Motivation for r.v.

Ex 1: Consider a random experiment of rolling a pair of dice. Then,

$$\Omega = \{(i,j) : i = 1,..,6;\ j = 1,..,6\}.$$

We are often interested in the sum of the values obtained after rolling a pair of dice

$$S = \{2, 3, 4, \ldots, 12\}$$

Random Variable

Let a random experiment be performed and Ω be the associated sample space.

Then, a random variable (r.v.), say X, is a real-valued function defined on Ω.

That is, $X : \Omega \to \mathbb{R}$, which associates a numerical value to each event.

The possible sum are $2, 3, \ldots, 12$.

Our interest in the set $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Define r.v. $X : \Omega \to \mathbb{R}$ as follows:

$$X(\omega) = 2, \quad \omega \in \{(1,1)\} \longrightarrow P[X=2] = \frac{1}{36}$$

$$X(\omega) = 3, \quad \omega \in \{(1,2), (2,1)\} \rightarrow P[X=3] = \frac{2}{36}$$

$$X(\omega) = 4, \quad \omega \in \{(1,3), (2,2), (3,1)\} \rightarrow P[X=4] = \frac{3}{36}$$

$$X((i, j)) = i + j.$$

We associate probability to X, as follows,

$$P(X = s) = P(\text{all possible pairs } (i,j) \in \Omega \text{ such that } i + j = s)$$

$$\boxed{\text{Note that "} X = s \text{" means } \{ \omega \in \Omega \text{ s.t. } X(\omega) = s \}}$$

In the above example of rolling a pair of dice, we have $S = \{2, 3, \ldots, 12\}$

$$P(X=2) = P(\{(1,1)\}) = \frac{1}{36}$$

$$P(X=3) = P(\{(1,2),(2,1)\}) = \frac{2}{36}$$

$$P(X=4) = P(\{(1,3),(2,2),(3,1)\}) = \frac{3}{36}$$

$$P(X=5) = P(\{(1,4),(2,3),(3,2),(4,1)\}) = \frac{4}{36}$$

$$P(X=6) = P(\{(1,5),(2,4),(3,3),(4,2),(5,1)\}) = \frac{5}{36}$$

$$\boxed{P(X=7) = \frac{6}{36}}$$

$$P(X=8) = \frac{5}{36}$$

$$P(X=9) = \frac{4}{36}$$

$$P(X=10) = \frac{3}{36}$$

$$P(X=11) = \frac{2}{36}$$

$$P(X=12) = \frac{1}{36}$$

$$P(S) = P\left(\bigcup_{i=2}^{12} \{X = i\}\right)$$

$$= \sum_{i=2}^{12} P\{X = i\}$$

[Events are mutually exclusive]

$$\boxed{= 1}$$

Ex 2: Suppose a person purchases two electronic components, each of which may be either defective or acceptable.

d, a

$(d, d) \longrightarrow 0.09$

$(d, a) \longrightarrow 0.21$

$(a, d) \longrightarrow 0.21$

$(a, a) \longrightarrow 0.49$

X : no. of acceptable components obtained during the purchase.

$$X \in \{0, 1, 2\} = S = X(\Omega)$$

$$\left.\begin{aligned} P[X=0] &= 0.09 \\ P[X=1] &= 0.42 \\ P[X=2] &= 0.49 \end{aligned}\right\} \longrightarrow \underline{1}$$

Let I : atleast one acceptable component.

$I \in \{0, 1\}$

$$I = \begin{cases} 0 & \text{if } X = 0 \\ 1 & \text{if } X = 1 \text{ or } X = 2. \end{cases}$$

A : atleast one acceptable component is obtained

I : indicator r.v.

$$\left.\begin{array}{l} P[I=0] = 0.09 \\ P[I=1] = 0.91. \end{array}\right\} 1$$

Ex: Toss a coin repeatedly until head occurs.

$$\Omega = \{ H, TH, TTH, TTTH, \dots \}$$

$$P(H) = \frac{1}{2}, \quad P(T) = \frac{1}{2}.$$

$\omega \in \Omega$, $\quad X(\omega) =$ no. of tails in ω

$$\in \{0, 1, 2, 3, \dots\}$$

$$P[X=0] = \frac{1}{2}, \quad P[X=1] = \frac{1}{2^2}, \dots, \quad P[X=i] = \frac{1}{2^{i+1}}$$

A r.v. whose range $(X(\Omega))$ is either a finite set or a countably infinite set, is called a discrete r.v.

