Exponential Random Variable

A continuous r.v. X whose p.d.f. is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

, for some $\lambda > 0$, is called exponential r.v. with parameter λ.

We write $X \sim Exp(\lambda)$.

$$\int_0^{\infty} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} e^{-\lambda x}\, dx = \lambda \times \frac{1}{\lambda} = 1.$$

Let $X \sim Exp(\lambda)$.

$$F_X(x) = P[X \leq x]$$

$$= \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \begin{cases} 0 & \text{if } x \leq 0 \\ \int_{0}^{x} f_X(x)\,dx & \text{if } x \geq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0 \end{cases}$$

$$\Rightarrow P[X > x] = 1 - F_X(x)$$

$$= e^{-\lambda x} \quad \text{if } x > 0.$$

"λ is called the rate of the Exponential distribution".

<u>Ex:</u> Let $X \sim Exp(\lambda)$. Then

(i) $M_X(t) = \dfrac{\lambda}{\lambda - t}$, $t < \lambda$

(ii) $E[X] = \dfrac{1}{\lambda}$

(iii) $Var(X) = \dfrac{1}{\lambda^2}$

Sol: Given $X \sim Exp(\lambda)$

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x}\, dx$$

$$= \begin{cases} \dfrac{\lambda}{\lambda - t} & \text{if } t < \lambda \\ \text{does not exist.} & \text{if } t \geq \lambda \end{cases}$$

$$M_X(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

$$M_X'(t) = \frac{\lambda}{(\lambda - t)^2}, \quad t < \lambda$$

$$M_X''(t) = \frac{2\lambda}{(\lambda - t)^3}, \quad t < \lambda$$

$X \sim Geom(p)$

$E[X] = \frac{1}{p}$

$Var(X) = \frac{(1-p)}{p^2}$

$np = \lambda$

$$E[X] = M_X'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$E[X^2] = M_X''(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\Rightarrow Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Memoryless Property

We say that a non-negative r.v. X is memoryless if

$$P[X > t+s \mid X > t] = P[X > s] \quad -(1)$$

$$\forall\, s, t \geq 0.$$

1 $\Leftrightarrow$ $$\frac{P[X>t+s \ \& \ X>t]}{P[X>t]} = P[X>s]$$

$\Leftrightarrow$ $$P[X>t+s, X>t] = P[X>t]\,P[X>s]$$

$\Leftrightarrow$ $$P[X>t+s] = P[X>t]\,P[X>s]$$

Ex: $X \sim \text{Exp}(\lambda)$,

$$P[X>t+s] = e^{-\lambda(t+s)} = e^{-\lambda t} e^{-\lambda s} = P[X>t]\,P[X>s]$$

Ex: If $X \sim \mathrm{Exp}(\lambda)$, then X is memoryless.

Ex: If $X \sim \mathrm{Exp}(\lambda)$, then $cX \sim \mathrm{Exp}\left(\frac{\lambda}{c}\right)$ for any constant $c > 0$. λ is called the rate of the exponential distribution.

Sol: Let $Y = cX$,

$$F_Y(y) = P[Y \le y]$$
$$= P[cX \le y]$$
$$= P[X \le y/c]$$
$$= \begin{cases} 1 - e^{-\lambda y/c} & y > 0 \\ 0 & y \le 0 \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$\Rightarrow \quad F_Y(y) = 1 - e^{-\frac{\lambda}{c} y}$$

$$f_Y(y) = \begin{cases} \frac{\lambda}{c} e^{-\frac{\lambda}{c} y} & y > 0 \\ 0 & \text{ow} \end{cases}$$

$$\Rightarrow \quad Y \sim \text{Exp}\left(\frac{\lambda}{c}\right).$$

Ex: Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10000 miles.

If a person desires to take a 5000-miles trip, what is the prob. that she will be able to complete her trip without having to replace her car battery?

Sol: Exp(λ) is memoryless.

Let X = remaining lifetime of the battery (in thousand miles)

$$\sim \text{Exp}\left(\lambda = \frac{1}{10}\right) = \text{Exp}(0.1)$$

$$P[X > 5] = e^{-\lambda 5} = e^{-0.5}$$

Ex: If $X_1, X_2, \ldots, X_n$ are independent exponential random variables having respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$Y = \min(X_1, \ldots, X_n) \sim \text{Exp}\left(\sum_{i=1}^{n} \lambda_i\right).$$

Sol: Given that (i) $X_i \sim Exp(\lambda_i)$, $i = 1, 2, \ldots, n$.

