Assigning Probabilities: Axiomatic Approach

Definition

Let Ω be a sample space associated with a random experiment. [redacted]. A probability [redacted] (or [redacted]) is a set function P, [redacted], satisfying the following three axioms:

$P: 2^\Omega \to [0,1]$

(i) $P(E) \geq 0$ for all E [redacted] $\subseteq \Omega$ (Axiom 1: Non-negativity);

(ii) If $E_1, E_2, \ldots$ is a countable infinite collection of mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i); \text{ (Countable infinite additivity);}$$

$$\boxed{E_i \cap E_j = \phi \quad \forall i \neq j}$$

$$\boxed{P(E_i) = \frac{1}{2^i}}$$

$\mathbb{N}$

$E_i = \{i\}, \; i \in \mathbb{N}$

$\bigcup_{i=1}^{\infty} E_i = \mathbb{N}$

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(iii) $\boxed{P(\Omega) = 1.}$

$P : \mathcal{P}(\mathbb{N}) \to [0,1]$

$P(\{i\}) = \frac{1}{2^i}, \quad i \in \mathbb{N}.$

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(iii) $P(\Omega) = 1$.

Note the followings:

(1) Ω is also called the event space.

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(2) The members of Ω are called events.

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(1) [illegible] is also called the event space.

(2) The members of [illegible] are called events.

(3) A countable collection $\{E_i : i \in \Lambda\}$ of events is said to be exhaustive if $\cancel{P\left(\bigcup_{i\in\Lambda} E_i\right) = 1}$

$$\boxed{\bigcup_{i\in\Lambda} E_i = \Omega}\ ,$$

Questions

(1) Is it possible to assign probabilities to all subsets of Ω, when Ω is countable?

$x : \underline{\Omega} \to \underline{\mathbb{N}}$

$P(\{p_i\})$

$\{p_1, p_2, p_3, \ldots\}$

$P(\{p_3\}) = \frac{1}{2^3}$; $P(\{p_i\}) = \frac{1}{2^i}$.

Questions

(1) Is it possible to assign probabilities to all subsets of Ω, when Ω is countable?

Ex; (2) In any probability space (Ω, [illegible] P), we have $\boxed{P(\Omega) = 1.}$ Whether $P(A) = 1$ implies $A = \Omega$ or not?

Problem

Suppose that five cards are drawn at random and without replacement from a deck of 52 cards. Assuming that the outcomes are equally likely, find the probabilities for the following events:

(1) E_1 : the event that each card is spade.

(2) E_2 : the event that at least one of the drawn cards is spade.

(3) E_3 : the event that among the drawn cards three are kings and two are queens.

(4) E_4 : the event that among the drawn cards two are kings, two are queens and one is Jack.

(1) E_1 : Each card is spade

$$P(E_1) = \frac{^{13}C_5}{^{52}C_5}$$

(2) E_2 : Atleast one of the drawn cards is spade.

E_2^c : None of the drawn card is spade

$$P(E_2^c) = \frac{^{39}C_5}{^{52}C_5}$$

$$\Omega = E_2 \cup E_2^c$$

$$\Rightarrow P(\Omega) = P(E_2) + P(E_2^c) \Rightarrow P(E_2) = 1 - P(E_2^c)$$

(3) E_3: Among the drawn cards three are kings and two are queens.

$$P(E_3) = \frac{{}^4C_3 \times {}^4C_2}{{}^{52}C_5}$$

(4) E_4: Two kings + Two Queens + One Jack.

$$P(E_4) = \frac{{}^4C_2 \times {}^4C_2 \times {}^4C_1}{{}^{52}C_5}$$

$\boxed{\text{Conditional}}$ Probability

$$\Omega = \{ (i, j) \; ; \; i = 1, 2, \ldots, 6 \; ; \; j = 1, 2, \ldots, 6 \}$$

i: 1st dice shows
j: 2nd dice shows

Let $P(\{(i,j)\}) = \dfrac{1}{36}$

Let E_1 : The sum is 8.

