Ex: The joint p.d.f. of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & 0<x<\infty,\ 0<y<\infty \\ 0, & \text{otherwise} \end{cases}$$

Find (i) $P[X>1, Y<1]$

(ii) $P[X<Y]$

(iii) $P[X<a]$

(1) $P[X>1, Y<1] = P[X \in (1,\infty), Y \in (-\infty, 1)]$

$$= P[(X,Y) \in \underbrace{(1,\infty) \times (-\infty, 1)}_{B}]$$

$$= \iint_{B} f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{1} \int_{1}^{\infty} f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{0}^{1} \int_{1}^{\infty} f_{X,Y}(x,y)\, dx\, dy$$

$$= 2\int_0^1 \int_1^\infty \left(e^{-x} e^{-2y}\right) dx\, dy$$

$$= \cancel{2}\underbrace{\left(\int_0^1 e^{-2y}\, dy\right)}\left(\int_1^\infty e^{-x}\, dx\right)$$

$$\left[\frac{e^{-2y}}{\cancel{-2}}\right]_0^1$$

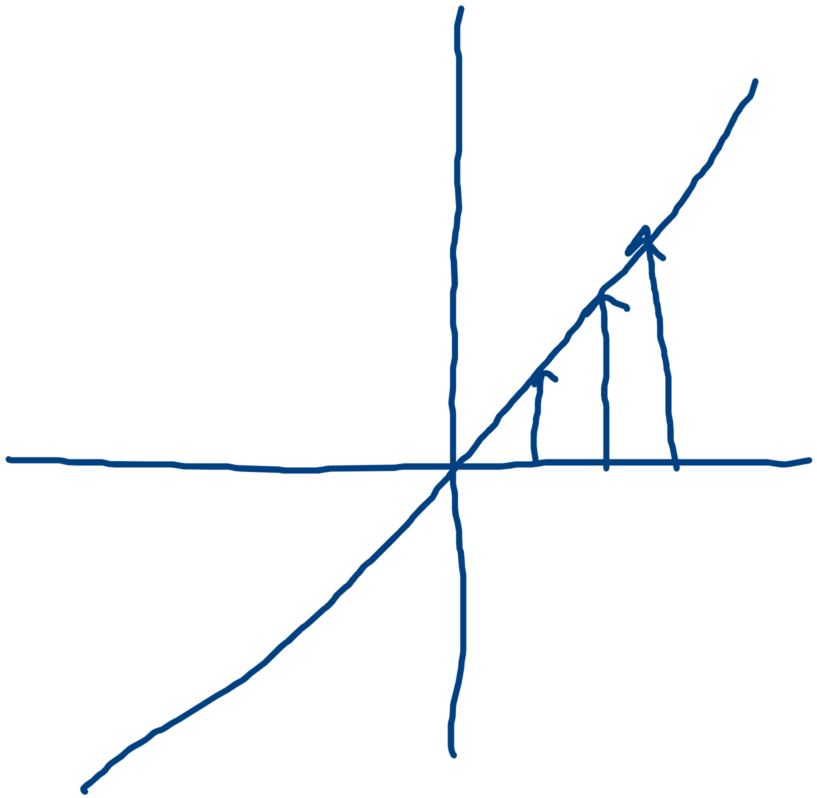
$$= \boxed{\left[1 - e^{-2}\right]}$$

$$\lim_{a\to\infty}\int_1^a e^{-x}\, dx$$

$$= -\lim_{a\to\infty}\left[e^{-x}\right]_1^a$$

$$= \lim_{a\to\infty}\left[1 - e^{-a}\right] = 1$$

(ii) $P[X < Y] = \iint_{\{(x,y):\, x<y\}} f_{X,Y}(x,y)$



$$= \int_0^\infty \int_0^y 2e^{-x} e^{-2y} \, dx \, dy$$

$$= 2\int_0^\infty e^{-2y} \left[1 - e^{-y}\right] dy$$

$$= 2\int_0^\infty \left(e^{-2y} - e^{-3y}\right) dy =$$

$$\int_0^\infty e^{-ay}\,dy = \lim_{t\to\infty}\int_0^t e^{-ay}\,dy = -\frac{1}{a}\lim_{t\to\infty}\left[e^{-at}-1\right]$$

$$= \frac{1}{a}$$

$$P[X<Y] = 2\left[\frac{1}{2}-\frac{1}{3}\right] = \cancel{2}\,\frac{1}{\cancel{6}\,3} = \frac{1}{3}$$

(iii) $P[X<a] = P[X<a,\ Y\in(-\infty,\infty)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a} f_{X,Y}(x,y)\,dx\,dy$$

$$F_X(a) = P[X \le a] = P[X \le a,\ Y < \infty]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a} f_{X,Y}(x,y)\, dx\, dy$$

$$= 2 \int_0^{\infty} \int_0^{a} e^{-x} e^{-2y}\, dx\, dy$$

$$= 2 \left(\int_0^{\infty} e^{-2y}\, dy \right) \left(\int_0^{a} e^{-x}\, dx \right)$$

$$= \cancel{2}\left[1 - e^{-a}\right] \frac{1}{\cancel{2}} = 1 - e^{-a}$$

Independent r.v.