(ii) X_i's are independent r.vs.

Let $Y = \min(X_1, X_2, \ldots, X_n)$

Now, $P[Y > y] = P[\min(X_1, \ldots X_n) > y]$

$= P[X_1 > y, X_2 > y, \ldots, X_n > y]$

$= \prod_{i=1}^{n} P[X_i > y]$

$= \prod_{i=1}^{n} e^{-\lambda_i y} = e^{-\lambda y}$, $\lambda = \sum_{i=1}^{n} \lambda_i$

$$\Rightarrow \quad Y \sim \mathrm{Exp}\left(\sum_{i=1}^{n} \lambda_i\right).$$

Ex: A series system is one that needs all of its components to function in order for the system itself to be functional.

For an n-component series system in which the component lifetimes are independent exponential random variables with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, what is the prob. that the system survives for a time t.

Sol: Since the system lifetime is equal to the minimal component life, therefore

$$P[\text{system life exceeds } t]$$

$$= P[\min(X_1, \ldots, X_n) > t]$$

$$= e^{-\sum_{i=1}^{n} \lambda_i t}$$

Gamma Distribution :

A random variable X is said to have a gamma distribution with parameters (α, λ), $\alpha > 0$, $\lambda > 0$, if its p.d.f. is given by

$$f_X(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

We write $X \sim \text{Gamma}(\alpha, \lambda)$.

