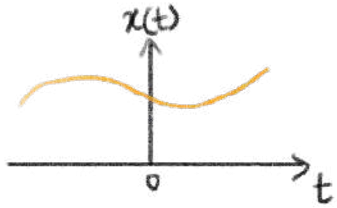
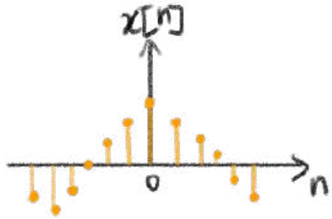
- Signal & system

Signal

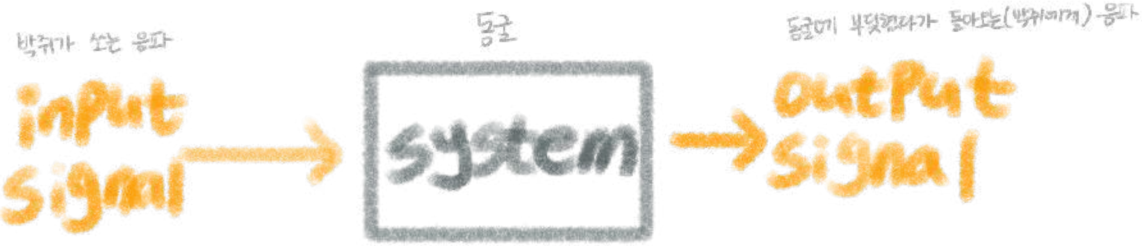
- continuous
  - analog signal
  - 독립 변수 : $t \rightarrow x(t)$



- discrete
  - digital signal
  - 독립 변수 : $n \rightarrow x[n]$



system

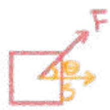


- connection
  - 종속 연결 : 각 부시스템 직렬 연결
  - 병렬 연결 : 각 부시스템 병렬 연결
  - 궤환 연결 : 시스템 출력이 다시 입력으로(feedback)

- Energy & power

$t_1 \le t \le t_2$일 때, t 구간에서의 total energy

$$\boxed{\int_{t_2}^{t_1} |x(t)|^2 dt}$$

- energy : 물체에 '일'을 했을 때 발생하는 물리량(에너지 보존)
- 일·운동 에너지 정리 : 일의 양만큼 에너지가 변함
- energy 단위 : $[J] = [\underbrace{N}_{\text{힘}} \cdot \underbrace{m}_{\text{거리}}]$
- 일 : 물체 이동 방향으로의 힘 × 물체 이동 거리  $\rightarrow \underbrace{F}_{F=m\cdot a} \cdot \cos\theta$
- 일률(Power) : 일의 미분 [J/sec] = [W]

E

continuous time Energy

$$E = \int_{t_2}^{t_1} |x(t)|^2 dt \;,\quad E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

discrete time Energy

$$E = \sum_{n_1}^{n_2} |x[n]|^2 \;,\quad E_\infty = \lim_{n\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

P

- $E_\infty < \infty \Rightarrow P_\infty = 0 \left(P_\infty = \lim_{T\to\infty} \frac{E_\infty (<\infty)}{2T (=\infty)} = 0 \right)$
- $P_\infty < \infty \Rightarrow E_\infty = \infty \left(P_\infty = \lim_{T\to\infty} \frac{E_\infty (=\infty)}{2T (=\infty)} < \infty \right)$

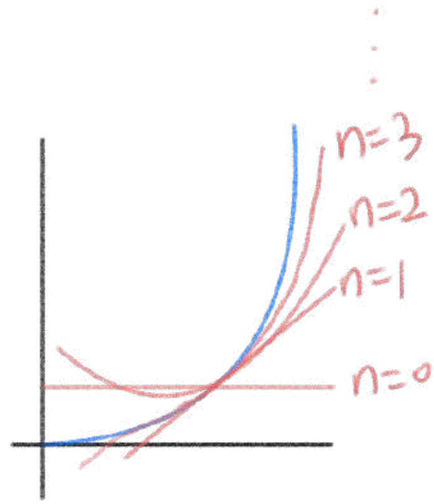
오일러 공식

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

증명

다항 함수의 합으로 어떤 함수를 표현하는 매클로린

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots \rightarrow$$



$$\downarrow$$

$$f(0) = c_0$$

$$f'(0) = c_1$$

$$f''(0) = 2c_2$$

$$f'''(0) = 3\cdot 2c_3$$

$$\vdots$$

$$f^n(0) = n!\,c_n$$

$$\boxed{\begin{aligned} \cos\theta &= 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \cdots \\ \sin\theta &= \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots \end{aligned}}$$

$e^{j\theta}$ 미분

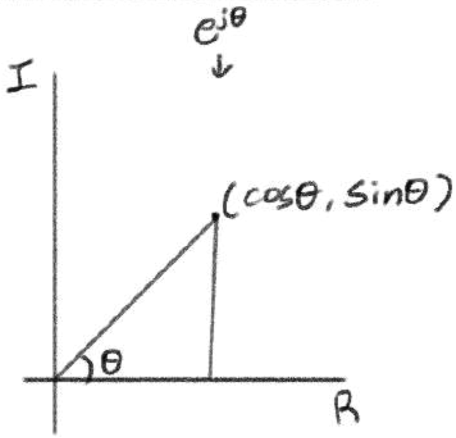
$$\boxed{e^{j\theta} \rightarrow je^{j\theta} \rightarrow -e^{j\theta} \rightarrow -je^{j\theta} \cdots}$$

$$\therefore e^{j\theta} = 1 + j\theta - \frac{1}{2!}\theta^2 - j\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + j\frac{1}{5!}\theta^5 + \cdots$$

$$\downarrow$$

$$\cos\theta + j\sin\theta$$

오일러 공식 좌표 표현



$$2e^{j\theta} = 2\cos\theta + 2j\sin\theta$$

$\downarrow$

$A \times e^{j\theta}$ (A는 양수, θ는 $0 \sim 2\pi$)

↳ 2D 평면의 모든 수 표현 가능

ex1) $-1 = 1 \times e^{j\pi}$

ex2) $z_1 = a_1 + b_1 j = A_1 e^{j\theta_1}$

$z_2 = a_2 + b_2 j = A_2 e^{j\theta_2}$ $\rightarrow$ $z_1 z_2 z_3 = A_1 A_2 A_3 e^{j(\theta_1+\theta_2+\theta_3)}$. 곱셈이 간단해진다.

$z_3 = a_3 + b_3 j = A_3 e^{j\theta_3}$

- Signals

주기 신호

임의의 t에 대해 $x(t)=x(t+T)$ 을 만족하는 양수 T or N이 존재 (N은 정수)
$x[n]=x[n+N]$

- fundamental period : T/N 주기 중 0보다 큰 가장 작은 값

even & odd signal

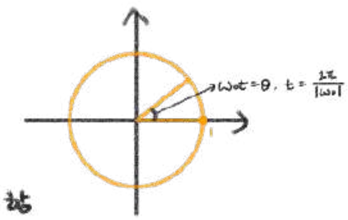
- even : $x(-t)=x(t)$, $x[-n]=x[n]$
- odd : $x(-t)=-x(t)$, $x[-n]=-x[n]$
- even-odd decomposition : 임의의 신호 $x(t)$는 even과 odd의 합으로 표현 가능
  ↳ $x(t)=x_{even}(t)+x_{odd}(t)$

exponential Signal (in CT)

$$x(t)=Ce^{at}$$

- a, C가 실수

- a가 허수 ($a=j\omega_0$) — 각속도
  - $x(t)=e^{j\omega_0 t}=\underbrace{\cos\omega_0 t+j\sin\omega_0 t}_{\text{오일러 공식 (원의 정의)}}$
  - 주기 : $e^{j\omega_0(t+T)}=e^{j\omega_0 t}\cdot \underbrace{e^{j\omega_0 T}}_{}=e^{j\omega_0 t}$ 여야 함
    ↳ 1이 되야 함



$T_0=\frac{2\pi}{|\omega_0|}$ 이므로, $e^{j\omega_0 T}=e^{j\cancel{\omega_0}\frac{2\pi}{\cancel{\omega_0}}}=1$ ∴ $x(t)$는 주기 함수

— a, C가 복소수

• $C = |C|e^{j\theta}$, $a = r + j\omega_0$

$\rightsquigarrow Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = \underbrace{\overset{\text{실수}}{\boxed{|C|e^{rt}}}}_{\text{신호의 크기 결정}} e^{j(\omega_0 t \boxed{+\theta})}$ → 시작점이 θ로 이동

$= |C|e^{rt} \cdot \cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$

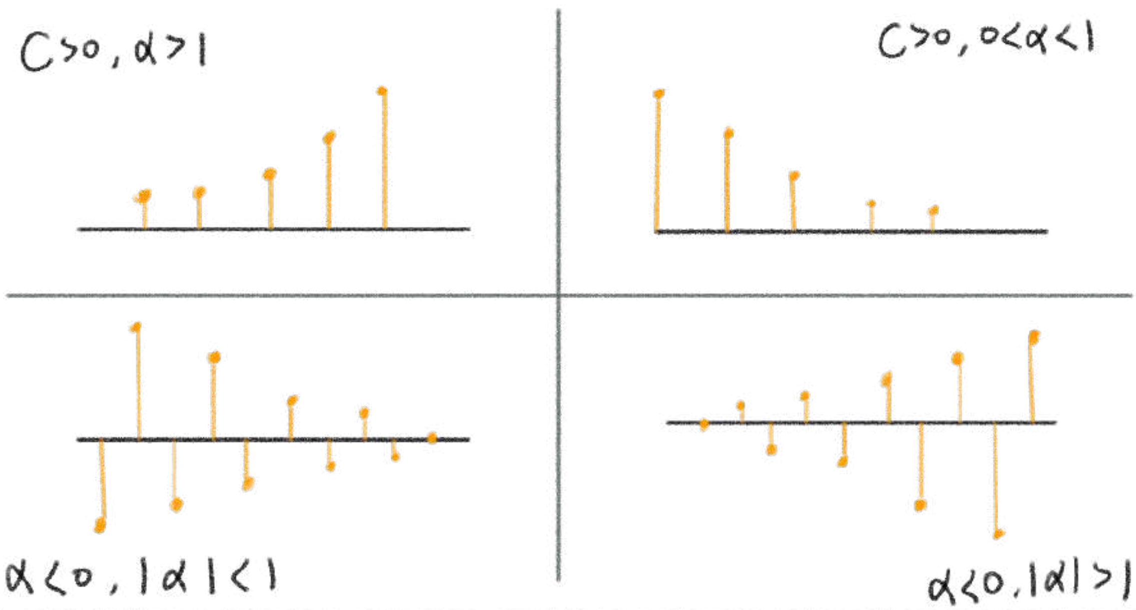
* E_{period}, P_{period}

$$E_{period} = \int_0^{T_0} \left| e^{j\omega_0 t} \right|^2 dt = \int_0^{T_0} dt = 1$$

$$P_{period} = \frac{1}{T_0} E_{period} = 1$$

exponential Signal (in DT)

$x[n] = C \cdot \alpha^n$ ($C \cdot e^{an}$은 α^n보다 표현력↓)



- 주기 (fundamental period)

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}\underbrace{e^{j\omega_0 N}}_{1} = e^{j\omega_0 n}$$

$$\therefore \omega_0 N = 2\pi m$$

$$N = \frac{2\pi}{\omega_0} m$$

⇒ N이 정수가 되는 최솟값 m

- 주기와 주파수 관계

ⓐ $\frac{2\pi}{12} n = 2\pi m$, $n = 12m \rightarrow N_0 = 12$

ⓑ $\frac{8\pi}{31} n = 2\pi m$, $n = \frac{31}{4} m \rightarrow N_0 = 31$ ($\frac{31}{4}$은 정수가 아니므로, $m = 4$)

주파수 : a < b , 주기 : a < b (CT는 반대)

