Ex: If $X \sim N(3, 16)$, find

$\mu = 3$, $\sigma^2 = 16$ $\quad [\mu = 3,\ \sigma = 4]$

(a) $P[X < 11]$

$$= P\left[\frac{X-3}{4} < \frac{11-3}{4}\right] = P\left[\frac{X-3}{4} < 2\right] \qquad \left[Z = \frac{X-\mu}{\sigma}\right]$$

$$= \Phi(2) \simeq 0.9772.$$

(b) $P[X > -1] \simeq 0.8413.$

$$= P\left[\frac{X-3}{4} > \frac{-1-3}{4}\right] = P\left[\frac{X-3}{4} > -1\right] = 1 - \Phi(-1)$$

$$= 1 - P[Z > 1] = 1 - [1 - P(Z \le 1)] = \Phi(1)$$

(c) $P[2 < X < 7]$

$= P\left[\frac{2-3}{4} < \frac{X-3}{4} < \frac{7-3}{4}\right]$

$= P\left[-0.25 < \frac{X-3}{4} < 1\right]$

$= \Phi(1) - \Phi(-0.25)$

$= \Phi(1) - [1 - \Phi(0.25)]$

$= \Phi(1) + \Phi(0.25) - 1 \approx 0.8413 + 0.5987 - 1 = 0.44.$

$P[a < X < b]$

$= F_X(b) - F_X(a)$

Ex: The power W dissipated in a resistor is proportional to the square of the voltage V. That is,

$$\boxed{W = rV^2},$$

where r is a constant.

If $r = 3$, and $V \sim N(6, 1)$, find (a) $E[W]$; (b) $P[W > 120]$.

(a) $E[W] = E[3V^2] = 3E[V^2]$

$= 3 \times 37 = 111$

$V \sim N(6, 1)$

$E[V] = 6$

$Var[V] = 1$

$E[V^2] = 1 + (E[V])^2$

$= 1 + 36 = 37$

(b) $P[W > 120] = P[3V^2 > 120]$

$= P[V^2 > 40]$

$= P[V > \sqrt{40}]$

$= P\left[\frac{V-6}{1} > \frac{\sqrt{40}-6}{1}\right] = P\left[\frac{V-6}{1} > 0.3246\right]$

$= 1 - \Phi(0.3246) = 1 - 0.6293 = 0.3707.$

Ex Let X be the number of times that a fair coin that is flipped 40 times lands on heads. Find the prob. that $X = 20$.

Sol:

$$B(40, 0.5)$$

$$P[X=20] = {}^{40}C_{20}\,(0.5)^{20}\,(0.5)^{20} \approx 0.1254.$$

$$P[X=20] = P\Big[\underbrace{20-\tfrac{1}{2} < X < 20+\tfrac{1}{2}}_{\text{continuity correction}}\Big]$$

$n = 40, \; p = 0.5$.

$E[X] = np = 40 \times 0.5 = 20$

$Var[X] = np(1-p) = 40 \times 0.25 = 10$.

$X \sim N(20, 10)$

$$P[19.5 < X < 20.5] = P\left[\frac{19.5-20}{\sqrt{10}} < \frac{X-20}{\sqrt{10}} < \frac{20.5-20}{\sqrt{10}}\right]$$

$$= P\left[-0.16 < \frac{X-20}{\sqrt{10}} < 0.16\right] = \Phi(0.16) - \Phi(-0.16)$$

$$= 2\Phi(0.16) - 1 = 2 \times 0.5636 - 1 = 0.1272.$$

Ex: The ideal size of a I^{st} year class at a particular college is 150 students. The college, knowing from past experience that, on the average, only 30% of those accepted for admission will actually attend, uses a policy of approving the application of 450 students. Compute the probability that more than 150 I^{st} year studends attend this college.

Sol: X = number of students that attend this college.

