Till now: point estimation

$$\bar{x} = \frac{\sum x_i}{n}$$

μ will, in general, not be equal to $\bar{x}$

Interval estimates

Specify an interval around $\bar{x}$, where in μ has a high chance of being found.

Point estimate $\bar{x}$ Known

population

$$\bar{x} = \frac{\sum x_i}{n}$$

Mean of $\bar{x} = \underline{\mu}$

variance

$$\text{variance of } \bar{x} = \underline{\frac{\sigma^2}{n}}$$

$$Var(\bar{x}) =$$

$$Var\left(\frac{\sum x_i}{n}\right) = \frac{1}{n^2} Var\left(\sum x_i\right)$$

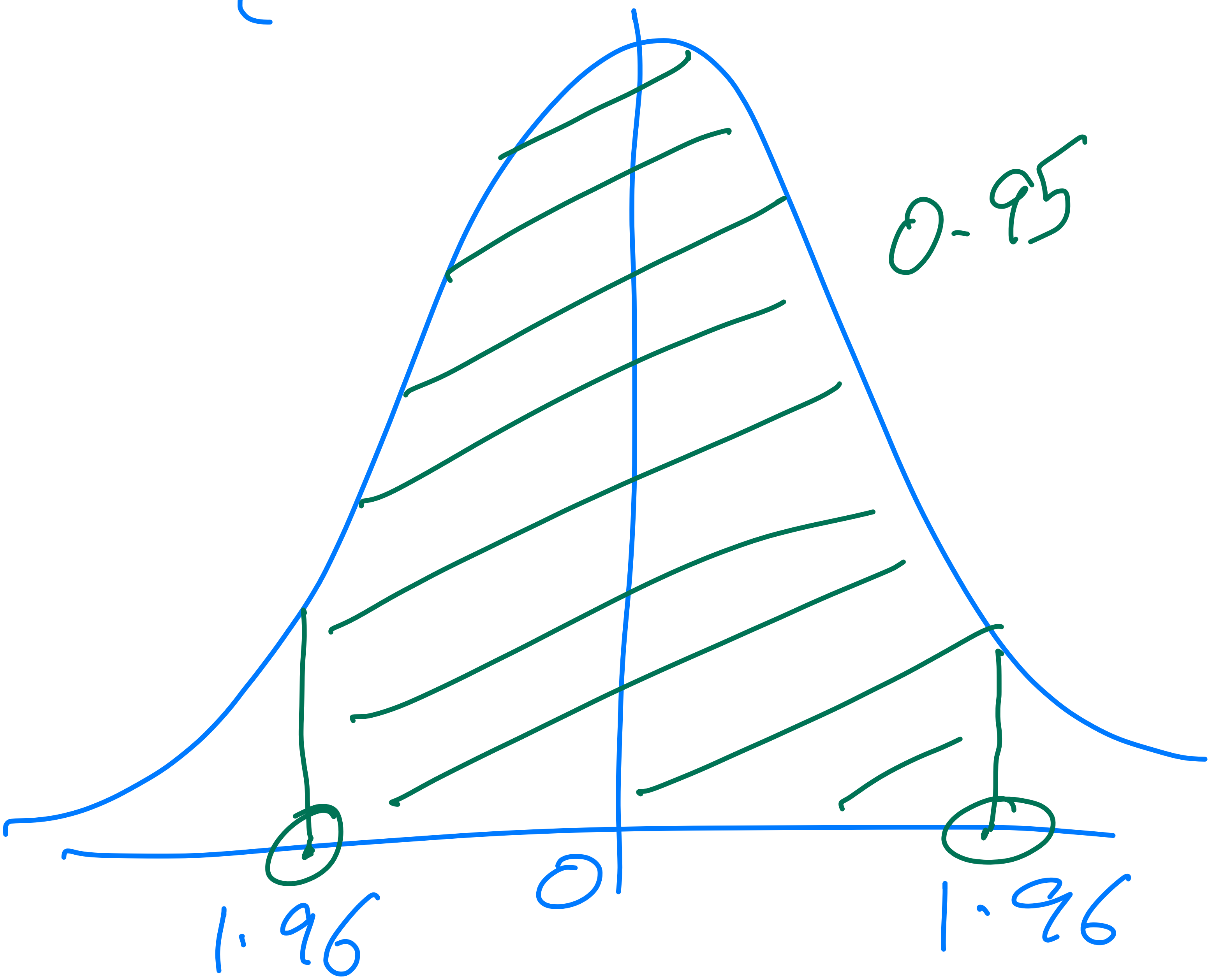
$$= \frac{1}{n^2} \sum Var(x_i)$$

$$= \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

$\frac{\sigma^2}{n}$ is the variance of sample mean

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$ is std. normal

$$\sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right)$$ is $N(0,1)$



$$P\left\{-1.96 < \frac{\sqrt{n}\,(\bar{X}-\mu)}{\sigma} < 1.96\right\} = 0.95$$

$$P\left\{-\frac{1.96\sigma}{\sqrt{n}} < \bar{X}-\mu < \frac{1.96\sigma}{\sqrt{n}}\right\} = 0.95$$

