Chapter 8: Inverse Circular Functions

Exercise 8

1. Solution:

a. Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$

Then, $\sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\theta}{4}$

$\Rightarrow \quad \theta = \frac{\theta}{4}$

$\therefore \quad \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\theta}{4}$

b. Let $\text{cosec}^{-1}(2) = \theta$

Then $\text{cosec}\theta = 2$

$\frac{1}{\sin\theta} = 2$

$\sin\theta = \frac{1}{2}$

$\sin\theta = \sin\frac{\theta}{6}$

$\Rightarrow \quad \theta = \frac{\theta}{6}$

$\therefore \quad \text{cosec}^{-1}(2) = \frac{\theta}{6}$

c. Let $\cot^{-1}(-\sqrt{3}) = \theta$

Then $\cot\theta = -\sqrt{3}$

$\therefore \quad \tan\theta = -\frac{1}{\sqrt{3}}$

$\tan\theta = \tan\frac{5\theta}{6}$

$\therefore \quad \theta = \frac{5\pi}{6}$

d. Let arc $\tan\left(\frac{2}{\sqrt{3}}\right) = \theta$

$\tan^{-1}\left(\frac{2}{\sqrt{3}}\right) = \theta$

$\Rightarrow \quad \tan\theta = \frac{2}{\sqrt{3}}$

e. Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \theta$

Then, $\sec\theta = \frac{2}{\sqrt{3}}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = \cos\frac{\pi}{6}$

$\therefore \quad \theta = \frac{\pi}{6}$

$\therefore \quad \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

2. Evaluate:

a. $\cos\left(\tan^{-1}\frac{3}{4}\right)$

Let $\tan^{-1}\frac{3}{4} = \theta$

$\tan\theta = \frac{3}{4}$

$\cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right) = \cos\theta = \frac{4}{5}$

b. $\sin(\cot^{-1} x)$

Let $co^{-1} x = \theta$

$\therefore \quad \cot\theta = x$

$\sin(\cot^{-1} x)$

$\sin\theta = \frac{p}{h} = \frac{1}{\sqrt{1 + x^2}}$

c. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

d. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

e. $\cos(2\cot^{-1} x)$

Let $\cot^{-1}x = \theta$

$\therefore \quad \cot\theta = x$

Now, $\cos(2\cot^{-1} x)$

$= \cos 2\theta$

$= \frac{\cot^2\theta - 1}{\cot^2\theta + 1} = \frac{x^2 - 1}{x^2 + 1}$

f. $\sin(2 \text{ arc } \tan x)$

$= \sin(2\tan^{-1}x)$

Let $\tan^{-1}x = \theta$

$\tan\theta = x$

$\sin(2\tan^{-1}x) = \sin 2\theta$

$$= \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2x}{1 + x^2}$$

3. Solution:

a. $\cos\left[\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right]$

Let $\sin^{-1}\frac{4}{5} = 4$

$\therefore \quad \sin A = \frac{4}{5}$

$\Rightarrow \quad \cos A = \sqrt{1 - \sin^2 A} = \frac{3}{5}$

and $\tan^{-1}\frac{5}{12} = B$

$\therefore$ $\tan B = \frac{5}{12}$

Now, $\cos\left[\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right]$

$= \cos(A + B)$

$= \cos A.\ \cos B - \sin A\ .\ \sin B$

$= \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13}$

$= \frac{36 - 20}{65}$

$= \frac{16}{65}$

b. $\tan[\tan^{-1} x - \tan^{-1}2y]$

$= \tan\left[\tan^{-1}\left(\frac{x - 2y}{1 + x.2y}\right)\right]$ $\left[\tan^{-1}A\ \ \tan^{-1}B = \tan^{-1}\left(\frac{A - B}{1 + AB}\right)\right]$

