Chapter–5
Quadratic Equations

Exercise 5.1

1. Solution:

a. Here, $x^2 - 12x + 40 = 0$ ... ... (i)
Comparing equation (i) with $ax^2 + bx + c = 0$, we get
$\therefore$ a = 1, b = –12, c = 40
Now,
$b^2 - 4ac = (-12)2 - 4 \times 1 \times 40 = 144 - 160 = -16 < 0$
Here, Roots are imaginary and unequal.

b. Here,
$x^2 - 14x - 3 = 0$ ... ... (i)
Comparing equation (i) with $ax^2 + bx + c = 0$, we get
$\therefore$ a = 1, b = –4, c = –3
Now,
$b^2 - 4ac = (-4)^2 - 4\times1\times(-3) = 16 + 12 = 28 > 0$
Hence, Roots are unequal, real and irrational.

c. Here, $2x^2 - 12x + 18 = 0$ ... ... (i)
Comparing equation (i) with $ax^2 + bx + = 0$, we get
$\therefore$ a = 2, b = –12, c = 18
Now,
$b^2 - 4ac = (-12)^2 - 4\times2\times18 = 144 - 144 = 0$
Hence, roots are real, equal and rational.

d. Here, $4x^2 + 8x - 5 = 0$ ... ... (i)
Comparing equation (i) with $ax^2 = bx + c = 0$, we get,
$\therefore$ a = 4, b = 8, c = –5
Now,
$b^2 - 4ac = (8)2 - 4 \times 4 \times (-5) = 64 + 80 = 144 > 0$ and perfect square
Hence,
Roots are real, unequal is rational.

e. Here, $x^2 - 16 = 0$ ... ... (ii)
Comparing equation (i) with $ax^2 + bx + c = 0$
$\therefore$ a = 1, b = 0, c = –16
Now,
$b^2 - 4ac = 0 - 4 \times 1 \times (-16) = 64 > 0$ and perfect square
Hence, roots are real, unequal is rational.

2. Solution:

Given equation is $5x^2 - px + 45 = 0$ ... (i)
Comparing equation (i) with
$ax^2 + bx + c = 0$
$\therefore$ a = 5, b = –p, c = 45
Now, for being equal roots;
$b^2 = 4ac = 0$
or, $(-p)^2 - 4\times5\times45 = 0$
or, $p^2 = 900$
or, $(p)^2 = (\pm 30)^2$
$\therefore$ $p = \pm 30$

3. Solution:

a. Here,
Comparing equation $x^2 + (k + 2)x + 2k = 0$ with $ax^2 + bx + c = 0$
$\therefore$ a = 1, b = k+2, c = 2k

Now, for being equal roots;
$b^2 - 4ac = 0$
or, $(k+2)^2 - 4\times1\times2k = 0$
or, $k^2 + 4k + 4 - 8k = 0$
or, $k^2 - 4k + 4 = 0$
or, $(k - 2)^2 = 0$
$\therefore$ $k = 2$

b. Here, Comparing equation $x^2 - (2k-1).\ x - (k - 1) = 0$ with $ax^2 + bx + c = 0$. we get,
$\therefore$ $a = 1, b = -(2k -1), c = -(k - 1)$
Now, for being equal roots;
$b^2 - 4ac = 0$
or, $\{-(2k - 1)\}^2 - 4\times1\times\{-(k - 1)\} = 0$
or, $4k^2 - 4k + 1 + 4k - 4 = 0$
or, $4k^2 - 3 = 0$
or, $k^2 = \frac{3}{4}$
$\therefore$ $k = \pm\frac{\sqrt{3}}{2}$

4. Solution:

a. Here, comparing equation $(1 + m^2).\ x^2 + 2mc.\ x + (c^2 - a^2) = 0$ with $ax^2 + bx + c = 0$, we get,
$\therefore$ $a = 1+m^2, b = 2mc, c = c^2 - a^2$
Now,
For being equal roots;
$b^2 - 4ac = 0$
or, $(2mc)^2 - 4(1 + m^2) . (c^2 - a^2) = 0$
or, $4m^2c^2 - 4\{1(c^2 - a^2) + m^2 (c^2 - a^2)\} = 0$
or, $m^2c^2 - (c^2 - a^2) - m^2c^2 + m^2a^2 = 0$
or, $-(c^2 - a^2) = - m^2a^2$
or, $c^2 - a^2 = m^2a^2$
or, $c^2 = m^2a^2 + a^2$
or, $c^2 = a^2(1 + m^2)$ proved.