- DT의 주기적 특징

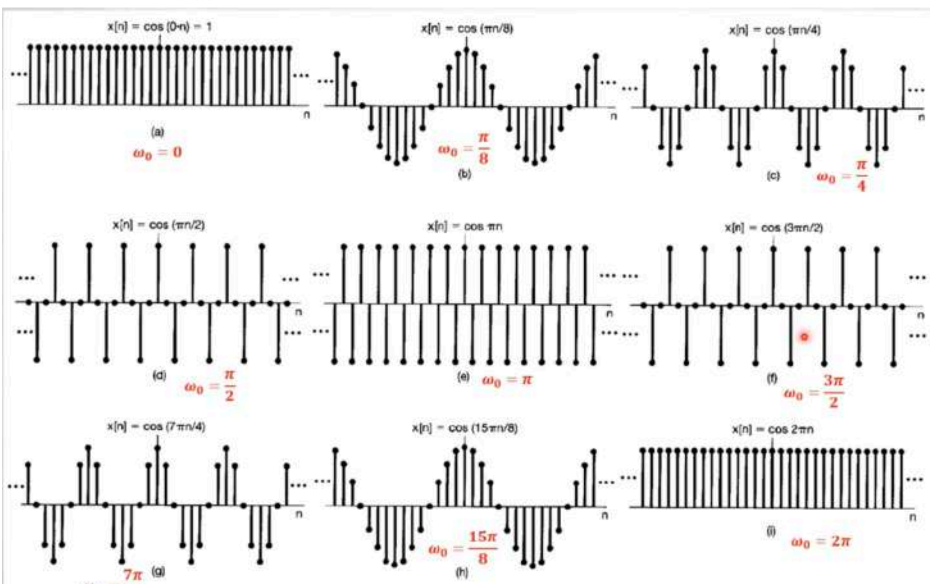


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

↳ π를 기준으로 신호가 반복되므로 0 ~ π까지만 물리적 의미를 가짐

_system properties(LTI 제외)

- memoryless : 출력이 동일(현재 시간에만 관계)

  ① $y[n] = (2x[n] - x^2[n])^2$

  ② $y(t) = Bx(t)$

- Invertible : 출력이 다시 입력으로 돌아올 수 있는 시스템

  $x(t) \rightarrow \boxed{y(t)=2x(t)} \rightarrow y(t) \rightarrow \boxed{w(t)=\frac{1}{2}y(t)} \rightarrow w(t) = x(t)$

- Causal : 현재를 포함하여 과거의 입력에 관계돼 출력되는 것

  ① $y(t) = x(3t)$ 3초에서의 입력은 1초의 출력에 영향을 줄 수 없음.

  $\quad \underset{t=1}{\rightarrow}\ y(1) = x(3)$ $\quad \therefore$ non-causal

  ② $y(t) = x(t)\cos(t+1)$ cos(2)는 그냥 '값'임

  $\quad \underset{t=1}{\rightarrow}\ y(1) = x(1)\cos(2)$ $\quad \therefore$ causal

  ③ $y(t) = x(-t)$

  $\quad \underset{t=1}{\rightarrow}\ y(1) = x(-1)$ ~> causal인가?

  $\quad \underset{t=-1}{\rightarrow}\ y(-1) = x(1)$ $\quad \therefore$ non-causal

- BIBO (Bounded Input Bounded Output) Stable

  $|x(t)| < \infty \rightarrow \boxed{\text{system}} \rightarrow |y(t)| < \infty$

  조건.
  $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ or $\int_{-\infty}^{\infty} |h(t)| < \infty$ 면 성립

  ex) $y(t) = e^{x(t)}$

  $|x(t)| < B < \infty \rightarrow -B < x(t) < B$

  $\rightarrow e^{-B} < |y(t)| < e^{B}$

  $\therefore |y(t)| = |e^{x(t)}| < B < \infty \Rightarrow$ BIBO

- LTI (Linear Time Invarient)

- Linear = Scaling + additivity
  = superposition

- Scale : input이 A배이면 output도 A배인 system

- additivity : $x_1(t) \rightarrow \boxed{\text{system}} \rightarrow y_1(t)$, $x_2(t) \rightarrow \boxed{\text{system}} \rightarrow y_2(t)$ 일 때, $x_1(t) + x_2(t) \rightarrow \boxed{\text{system}} \rightarrow y_1(t) + y_2(t)$ 를 만족하는 system

---

- Time - Invarient : input이 time-shift 되면, output도 time-shift 되는 시스템
  ↳ input이 1초, 2초, 3초일 때 언제나 output이 같다는 이야기 X ($x(t-t_0) \rightarrow y(t)$)
  ↳ $x(t-t_0) \rightarrow y(t-t_0)$

ex) $y(t) = x(-2t+2)$는 TI인가?

먼저, $x(t)$ 변화 생각해보기

$$x(t) \xrightarrow[\text{대칭}]{} x(-t) \xrightarrow[\text{2배로 shrink}]{} x(-2t) \xrightarrow[\text{1 이동}]{} x(-2(t-1)) = x(-2t+2)$$

$$x_1(t) \longrightarrow \boxed{\text{system}} \longrightarrow x_1(-2t+2) \text{을 } y_1(t)$$

$$x_1(t-t_0) \rightarrow x_1(-t-t_0) \rightarrow x_1(-2t-t_0) \rightarrow x_1(-2(t-1)-t_0)$$
$$= x_1(-2t-2-t_0)$$

$$y_1(t-t_0) = x_1(-2(t-t_0)+2) = x_1(-2t+2t_0+2)$$

∴ TI 아님

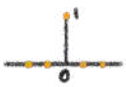
> 결론 : LTI는 예측 가능함을 다루는 것이다

– 컨볼루션 (convolution)

convolution impulse response

$$\boxed{y(t) = f(x(t), h(t)) = x(t) * h(t)}$$

- Impulse function
  - discrete domain에서

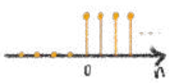


$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n=\text{o.w.} \end{cases}$ 조건 $\left(\begin{array}{l} \sum_{n=-\infty}^{\infty} \delta[n] = 1 \\ \sum_{n=-\infty}^{\infty} f[n]\delta[n] = f(0) \end{array} \right.$ 을 만족해야 함

  - continuous domain에서

$\delta(n) = \begin{cases} \infty & t=0 \\ 0 & t=\text{o.w.} \end{cases}$ 조건 $\left(\begin{array}{l} \int_{-\infty}^{\infty} \delta(t) = 1 \\ \int_{-\infty}^{\infty} f(t)\delta(t) = f(0) \end{array} \right.$ 을 만족해야 함

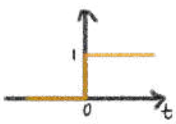
- unit step



$u[n] = \begin{cases} 0, & n<0 \\ 1, & n\geq 0 \end{cases} \rightarrow$

$\delta[n] = u[n] - u[n-1]$

$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$



$u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases} \rightarrow$

$\delta(t) = \dfrac{du(t)}{dt} = \begin{cases} \infty, & t=0 \\ 0, & t\neq 0 \end{cases}$

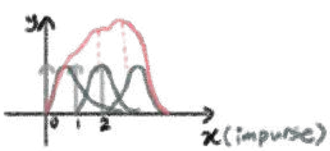
$u(t) = \int_0^{\infty} \delta(t-\sigma)\,d\sigma$

- Impulse response

$x_1(t) = \delta(t) \rightarrow \boxed{\text{system}} \rightarrow y_1(t) = h(t)$ → $h(t)$가 있으면,

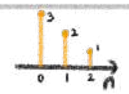
임의의 $x(t)$에 대한 output $y(t)$를 알 수 있다

- Convolution



- discrete time Convolution

$x[n]$:



$\delta[n]$:

$x[n] = 3\delta[n] + 2\delta[n-1] + \delta[n]$ (like taylor)

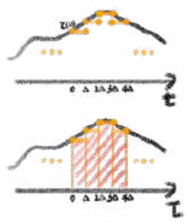
system

$y[n] = 3h[n] + 2h[n-1] + h[n]$

$\therefore\ y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

- Continuous time Convolution

$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$



$$x(t) = \lim_{\Delta \to 0} \cdots + x(0)\delta_\Delta(t)\Delta + x(\Delta)\delta_\Delta(t-\Delta)\Delta + x(2\Delta)\delta_\Delta(t-2\Delta)\Delta + \cdots$$

$$= \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad \left(\lim_{\Delta \to 0} k\Delta = \tau\right)$$

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- τ-domain

τ 도메인 변화

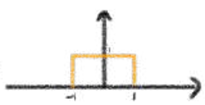
① $x(t) \to x(\tau)$

② $h(t) \to h(\tau)$

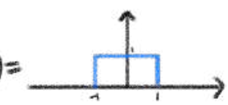
③ $h(\tau) \to h(-\tau)$

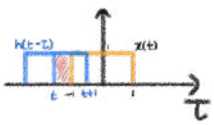
④ $h(-(\tau-t)) = h(t-\tau)$

ex) if $x(t) =$

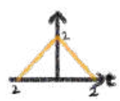


, $h(t) =$

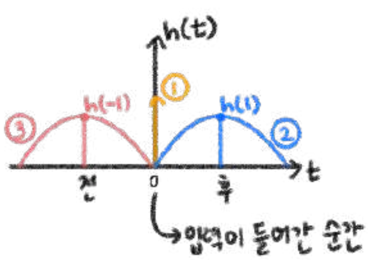




$$\rightarrow y(t) = \begin{cases} t+2 & t > -2 \\ 2 & t = 0 \\ 2-t & 2 > t > 0 \end{cases} \rightarrow$$



- system determination

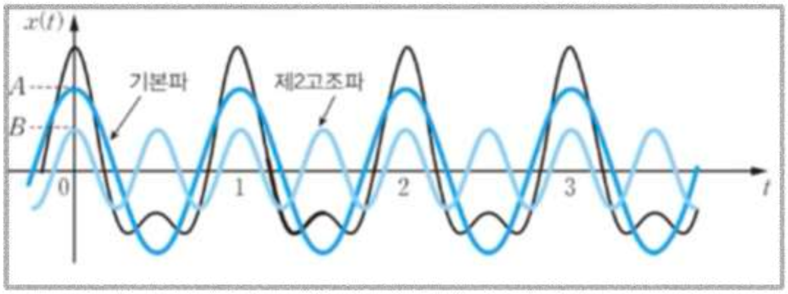


— memoryless : ①

— causal : ①, ②

- Signal & frequency

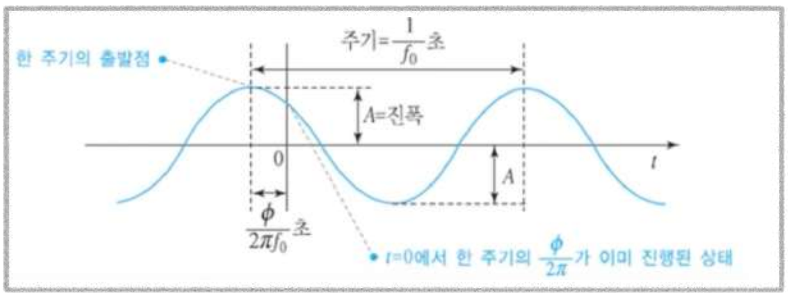
• 고조파 : 기본 주파수와 그 정수배 주파수의 합으로 표현



- 신호는 기본 신호들로 나눌 수 있다.

