Nuclear Processes

At one point in time matter was considered to be made up of "atoms" only, which were considered to be the fundamental particles. It was soon discovered that the atoms are made up of elementary particles. Thus the physical world is composed of a combination of various sub-atomic or fundamental particles.

Thus an atom is denoted as ${}_{Z}^{A}X$ where

- "Z" is the proton number, or the atomic number. Just a reminder that protons are positively charged.
- "A" is the number of nucleons, the number of protons plus the neutrons and is known as the mass number.

The **electron**(e^{-}) are negatively charged particles and carry a charge of 1.6×10^{-19} Coulomb. The mass of an electron is approximately 9.1×10^{-31} Kg. Electrons are Fermions *i.e* they have an half integer spin and hence they open Fermi-Dirac statistics / distribution.

We do encounter a "positive electron" known as the "positron", which is identical to an electron in all respects, except that it is positively charged.

From Einstein's energy-mass equivalence, we know that the rest-mass of an electron / positron is 511 keV. According to Einstein's theory of relativity, the mass of a particle depends upon it's speed (relative to a stationary observer). The rest mass corresponds to the situation wherein the particle is at rest. "keV" refers to kilo-electron-volt, and is the unit of energy conventionally used in nuclear physics.

It corresponds to the energy gained by an electron when accelerated across a potential difference of 1 volt.

$$1\ eV = (1.6 \times 10^{-19}) \times 1 \text{ Coulomb –Volt} = 1.6 \times 10^{-19} \text{ Joule}$$

It is customary to use the following larger units

$$1\ keV = 10^{3}\ eV\ ;\ 1\ MeV = 10^{6}\ eV$$

In the above context, the rest mass of an electron is

$$E_o = m_0 \times c^2 = (9.11 \times 10^{-31}) \times (3 \times 10^8)^2 = 8.2 \times 10^{-14} \text{Joule}$$

$$E_o = \frac{8.2 \times 10^{-14}}{1.6 \times 10^{-19}} = 0.511 \text{ MeV}$$

Protons$(\boldsymbol{\pi})$ are positively charged, with a charge equal in magnitude to that of an electron. It's mass is $m_p = 1.6 \times 10^{-27}$ kg.

The **neutron**$(\boldsymbol{\nu})$ is an electrically neutral particle, and has a mass slightly heavier than that of proton. This small mass difference between the neutron and the proton has a significant bearing on us being what we are today. The neutron is not a stable particle *i.e* it is stable inside the nucleus, however a free neutron has a mean life of about 14 minutes and 42 seconds or a half life of about 10 minutes and 11 seconds.

A neutron decays to a proton and the conservation laws demand that this reaction proceed as

$$n_1 \longrightarrow p^1 + e^{-1} + \bar{\nu}, \text{ where} \bar{\nu}, \text{ is an anti-neutrino}$$

The neutrino is a particle with rest-zero mass and no charge and is required from conservation laws.

The atomic dimensions are of the order of Angstrom $(1A^0 = 10^{-10}\ m)$, whereas the nuclear dimensions are of the order of Fermi $(1F = 10^{-15}\ m)$.

The radius of the nucleus is related to the total number of nucleons present viz. the mass number as

$$R = R_o \times A^{1/3} \text{ Fermi, where } R_o = 1.2 \text{ fm.}$$

The nuclear density is constant across the nuclear landscape, and

$$\rho_{nucl} = \frac{mass}{volume} = \frac{A(amu)}{\frac{4}{3}\pi R^3} = \frac{A \times 1.6 \times 10^{-27}}{\frac{4}{3} \times \pi \times \left(R_o \times A^{1/3}\right)^3} \frac{kg}{m^3} = \frac{A \times 1.6 \times 10^{-19}}{\frac{4}{3} \times \pi \times R_o \times A}$$

Hence,

$$\rho_{nucl} \approx 10^{18}\, kg/m^3$$

Thus we can conclude that

- The density of nuclear matter is very high
- The nuclear density is independent of A, i.e all nuclei have almost identical density.

Therefore, this suggests that nuclei are very similar to liquid drops, which have the same density irrespective of the size of the drop. This has resulted in the development of a macroscopic model known as the "Liquid Drop Model" of the nucleus (as described /discussed later)

Alternatively we know that

$$Volume \;\propto R^3 \propto \left[A^{1/3}\right]^3 \propto A$$

Hence, $\frac{Volume}{A} = Constant$

Therefore, number of nucleons per unit volume is constant for all nuclei.

Atomic Mass Unit (AMU, u)

The mass of nucleons is usually expressed as Atomic Mass Unit (AMU, u). It is a relative scale, where the mass of a neutral ^{12}C atom is taken precisely as 12 units. *i.e*

$$m({}^{12}_{6}C) = 12\, u$$

1 *amu* is hence considered as $1/12^{th}$ the mass of ^{12}C atom, 1 $amu = \frac{m({}^{12}_{6}C)}{12}$

We know that 1 mole of a substance contains Avagrados number of atoms (6.023×10^{23}). *i.e* one mole of a substance contains the same number of atoms / molecules.

∴ 1 mole of ^{12}C contains 12 gm of the isotope, on the other hand 1 mole of natural O_2 contains $2 \times 15.99938 = 31.99876$ gm.

∴ 12 gm of ^{12}C has 6.023×10^{23} atoms.

∴ 1 atom of ^{12}C weighs $\frac{12 \times 10^{-3}}{6.023 \times 10^{23}}\, kg$

$$\therefore 1\, amu = \frac{12 \times 10^{-3}}{6.023 \times 10^{23}} \times \frac{1}{12} = 1.66053 \times 10^{-27}\, kg$$

∴ from energy-mass equivalence, we have the energy associated with 1 u as

$E_{1u} = 1.66053 \times 10^{-27} \times (3 \times 10^{8})^2$ Joule

$\therefore\ E_{1u} = 1.49447\ \times\ 10^{-10}$ Joule

$\because 1\ eV = 1.6\ \times\ 10^{-19}$ Joule $\Longrightarrow$ 1 Joule = $\frac{1}{1.6 \times 10^{-19}}$ eV

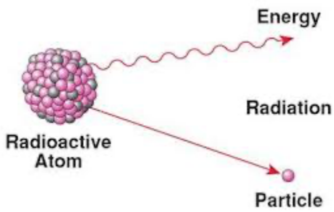
$\therefore\ E_{1u} = \frac{1.49447 \times 10^{-10}}{1.6 \times 10^{-19}} \approx 931\ \times\ 10^{6}$MeV $= 931$ MeV

Hence for a neutron

$m_{\nu} = 1.008665\, u = 1.008665\ \times 931$ MeV = 939.0671 MeV ≈ 1000 MeV

Radioactivity

It is that process wherein a nucleus attempts to attain stability by emission of either α or β particles or γ rays.