5. Solution:

Here, comparing $(a^2 - bc) . x^2 + 2(b^2 - ca) . x + c^2 - ab = 0$ with $Ax^2 + Bx + C = 0$
$\therefore$ $A = a^2 - bc, B = 2(b^2 - ca), c = c^2 - ab$
For equal roots,
$B^2 - 4AC = 0$
or, $\{2(b^2 - ca)\}^2 - 4(a^2 - bc) . (c^2 - ab) = 0$
or, $(b^2 - ca)^2 - (a^2 - bc) (c^2 - ab) = 0$
or, $b^4 = 2ab^2c + c^2a^2 - a^2c^2 + a^3b + bc^3 - ab^2c = 0$
or, $a^3b + b^4 + bc^3 - 3ab^2c = 0$
or, $b(a^3 + b^3 + c^3 - 3abc) = 0$
Either, $b = 0$,
$a^3 + b^3 + c^3 - 3abc = 0$

6. Solution:

Here, given equation is
$(x - a) (x - b) + (x - b) (x - c) + (x - c) (x - a) = 0$
or, $x^2 - bx - ax + ab + x^2 - cx - bx + bc + x^2 - ax - cx + ca = 0$
or, $3x^2 - 2(a + b + c).\ x + (ab + bc + ca) = 0$ ... ... (i)
Comparing equation (i) with $Ax^2 + Bx + C = 0$
$A = 3, B = -2 (a + b + c), c = ab + bc + ca$
Now,
$B^2 - 4ac = 0$

or, $\{-2(a + b + c)\}^2 - 4\times3 (ab + bc + ca) = 0$
or, $4(a^2 + b^2 + c^2 + ab + bc + ca) - 12 (ab + bc + ca) = 0$
or, $(a^2 + b^2 + c^2 + ab + bc + ca - 3ab - 3bc - 3ca) = 0$
or, $(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca) = 0$
or, $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$
Either,
$a = b, b = c, c = a$
$\therefore$ $a = b = c$

7. Solution:
Here, comparing $(a^2 + b^2) x^2 - 2(ac + bd) x + (c^2 + d^2) = 0$ with $Ax^2 + BX + C = 0$, we get,
$A = a^2 + b^2$
$B = -2(ac + bd)$
$C = c^2 + d^2$
The roots are equal if
$B^2 - 4AC = 0$
or, $\{-2(ac + bd)\}^2 - 4\times(a^2 + b^2) (c^2 + d^2) = 0$
or, $4(ac + bd)^2 - 4(a^2 + b^2) (c^2 + d^2) = 0$
or, $a^2c^2 + 2abcd + b^2d^2 - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2 = 0$
or, $-a^2d^2 + 2abcd - b^2c^2 = 0$
or, $-(ad - bc)^2 = 0$
or, $ad - bc = 0$
or, $ad = bc$
or, $\frac{a}{b} = \frac{c}{d}$
$\therefore$ $\frac{a}{b} = \frac{c}{d}$ proved.

8. Solution:
Here, given equation is $(b + c - a) . x^2 + (c + a - b) . x + (a + b - c) = 0$ ... ... (i)
If $a + b + c = 0$
Comparing equation (i) with $Ax^2 + Bx + C = 0$
$A = (b + c - a)$
$B = (c + a - b)$
$C = (a + b - c)$
Now,
$B^2 - 4AC$
$= (c + a - b)^2 - 4(b + c - a) . (a + b - c)$
$= (-b - b)^2 - 4(-2a) . (-2c)$
$= 4b^2 - 16ac$
$= 4(b^2 - 4ac)$
$= 4\{b^2 - 4a (-a - b)\}$
$= 4(b^2 + 4a^2 + 4ab)$
$= 4(b + 2a)^2 > 0$ and a perfect square
Hence, roots are rational.

9. Solution:
Here, given equation is $(x - a) (x - b) = k^2$ ... ... (i)
or, $x^2 - bx - ax + ab - k^2 = 0$
or, $x^2 - (a + b) . x + (ab - k^2) = 0.$
Comparing equation (i) with $Ax^2 = BX + C = 0$, we get,
$A = 1, B = -(a + b), C = ab - k^2$
Now,
$B^2 - 4AC$
$= \{-(a + b)\}^2 - 4\times1(ab - k^2)$
$= a^2 + 2ab + b^2 - 4(ab - k^2)$
$= a^2 - 2ab + b^2 + 4ab + 4k^2$

$= a^2 - 2ab + b^2 + 4k^2$

$= (a - b)^2 + 4k^2 > 0$ for all k

Hence, roots are real.

10. Solution:

Comparing equation $(b - c)\, x^2 + 2(c - a) . x + (a - b) = 0$ with $Ax^2 + BX + C = 0$.

$A = (b - c)$

$B = 2(c - a)$

$C = (a - b)$

Now,

$B^2 - 4AC$

$= 4(c -)^2 - 4(b - c)\,(a - b)$

$= 4\{(c - a)^2 - (b - c)\,(a - b)^2\}$

$= 4\{c^2 + a^2 - 2ca - ab + b^2 + ca - bc\}$

$= 4\{a^2 + b^2 + c^2 - ab - bc - ca^2\}$

$= 2\{2a^2 + 2b^2 + 2c - 2ab - 2bc - 2ca^2\}$

$= 2\{(a - b)^2 + (b - c)^2 + (c - a)^2\} > 0$

Hence, roots are always real.