$= \frac{x - 2y}{1 + 2xy}$

c. $\sin^{-1}x - \cos^{-1}(-x)$

$\sin^{-1}x - (\pi - \cos^{-1}x)$

$= \sin^{-1}x - \pi + \cos^{-1}x$

$= \sin^{-1}x + \cos^{-1}x - \pi$

$= \frac{\pi}{2} - \pi \Rightarrow -\frac{\pi}{2}$

d. Let $\sin^{-1}\frac{4}{5} = A$

$\sin A = \frac{4}{5}$ $\qquad\qquad$ $\therefore\ \cos A = \frac{3}{5}$

and $\cot^{-1}3 = B$

$\cot B = 3$

$\therefore$ $\sin B = \frac{1}{\sqrt{10}}$ $\cos B = \frac{3}{\sqrt{10}}$

Now, $\sin\left(\sin^{-1}\frac{4}{5} + \cot^{-1}3\right)$

$\sin(A + B)$

$= \sin A\ .\ \cos B + \cos A\ .\ \sin B$

$= \frac{4}{5}\cdot\frac{3}{\sqrt{10}} + \frac{3}{5}\cdot\frac{1}{\sqrt{10}}$

$= \frac{15}{5\sqrt{10}}$

$= \frac{3}{\sqrt{10}}$

e. Let $\cos^{-1}\frac{4}{5} = A$ $\qquad\qquad$ and $\tan^{-1}\frac{2}{3} = B$

$\therefore$ $\cos A = \frac{4}{5}$ $\qquad\qquad$ $\tan B = \frac{2}{3}$

$\sin A = \frac{3}{5}$ $\qquad\qquad$ $\sin B = \frac{2}{\sqrt{13}}$ and $\cos B = \frac{3}{\sqrt{13}}$

Now, $\tan\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right]$

$$= \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \, . \, \tan B} = \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}} = \frac{\frac{9+8}{12}}{\frac{12-6}{12}} = \frac{17}{6}$$

4. Solution:

a. Prove that $\sin^{-1}(3x - 4x^3) = 3\sin^{-1}x$

Let $x = \sin\theta$ then $\sin^{-1}x = \theta$

LHS $\sin^{-1}(3x - 4x^3) = \sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin(\sin 3\theta) = 3\theta$

$= 3\theta$

$= 3\sin^{-1}x$ RHS

b. $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x$

Let $x = \cos\theta$

$\therefore \quad \cos^{-1}x = \theta$

Taking LHS:

$\cos^{-1}(4x^3 - 3x) = \cos{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos^3\theta)$

$= 3\theta$

$= 3\cos^{-1}x$ RHS

c. $\tan^{-1}2 - \tan^{-1}1 = \tan^{-1}\left(\frac{1}{3}\right)$

LHS $\tan^{-1}2 = \tan^{-1}1$

$= \tan^{-1}\left(\frac{(2-1)}{1 + 2.1}\right) \qquad \left[\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$

$= \tan^{-1}\left(\frac{1}{3}\right)$ RHS

d. $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$

Let $\sec^{-1}x = \theta$ then $x = \sec\theta$

$x = \text{cosec}\left(\frac{\pi}{2} - \theta\right)$

$\text{cosec}^{-1}x = \frac{\pi}{2} - \theta$

$\Rightarrow \quad \theta + \text{cosec}^{-1}x = \frac{\pi}{2}$

$\therefore \quad \sec^{-1}x + \text{cosce}^{-1}x = \frac{\pi}{2}$

e. $\text{Tan}^{-1}x = \frac{1}{2}\sin^{-1}x\frac{2x}{1 + x^2}$

Let $x = \tan\theta$

$\therefore \quad \tan^{-1}x = \theta$

Now, $\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) = \frac{1}{2}\sin^{-1}(\sin^2\theta)$

$= \frac{1}{2} \, . \, 2\theta$

$= \theta$

$= \tan^{-1}x$

f. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

LHS $\left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5}\right) + \left(\tan+\frac{1}{7} + \tan^{-1}\frac{1}{8}\right)$

$\tan^{-1}\left(\dfrac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}}\right) + \tan^{-1}\left(\dfrac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}\right)$

$= \tan^{-1}\left(\frac{8}{14}\right) + \tan^{-1}\left(\frac{15}{55}\right)$

$= \tan^{-1}\left(\frac{4}{7}\right) + \tan^{-1}\left(\frac{3}{11}\right)$

$= \tan^{-1}\left(\dfrac{\frac{4}{7}+\frac{3}{11}}{1-\frac{12}{77}}\right) \Rightarrow \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1} 1 = \frac{\pi}{4}$

g. $\cot^{-1}3 + \text{cosec}^{-1}\sqrt{4}$

$= \tan^{-1}\left(\frac{1}{3}\right) + \text{cosec}^{-1}\sqrt{(5)}$ ... ... (i)

Let $\text{cosec}^{-1}\sqrt{5} = \theta$ then, $\text{cosec}\theta = \sqrt{5}$

$\therefore \quad \frac{h}{p} = \frac{\sqrt{5}}{1}$

$\therefore \quad h = \sqrt{5}$, $p = 1$ then $b = 2$

from fig.

