Simulation laboratory 4: Variance reduction

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TRANSP-OR

Goals

Variance reduction:

- Understand two different variance reduction techniques.
- Increase the precision of vehicle queue simulation result.

Implementation:

1. Antithetic draws
2. Control variates

Antithetic draws

Normal run:

$r \sim U(0,1)$ ⟶
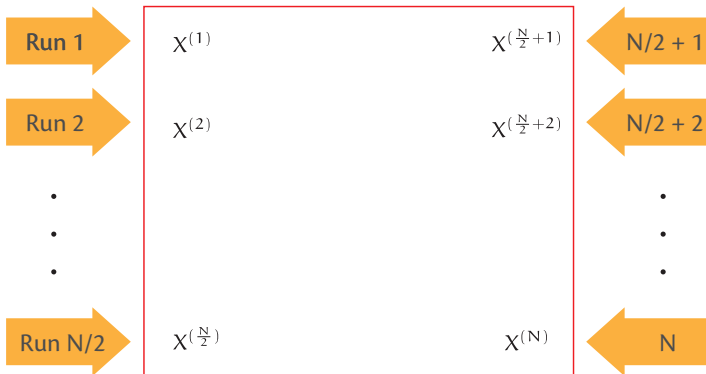
Antithetic run:

$1 - r \sim U(0,1)$

Run 1 ➤ $X^{(1)}$

$X^{(\frac{N}{2}+1)}$ ◀ N/2 + 1

Run 2 ➤ $X^{(2)}$

$X^{(\frac{N}{2}+2)}$ ◀ N/2 + 2

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Run N/2 ➤ $X^{(\frac{N}{2})}$

$X^{(N)}$ ◀ N

Calculate statistics

Antithetic draws

Implementation:

- Modify the function `simulate`:
    - Keywords: `scenario`, `u`
    - Return: `times`, `queues`
- Conduct statistical analysis of maximum queue length using code developed in the previous lab.

Antithetic draws workflow

1. For $r = 1, \ldots, N/2$:
   1. Independent simulation:
      1. Generate array of uniform random numbers u.
      2. Run `simulate(scenario, u)` and obtain max-queue-length q_{max}^{ind}.
   2. Antithetic simulation:
      1. Set $u = 1 - u$.
      2. Run `simulate(scenario, u)` and obtain max-queue-length q_{max}^{ant}.
      3. Compute $q_{max}^{(r)} = \frac{q_{max}^{ind} + q_{max}^{ant}}{2}$.
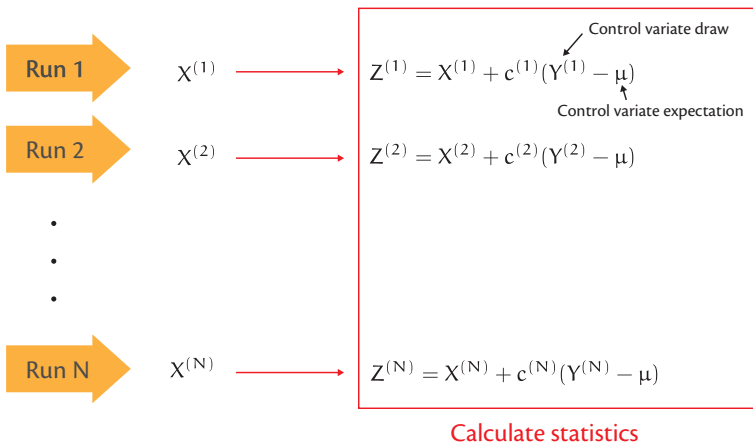2. Analyse the statistics of $[q_{max}^{(1)}, \ldots, q_{max}^{(\frac{N}{2})}]$

Keep draws

- The negative correlation between a run and its antithetic should be as big as possible.
- The order of events could change between the two runs.
- **Different streams of random numbers for the different event types should be used**.

Control variates

Run 1 $X^{(1)}$ $\longrightarrow$ $Z^{(1)} = X^{(1)} + c^{(1)}(Y^{(1)} - \mu)$

Control variate draw

Control variate expectation

Run 2 $X^{(2)}$ $\longrightarrow$ $Z^{(2)} = X^{(2)} + c^{(2)}(Y^{(2)} - \mu)$

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Run N $X^{(N)}$ $\longrightarrow$ $Z^{(N)} = X^{(N)} + c^{(N)}(Y^{(N)} - \mu)$

Calculate statistics

Control variates

Implementation:

- Modify the function `simulate`:
    - Keywords: `scenario`,
    - Return: `times`, `queues`, `service_time_mean`
- Conduct statistical analysis of maximum queue lengths.
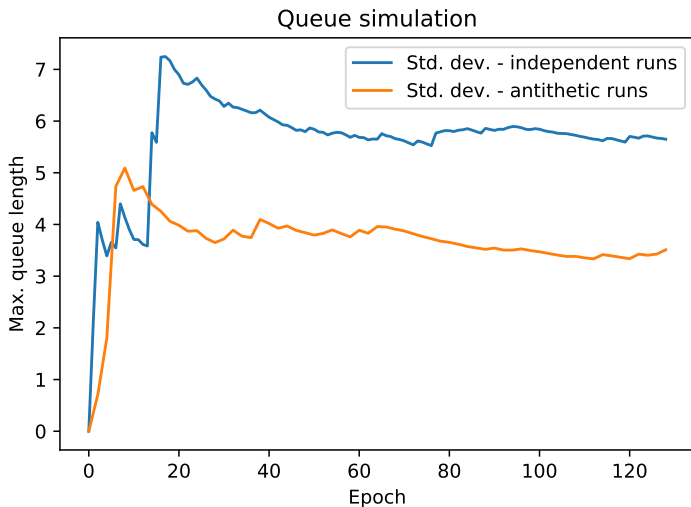
Control variates workflow

1. For $r = 1, \ldots, N$:
    1. Run `simulate` and output the control variate `control` $(= Y^{(r)})$.
    2. Obtain max-queue-length q_{max} $(= X^{(r)})$.
    3. Execute the function `controlled_mean`:
        1. Calculate variance $\text{Var}(Y)$ and covariance $\text{Cov}(X, Y)$
        2. Define the constant $c^* = -\text{Cov}(X, Y)/\text{Var}(Y)$
        3. Calculate $Z = X + c^*(Y - \mu)$
        4. Obtain the average and variance of $Z = [Z^{(1)}, \ldots, Z^{(r)}]$

Control variates

- Choose the control variate (Y) as you prefer. The higher the correlation is, the bigger the variance reduction is.
- Example: $Y =$ mean of service time at bottleneck

Antithetic draws

Queue simulation

Control variates