Review

Let Ω be a sample space associated with a random experiment.

A real-valued function

$$X : \Omega \to \mathbb{R}$$

is called random-variable.

We say that a r.v. $\underbrace{X : \Omega \to \mathbb{R}}$ is discrete if

$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$

"$\boxed{X(\Omega)}$ is either finite

or countably infinite"

→ Its p.m.f $p_X(\cdot)$ is defined as :

$$p_X(x) = P[X = x] \quad \text{for } x \in X(\Omega)$$

Properties of p.m.f:

p

(i) $p_X(\cdot) \geq 0$

(ii) $$\sum_{x \in X(\Omega)} p_X(x) = 1.$$

c.d.f. of X will be

$$\boxed{F_X(a) = P[X \leq a]} = \sum_{\substack{x \in X(\Omega) \\ x \leq a}} p_X(x).$$

In case of discrete r.v., c.d.f will behave like a step function.

We say that a r.v. $X : \Omega \to \mathbb{R}$ is continuous if $\exists$ $f_X(x) \geq 0$ on $(-\infty, \infty)$ such that $P[X \in B] = \int_B f_X(x)\,dx$, $B \subseteq \mathbb{R}$.

$\{1, 2, 3, 4\}$

$f(x)$ is called p.d.f. of X and is such that

✓ i) $f_X(x) \geq 0$ on $(-\infty, \infty)$

✓ ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

$f_X(x)\,dx \approx P[x < X < x+dx]$

Note: If a function f satisfies the above properties, then it will be a p.d.f. of some cts r.v.

The c.d.f of a cont. r.v. X will be

$$F_X(x) = P[X \le x] = P[X \in (-\infty, x]]$$

$$= \int_{-\infty}^{x} f_X(x)\,dx$$

The properties of c.d.f. of a r.v.

(i) $\boxed{F_X(-\infty)} = P[X \leq -\infty] = 0$

(ii) $F_X(\infty) = P[X \leq \infty] = 1$

(iii) F_X is non-decreasing

(iv) F_X is right-cont. $\left(F_X(x) = F_X(x^-) \right)$

$x_1 < x_2$

$F_X(x_1) \leq F_X(x_2)$

$\{X \leq x_1\} \subseteq \{X \leq x_2\}$

Jointly Distributed r.v.

Ex: 1 Suppose that 3 batteries are chosen from a group of 3 new, 4 used but still working, and 5 defective batteries.

Let X = number of new batteries that are chosen

and Y = number of used batteries that are chosen.

Then, $X \in \{0,1,2,3\}$ and $Y \in \{0,1,2,3\}$.

Q. How do we define the relationship between X and Y?

$$F_X(x) = P[X \le x]$$

$$F_Y(y) = P[Y \le y]$$

$$F_{X,Y}(x, y) = P[\{X \le x\} \text{ and } \{Y \le y\}]$$

We define the joint c.d.f of X and Y as:

$$F_{X,Y}(x,y) = P[X \le x,\ Y \le y]$$

$$= P[\{X \le x\} \text{ and } \{Y \le y\}]$$

One can obtain individual c.d.f of X and Y as follows:

✓ $F_X(x) = F_{X,Y}(x,\infty) = P[X \le x, Y < \infty]$

✓ $F_Y(y) = F_{X,Y}(\infty, y) = P[X < \infty, Y \le y]$

$\{X \le x\} \cap \{Y < \infty\} = \{X \le x\} \cap \Omega = \{X \le x\}$

In case, both X and Y are discrete r.v.

Joint p.m.f. of X and Y will be defined as

$$p_{X,Y}(x_i, y_j) = P[X = x_i, Y = y_j].$$

$$p_X(x_i) = P[X = x_i], \quad p_Y(y_j) = P[Y = y_j]$$

$$\bigcup_j \{Y = y_j\} = \Omega \quad \& \quad \bigcup_i \{X = x_i\} = \Omega.$$

Note that

$\{X = x_i\} \cap \Omega = \{X = x_i\} \cap \left(\bigcup_j \{Y = y_j\}\right)$

$$\{X = x_i\} = \bigcup_j \left[\{X = x_i\} \text{ and } \{Y = y_j\}\right]$$

$$\Rightarrow \quad p_X(x_i) = \sum_j p_{X,Y}(x_i, y_j)$$

$||^{ly}$

$$p_Y(y_j) = \sum_i p_{X,Y}(x_i, y_j)$$

p.m.f for Example 1

$X \in \{0, 1, 2, 3\}$, $Y \in \{0, 1, 2, 3\}$

$$p_{X,Y}(0,0) = P[X=0, Y=0] = \frac{{}^5C_3}{{}^{12}C_3} = \frac{\frac{5 \cdot 4 \cdot 2}{\cancel{2}}}{\frac{2\,\cancel{12} \times 11 \times 10}{\cancel{6}}} = \frac{10}{220}$$

$$p_{X,Y}(0,1) = P[X=0, Y=1] = \frac{{}^5C_2 \times {}^4C_1}{{}^{12}C_3} = \frac{40}{220}$$

$$p_{X,Y}(0,2) = P[X=0, Y=2] = \frac{{}^5C_1 \times {}^4C_2}{{}^{12}C_3} = \frac{30}{220}$$

$$p_{X,Y}(0,3) = P[X=0, Y=3] = \frac{{}^{4}C_3}{{}^{12}C_3} = \frac{4}{220}$$

$$p_{X,Y}(1,2) = P[X=1, Y=2] = \frac{{}^{3}C_1 \times {}^{4}C_2}{{}^{12}C_3} = \frac{18}{220}$$

