} + 9$

$= \frac{9}{2} - 9 + 9 = \frac{9}{2}$

$\therefore$ Required point is $\left(\frac{3}{2}, \frac{9}{2}\right)$

c. $x^2 + y^2 = 16$

Solution: Differentiate both sides w.r.t. to x, we get,

$2x + 2y\frac{dy}{dx} = 0$

or, $2y\frac{dy}{dx} = -2x$

or, $\frac{dy}{dx} = -\frac{x}{y}$

Now, $\frac{dy}{dx} = 0 \Rightarrow -\frac{x}{y} = 0 \Rightarrow -x = 0$

$\Rightarrow x = 0$

If x = 0 then y = ±4

Therefore the required point (0, ±4)

5. Find the points are the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangent are 0.

   a. Parallel to x–axis   b. Parallel to γ–axis.

Solution: Given, $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Differentiate both sides w.r. to x, we get,

$\frac{2x}{9} + \frac{9y}{16}\frac{dy}{dx} = 0$

$\Rightarrow \frac{y}{16}\frac{dy}{dx} = -\frac{x}{9}$

$\Rightarrow \frac{dy}{dx} = -\frac{16x}{9y}$

a. For, parallel to x–axis

$\frac{dy}{dx} = 0$

$\Rightarrow \frac{-16x}{9y} = 0$

$\Rightarrow x = 0$

If x = 0 then, $y^2 = 16$

$y = \pm 4$

Therefore required points (0, ± 4)

b. For parallel to γ-axis,

$\frac{dy}{dx} = \infty$

$\Rightarrow \frac{-16x}{9y} = \infty$ which is possible only when y = 0

Then we have,

$\frac{x^2}{9} = 0$

$\therefore$ x = ±3

Therefore required points (± 3, 0)

6. Show that the tangents to the curve $y = 2x^3 - 3$ at the point where x = 2 and x = –2 are parallel.

Solution: Given, $y = 2x^3 - 3$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 6x^2$

Slop at x = 2 i.e, $\frac{dy}{dx}_{x=2}$ is $\frac{dy}{dx} = 6\times2^2 = 24$.

Again slope at x = –2 is, $\frac{dy}{dx}_{x=-2} = 6\times(-2)^2 = 24$

Hence, if x = 2 and x = –2 slope are equal it means the tangent are parallel.

7.a. Find the equation of tangent line to the curves $y = x^2 - 2x + 7$ which is parallel to the line $2x - y + 9 = 0$

Solution: Given, curve, $y = x^2 - 2x + 7$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 2x - 2$(slop of tangent)

Again slop of the line $2x - y + 9 = 0$, obtained by comparing to $y = mx + c$ i.e. $y = 2x + 9$. Therefore the slope of given line?

If the required tangent is parallel to the given line then slope must be equal

$\therefore \quad 2x - 2 = 2$

or, $\quad x = 2$

Put, $x = 2$ in $y = x^2 - 2x + 7$ we get,

$y = 2^2 - 2\times2 + 7 = 7$

$\therefore$ Required point is, (2, 7)

Now, the equation at tangent is,

$y - 7 = 2(x - 2)$

or, $\quad y-7 = 2x - 4$

or, $\quad 2x - y + 3 = 0$

b. Find the point on the curve $y^2 = 4x + 1$ at which the tangent is perpendiculars to the line $7x + 2y = 1$.

Solution: Given, curve, $y^2 = 4x + 1$,

$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

Again slope of $7x + 2y = 1$

or, $\quad 2y = -7x + 1$

$y = \frac{-7}{2}x + \frac{1}{2}$ is $\frac{-7}{2}$

If the tangent to the curve $y^2 = 4x+1$ is perpendicular to the line $7x + 2y = 1$, product of slope must be -1.

$\therefore \quad \frac{2}{y} \times -\frac{7}{2} = -1 \Rightarrow 14 = 2y \Rightarrow y = 7$

Putting $y = 7$ in $y^2 = 4x + 1$ we get,

$49 = 4x + 1$

$48 = 4x \Rightarrow x = 12$

$\therefore$ The required point is (12, 7)

8. Show that equation of tangent to the curve $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 - 2$ at (a, b) is $\frac{x}{a} + \frac{y}{b} = 2$

Solution: Given, curve is, $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 2$

Differentiate both sides w.r. to x, we get,

$\frac{3x^2}{a^3} + \frac{3y^2}{b^3}\frac{dy}{dx} = 0$

or, $\quad \frac{dy}{dx} = \frac{x^2}{a^3} \times \frac{b^3}{y^2}$

Now, $\left.\frac{dy}{dx}\right|(a, b) = -\frac{a^2b^3}{a^3b^2} = -\frac{b}{a}$

Now, equation of tangent at (a, b) is,

$y - b = -\frac{b}{a}(x - a)$

$\Rightarrow \quad ay - ab = -bx + ab$

$\Rightarrow \quad ay + bx = 2ab$. Dividing both sides by ab

We get, $\frac{x}{a} + \frac{y}{b} = 1$ proved.

9. Find the angle of intersection of the following curves:

a. $y = x^3$ and $6y = 7–x^2$ at (1, 1)

Solution: Solving we get,

$6x^3 + x^2 – 7 = 0$

or, $6x^3 – 6x^2 + 7x2 – 7x + 7x – 7 = 0$

or, $6x^2 (x – 1) + 7x(x – 1) + 7 = (x – 1) = 0$

or, $(x – 1) (6x^2 + 7x + 7) = 0$

$\therefore$ x = 1 as $6x^2 + 7x + 7 = 0$ does not have any real values.