• 정현파 : 진폭, 주파수, 위상에 대한 정보 표현

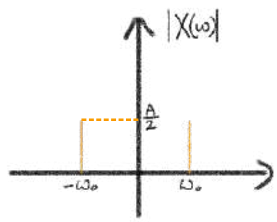
$$- \; x(t) = \underbrace{A}_{\text{진폭}}\cos(\omega_0 t + \underbrace{\varphi}_{\text{위상}}) = A\cos(2\pi \underbrace{f_0}_{\text{주파수}} t + \varphi)$$

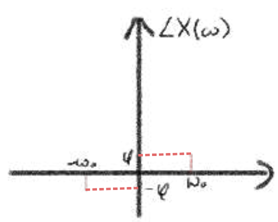


• 스펙트럼: 정현파를 주파수로 취급 → 주파수 축 상에 진폭과 위상을 나타냄

$$- \; x(t) = A\cos(\omega_0 t + \varphi) = A\cos(2\pi f_0 t + \varphi)$$

$$x(t) = \frac{A}{2}e^{j\varphi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\varphi}e^{-j\omega_0 t} = \frac{A}{2}e^{j\varphi}e^{j2\pi f_0 t} + \frac{A}{2}e^{-j\varphi}e^{-j2\pi f_0 t}$$





- fourier series

- 푸리에 급수 : 고조파를 basic으로 하는 신호의 선형 결합된 주기 신호

- $x(t) = e^{st}, e^{j\omega t}$ / Impulse response : $h(t)$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{j\omega t}e^{-j\omega\tau}d\tau$$

$$= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}dt$$

$\therefore$ fourier transform : $y(t) = H(j\omega)e^{j\omega t}$

- composite input

$$x(t) = a_1 e^{j1\omega t} + a_2 e^{j2\omega t} + \cdots$$

$$y(t) = a_1 H(j1\omega)e^{j1\omega t} + a_2 H(j2\omega)e^{j2\omega t} + \cdots$$

$\therefore x(t) = \sum_k a_k e^{jk\omega t}$, $y(t) = \sum_k a_k H(jk\omega)e^{j}$ ^{t}

k는 $-\infty \sim \infty$, 0이면 DC, $e^{j\omega t}$는 주어졌을 때 a_k를 구하는 것

- representation of CT FS

↳ 푸리에 급수에서 $x(t)$가 실수 → coefficient의 켤레 복수가 같다

표현형 ① $x(t) = x^*(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_{-k} e^{-jk\omega_0 t}$$

↳ $k \to -k$

$\therefore a_k^* = a_{-k}$면 됨

표현형 ②

표현형 ①에 의해,

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}]$$

$$= a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{a_k e^{jk\omega_0 t}\}$$

표현형 ③

$a_k = A_k e^{j\theta_k}$ 대입. 표현형 ②에 의해,

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{A_k e^{j(k\omega_0 t + \theta_k)}\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

표현형 ④

$a_k = B_k + jC_k$ 대입. 표현형 ②에 의해,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]$$

- 계수 a_k 구하기

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\times e^{-jn\omega_0 t}$ & $\int_0^T$

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t} dt \right.$$

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos\{(k-n)\omega_0 t\} dt$$

$$+ \cancel{\int_0^T \sin\{(k-n)\omega_0 t\} dt} \to 0$$

$$= \begin{cases} T & k=n \\ 0 & k \neq n \end{cases}$$

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = a_n T$$

$$\boxed{\therefore a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt}$$

— Convergence of FS

- truncation

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$ → N을 ∞로 보내자

- error approximation

$$e_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \neq 0$$

- minimize Energy of approximation error

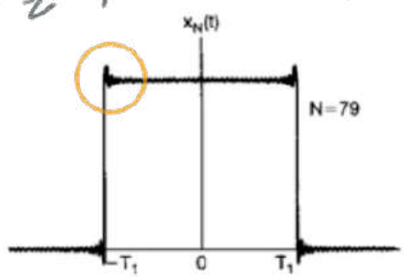
$$E_N = \int_T |e_N(t)|^2 dt$$

- same with FS coefficient

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

∴ FS 표현 가능 → 회선의 근사화 가능

— Gibbs 현상 : 불연속 지점에서 발생하는 error



-properties of CT FS

① linearity (주기가 T로 같을 때만)

$$x(t) \xleftrightarrow{fs} a_k \quad , \quad y(t) \xleftrightarrow{fs} b_k$$

$$\underbrace{z(t)}_{\text{↳주기 } T} = Ax(t) + By(t) \xleftrightarrow{fs} c_k = Aa_k + Bb_k$$

② time shifting

$$x(t) \xleftrightarrow{fs} \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

$$x(t-t_0), \quad b_k = \frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt$$

$$\tau = t - t_0, \quad b_k = \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau + t_0)} d\tau$$

$$= e^{-jk\omega_0 t_0} \boxed{\frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau}\; a_k$$

$$= e^{-jk\omega_0 t_0} a_k$$ (크기는 같고 위상 값만 달라짐)

③ time reversal

$$b_k = a_{-k}$$

④ time scaling

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

⑤ multiplication

$$x(t) \xleftrightarrow{fs} a_k \quad , \quad y(t) \xleftrightarrow{fs} b_k$$

$$x(t)y(t) \xleftrightarrow{fs} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$ (convolution)

⑥ Conjugation

$x(t) \xleftrightarrow{fs} a_k$, $x^*(t) \xleftrightarrow{fs} a^*_{-k}$

if $x(t)$ is real, $x(t) = x^*(t)$, $a_k = a^*_{-k}$, $|a_k| = |a^*_{-k}|$

⑦ Pasval's relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 dt$$

$$\frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = \underline{|a_k|^2}$$

↳ E of kth halmonic

- representation of DT FS

- $x[n] = \sum_{k=\langle N \rangle} a_k \varphi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$

- 계수 구하기

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$$

$$\times e^{-jr(\frac{2\pi}{N})n} \quad \& \quad \sum_{n=\langle N \rangle}$$

$$\sum_{n=\langle N \rangle} x[n] e^{-jr(\frac{2\pi}{N})n} = \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{j(k-r)(\frac{2\pi}{N})n}$$

$$\sum_{k=\langle N \rangle} e^{jk(\frac{2\pi}{N})n} = \begin{cases} N, & k=0, \pm N, \pm 2N \cdots \\ 0, & \text{o.w} \end{cases}$$

$$\therefore a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

* DT는 유한한 샘플을 다루기 때문에 수렴 조건 X

- properties of CT FS

↳ CT와 다른 점만

① multiplication

$$x[n]y[n] \xleftrightarrow{FS} d_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$$

② first difference (DT에서는 one-step shift)

$$a_k = \frac{1}{N}\sum_{n=\langle N \rangle} x[n]e^{-jk(\frac{2\pi}{N})n}$$

$$b_k = \frac{1}{N}\sum_{n=\langle N \rangle} x[n-1]e^{-jk(\frac{2\pi}{N})n} = \underbrace{\frac{1}{N}e^{-jk(\frac{2\pi}{N})} \sum_{n=\langle N \rangle} x[n-1]e^{-jk(\frac{2\pi}{N})(n-1)}}_{a_k}$$

$$= e^{-jk(\frac{2\pi}{N})}a_k$$

$$\therefore x[n]-x[n-1] \xleftrightarrow{FS} a_k(1-e^{-jk(\frac{2\pi}{N})})$$

③ Pasval's relation

$$\frac{1}{N}\sum_{n=\langle N \rangle}|x[n]|^2 = \sum_{k=\langle N \rangle}|a_k|^2$$

- frequency response

- $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$x[n] = e^{j\omega n}$, $h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} = e^{j\omega n} H(e^{j\omega})$$

Filtering

3.9.1 Frequency shaping filter

Weighting Frequency Response

Ex: Equalizer, Bass control, Treble control

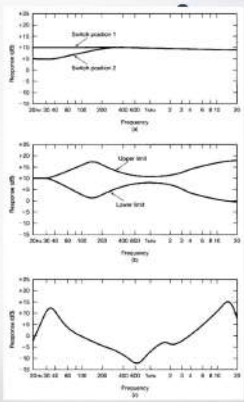
$H(j\omega)$ $H(e^{j\omega})$

$y[n] = x[n] * h[n]$

$= x[n] H(e^{j\omega})$

$b_k = a_k \cdot H(e^{j\omega})$

$x[n] \xrightarrow{FS} a_k$



Filtering

differentiator (미분)

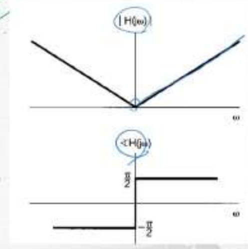
$$y(t) = \frac{dx(t)}{dt}$$

$$|H(j\omega)| = j\omega$$

$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$y(t) = jk\omega_0 \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$y(t) = H(j\omega) x(t)$



High-pass filter (HPF)

$$\cos \omega t \rightarrow -\omega \sin \omega t = \omega \cos(\omega t + \frac{\pi}{2})$$

$$\sin \omega t \rightarrow \omega \cos \omega t = \omega \sin(\omega t + \frac{\pi}{2})$$

Filtering

$h[n] \xrightarrow{FT} H(e^{j\omega})$

For Discrete-time systems

$d[n] - d[n-1] = h[n]$

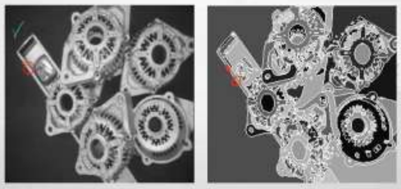
$$x[n] = \sin \omega n \quad y[n] = x[n] - x[n-1]$$

$$y[n] = \sin \omega n - \sin \omega (n-1) = 2\cos(\omega n - \frac{\omega}{2})\sin(\frac{\omega}{2})$$

$\omega \to \pi$

$$H(e^{j\omega}) = 1 - e^{-j\omega} = e^{-\frac{j\omega}{2}}(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}) = 2j\sin\left(\frac{\omega}{2}\right)e^{-\frac{j\omega}{2}}$$

Edge detection



Filtering

Discrete-time low-pass filter

$d[n] + d[n-1]$

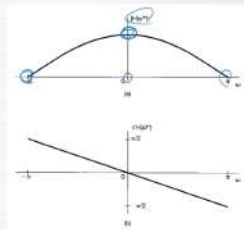
$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

In frequency domain

$$H(e^{j\omega}) = \frac{1}{2}[1 + e^{-j\omega}] = e^{-\frac{j\omega}{2}}\cos\left(\frac{\omega}{2}\right)$$

$\omega \to \pi$ 0

$\omega \to 0$ 1



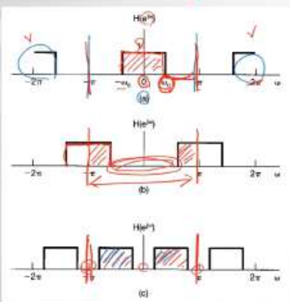
In time domain

$$y[n] = \frac{1}{2}(\sin \omega n + \sin \omega (n-1)) = \sin(\omega n - \omega/2)\cos(\omega/2)$$

Filtering

3.9.2 Frequency selective filters

ω_c : cut off



Lowpass

Highpass

Bandpass filters

Examples of discrete-time filters described by difference equations

3.11.1 First-order recursive discrete-time filters

$$y[n] = ay[n-1] + x[n] \qquad y[n] - ay[n-1] = x[n]$$

$e^{j\omega n}$

$$H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

$y[n] = e^{j\omega n} H(e^{j\omega})$

$$(1 - ae^{-j\omega})H(e^{j\omega})e^{j\omega n} = e^{j\omega n}$$

$$H(e^{j\omega}) = \boxed{\frac{1}{(1 - ae^{-j\omega})}}$$

+ impulse 응답을 이용한 풀이 방법

Examples of discrete-time filters described by difference equations

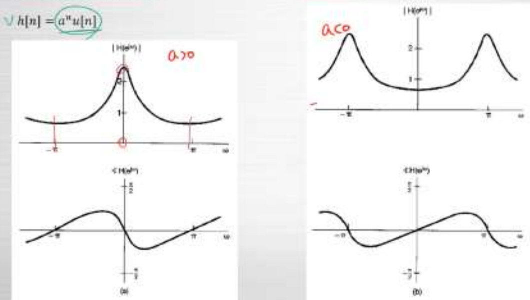
impulse 응답을 이용한 풀이 방법

$y[n] = ay[n-1] + x[n]$

$h[n] = ah[n-1] + \delta[n]$

√ $h[n] = a^n u[n]$

$$H(e^{j\omega}) = 1 + ae^{-j\omega} + a^2 e^{-j2\omega} + \cdots = \frac{1}{1 - ae^{-j\omega}}$$

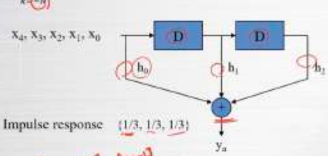


Examples of discrete-time filters described by difference equations

3.11.2 Nonrecursive discrete-time filters

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]$$

Moving Average



Impulse response {1/3, 1/3, 1/3}

√ $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$ $\quad H(e^{j\omega}) = \frac{1}{3}[e^{j\omega} + 1 + e^{-j\omega}] = \frac{1}{3}(1 + 2\cos\omega)$

√ $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$ $\quad H(e^{j\omega}) = \frac{1}{3}[1 + e^{-j\omega} + e^{-j2\omega}] = \frac{1}{3}e^{-j\omega}(e^{j\omega} + 1 + e^{-j\omega}) = \frac{1}{3}e^{-j\omega}(1 + 2\cos\omega)$

Examples of discrete-time filters described by difference equations

moving Av.

