# Review of Selected Mathematical Techniques

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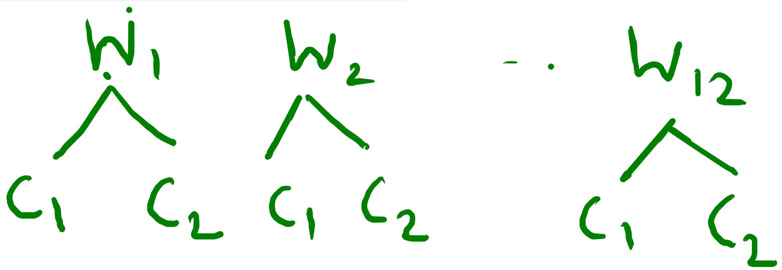
- Algebra of Sets.

# Basic Counting Principle

**Example**

A small community consists of 12 women, each of whom has 2 children. If one women and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

$\underline{12 \times 2 = 24}$



# Basic Counting Principle

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$S_1$

$S_2$

$S_3$

**Example**

Suppose a person has 3 shirts and 5 ties. How many ways he can choose a shirt with a tie?

$1 \times 5 + 1 \times 5 + 1 \times 5 = \underline{15}$

# Basic Counting Principle

1) $12 \times 2$

2) $3 \times 5 = 15.$

Suppose two experiments $E_1$ and $E_2$ are to be performed in order/simultaneously, with $N_1$ possible outcomes for $E_1$ and $N_2$ possible outcomes for $E_2$. Then, together there are

$$N_1 . N_2$$

possible outcomes of two experiments.

# Basic Counting Principle

In general, if $n$ experiments $E_1, E_2, ..., E_n$ are to be performed in order/simultaneously, with possible number of outcomes $N_1, N_2, ..., N_n$ respectively. Then, together there are

$$N_1.N_2...N_n$$

possible outcomes of $n$ experiments in given order.

## Basic Counting Principle

**Example**

How many 8 place License plates are possible if first four places are to be occupied by alphabets and last four by numbers?

A

$$26 \times 26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^4 \times 10^4.$$

$$\underbrace{\square\ \square\ \square\ \square}_{A-Z}\ \underbrace{\square\ \square\ \square\ \square}_{0-9}$$

# Basic Counting Principle

**Example**

How many 8 place License plates are possible if first four places are to be occupied by alphabets and last four by numbers?

**Example**

Repetition of letters and numbers is not permitted.

$\square\ \square\ \square\ \square\ \square\ \square\ \square\ \square$

$26 \cdot 25 \cdot 24 \cdot 23 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

## Permutation and Combination

**Example 5**

Consider collection of 3 novels by authors $A$, $B$ and $C$, 2 mathematics books by authors $D$ and $E$, and 1 physics book by author $P$. How many arrangements are possible if the books are to be distinguished by the authors.

6

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720.$

# Permutation and Combination

**Example**

Consider collection of 3 novels by authors $A$, $B$ and $C$, 2 mathematics books by authors $D$ and $E$, and 1 physics book by author $P$. How many arrangements are possible if the books are to be distinguished by the authors.

$3! \times 3! \times 2! \times 1!$

$= 72$

**Example**

How many arrangements are possible if the books are to be distinguished by the subjects?

$3 \times 2 \times 1$

N M P

$3! \times 2! \times 1! + 2! \times 3! \times 1! +$

MNP

PNM

## Permutation and Combination Contd...

(If the order is important)

Number of arrangements of $n$ objects $= n!$

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(If the order is important)

Number of arrangements of $n$ objects $= n!$

(If the order is not important)

Number of arrangements of $n$ objects where $n_1$ are alike, $n_2$ are alike,..., $n_r$ are alike such that $n_1 + n_2 + ... + n_r = n$

$$= \boxed{\frac{n!}{n_1!n_2!...n_r!}}$$

# Permutation and Combination Contd...

$$\frac{9!}{3!\,2!\,4!}$$
$= 1260.$

**Example**

A tennis tournament has 9 competitions, 3 from India, 2 from Japan and 4 from Australia. Results of the tournament are announced by nationalities of the players in the order in which they are placed. How many such lists are possible?

ors

# Permutation and Combination Contd...

In a collection of $n$ objects, in how many way can we select $r$ objects?

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In a collection of $n$ objects, in how many way can we select $r$ objects?

$$\underbrace{n \times n \times n \times n \cdots \times n}_{n\text{-times}}$$

**If the order is important**:

$$\frac{n!}{(n-r)!} = \boxed{{}^{n}P_{r}}.$$

# Permutation and Combination Contd...

In a collection of $n$ objects, in how many way can we select $r$ objects?

$\boxed{{}^nC_r}$

<u>**If the order is important**</u>:

$$\frac{n!}{(n-r)!} = {}^nP_r.$$

<u>**If the order is immaterial**</u>:

$$\frac{n!}{r!(n-r)!} = {}^nC_r.$$

# Permutation and Combination Contd...

Example

A jury of 7 is to be formed from a group of 30 people. How many different juries can be formed?