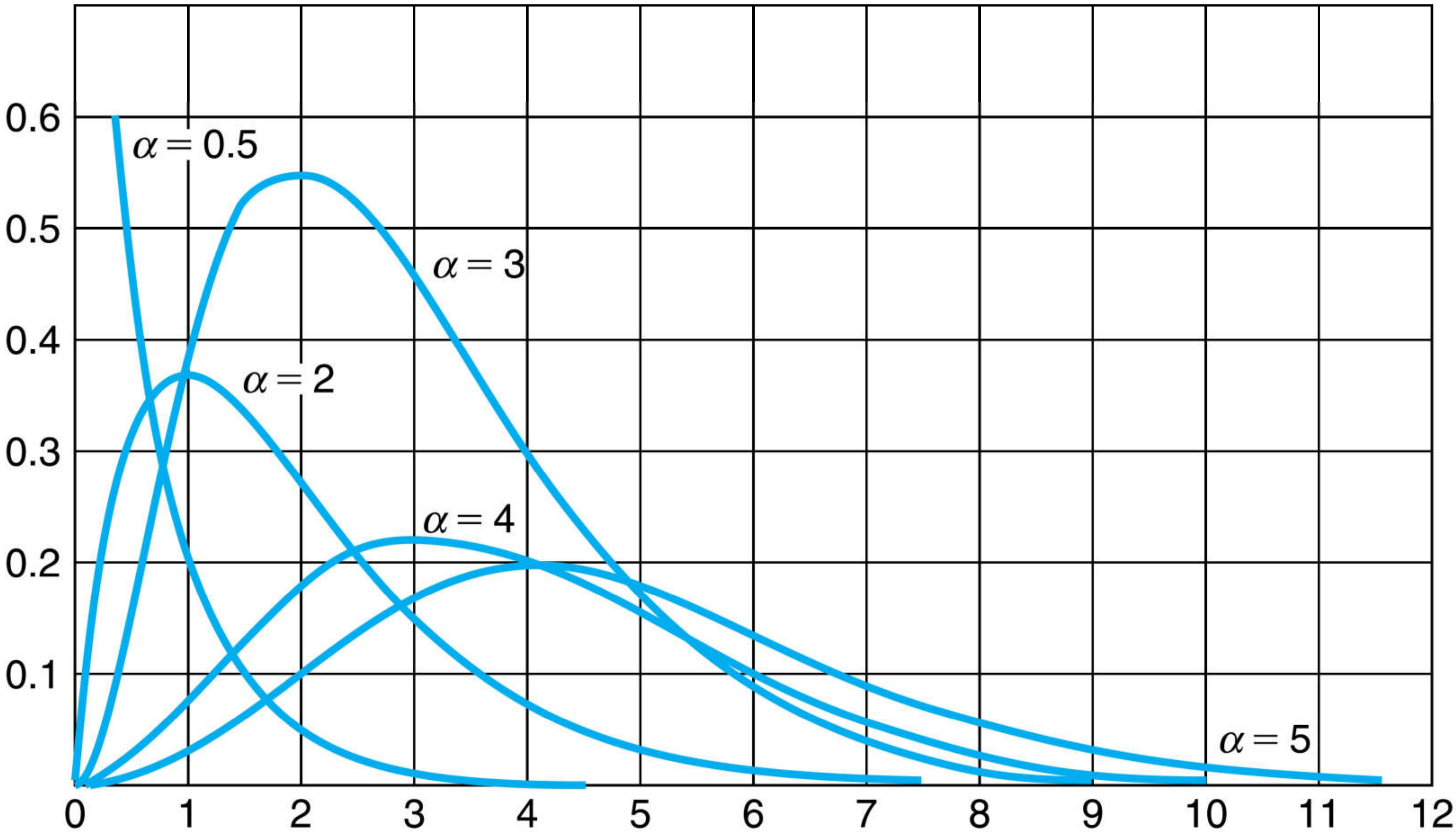
It is easy to verify that

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

whenever $X \sim \text{Gamma}(\alpha, \lambda)$.

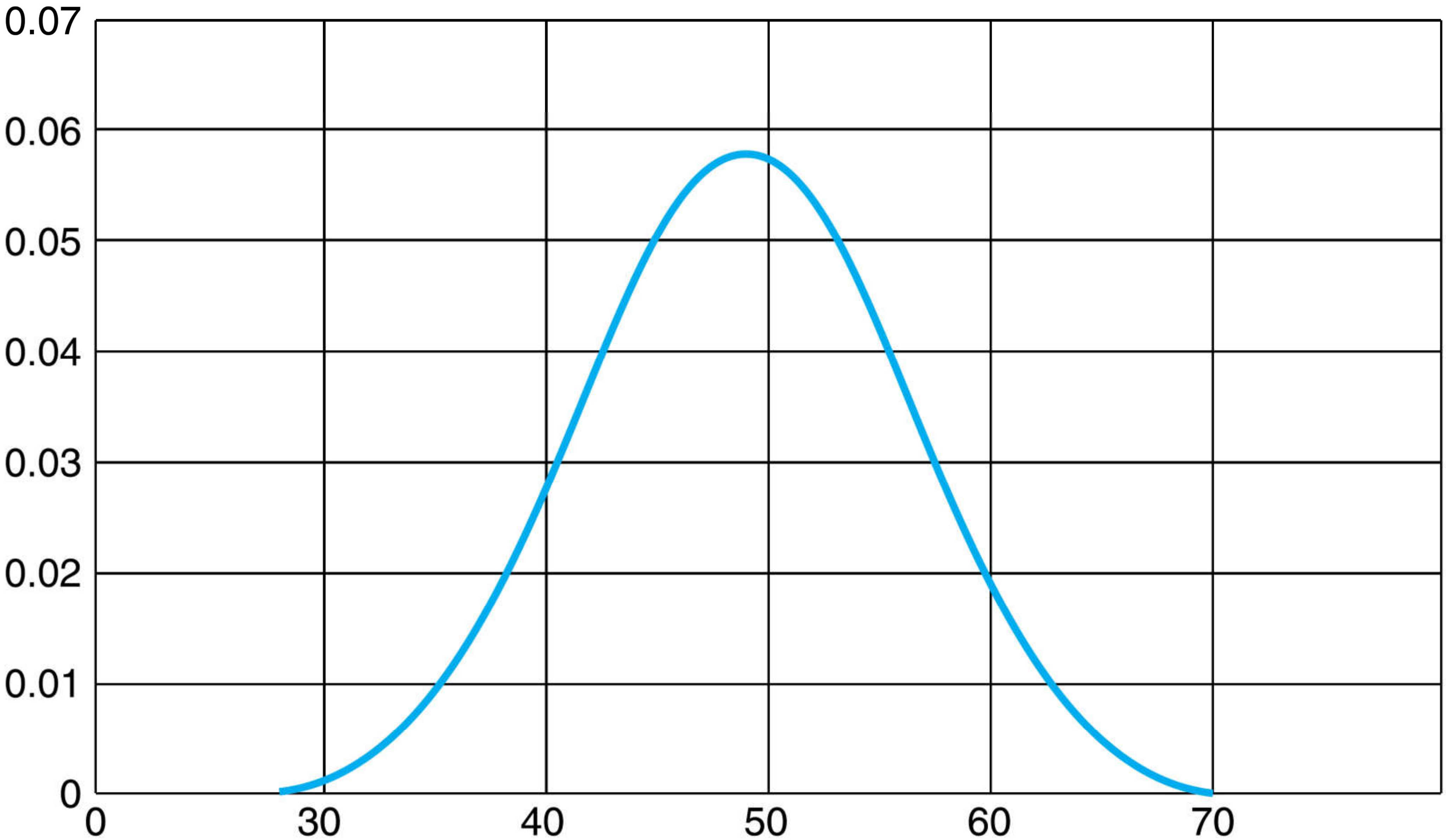
$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y}\,dy = \int_0^{\infty} \lambda (\lambda x)^{\alpha-1} e^{-\lambda x}\,dx$$