Radioactivity is a "statistical / probabilistic" process. Since, it is a statistical process, the law of radioactive decay states that the probability for decay at any given instant depends on (is proportional to) the number of radioactive atoms present at that given instant.

If at a given instant t, N, represents the number of radioactive atoms present, then the rate of decay (probability for decay) is

$$\frac{dN}{dt} \propto -N$$

Where the negative sign indicates that with passage of time the number the number of radioactive atoms left in the sample will decrease with time.

$$\frac{dN}{dt} = -\lambda N$$

Where λ is the constant of proportionality and is known as the "decay constant" and has units of $second^{-1}$.

$$\frac{dN}{N} = -\lambda dt$$

$$\int \frac{dN}{N} = -\lambda \int dt$$

$$\ln N = -\lambda t + k$$

Where k is the constant of intergration, and is determined from the initial conditions, i.e at $t = 0$, if N_o , is the number of radioactive atoms present, then

$$\ln N_o = -\lambda * 0 + k \implies \ln N_o = k$$

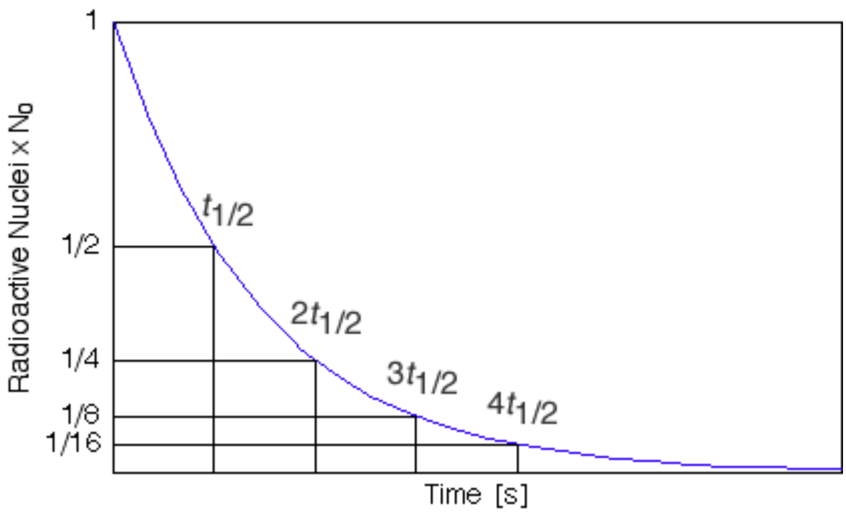
$$\therefore \ln N = -\lambda t + \ln N_o$$

$$\therefore \ln N - \ln N_o = -\lambda t$$

$$\therefore \ln \frac{N}{N_o} = -\lambda t$$

$$\therefore \frac{N}{N_o} = e^{-\lambda t}$$

$$\therefore N = N_o e^{-\lambda t}$$



Hence, with passage of time the number of nuclei "un-decayed" decreases.

We now, define a quantity, known as the **"half-life $T_{1/2}$",** which is the time interval in which the half the number of radioactive nuclei may have decayed, *i.e*

$$\frac{N_o}{2} = N_o e^{-\lambda T_{1/2}}$$

$$\frac{1}{2} = e^{-\lambda T_{1/2}}$$

$$\frac{1}{2} = \frac{1}{e^{\lambda T_{1/2}}}$$

$$\ln 2 = e^{\lambda T_{1/2}}$$

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.693}{T_{1/2}}$$

$$N(n) = N_o \left(\frac{1}{2}\right)^n$$

Where, $N(n)$ represents the number un-decayed of radioactive atoms, after "n" half-lives.

The fraction of radioactive isotope remaining after "n" half-lives are

Number of half-lives (n)	Fraction remaining
1	$1/2$
2	$1/4$
3	$1/8$
4	$1/16$
5	$1/32$
6	$1/64$

Therefore, after about 7 half-lives, we have about $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$, *i.e* less than 1 percent of the original atoms would remain.

We define a term known as **"Activity",** which provides us with a measure of the intensity of the radioactivity (radioactive transformations), and it is defined as

$$\frac{dN}{dt} = -\lambda N$$

Where the left hand side represents "Activity". The units for this are "counts(**d**isintegrations) **p**er **s**econd", and ignoring the negative sign , which has no numerical significance, we have

$$A(t) = \lambda N(t) = \lambda \times N_o e^{-\lambda t} = A_o e^{-\lambda t}$$

The conventional units for Activity is Curie, where

$$1\ Curie = 3 \times 10^{10}\ dps$$

The half-life and correspondingly the decay constant can be determined as

$$\frac{A(t)}{A_o} = e^{-\lambda t}$$

$$\ln \frac{A(t)}{A_o} = -\lambda t$$

$$\ln A(t) = \ln A_o - \lambda t$$

$$y(t) = y_o - mt$$

Hence, if the natural logarithm of the measured activity is plotted as a function of time, then we except a straight line whose slope is $-\lambda$.