$E_1 = \{ (2,6), (3,5), (4,4), (5,3), (6,2) \}$

$P(E_1) = \dfrac{5}{36}$

Given that E_1 has occured, find the probability that the first dice shows 3.

$$P(\underbrace{\text{The } 1^{st} \text{ dice shows 3}}_{E_2} / E_1) = \frac{1}{5}.$$

$P(E_2) = \dfrac{1}{6}$, $P(E_2 \cap E_1) = \dfrac{1}{36}$

Now, $\boxed{\dfrac{P(E_2 \cap E_1)}{P(E_1)}} = \dfrac{1/36}{5/36} = \dfrac{1}{5}$

Def:

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

provided $P(B) > 0$.

(i) $P(A|B) \geq 0$

(ii) $P(A_1 \cup A_2 | B) = P(A_1|B) + P(A_2|B)$, $A_1 \cap A_2 = \phi$.

(iii) $P(B \mid B) = 1.$

Ex: A bin contains 5 defective, 10 partially defective, and 25 acceptable transistors. A transistor is chosen at random from the bin and put into use. If it does not immediately fail, what is the probability it is acceptable.

E_1: Chosen transistor does not immediately fail.

$P(E_1) = 35/40$.

E_2: chosen transistor is acceptable

$$P(E_2) = \frac{25}{40}$$

$$P(E_2 \mid E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{25/40}{35/40} = \frac{25}{35}$$

Example

Six cards are dealt at random (without replacement) from a deck of 52 cards. Find the probability of getting all cards of heart in a hand given that there are at least 5 cards of heart in the hand.

E_1 : at least 5 cards of heart in the hand

E_2 : getting all cards of ~~~~

$P(E_2|E_1)$; $P(E_1) = \frac{{}^{13}C_5 \times {}^{39}C_1}{{}^{52}C_6} + \frac{{}^{13}C_6}{{}^{52}C_6}$; $P(E_2) = \frac{{}^{13}C_6}{{}^{52}C_6}$

$= \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{P(E_2)}{P(E_1)}$

Conditional Probability Contd...

Example

Six cards are dealt at random (without replacement) from a deck of 52 cards. Find the probability of getting all cards of heart in a hand given that there are at least 5 cards of heart in the hand.

Example

An urn contains 4 red and 6 black balls. 2 balls are drawn successively, at random and without replacement, from the urn. Find the probability that the first draw resulted in a red ball and the second draw resulted in a black ball.

4R + 6B

$^{10}C_2$

~~3R + 6B~~

E_1 : The 1^{st} drawn ball is red.

$$P(E_1) = \frac{4}{10} \times \frac{3}{9} + \boxed{\frac{4}{10} \times \frac{6}{9}} = \frac{4}{10}$$

E_2 : The second drawn ball is black.

$$P(E_2 | E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{\cancel{\frac{4}{10}} \times \frac{6}{9}}{\cancel{\frac{4}{10}}} = \frac{2}{3}.$$

Theorem of Total Probability

Theorem

Let $(\Omega, \mathcal{A}, P)$ *be a probability space and let* $\{E_i : i \in \Lambda\}$ *be a countable collection of mutually exclusive and exhaustive events such that* $P(E_i) > 0$ *for all* $i \in \Lambda$*. Then, for any event* $E \cancel{\in \mathcal{F}} \subseteq \Omega$*, we have*

$$\boxed{P(E) = \sum_{i \in \Lambda} P(E \cap E_i) = \sum_{i \in \Lambda} P(E_i)P(E|E_i).}$$

Let $E_1, E_2, \ldots$ be a countable collection of mutually exclusive and exhaustive events

That is, $\bigcup_{i=1}^{\infty} E_i = \Omega$

& $E_i \cap E_j = \phi \quad \forall\, i \neq j$.

Let E be an event.

