Markov's Inequality

If X is a random variable that takes only non-negative values, then for any value $a > 0$,

$$P[X \geq a] \leq \frac{E[X]}{a}$$

Proof:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{0}^{\infty} x f_X(x)\, dx$$

$$= \int_{0}^{a} x f_X(x)\, dx + \int_{a}^{\infty} x f_X(x)\, dx$$

$$\geq \int_{a}^{\infty} x f_X(x)\, dx \geq a \int_{a}^{\infty} f_X(x)\, dx$$

$$= a P[X \geq a]$$

$\forall\, a > 0,$

$$P[X \geq a] \leq \frac{E[X]}{a}.$$

Chebyshev's inequality

If X is a r.v. with mean μ_X and variance σ_X^2, then for any $k>0$,

$$P\left[|X-\mu_X| \geq k\right] \leq \frac{\sigma^2}{k^2}$$

Sol:

$$\{|X-\mu_X| \geq k\} \Leftrightarrow \{(X-\mu_X)^2 \geq k^2\}$$

Now,

$$P\left[(X-\mu_X)^2 \geq k^2\right] \leq \frac{E\left[(X-\mu_X)^2\right]}{k^2}$$

$$= \frac{\sigma_X^2}{k^2}$$

$$\Rightarrow P\left[|X-\mu_X| \geq k\right] \leq \frac{\sigma_X^2}{k^2}$$

The weak law of large numbers

Let $X_1, X_2, \ldots$ be a sequence of identically distributed independent (i.i.d.) r.v s each having mean $\mu = E[X]$.

Then, for any $\varepsilon > 0$,

$$P\left[\left|\underbrace{\frac{X_1+X_2+\cdots+X_n}{n}}_{\text{Sample mean}}-\mu\right|>\varepsilon\right]\to 0 \quad \text{as } n\to\infty.$$

Proof: $\sigma_X^2<\infty$.

$$E\left[\frac{X_1+\cdots+X_n}{n}\right]=\frac{1}{n}\left[E[X_1]+\cdots+E[X_n]\right]=\mu$$

$$\mathrm{Var}\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n)$$

$$= \frac{1}{n^2}\sum_i \mathrm{Var}(X_i) = \frac{\cancel{n}\,\sigma_X^2}{n^{\cancel{2}}}$$

$$= \frac{\sigma_X^2}{\sqrt{n}}$$

Chebyshev's inequality:

$$P\left[|X-\mu| \geq k\right] \leq \frac{\sigma_X^2}{k^2}$$

Replacing

$X = \dfrac{X_1 + \cdots + X_n}{n}$, $\mu = \mu$, σ_X^2 by $\dfrac{\sigma_X^2}{n}$, $k = \varepsilon$.

$$P\left[\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right] \leq \frac{\sigma_X^2}{n\varepsilon^2} \longrightarrow 0 \text{ as } n \to \infty$$

Ex Suppose that it is known that the number of items produced in a factory during a week is a r.v. with mean 50.

(a) What can be said about the probability that this week's production will exceed 75?

$$P[X > 75] \leq \frac{E[X]}{75} = \frac{50}{75} = \frac{2}{3}.$$

(b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this ~~[illegible]~~ week's production will be between 40 and 60?

$$P[40 < X < 60] = P[|X-50| < 10]$$

$$= 1 - P[|X-50| \geq 10]$$

$$P\left[|X-50| \geq 10\right] \leq \frac{\sigma_x^2}{10^2} = \frac{25}{100} = \frac{1}{4}$$

$$\Rightarrow \quad P\left[|X-50| < 10\right] \geq 1 - P\left[|X-50| \geq 10\right]$$

$$\geq 1 - \frac{1}{4} = \frac{3}{4}.$$

Ex: Suppose that a sequence of i.i.d.

[$X_1, X_2, \ldots, X_n, \ldots$ are independent and share the same c.d.f.]

is performed. Let E be a fixed event and $P(E)$ denotes the probability that E occurs on a given trial.

Let $X_i = \begin{cases} 1 & \text{if } E \text{ occurs on a trial } i \\ 0 & \text{if } E \text{ does not occur on a trial } i; \end{cases}$

$i = 1, 2, 3, 4, \ldots$

$X_1 + X_2 + \ldots + X_n$: no. of times that E occurs in the 1^{st} n trials.

$E[X_i] = 1\,P[E] + 0\,P[E^c] = P[E] \quad \forall i$

$\Rightarrow \quad \mu = P[E].$

$$\mathrm{Var}(X_i) = P[E]\left[1 - P[E]\right] = \sigma_X^2 \quad \forall i.$$

Now, let $\overline{X} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$

$$E[\overline{X}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \mu = P[E].$$

$$\mathrm{Var}(\overline{X}) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{\sigma_X^2}{n}$$

$$P\left[\,|\bar{X} - \mu| > \varepsilon\,\right] \to 0 \text{ as } n \to \infty$$

for every $\varepsilon > 0$.