√ $$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]$$

$x[n] = e^{j\omega n}$

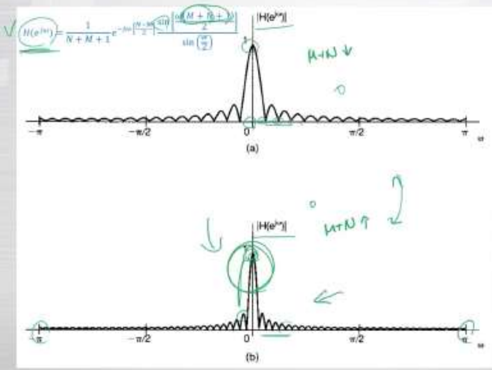
$e^{j\omega(n-k)} = e^{j\omega n} \cdot e^{j\omega(-k)}$

$H(e^{j\omega}) \cdot e^{j\omega n}$

$$H(e^{j\omega}) = \frac{1}{N+M+1}\sum_{k=-N}^{M} e^{-j\omega k}$$

$$H(e^{j\omega}) = \frac{1}{N+M+1} e^{-j\omega\left[\frac{N-M}{2}\right]} \frac{\sin\left[\frac{\omega(M+N+1)}{2}\right]}{\sin\left(\frac{w}{2}\right)}$$

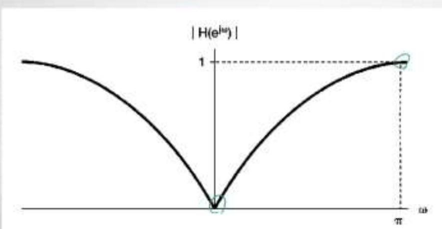
Examples of discrete-time filters described by difference equations



Examples of discrete-time filters described by difference equations

$$y[n] = \frac{x[n] - x[n-1]}{2}$$

$$H(e^{j\omega}) = \frac{1}{2}[1 - e^{-j\omega}] = je^{\frac{j\omega}{2}}\sin\left(\frac{\omega}{2}\right)$$



The Continuous-Time Fourier Transform

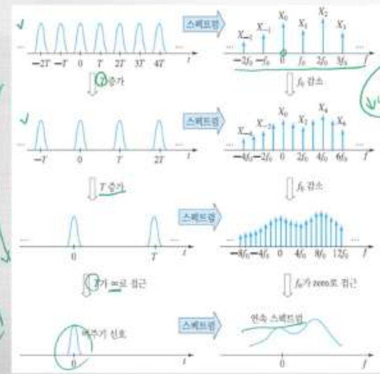
- 주파수 해석이란?
  - 신호에 포함되는 있는 여러 주파수 성분을 구하는 과정
- 코사인 신호를 주파수에 대한 기본 신호(basis signal)로 사용
  - 코사인 신호는 하나의 주파수로만으로 구성
  - 주파수 1Hz 신호 = 1Hz 코사인 신호
- 코사인 신호의 3가지 정보
  - 진폭, 주파수, 위상

$A\cos(\omega_0 t + \varphi)$

- 주기 신호(특별)의 주파수 성분 – 푸리에 급수를 통해 획득
- 비주기 신호(일반)의 주파수 성분 – 푸리에 변환을 통해 획득
- 비주기 신호를 주기 $T = \infty$ 인 주기 신호로 취급 → 푸리에 급수 확장
- 주기 T 가 커질수록 스펙트럼은 조밀해짐 → $T = \infty$ 이면 연속 파형이 됨
- 푸리에 표현은 급수 전개에서 적분 형태로 바뀜 → 푸리에 변환

$T \to \infty$ $\omega_0 = \frac{2\pi}{T}$

The Continuous-Time Fourier Transform



The Continuous-Time Fourier Transform

Aperiodic signals 비주기

Approaches to derive Fourier Transform

- Extending Fourier series by letting the period become infinite (주기∞ 인 주기신호)

$y = x * h$ $\quad e^{j\omega t} = x(t)$ $\quad e^{j2t}, e^{-j3t}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} h(\tau)e^{j\omega(t-\tau)}d\tau = e^{j\omega t}\boxed{\int_{-\infty}^{+\infty} h(\tau)e^{-j\omega\tau}d\tau}$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)e^{-j\omega\tau}d\tau$$

$y(t) = H(j\omega)x(t)$

$$y(t) = H(j\omega)e^{j\omega t}$$

$$X(j\omega) = \boxed{\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau}$$

The Continuous-Time Fourier Transform

- 푸리에 변환의 유도
  - 주기 신호의 푸리에 급수 전개에서

$x(t) \rightleftarrows a_k$ $\quad T \to \infty$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\,dt = \boxed{\frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\,dt}$$

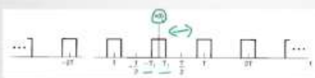
$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty}\left(\frac{\omega_0}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}\,dt\right)e^{jk\omega_0 t}$$

$T \to \infty$, $\omega_0 \to d\omega$, $k\omega_0 \to \omega$, $\Sigma \to \int$ 로 치환하면

$$x(t) = \int_{-\infty}^{+\infty}\left(\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right)e^{j\omega t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\boxed{\left(\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right)}e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \qquad X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

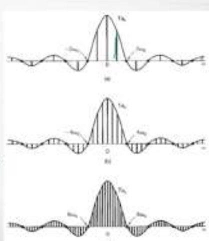
Representation of aperiodic signals : the continuous-time Fourier transform



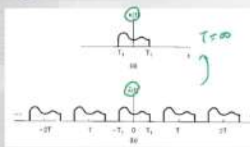
$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$Ta_k = \boxed{\frac{2\sin\omega T_1}{\omega}}\Bigg|_{\omega = k\omega_0}$$

Sampled values of a specific function



Representation of aperiodic signals : the continuous-time Fourier transform



$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$X(j\omega)$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\,dt = \boxed{\frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\,dt}$$

$a_k = \frac{1}{T}X(j\omega)$

$$Ta_k = X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt \qquad a_k = \frac{1}{T}X(j\omega) = \frac{1}{T}X(jk\omega_0)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \qquad T = \frac{2\pi}{\omega_0}$$

If T goes to infinite ω_0 becomes 0 ($T \to \infty, \omega_0 \to 0$).

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \qquad X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

4.1.2 Convergence of Fourier transform

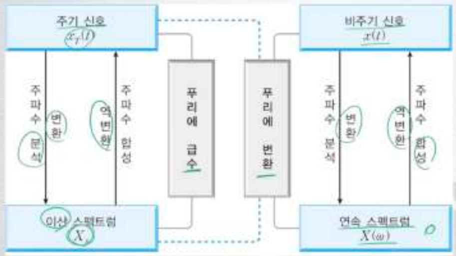
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \qquad X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

- 푸리에 변환(주파수 분석(분해)) :
  - 비주기 신호를 연속적인 주파수를 갖는 복소 정현파 $e^{j\omega t}$의 성분별로 표시한 것,
  - 즉 신호를 주파수 성분으로 나눔
  - 시간 함수 신호 $x(t)$에 대해 주파수 함수 스펙트럼 $X(j\omega)$를 구함
  - $X(j\omega)$는 ω에 대한 연속 함수
- Square integrable
- Dirichlet conditions

FS와 동일

√ 비주기 신호 대상이므로 Dirichlet 조건에서 주기성이 삭제되어 바뀜

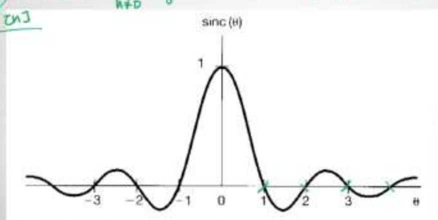
$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$



Representation of aperiodic signals : the continuous-time Fourier transform

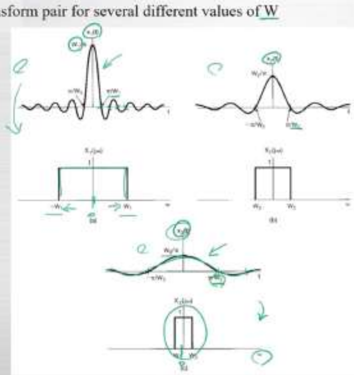
Sinc function

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta} \qquad \frac{\sin Wt}{\pi t} = \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$



Representation of aperiodic signals : the continuous-time Fourier transform

Fourier transform pair for several different values of W



- 주기신호: 에너지가 finite X → FT가 converge X
- The Fourier transform of a periodic signal can be represented by using delta functions.
- 주기신호의 푸리에 변환은 주파수 영역에서 임펄스 함수를 포함

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$

$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$

Ex) $X(j\omega) = 2\pi\delta(\omega)$ $\quad \omega = 0$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega)e^{j\omega t}\,d\omega = e^{j0t} = 1$$

$\frac{1}{2\pi}\cdot 2\pi\cdot \delta(0)$ $\quad e^{j0t}$

Ex) $X(j\omega) = 2\pi\delta(\omega - \omega_o)$ $\quad \omega = \omega_0$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

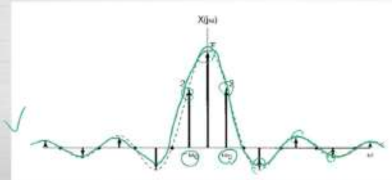
$T = \frac{2\pi}{\omega_0}$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k\delta(\omega - k\omega_o) \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Ex 4.6 $a_k = \frac{\sin(k\omega_o T_1)}{\pi k}$ $\qquad X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k\delta(\omega - k\omega_0)$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin k\,\omega_0 T_1}{k}\,\delta(\omega - k\omega_0)$$

$\omega = k\omega_0$



Ex 4.7 sine and cosine functions

$\sin(\omega_0 t)$ $\quad \cos(\omega_0 t)$

$F(e^{j\omega_0 t}) = 2\pi\,\delta(\omega - \omega_0)$

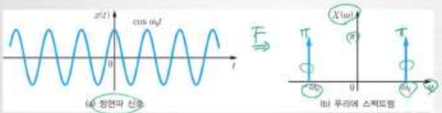
$F(e^{-j\omega_0 t}) = 2\pi\,\delta(\omega + \omega_0)$

$$\mathcal{F}\{\cos\omega_0 t\} = \mathcal{F}\left\{\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$\cos\omega_0 t \overset{F}{\Leftrightarrow} \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$\omega = \omega_0$ $\quad \omega = -\omega_0$

$T = \frac{2\pi}{\omega_0}$



(a) 정현파 신호 (b) 푸리에 스펙트럼

$X(j\omega) = j\pi\delta(\omega + \omega_0) - j\pi\delta(\omega - \omega_0) \overset{F}{\Leftrightarrow} \sin\omega_0 t$

$X(j\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0) \overset{F}{\Leftrightarrow} \cos$

$\mathcal{F}$ $\quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$ $\qquad$ F $\quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$

4.3.1 Linearity

$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$

$y(t) \overset{F}{\longleftrightarrow} Y(j\omega)$

$ax(t) + by(t) \overset{F}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$

4.3.2 Time shifting

Phase만 linear 하게 변한다

$x(t - t_0) \overset{F}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega)$ $\quad -\omega t_0$