$X \sim B(450, 0.3)$

$P[X > 150] = P[X \geq 150.5]$

$E[X] = 450 \times 0.3 = 135$, $Var(X) = \sigma^2 = 450 \times 0.3 \times 0.7 = 94.5$

$= P\left[\frac{X-135}{\sqrt{94.5}} \geq \frac{150.5-135}{\sqrt{94.5}}\right] = P\left[\frac{X-135}{\sqrt{94.5}} \geq 1.5945\right]$

$$= 1 - \Phi(1.59)$$

$$= 1 - 0.9441 = 0.0559.$$

Ex: Sum of independent normal random variables is also a normal random variable.

Sol: Let $X_1, X_2, \ldots, X_n$ be n independent normal random variables s.t.

$$X_i \sim N(\mu_i, \sigma_i^2), \quad i = 1, 2, \ldots, n.$$

Define $X = X_1 + X_2 + \cdots + X_n$

<u>Claim:</u> $X \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$

Let $Y \sim N(\mu, \sigma^2)$. Then

$$M_Y(t) = E[e^{tY}] = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$
\begin{aligned}
M_X(t) &= E\left[e^{tX}\right] \\
&= E\left[e^{t(X_1+X_2+\cdots+X_n)}\right] \\
&= E\left[e^{tX_1} e^{tX_2} \cdots e^{tX_n}\right] \\
&= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] \cdots E\left[e^{tX_n}\right] \\
&= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \, e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} \cdots e^{\mu_n + \frac{\sigma_n^2 t^2}{2}} \\
&= e^{\mu t + \frac{\sigma^2 t^2}{2}}, \quad \mu = \sum_{i=1}^{n} \mu_i, \quad \sigma^2 = \sum_{i=}^{n} \sigma_i^2
\end{aligned}
$$

$$\Rightarrow X \sim N\left(\mu = \sum_{i=1}^{n} \mu_i \,,\ \sigma^2 = \sum_{i=1}^{n} \sigma_i^2\right).$$

Ex: Data from the National Oceanic and Atmospheric Administration (NOAA) indicate that the yearly precipitation in Los Angles is a normal random variable with a mean of $\underset{}{\overset{\mu}{12.08}}$ inches and a standard deviation of $\underset{\sigma}{3.1}$ inches

(a) Find the probability that the total precipitation during the next 2 years will exceed 25 inches.

(b) Find the prob. that next years precipitation will exceed that of the following year by more than 3 inches.

Assume that the precipitation totals for the next 2 years are independent.

(a) Let X_1 & X_2 be the precipitation totals for the next 2 years.

$$X_1 + X_2 \sim N\left(12.08 + 12.08,\ (3.1)^2 + (3.1)^2\right)$$

$$= N(24.16,\ 19.22)$$

$$P[\overbrace{X_1 + X_2}^{X} > 25] = P\left[\frac{X-\mu}{\sigma} > \frac{25-\mu}{\sigma}\right] = P\left[\frac{X-24.16}{\sqrt{19.22}} > 0.1916\right]$$

$$\Rightarrow P[X > 25] = P[Z > 0.1916]$$
$$= 1 - \Phi(0.1916)$$
$$\approx 0.4240$$

(b) $P[X_1 > X_2 + 3] = P[X_1 - X_2 > 3] = P\left[\frac{Y-0}{\sqrt{19.22}} > \frac{3-0}{\sqrt{19.22}}\right]$

≈ 0.2469

$X_1 \sim N(12.08, 9.61)$

$X_2 \sim N(12.08, 9.61)$

$-X_2 \sim N(-12.08, 9.61)$

$X_1 - X_2 \sim N(0, 19.22)$

$Y = X_1 - X_2$

Def: For $\alpha \in (0,1)$, let z_α be such that

$$P[Z > z_\alpha] = \alpha$$

$$\Rightarrow \quad \Phi(z_\alpha) = 1-\alpha \; = P[Z \le z_\alpha]$$

We call "z_α" the $100(1-\alpha)$ percentile of the standard normal variate.