Linear Algebra and Logic Gates

- Hilbert Space
- Inner Products and Tensor Products
- Hermitian and Unitary Matrices
- Pauli-gates
- The Hadamard Gate
- Simulations

" Quantum mechanics is a mathematical framework or set of rules for the construction of physical theories "

Hilbert Space

In the finite dimensional case, a Hilbert Space refers to an inner product space, i.e., a vector space equipped with an inner product function.

Vector Space, $V(+, x_{scalar})$

- + : Closure, Commutativity, Associativity, Identity Element (0), Inverse Element (-a)
- x : Closure, Associativity, Identity Element (1)
- Distributivity of scalar and vector sums

Vector Space -- from Wolfram MathWorld

Inner Product, $(-,-)$

- Linear in second argument
- $(x,y) = (y,x)^*$
- (x,x) *is* non-negative, zero iff $x = 0$

Quantum Computation and Quantum Information,
Nielsen Chunag, 10th Anniversary Edition, Page 65

Hilbert Space

We will use the following:

Vector Space, $V(+, x_{\text{scalar}})$

$$\mathbb{C}^n$$

Inner Product, $(-,-)$

$$((y_1, \ldots, y_n), (z_1, \ldots, z_n)) \equiv \sum_i y_i^* z_i = [y_1^* \ldots y_n^*] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

Hermitian and Unitary Matrices

Adjoint or Hermitian Conjugate

$$A^\dagger \equiv (A^*)^T$$

$$(|v\rangle, A|w\rangle) = (A^\dagger|v\rangle, |w\rangle)$$

Bra and Ket are Adjoints of each other

Self-adjoint or Hermitian Matrix

A matrix which is equal to its Hermitian conjugate

All eigenvalues are real

Unitary Matrix

$$U^\dagger U = UU^\dagger = I$$

All eigenvalues have a modulus of 1

Any Unitary matrix represents a valid quantum gate.

Unitary matrices are always invertible, and their inverse is also Unitary. Hence, every quantum gate is reversible

Inner Products and Tensor Products

This section is mainly concerned with the notations encountered frequently

Notation	Description
z^*	Complex conjugate of the complex number z. $(1+i)^* = 1 - i$
$\|\psi\rangle$	Vector. Also known as a *ket*.
$\langle\psi\|$	Vector dual to $\|\psi\rangle$. Also known as a *bra*.
$\langle\varphi\|\psi\rangle$	Inner product between the vectors $\|\varphi\rangle$ and $\|\psi\rangle$.
$\|\varphi\rangle \otimes \|\psi\rangle$	Tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$.
$\|\varphi\rangle\|\psi\rangle$	Abbreviated notation for tensor product of $\|\varphi\rangle$ and $\|\psi\rangle$.
A^*	Complex conjugate of the A matrix.
A^T	Transpose of the A matrix.
$A^\dagger$	Hermitian conjugate or adjoint of the A matrix, $A^\dagger = (A^T)^*$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$.
$\langle\varphi\|A\|\psi\rangle$	Inner product between $\|\varphi\rangle$ and $A\|\psi\rangle$. Equivalently, inner product between $A^\dagger\|\varphi\rangle$ and $\|\psi\rangle$.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2 2

$\rightarrow$ 4

Pauli Gates

The most frequently used quantum logic gates

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$Z \quad |1\rangle \quad -|1\rangle$