We then define a term known as the "Mean (Average) life", which provides the average life expectancy of the atoms of a radioactive species. This average life is found from the sum of the times of existence of all the atoms divided by the initial number. If we consider N to be a very large number, we may approximate this sum by an equivalent integral, finding for the average life τ

$$\tau = \frac{1}{N_0}\int_0^\infty t dN = \frac{1}{N_0}\int_0^\infty \lambda t N dt = \lambda\int_0^\infty t e^{-\lambda t} dt = -\left[\frac{\lambda t + 1}{\lambda} e^{-\lambda t}\right]_0^\infty = \frac{1}{\lambda}$$

Alternatively, we can obtain the same expression, as

$$\tau \times \lambda = 1 \leftrightarrow \tau = \frac{1}{\lambda} = \frac{1}{0.693/T_{1/2}}$$

$$\tau = \frac{T_{1/2}}{0.693} = 1.141 \times T_{1/2}$$

Successive Radioactive Decay (transformations)

We usually encounter a situation, wherein the daughter of a radioactive decay is also Radioactive, for example consider the case of a radioactive isotope 1 decaying into another isotope 2, which in turn decays into a stable isotope 3, *i.e*

$$1 \longrightarrow 2 \longrightarrow 3$$

And let N_1, N_2 represent the number of atoms in 1 and 2 respectively at a given time, and the equations would be

$$\frac{dN_1(t)}{dt} = -\lambda_1 N_1(t), \ \& N_1(t) = N_{10} e^{-\lambda_1 t}$$

$$\frac{dN_2(t)}{dt} = formation\ rate - decay\ rate$$

$$\frac{dN_2(t)}{dt} = +\lambda_1 N_1(t) - \lambda_2 N_2(t)$$

$$\frac{dN_3(t)}{dt} = +\lambda_2 N_2(t) - \ decay$$

$$\frac{dN_2(t)}{dt} = +\lambda_1 N_1(t) - \lambda_2 N_2(t) = +\lambda_1 N_{10} e^{-\lambda_1 t} - \lambda_2 N_2(t)$$

$$\frac{dN_2(t)}{dt} + \lambda_2 N_2(t) = \lambda_1 N_{10} e^{-\lambda_1 t}$$

Multiply both sides of the equation by $e^{\lambda_2 t}$, we have

$$e^{\lambda_2 t}\,\frac{dN_2(t)}{dt} + \lambda_2 N_2(t) e^{\lambda_2 t} = \lambda_1 N_{10} e^{-\lambda_1 t} * e^{\lambda_2 t}$$

Now we know that the LHS of the above equation is

$$\frac{d\left[e^{\lambda_2 t}\, N_2(t)\right]}{dt} = e^{\lambda_2 t}\,\frac{dN_2(t)}{dt} + \lambda_2 N_2(t) e^{\lambda_2 t}$$

$$\frac{d\left[e^{\lambda_2 t}\, N_2(t)\right]}{dt} = \lambda_1 N_{10} e^{-\lambda_1 t} * e^{\lambda_2 t} = \lambda_1 N_{10} e^{(\lambda_2 - \lambda_1)t}$$

$$d\left[e^{\lambda_2 t}\, N_2(t)\right] = \lambda_1 N_{10} e^{(\lambda_2 - \lambda_1)t}\, dt$$

$$N_2\,(t) e^{\lambda_2 t} = \frac{\lambda_1 N_{10} e^{(\lambda_2 - \lambda_1)t}}{\lambda_2 - \lambda_1} + C$$

Where "C" is the constant of integration, and can be determined from the initial condition *i.e* $t = 0$, where $N_2(t) = 0$,

$$C = \frac{\lambda_1 N_{10}}{\lambda_2 - \lambda_1}$$

Hence,

$$N_2\,(t) e^{\lambda_2 t} = \frac{\lambda_1 N_{10} e^{(\lambda_2 - \lambda_1)t}}{\lambda_2 - \lambda_1} + \frac{\lambda_1 N_{10}}{\lambda_2 - \lambda_1}$$

$$N_2\,(t) = \left\{\frac{\lambda_1 N_{10} e^{(\lambda_2 - \lambda_1)t}}{\lambda_2 - \lambda_1} + \frac{\lambda_1 N_{10}}{\lambda_2 - \lambda_1}\right\} \times \frac{1}{e^{\lambda_2 t}}$$

$$N_2\,(t) = \left\{\frac{\lambda_1 N_{10} e^{(-\lambda_1)t}}{\lambda_2 - \lambda_1} + \frac{\lambda_1 N_{10} e^{-\lambda_2 t}}{\lambda_2 - \lambda_1}\right\}$$

$$N_2(t) = \frac{\lambda_1}{(\lambda_2 - \lambda_1)} N_{10}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

Further, when this equation is substituted in the third equation, and assuming the initial number of nuclei N_{30} as zero, we have

$$N_3(t) = N_{10}\left[1 + \frac{\lambda_1}{(\lambda_2 - \lambda_1)} e^{-\lambda_2 t} - \frac{\lambda_2}{(\lambda_2 - \lambda_1)} e^{-\lambda_1 t}\right]$$

If the half life of the first isotope is less than the half life of the second isotope, the overall result is that the number of nuclei of 1 will decrease exponentially as governed by its own half-life. The second isotope number which is initially zero, increase to a maximum and then decreases gradually. The third isotope which is the end product will increase in number and then approaches N_{10} as all the nuclei of 1 shall eventually decay to this stable product.

Radioactive Equilibrium

Let us consider the chain decay $1 \rightarrow 2 \rightarrow 3$, where the parent isotope (1), decays into a daughter isotope (2), which itself is radioactive and undergoes further decay to the product (3), which may or may not be stable. Radioactive Equilibrium is said to be attainted when the decay rate of a daughter radioactive isotope, equals the production rate of that isotope, primarily by the parent.

We know that

$$N_2(t) = \frac{\lambda_1}{(\lambda_2 - \lambda_1)} N_{10}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

And, at $t = 0$ & $\infty, N_2(t) = 0$. Hence, we should have a time t_m, when $N_2(t) = max$, which is obtained from the condition :

$$\frac{dN_2(t)}{dt} = 0$$

$$\frac{dN_2(t)}{dt} = \frac{\lambda_1 \cdot N_{10}}{(\lambda_2 - \lambda_1)} \left[-\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}\right] = 0 \; at \; t = t_m$$

Since λ_1, λ_2 & $N_{10} \neq 0$, the above condition implies

$$-\lambda_1 e^{-\lambda_1 t_m} + \lambda_2 e^{-\lambda_2 t_m} = 0$$

$$\lambda_1 e^{-\lambda_1 t_m} = \lambda_2 e^{-\lambda_2 t_m}$$

$$e^{(\lambda_2 - \lambda_1) t_m} = \frac{\lambda_2}{\lambda_1}$$

$$t_m = \frac{\ln\left[\frac{\lambda_2}{\lambda_1}\right]}{\lambda_2 - \lambda_1}$$