11. Solution:

Here, given equation is $x^2 + (2m - 1). x + m^2 = 0$ .. .. (i)

Comparing equation (i) with $ax^2 + bx + c = 0$, we get,

$a = 1,\ b = (2m - 1),\ c = m^2$

Now,

$b^2 - 4ac$

or, $(2m - 1)^2 - 4\times1\times m^2$

or, $4m^2 - 4m + 1 - 4m^2$

or, $-(um - 1)$

or, $-(4m - 1)$

The roots will be real if $b^2 - 4ac$

or, $-4m + 1 \geq 0$

or, $1 \geq 4m$

$\therefore\ m \leq \frac{1}{4}$

12. Solution:

Comparing $x^2 + 4abx + (a^2 + 2b^2)^2 = 0$ with $Ax^2 + Bx + C = 0$. We get,

$A = 1,\ B = 4ab,\ C = (a^2 + 2b^2)^2$

Now,

$B^2 - 4AC$

$= (4ab)^2 = 4\times1\times(a^2 + 2b^2)^2$

$= 16a^2b^2 - 4(a^4 + 2a^2b^2 + 4b^4)$

$= 4(4a^2b^2 - a^4 - 2a^2b^2 - 4b^4)$

$= 4(-a^4 + 2a^2b^2 - 4b^4)$

$= -4(a^4 - 2a^2b^2 + 4b^4)$

$= -4(a^2 - 2b^2)^2 < 0$

Hence, roots are imaginary.

13. Solution:

Here, $qx^2 + 2px + 2q = 0$

$b^2 - 4ac = (2p)^2 - 4.q.\ 2q]$

$= 4p^2 - 8q^2$

$= 4(p^2 - 2q)^2 > 0$ ... ... (i)

$(p + q)\, x^2 + 2qx + (p - q) = 0$

$b^2 - 4ac = (2q)^2 - 4(p + q) . (p - q)$

$= 4q^2 - 4(p^2 - q^2)$

$= 4(q^2 - p^2 + q^2)$

$= -4(p^2 - 2q^2) < 0$ ... ... (ii)

The roots of second equation (ii) are imaginary if the roots of first equations are real.

14. Solution:

Here, comparing (ab – ac) x^2 + (bc - ab) x + ca - ab = 0 with $Ax^2 + BX + C = 0$

$\therefore$ A = (ab – ac)

$\therefore$ B = (bc – ab)

$\therefore$ c = (ca – ab)

Now, $B^2 - 4AC = 0$

or, $(bc - ab)^2 - 4(ab - ac)(ca - ab) = 0$

or, $b^2c^2 - 2ab^2c + a^2b^2 - 4\{a^2bc - a^2b^2 - a^2c^2 + a^2bc\} = 0$

or, $b^2c^2 - 2ab^2c + a^2b^2 - 4a^2bc + 4a^2b^2 + 4a^2c^2 - 4a^2bc$

Exercise: 5.2

1. Solution:

a. Let, α and β be the two roots i.e. $\alpha = 3$, $\beta = -5$

Now, x^2 – (sum of roots) . x + product of roots = 0

or, $x^2 - (3 - 5) . x + 3 \times (-5) = 0$

or, $x^2 + 2x - 15 = 0$

Hence, The required quadratic equation is $x^2 + 2x - 15 = 0$.

b. Here, let, α and β be the two roots i.e. $\alpha = 2$, $\beta = \frac{1}{2}$.

Now, x^2 – (sum of roots) x + product of roots = 0

or, $x^2 - \left(2 + \frac{1}{2}\right) x + 2 \times \frac{1}{2} = 0$

or, $x^2 - \frac{5x}{2} + 1 = 0$

or, $2x^2 - 5x + 2 = 0$

Hence, the required equation is $2x^2 - 5x + 2 = 0$

c. Here, let α and β be the two roots i.e. $\alpha = 2 - 3i$, $\beta = 2 + 3i$

Now, x^2 – (sum of roots) x + product of roots = 0

or, $x^2 - (2 - 3i + 2 + 3i) x + (2 - 3i)(2 + 3i) = 0$

or, $x^2 - 4x + 4 + 9 = 0$

or, $x^2 - 4x + 13 = 0$

Hence, the required quadratic equation is $x^2 - 4x + 13 = 0$

2. Solution:

a. Here, one root (α) = $3 - \sqrt{5}$

Other root (β) = $3 + \sqrt{5}$

Quadratic equation is x^2 – (sum of roots) x + product of roots = 0

or, $x^2 - (3 - \sqrt{5} + 3 + \sqrt{5}) x + 9 - 5 = 0$

or, $x^2 - 6x + 4 = 0$

$\therefore$ $x^2 - 6x + 4 = 0$

b. Here, one root (α) = $-2i$

Other root (β) = 2i

The required equation is $x^2 - (\alpha + \beta) x + \alpha\beta = 0$

or, $x^2 - (2i - 2i) . x - 4i^2 = 0$

or, $x^2 + 4 = 0$

c. Here, one root (α) = $1 + \sqrt{3}i$

Other root (β) = $1 - \sqrt{3}i$

The required quadratic equation is

$x^2 - (\alpha + \beta) x + \alpha\beta = 0$

or, $x^2 - (1 + \sqrt{3}i + 1 - \sqrt{3}i) x + 1^2 + (\sqrt{3})^2 = 0$

or, $x^2 - 2x + 1 + 3 = 0$

or, $x^2 - 2x + 4 = 0$

d. Here,

One root (α) = $\frac{1}{3+\sqrt{5}}$

Other root (β) = $\frac{1}{3-\sqrt{5}}$

The required quadratic equation is,

$x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$

or, $x^2 - \left(\frac{1}{3+\sqrt{5}} + \frac{1}{(3-\sqrt{5})}\right) x + \frac{1}{3+\sqrt{5}}\,\frac{1}{3-\sqrt{5}} = 0$