$$\bar{X}-\mu > -\frac{1.96\sigma}{\sqrt{n}}$$

$$\mu < \bar{X} + \frac{1.96\sigma}{\sqrt{n}}$$

$$\mu > \bar{X} - \frac{1.96\sigma}{\sqrt{n}}$$

$$P\left\{\bar{X}-\frac{1.96\sigma}{\sqrt{n}} < \mu < \bar{X}+\frac{1.96\sigma}{\sqrt{n}}\right\} = 0.95$$

$$\left(\bar{X} - \frac{1.96\sigma}{\sqrt{n}},\ \bar{X} + \frac{1.96\sigma}{\sqrt{n}}\right)$$

μ belongs to this interval with 95% confidence.

High σ will imply wider interval

Higher n will lead to narrower intervals

E.g. loc A $\xrightarrow{\text{signal}}$ loc B

$\mu \longrightarrow \mu + N$

N is Noise is normal

N has mean 0, $\sigma = 2$

variance of the population ← variance $= 4$

To reduce errors, you transmit the same value 9 times. You receive

5, 8.5, 12, 15, 7, 9, 7.5, 6.5, 10.5. You want to estimate 95% confidence interval for μ

what is the point estimate for μ

$$\mu = \frac{\sum x_i}{9} = 9$$

95% Confidence interval is

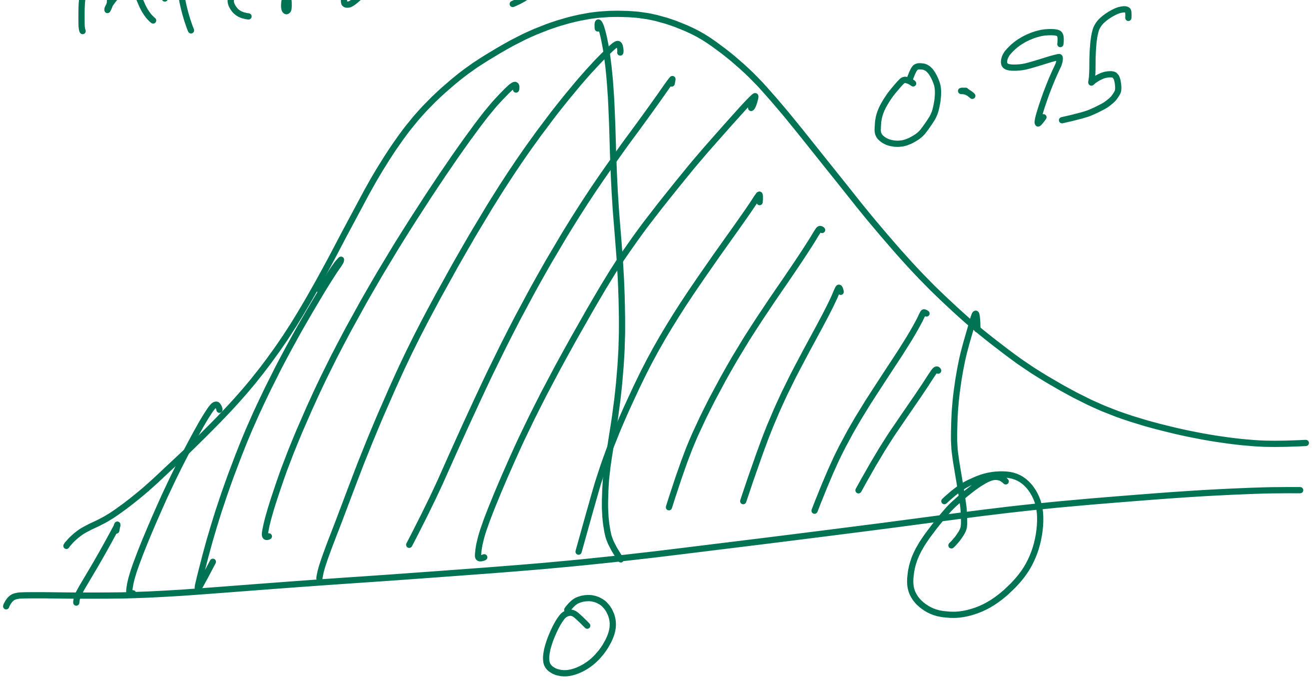
$$\left(9 - 1.96 * \frac{2}{3},\ 9 + 1.96 * \frac{2}{3}\right)$$