<u>Markov's</u> <u>Inequality</u>

$$P\left[\bar{X} > a\right] \leq \frac{E[X]}{a} = \frac{\mu}{a}, \quad a > 0.$$

$a = \mu$, $a = \sqrt{\mu}$;

Chebyshev's inequality

$$P[|\bar{X}-\mu|>k] \leq \frac{\sigma_X^2}{k^2}$$

Suppose that a trial, or an experiment, whose outcome can be classified as either a "success" or as a "failure" is performed.

Let $X = \begin{cases} 1 & \text{when the outcome is a success} \\ 0 & \text{failure} \end{cases}$

and be s.t. $P[X=0] = 1-p$, $P[X=1] = p$, $p \in (0,1)$

Def:

A r.v., say X, is said to be a Bernoulli r.v. if it assumes the values 0 or 1 and its p.m.f. is given by

$$P[X=0] = 1-p$$

$$P[X=1] = p, \quad \text{for some } p \in (0,1)$$

Suppose that "n" independent Bernoulli trials are to be performed, (say X_i, $i=1,2,\dots,n$). Assume that each trial results in a success with probability "p" and in a failure with prob. $1-p$.

Let $X = X_1 + X_2 + \dots + X_n$

$=$ no. of successes that occurs in the n-trials.

$X \in \{0, 1, 2, \ldots, n\}$

$X \sim B(n, p)$ =: Binomial r.v. with parameters (n, p).

n: no. of times independent trials performed.

p: probability of success in each trial.

$P[X=i]$ = Probability of n outcomes containing i-successes and $(n-i)$-failures.

$\square\ \square\ \square\ \ldots\ \square$ $= {}^{n}C_{i}\ p^{i}(1-p)^{n-i}$; $i=0,1,\ldots,n$.

$[p+(1-p)]^{n}$

Let $X \sim B(n,p)$. Show that

$$E[X] = np$$

& $$Var(X) = np(1-p)$$

$$\left[\quad Var(X) < E[X] \quad \right]$$

Sol

$$E[X] = \sum_{i=0}^{n} i\, P[X=i]$$

$$= \sum_{i=1}^{n} i\, {}^{n}C_{i}\, p^{i} (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{n!\, i}{(n-i)!\, i!} p^{i} (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} {}^{n-1}C_{i-1} \, p^{i-1} (1-p)^{n-i}$$

$$= np \left[p + (1-p) \right]^{n-1}$$

$$= np$$

$X \sim B(n, p)$.

$X = X_1 + X_2 + \dots + X_n$

= sum of n-independent Bernoulli trials with probability of success p in each trial.

$$\Rightarrow \quad E[X] = \sum_{i=1}^{n} E[X_i] = np$$

$$Var(X) = \sum_{i=1}^{n} Var(X_i) = np(1-p).$$

Ex: It is known that disks produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the disks in packages of 10 and offer a money-back guarantee that at most 1 of the 10 disks is defective. What proportion of package is returned?

Sol: Let X denotes the number of defective disks in a Package.

$$X \in \{0, 1, \ldots, 10\}.$$

Since each disk will be defective with prob. 0.01 and non-defective with prob. 0.99, therefore, it is a Bernoulli trial.

Also, ~~each~~ Bernoulli trials ~~is~~ are independent of each other.

$\Rightarrow \quad X \sim B(10, 0.01)$.

$$P[X>1] = 1 - P[X \le 1]$$

$$= 1 - P[X=0] - P[X=1]$$

$$= 1 - {}^{10}C_0 (.01)^0 (.99)^{10} - {}^{10}C_1 (0.01)^1 (.99)^9$$

$$\approx 0.005.$$

Since each package will, independently, have to be replaced with probability 0.005. therefore, from ~~th~~ the law of large numbers it follows that in the long ~~run~~ run 0.5% of the packages will have to be replaced.

Ques: If someone buys three packages, what is the probability that exactly one of them will be returned?

Ans:

Y = no. of packages that the person will have to return.

$Y \sim B(3, 0.005)$

$P[Y=1] = {}^3C_1 (0.005)(.995)^2$

$= 0.015.$

Ex: If X_1 and X_2 are independent binomial r.v.s such that

$$X_i \sim B(n_i, p), \quad i = 1, 2.$$

then,

$$X_1 + X_2 \sim B(n_1 + n_2, p).$$