Replace $\lambda x = y \Rightarrow dy = \lambda\,dx$



0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
8
9
10
11
12
$\alpha = 0.5$
$\alpha = 3$
$\alpha = 2$
$\alpha = 4$
$\alpha = 5$

$\alpha = 50$



Ex: Let $X \sim \text{Gamma}(\alpha, \lambda)$. Find its moment generating function. Hence, find its mean and variance.

Sol:

$$M_X(t) = E[e^{tX}]$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda e^{tx} (\lambda x)^{\alpha-1} e^{-\lambda x} \, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} \, dx$$

Let $(\lambda - t)x = y$

$$= \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \boxed{\frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y} \, dy}$$

$$= \frac{\lambda^\alpha}{(\lambda-t)^\alpha}, \quad \lambda > t$$

$$E[X] = M_X'(0), \quad E[X^2] = M_X''(0)$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

Ex: Let $X_i \sim \text{Gamma}(\alpha_i, \lambda)$, $i=1,2$.

Further, suppose that X_1 & X_2 are independent.

Then, show that

$$X = X_1 + X_2 \sim \text{Gamma}(\alpha_1+\alpha_2, \lambda).$$

Sol:

$$M_X(t) = E[e^{tX}] = E[e^{tX_1} e^{tX_2}]$$

$$= E[e^{tX_1}]\, E[e^{tX_2}]$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_1} \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_2} = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_1+\alpha_2}.$$

Ex: If $X_1, X_2, \ldots, X_n$ are independent exponential r.v.s, each having rate λ, then

$$X = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda).$$

Ex: The lifetime of a battery is exponentially distributed with rate λ.

If a stereo cassette requires one battery to operate, then the total playing time one can obtain from a total of n batteries is a gamma r.v. with parameters (n, λ).

The Chi-Square distribution

If $Z_1, Z_2, \ldots, Z_n$ are independent standard normal random variables, then

$$X = \sum_{i=1}^{n} Z_i^2 \qquad \left[\begin{array}{c} Z_i \sim N(0,1) \\ i = 1, 2, \ldots, n \end{array} \right]$$