30

7

$\boxed{{}^{30}C_7}$

# Permutation and Combination Contd...

### Example

A jury of 7 is to be formed from a group of 30 people. How many different juries can be formed?

10W + 20M

### Example

In case the group of 30 people consists of 10 women and 20 men and it is required that 2 women and 5 men should form the jury. How many juries can be formed?

$^{10}C_2 \times {}^{20}C_5$

$= 697680$

# Algebra of Sets

### Definition (Set)

A set is a collection of well defined objects, called elements.

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**Set Operations:** Let $\boxed{\Omega}$ is an abstract/universal set.

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(iii) **Complementation:** Let $\boxed{A \subset \Omega.}$ The complement of set $A$, denoted by $\boxed{\overline{A}}$ or $\boxed{A^c}$, is the set containing all elements in $\Omega$ but not in $A$.

## Set Operations Contd...

(iv) **Union:** The union of sets $A$ and $B$, denoted by $A \cup B$, is the set containing all elements in either $A$ or $B$ or both.

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

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(v) **Intersection:** The intersection of sets $A$ and $B$, denoted by $A \cap B$, is the set containing all elements in both $A$ and $B$.

$$A \cap B = \{ x \in \Omega \mid x \in A \text{ \& } x \in B \}$$

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(vi) **Null/Empty Set:** The set containing no elements is called the null set, denoted by $\phi$.

(vii) **Disjoint/Mutually Exclusive Sets:** Two sets $A$ and $B$ are called disjoint or mutually exclusive if $\boxed{A \cap B = \phi.}$

## Set Operations Contd...

(viii) **De Morgan's Laws:**

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$.


(ix)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

# Random Experiment and Sample Space

**Definition (Random Experiment)**

A random experiment is an experiment in which:

(a) the set of all possible outcomes of the experiment is known in advance;

(b) the outcome of a particular trial of the experiment can not be predicted in advance;

(c) the experiment can be repeated under identical conditions.

## Random Experiment and Sample Space

$\Omega = [0,1] \times [0,1]$

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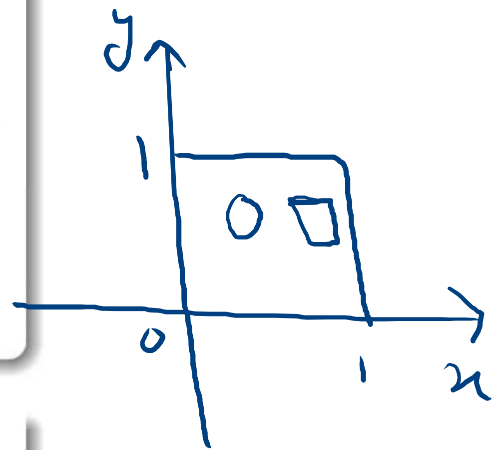

$\{H, TH, TTH, TTTH, \ldots\}$

**Definition (Sample Space: $\Omega$)**

The collection of all possible outcomes of a random experiment is called the sample space.

## Random Experiment and Sample Space: Example

**Example**

Write down the sample space for each of the following random experiments:

(i) Rolling a die. $= \{1, 2, 3, 4, 5, 6\}$

(ii) Simultaneously flipping a coin and rolling a die.

(iii) Coin is tossed repeatedly until a head is observed. $= \{H, TH, TTH, TTTH, \ldots\}$

(ii) $\{(H,1), (H,2), \ldots, (H,6),$
$(T,1), (T,2), \ldots, (T,6)\}$

# Events and Mutually Exclusive

$E = \{1, 2, 3\} \subseteq \Omega$

**Definition (Event)**

An event $E$ is a set of outcomes of the random experiment, i.e.,

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**Definition (Mutually Exclusive)**

Two events $E_1$ and $E_2$ are said to be mutually exclusive if they can not occur simultaneously, i.e., if $E_1 \cap E_2 = \phi$.

$\Omega = \{1, 2, .., 6\}$

$E_1 = \{1, 3, 5\}$, $E_2 = \{2, 4, 6\}$, $E_3 = \{3, 6\}$

# Assigning Probabilities: Classical Method

- It is used for random experiments which result in a finite number of equally likely outcomes.
- For an instance, let $\Omega = \{w_1, w_2, ..., w_n\}$ with $n \in \mathbb{N}$.
- For $E \subseteq \Omega$, let $\mid E \mid$ denotes the number of elements in $E$.

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- For an instance, let $\Omega = \{w_1, w_2, ..., w_n\}$ with $n \in \mathbb{N}$.
- For $E \subseteq \Omega$, let $| E |$ denotes the number of elements in $E$.
- The probability of an event $E$ is given by

$$P(E) = \frac{\text{number of outcomes favourable to E}}{\text{total number of outcomes}} = \frac{| E |}{| \Omega |} = \frac{| E |}{n}.$$

**Note:** An outcome $w \in \Omega$ is said to be favorable to an event $E$ if $w \in E$.

$\Omega = \{1, 2, \ldots, 6\}$

$E_1 = \{1, 3, 5\}$

$E_2 = \{2, 4, 6\}$

$E_3 = \{3, 6\}$

$P(E_1) = \frac{3}{6} = \frac{1}{2}$

$P(E_2) = \frac{3}{6} = \frac{1}{2}$

$P(E_3) = \frac{2}{6} = \frac{1}{3}$.