Now, $E = E \cap \Omega = E \cap \left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} (E \cap E_i)$

Also, $B_i = E \cap E_i$, $i = 1, 2, \ldots$

$\boxed{\text{verify that } B_i \cap B_j = \phi \quad \forall\, i \neq j.}$

Countable infinite additivity, implies

$$P(E) = P(E \cap \Omega) = P\left(E \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right)$$

$$= P\left(\bigcup_{i=1}^{\infty} (E \cap E_i)\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right)$$

$$= \sum_{i=1}^{\infty} P(B_i) = \sum_{i=1}^{\infty} P(E \cap E_i)$$

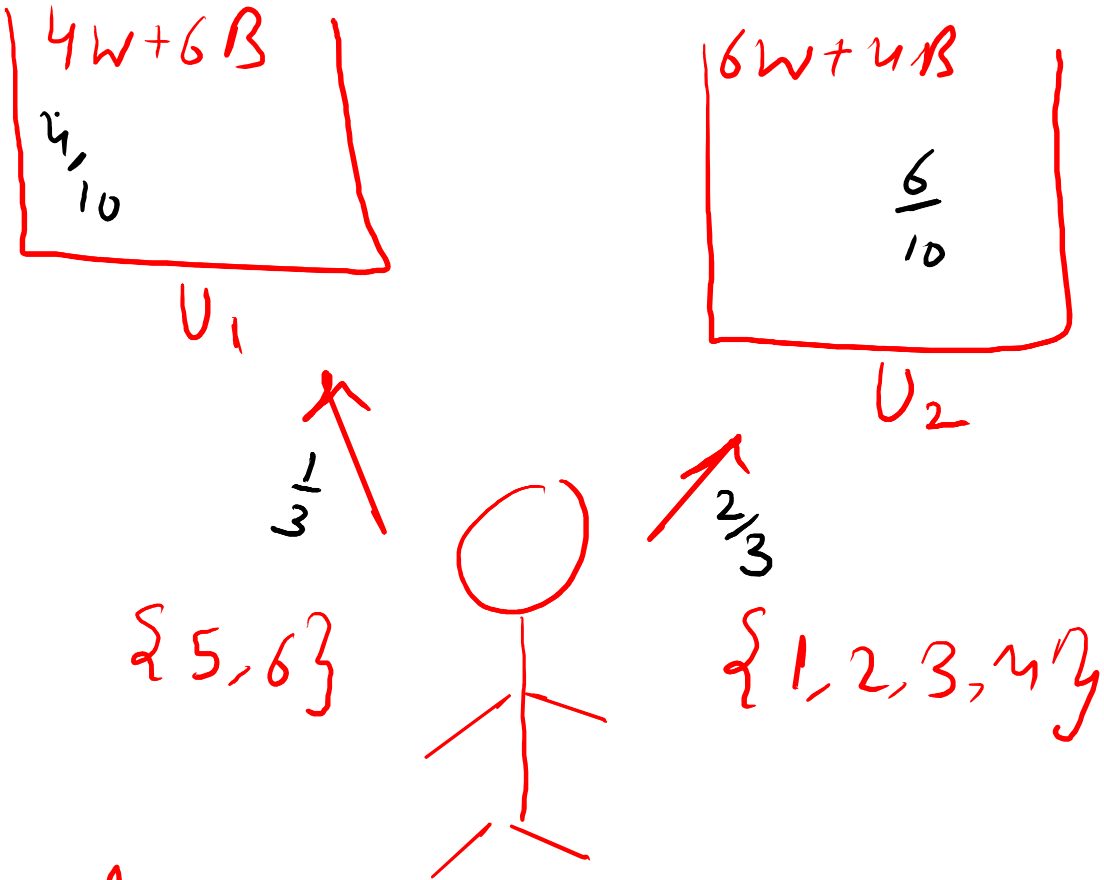
$$\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)} \qquad P(B) > 0} \Rightarrow P(A\cap B) = P(A|B)\,P(B)$$

$$\Rightarrow P(E) = \sum_{i=1}^{\infty} P(E\cap E_i) = \sum_{i=1}^{\infty} P(E|E_i)\,P(E_i)$$

Theorem of Total Probability: Example

Example

Urn $\boxed{U_1}$ contains 4 white and 6 black balls and urn $\boxed{U_2}$ contains 6 white and 4 black balls. A fair die is cast and urn U_1 is selected if the upper face of die shows 5 or 6 dots, otherwise urn U_2 is selected. If a ball is drawn at random from the selected urn, find the probability that the drawn ball is white.