or, $x^2 - \frac{(3-\sqrt{5}+3+\sqrt{5})}{9 - 5} . x + \frac{1}{9-5} = 0$

or, $x^2 - \frac{6x}{4} + \frac{1}{4} = 0$

or, $4x^2 - 6x + 1 = 0$

e. Here,

One root (α) = $\frac{1}{3!}$

Other root (β) = $-\frac{1}{3!}$

The required quadratic equation is,

$x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$

or, $x^2 - \left(\frac{1}{3i} - \frac{1}{3i}\right) . x - \frac{1}{3i}\,\frac{1}{3i} = 0$

or, $x^2 + \frac{1}{9} = 0$

or, $9x^2 + 1 = 0$

3. Solution:

a. Here, let α and β be the roots of $4x^2 + 8x - 5 = 0$

$\alpha + \beta = \frac{-8}{4} = -2$, $\alpha\beta = \frac{-5}{4}$

and α^2 and β^2 be the roots of required equation.

$\alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= 4 - 2. \frac{-5}{2} = \frac{13}{2} = 6.5$

$\alpha^2\, \beta^2$

$= (\alpha\beta)2$

$= \left(\frac{-5}{4}\right)^2 = \frac{25}{16}$

The required quadratic equation is,

$x^2 - \frac{13}{2} . x + \frac{25}{16} = 0$

or, $16x^2 - 104x + 25 = 0$

b. Here, let α and β be the two roots of $3x^2 - 5x - 2 = 0$

$\alpha + \beta = \frac{5}{3}$, $\alpha\beta = \frac{-2}{3}$

and $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ be the two roots of required equation,

$\frac{1}{\alpha} + \frac{1}{\beta}$

$= \frac{\beta + \alpha}{a\beta} = \frac{1}{\alpha\beta}$

$= \frac{\frac{5}{3}}{\frac{-2}{3}}$

$\frac{1}{\alpha} \times \frac{1}{\beta}$

$= \frac{1}{\frac{-2}{3}}$

$= \frac{-5}{3} \times \frac{3}{2}$ $= \frac{1 \times 3}{-2}$

$= -\frac{5}{2}$ $= -\frac{3}{2}$

The required quadratic equation is, $x^2 - \frac{5}{2} . x - \frac{3}{2} = 0$

or, $2x^2 + 5x - 3 = 0$

c. Here, let, α and β be the two roots of $x^2 - bx + c = 0$

$\alpha + \beta = \frac{b}{1}$, $\alpha\beta = c$

or, $\alpha + \beta = b$

and $m\alpha$ and $m\beta$ be the two roots of required equation,

$m\alpha + m\beta$ $m\alpha . m\beta$

$= m(\alpha + \beta)$ $= m^2\alpha\beta$

$= mb$ $= m^2 \times c$

The required quadratic equation is $x^2 - mbx + m^2c = 0$

d. Here, let α and β be the roots of $x^2 - px + q = 0$

Then,

sum of roots $= \alpha + \beta$

$= \frac{-p}{-1} = p$

Product of roots $= \alpha\beta$

$= \frac{q}{1} = q$

Since,

The roots of the required equation are by h, so

Sum of roots $= (\alpha + h) + \beta + h$

$= (\alpha + \beta) + 2h = p + 2h$

Product of roots $= (\alpha + h) (\beta + h)$

$= \alpha\beta + (\alpha + \beta) h + h^2$

$= q + ph + h^2$

The required equation is $x^2 - (p + 2h) . x + (q + ph + h^2) = 0$

4. Solution:

a. Here, let α and 3α be the two roots of $ax^2 + bx + c = 0$

Sum of roots, $\alpha + 3\alpha = \frac{b}{a}$

or, $4\alpha = \frac{b}{a}$ or, $\alpha = \frac{b}{4a}$

Production of roots, $\alpha . 3\alpha = \frac{c}{a}$

or, $3\alpha^2 = \frac{c}{a}$

or, $\frac{3b^2}{16a} = \frac{c}{a}$

or, $3b^2 = 16ac$

$\therefore$ $3b^2 = 16ac$ proved.