Secular Equilibrium

Equilibrium is attainted when the time derivatives in the rate equations are equal to zero.

$$\frac{dN_1(t)}{dt} = -\lambda_1 N_1(t) = 0$$

$$\frac{dN_2(t)}{dt} = +\lambda_1 N_1(t) - \lambda_2 N_2(t) = 0$$

$$\therefore\ \lambda_1 N_1(t) = \lambda_2 N_2(t)$$

$$\therefore\ \lambda_2 N_2(t) = \lambda_3 N_3(t)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore\ \lambda_{n-1} N_{n-1}(t) = \lambda_n N_n(t)$$

This cannot be realized in principle, since it would result in $\lambda_1 = 0$. However, this can be achieved if the parent has along half life compared to the daughter half-life. Then, the number of atoms of the initial parent is N_1 as a constant and the value of the decay constant is much smaller than the other decay constants in the chain, *i.e* when a succession of short lived isotopes arises from the decay or a relatively long lived parent. Such a situation is referred to as "secular equilibrium".

$$\therefore\ \lambda_1 N_1(t) = \lambda_2 N_2(t) = \lambda_3 N_3(t) = \cdots = \lambda_n N_n(t)$$

$$\therefore\ A_1(t) = A_2(t) = A_3(t) = \cdots = A_n(t)$$

$$\therefore\ \frac{N_1(t)}{T_1} = \frac{N_2(t)}{T_2} = \frac{N_3(t)}{T_3} = \cdots = \frac{N_n(t)}{T_n}$$

This helps us determine the half-life of the long lived parent (T_1), is the ratio of two isotopes in secular equilibrium is known along with the half-life of the daughter, *i.e*

$$T_1 = \left[\frac{N_1}{N_2}\right] T_2$$

$$N_2(t) = \frac{\lambda_1}{(\lambda_2 - \lambda_1)} N_{10}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

We know that,

Now, since the parent is long lived $T_2 \ll T_1$ & $\lambda_1 \ll \lambda_2$, hence,

$$(\lambda_2 - \lambda_1) \sim \lambda_2\ ;\ e^{-\lambda_1 t} = 1$$

$$N_2(t) = \frac{\lambda_1 \cdot N_{10}}{(\lambda_2)}\left(1 - e^{-\lambda_2 t}\right)$$

Now for $t \gg T_2$; $e^{-\lambda_2 t} \to 0$, hence

$$N_2(t) = \frac{\lambda_1 \cdot N_{10}}{(\lambda_2)}$$

Hence, in this situation, we have the decay rate of (1), and hence the production rate of (2), is approximately constant.

We know that since λ_1 is very small, we have the parent activity as $A_1 = R$, i.e the parent activity does not decrease measurably during several daughter half-lives. Now for small values of t,

$$N_2(t) = \frac{\lambda_1 N_{10}}{(\lambda_2 - \lambda_1)}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \sim \frac{R}{\lambda_2}\left(1 - e^{-\lambda_2 t}\right)$$

$$N_2(t) \cdot \lambda_2 = A_2 = R\ \left(1 - e^{-\lambda_2 t}\right) = R - R\, e^{-\lambda_2 t}$$

Hence, the total activity is

$$A = A_1 + A_2$$

At, $t = 0, e^{-\lambda_2 t} \rightarrow 1$;

$$A = R + R - R = R\ (A_1)$$

And for large $t, e^{-\lambda_2 t} \rightarrow 0$;

$$A = R + R - 0 = 2R$$

Transient Equilibrium

If the parent is long lived, but not too long lived, then we know that

$$T_1 > T_2 \ \Rightarrow\ \lambda_1 < \lambda_2$$

Hence, the equation,

$$N_2(t) = \frac{\lambda_1}{(\lambda_2 - \lambda_1)} N_{10}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

Can be rewritten as

$$N_2(t) = \frac{\lambda_1}{(\lambda_2 - \lambda_1)} N_{10}\left(e^{-\lambda_1 t}\right) = \frac{\lambda_1}{(\lambda_2 - \lambda_1)} N_1(t)$$

Hence,

$$\frac{N_2(t)}{N_1(t)} = \frac{\lambda_1}{(\lambda_2 - \lambda_1)}$$

And in terms of activities ($A[t] = \lambda\, N[t]$), we can rewrite the above equation as

$$\frac{A_2(t)}{A(t)} = \frac{\lambda_2\, N_2(t)}{\lambda_1\, N_1(t)} = \frac{\lambda_2}{\lambda_1} * \frac{\lambda_1}{(\lambda_2 - \lambda_1)} = \frac{\lambda_2}{(\lambda_2 - \lambda_1)}$$

Hence, the activity of the daughter is less than that of the parent by a factor of $\frac{\lambda_2}{(\lambda_2 - \lambda_1)}$

Naturally occurring Radioactive Decay :

Alpha Decay

The alpha decay is a nuclear transformation, wherein a heavy nucleus (exhibiting instability due to Coulomb repulsion) attempts to achieve a stable configuration, following the emission of an alpha particle (Helium nucleus ${}_2^4He$) :

$$_{Z}X_{N}^{A} \longrightarrow {}_{Z-2}Y_{N-2}^{A-4} + {}_{2}He_{2}^{4}$$

$$_{Z}X_{N}^{A} \longrightarrow {}_{Z-2}Y_{N-2}^{A-4} + \alpha$$

We can convince ourselves that the emission of an alpha particle is energetically favourable. Consider the following reaction :

$${}_{92}^{232}U \rightarrow {}_{90}^{228}U + \alpha$$

$$\Delta m = [(232.037156) - (228.028715 + 4.0026003)]\,(u)$$

$$= 5.838 \times 10^{-3}\,(u) = 5.838 \times 10^{-3}\,(u) \times 931.5\,\frac{MeV}{u}$$

$$= 5.458\, MeV$$

On the other hand, for the reaction :

$${}_{92}^{232}U \rightarrow {}_{91}^{231}U + p$$

$$\Delta m = [(232.037156) - (221.035880 + 1.007825)]\,(u)$$

$$= -\,6.549 \times 10^{-3}\,(u) = 6.549 \times 10^{-3}\,(u) \times 931.5\,\frac{MeV}{u}$$

$$= \boldsymbol{-6.123\, MeV}$$

The negative Q value for the second reaction indicates that the reaction can only proceed if we were to provide the energy from external source. On the other hand the positive Q value for the α- decay, indicates that the reaction would proceed on it's own, and the emitted alpha particle would have an energy equal to this Q value.