4.3.3 Conjugation and conjugate symmetry (대칭성)

$a+jb \quad a-jb$

① $x^*(t) \overset{F}{\longleftrightarrow} X^*(-j\omega)$ $\qquad x(t) = x^*(t) \Leftrightarrow X(j\omega) = X^*(-j\omega)$

$$\because\ \mathcal{F}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\,dt = \left[\int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt\right]^* = X^*(-j\omega)$$

$X(-j\omega) = X^*(j\omega)$ or $X(j\omega) = X^*(-j\omega)$ $\quad [x(t)\ \ real]$

② For x(t) even $\quad x(t) = x(-t)$ $\qquad x(t)$ even → $X(j\omega)$ even

$$X(-j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = X(j\omega)$$

③ For x(t) real & even

$X(-j\omega) = X(j\omega) = X^*(j\omega)$

$X(j\omega) = X^*(j\omega)$ $\quad$ real function

4.3.3 Conjugation and conjugate symmetry (대칭성)

$a+jb \quad a-jb$

① $x^*(t) \overset{F}{\longleftrightarrow} X^*(-j\omega)$ $\qquad x(t) = x^*(t) \Leftrightarrow X(j\omega) = X^*(-j\omega)$

$$\because\ \mathcal{F}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\,dt = \left[\int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt\right]^* = X^*(-j\omega)$$

$X(-j\omega) = X^*(j\omega)$ or $X(j\omega) = X^*(-j\omega)$ $\quad [x(t)\ \ real]$

② For x(t) even $\quad x(t) = x(-t)$ $\qquad x(t)$ even → $X(j\omega)$ even

$$X(-j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = X(j\omega)$$

③ For x(t) real & even

$X(-j\omega) = X(j\omega) = X^*(j\omega)$

$X(j\omega) = X^*(j\omega)$ $\quad$ real function

4.3.4 Differentiation and integration

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\cdot e^{j\omega t}\, d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{j\omega X(j\omega)}\, e^{j\omega t}\, d\omega$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \boxed{\pi X(0)\delta(\omega)}$$

$\omega = 0$

Ex 4.11 Fourier Transform of the Unit step function $\leftarrow \delta(t) \xrightarrow{F} 1$

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Ex 4.12

4.3.5 Time and frequency scaling

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- 시간 영역에서 신호를 압축하면 주파수 스펙트럼은 신장됨
- 시간 영역에서 신호를 신장시키면 주파수 스펙트럼은 압축됨

4.3.6 Duality

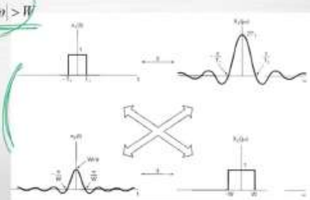
F.T. I.F.T

$$x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \xleftrightarrow{F} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

$$x_2(t) = \frac{\sin Wt}{\pi t} \xleftrightarrow{F} X_2(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

변환과 역변환 연산의 수식 구조가 동일

→ 시간 변수와 주파수 변수를 서로 바꾸어도 변환쌍 관계가 유효함



4.3.7 Parseval's relation

신호의 시간 영역 에너지 총합 = 각 스펙트럼 성분의 에너지 총합

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\underbrace{\left\{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right\}}_{X(j\omega)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Ex 4.14

4.3 Properties of the continuous-time Fourier transform

4.4 The convolution property

$$y(t) = h(t) * x(t) \xleftrightarrow{F} Y(j\omega) = H(j\omega)X(j\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \lim_{\omega_0 \to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$H(jk\omega_0) = \int_{-\infty}^{\infty} h(t)e^{-jk\omega_0 t}dt$$

$y(t) = H(j\omega)\cdot x(t)$

$Y(j\omega)$

$$y(t) = \lim_{\omega_0 \to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)H(j\omega)e^{j\omega t}d\omega$$

$$\Rightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

4.3 Properties of the continuous-time Fourier transform

4.4 The convolution property

$$\int_{-\infty}^{\infty} h(t')e^{-j\omega t'}e^{-j\omega\tau}dt'$$

$$= e^{-j\omega\tau}\underbrace{\int_{-\infty}^{\infty} h(t')e^{-j\omega t'}dt'}_{H(j\omega)}$$

$$= e^{-j\omega\tau}\cdot H(j\omega)$$

$= x(t) * h(t)$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau$$

$y(t)$

$$Y(j\omega) = F\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\boxed{\left[\int_{-\infty}^{+\infty} h(\underbrace{t-\tau}_{=t'})e^{-j\omega t}dt\right]}d\tau$$

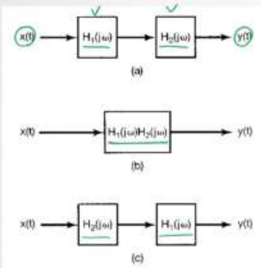
$// X(j\omega)$

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\boxed{e^{-j\omega\tau}H(j\omega)}d\tau = H(j\omega)\boxed{\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau} - H(j\omega)\cdot X(j\omega)$$

$$y(t) = h(t) * x(t) \xleftrightarrow{F} Y(j\omega) = H(j\omega)X(j\omega)$$

4.3 Properties of the continuous-time Fourier transform

$h_1(t) * h_2(t)$



$$H(j\omega) = H_1(j\omega)H_2(j\omega)$$
$$h(t) = h_1(t) * h_2(t)$$

4.3 Properties of the continuous-time Fourier transform

Ex)

$h(t) = e^{-at}u(t)\ ,\ a > 0$

$x(t) = e^{-bt}u(t)\ ,\ b > 0$

$Y(j\omega) = X(j\omega)H(j\omega) \xrightarrow{IFT} y(t)$

$y(t) = x(t) * h(t)$

$$X(j\omega) = \frac{1}{b+j\omega}\ ,\ H(j\omega) = \frac{1}{a+j\omega}$$

$$\Rightarrow Y(j\omega) = \frac{1}{(b+j\omega)(a+j\omega)}$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

$$\omega \rightarrow \frac{dY(j\omega)}{d\omega} = \int_{-\infty}^{\infty}\boxed{-jty(t)}e^{-j\omega t}dt$$

$$\Big(\underbrace{\frac{1}{a+j\omega}}_{Y(j\omega)}\Big)' = \frac{-j}{(a+j\omega)^2} \overset{F}{\leftrightarrow} -jty(t)$$

$$\frac{1}{(a+j\omega)^2} \overset{F}{\leftrightarrow} ty(t)$$

using partial fraction

If $a \neq b$

$$Y(j\omega) = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right] \Rightarrow y(t) = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]u(t)$$

If $a = b$

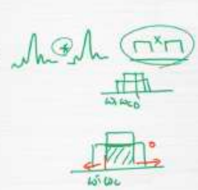
$$Y(j\omega) = \boxed{\frac{1}{(a+j\omega)^2}} \Rightarrow y(t) = te^{-at}u(t)$$

4.3 Properties of the continuous-time Fourier transform

Ex)

$-\omega_i \sim \omega_i$ $\qquad$ $-\omega_c \sim \omega_c$

$$x(t) = \frac{\sin\omega_i t}{\pi t}\ ,\ h(t) = \frac{\sin\omega_c t}{\pi t}\ \text{(LPF)}$$



$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$Y(j\omega) = \begin{cases} 1 & |\omega| \le \min[\omega_i, \omega_c] \\ 0 & elsewhere \end{cases}$$

$$y(t) = \begin{cases} \dfrac{\sin\omega_c t}{\pi t} & \omega_c \le \omega_i \\ \dfrac{\sin\omega_i t}{\pi t} & \omega_i \le \omega_c \end{cases}$$

4.3 Properties of the continuous-time Fourier transform

4.5 The multiplication property

Convolution property in the frequency domain (duality):

$$r(t) = s(t)p(t) \xleftrightarrow{F} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

Ex 4.21

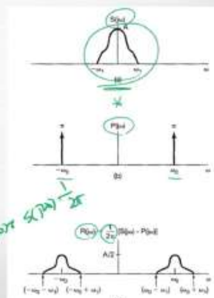
$p(t) = \cos\omega_0 t$

$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$

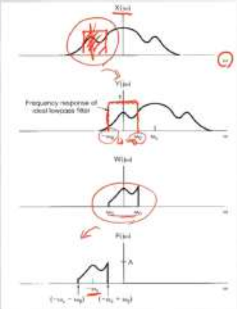
$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$
$$= \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0)).$$

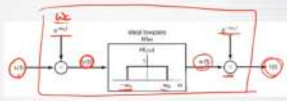
$= p(j\omega) * S(j\omega)\frac{1}{2\pi}$

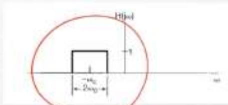
$$p(t)s(t) \xleftrightarrow{F} 0.5S(j\omega - j\omega_0) + 0.5S(j\omega + j\omega_0)$$



4.5.1 Frequency-selective filtering with variable center frequency

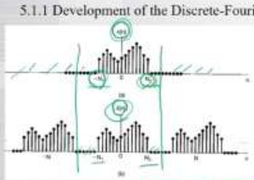






Representation of Aperiodic Signals : The Discrete-time Fourier Transform

5.1.1 Development of the Discrete-Fourier Transform



$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Let us define $\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}}$ $\quad \omega_0 = \frac{2\pi}{N}, \quad \boxed{\frac{1}{N} = \frac{\omega_0}{2\pi}}$

$$a_k = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N} X(e^{jk\omega_0}) = \frac{1}{N} X(e^{j\omega})$$

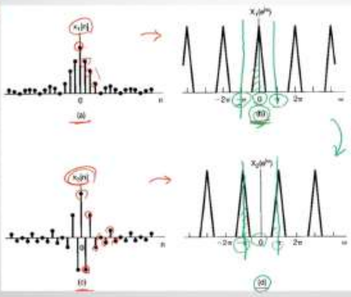
$T \to \infty \quad \omega_0 \to 0$

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} = \sum_{k=\langle N\rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi}\sum_{k=\langle N\rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n}\omega_0$$

$N \to \infty, \omega_0 \to 0$ $\quad \boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{jk\omega n} d\omega}$

Representation of Aperiodic Signals : The Discrete-time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{jk\omega n} d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$



5.1.3 Convergence issues associated with the discrete-time Fourier transform

① $\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$ or ② $\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$

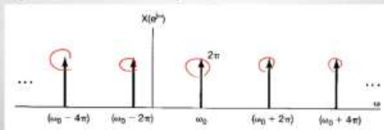
5.2 The Fourier transform for periodic signals

$$x[n] = e^{j\omega_0 n} \qquad X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

$X(e^{j\omega}) = \sum_{-\infty}^{\infty} e^{j\omega_0 n} \cdot e^{-j\omega n}$ $\quad$ 1) $\omega_0 \neq \omega$ $= 0$

$= \sum_{-\infty}^{\infty} e^{j(\omega_0 - \omega)n}$ ① $\quad$ 2) $\omega_0 = \omega$ ∞

$\omega = \omega_0 + 2\pi \leftarrow \omega - \omega_0 = 2\pi$



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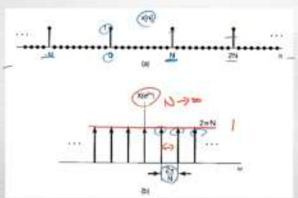
5.2 The Fourier transform for periodic signals

Ex 5.6 n 의 개수 1

$$x[n] = \sum_{-\infty}^{+\infty} \delta(n - kN) \qquad a_k = \frac{1}{N} \qquad X(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

$a_k = \frac{1}{N}\sum_{\langle N\rangle} x[n] \boxed{e^{-jk(2\pi/N)n}} = \frac{1}{N}$ $\quad \omega = \frac{2\pi k}{N}$

$X(e^{j\omega}) = \sum_{-\infty}^{\infty} 2\pi \cdot a_k \cdot \delta(\omega - \frac{2\pi}{N}k) = \sum_{-\infty}^{\infty} \frac{2\pi}{N} \cdot \delta(\omega - \frac{2\pi}{N}k)$



5.3 Properties of the discrete-time Fourier transform

5.3.1 Periodicity of the discrete-time Fourier transform (different)

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

5.3.2 Linearity of the Fourier transform (same)

$$ax_1[n] + bx_2[n] \overset{\mathfrak{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