If x = 1 then y = 1

Now, from $y = x^3$

$\Rightarrow \frac{dy}{dx} = 3x^2$ $\qquad \therefore \frac{dy}{dx\,(1, 1)} (6ay\ m_1) = 3$

Again, from, $6y – 7+x^2 = 0$

$$6\frac{dy}{dx} + 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{–2x}{6} = –\frac{x}{3}$$

$\therefore \frac{dy}{dx}$ at (1, 1) (say m_2) $= \frac{–1}{3}$

If θ be the angle between two curves, then,

$$\tan\theta = \left|\frac{m_1 – m_2}{1 + m_1m_2}\right| = \left|\frac{3 + \frac{1}{3}}{1 + 3 \times \left(–\frac{1}{3}\right)}\right| = \left|\frac{\frac{10}{3}}{0}\right| = \infty$$

$\therefore \theta = \tan^{–1}\infty$

$\therefore \theta = \frac{\pi}{2}$

b. $y = x^3$ and $y = 2x$ at (4, 8)

Solution: Solving we get,

$x^3 = 2x$ $\qquad$ [$\because x \neq 0$ otherwise it doesn't remains curves]

$\Rightarrow x^2 = 2 \quad \therefore x = \pm\sqrt{2}$

Now, if $x = \sqrt{2}$. Then $y = \pm 2\sqrt{2}$

From $y = x^3$

$\frac{dy}{dx} = 3x^2 = 48$

$\left.\frac{dy}{dx}\right|_{(\sqrt{2} + 2\sqrt{2})}$ (m_1 say) $= 3(\sqrt{2})^2 = 6$

From, y = 2x

$\frac{dy}{dx} = 2 \qquad \therefore \left.\frac{dy}{dx}\right|_{(\sqrt{2} + 2\sqrt{2})}$ (m_2 say) = 2

Now, by using the formula,

$$\tan\theta \left|\frac{m_1 – m_2}{1 + m_1m_2}\right| = \left|\frac{6 – 2}{1 + 6\times2}\right| = \left|\frac{4}{1 + 12}\right| = \left|\frac{4}{13}\right|$$

$\therefore \theta = \tan^{–1}\frac{4}{13}$

c. $y = 6 –x^2$ and $x^3 = 4y$ at (2, 4)

Solution: Here, $y = 6 – x^2$

$\frac{dy}{dx} = –2x$

Say $m_1 = \frac{dy}{dx\,(2, 4)} = –2\times2 = –4$

and, $x^3 = 4y$

$3x^2 = \frac{4dy}{dx} \qquad \therefore \frac{dy}{dx} = \frac{3}{4}x^2$

say $m_2 = \frac{dy}{dx}(2, 4) = \frac{3}{4} \times 4 = 3$

$$\frac{d}{dx}(f(x)) = \lim_{n\to 0} \frac{\log(y+k) - \log y}{h}$$

$$= \lim_{n\to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{n\to 0} \frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}} \times \frac{\frac{k}{y}}{h}$$

$$= \lim_{n\to 0} \times \frac{k}{yh} \qquad \left[\because \lim_{n\to 0} \frac{\log(1+x)}{x} = 1\right]$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\sin(x+h).\cos x - \sin x.\cos(x+h)}{h.\cos x.\cos(x+h)}$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\sin(x+h-x)}{h\cos x\cos(x+h)}$$

$$= \frac{1}{y} \lim_{n\to 0} \frac{\sin h}{x} \times \lim_{n\to 0} \frac{1}{\cos x\cos(x+h)}$$

$$= \frac{1}{\tan x} \times 1 \times \frac{1}{\cos^2 x}$$

$$= \frac{\sec^2 x}{\tan x}$$

If θ be the angle then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-4-3}{1+(-4)(3)}\right| = \left|\frac{-7}{11}\right|$$

$$\therefore \quad \theta = \tan^{-1}\frac{7}{11}$$

d. $x^2 + y^2 = 5$ and $y^2 = 4x$

Solution: Solving we ge,t

$x^2 + 4x - 5 = 0$

or, $x^2 + 5x - x - 5 = 0$

or, $x(x+5) - 1(x+5) = 0$

$\therefore$ $x = 1, -5$

If $x = 1$ then $y^2 = 4 \Rightarrow y = 2$

If $x = -5$ then $y^2 = -20$ (does not give real values)

Now from $x^2 + y^2 = 5$

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

m_1 (say) $= \frac{dy}{dx}(1, 2) = -\frac{1}{2}$

Again, $y^2 = 4x$

$dy\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

or, $m_2(\text{say}) = \frac{dy}{dx}_{(1,2)} = \frac{2}{2} = 1$

$$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\frac{-1}{2} - 1}{1 + \left(-1\frac{1}{2}\right)(1)}\right| = \left|\frac{\frac{-3}{2}}{\frac{1}{2}}\right| = |3|$$

$\therefore \quad \theta = \tan^{-1}(3)$

Exercise 15.5

1. Verify the Rolle's theorem for each of the following functions.

a. $f(x) = x^2 + 2$ in $[-2, 2]$

Solution: Here, $f(x) = x^2 + 2$

i. Since the polynomial function is continuous. Hence given function is continuous in $[-2, 2]$.

ii. Again, $f(x) = x^2 + 2$
$f'(x) = 2x$ which gives real values for all values of x in (2, 2)
Hence f(x) is differentiable in $(-2, 2)$

iii. Since $f(a) = f(-2) = (-2)^2 + 2 = 6$
and $f(b) = f(2) = 2^2 + 2 = 6$

$\therefore \quad f(a) = f(b)$.