5. Solution:

Here,

α and β be the roots of $ax^2 + bx + b = 0$

Then, $\alpha + \beta = \frac{-b}{a}$

$\alpha\beta = \frac{c}{a}$

Now,

or, $\dfrac{\alpha + \beta}{\sqrt{\alpha\beta}} = \dfrac{-b/a}{\sqrt{\dfrac{c}{a}}}$

or, $\dfrac{\alpha}{\sqrt{\alpha\beta}} = \dfrac{\beta}{\sqrt{\alpha\beta}} = -\sqrt{\dfrac{b}{a}}$

or, $\sqrt{\dfrac{\alpha}{\beta}} + \sqrt{\dfrac{\beta}{\alpha}} = -\sqrt{\dfrac{b}{a}}$

or, $\sqrt{\dfrac{\alpha}{\beta}} + \sqrt{\dfrac{\beta}{\alpha}} + \sqrt{\dfrac{b}{a}} = 0$

i.e., $\sqrt{\dfrac{p}{q}} + \sqrt{\dfrac{q}{p}} + \sqrt{\dfrac{b}{a}} = 0$ ($\because$ roots α, β are in the ratio p:q)

6. Solution:

a. Here,

If α, β be the roots of $px^2 + qx + q = 0$

Then,

$\alpha + \beta = \dfrac{-q}{p}$

$\alpha\beta = \dfrac{q}{p}$

L.H.S. $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + 1$

$= \dfrac{\beta + \alpha}{\alpha\beta} + 1$

$= \dfrac{\frac{-q}{p}}{\frac{q}{p}} + 1$

$= -1 + 1$

$= 0$ R.H.S.

b. Here, $\alpha + \beta = -\dfrac{q}{p}$

$\sqrt{\alpha\beta} = \sqrt{\dfrac{q}{p}}$

Now, $\dfrac{\alpha + \beta}{\sqrt{\alpha\beta}} = \dfrac{\frac{-q}{p}}{\sqrt{\dfrac{q}{p}}}$

or, $\dfrac{\alpha}{\sqrt{\alpha\beta}} + \dfrac{\beta}{\sqrt{\alpha\beta}} = -\sqrt{\dfrac{q}{p}}$

or, $\sqrt{\dfrac{\alpha}{\beta}} + \sqrt{\dfrac{\beta}{\alpha}} + \sqrt{\dfrac{q}{p}} = 0$

7. Solution:

Let, α be the one root of $ax^2 + bx + c = 0$ then other root be α^2.

$\alpha + \alpha^2 = \dfrac{-b}{a}$ .... (i)

or, $\alpha \cdot \alpha^2 = \dfrac{c}{a}$

or, $\alpha^3 = \dfrac{c}{a}$ .... (ii)

Cubing on both side of equation (i)

$(\alpha + \alpha^2)^3 = \left(\frac{-b}{a}\right)^3$

or, $\alpha^3 + (\alpha^2)^3 + 3\alpha . \alpha^2 (\alpha + \alpha^2) = \frac{-b^3}{a^3}$

or, $\alpha^3 + (\alpha^3)^2 + 3\alpha^3 (\alpha + \alpha^2) = \frac{-b^3}{a^3}$

or, $\frac{c}{a} + \frac{c^2}{a^2} \frac{-3bc}{a^2} = \frac{-b^3}{a^3}$

Multiplying each term by a^3

Then,

$a^2c + ac^2 - 3abc = -b^3$

or, $b^3 + a^2c + ac^2 = 3abc$ proved.

8. Solution:

Let, α and β be the two roots of $x^2 + px + q = 0$ $\alpha + \beta = -p$, $\alpha\beta = q$

a. The roots of required equation are

$\alpha\beta - 1$ and $\beta\alpha - 1$

Sum of roots = $\alpha\beta - 1 + \beta\alpha - 1$

$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{p^2 - 2q}{q}$

Product of roots = $\alpha\beta - 1 \times \beta\alpha - 1$

$= \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$

The required equation is

x^2 – (sum of roots) . x + product of roots = 0

or, $x^2 - \frac{(p^2 - 2q)}{q} . x + 1 = 0$

or, $qx^2 - (p^2 - 2q) x + q = 0$

b. Here,

$(\alpha - \beta)^2$ and $(\alpha + \beta)^2$ are the roots of required equation,

Sum of roots = $(\alpha - \beta)^2 + (\alpha + \beta)^2$

$= (\alpha + \beta)^2 - 4\alpha\beta + (\alpha + \beta)^2$

$= 2(\alpha + \beta)^2 - 4\alpha\beta$

$= 2p^2 - 4q$

Product of roots = $(\alpha - \beta)^2 . (\alpha + \beta)^2$

$= \{(\alpha + \beta)^2 - 4\alpha\beta\} (\alpha + \beta)^2$

$= (p^2 - 4q) . p^2$

The required equation is

x^2 – (Sum of roots) x + product of roots = 0

or, $x^2 - (2p^2 - 4q) . x + (p^4 - 4p^2q) . (2p^2 - 4q) = 0$

or, $x^2 - (2p^2 - 4q) . x + p^2 (p^2 - 4q) = 0$

or, $x^2 - 2(p^2 - 2q) . x + p^2 (p^2 - 4q) = 0$

c. $\alpha^2\beta - 1$ and $\beta^2\alpha - 1$ be the two roots of required equation,