5.3.3 Time shifting and frequency shifting (same)

$$x[n - n_0] \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

5.3 Properties of the discrete-time Fourier transform

5.3.4 Conjugation and conjugate symmetry (same)

$$x^*[n] \xleftrightarrow{3} X^*(e^{-j\omega})$$

$$X(e^{j\omega}) = X^*(e^{-j\omega}),\ [x[n] real]$$

5.3.5 Differencing and accumulation (different)

$$x[n] - x[n-1] \xleftrightarrow{3} (1 - e^{-j\omega})X(e^{j\omega})$$

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

$$\boxed{\sum_{m=-\infty}^{n} x[m]} \xleftrightarrow{3} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \boxed{\sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)}$$

5.3 Properties of the discrete-time Fourier transform

5.3.6 Time Reversal (same) $x[-n]$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[-n] e^{-j\omega n}$$ $m=-n$

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[m] e^{-j(-\omega)m} = X(e^{-j\omega})$$

$$x[-n] \xleftrightarrow{3} X(e^{-j\omega})$$

5.3 Properties of the discrete-time Fourier transform

5.3.7 Time Expansion (different)

✓For Continuous-time signals $x(at) \xleftrightarrow{3} \frac{1}{|a|} X(\frac{j\omega}{a})$

It is different for discrete-time signal.

$$x_{(k)}[n] = \begin{cases} x\left[\frac{n}{k}\right], & if\ n\ is\ a\ multiple\ of\ k \\ 0, & if\ n\ is\ a\ multiple\ of\ k \end{cases}$$

$$\boxed{x_{(k)}[n] \xleftrightarrow{3} X(e^{jk\omega})}$$ $x[\frac{n}{k}]$

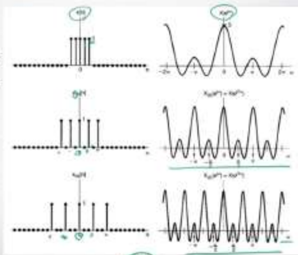
$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\omega n}$

$n = mk$

$= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega mk}$

$= X(e^{j\omega \cdot k})$

$k\omega$



5.3 Properties of the discrete-time Fourier transform

5.3.8 Differentiation in frequency (similar)

$$n x[n] \xleftrightarrow{3} j \frac{dX(e^{j\omega})}{d\omega}$$

$X(e^{j\omega}) = \sum x[n] e^{-j\omega n}$

$\frac{dX(e^{j\omega})}{d\omega} = -j \sum n x[n] e^{-j\omega n}$

$j \cdot \frac{dX(e^{j\omega})}{d\omega} = \sum \boxed{n x[n]} e^{-j\omega n}$

5.3.9 Parseval's relation (similar)

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

5.3 Properties of the discrete-time Fourier transform

Ex 5.10

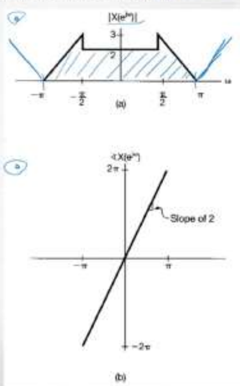
Is the time-domain signal $x[n]$

Periodic? X

Real? - Conjugate Symmetric

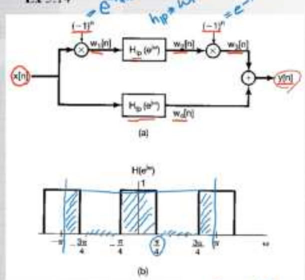
Even? $x[n] = x[-n] \leftrightarrow X(e^{j\omega}) = X(e^{-j\omega})$ X

Of finite energy?



5.4 The Convolution Property

Ex 5.14



$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega}) X(e^{j(\omega-\pi)})$$

$$W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}) X(e^{j(\omega-2\pi)})$$

hpF

LPF

✓ All pass filter

5.7 Duality (쌍대)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

5.7 Duality (쌍대)

IFT
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \cdots ①$$

FT
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$ 2π주기 → FS

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\frac{2\pi}{2\pi}\omega}$

$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega}$

$a_k = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{-jk\frac{2\pi}{2\pi}\omega}d\omega$

$T = 2\pi$

$k \to n$

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a_n \cdot e^{j\omega n}$

② $a_n = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{-j\omega n}d\omega$

$x[n] = a[-n]$

5.8 Systems characterized by linear constant-coefficient difference eq.

① $y[n] - ay[n-1] = x[n]$ $\overset{F}{\leftrightarrow}$ $Y(e^{j\omega}) - ae^{-j\omega}\cdot Y(e^{j\omega}) = X(e^{j\omega})$ $\quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$ $\quad \checkmark\ h[n] = a^n u[n]$

$Y(e^{j\omega})\{1 - ae^{-j\omega}\} = X(e^{j\omega})$

② $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$ $\quad Y(e^{j\omega})\{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}\} = 2X(e^{j\omega})$

$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-j2\omega}\cdot Y(e^{j\omega}) = 2X(e^{j\omega})$

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}} = \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{4}{\left(1 - \frac{1}{2}e^{-j\omega}\right)} - \frac{2}{\left(1 - \frac{1}{4}e^{-j\omega}\right)}$$

$$\checkmark\ h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

Time and Frequency Characterization of Signals and Systems

6.1 The magnitude-phase representation of the Fourier Transform

$X(jw) = |X(jw)|e^{j\angle X(jw)}$

$H(e^{jw}) = |H(e^{jw})|e^{j\angle H(e^{jw})}$

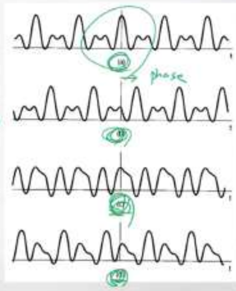
DC, ω_1

$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \varphi_1)$

ω_2, ω_3

$+ \cos(4\pi t + \varphi_2) + \frac{2}{3}\cos(6\pi t + \varphi_3)$

Amplitude is perceptually more important



6.2 The magnitude-phase representation of the frequency response of LTI system

$X(j\omega)$ $H(j\omega)$ $\rightarrow Y(j\omega)$

$|Y(jw)| = |H(jw)||X(jw)|$

$y = x * h$ 두 $Y(j\omega) = X(j\omega)H(j\omega)$

$\angle Y(jw) = \angle H(jw) + \angle X(jw)$

6.2.1 Linear and nonlinear phase

Linear Phase $\begin{cases} \angle H(j\omega) = -\omega t_0 \\ \angle H(e^{j\omega}) = -\omega n_0 \end{cases}$

Ex : $\cos 10t \rightarrow \cos(10t - 20) = \cos(10(t - 2)) \rightarrow 2$ phase delay

$\cos 30t \rightarrow \cos(30t - 60) = \cos(30(t - 2)) \rightarrow 2$

6.2 The magnitude-phase representation of the frequency response of LTI system

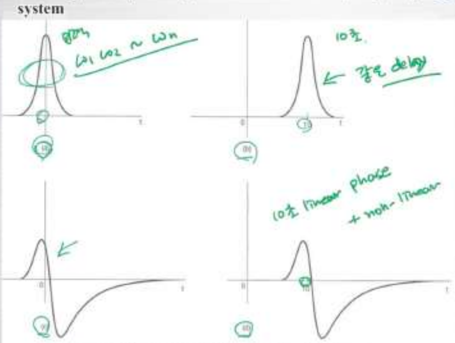


Figure 6.3 (a) Continuous-time signal that is applied as the input to several systems for which the frequency response has unity magnitude; (b) response for a system with linear phase; (c) response for a system with nonlinear phase; and (d) response for a system with phase equal to the nonlinear phase of the system in part (c) plus a linear phase term.

6.2 The magnitude-phase representation of the frequency response of LTI system

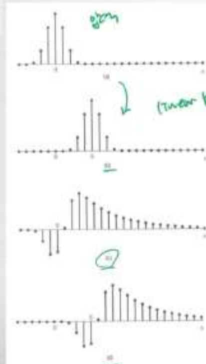


Figure 6.4 (a) Discrete-time signal that is applied as input to several systems for which the frequency response has unity magnitude; (b) response for a system with linear phase with slope of −5; (c) response for a system with nonlinear phase; and (d) response for a system whose phase characteristic is that of part (c) plus a linear phase term with integer slope.

6.2.2 Group Delay

For narrow band system

$\angle H(j\omega) \approx -\varphi - \omega\alpha \rightarrow \alpha$ time delay

$Y(j\omega) \approx X(j\omega)|H(j\omega)|e^{-j\varphi}e^{-j\omega\alpha}$

α: Group Delay

$\tau(w) = -\frac{d}{dw}\{\angle H(jw)\}$

6.2.2 Group Delay

Ex 6.1 : All-pass filter

$H(j\omega) = \prod_{i=1}^{3} H_i(j\omega)$

$H_i(jw) = \frac{1 + (jw/w_i)^2 - 2j\zeta_i(w/w_i)}{1 + (jw/w_i)^2 + 2j\zeta_i(w/w_i)}$ — mag = 1

$\omega_1 = 315$ $\varsigma_1 = 0.0666$ $f_1 \approx 50Hz$

$\omega_2 = 943$ $\varsigma_2 = 0.033$ $f_2 \approx 150Hz$

$\omega_3 = 1888$ $\varsigma_3 = 0.058$ $f_3 \approx 300Hz$

ex) $15\pi = \pi$

$|H(j\omega)| = 1$

$\angle H_i(j\omega) = -2\,\mathrm{atan}\left[\frac{2\varsigma_i(\omega/\omega_i)}{1 - (\omega/\omega_i)^2}\right]$ imag / real

- principal phase $\rightarrow 0 \sim 2\pi$
- phase unwrapping

$\angle H(j\omega) = \sum_{i=1}^{3} \angle H_i(j\omega)$

6.2.2 Group Delay

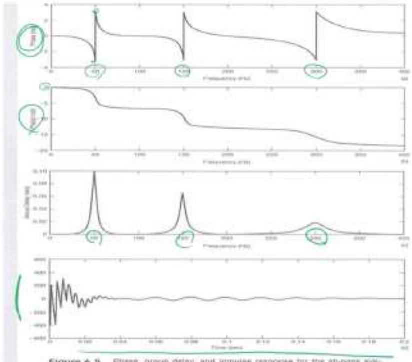


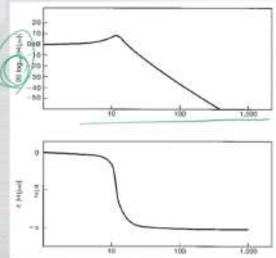
Figure 6.5 Phase, group delay, and impulse response for the all-pass system of Example 6.1: (a) principal phase; (b) unwrapped phase; (c) group delay; (d) impulse response. Each of these quantities is plotted versus frequency measured in Hertz.