Here f(x) satisfies all the condition of Rolle's theorem so these exists $C \in (a, b)$ such that $f'(c) = 0$

Now, $f(c) = 2 = 0$

$\Rightarrow \quad c = 0 \in (-2, 2)$

b. $f(x) = x^3 - 4x$ in $[0, 2]$

Solution: Here $f(x) = x^3 - 4x$

Here, $f'(x) = 3x^2 - 4$. Which is defined for all values in (0, 2). Hence the given function is differentiable in (0, 2). Again since differentiable function is continuous. So the given function is continuous on [0, 2].

Now, $f(a) = f(0) = o3 - 4\times0 = 0$

and $f(b) = f(2) = 2^3 - 4\times2 = 8 - 8 = 0$

$\therefore \quad f(a) = f(b)$

So by Rolle's theorem these exits $c \in (a, b)$ s.t. $f'(c) = 0$

Now, $f'(x) = 3x^2 - 4$

$f'(c) = 3x^2 - 4$

$f'(c) = 0 \Rightarrow 3c^2 - 4 = 0 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm\sqrt{\frac{4}{3}}$

$\therefore \quad c = \sqrt{\frac{4}{3}} \in (0, 2)$.

c. $f(x) = \sin2x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Solution: Since we know sine function is continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and differentiable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $f(a) = f\left(\frac{\pi}{2}\right) = \sin2 \times \frac{\pi}{2} = \sin(-\pi) = -\sin\pi = 0$

and $f(b) = f\left(\frac{\pi}{2}\right) = \sin2\frac{\pi}{2} = \sin\pi = 0$

Thus, by Rolle's theorem there exists $c \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in which $f'(c) = 0$

$\Rightarrow 2\cos2c = 0$

$\Rightarrow \cos2c = 0 = \cos\frac{\pi}{2}$

$\Rightarrow 2c = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

d. $f(x) = \cos 2x$ in $[-\pi,\pi]$

Solution: Since cosine function is continuous in $[-\pi,\pi]$ and is also differentiable $(-\pi,\pi)$

Now, $f(a) = f(-\pi) = \cos 2(-\pi) = \cos 2\pi = 1$

$f(b) = f(\pi) = \cos 2\pi = 1$

Hence by Rolle's Theorem $c \in (-\pi, \pi)$ such that $f'(c) = l$

$\Rightarrow \quad f'(c) = -\sin 2c = 0$

$\Rightarrow \quad \sin 2c = \sin 0$

$\Rightarrow \quad 2c = 0$

$\Rightarrow \quad c = 0 \in (-\pi, \pi)$

e. $f(x) = \sqrt{25 - x^2 \text{ in } [-5, 5]}$

Solution: Since given function is continuous in $[-5, 5]$ and also gives real values for all values in $(-5, 5)$ and hence differentiable.

Now, $f(-5) = \sqrt{25 - 25} = 0$, $f(b) = f(5) = \sqrt{25 - 25} = 0$

$\therefore \quad f(a) = f(b)$

Now, by Rolle's theorem, $\ni c \in (-5, 5)$ s.t. $f'(c) = 0$

$\Rightarrow \quad f'(c) = \dfrac{-2x}{2\sqrt{25 - x^2}} = 0$

$\Rightarrow \quad -2x = 0 \Rightarrow x = 0 \in (-5, 5)$

f. $f(x) = (x - 1)(x - 2)(x - 3) = (x - 1)(x^2 - 5x + 6) = x^3 - x^2 - 5x^2 + 5x + 6x - 6 = x^3 - 6x^2 + 6x$

Solution: Since polynomial function is continuous. So given function is continuous in $[1, 3]$ and is also defined for all values in $(1, 3)$ so is differentiable.

Now, $f(1) = 0$ $f(b)$

Now by Rolle's theorem,

$f'(c) = 3c^2 - 12c + 11 = 0$

$\Rightarrow \quad c = \dfrac{6 \pm \sqrt{3}}{3}$

$\therefore \quad c = \dfrac{6 + \sqrt{3}}{3} \in (1, 3)$

g. $f(x) = \sin x + \cos x$ in $[0, 2\pi]$

Solution: Since the sum of two continuous function is continuous. So given function is continuous in $[0, 2\pi]$.