We say that r.v. X and Y are independent if

$$P[X \in A, Y \in B] = P[X \in A] \, P[Y \in B].$$

for any $A, B \subseteq \mathbb{R}$.

In terms of joint c.d.f.

X and Y are independent if

$$\Rightarrow F_{X,Y}(x,y) = P[\overbrace{X \le x}^{X \in (-\infty, x]},\ \overbrace{Y \le y}^{(-\infty, y]}]$$

$$= P[X \le x]\, P[Y \le y]$$

$$= F_X(x)\, F_Y(y)$$

In terms of joint p.m.f

X and Y are independent if

$$\rightarrow p_{X,Y}(x_i, y_j) = P[X = x_i, Y = y_j]$$

$$= P[X = x_i]\, P[Y = y_j]$$

$$= p_X(x_i)\, p_Y(y_j).$$

In terms of joint p.d.f.

X and Y are independent if

$$\boxed{f_{X,Y}(x,y) = f_X(x)\, f_Y(y)}$$

Ex: Let X and Y be independent r.v. that have common density fun^n.

$\rightarrow$ $$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\rightarrow$ Find p.d.f. of $\boxed{X/Y}$.

$$\frac{X}{Y}$$

$$F_{X/Y}(a) = 0 \quad \text{if} \quad a \leq 0$$

For $\cancel{0<a}$ $a>0$,

$$F_{X/Y}(a) = P\left[\frac{X}{Y} \leq a\right] = P\left[X \leq aY\right]$$

$$= \iint\limits_{(x,y):\, x \leq ay} f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^{\infty} \int_0^{ay} \left[f_X(x)\, f_Y(y)\right] dx\, dy$$

$$F_{X/Y}(a) = \int_0^\infty \int_0^{ay} (e^{-x})(e^{-y})\, dx\, dy$$

$$= \int_0^\infty e^{-y}\left[1 - e^{-ay}\right] dy$$

$$= \int_0^\infty \left[e^{-y} - e^{-(1+a)y}\right] dy$$

$$= 1 - \frac{1}{1+a}$$

$$\Rightarrow f_{X/Y}(a) = F'_{X/Y}(a) = \frac{1}{(1+a)^2}, \quad 0 < a < \infty$$

Expectation of random Variable

$x f_X(x)\,dx$

$\approx x P[x < X < x+dx]$

$x P[x < X < x+dx]$

Let X be a discrete r.v. that assumes the values $x_1, x_2, x_3, \ldots$

Then, the expected value of X is the weighted average of the values of X, defined by

$$E[X] = \sum_{x \in X(\Omega)} x\, p_X(x).$$

Ex 1 $X \in \{0, 1\}$, $p_X(0) = \frac{1}{2}$, $p_X(1) = \frac{1}{2}$

$$E[X] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2} = \frac{0+1}{2}$$

$$\boxed{\begin{gathered} X \in \{a, b\}, \quad p_X(a) = \frac{1}{2} = p_X(b) = \frac{1}{2} \\ E[X] = \frac{a+b}{2} \end{gathered}}$$

2 $X \in \{0, 1\}$, $p_X(0) = \frac{1}{4}$, $p_X(1) = \frac{3}{4}$

$$E[X] = 0 \times \frac{1}{4} + 1 \times \frac{3}{4} = \frac{3}{4} = 0.75$$

Ex: Let a r.v. X denotes the outcome when we roll a fair dice. Find $E[X]$.

$$X \in \{1, 2, 3, 4, 5, 6\}, \quad p_X(i) = \frac{1}{6}$$

$$E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6}$$

$$= \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}.$$

$$X^2 = i^2 \in \{1, 4, 9, 16, 25, 36\}$$

Ex: Let I is an indicator random variable for the event A, i.e.,

$$I = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

Then, $E[I] = ?$

$$= 0 \times P(A^c) + 1 \times P[A] = P[A]$$

Let X be a cont. r.v. Then, the expectation of X will be defined as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Note: For small "dx", we have

$$f(x)\, dx = P[x < X < x + dx].$$

$X \sim U(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & ; a < x < b \\ 0 & \text{ow} \end{cases}$$

is called uniform random variable.