# Example

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots$$

**Example**

Suppose that in a classroom we have 25 students (with registration number 1,2,...,25) born in the same year having 365 days. Suppose that we want to find the probability of the event that they all are born on different days of the year. Assume that outcomes are equally likely.

$$\frac{365 \times 364 \times 363 \times \cdots \times (365 - 25 + 1)}{(365)^{25}}$$

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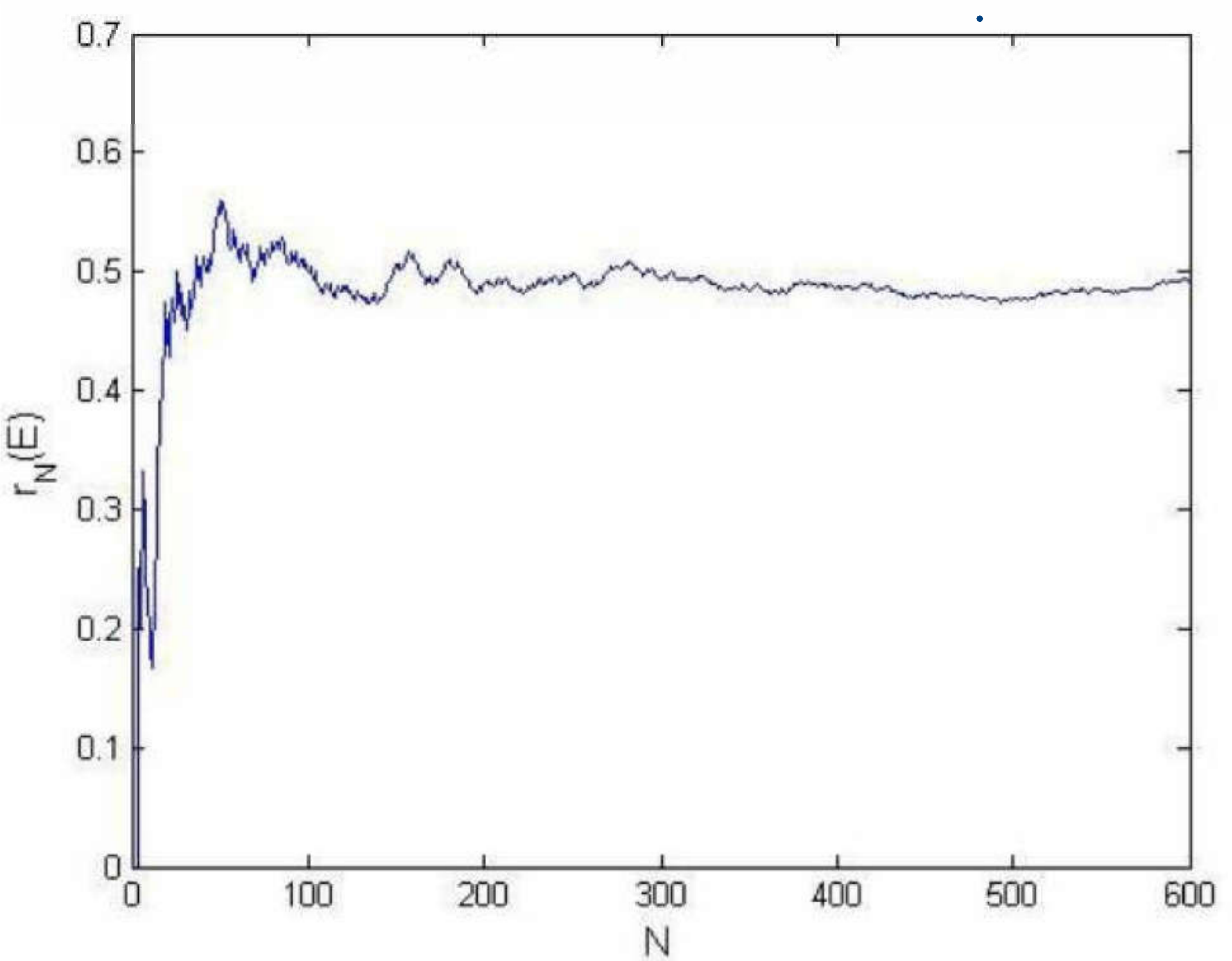
$$r_N(E) = \frac{f_N(E)}{N}.$$

- The probability of an event $E$ is given by

$$\boxed{P(E) = \lim_{N\to\infty} \frac{f_N(E)}{N}.}$$

# Example

Plot of relative frequencies of number of heads against number of trials in the random experiment of tossing a fair coin.




$\boxed{0.5}$

EA