Since, the parent was initially at rest, the conservation of momentum demands,

$$m_\alpha * v_\alpha = M_d * V_d \Rightarrow V_d = \frac{m_\alpha}{M_d} * v_\alpha = \frac{4}{A-4} * v_\alpha$$

Hence, the recoil energy of the daughter nucleus, following the alpha emission would be,

$$E_{recoil} = \frac{1}{2} * M_d * {V_d}^2 = \frac{1}{2} * M_d * \left[\frac{m_\alpha}{M_d}\right]^2 * {v_\alpha}^2$$

$$= \frac{1}{2} * m_\alpha * {v_\alpha}^2 * \frac{m_\alpha}{M_d} = E_\alpha * \frac{m_\alpha}{M_d}$$

Beta Decay

For a majority of unstable nuclides, the dominant pathway of transformation is via the beta-decay process. Qualitatively, beta-decay converts a neutron to a proton or vice versa. Thus beta decay is characterized by $\delta A = 0$ & $\delta Z = \pm 1$.

Following the conservation of charge and momentum beta decay can be represented as :

$$\begin{aligned}
{}_0\mathrm{n}_1^1 &\longrightarrow {}_1\mathrm{p}_0^1 + e^- + \bar{\nu} \\
{}_\mathrm{Z}\mathrm{X}_\mathrm{N}^\mathrm{A} &\longrightarrow {}_\mathrm{Z+1}\mathrm{Y}_\mathrm{N-1}^\mathrm{A} + e^- + \bar{\nu} \\
{}_1\mathrm{p}_0^1 &\longrightarrow {}_0\mathrm{n}_1^1 + e^+ + \nu \\
{}_\mathrm{Z}\mathrm{X}_\mathrm{N}^\mathrm{A} &\longrightarrow {}_\mathrm{Z-1}\mathrm{Y}_\mathrm{N+1}^\mathrm{A} + e^+ + \nu
\end{aligned}$$

Wherein, υ, is a massless particle know as "neutrino", for momentum conversation (sharing of energy, which results in a continuous energy spectrum for the beta particle).

Gamma Decay

Following the alpha and or beta decay, the parent and daughter nuclei are different. However, gamma decay results in the energetic colling down of an excited (X^* : energetically higher state) nucleus, to it's ground state (X : stable configuration, characterized by the lowest energy state). Thus gamma decay could be schematically represented as

$$X^* \rightarrow X + \gamma$$

Gamma rays are nuclear in origin, and are electromagnetic rays, travelling with speed of light, and can propagate through vacuum as well. If we were to assume a gamma ray having energy $E_\gamma = 439\ keV$, (this gamma ray de-excites two levels, which are an energy difference of 439 keV) , we know that

$$E_\gamma = h\,\upsilon = h\frac{c}{\lambda}$$

Hence, the wavelength associated with this gamma photon (quanta of radiation) is,

$$\lambda = \frac{h\,c}{E_\gamma} = \frac{(6.6\ \times 10^{-34})\ \times\ (3\ \times\ 10^{8})}{(439\ \times\ 10^{3}\ \times\ 1.6\ \times 10^{-19})} = 2.8\ \times\ 10^{-12}\ m$$

We know that the momentum of the gamma ray (p_γ), is related to it's energy (E_γ), as

$$p_\gamma = \frac{E_\gamma}{c}$$

Since, the parent nucleus was initially at rest, conservation laws demand that ,

$$P_d = p_\gamma$$

$$T_d = \frac{{P_d}^2}{2\ \times\ M_d} = \frac{{P_\gamma}^2}{2\ \times\ M_d} = \frac{{E_\gamma}^2}{2\ \times\ M_d\ \times\ c^2}$$

Where, T_d, is the recoil energy of the daughter nucleus.

Mass Defect

It is observed that when a neutron and a proton combine to form a deuteron, the process is accompanied by the emission of a γ gamma with $E_\gamma \sim 2.33$ MeV.

$$\nu + \pi \rightarrow d + \gamma$$

The recoil energy of the deuteron is very small and can be neglected. Since conservation laws are never violated, we can conclude that the mass of the deuteron (in energy units, as mass and energy are equivalent) would be less than the mass of the proton and neutron (the reactants) by an amount of 2.33 MeV.

Hence, it is observed that the mass of the nucleus is always less than the mass of its constituent nucleons. This difference in the mass is referred to as **Mass Defect** (Δm) and is customary expressed in amu

If Z number of protons and N number of neutrons combine to form a nucleus say X, then

$$Z\pi + N\upsilon \rightarrow X \; ; \qquad Z * m_p + N * m_n \;\rightarrow\; M_x \,,$$

where : m_p, m_n, M_x : represent the mass of a proton, neutron and the atom in amu.

$$\therefore\; \Delta m = \; [mass\ of\ consituent\ nuceons] - \; [mass\ of\ the\ atom]$$

$$\therefore \Delta m = \; \left[\left(Z * m_p\right) + \; (N * \; m_n)\right] - \; [M_x]$$

Now, from the energy-mass equivalence, we shall have an energy equivalent of this mass defect. This is referred to as **Binding Energy**,

$$Binding\ Energy = \; \Delta m \; \times \; 931 \text{ MeV}$$

This represents the energy required to break / disassemble the nucleus into its independent constituent nucleons.