$\tan\theta = \frac{1}{2}$

$\therefore \quad \theta = \tan^{-1}\frac{1}{2}$

$\text{cosec}^{-1}\sqrt{5} = \tan^{-1}\left(\frac{1}{2}\right)$

from (i),

$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right)$

$= \tan^{-1}\left(\dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{6}}\right) = \tan^{-1}\left(\dfrac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}1 = \frac{\pi}{4}$

h. $\tan^{-1}\frac{m}{n} - \tan^{-1}\left(\frac{m-n}{m+n}\right)$

$\tan^{-1}\left(\dfrac{\frac{m}{n} - \frac{m-n}{m+n}}{1 + \frac{m}{n}\cdot\left(\frac{m-n}{m+n}\right)}\right) = \tan^{-1}\left(\dfrac{\frac{m^2+mn-mn+n^2}{n(m+n)}}{\frac{mn+n^2+m^2-mn}{n(m+n)}}\right) = \tan^{-1}1 = \frac{\pi}{4}$

i. Let $\cos^{-1}x = A \Rightarrow \cos A = x$

$\therefore \quad \sin A = \sqrt{1-x^2}$

$\cos^{-1}y = B \Rightarrow \cos B = y$

$\therefore \quad \sin B = \sqrt{1-y^2}$

We know that,

$\cos(A - B) = \cos A . \cos B + \sin A . \sin B$

$\cos(A - B) = x.y + \sqrt{1 - x^2} . \sqrt{1 - y^2}$

$\therefore$ $A–B = \cos^{-1} \left(xy + \sqrt{1–x^2} . \sqrt{1 - y^2}\right)$

$\cos^{-1}x = \cos^{-1}y = \cos^{-1} (xy + \sqrt{1–x^2} . \sqrt{1–y^2})$

j. We have,

$\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{5}{13}$

$= \sin^{-1} \left\{\frac{3}{5}\sqrt{1 - \frac{25}{169}} + \frac{5}{13}\sqrt{1 - \frac{9}{25}}\right\}$

$= \sin^{-1} \left\{\frac{3}{5} \cdot \frac{12}{13} + \frac{5}{13} \cdot \frac{4}{5}\right\}$

$= \sin^{-1} \left(\frac{36}{65} + \frac{20}{65}\right)$

$= \sin^{-1} \left(\frac{56}{65}\right)$

k. Let $\sin^{-1} \frac{12}{13} = \theta$ and $\cos^{-1} \frac{4}{5} = \beta$

$\therefore$ $\sin\theta = \frac{12}{13}$ then $\cos\beta = \frac{4}{5}$

$\therefore$ $\tan\theta = \frac{12}{5}$ $\therefore$ $\tan\beta = \frac{3}{4}$

$\theta = \tan^{-1} \left(\frac{12}{5}\right)$ $\therefore$ $\beta = \tan^{-1} \left(\frac{3}{4}\right)$

$\therefore$ $\sin^{-1} \left(\frac{12}{13}\right) = \tan^{-1} \left(\frac{12}{5}\right)$ $\cos^{-1} \left(\frac{4}{5}\right) = \tan^{-1} \left(\frac{3}{4}\right)$

Now, $\sin^{-1} \left(\frac{12}{13}\right) + \cos^{-1} \left(\frac{4}{5}\right) + \tan^{-1} \left(\frac{63}{16}\right)$

$\tan^{-1} \left(\frac{12}{5}\right) + \tan^{-1} \left(\frac{3}{4}\right) + \tan^{-1} \left(\frac{63}{16}\right)$

$= \tan^{-1} \left(\frac{\frac{12}{5} + \frac{3}{4}}{1 - \frac{36}{20}}\right) + \tan^{-1} \left(\frac{63}{16}\right)$

$= \tan^{-1} \left(\frac{63}{-16}\right) + \tan^{-1} \left(\frac{63}{16}\right)$

$= \tan^{-1} \left(\frac{-63}{16}\right) + \tan^{-1} \left(\frac{63}{16}\right)$

$= -\tan^{-1} \left(\frac{63}{16}\right) + \tan^{-1} \left(\frac{63}{16}\right)$ $[\because \tan^{-1} (-x) = -\tan^{-1} (x)]$