$\vdots$

$X \leftarrow i$ \ $Y \leftarrow j$	0	1	2	3	$P[X=i]$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$4/220$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
$P[Y=j]$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Ex 2 Suppose that 15% of the families in a certain community have no children, 20% have 1, 35% have 2, and 30% have 3.

Suppose that each child is equally likely to be a boy or a girl.

If a family is chosen at random from this community, then B, the number of boys, and G, the number of girls in this family. Write their joint p.m.f.

$B \in \{0, 1, 2, 3\}$

$G \in \{0, 1, 2, 3\}$

Sol: Clearly,

$$B \in \{0, 1, 2, 3\}$$

$$G \in \{0, 1, 2, 3\}.$$

$$P_{B,G}(0,0) = P[B=0, G=0] = 0.15$$

$$P_{B,G}(0,1) = P[B=0, G=1] = (0.20)\frac{1}{2} = 0.10$$

$F_0 \quad F_1 \quad F_2 \quad F_3$

$F_3 \rightarrow 3$ (0.30)

$3 \rightarrow 1, \; 2$

$G_1 G_2 B$

$G_1 B G_2$

$B G_1 G_2$

${}^3C_2$

$$p_{B,G}(1,2) = P[B=1, G=2]$$

$$= P[3\text{ child}]\; P[B=1, G=2 \mid 3\text{ child}]$$

$$= (0.30)\left({}^3C_2\right)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)$$

$$= 0.30 \times 3 \times \frac{1}{8}$$

$$= \frac{0.90}{8} = 0.1125$$

$$p_{B,G}(2,1) = P[B=2, G=1] = P[3\ \text{child}]\, P[B=2, G=1 \mid 3\ \text{child}]$$

$$= 0.1125$$

$$\boxed{\begin{matrix} G\,G\,G \\ B\,G\,B \\ B\,B\,G \\ G\,B\,B \end{matrix}}$$

$$\begin{matrix} \boxed{} & \boxed{} & \boxed{} \\ C_1 & C_2 & C_3 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix}$$

$$\frac{{}^3C_2\left(\frac{1}{8}\right)}{{}^3C_2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

$$\frac{3}{4} \times \frac{1}{2}$$

B \ $G = j$	0	1	2	3	$P[B=i]$
0	0.15	0.10	0.0875	0.0375	X
1	0.10	0.175	0.1125	0	X
2	0.0875	0.1125	0	0	X
3	0.0375	0	0	0	X
$P[G=j]$	X	X	X	X	

Jointly Continuous r.v.

We say that X and Y are jointly continuous if $\exists$ a non-negative function $f_{X,Y}(x,y)$ on $\mathbb{R}^2$ such that

$$P\{(X,Y)\in C\} = \iint\limits_{(x,y)\in C} f_{X,Y}(x,y)\,dx\,dy.$$

$C \subseteq \mathbb{R}^2$

$f_{X,Y}$ is called joint p.d.f. of X and Y.

$F_{X,Y}(a,b) = P[X \le a, Y \le b]$

$x \in (-\infty, a]$, $y \in (-\infty, b]$

$= P\{(X,Y) \in (-\infty,a] \times (-\infty,b]\}$

$$= \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x,y)\,dy\,dx.$$

$A \times B = \{(a,b) : a \in A, b \in B\}$

$$F_{X,Y}(a,\infty) = \int_{-\infty}^{a} \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \int_{-\infty}^{a} f_X(x)\, dx = F_X(a)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \rightarrow \text{p.d.f. of } X$$

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$$F_{X,Y}(\infty, b) = \int_{-\infty}^{b} \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \right) dy$$

$$= \int_{-\infty}^{b} f_Y(y)\, dy = F_Y(b)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \rightarrow \text{p.d.f. of } Y.$$

If X and Y are jointly continuous, then they are individual continuous. Also,

$$P[X \in B] = P[X \in B, Y \in (-\infty, \infty)]$$

$$= \int_B \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \int_B f_X(x)\, dx$$

where

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$

||ly

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$$

Ex: The joint p.d.f. of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & 0<x<\infty,\ 0<y<\infty. \\ 0, & \text{otherwise.} \end{cases}$$

Find (i) $P[X>1, Y<1]$

(ii) $P[X<Y]$

(iii) $P[X<a]$