$\underline{\alpha|0\rangle + \beta|1\rangle}$

0 1 $\alpha^2 + \beta^2 = 1$

$\psi = \alpha|0\rangle + \beta|1\rangle \longrightarrow \boxed{\uparrow}$

$\alpha^2 \longrightarrow |0\rangle$

$\beta^2 \longrightarrow |1\rangle$

$\alpha^2 + \beta^2 = 1$

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

also known as the quantum NOT gate

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

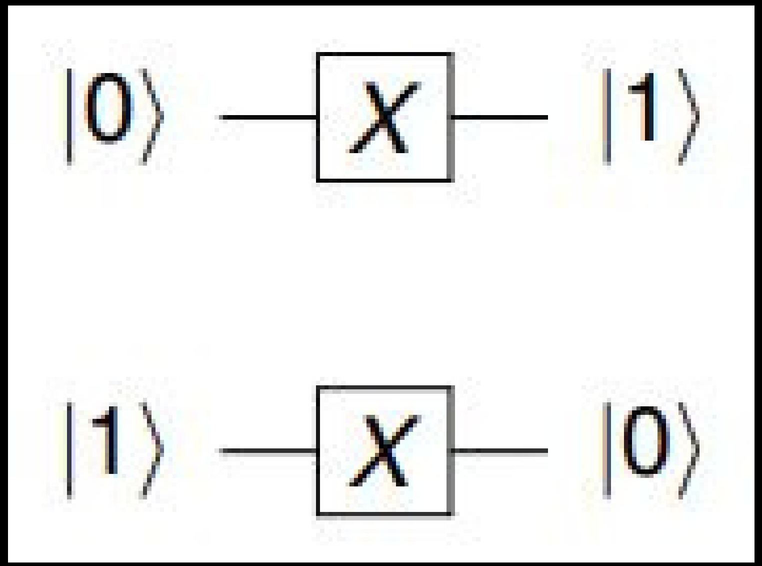
- I : Identity operator; does not affect the qubit
- X : Rotation by π around x-axis (on the bloch sphere)
- Y : Rotation by π around y-axis (on the bloch sphere)
- Z : Rotation by π around y-axis (on the bloch sphere)

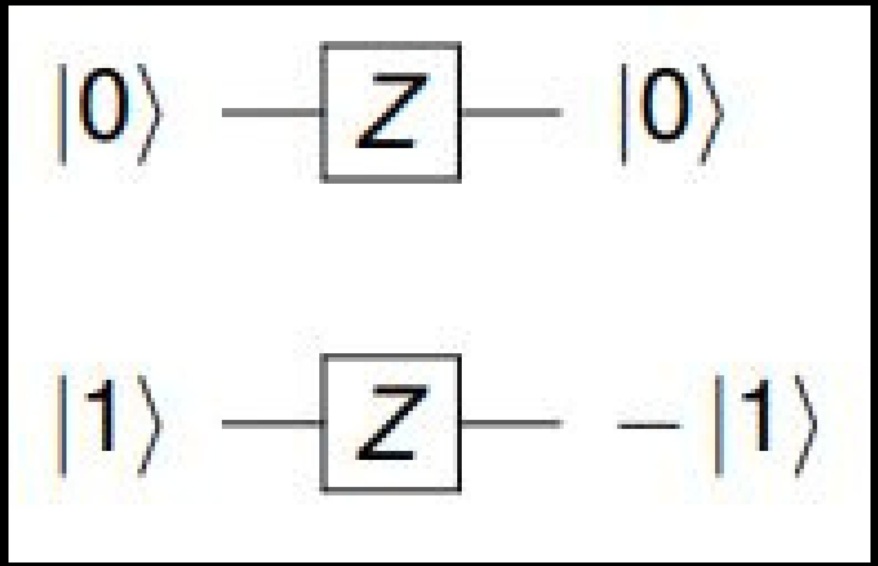
All four Pauli gates are Unitary and Hermitian!

Pauli Gates

The most frequently used quantum logic gates

"Truth Tables"