$= 0$

l. $\tan^{-1} \left(\frac{1}{3}\right) + \sec^{-1} \frac{\sqrt{5}}{2}$

Let $\sec^{-1} \left(\frac{\sqrt{5}}{2}\right) = \theta$

$\therefore$ $\sec\theta = \frac{\sqrt{5}}{2}$

$\therefore \quad \frac{h}{b} = \frac{\sqrt{5}}{2}$

$\therefore \quad p = 1$

$\tan\theta = \frac{p}{b} = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right)$

or, $\sec^{-1}\left(\frac{\sqrt{5}}{2}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Therefore, $\tan^{-1}\left(\frac{1}{3}\right) + \sec^{-1}\left(\frac{\sqrt{5}}{2}\right)$

$= \tan^{-1}\left(\frac{1}{3}\right) + \sec^{-1}\left(\frac{\sqrt{5}}{2}\right)$

$= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{6}}\right)$

$= \tan^{-1}1 = \frac{\pi}{4}$

5. Solution:

a. $\cos^{-1}x = \sin^{-1}x = 0$

$\cos^{-1}x = \sin^{-1}x$

or, $\sin^{-1}\sqrt{1 - x^2} = \sin^{-1}x$

or, $\sqrt{1 - x^2} = x$

Squaring both sides

$1 - x^2 = x^2$

$1 = 2x^2$

$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$

b. $\sin^{-1}\frac{x}{2} = \cos^{-1}x$

or, $\sin^{-1}\frac{x}{2} = \sin^{-1}\sqrt{1 - x^2})$

$\therefore \quad \frac{x}{2} = \sqrt{1 - x^2}$

Squaring both sides

or, $\frac{x^2}{4} = 1 - x^2$

or, $5x^2 = 4$

$\therefore \quad x = \pm\frac{2}{\sqrt{5}}$

c. $\cos^{-1}x = \cos^{-1}\frac{1}{2x}$

$\therefore \quad x = \frac{1}{2x}$

$x^2 = \frac{1}{2}$

$\therefore\ x = \pm\frac{1}{\sqrt{2}}$

d. $\tan^{-1}x - \cot^{-1}x = 0$

$\tan^{-1}x = \cot^{-1}x$

or, $\tan^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore\ x = \frac{1}{x}$

$\therefore\ x = \pm 1$

e. $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}1$

or, $\tan^{-1}\left\{\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right\} = \tan^{-1}1$

or, $\tan^{-1}\left(\frac{x^2 + 2x - x - 2 + x^2 + x - 2x - 2}{x^2 - 4 - x^2 + 1}\right) = \tan^{-1}1$

$\tan–1\left(\frac{2x^2 - 4}{-3}\right) = \tan^{-1}1$

$\therefore\ \frac{2x^2 - 4}{-3} = 1$

$2x^2 - 4 = -3$

$2x^2 = 1$

$\therefore\ x = \pm\frac{1}{\sqrt{2}}$

f. $\sin^{-1}2x - \sin^{-1}\sqrt{3}x = \sin^{-1}x$

or, $\sin^{-1}2x - \sin^{-1}x = \sin^{-1}x\sqrt{3}x$

$\sin^{-1}\left\{2x \, . \, \sqrt{1-x^2} - x \, . \, \sqrt{1-4x^2}\right\} = \sin^{-1}\sqrt{3}x$

$2x\sqrt{1-x^2} - x\sqrt{1-4x^2} = \sqrt{3}\,x$

$x\left(2\sqrt{1-x^2} - \sqrt{1-4x^2}\right) = \sqrt{3}\,x$

$\therefore\ x\left(2 - \sqrt{1-x^2} - \sqrt{1-4x^2} - \sqrt{3}\right) = 0$

Either $x = 0$ Or, $2\sqrt{1-x^2} - \sqrt{1-4x^2} - \sqrt{3} = 0$

$2\sqrt{1-x^2} - \sqrt{3} = \sqrt{1-4x^2}$

Squaring both sides, we get,

$x = \frac{1}{2}$

Hence, $x = 0, \frac{1}{2}$. Since $x > 0$, therefore required $x = \frac{1}{2}$

g. The given equation is

$3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right) - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$

or, $3\tan^{-1}(2 - \sqrt{3}) - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$

or, $\tan^{-1}\left\{\dfrac{3(2-\sqrt{3})\ \ (2-\sqrt{3}^3)}{1-3(2-\sqrt{3})^2}\right\} - \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{x}$