E_1 : Select the urn U_1 $\Rightarrow P(E_1) = \frac{2}{6} = \frac{1}{3}$

E_2 : Select the urn U_2 $\Rightarrow P(E_2) = \frac{4}{6} = \frac{2}{3}$

A : The drawn ball is white.

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2)$$

$$= \frac{1}{3} \times \frac{4}{10} + \frac{2}{3} \times \frac{6}{10}$$

$$= \frac{16}{30} = \frac{8}{15}$$

Baye's Theorem

Theorem

Let $(\Omega, \cancel{\mathfrak{F}}, P)$ *be a probability space and let* $\{E_i : i \in \Lambda\}$ *be a countable collection of mutually exclusive and exhaustive events such that* $P(E_i) > 0$ *for all* $i \in \Lambda$*. Then, for any event* $E \cancel{\in \mathfrak{F}}$ *with* $P(E) > 0$*, we have*

$\subseteq \Omega$

$$P(E_j|E) = \frac{P(E|E_j)P(E_j)}{\sum\limits_{i \in \Lambda} P(E_i)P(E|E_i)}, \quad i \in \Lambda.$$

$$P(E_j \mid E) = \frac{P(E_j \cap E)}{P(E)}$$

$$= \frac{P(E_j)\, P(E \mid E_j)}{\sum_{i=1}^{\infty} P(E \mid E_i)\, P(E_i)}$$

Baye's Theorem: Example

Example

Urn U_1 contains 4 white and 6 black balls and urn U_2 contains 6 white and 4 black balls. A fair die is cast and urn U_1 is selected if the upper face of die shows 5 or 6 dots, otherwise urn U_2 is selected. A ball is drawn at random from the selected urn.

(i) Given that the drawn ball is white, find the conditional probability that it came from urn U_1;

(i) Given that the drawn ball is white, find the conditional probability that it came from urn U_2;

$$P(A) = \frac{8}{15}$$

$$P(E_1|A) = \frac{P(E_1 \cap A)}{P(A)} = \frac{P(A|E_1)\,P(E_1)}{P(A)}$$

$$\Rightarrow P(E_1|A) = \frac{\frac{1}{3} \times \frac{4}{10}}{8/15} = \frac{1}{4}$$

Ex: At a certain stage of a criminal investigation, the inspector incharge is 60% convinced of the guilt of a certain suspect.

Suppose now a new piece of evidence that shows that the criminal has a certain characteristics is uncovered.

If 20% of the population possesses this characteristic, how certain of the guilt of the suspect should the inspector now be if it turn out that the suspect is among this group?

Sol;

G : The suspect is guilty

C : He possesses the characteristics of the criminal.

$$P(G|C) = \frac{P(G\cap C)}{P(C)} = \frac{(0.6)1}{P(C|G)P(G)+P(C|G^c)P(G^c)}$$

$$= \frac{0.6}{0.6+(0.2)(0.4)} = \frac{60}{68} = 15/17.$$

Independence of Two Events

$$P(A \cap B) \quad < \quad P(A)P(B)$$
$$>$$
$$=$$

Definition

Let $(\Omega,$ [illegible]$, P)$ be a probability space and let A and B be two events. Events A and B are said to be

(i) **negatively associated** if $P(A \cap B) < P(A)P(B)$;

(ii) **positively associated** if $P(A \cap B) > P(A)P(B)$;

(iii) **independent** if $\boxed{P(A \cap B) = P(A)P(B);}$

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

Ex: Two fair dice are thrown;

E_1: Sum of the dice is 7. $= \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$

E_2: The I^{st} die equals 4.

E_3: The 2^{nd} die equal 3.

$\{E_1, E_1 \cap E_2\}$

$\{E_2, E_1 \cap E_2\}$

$P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{1}{6}$, $P(E_3) = \frac{1}{6}$

$P(E_i \cap E_j) = P(E_i)P(E_j)$, $i = 1,2,3$, $j = 1,2,3$, $i \neq j$