Sum of roots = $\alpha^2\beta - 1 + \beta^2\alpha - 1$

$= \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

$= \frac{\alpha^3 + \beta^3}{\alpha\beta}$

$= \frac{(\alpha + \beta) (\alpha^2 + \beta^2 - \alpha\beta)}{\alpha\beta}$

$= \frac{-p \{(\alpha + \beta)^2 - 3\alpha\beta}{q}$

$= \frac{-p\{p^2 - 3q\}}{q}$

$= \frac{-p^3 + 3pq}{q}$

Product of roots $= \frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha} = \frac{(\alpha\beta)^2}{\alpha\beta} = \alpha\beta = q$

The required equation is,

x^2 – (sum of roots) x + product of roots = 0

or, $x^2 + \frac{(p3 - 3pa) - x}{q} + q = 0$

or, $qx^2 - (p^3 - 3pq)x + q^2 = 0$

9. Solution:

a. Let, the other root be α then,

$\alpha \,.\, 3 = \text{Product of roots} = \frac{-15}{2}$

or, $\alpha = -\frac{5}{2}$

Sum of roots $= \frac{-k}{2}$

or, $\alpha + 3 = \frac{-k}{2}$

or, $\frac{-k}{2} + 3 = \frac{-k}{2}$

or, $\frac{-5 + 6}{2} = \frac{-k}{2}$

or, $k = -1$

$\therefore$ $k = -1$

b. Given, equation is $3x^2 + kx - 2 = 0$

Sum of roots $= \frac{-k}{3}$

or, $6 = \frac{-k}{3}$

$\therefore$ $k = -18$

c. Given, equation is $2x^2 + (4 - k).\, x - 17 = 0$

If one root = α then other root = $-\alpha$

So that sum of the roots = 0

Sum of roots $= -\frac{4 - k}{2}$

or, $0 = -\frac{4 - k}{2}$

or, $0 = -4 - k$

$\therefore$ $k = 4$

d. Let, one root = α

Another root $= \frac{1}{\alpha}$

Product of roots $= \frac{c}{a}$

or, $\alpha \,.\, \frac{1}{\alpha} = \frac{-21}{7k}$

or, $7k = -21$

$\therefore$ $k = -3$

e. Let, α and α^2 be the two roots of $x^2 - kx + 1 = 0$

Sum of roots = $\frac{k}{1}$

or, $\alpha + \alpha^2 = k$

Products of roots = $\frac{1}{1}$

or, $\alpha . \alpha^2 = 1$

or, $\alpha^3 = 1$, or, $\alpha = 1$, i.e. $1 \times 1^2 = k \Rightarrow k = 2$

10. Solution:

Here, if α and β be the two roots of equations, then,

$\alpha + \beta = 1$, and $\alpha^2 + \beta^2 = 13$

or, $(\alpha + \beta)^2 - 2\alpha\beta = 13$

or, $1 - 2\alpha\beta = 13$

or, $2\alpha\beta = -12$

or, $\alpha\beta = -6$

Now,

Sum of roots = $\alpha + \beta = 1$

Products of roots = -6

The required equation is

x^2 – (sum of roots) . x + product of roots = 0

or, $x^2 - 1.x - 6 = 0$

$\therefore$ $x^2 - x - 6 = 0$

11. Solution:

Let, α and β be the equation of $x^2 + px + q = 0$

$\alpha + \beta = -p$

$\alpha\beta = q$

If the roots of $x^2 + lx + m = 0$ are in same ratio. Let $k\alpha$ and $k\beta$ be the roots of $x^2 + \ell x + m =$ 0

Then,

$k\alpha + k\beta = -\ell, \Rightarrow k = \frac{-\ell}{-p} = \frac{\ell}{p}$

$k\alpha . k\beta = m$

or, $k^2 = \frac{m}{q}$

Now, $\frac{\ell^2}{p^2} = \frac{m}{q}$

or, $p^2m = \ell^2q$

$\therefore$ $p^2m = \ell^2q$ proved.

12. Solution:

Let, α and β be the roots of $\ell x^2 + mx + n = 0$

Then, $\alpha + \beta = \frac{-m}{\ell}$

$\alpha\beta = \frac{n}{\ell}$

Again, Let, α' and β' be the roots of $\ell_1x^2 + m_1x + n_1 = 0$

Then,

$\alpha'\beta' = \frac{-m_1}{\ell_1}$ $\qquad$ $\alpha'\beta' = \frac{n_1}{\ell_1}$