$\therefore$ $3\tan^{-1}x = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

or, $\tan^{-1}\left(\dfrac{12\sqrt{3}-20}{12\sqrt{3}-20}\right) - \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{x}$

or, $\tan^{-1}1 - \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{x}$

or, $\tan^{-1}\left(\dfrac{1-\dfrac{1}{3}}{1+1\,\dfrac{1}{3}}\right) = \tan^{-1}\dfrac{1}{x}$

or, $\tan^{-1}\left(\dfrac{1}{2}\right) = \tan^{-1}\dfrac{1}{x}$

$\therefore$ $\dfrac{1}{2} = \dfrac{1}{x}$

$\therefore$ $x = 2$

h. $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right) - \cos^{-1}\left(\dfrac{1-b^2}{1+b^2}\right) = 2\tan^{-1}x$

or, $2\text{tn}^{-1}a - 2\tan^{-1}b = 2\tan^{-1}x$

or, $\tan^{-1}a - \tan^{-1}b = \tan^{-1}x$

or, $\tan^{-1}\left(\dfrac{a-b}{1+ab}\right) = \tan^{-1}x$

$\therefore$ $x = \dfrac{a-b}{1+ab}$

i. $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\dfrac{8}{31}\right)$

or, $\tan^{-1}\left\{\dfrac{x+1+x-1}{1-(x+1)(x-1)}\right\} = \tan^{-1}\left(\dfrac{8}{31}\right)$

$\tan^{-1}\left(\dfrac{2x}{1-x^2+1}\right) = \tan^{-1}\left(\dfrac{8}{31}\right)$

$\therefore$ $\dfrac{2x}{2-x^2} = \dfrac{8}{31}$

or, $8-4x^2 = 31x$

or, $4x^2+31x-8 = 0$

or, $4x^2+32x-x-8 = 0$

or, $4x(x+8)-1(x+8) = 0$

$\therefore$ $x = -8$ or $\dfrac{1}{4}$

6. Solution:

a. Let $x = \tan\theta$ then $2\tan^{-1}x = 2\tan^{-1}\tan\theta = 2\theta$

$\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin^2\theta) = 2\theta$

$\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos^2\theta) = 2\theta$

$\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}\tan^2\theta) = 2\theta$

Combining the above results, we get the required result.

Hence, $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

b. $\tan(2\tan^{-1}x) = \tan\left(\tan^{-1}\frac{2x}{1-x^2}\right) = \frac{2x}{1-x^2}$

$2\tan(\tan^{-1}x + \tan^{-1}x^3)$

$= 2\tan\tan^{-1}\left(\frac{x + x^3}{1 - x^4}\right)$

$= 2\frac{x(1 + x^2)}{(1 - x^2)(1 + x^2)}$

$= \frac{2x}{1 - x^2}$

Hence, $tn(2\tan^{-1}x) = 2\tan(\tan^{-1}x + \tan^{-1}x^3)$

c. LHS $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$

$= \tan^{-1}1 + \tan^{-1}\left(\frac{2+3}{1-6}\right)$

$= \tan^{-1}1 + \tan^{-1}(-1)$

$= \frac{\pi}{4} + \frac{3\pi}{4}$

$= \pi$

RHS, $2\left(\tan^{-1}1 + \tan^{-1} eq\frac{1}{2} + \tan^{-1}\frac{1}{3}\right)$

$= 2\left\{\tan^{-1}1 + \tan{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}\right)\right\}$

$= 2\{\tan^{-1}1 + \tan^{-1}1\}$

$= 2\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$

$= \pi$

Hence, LHS = RHS

7. We have,

$\cot^{-1}x + \cot^{-1}y + \cot^{-1}z = \pi$

$\cot^{-1}x + \cot^{-1}y = \pi - \cot^{-1}z$

$\cot^{-1}\left(\frac{xy - 1}{x + y}\right) = \pi - \cot^{-1}z$

or, $\frac{xy - 1}{x + y} = \cot(\pi - \cot^{-1}z)$

or, $\frac{xy - 1}{x + y} = -\cot\cot^{-1}z$

or, $\frac{xy - 1}{x + y} = -z$

$xy - 1 = -xz - yz$

$\therefore\ xy + yz + zx = 1$

8. Given,

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

$\tan^{-1}\left(\frac{x + y}{1 - x}\right) = \pi - \tan^{-1}z$

or, $\dfrac{x + y}{1 - xy} = \tan(\pi - \tan^{-1}z)$

or, $\dfrac{x + y}{1 - xy} = -\tan\tan^{-1}z$ $\quad(\because \tan(\pi - \theta) = -\tan\theta)$

$\dfrac{x + y}{1 - xy} = -z$

$x + y = -z + xyz$

$\therefore\ x+y+z = xyz$

9. Let $\sin^{-1}x = A \Rightarrow \sin A = x$ $\quad\therefore \cos A = \sqrt{1 - x^2}$

$\sin^{-1}y = B \Rightarrow \sin B = y$ $\quad\therefore \cos B = \sqrt{1 - y^2}$