The given function defined all values in $[0, 2\pi]$. Hence differentiable in $(0, 2\pi)$

Again, $f(0) = \sin 0 + \cos 0$

$= 0 + 1 = 1$

and $f(2\pi) = \sin 2\pi + \cos 2\pi = 0+1 = 1$

Now by Rolle's theorem $\ni c \in (0, 2\pi)$ s.l. $f'(c) = 0$

$\Rightarrow \quad f'(c) = \cos c - \sin c = 0$

$\Rightarrow \quad \cos c = \sin c$

Which is positive for $c = \frac{\pi}{4} \in (0, 2\pi)$

2. By using Rolle's theorem find a point on each of the curves given by the following. Where the tangent is parallel to x–axis.

a. $f(x) = 6x - x^2$ in $(0, 6)$

Solution: Being a polynomial function continuous in $[0,6]$. lso gives real values in $(0, 6)$. Therefore the given function is differentiable in $(0, 6)$

$f(a) = 6\times 0 - 0^2 = 0$

and $f(b) = 6\times 6 - 6^2 = 0$

$\therefore \quad f(a) = f(b)$

So by Rolle's theorem, $\exists c \in (0, 6)$ s.t. $f'(c) = 6 - 2c$

$\Rightarrow$ f'(c) = 6 – 2c = 0

$\Rightarrow$ 2c = 6 $\Rightarrow$ c = 3

Thus the tngent to the given curve is parallel to x–axis t x = 3.

$\therefore$ If x = 3 . Then y = $6\times3 - 3^2 = 18 - 9 = 9$

Therefore the required points is (3, 9)

b. $f(x) = 2x^2 - 4x$ in [0, 2]

Solution: Given $f(x) = 2x^2 - 4x$

Since the polynomial function is continuous in [0, 2]

Also the given function gives definite values for all values in (0, 2). Hence the function is also differentiable in (0, 2)

Now, $f(0) = 2\times0^2 = -4\times0 = 0$

$f(2) = 2\times2^2 - 4\times2 = 8 - 8 = 0$

$\therefore$ f(0) = f(2)

Here, all the conditioned of Rolle's theorem is satisfied os $\ni$ c$\in$(0, 2) s.t. f'(c) = 0

$\Rightarrow$ f'(c) = 4x – 4 = 0

$\Rightarrow$ 4x = 4 $\Rightarrow$ x = 1$\in$(0, 2)

Thus, the tangent to the curve $2x^2 - 4x$ is parallel to x–axis at the point x=1.

$\therefore$ When x=1, $y = 2x^2 = 4x$

$= 2\times1 - 4\times1$

$= -2$

So the required point is (1, –2)

3. Verify the mean value theorem for each of the following function on the given interval.

a. $f(x) = 3x^2 - 2$ in [2, 3]

Solution: Since, being the polynomial function $f(x) = 3x^2 - 2$ is continuous in [2, 3]

Also $f(x) = 3x^2 - 2$ is defined for all values in (2, 3). Hence is differentiable in (2, 3).

Again, $f(2) = 3\times2^2 - 2 = 12 - 2 = 10$

and $f(3) = 3\times3^2 - 2 = 27 - 2 = 25$

$\therefore$ f(2) $\neq$ f(3)

Hence all the condition of mean value theorem satisfied. So by the theorem $\exists$ c$\in$(a, b) s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(2)}{3–2} = \frac{25 - 10}{1} = 15 \text{ ... ... } (*)$$

We have, $f(x) = 3x^2 - 2$ ... ... (**)

f'(c) = 6c

$\therefore$ from (*) and (**)

$$6c = 15 \Rightarrow c = \frac{15}{6} \in (2, 3)$$

b. $f(x) = x^2$ in [1, 2]

Solution: Since quadratic function is continuous for all values of x so the given function is continuous in [1, 2]

$f(x) = x^2$ have a definite values in [1,2] so is differentiable in (1, 2)

Now, f(1) = 1 and f(2) = 4

$\therefore$ f(1) $\neq$ f(2)

All condition of M.V.T satisfied so $\exists$ c$\in$(a, b)

$$\text{Such that, } f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(2) - f(1)}{2–1} = \frac{4–1}{1} = 3 \text{ ... ... } (*)$$

Again, f'(x) = 2x $\therefore$ f'(c) = 2c ... ... (**)

from (*) and (**)

$$2c = 3 \Rightarrow c = \frac{3}{2} = 1.5 \in (1, 2)$$

c. f(x) = x(x – 1) (x – 2) in [0, ½]

Solution: $f(x) = x(x^2 - 3x + 2) = x^3 - 3x^2 + 2x$

Since being polynomial function is continuous so the given function is continuous in [0, ½]. All gives definite values in (0, ½). So is differentiable in (0, ½)

Now, $f(0) = 0$ and $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2\times\frac{1}{2}$

$= \frac{1}{8} - \frac{3}{4} + 1 = \frac{1-6+8}{8} = \frac{3}{8}$

$\therefore \quad f(0) \neq f\left(\frac{1}{2}\right)$

Therefore M.V. theorem applicable, so $\exists$ c$\in$ (a, b) such that f'(c)

$= \frac{f(b)-f(a)}{b-a} = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} = \frac{\frac{3}{8} - 0}{\frac{1}{2}} = \frac{3}{8} \times 2 = \frac{3}{4}$

Also, $f'(x) = 3x^2 - 6xc + 2$

$f'(c) = 3c^2 - 6c + 2$ ... ... (**)

from (*) and (**) we have,

$3c^2 - 6c + 2 = \frac{3}{4}$

$\Rightarrow \quad 12c^2 - 24c + 8 = 3$

$\Rightarrow \quad 12c^2 - 24c + 5 = 0$

$\Rightarrow \quad c = \frac{24 \pm \sqrt{576-240}}{24} = \frac{24 \pm \sqrt{16 \times 21}}{24}$

$= \frac{4(6 \pm \sqrt{21})}{24} = \frac{6 \pm 4.58}{6}$ (Appro.)