There does exist r.v. that take on a continuum of possible values.

<u>Defⁿ</u> (Cummulative Distribution Function or c.d.f)

The c.d.f $\boxed{F_X}$ of the r.v. X is defined for any real values x by

$$F_X(x) = P[X \le x]$$

$$\boxed{F_X : \mathbb{R} \to [0,1].}$$

Ex: Write the c.d.f for the r.v. defined in Example 1. [Rolling two dice]

Solution: c.d.f of X will be $S = \{2, 3, \dots, 12\}$

$$F_X(x) := P[X \leq x]$$

$$= \begin{cases} 0 & \text{if } x < 2 \\ 1/36 & \text{if } 2 \leq x < 3 \\ 3/36 & \text{if } 3 \leq x < 4 \\ 6/36 & \text{if } 4 \leq x < 5 \end{cases}$$

$P[X=2] + P[X=3]$

$P[X=2] + P[X=3] + P[X=4]$

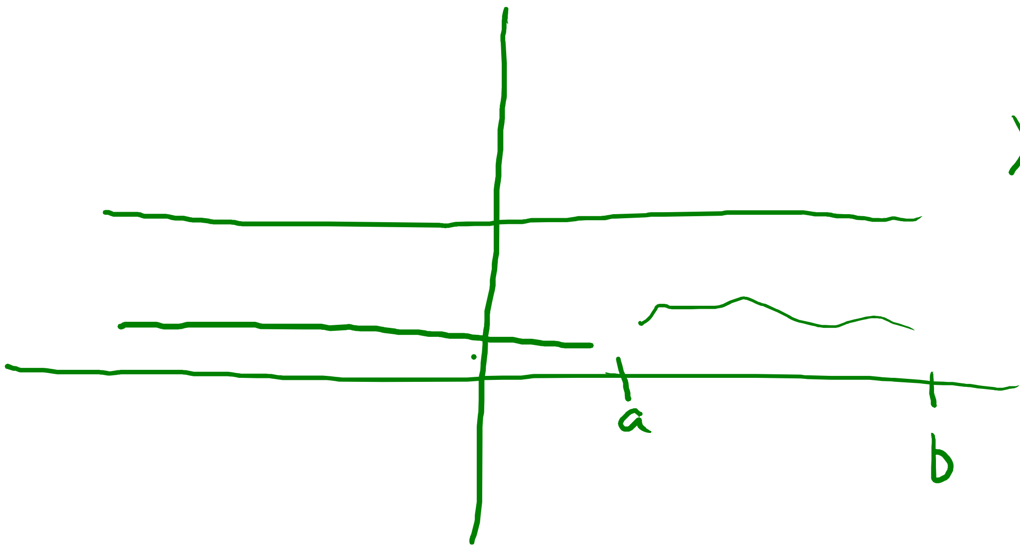
$$F_X(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{1}{36} & \text{if } 2 \le x < 3 \\ \frac{3}{36} & \text{if } 3 \le x < 4 \\ \frac{6}{36} & \text{if } 4 \le x < 5 \\ \vdots & \vdots \qquad \vdots \\ \frac{35}{36} & \text{if } 11 \le x < 12 \\ 1 & \text{if } x \ge 12 \end{cases}$$

Ex: $P[a < X \le b] = F_X(b) - F_X(a)$?

$$a < X \le b = \{\omega \in \Omega \mid a < X(\omega) \le b\}$$

$X \le b$

$- \; X \le a$



$$X \le b = \{a < X \le b\} \cup \{X \le a\}$$

$$\{X \le b\} = \{X \le a\} \cup \{a < X \le b\}$$

$$\Rightarrow P[X \le b] = P[X \le a] + P[a < X \le b]$$

$$\Rightarrow P[a < X \le b] = P[X \le b] - P[X \le a]$$

$$= F_X(b) - F_X(a)$$

Ex: Let X be a r.v. whose c.d.f is

$$F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1-e^{-x^2} & \text{if } x > 0. \end{cases}$$

Then, find $P[X>1]$.

Sol: $P[X>1] = 1-P[X\le 1] = 1-F_X(1)$

$= 1-(1-e^{-1}) = e^{-1}.$

$$P[1 < X \leq 2] = F_X(2) - F_X(1)$$

$$= (\cancel{1} - e^{-4}) - (\cancel{1} - e^{-1})$$

$$= e^{-1} - e^{-4}$$

Types of r.v.