$\sin^{-1}z = C \Rightarrow \sin C = z$ $\quad\therefore \cos C = \sqrt{1 - z^2}$

Since,

$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$

i.e. $A + B + C = \pi$

$\therefore\ A + B = \pi - C$

$\therefore\ \text{Sin}(A + B) = \sin(\pi - c) = \sin C$

$\cos(A + B) = \cos(\pi - C) = -\cos C$

Now,

Taking, LHS, $x\sqrt{1 - x^2} + y\sqrt{1 - y^2} + z\sqrt{1 - z^2}$

$= \sin A.\ \cos A + \sin B.\ \cos B + \sin C\ .\ \cos C$

$= \dfrac{1}{2}(\sin 2A + \sin 2B) + \sin C\ .\ \cos C$

$= \dfrac{1}{2}\, 2\sin\dfrac{2A + 2B}{2}\ .\ \cos\dfrac{2A - 2B}{2} + \sin C.\ \cos C$

$= \sin(A + B)\ .\ \cos(A - B) + \sin C\ .\ \cos C$

$= \sin C.\ \cos(A - B) + \sin C.\ \cos C$

$= \sin C\ \{\cos(A - B) + \cos C\}$

$= \sin C\ \{\cos(A - B) - \cos(A + B)\}$ $\quad[\because \cos(A+B) = -\cos C]$

$= \sin C\ .\ 2\sin A.\ \sin B$

$= 2\sin A.\ \sin B.\ \sin C$

$= 2\ x.\ y.\ z = 2xyz$

Hence, $x\sqrt{1 - x^2} + y\sqrt{1 - y^2} + z\sqrt{1 - z^2} = 2xyz$

10. We have,

$\cos^{-1}a + \cos^{-1}b + \cos^{-1}c = \pi$

or, $\cos^{-1}a + \cos^{-1}b = \pi - \cos^{-1}c$

or, $\cos^{-1}\left\{ab - \sqrt{1-a^2}\ .\ \sqrt{1-b^2}\right\} = \pi - \cos^{-1}c$

or, $ab - \sqrt{1 - a^2}\ .\ \sqrt{1 - b^2} = \cos(\pi - \cos^{-1}c)$

$ab - \sqrt{1 - a^2}\ .\ \sqrt{1 - b^2} = -\cos\cos^{-1}c$

$ab - \sqrt{1 - a^2}\sqrt{1 - b^2} = -c$

$ab + c = \sqrt{(1 - a+)(1 - b^2)}$

Squaring both sides

$a^2b^2 + 2ac + c^2 = (1 - a^2)(1 - b^2)$

$a^2b^2 + 2abc + c^2 = 1 - b^2 - a^2 + a^2b^2$

$a^2 + b^2 + c^2 + 2abc = 1$

11. LHS $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c$

$= \tan^{-1}\left(\dfrac{a + b}{1 - ab}\right) + \tan^{-1}c$

$$= \tan^{-1}\left\{\frac{\frac{a+b}{1-ab}+c}{1-\left(\frac{a+b}{1-ab}\right).c}\right\}$$

$$= \tan^{-1}\left\{\frac{\frac{a+b+c-abc}{1-ab}}{\frac{1-ab-ac-bc}{1-ab}}\right\}$$

$$= \tan^{-1}\left(\frac{a+b+c-abc}{1-ab+bc-ca}\right) \text{ RHS.}$$

12. LHS $\sin^{-1}(\cos\sin^{-1}x) + \cos^{-1}(\sin\cos^{-1}x)$

$$= \sin^{-1}\left(\cos\cos^{-1}\sqrt{1-x^2}\right) + \cos^{-1}\left(\sin\sin^{-1}\sqrt{1-x^2}\right)$$

$$= \sin^{-1}\sqrt{1-x^2} + \cos^{-1}\sqrt{1-x^2}$$

$$= \frac{\pi}{2} \qquad \left(\because \sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}\right) \text{ RHS}$$

13. LHS $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$$= \tan^{-1}\left\{\frac{\cos + \frac{x}{2} - \sin^2\frac{x}{2}}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\right\}$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right) \qquad (\because \text{Dividing by } \cos x/2)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1+\tan\frac{\pi}{4}.\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$= \frac{\pi}{4} - \frac{x}{2} \text{ RHS}$$