Taking positive sing,

$c = \frac{1.42}{6} = 0.23 \in (0, ½)$

d. $f(x) = e^x$ in [0, 1]

Solution: Since exponential function is continuous

$\therefore$ The given function is continuous in [0, 1]

Also differential in (0, 1)

Now, $f(0) = e^0 = 1$

$f(1) = e^1 = 2.718$ (Approx)

$\therefore \quad f(0) \neq f(1)$

Now by M.V. theorem $\ni$ c$\in$(0,b) s.t.

$f'(c) = \frac{f(b)-f(a)}{b-} = \frac{f(1)-f(0)}{1-0} = \frac{2.718-1}{1} = 1.718$ ... ... (*)

and $f'(x) = e^x$

$f'(c) = e^c$ ... ... (**)

$\therefore$ from (*) and (**)

$e^c = 1.718$

Taking log on both sides,

c loge = log (1.718)

$\therefore \quad c = 0.236 \in (0, 1)$

e. $f(x) = \sqrt{a^2-4}$ in [2, a]

Solution: Given function is continuous in [2, a]

Also differentiable in (2, a)

Now, $f(2) = 0$

$f(a) = \sqrt{a^2-4}$

$\therefore \quad f(2) \neq f(a)$ for all $a > 2$

So by M.V.T $\exists$ c$\in$(2, a) for which $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\sqrt{a^2-4} - 0}{a-2}$ ... ... (*)

Also, $f'(x) = \dfrac{2x}{2\sqrt{x^2-4}}$

$\therefore \quad f'(c) = \dfrac{c}{\sqrt{c^2-4}}$ ... ... (**)

from (*) and (**)

$$\frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{a^2-4}}{a-2}$$

or, $\dfrac{c^2}{c^2-4} = \dfrac{a^2-4}{(a-2)\,(a-2)}$

or, $\dfrac{c^2}{c^2-4} = \dfrac{a+2}{a-2}$

or, $\dfrac{c^2-4}{c^2} = \dfrac{a+2}{a-2}$

or, $1 - \dfrac{4}{c^2} = \dfrac{a-2}{a+2}$

or, $\dfrac{4}{c^2} = 1 - \dfrac{a-2}{a+2}$

or, $\dfrac{4}{c^2} = \dfrac{a+2-a+2}{a+2}$

or, $\dfrac{4}{c^2} = \dfrac{4}{a+2}$

or, $c^2 = \ + 2$

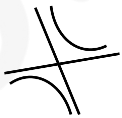
$c = \pm\sqrt{a+2}$

If $c = \sqrt{a+2}$ for all $a > 2$ then, $c = \sqrt{a+2} \in (2, a)$

4. Show that the mean value theorem is not applicable to the function $f(x) = \dfrac{1}{x}$ in (–1, 1).

Solution: Given function $f(x) = \dfrac{1}{x}$ in (–1, 1) since the given function is not defined at $x = 0 \in (-1, 1)$. Hence the function is not differentiable at $x = 0 \in (-1, 1)$. To satisfy the M.V.T, f(x) should be differentiable for all $x \in (-1, 1)$

Moreover the graph of $f(x) = \dfrac{1}{x}$ is,



Here, we cannot draw a tangent at x = 0. So, the function is not differentiable. Hence M.V. theorem for the underlying function in the defined interval is not applicable.

5. Find the points on the curve $f(x) = (x-3)^2$ where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Solution: Let, the chord joining the ending points be, (a, f(a)) = (3, 0) and (b, f(b)) = (4, 1)

Since $f(x) = (x-2)^2$ is continuous in [3, 4]

Also exist for all values in (3, 4) and hence differentiable.

Also, $f(a) \neq f(b)$

By M.V. theorem $\ni c \in (3, 4)$ s.l.

$f'(c) = \dfrac{f(b) - f(a)}{b-a}$

Now the slope of the chord joining (3, 0), (4, 1) is, $\dfrac{f(b)-f(a)}{b-a} = \dfrac{1-0}{4-3} = 1$ ... ... (*)

Since, $f(x) = (x-3)^2$

$f'(x) = \dfrac{dy}{dx} = 2(x-3)$

$f'(c) = 2(c-3)$ ... ... (**)

from (*) and (**)

$2c - 6 = 1 \Rightarrow 2c = 7 \Rightarrow c = \frac{7}{2} \in (3, 4)$

If $x = \frac{7}{2}$ then $y = \left(\frac{7}{2} - 3\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\therefore$ The tangent at $\left(\frac{7}{2}, \frac{1}{4}\right)$ is parallel to be chord joining (3, 0) and (4, 1).

6. Find the point on the curve $f(x) = x^3 - x^2 + 2$ where the tangent is parallel to the line joining the points (1, 2) and (3, 20).

Solution: Since the given function is continuous on [1, 3] being polynomial and $f'(x) = 3x^2 - 2x$ exist for all (1, 3) and f(a) = 2 and f(b) = 20

$\therefore$ $f(a) \neq f(b)$

So by M.V. theorem $\ni c \in (1, 3)$ s.t.