Now the Binding Energy of the last neutron in ^{13}C, is

$$C^{12} + \;\; n \;\rightarrow C^{13}$$

$$\Delta m = \; [(M_{12C}) + \; (m_n)] - \; [M_{13C}]$$

$$\Delta m = \; [(12) + \; (1.00866)] - \; [13.00335] = 0.00531\, u$$

$$B.E = 0.00531 \; \times \; 931 \text{ MeV} = 4.943 \text{ MeV}$$

Another useful quantity is the **binding energy per nucleon**, this is the average energy needed to **"remove a single nucleon"** from the nucleus. It is defined as:

Binding Energy per nucleon = Binding Energy/ Nucleon Number

It is denoted by B.E/A, and this quantity represents the "stability" of the nucleus. Higher the B.E/A, higher is its stability. Elements with a high binding energy per nucleon are very difficult to break apart. This is also referred to as Packing Fraction.

The graph below illustrates the Binding Energy per nucleon as a function of the nucleon number (mass number) “A”.

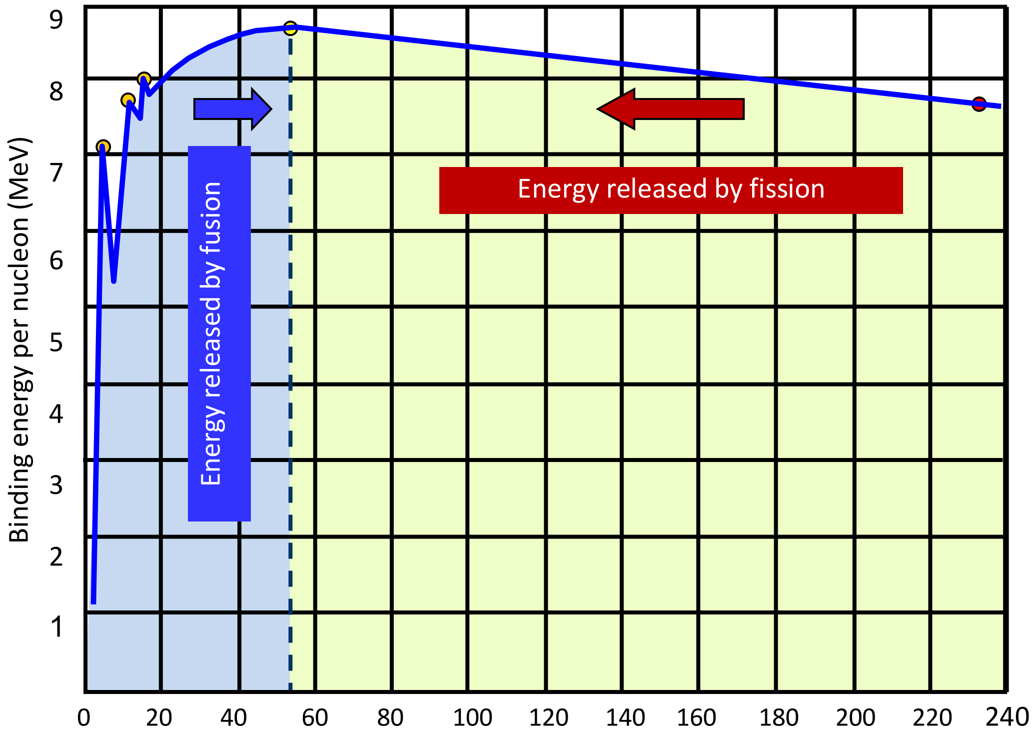


Figure 1

As seen from the graph, ^{56}Fe, has the highest binding energy per nucleon, and hence has the highest abundance (due to its stability)

The B.E/A remains almost constant at ~ 7-8 MeV/nucleon, which is indicative of the short range of nuclear force, as the nucleon is said to interact with only its immediate neighbours a quantity which is approximately constant across the nuclear landscape.

Nuclei would like to attain a configuration which is has a higher stability (higher BE/A) and hence light nuclei would prefer to “fuse”, while heavier nuclei would like to “fission”. The fission process consists of splitting a nucleus into roughly equal parts

The difference in the B.E/A (between the reactants and the final product(s)) is released and hence fusion of light nuclei and fission of heavier nuclei results in the release of energy.

Representative B.E/A are presented in the table below.

Element	Mass of nucleons (u)	Nuclear Mass (u)	Binding Energy (MeV)	Binding Energy per Nucleon (MeV)
Deuterium	2.01594	2.01355	2.23	1.12
Helium 4	4.03188	4.00151	28.29	7.07
Lithium 7	7.05649	7.01336	40.15	5.74
Beryllium 9	9.07243	9.00999	58.13	6.46
Iron 56	56.44913	55.92069	492.24	8.79
Silver 107	107.86187	106.87934	915.23	8.55
Iodine 127	128.02684	126.87544	1072.53	8.45
Lead 206	207.67109	205.92952	1622.27	7.88
Polonium 210	211.70297	209.93683	1645.16	7.83
Uranium 235	236.90849	234.99351	1783.80	7.59
Uranium 238	239.93448	238.00037	1801.63	7.57

Now, just a back-of-the envelope calculation for the energy released in the fission of a hypothetical heavy nucleus, with A = 240, which splits into two medium mass nuclei with $A_1 \sim 100$ and $A_2 \sim 140$ would be, the difference in the B.E/A_{240} and the B.E/A_{100}, ie

$$\Delta BE = 7.57 - 8.55 \sim 1\ MeV\ per\ nucleon$$

Since in one atom of this heavy nucleus we have 240 nucleons, the total energy released in the fission of one atom is 240 MeV.

In 240 grams of the material (1 mole) we have $6\ \times 10^{23}$number of atoms. Hence the energy released in the fission of 1 mole of a heavy nucleus is

$$\sim 6\ \times 10^{23} \times 240 \text{ MeV} = 6\ \times 10^{23} \times\ 240\ \times 10^{6} \times\ 1.6\ \times 10^{-19} \text{ Joule.}$$

Therefore fission of a heavy nucleus results in a release of considerable amount of energy. This is the basic underlying principle of a nuclear reactor.

Nuclear Reactions : Q Value

Let us consider a nuclear reaction wherein the target (a), which is at rest (hence only has rest mass energy), and an energetic projectile (b), (which would have both kinetic as well as rest mass energy) fuse / combine to form two product nuclei $(c\ , d)$, *i.e*

$$a + b \rightarrow c + d$$

We now define a quantity known as the Q-value which is essentially related to / analogous the mass defect, and is defined as

$$Q = [(m_a) + (m_b)] - [(m_c) + (m_d)], \text{ where the masses are in amu.}$$

Following the convention used in Chemical reactions a positive Q value results in the release of energy (exothermic reaction), whereas a negative Q value requires energy to be supplied for the reaction to occur (endothermic reaction).

