If X is a discrete r.v. with p.m.f. $p_X(x)$, then for any real-valued function g,

$$\boxed{E[g(X)] = \sum_{x \in X(\Omega)} g(x)\, p_X(x).}$$

$$E[\sqrt{X}] = \sum_{x \in X(\Omega)} \boxed{\sqrt{x}\, p_X(x)}$$

If X is a cont. r.v. with p.d.f. $f_X(x)$, then for any real-valued funⁿ g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx.$$

Property

Ex: (1) $E[aX+b] = aE[X]+b$, $a, b \in \mathbb{R}$

(2) $E[b] = b$

Proof: (1) $E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$

$= a\int_{-\infty}^{\infty} x f_X(x)\, dx + b\int_{-\infty}^{\infty} f_X(x)\, dx$

$= aE[X]+b$

(2) $\boxed{a=0}$

Let X and Y be discrete r.v. Further let $p_{X,Y}(x,y)$ be joint p.m.f. of X and Y.

Then,

$$E[g(X,Y)] = \sum_{x}\sum_{y} g(x,y)\, p_{X,Y}(x,y)$$

<u>example</u>: $E[X+Y] = \sum_{x}\sum_{y} (x+y)\, p_{X,Y}(x,y)$

$$= \sum_{x}\sum_{y} \left[x\, p_{X,Y} + y\, p_{X,Y} \right] = E[X] + E[Y].$$

$$E[X_1 + X_2 + \cdots + X_n] = \sum_{i=1}^{n} E[X_i]$$

Ex: A construction firm has recently sent in bids for 3 jobs worth (in profit) 10, 20, and 40 (Thousand) dollars.

If its probability of winning the jobs are respectively 0.2, 0.8 and 0.3, what is the firm's expected total profit.

Sol:

Let X_i denotes the firm profit from the job i, then $(i = 1, 2, 3)$

$$X_1 = \begin{cases} 10 & \text{with probability } 0.2 \\ 0 & \text{〃} \quad 0.8 \end{cases}$$

$$X_2 = \begin{cases} 20 & \text{〃} \quad 0.8 \\ 0 & \text{〃} \quad 0.2 \end{cases}$$

$$X_3 = \begin{cases} 40 & \text{〃} \quad 0.3 \\ 0 & \text{〃} \quad 0.7 \end{cases}$$

$$E[X_1+X_2+X_3] = \sum_{i=1}^{3} E[X_i]$$

$$E[X_1] = 10(.2) + 0\cdot(0.8) = 2$$

$$E[X_2] = 20(.8) + 0(0.2) = 16$$

$$E[X_3] = 40(0.3) + 0(0.7) = 12$$

$$E[X_1+X_2+X_3] = 30.$$

Ex: Suppose there are 20 coupons of different type. Assume that each time one obtains a coupon, it is equally likely to be any one of the types. Compute the expected number of different types that are obtained in a set of 10 coupons.

Sol: $X \sim$ no. of different types that are obtained in a set of 10 coupons.

Let

$$X_i = \begin{cases} 1 & \text{if atleast one type } i \text{ coupon is contained in the set of 10 coupons,} \\ 0, & \text{otherwise} \end{cases}$$

$i = 1, 2, 3, \ldots, 20.$

$$X = X_1 + X_2 + \cdots + X_{20}$$

$$\Rightarrow E[X] = \sum_{i=1}^{20} E[X_i].$$

$$E[X_i] = 1 \cdot P[X_i = 1] + 0 \cdot P[X_i = 0]$$

$$= \boxed{P[X_i = 1]}$$

$$P[X_i = 1] = 1 - P[\text{no type } i \text{ coupons are contained in the set of 10 coupons}]$$

20

$\boxed{19}$

$$= 1 - \underbrace{\frac{19}{20} \times \frac{19}{20} \times \cdots \times \frac{19}{20}}_{10\text{-times}}$$

$$= 1 - \left(\frac{19}{20}\right)^{10} \qquad \forall\, i = 1, 2, \ldots, 20$$

$$\therefore\ E[X] = \sum_{i=1}^{20} E[X_i] = 20\left[1 - \left(\frac{19}{20}\right)^{10}\right] = 8.025$$