$f'(c) = \frac{f(b) - f(a)}{b - a}$

Now the slope of chord joining (1, 2) and (3, 20)

i.e, $\frac{f(b) - f(a)}{b - a}$ is, $\frac{20 - 2}{3 - 1} = \frac{18}{2} = 9$ ... ... (*)

And, $f'(c) = 3c^2 - 2c$ ... ... (**)

From (*) and (**)

$3c^2 - 2c = 9 \Rightarrow 3c^2 - 2c - 9 = 0$

Solving $c = \frac{1 \pm 2\sqrt{7}}{3} = \frac{1 + 2 \times 2.64}{3}$ (Appro.)

$= \frac{1+5.29}{3}$ (Appro.) (Taking positive sign)

$= \frac{6.29}{3} \in (1, 3)$

If x = 2.1 then, $y = (2.1)^3 - (2.1)^2 + 2$

$= 9.26 - 4.41 + 2$

$= 6.85$

$\therefore$ The required point is (2.1, 6.85)

Exercise - 15.6

1. By using L Hospital's rule, evaluate:

a. $\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$

Solution: Here, $\lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$ (% form)

$= \lim_{x \to 3} \frac{3x^2}{2x}$ (Differentiate w.r.to x) $= \frac{3 \times 3^2}{2 \times 3} = \frac{9}{2}$

b. $\lim_{x \to a} \frac{x+- a^n}{x - a}$

$= \lim_{x \to a} \frac{nx^{n-1}}{1}$ [Differentiate w.r.to x]

$= na^{n-1}$

c. $\lim_{x \to 0} \frac{3x - \sin x}{x}$

Solution: $\lim_{x \to 0} \frac{3x - \sin x}{x}$ = (% form)

$= \frac{3 - \cos x}{1}$ [Differentiate w.r. to x]

$$= \frac{3 - \cos 0}{1} = \frac{3 - 1}{1} = 2$$

d. $\lim\limits_{x \to \infty} \dfrac{5x++ 4x - 3}{2x^2 - 3x + 5}$

Solution: $\lim\limits_{x \to \infty} \dfrac{5x^2 + 4x - 3}{2x^2 - 3x + 5} = \left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x \to \infty} \frac{10x + 4}{4x - 3} \text{ (Differentiate w.r.to x] } \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \frac{10}{-4} = \frac{5}{2}$$

e. $\lim\limits_{x \to 0} \dfrac{e^x + e^{-x} - 2\cos x}{\sin^2 x}$

Solution: $\lim\limits_{x \to 0} \dfrac{e^x + e^{-x} - 2\cos x}{\sin^3 c} = (\% \text{ form})$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + 2\sin x}{2\sin x \cos x} (\% \text{ form})$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} + 2\cos x}{2\cos 2x} = \frac{1 + 1 + 2}{2} = \frac{4}{2} = 2$$

f. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{\tan^3 x}$

Solution: Since, $\lim\limits_{x \to 0} \dfrac{x - \sin x}{\tan^3 x} = (\% \text{ form})$

$$= \lim_{x \to 0} \frac{x - \sin x}{x^3 \times \left(\dfrac{\tan x}{x}\right)^3}$$

$$= \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^3 \times \lim_{x \to 0} \frac{x - \sin x}{x^3}$$

$$= 1 \times \lim_{x \to 0} \frac{x - \sin x}{x^3} (\% \text{ form})$$

$$= \lim_{x \to 0} \frac{1 \cos x}{3x^2} (\% \text{ form}) \text{ [Differentiate w.r. to x]}$$

$$= \lim_{x \to 0} \frac{0 + \sin x}{6x} \text{ [Differentiate w.r.to x]}$$

$$= \lim_{x \to 0} \left[\frac{\sin x}{x}\right] \times \frac{1}{6}$$

$$= 1 \times \frac{1}{6} = \frac{1}{6}$$

g. $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x^3}$

Solution: Here, $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x^3} = (\% \text{ form})$

By L–Hospital rule, differentiate numerator and denominator w.r.to x, we get,

$$\lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2} (\% \text{ form}) \text{ [Differentiate w.r. to x]}$$

$$= \lim_{x \to 0} \frac{\tan^2 x}{3x^2}$$

$$= -\frac{1}{3} \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^2$$

$= -\frac{1}{3} \times 1 = -\frac{1}{3}$

h. $\lim_{x\to\pi/2} \frac{\tan^5 x}{\tan x}$

Solution: Here, $\lim_{x\to\pi/2} \frac{\tan^5 x}{\tan x} \left(\frac{\infty}{\infty} \text{ form}\right)$

Using L-Hospital rule,

$\lim_{x\to\pi/2} \frac{5\sec^2 5x}{\sec^2 x}$ [Differentiate w.r. to x]

$= \lim_{x\to\pi/2} \frac{5 \cos^2 x}{\cos^2 5x}$ (% form)

$= \lim_{x\to\pi/2} \frac{-5 \times 2 \cos x \sin x}{-5\ 2\cos 5x \sin 5x}$ [Differentiate w.r. to x]

$= \lim_{x\to\pi/2} \frac{\sin 2x}{\sin 10x}$

$= \lim_{x\to\pi/2} \frac{2\cos 2x}{10 \cos 10x}$

$= \frac{1}{5}\left(\frac{\cos\pi}{\cos 5\pi}\right) = \frac{1}{5}\left(\frac{-1}{-1}\right) = \frac{1}{5}$

i. $\lim_{x\to 0} \frac{(e^x - 1) \tan x}{x^2}$

Solution: Here, $\lim_{x\to 0} \frac{(e^x - 1) \tan x}{x^2}$ (% form)