By the question,

$\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}$

By componendo and dividendo,

$$\frac{\alpha + \beta}{\alpha - \beta} = \frac{\alpha' + \beta'}{\alpha' - \beta'}$$

or, $\dfrac{(\alpha + \beta)^2}{(\alpha - \beta)^2} = \dfrac{(\alpha' + \beta')^2}{(\alpha' - \beta')^2}$

or, $\dfrac{(\alpha + \beta)^2}{(\alpha + \beta)^2 - (\alpha - \beta)^2} = \dfrac{(\alpha' + \beta')^2}{(\alpha' + \beta')^2 - (\alpha' - \beta')^2}$

or, $\dfrac{(\alpha + \beta)^2}{4\alpha\beta} = \dfrac{(\alpha' + \beta')^2}{4\alpha'\beta'}$

or, $\dfrac{\left(\dfrac{-m}{\ell}\right)^2}{4\dfrac{n}{\ell}} = \dfrac{\left(\dfrac{-m_1}{\ell_1}\right)^2}{\dfrac{n_1}{\ell_1}}$

or, $\dfrac{m^2}{4\ell n} = \dfrac{m1^2}{4\ell_1 n_1}$

or, $\dfrac{m^2}{ln} = \dfrac{m\ell^2}{\ell_1 n_1}$, or, $\dfrac{m^2}{m\ell^2} = \dfrac{\ell n}{\ell_1 n_1}$ proved.

Exercise 5.3

1. Solution:

a. Given, equations are

$2x^2 + x - 3 = 0$ and $3x^2 - 4x + 1 = 0$

Writing the coefficients of order and repeating the first one.

$$\begin{array}{cccc} 2 & 1 & 3 & 2 \\ 3 & -4 & 1 & 3 \end{array}$$

The left hand expression of the condition

$(2 \times (-4) - 3\times1) . (1\times1 - (-4) \times (-3))$

$= (-8 - 3) . (1 - 12)$

$= -11 -11 = 121$

The right hand expression of the condition,

$\{(-3\times3) - 1\times2\}^2 = (-9 - 2)$

$= (-11)^2 = 121$

Since, two results are equal, they have common root.

b. Here, given equations are

$3x^2 - 8x + 4 = 0$ and $4x^2 - 7x - 2 = 0$

Writing the coefficients of order and repeating the first one

$$\begin{array}{cccc} 3 & -8 & 4 & 3 \\ 4 & -7 & -2 & 4 \end{array}$$

The left hand expression of the condition,

$= (3\times(-7) - 4\times(-8) . (-8) . (-2) - (-7) \times 4$

$= (-21 + 32) . (16 + 28)$

$= 11.44 = 484$

The right hand expression of the condition

$(4\times4 - (-2) \times 3)^2$

$= (16 + 6)^2 = (22)^2 = 484$

Since, two results are equal, they have common root.

2. Solution:

Here, given equations are

$3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$

Writing the coefficients of order and repeating the first one

$$\begin{array}{cccc} 3 & 4m & 2 & 3 \\ 2 & 3 & -2 & 2 \end{array}$$

The left hand expression of the condition,

$= (3\times3 - 2\times4m) . (4m . (-2)) - (3\times2)$

$= (9 - 8m) . (-8m - 6)$

$= -72m - 54 + 64m^2 + 48m$

$= -24m + 64m^2 - 54$

The right hand expression of the condition,

$(2\times2 - (-2) \times 3)^2 = (4 + 6)^2 = 100$

$\therefore \quad 64m^2 - 24m - 54 = 100$

or, $\quad 64m^2 - 24m - 154 = 0$

or, $\quad 32m^2 - 12m - 77 = 0$ ... ... (i)

or, Comparing equation (i) with $ax^2 + bx + c = 0$

$\therefore \quad a = 32,\ b = -12,\ c = -77$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{12 \pm \sqrt{144 - 4 \times 32 \times (-77)}}{2\times32}$$

$$= \frac{12 \pm 100}{64}$$

Taking +ve,

$$x = \frac{12 + 100}{64} = \frac{7}{4}$$

Taking –ve,

$$x = \frac{12 - 100}{64} = \frac{-11}{8}$$

Here, x is the value of m

So, $m = \frac{7}{4}$ and $\frac{-11}{8}$

3. Solution:

a. Here, given equation are

$4x^2 + px - 12 = 0$ and $4x^2 + 3px - 4 = 0$

Writing the coefficients of order and repeating the first one.

$$\begin{array}{cccc} 4 & p & -12 & 4 \\ 4 & 3p & -4 & 4 \end{array}$$

The left hand expression of the condition,

$= (4 \times 3p - 4p) . (-4p + 36p)$

$= (12p - 4p) . (32p)$

$= 8p . 32p = 256p^2$

The right hand expression of the condition,

$= (-12\times4 - (-4) \times 4)^2$

$= (-48 + 16)^2 = (32)^2 = 1024$

Now, $256p^2 = 1024$

or, $\quad p^2 = 4$

$\therefore \quad p = \pm2$

b. Here,

Given equations are

$2x^2 + px - 1 = 0$ and $3x^2 - 2x - 5 = 0$

Writing the coefficients of order and repeating the first one,

$$\begin{array}{cccc} 2 & p & -1 & 2 \\ 3 & -2 & -5 & 3 \end{array}$$

The left hand expression of the condition,

$= (2\times(-2) - 3p) . (-5p - 2)$

$= (-4 - 3p) . (-5p - 2)$

$= 20p + 8 + 15p^2 + 6p$

$= 26p + 8 + 15p^2$

The right hand expression of the condition,

$= ((-1) \times 3 - (-5) \times 2)^2$

$= (-3 + 10)^2 = 49$

Now,

$15p^2 + 26p + 8 = 49$

or, $15p^2 + 26p - 41 = 0$

or, $15p^2 + 41p - 15p - 41 = 0$

or, $p(15p + 41) - 1p(15p + 41) = 0$

or, $(15p + 41)(p - 1) = 0$

either,

$p = 1$,

or, $p = \frac{-41}{15}$

4. Solution:

Let, α be the common root of the given equations,

$\alpha^2 + p\alpha + q = 0$

$\alpha^2 + p'\alpha + q' = 0$

By using cross multiplication method;