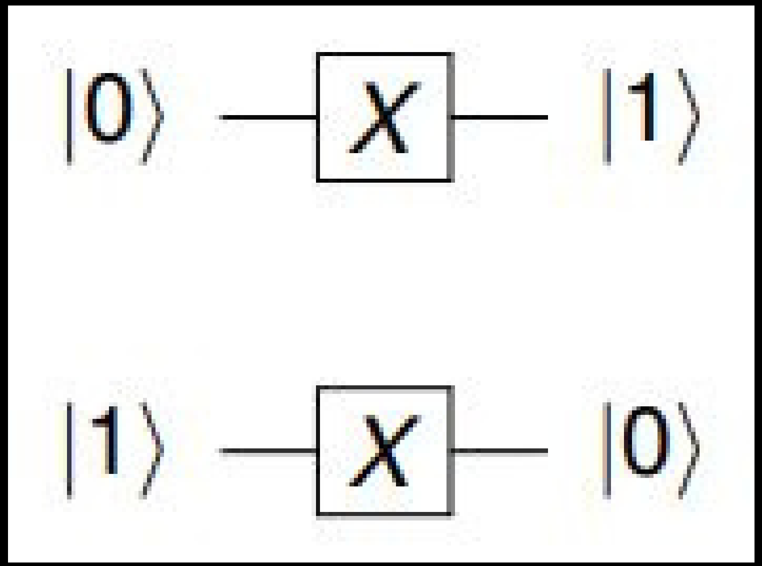


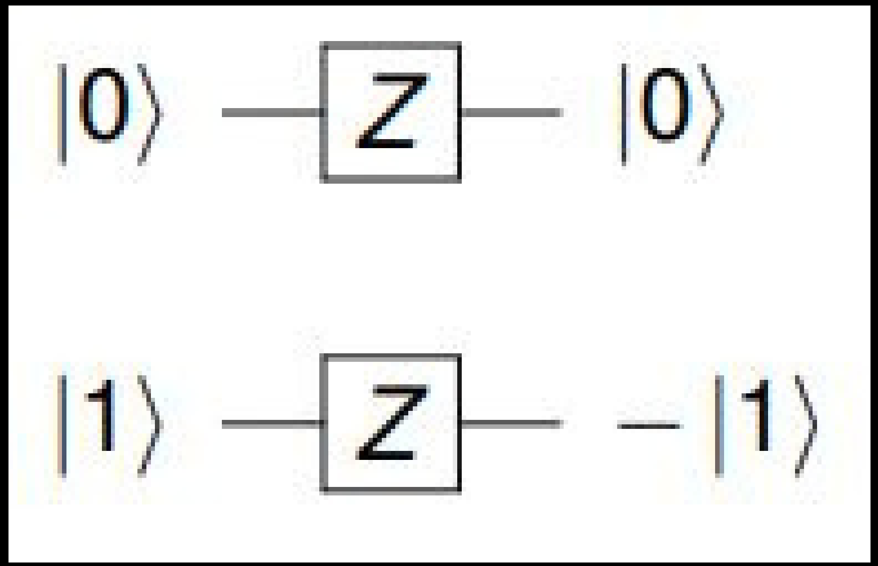
- $Y = iXZ$

Pauli Gates

The most frequently used quantum logic gates

"Truth Tables"





- *Y = iXZ*

  |0>   --> -i |1>
  |1>   --> i|0>

The Hadamard Gate

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

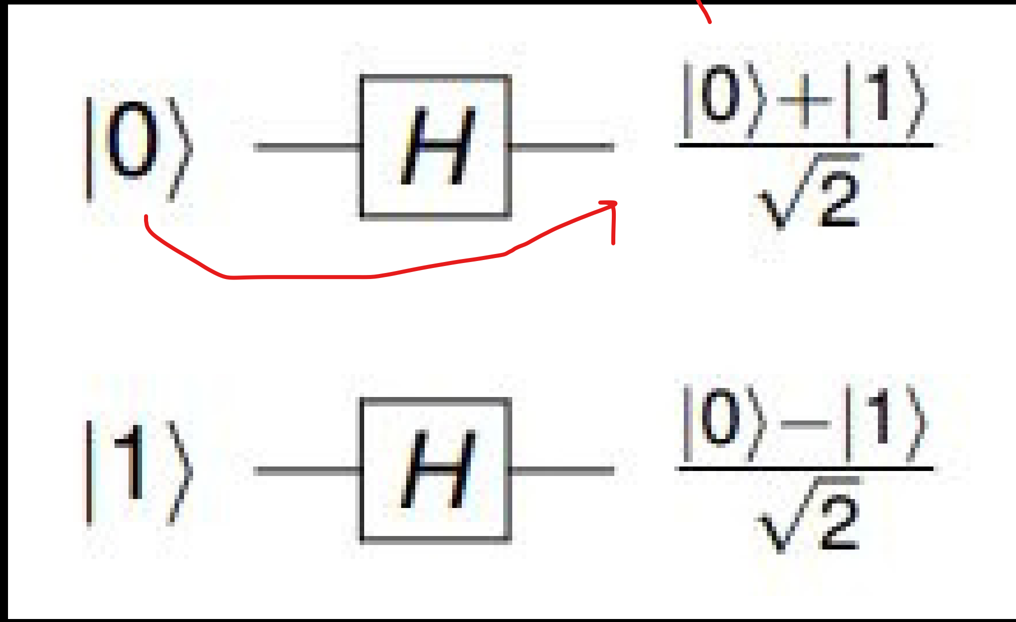
$H \quad |0\rangle$

$$= \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\right)$$

Used extensively for state preparation

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Takes a pure computational state into a superposition



$$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle := \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

These states appear often, and hence have their own representation. They can also be used as an alternative basis, called the 'Hadamard basis', in place of using the computational basis

Utilizing Superposition and Entanglement are two of the ways quantum computers attain time and space efficiency as compared to classical computers!

$$= \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

Simulations

Please head over to the following link: https://algassert.com/quirk

Thank you!!