$= \lim_{x\to 0} \frac{(e^x - 1)}{x} \times \lim_{x\to 0} \frac{\tan x}{x}$

$= \lim_{x\to 0} \frac{e^x - 1}{x} \times 1$ [% form]

$= \frac{e^x}{1}$ [Differentiate w.r. to x]

$= e^0 = 1$

j. $\lim_{x\to 0} \frac{\log \tan x}{\log x}$

Solution: Since, $\lim_{x\to 0} \frac{\log \tan x}{\log x} \left(\frac{-\infty}{\infty} \text{ form}\right)$

By using L–Hospital rule

$= \lim_{x\to 0} \frac{x - x \sec^2 x}{\tan x}$ [Differentiate w.r. to x]

$= \lim_{x\to 0} \sec^2 x \times \lim_{x\to 0} \frac{x}{\tan x}$

$= 1\times 1 = 1$

2. Solution

a. $\lim_{x\to\infty} \frac{x^4}{e^x}$

Solution: Since, $\lim_{x\to\infty} \frac{x^4}{e^\infty} \left(\frac{\infty}{\infty} \text{ form}\right)$

Using L–Hospital rule,

$= \lim_{x\to\infty} \frac{4x^3}{e^x}$ [Differentiate num and dere. w.r.to x]

Again $\frac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{12x^2}{e^x} \text{ [Differentiate w.r. to x]}$$

Again $\frac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{24x}{e^x}$$

Again $\frac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{24}{e^x} = \frac{24}{e^\infty} = \frac{24}{\infty} = 0$$

b. $\lim\limits_{x\to\infty} \frac{\log (x^2 + 1)}{\log (x^3 + 1)}$

Solution: Since, $\lim\limits_{x\to\infty} \frac{\log (x^2 + 1)}{\log (x^3 + 1)} \left(\frac{\infty}{\infty} \text{ form}\right)$

Using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{2x(x^3 + 1)}{3x^2(x^2 + 1)} \text{ [Differentiate w.r. to x]}$$

$$= \frac{2}{3} \lim_{x\to\infty} \frac{x^3 + 1}{x(x^2 + 1)}$$

Again $\frac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \frac{2}{3} \frac{3x^2}{3x^2 + 1}$$

$$= \frac{2}{3} \lim_{x\to\infty} \left(\frac{3x^2 + 1}{3x^2 + 1} - \frac{1}{3x^2 + 1}\right)$$

$$= \frac{2}{3} \lim_{x\to\infty} \left(1 - \frac{1}{3x^2 + 1}\right)$$

$$= \frac{2}{3} (1 - 0) = \frac{2}{3}$$

c. $\lim\limits_{x\to 0} x^x$

Solution: Since, $\lim\limits_{x\to 0} x^x$ (0^0 forms)

Using L–Hospital rule, for this let,

$y = x^x \Rightarrow \log y = x\log x$

Taking limit as x tends to 0

$$\lim_{x\to 0} \log y = \lim_{x\to 0} x \log x$$

$$\text{or, } \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log x}{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{1/x}{-1/x^2} \text{ [Differentiate w.r. to x]}$$

$$= \lim_{x\to 0} -x$$

$$\therefore \lim_{x\to 0} \log y = 0$$

$$\therefore \lim_{x\to 0} y = e^0$$

$\therefore \lim_{x\to 0} x^x = 1$

d. $\lim_{x\to 0} \sin x \log x^2$

Solution: Since, $\lim_{x\to 0} \sin x \log x^2$ (0.∞ forms)

$= \lim_{x\to 0} \frac{\log x^2}{\operatorname{cosec} x}$

$= \lim_{x\to 0} \frac{2x/x^2}{-\text{coxcx}.\cot x}$ [Differentiate w.r. to x]

$= \lim_{x\to 0} -\frac{2}{x} \times \frac{\tan x}{\operatorname{cosec} x}$

$= -2 \times \lim_{x\to 0} \left(\frac{\tan x}{x}\right) \times \lim_{x\to 0} \sin x$

$= -2\times1\times0 = 0$

e. $\lim_{x\to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$ (∞ – ∞ forms)

$= \lim_{x\to 0} \left[\frac{\sin^2 x - x^2}{x^2 \sin^2 x}\right]$

$= \lim_{x\to 0} \frac{\left(\frac{\sin x}{x}\right)^2 - 1}{1 - \cos^2 x}$

$= \lim_{x\to 0} \frac{\left(\frac{\sin x}{x} + 1\right)\left(\frac{\sin x}{x} - 1\right)}{(1 + \cos x)(1 - \cos x)}$

$= \lim_{x\to 0} \frac{\sin x - x}{x(1 - \cos x)}$

$= \lim_{x\to 0} \frac{\cos x - 1}{x\sin x + (1 - \cos x)}$ [Differentiate w.r. to x]

$= \lim_{x\to 0} \frac{-\sin x}{x\cos x + \sin x + \sin x} = \lim_{x\to 0} \frac{-\sin x}{x\cos x + 2.\sin x}$

$= \lim_{x\to 0} \frac{-\cos x}{-x\sin x + \cos x + 2\cos x}$

$= \frac{-\cos 0^o}{-0 + 1+2\times1} = -\frac{1}{3}$

Exercise – 15.7

1. Find Δy and dy of the following.

a. $y = x^3 + 3$ for $x = 2$ and $\Delta x = 0.1$

Solution: Since we know, $\Delta y = f(x + \Delta x) - f(x)$ and $dy = f'(x)\,dx$.