$$\frac{\alpha^2}{pq' - qp'} = \frac{\alpha}{q - q'} = \frac{1}{p' - p}$$

$\therefore \quad \alpha = \frac{pq' - p'q}{q - q'}$, $\alpha = \frac{q - q'}{p' - p}$

$\therefore$ The common root is $\frac{pq' - p'q}{q - q'}$ or $\frac{q - q'}{p' - p}$

5. Solution:

Let, α be the common root of the given equations,

$\alpha^2 + q\alpha + pr = 0$

$\alpha^2 + r\alpha + pq = 0$

By the rule of cross–multiplication method'

$$\frac{\alpha^2}{pq^2 - pr^2} = \frac{\alpha}{pr - pq} = \frac{1}{r - q}$$

$\alpha = \frac{pq^2 - pr^2}{pr - pq}$,

$= \frac{p(q - r)(q + r)}{p(r - q)}$

$= -q - r$

$\alpha = \frac{p(r - q)}{(r - q)}$

$\alpha = p$

Now,

$-q - r = p$

or, $p = -q - r$

or, $p + q + r = 0$

$\therefore \quad p + q + r = 0$

6. Solution:

Let, α be the common root of the given equations,

$\alpha^2 + b\alpha + ca = 0$

$\alpha^2 + c\alpha + ab = 0$

By the rule of cross multiplication method,

$$\frac{\alpha^2}{ab^2 - ac^2} = \frac{\alpha}{ca - ab} = \frac{1}{c - b}$$

or, $\frac{\alpha^2}{a(b + c)(b - c)} = \frac{\alpha}{-a(b - c)} = \frac{1}{-(b - c)}$

$\therefore \quad \alpha = \frac{-a(b - c)}{-(b - c)}$

Also, $\alpha = \frac{a(b + c)(b - c)}{-a(b - c)} = -(b + c)$

$\therefore \quad a = -(b + c)$

or, $a + b + c = 0$

If β be the other roots of $x^2 + bx + ca = 0$, then $\alpha\beta = \frac{ca}{1}$.

or, $a\beta = ca$

$\therefore$ $\beta = c$

Again,

If γ be the other root of $x^2 + cx + ab = 0$, then $\alpha . \gamma = \frac{ab}{1} = ab$

or, $a . \gamma = \frac{1}{ab}$

$\therefore$ $\gamma = b$

The quadratic equation whose roots are β and γ is

$x2 - (\beta + \gamma) x + \beta\gamma = 0$

or, $x^2 - (c + b) . x + cb = 0$

or, $x^2 - (-a) . x + bc = 0$ $[\because a + b + c = 0]$

or, $x^2 + ax + bc = 0$

$\therefore$ $x^2 + ax + bc = 0$

7. Solution:

Let, α be the common root of the equation,

$\alpha^2 + 2b\alpha + c = 0$

$a\alpha^2 + 2c\alpha + b = 0$

By the rule of cross multiplication method,

$$\frac{\alpha 2}{2b^2 - 2c^2} = \frac{\alpha}{ac - ab} = \frac{1}{2ac - 2ab}$$

$\alpha = \frac{2(b - c)(b + c)}{a(c - b)}$, $\alpha = \frac{a(c - b)}{2a(c - b)}$

$= \frac{2(-b - c)}{a \times 1}$ $\alpha = \frac{1}{2}$

$= \frac{2(-b - c)}{a}$

Now, $\frac{-2b - 2c}{a} = \frac{1}{2}$

or, $-4b - 4c = a$

or, $a = -4b - 4c$

or, $a + 4b + 4c = 0$

8. Solution:

Here, α be the common roots of the given equations, then

$\alpha^2 + p\alpha + q = 0$

$\alpha^2 + q\alpha + p = 0$

By the rule of cross multiplication,

$$\frac{\alpha^2}{p^2 - q^2} = \frac{\alpha}{q - p} = \frac{1}{q - p}$$

$\alpha = \frac{q - p}{q - p}$ and $\alpha = \frac{p^2 - q^2}{q - p}$

$\therefore$ $\frac{p^2 - q^2}{p - p} = \frac{q - p}{q - p}$

or, $p^2 - q^2 = -(p - q)$

or, $(p + q)(p - q) + (p - q) = 0$

or, $(p - q)(p + q + 1) = 0$

either,

$p - q = 0 \therefore p = q$

$p + q + 1 = 0$ P