$$= (7.69,\ 10.31)$$

we are 95% confident that the message that was sent was between 7.69 and 10.31.

These are called two-sided confidence intervals

One-sided confidence intervals



$$P(Z < 1.645) = 0.95$$

$\therefore$

$$P\left(\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu) < 1.645\right) = 0.95$$

$$P\left(\bar{X} - \mu < \frac{1.645\sigma}{\sqrt{n}}\right) = 0.95$$

$$P\left(\mu > \bar{X} - \frac{1.645\sigma}{\sqrt{n}}\right) = 0.95$$

one-sided upper confidence interval

$$\left(\bar{X} - \frac{1.645\sigma}{\sqrt{n}}, \infty\right)$$

---

one-sided lower confidence interval

$$\left(-\infty, \bar{X} + \frac{1.645\sigma}{\sqrt{n}}\right)$$

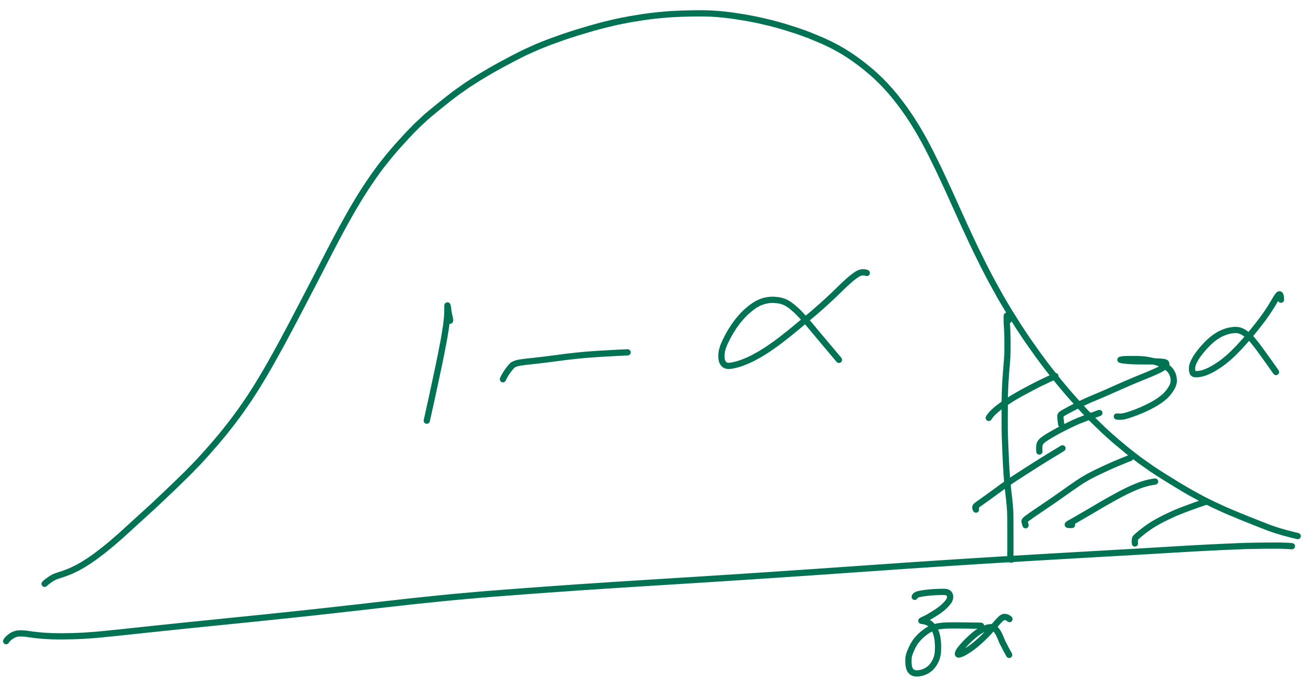
Apply this on the previous example

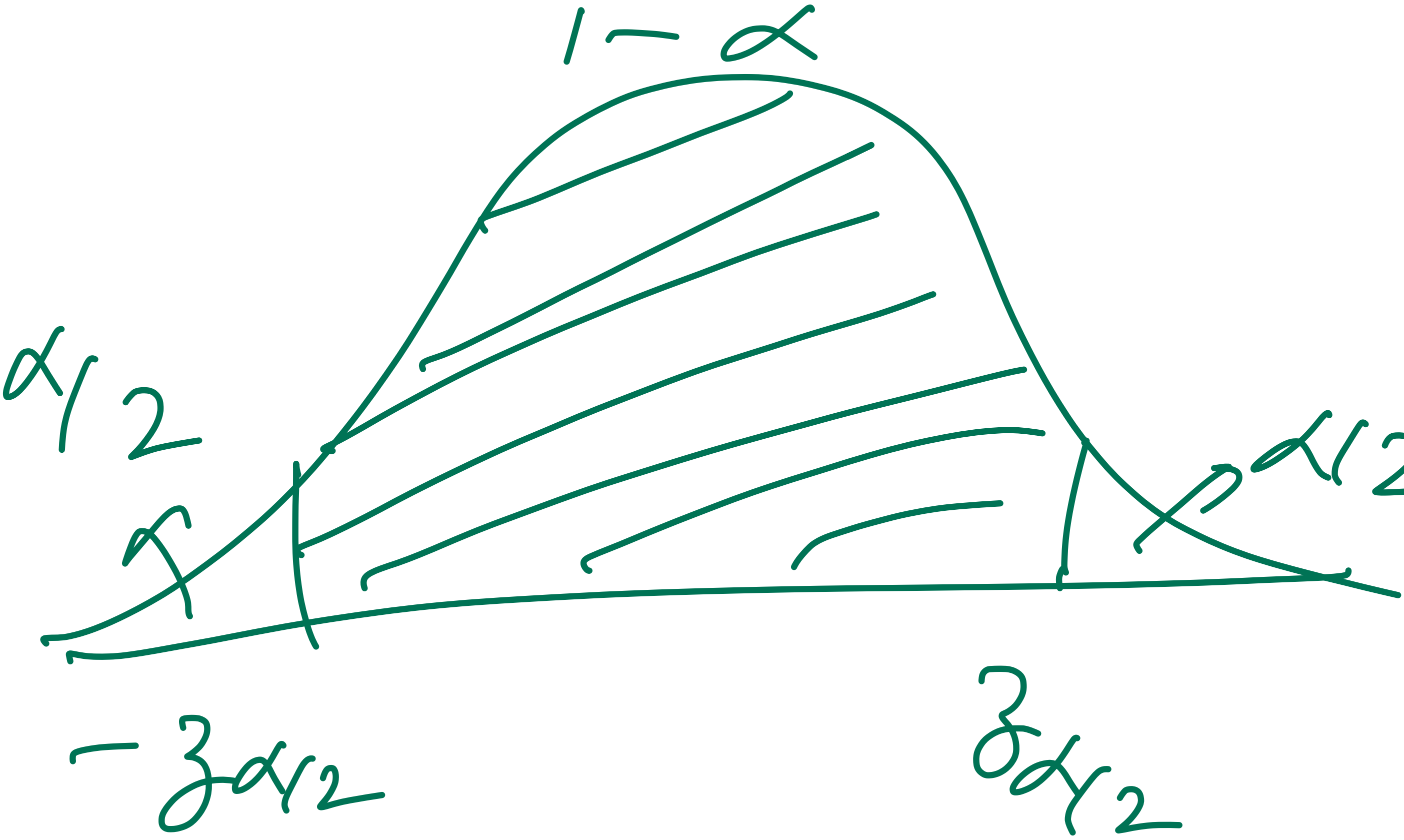
$(7.903, \infty)$: one-sided upper confidence interval