Ex: Suppose that you are expecting a message at some time past 5 P.M. From experience you know that X, the number of hours after 5 P.M., until the message arrives, is a r.v. with p.d.f

$$f_X(x) = \begin{cases} \frac{1}{1.5} & 0 < x < 1.5 \\ 0 & \text{ow} \end{cases}$$

Find the expected amount of time past 5 P.M, untill the message arrives.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{1.5} x f_X(x)\, dx$$

$$= \frac{1}{1.5} \int_0^{1.5} x\, dx = \frac{1}{3}\left[2.25 - 0\right]$$

$$= 0.75 \approx 45 \text{ mins.}$$

Expectation of a function of r.v.

X is a r.v.

$E[g(X)]$

Ex: Let X be a discrete r.v. that takes the values 0, 1, 2.

The p.m.f. of X is

$$p_X(0) = 0.2$$

$$p_X(1) = 0.5$$

$$p_X(2) = 0.3$$

Find $E[X^2]$,

$E[X^3]$,

Sol 1

$X \in \{0, 1, 2\}, \quad p_X(0) = 0.2, \quad p_X(1) = 0.5, \quad p_X(2) = 0.3.$

$X^2 \in \{0, 1, 4\}, \quad P[X^2 = 0] = P[X = 0] = 0.2$

$P[X^2 = 1] = P[X = 1] = 0.5$

$P[X^2 = 4] = P[X = 2] = 0.3$

$E[X^2] = 0\, p_{X^2}(0) + 1 \times p_{X^2}(1) + 4 \times p_{X^2}(4)$

$= 0.5 + 1.2 = 1.7$

$X^3 \in \{0, 1, 8\}, \quad p_{X^3}(0) = 0.2, \quad p_{X^3}(1) = 0.5, \quad p_{X^3}(8) = 0.3$

$E[X^3] = 0 \times 0.2 + 1 \times 0.5 + 8 \times 0.3 = 2.9.$

$$E[X^2] = \sum_{x \in X(\Omega)} x^2 p_X(x)$$

$$E[X^3] = \sum_{x \in X(\Omega)} x^3 p_X(x)$$

Ex: The time, in hours, it takes to locate and repair an electrical breakdown in a certain factory is a r.v. with p.d.f

$$f_X(x) = \begin{cases} \frac{1}{1-0} & \text{if } 0 < x < 1 \\ 0 & \text{ow.} \end{cases}$$

If the cost involved in breakdown of duration x is x^3, what is the expected cost of such breakdown?

$X \sim U(0,1)$, $X^3 \sim$

$$F_{X^3}(a) = P[X^3 \leq a] = 0 \quad \text{if } a \leq 0.$$

$$= P[X^3 \leq a] \quad \text{if } a > 0.$$

$$= P[X \leq a^{1/3}]$$

$$= \int_{-\infty}^{a^{1/3}} f_X(x)\,dx = \int_0^{a^{1/3}} f_X(x)\,dx$$

$$F_{X^3}(a) = \begin{cases} 0 & \text{if } a \leq 0 \\ a^{1/3} & \text{if } 0 < a < 1 \\ 1 & \text{if } a > 1 \end{cases}$$

$$f_{X^3}(a) = F'_{X^3}(a) = \frac{1}{3} a^{-2/3} \quad , \; 0 < a < 1$$

and $f_{X^3}(a) = 0 \quad a < 0$ and $a \geq 1$

Let $Y = X^3$

$$E[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$= \frac{1}{3} \int_0^1 y \, y^{-2/3} \, dy$$

$$= \frac{1}{3} \int_0^1 y^{1/3} \, dy = \frac{1}{3} \times \frac{3}{4} \left[y^{4/3} \right]_0^1$$

$$= \frac{1}{4}.$$

Verify that

$$\int_{-\infty}^{\infty} x^3 f_X(x)\,dx = \frac{1}{4}.$$

for this example.

$$E[X^3] = \int_{-\infty}^{\infty} x^3 f_X(x)\,dx.$$

X

X^2

X^3

X^4

It can be shown that

$$\underbrace{E\left[X^n\right]}_{n\text{th moment of } X} = \int_{-\infty}^{\infty} x^n f_X(x)\,dx \qquad n \in \mathbb{N}$$

$$= \sum_{x \in X(\Omega)} x^n p_X(x)$$

nth moment of X.

If X is a discrete r.v. with p.m.f. $p_X(x)$, then for any real-valued function g,

$$\boxed{E[g(X)] = \sum_{x \in X(\Omega)} g(x)\, p_X(x).}$$

$$E[\sqrt{X}] = \sum_{x \in X(\Omega)} \sqrt{x}\, p_X(x)$$

If X is a cont. r.v. with p.d.f. $f_X(x)$,

then for any real-valued fun^n g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$