is said to have a chi-square distribution with n-degrees of freedom. We write

$$X \sim \chi_n^2$$

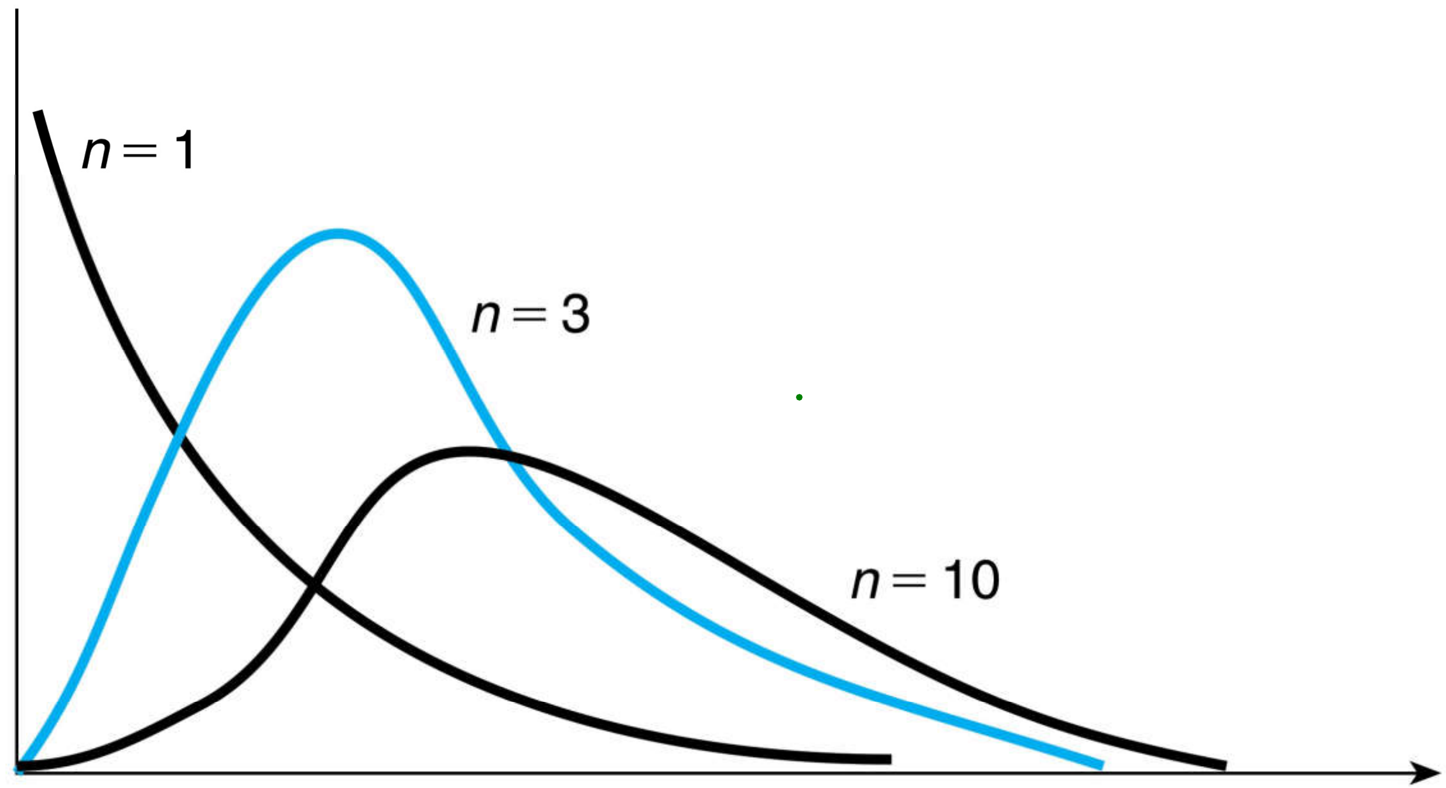


FIGURE 5.13 *The chi-square density function with n degrees of freedom.*

Ex: Let $X \sim \chi^2_n$. Find its m.g.f. Hence, find its mean and variance.

Sol: $\chi^2_n = Z_1^2 + Z_2^2 + \cdots + Z_n^2$

$$M_{\chi^2_n}(t) = E\left[e^{t\chi^2_n}\right] = \prod_{i=1}^{n} E\left[e^{tZ_i^2}\right]$$

$$= \left[[1-2t]^{-\frac{1}{2}}\right]^n$$

$\boxed{Z \sim N(0,1)}$

$$E\left[e^{tZ^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2} e^{-z^2/2} \, dz$$

$$\therefore = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2\left[\frac{1-2t}{2}\right]}\, dz$$

let $\sigma^2 = \dfrac{1}{1-2t}$

$$E[e^{tZ^2}]\cancel{M} = \frac{\sigma}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2}\, dz$$

$N(0, \sigma^2)$

$$= \sigma = (1-2t)^{-1/2}$$

$$\therefore \quad M_{\chi^2_n}(t) = (1-2t)^{-n/2}$$

$$= \frac{1}{(1-2t)^{n/2}}$$

$$= \frac{\left(\frac{1}{2}\right)^{n/2}}{\left(\frac{1}{2}-t\right)^{n/2}}$$

$$\Rightarrow \quad \chi^2_n \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right).$$