6.2.3 Log-Magnitude and Bode Plots

$\log|Y(jw)| = \log|H(jw)| + \log|X(jw)|$

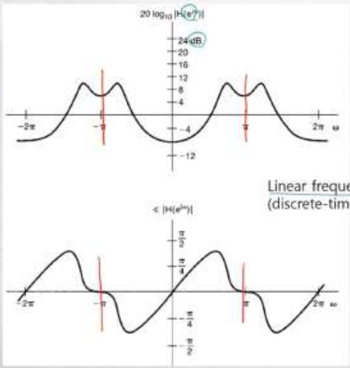
dB : decibel

1. Additive relation
2. Dynamic range



Frequency
in log scale
(continuous-time case)

6.2.3 Log-Magnitude and Bode Plots

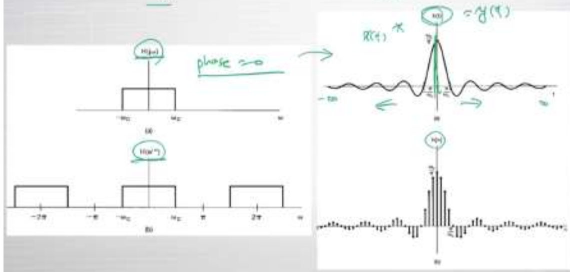


Linear frequency scale
(discrete-time case)

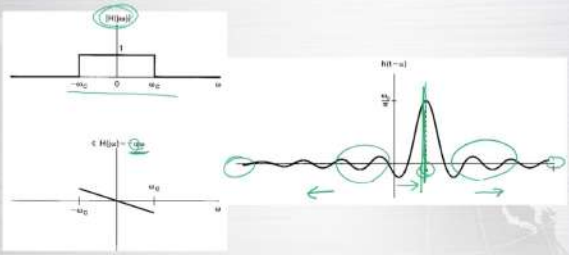
6.3 Time-domain Properties of Ideal Frequency-selective Filters

Ideal LPF $H(jw) = \begin{cases} 1, & |w| \le w_c \\ 0, & |w| > w_c \end{cases}$ $\qquad$ $H(e^{jw}) = \begin{cases} 1, & |w| \le w_c \\ 0, & w_c < |w| \le \pi \end{cases}$

$h(t) = \dfrac{\sin w_c t}{\pi t}$ $\qquad$ $h[n] = \dfrac{\sin w_c n}{\pi n}$



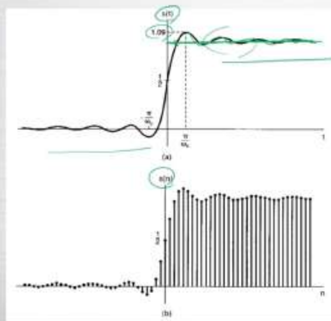
6.3 Time-domain Properties of Ideal Frequency-selective Filters



6.3 Time-domain Properties of Ideal Frequency-selective Filters

Step response of the ideal low-pass filter

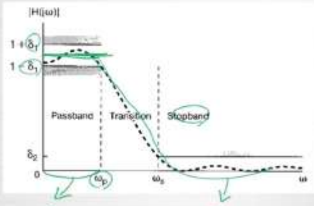
$h(t)$



Ringing
Rise time

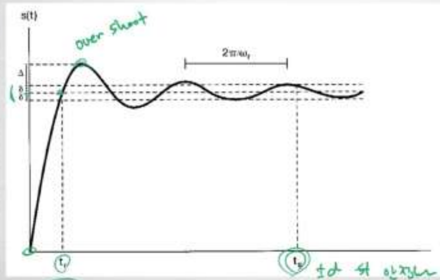
6.4 Time-domain and Frequency-domain Aspects of Nonideal Filters

* Ideal filter : Rectangular Function



6.4 Time-domain and Frequency-domain Aspects of Nonideal Filters

Step Response of LPF



6.4 Time-domain and Frequency-domain Aspects of Nonideal Filters

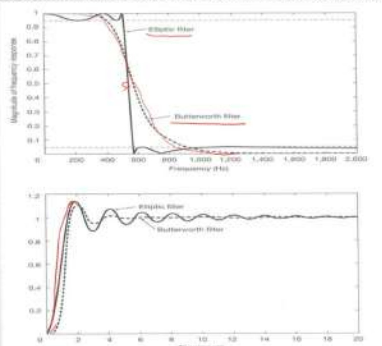


Figure 6.18 Example of a fifth-order Butterworth filter and a fifth-order elliptic filter designed to have the same passband and stopband ripple and the same cutoff frequency: (a) magnitudes of the frequency responses plotted versus frequency measured in Hertz; (b) step responses.

6.5 First-order and second-order continuous-time systems

6.5.1 First order continuous-time systems

$\tau \cdot j\omega Y(j\omega) + Y(j\omega) = X(j\omega)$

$(\tau \cdot j\omega + 1) Y(j\omega) = X(j\omega)$

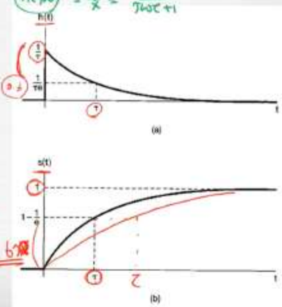
$H(j\omega) = \frac{Y}{X} = \frac{1}{j\omega\tau + 1}$

$$\tau \frac{dy(t)}{dt} + y(t) = x(t)$$

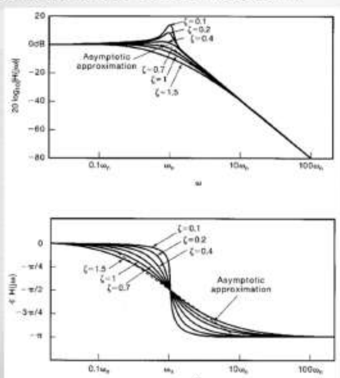
$$H(jw) = \frac{1}{jw\tau + 1}$$

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

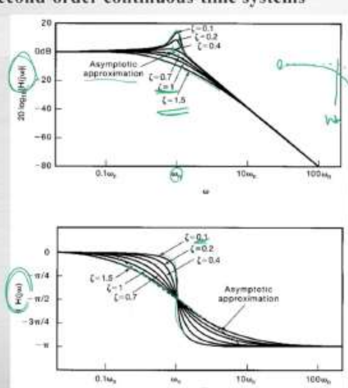
$$s(t) = h(t) * u(t) = [1 - e^{-t/\tau}]u(t)$$



6.5.2 Second-order continuous-time systems

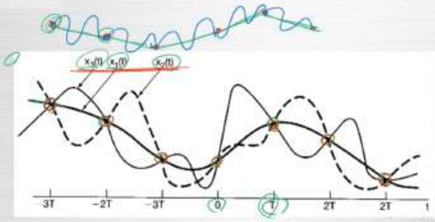


6.5.2 Second-order continuous-time systems



7. Sampling

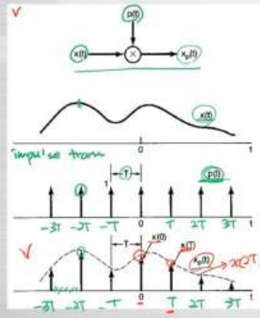
7.1 Representation of a continuous-time signal by its samples : The sampling theorem



Sample --> ~~Continuous-time~~ signal : not unique

* For a Band-limited signal → unique restoration

7.1.1 Impulse-train sampling



$$x_p(t) = x(t)p(t)$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

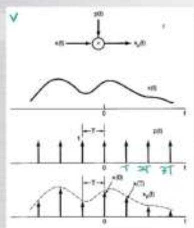
$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

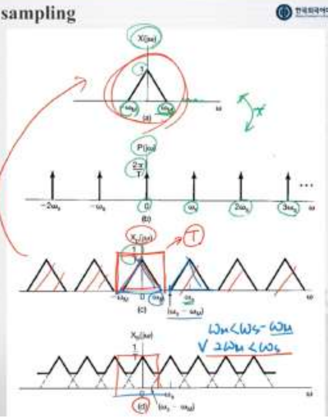
$$X_p(j\omega) = \frac{1}{2\pi}[X(j\omega) * P(j\omega)]$$

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))$$

7.1.1 Impulse-train sampling





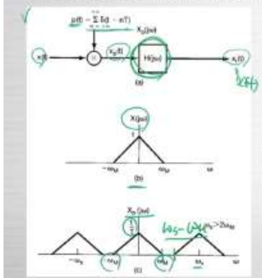
Nyquist Sampling Theorem

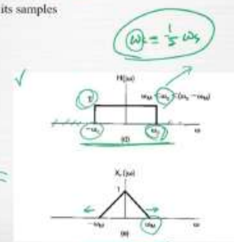
Let $x(t)$ be a band−limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$.

Then, $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \cdots$, if

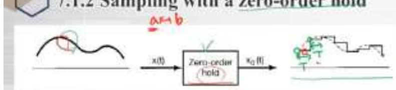
$$\omega_s > 2\omega_M, \text{ where } \omega_s = \frac{2\pi}{T}$$

Exact recovery of a continuous-time signal from its samples

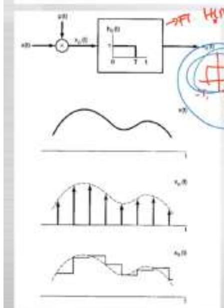


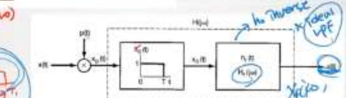


7.1.2 Sampling with a zero-order hold

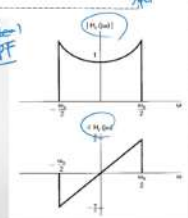


$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}\right]$$





$$H_r(j\omega) = \frac{e^{\frac{j\omega T}{2}} H(j\omega)}{\frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}}$$



7.2 Reconstruction of a signal from its samples using interpolation

- Linear interpolation

$h(t)$ T

$2\omega_M < \omega_s$

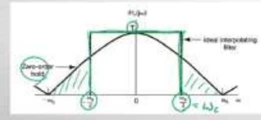
7.2 Reconstruction of a signal from its samples using interpolation

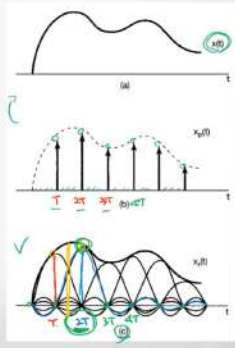
- Interpolation by ideal low-pass filtering

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t-nT)$$

$$h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t}$$

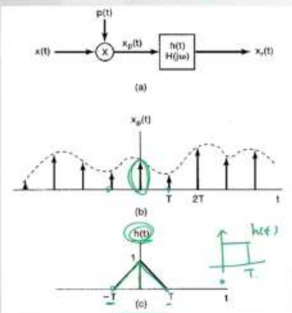
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)\frac{\omega_c T}{\pi}\frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}$$



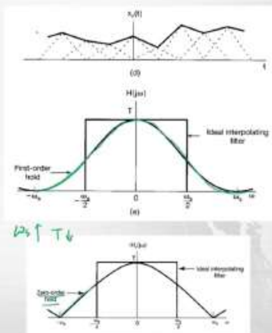


7.2 Reconstruction of a signal from its samples using interpolation

- Linear interpolation



$\omega_s \uparrow$ $T \downarrow$



7.3 The effect of under sampling : aliasing

$x(t) = \cos \omega_0 t$

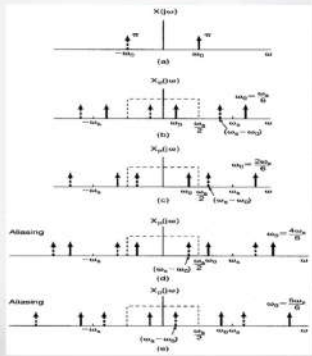
a) $\omega_0 = \frac{1}{6}\omega_s \quad x_r(t) = \cos \omega_0 t$

b) $\omega_0 = \frac{2}{6}\omega_s \quad x_r(t) = \cos \omega_0 t$

c) $\omega_0 = \frac{4}{6}\omega_s \quad x_r(t) = \cos(\omega_s - \omega_0) t$

d) $\omega_0 = \frac{5}{6}\omega_s \quad x_r(t) = \cos(\omega_s - \omega_0) t$

e) $\omega_0 = \omega_s \quad x_r(t) = 1$



7.3 The effect of under sampling : aliasing

$x(t) = \cos \omega_0 t$ $2\omega_0 < \omega_s$

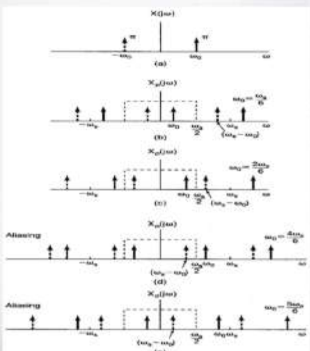
o a) $\omega_0 = \frac{1}{6}\omega_s \quad x_r(t) = \cos \omega_0 t$

o b) $\omega_0 = \frac{2}{6}\omega_s \quad x_r(t) = \cos \omega_0 t$

x c) $\omega_0 = \frac{4}{6}\omega_s \quad x_r(t) = \cos(\omega_s - \omega_0) t$

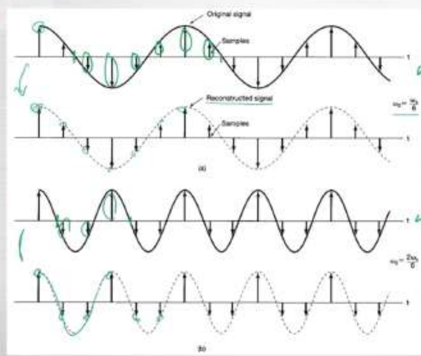
x d) $\omega_0 = \frac{5}{6}\omega_s \quad x_r(t) = \cos(\omega_s - \omega_0) t$

x e) $\omega_0 = \omega_s \quad x_r(t) = 1$



7.3 The effect of under sampling : aliasing

a)