Now let us consider the DT fusion reaction, where we have a deuterium and a triton (isotopes of Hydrogen) fuse /combine to form an alpha particle plus a neutron

$$_1D^2 + {}_1T^3 \rightarrow {}_0n^1 + {}_2He^4$$

Now,

$$Q = [(2.014102) + (3.016049)] - [(1.008665) + (4.002604)] = 0.018882\, u$$

$$\therefore Q = 0.018882 \times 931 = 17.58 \text{ MeV}$$

Now, let us consider the neutron induced fission of a heavy nucleus (such nuclei are referred to as “fissile” nuclei). We know that the absorption of a neutron by a fissile nucleus results in the formation of two medium mass nuclei, along with the emission of 2 energetic neutrons.

$$_0n^1 + {}_{92}U^{235} \rightarrow {}_{36}Kr^{97} + {}_{56}Ba^{137} + 2{}_0n^1$$

Now the Q value for this reaction is obtained as

$$Q = [(m_n) + (M_U)] - [(M_{kr}) + (M_{Ba}) + (2\, m_n)]$$

$$Q = [(1.00867) + (235.04390)] - [(96.92120) + (136.90610) + (2 \times 1.00867)]$$

$$Q = 0.20821\, u = 0.020821 \times 931 \text{ MeV} = 193.94 \text{ MeV}$$

Hence, from a single fission event we have about 1000 MeV of energy released as compared to the 17 MeV released in the DT fusion reaction.some heavy nuclei also exhibit spontaneous fission, which is very similar to the neutron induced fission process.

Mass & Energy

When a body is in motion, its mass increases relative to an observer at rest, according as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

Where m_0, m are the rest-mass, and mass of the body at any velocity v. At small (low) velocities, there is very little difference between m_0 & m $(m \rightarrow m_0\ as\ v \rightarrow 0)$. We know from the theory of relativity, that mass and energy are interchangeable and are related by

$$E = m\, c^2$$

Now, the $total\ energy$ of the particle would be given by the sum of its $rest - mass\ energy$ & its $kinetic\ energy,$

$$E_{total} = mc^2$$

Hence, the kinetic energy would be given by

$$E_{KE} = total\ energy - rest\ mass\ energy = m\,c^2 - \ m_0\,c^2$$

Substituting the value of m, from the above equation we have,

$$E_{KE} = \frac{m_0}{\sqrt{1 - \ v^2/c^2}}\,c^2 - \ m_0\,c^2 \ = \ m_0\,c^2\left[\left[\frac{1}{\sqrt{1 - \ v^2/c^2}}\right] - 1\right]$$

$$E_{KE} \sim \ m_0\,c^2\left\{\left[1 + \frac{1}{2} \times \frac{v^2}{c^2} \dots\dots\dots\dots\dots\dots.\right] - 1\right\}$$

$$E_{KE} \ = \ \frac{1}{2}\,m\,v^2$$

For $v \ll c$, when we neglect the higher order terms. This equation is valid only when the kinetic energy is small compared with the rest mass energy, *i.e*

$$\frac{1}{2}\ m_0 v^2 \ \ll \ m_0\,c^2$$

It has been demonstrated that the conventional equation for Kinetic Energy is valid for most cases when $v \ \leq 0.2\ c$ or

$$E \ \leq 0.02\ E_{rest}$$

We know that the rest mass of an electron is 0.511 MeV, hence the relativistic formula be used for all electron energies $\geq 0.02 \ \times 0.511\ MeV \ = 0.010\ MeV = 10\ keV$. On the other hand the rest mass of a neutron is almost 1.000 MeV, hence $0.02 \ \times 1\ MeV = 20\ MeV$. In practice rarely do we come across neutrons having such large kinetic energies, hence for all practical purposes, we use the non-relativistic relation while dealing with neutrons, i.e

$v = 1.383 \ \times \ 10^6\,\sqrt{E}$, where v is in cm/sec & E,the kinetic energy in eV.

These equations are valid only for particles with non-zero rest mass, and hence do not apply to photons, whose kinetic energy is given by

$$E = h\,\nu,$$

where h : $Planck's\ constant \ = 6.6 \ \times \ 10^{-34}\,Joule - second$ and ν : $frequency$

Some beta particles given off by radioactive nuclei have velocities as high as $0.99c$. Then these particles will have Kinetic energy

$$E_{KE} = \frac{m_0}{\sqrt{1 - v^2/c^2}} c^2 - m_0 c^2 = m_0 c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]$$

$$= m_0 c^2 \left[\frac{1}{\sqrt{1 - (0.99)^2}} - 1\right] = 6.07 \, m_0 c^2$$

For an electron $m_0 c^2 = (9.31 \times 10^{-31}) \times (3 \times 10^8)^2$

Hence, $E_{KE} = 6.07 \times 8.379 \times 10^{-14} \, Joule = \frac{5.08 \times 10^{-13}}{1.6 \times 10^{-19}} \, eV = 3.17 \times 10^6 \, eV$

Quantum theory

It is well known that energy can be transmitted through either "waves" or "particles". A particle is considered to be a consolidated (focused) structure having mass and inertia respectively. A particle in motion posses kinetic energy mv^2 and momentum mv, where the symbols have their usual meaning. In case of wave motion, the energy is transmitted by the vibrations and waves are characterized by frequency of vibration (ν), velocity of propagation (v) and the wavelength (λ).

Plank's hypothesis established that energy transfer could be a discrete process, where the transfer occurs in integral multiples of a fundamental unit known as the photon, whose magnitude is, $E = h\,\nu$, where $h = 6.6 \times 10^{-34} \, Joule - second$, is the Plank's constant.

It was de Brogile who inter-linked the particle and the wave picture, with his hypothesis, that matter (particulate in nature) has an associated wavelength (wave nature0, such that the wavelength of a particle of mass m, moving with a velocity v, is

$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2\,m\,E}}$, where E is the kinetic energy of the particle.