Suppose we predict that

$$X \approx c$$

$\mu_X = E[X]$

Then, "squared error" $= (X-c)^2$

Now, $E[(X-c)^2] = E[(X-\mu_X+\mu_X-c)^2]$

$$= E\left[(X-\mu_X)^2 + (\mu_X-c)^2 + 2(\mu_X-c)(X-\mu_X)\right]$$

$$\Rightarrow E[(X-c)^2] = E[(X-\mu_X)^2] + (\mu_X-c)^2 + 2(\mu_X-c)\,E[X-\mu_X]$$

$$= E[(X-\mu_X)^2] + (\mu_X-c)^2$$

$$\geq E[(X-\mu_X)^2]$$

$$\sum (x-\mu_X)^2 p_X(x)$$

$$E[X-\mu_X]$$

$$= E[X] - E[\mu_X]$$

$$= \mu_X - \mu_X = 0$$

$$E\left[(X-c)^2\right] \geq E\left[(X-\mu_X)^2\right]$$

Given r.v. X and its c.d.f. $F_X(x)$.

(i) I^{st} measure, $E[X]$.

Examples (i) $W_1 = 0$ with prob. 1.

(ii) $W_2 = \begin{cases} -1 & \text{with prob. } \frac{1}{2} \\ 1 & \text{"} \quad \frac{1}{2} \end{cases}$

(iii) $W_3 = \begin{cases} -100 & \text{"} \quad \frac{1}{2} \\ 100 & \text{"} \quad \frac{1}{2} \end{cases}$

$E[W_i] = 0$

$$E\left[\,|X-\mu_x|\,\right], \quad \mu_x = E[X].$$

$$\boxed{E\left[(X-\mu_x)^2\right] = Var(X)}$$

$$Var(X) = E\left[(X-\mu_x)^2\right] = E\left[X^2+\mu_x^2-2\mu_x X\right]$$

$$= E[x^2]+\mu_x^2-2\mu_x E[x]$$

$$= E[x^2]+\mu_x^2-2\mu_x^2 = E[x^2]-\mu_x^2$$

$$Var(X) = E[(X-\mu_X)^2]$$

$$= E[X^2] - (E[X])^2$$

$$\geq 0$$

Property (i) $Var(aX+b) = a^2 Var(X)$

(ii) $Var(b) = 0$

(iii) $Var(X+X) \neq 2Var(X)$.

(i) $Var(aX+b) = E\left[(aX+\cancel{b}-a\mu_X-\cancel{b})^2\right]$

$$= E\left[a^2(X-\mu_X)^2\right]$$

$$= a^2 E\left[(X-\mu_X)^2\right]$$

$$= a^2 Var(X).$$

$$E(aX+b) = a\mu_X + b$$

(ii) $Var(X+X) = Var(2X) = 2^2 Var(X)$

$$= 4Var(X) \neq 2Var(X).$$

Ex:

Let X represents the outcome when we roll a fair die.

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

Let $P[X = i] = \frac{1}{6} \quad \forall\, i$

$Var(X) = ?$

Sol:

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X] = \sum x \cdot p_X(x) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$$

$$E[X^2] = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6}$$

$$= \frac{\cancel{6}(6+1)(13)}{\cancel{6} \times 6} = \frac{91}{6}$$

$$\Rightarrow \quad \text{Var}(X) = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}.$$

<u>Ex:</u> let I be an indicator r.v.

$$I = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{''} \quad \text{does not occur} \end{cases}$$

$$I^2 = \begin{cases} 1 & \text{if ''} \\ 0 & \text{if event } A \text{ does not occur.} \end{cases}$$

$E[I^2] = P(A)$, $E[I] = P[A]$

$Var(I) = P(A) - (P[A])^2 = P(A)[1 - P(A)]$.

$Var(X+X) \neq 2Var(X)$.

Let X and Y be two r.v.. The covariance of X and Y will be defined as

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

$$\left[\; Y = X \;;\; Cov(X,X) = Var(X) \;\right]$$