$\omega_s = 6\omega_0$

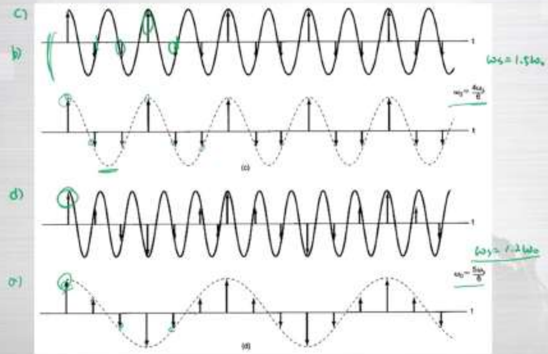
T. 6

b)

$\omega_s = 3\omega_0$

3

7.3 The effect of under sampling : aliasing



7.3 The effect of under sampling : aliasing

$x(t) = \cos \omega_0 t$

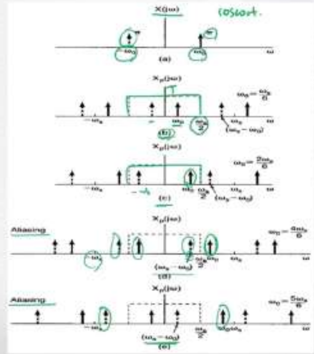
a) $\omega_0 = \frac{1}{6}\omega_s \quad x_r(t) = \cos \omega_0 t$

b) $\omega_0 = \frac{2}{6}\omega_s \quad x_r(t) = \cos \omega_0 t$

c) $\omega_0 = \frac{4}{6}\omega_s \quad x_r(t) = \cos(\omega_s - \omega_0) t$

d) $\omega_0 = \frac{5}{6}\omega_s \quad x_r(t) = \cos(\omega_s - \omega_0) t$

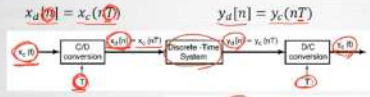
e) $\omega_0 = \omega_s \quad x_r(t) = 1$

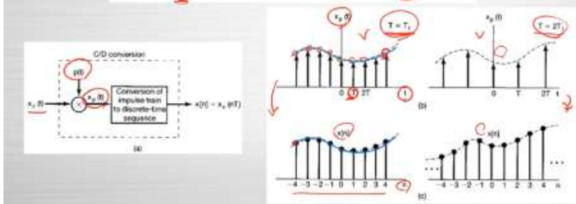


7.3 The effect of under sampling : aliasing

7.4 Discrete-time processing of continuous-time signals

$x_d[n] = x_c(nT) \qquad y_d[n] = y_c(nT)$





7.4 Discrete-time processing of continuous-time signals

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT)$$

$$X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\omega nT}$$

$\Omega = \omega T \Leftrightarrow \omega = \frac{\Omega}{T}$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n]e^{-j\Omega n}$$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n}$$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$X_d(e^{j\Omega}) = X_p(j\Omega/T)$$

$-\pi \sim \pi$

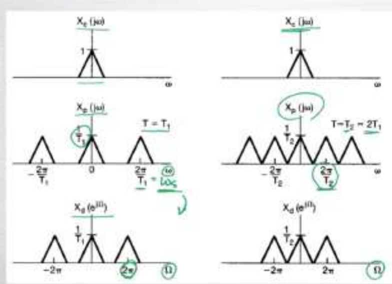
$-\frac{\pi}{T} \sim \frac{\pi}{T}$

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\Omega - 2\pi k)/T)$$

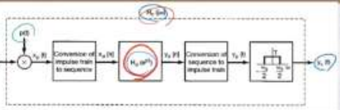
7.4 Discrete-time processing of continuous-time signals

$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\Omega - 2\pi k)/T) \qquad X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$

$\Omega = \omega T$



DA conversion

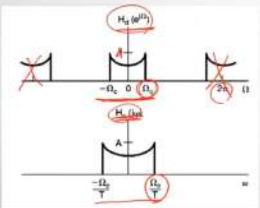


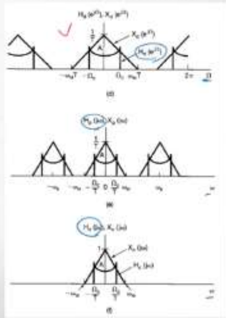
$$Y_c(j\omega) = X_c(j\omega)H_d(e^{j\omega T})$$

$= X_c(j\omega)H_c(j\omega)$ ωT

$= X_c(j\omega)H_d(e^{j\Omega})$

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$





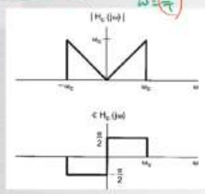
7.4.1 Digital differentiator

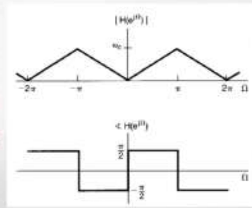
Continuous-time differentiator

$H_c(j\omega) = j\omega$ $\quad \Omega = \omega T$, $\omega = \frac{\Omega}{T}$

$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

$= \frac{\pi}{T} = \frac{\omega_s}{2}$





$-\frac{\pi}{T} < \omega < \frac{\pi}{T}$

$$H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right), \quad |\Omega| < \pi$$

$-\omega_c < \omega < \omega_c$

IFT

$h_d[n]$

7.4.1 Digital differentiator

Example 7.2 $\quad X_c(j\omega)$

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t} \qquad X_c(j\omega) = \begin{cases} 1, & |\omega| < \frac{\pi}{T} \\ 0, & \text{그외의 구간} \end{cases}$$

$$y_c(t) = \frac{d}{dt}x_c(t) = \frac{\cos(\pi t/T)}{Tt} - \frac{\sin(\pi t/T)}{\pi t^2}$$

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]$$

$$y_d[n] = y_c(nT) = \begin{cases} \frac{(-1)^n}{nT^2}, & n \neq 0 \\ 0, & n = 0 \end{cases} = \frac{1}{T}h_d[n]$$

: impulse response

$$h_d[n] = \begin{cases} \frac{(-1)^n}{nT}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

7.4.2 Half-sample delay

$T, \frac{T}{2}$ $\quad H_c(j\omega) = e^{-j\omega\frac{T}{2}}$

$H_c(j\omega) = e^{-j\omega\Delta}$

$$y_c(t) = x_c(t - \Delta)$$

$$Y_c(j\omega) = e^{-j\omega\Delta}X_c(j\omega)$$

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

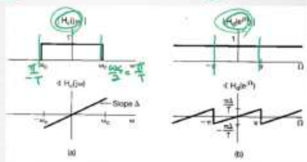
$\frac{\Omega}{T}$

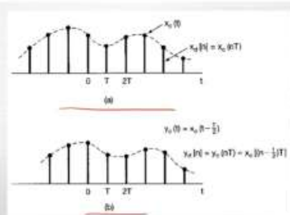
$$H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}, \quad |\Omega| < \pi$$

$$y_d[n] = x_d\left[n - \frac{\Delta}{T}\right]$$

IFT

$h_d[n]$





7.4.2 Half-sample delay

Example 7.3 $\quad X_c(j\omega)\begin{cases} -\frac{\pi}{T} \sim \frac{\pi}{T} & 1 \\ & 0 \end{cases}$

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t}$$

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]$$

$$y_c(t) = x_c(t - T/2) = \frac{\sin(\pi(t - T/2)/T)}{\pi(t - T/2)}$$

$$y_d[n] = y_c(nT) = \frac{\sin(\pi(n - \frac{1}{2}))}{T\pi(n - \frac{1}{2})}$$

$y_d[n] = \frac{1}{T}h_d[n]$

$$h[n] = \frac{\sin(\pi(n - \frac{1}{2}))}{\pi(n - \frac{1}{2})}$$

7.5 Sampling of discrete-time signals

7.5.1 Impulse-train sampling

$$x_p[n] = \begin{cases} x[n], & \text{if } n = \text{ an integer multiple of } N \\ 0, & \text{otherwise} \end{cases}$$

$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n - kN]$$

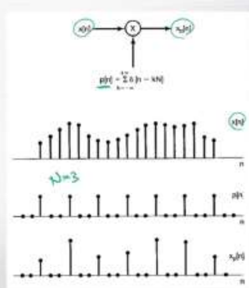
$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$

$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_s)$$

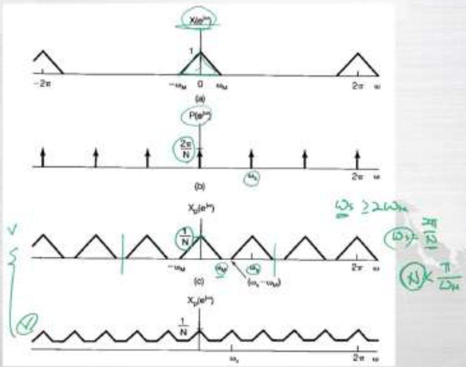
$= \frac{2\pi}{N}$

$\omega_s = \frac{2\pi}{T}, \frac{1}{T}$

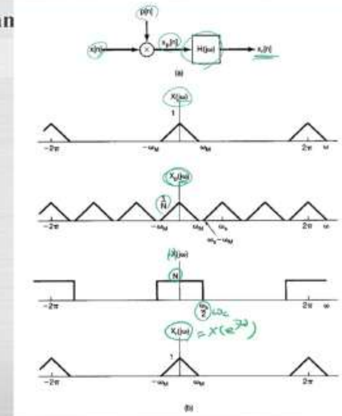
$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})$$



7.5 Sampling of discrete-time signals



7.5 San



7.5.2 Discrete-time decimation and interpolation

Decimation (sub-sampling) Down

$x_b[n] = x_p[nN]$ $\qquad$ $x_b[n] = x[nN]$

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_b[k]e^{-j\omega k}$$

$kN = n$

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_p[kN]e^{-j\omega k} \quad \longrightarrow \quad \sum_{n=-\infty}^{+\infty} x_p[n]e^{-j\omega n/N} = X_p(e^{j\omega/N})$$

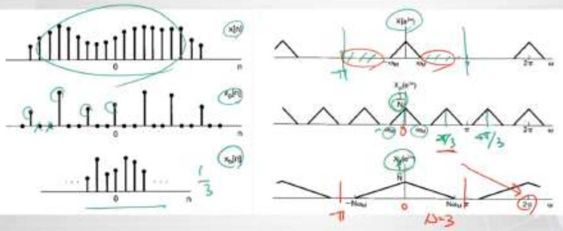
$k = \frac{n}{N}$

$\omega \to N\omega$

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_p[n]e^{-j\omega n/N}$$

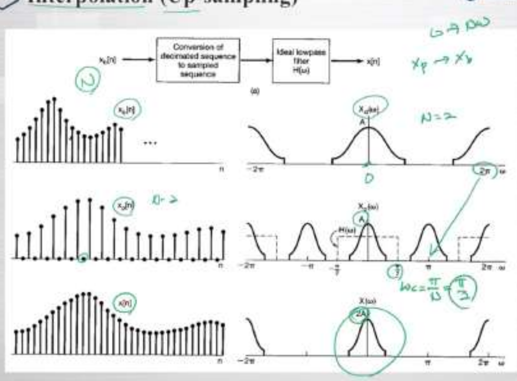
$$X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

7.5.2 Discrete-time decimation and interpolation



$\omega \to N\omega$

Interpolation (Up-sampling)



Interpolation (Up-sampling)