If, we were to express m in grams, E, in eV $(1.6 \times 10^{12} \, erg)$ then $h = 4.136 \times 10^{-15} \, eV - sec$, then for a neutron $(m = 1.67 \times 10^{-24} \, grams)$ we have

$$\lambda = \frac{2.8 \times 10^{-9}}{\sqrt{E}} \, cm$$

Hence, if we were to consider a slow neutron, a neutron with energy of about $0.03\ eV,$ then $\lambda \sim 10^{-8}\ cm$, which is approximately of the same order of magnitude as the diameter of an atom. On the other hand if we have fast neutrons, whose energy is $1\ MeV,$ then $\lambda \sim 10^{-12}\ cm.$ Hence, slow neutrons which behave as if they were as large as the atom, have a better chance / probability of interacting with the atomic nuclei, rather than the fast (energetic) neutrons.

Atomic & Molecular Weight

The atomic weight of an atom is defined as the mass of the neutral atom relative to the mass of a neutral ^{12}C atom, on a scale where the atomic weight of ^{12}C is taken to be 12 units.

However, in nature elements are found to have isotopes (same proton number but different number of neutrons). Then the atomic weight of the element is defined as the average atomic weight of the mixture. Hence, if γ_i is the isotopic abundance of the i^{th} isotope of atomic weight M_i, than the atomic weight of the element is

$$M = \sum_i {(\gamma_i\ M_i\ )}/{100}$$

The total mass of a molecule relative to the mass of a neutral ^{12}C atom is called the molecular weight.

For oxygen we have the following abundances

Isotope	Abundance (a/o)	Atomic weight
^{16}O	99.759	15.99492
^{17}O	0.037	16.99913
^{18}O	0.204	17.99916

Hence, $M\ (O) = \frac{1}{100}\ [(\gamma_{16}\ \times\ M_{16}) + \ (\gamma_{17}\ \times\ M_{17})] + \ (\gamma_{18}\ \times\ M_{18})$

$$M\ (O) = \frac{1}{100}\ [(99.759\ \times\ 15.999492) + \ (0.037\ \times\ 16.99913)] + \ (0.204\ \times\ 17.99916)$$

$$M\ (O) = \ 15.99938$$

Now, natural Uranium is composed of three isotopes $^{234,235,238}U$. Their abundances and atomic weights are

Isotope	Abundance	Atomic weight
^{234}U	0.0057	234.0409
^{235}U	0.72	235.0439
^{238}U	99.27	238.0508

Now $M_U = \frac{1}{100}\left[(\gamma_{234} \times M_{234}) + (\gamma_{235} \times M_{235}) + (\gamma_{238} \times M_{238})\right]$

$$M_u = \frac{1}{100}\left[(0.0057 \times 234.0409) + (0.72 \times 235.0439) + (99.27 \times 238.0508)\right]$$

$$= 238.0186$$

Atomic and molecular weights being ratios are unitless, but the gram atomic or gram molecular weights are defined as the amount of substance having a mass in grams equal to the atomic or molecular weight of the substance. Thus one gram atomic weight or one mole of a ^{12}C is 12 g of this isotope.

Atom Density

If the target material is an element of atomic/molecular weight M, then 1 mole (weight in grams) of the substance would contain $N_A = 6.02 \times 10^{23}$ number of atoms. We then define a quantity known as the atom density (N), also referred to as "number density" as the number of atoms per cubic meter, i.e

$$N = \frac{\rho\, N_A}{M} \text{ , where } \rho \text{ is the density in } gm/cc$$

For example, if the density of Na is $\rho_{Na} = 0.97\ gm/cc$, and its atomic weight is 22.990 then the atom (number) density is

$$N = \frac{\rho\, N_A}{M} = \frac{(0.97\ gm/cc) \times (6.022 \times 10^{23}\ atoms/mol)}{22.90\ gm\ /mol} = 0.0254 \times 10^{24}\ atoms/cc$$

If a reactor is fueled with 1500 kg of uranium rods enriched to 20 w/o in ^{235}U (atomic weight 235.043) and the remainder is ^{238}U (atomic weight 238.0508) compute the atom densities of both 235 & 238U.

$$N_{235} = \frac{w_{235} \times \rho \times N_A}{M_{235}} = \frac{0.20 \times 19.1 \times 6.022 \times 10^{23}}{235.0439} = 9.79 \times 10^{21}\ atoms/cc$$

$$N_{238} = \frac{w_{238} \times \rho \times N_A}{M_{238}} = \frac{0.80 \times 19.1 \times 6.022 \times 10^{23}}{238.0508} = 3.86 \times 10^{22}\ atoms/cc$$

Similarly, we can compute the number density for a compound, (mixture), Z, which is composed of elements X & Y, having density and molecular weight as ρ and M respectively, from the above equation as :

$$N_i = n_i \frac{\rho N_A}{M}$$

Where n_i is the number of atoms of the kind i, in the compound.

For example, if the density of UO_2 $\rho_{UO2} = 10.5\ gm/cc$, then the number density of natural uranium in UO2 is

$$N_U = \frac{\rho_{UO2} N_A}{M} = \frac{(10.5\ gm/cc)\ (6.02 \times 10^{23}\ atoms/mol)}{[238.0289 + \{2 \times (15.9994)\}]\ g/mol} = 2.34 \times 10^{22}\ atoms/cc$$

Sometimes, it is important to compute the concentration of an individual isotope i, given its fractional abundance γ_i, abbreviated as " a/o ",

$$N_i = \frac{\gamma_i \rho N_A}{M_e} = \frac{\gamma_i \rho N_A}{\sum_i \gamma_i M_i}$$

If ^{235}U has an abundance of 3 a/o, in UO_2, then the concentration of ^{235}U is ($\rho = 10.5\ gm/cc$)

$$N_{235} = \gamma_{235} \frac{\rho N_A}{\sum_i \gamma_i M_i}$$
$$= \frac{3}{100} \times \left\{\frac{10.5 \times 6.022 \times 10^{23}}{[238.05 \times 0.97] + [235.04 \times 0.03] + [2 \times (15.9994)]}\right\}$$
$$= 7.03 \times 10^{20}\ atoms/cc$$