$\therefore dy = 3x^2\,dx$
$= 3\times2^2 \times 0.1$
$= 12\times0.1$
$= 1.2$

Again, $\Delta y = f(x + \Delta x) - f(x)$
$= f(2 + 0.1) - f(2)$
$= f(2.1) - f(2)$
$= (2.1)^3 + 3 - (2^3 + 3)$
$= 9.261 + 3 - 11$
$= 12.261 - 11$

$= 1.261$

b. $y = \sqrt{x}$ for $x = 4$ and $\Delta x = 0.41$

Solution: Now, $dy = f'(x)\,dx$

$= \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2\sqrt{4}} \times 0.41 = \frac{1}{4} \times 0.41 = 0.1025$

and, $\Delta y = f(x|\Delta x) - f(x)$

$= f(4+0\ .41) - f(4)$

$= \sqrt{4.41} - \sqrt{4}$

$= 2.1 - 2$

$= 0.1$

c. $y = x^3 + 3x$ when $x = 2$ and $\Delta x = 0.2$

Solution: Since, $dy = f'(x)\,dx$

$= (2x + 3)dx$

$= (2\times2+3) \times 0.2$

$= -7 \times 0.2$

$= 1.4$

and, $\Delta y = f(x + \Delta x) - f(x)$

$= f(2 + 0.2) - f(2)$

$= f(2.2) - f(2)$

$= (2.2)^2 + 3\times2.2 - (2^2 + 3\times2)$

$= 4.84 + 6.6 - 10$

$= 11.44 - 10$

$= 1.44$

2. Find the approximate change in the volume of a cube of side xm caused by increasing the side by 2%.

Solution: Here, side of cube = xm

$\therefore \quad dx = 2\%$ of x

$= \frac{2}{100}x = 0.02x$

Now, the volume of cube having side x,

$v = x^3$

Now the change in volume,

$dv = 3x^2dx = 3x^2\,(0.02x) = 0.06x^3$

3. If $y = x^4 - 10$ and if x changes form 2 v 1.99. What is the exact and approximates change in y?

Solution: Since, $x = 2$ and $x + \Delta x = 1.99 \Rightarrow \Delta x = 1.99 - 2 = -0.01$

Now, $\frac{dy}{dx} = 4x^3$

$\Rightarrow dy = 4x^3dx$

At, $x = 2$, $dy = 4\times(2)^3 \times (-0.01)$

$= -0.32$

Again, if $x = 2$, then $y = x^4 - 10$

$= 2^4 - 10 = 6$

Since, $y + \Delta y = 6 + (-0.32) = 5.68$

4. If the radious of a sphere changes from 3cm to 3.01cm. Find the approximate increase in its volume.

Solution: Let, $x = 3$cm then $x + \Delta x = 3.01$

$\Rightarrow \quad \Delta x = 3.01 - 3$

$\Rightarrow \quad \Delta x = 0.01$

Since volume of sphere,

$v = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi \times 3(3)^2 \times 0.01$

$= 0.04\pi \times 9$

$= 0.36\pi$

5. Find the approximate increase in the surface area of a cube of the edge from 10 to 10.01. Calculate percent error in the surface area.

Solution: Let, $a = 10$ then $a + \Delta a = 10.01$

$\Delta a = 10.01 - 10 = 0.01$

Since surface area of cube is

$A = 6a^2$

$= 12a\ da$

$= 12 \times 10 \times 0.01$

$= 120 \times 0.01$

$= 1.2$

Again for percent error

Since we know that percentage error $= \frac{\text{Change}}{\text{orginal}} \times 100$

$= \frac{6(10.01 - 10)^2}{6 \times 10^2} \times 100$

$= (0.01)^2$

$= 0.0001\%$

6. A circular cupper plate is heated so that its radious increases from 5cm to 5.06 cm. Find the approximate increase in area and also the actual increase in area.

Solution: Let, $r = 5$. Then $r + \Delta r = 5.06$

$\Delta r = 5.06 - 5 = 0.06$

Now, $A = \pi r^2$

$dA = 2\pi r\ dr$

$= 2\pi \times 5 \times 0.06$

$= 0.6\pi$

Again, actual increase in area,

$= \pi(5.06)^2 - \pi(5)^2$

$= \pi(25.603 - 25)$

$= \pi \times 0.603$

$= 0.603\pi$

7. The radious of sphere is found by measurement to be 209cm with possible error of 0.02 of a centimeter. Find the consequent error in the surface.

Solution: Here, $r = 20$cm and $\Delta r = 0.02$

Then, $A = 4\pi r^2$

$= 4 \times \frac{22}{7} \times (20)^2 = \frac{3500}{7} = 5028.58$

Now since, $\frac{\Delta A}{A} = 2\frac{\Delta r}{r}$

$\Delta A = 2 \times \frac{0.02}{20} \times 5028.58 = 10.05$ sqcm