95%

$(-\infty, 10.097)$: 95%

one sided lower confidence interval





$$P\left\{-z_{\alpha/2} < Z < z_{\alpha/2}\right\} = 1-\alpha$$

$$P\left\{-z_{\alpha/2} < \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < z_{\alpha/2}\right\} = 1-\alpha$$

$$P\left\{\bar{X} - \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}} < \mu < \bar{X} + \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}\right\} = 1-\alpha$$

A $100(1-\alpha)$ percent confidence of interval for μ is

$$\left(\bar{X} - \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}},\ \bar{X} + \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}\right)$$

For one-sided

$$P(Z > z_\alpha) = \alpha$$

$$P(Z < -z_\alpha) = \alpha$$

$$\left(\bar{X} - \frac{z_\alpha \sigma}{\sqrt{n}},\ \infty\right) : \text{upper}$$

$$\left(-\infty,\ \bar{X} + \frac{z_\alpha \sigma}{\sqrt{n}}\right) : \text{lower}$$

E.g.: Obtain 99% confidence intervals for the previous example two-sided as well as one-sided

$$z_{0.005} = 2.58$$

$$z_{0.01} = 2.33$$

99% $(7.28, 10.72)$

95% $(7.69, 10.31)$

$(7.447, \infty) \rightarrow 99\%$

95% $(7.903, \infty)$

$(-\infty, 10.553) \rightarrow 99\%$

95% $(-\infty, 10.097)$

How to chose the sample size?

confidence is $1-\alpha = 99\%$

interval is 'a'

$$\left(\bar{x} - \frac{2.58\sigma}{\sqrt{n}},\ \bar{x} + \frac{2.58\sigma}{\sqrt{n}}\right)$$

$$2 \times \frac{2.58\sigma}{\sqrt{n}} = a$$

$$\sqrt{n} = \frac{2 \times 2.58 \times \sigma}{a}$$

$$n = \left(\frac{5.16 \times \sigma}{a}\right)^2$$

E.g. Salmon

$\sigma = 0.3$ pounds

95% certain that your estimate of this season's mean weight is correct to within $\pm$ 0.1 pounds, how many samples do you need to take?

$$\mu \in \left(\bar{X} - \frac{1.96\,\sigma}{\sqrt{n}},\ \bar{X} + \frac{1.96\,\sigma}{\sqrt{n}} \right)$$

$$\longleftarrow 0.2 \longrightarrow$$

$$\frac{1.96\sigma}{\sqrt{n}} = 0.1$$

$$\frac{1.96 * 0.3}{0.1} = \sqrt{n}$$

$$\sqrt{n} \geq 5.88$$

$$n \geq 34.57$$

35 sample readings will be sufficient

Section 7.5: ~~Self-Study~~ confidence interval for the mean of a Bernoulli

Put 2 examples in the scriber.✓.