(1) Discrete r.v.

A r.v. X is called discrete if its range, $X(\Omega)$, is either finite or countably infinite.

In this case, we define the probability mass function (p.m.f.) of X by

$$\boxed{p_X(a) = P(\{X = a\})}$$

Properties of p.m.f.

(i) $p_X(a) \geq 0 \quad \forall \quad a \in \mathbb{R}$

(ii) $\sum_{a \in \mathbb{R}} p_X(a) = 1$.

Ex: Write the p.m.f of Example 1 and Example 2.

Ex 1 (Rolling a dice)

$p_X(a) = 0$ if $a \notin S = \{2, 3, 4, \ldots, 12\}$

$p_X(2) = \frac{1}{36}$, $p_X(3) = \frac{2}{36}$, $p_X(4) = \frac{3}{36}$, $\ldots$,

$p_X(12) = \frac{1}{36}$.

Ex 2

$X \in \{0,1,2\}$; $I \in \{0,1\}$

$p_X(0) = 0.09$

$p_X(1) = 0.42$

$p_X(2) = 0.49$

$p_X(a) = 0$ if $a \notin \{0,1,2\}$

$p_I(0) = 0.09$

$p_I(1) = 0.91$.

$p_I(a) = 0$ if $a \notin \{0,1\}$.

Ex: Let X be a discrete r.v. that takes the values 1, 2 or 3. Assume that we know the following:

$$p_X(1) = \frac{1}{2}, \quad p_X(2) = \frac{1}{3}$$

Then, what is the value of $p_X(3)$?

$p_X(x) = 0$ if $x \notin \{1, 2, 3\}$.

$$\sum_{x \in \mathbb{R}} p_X(x) = 1 \Rightarrow p_X(1) + p_X(2) + p_X(3) = 1$$

$$\Rightarrow \frac{5}{6} + p_X(3) = 1$$

$$\Rightarrow p_X(3) = \frac{1}{6}.$$

Ex: Discuss the nature of c.d.f fun[ction of]
a discrete r.v. X.

Answer F_X is a step function.

Defⁿ (Continuous r.v.)

We say that a r.v. X is continuous if $\exists$ a non-negative function $f(x)$, defined for all $x \in (-\infty, \infty)$, having the property that

$$\boxed{P[X \in B] = \int_B f(x)\,dx\,, \quad B \subseteq \mathbb{R}}$$

The function 'f' is called the probability density function (p.d.f) of X.

<u>Properties</u> of <u>p.d.f</u>:

(i) $f(x) \geq 0 \quad \forall \; x \in (-\infty, \infty)$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = \underline{1}$.

Ex: (1) $P[a \leq X \leq b] = \int_a^b f(x)\,dx$ $\quad X \in [a,b]$

(2) $P[X = a] = 0$, if X is cts r.v.

$$\boxed{(3)\quad F_X'(a) = f(a)}$$

$$F_X(a) = P[X \leq a] = P[X \in (-\infty, a]] = \int_{-\infty}^{a} f(x)\,dx$$

$$F_X(a) = \lim_{b \to \infty} \left[\int_b^a f(x)\, dx \right]$$

Ex: Let X be a cts r. v. with p.d.f.

$$f(x) = \begin{cases} \boxed{C}(4x - 2x^2), & \text{if } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of C.

(b) $P[X > 1]$?

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{2} f(x)\,dx = 1$$

$$\Rightarrow \int_0^2 f(x)\,dx = 1$$

$$\Rightarrow \int_0^2 c(4x - 2x^2)\,dx = 1$$

$$\Rightarrow c\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = 1$$

$$\Rightarrow c\left[8 - \frac{16}{3}\right] = 1 \Rightarrow c = 3/8$$

$$P[X>1] = P[X \in (1,\infty)] = \int_1^\infty f(x)\,dx$$

$$= \frac{3}{8}\int_1^2 (4x-2x^2)\,dx$$

$$= \frac{3}{8}\left[2x^2 - \frac{2}{3}x^3\right]_1^2$$

$$= \frac{3}{8}\left[8 - \frac{16}{3} - 2 + \frac{2}{3}\right] = \frac{3}{8}\left[6 - \frac{14}{3}\right]$$

$$= \frac{\cancel{3}}{{}_2\cancel{8}} \times \frac{4}{\cancel{3}} = \frac{1}{2}.$$