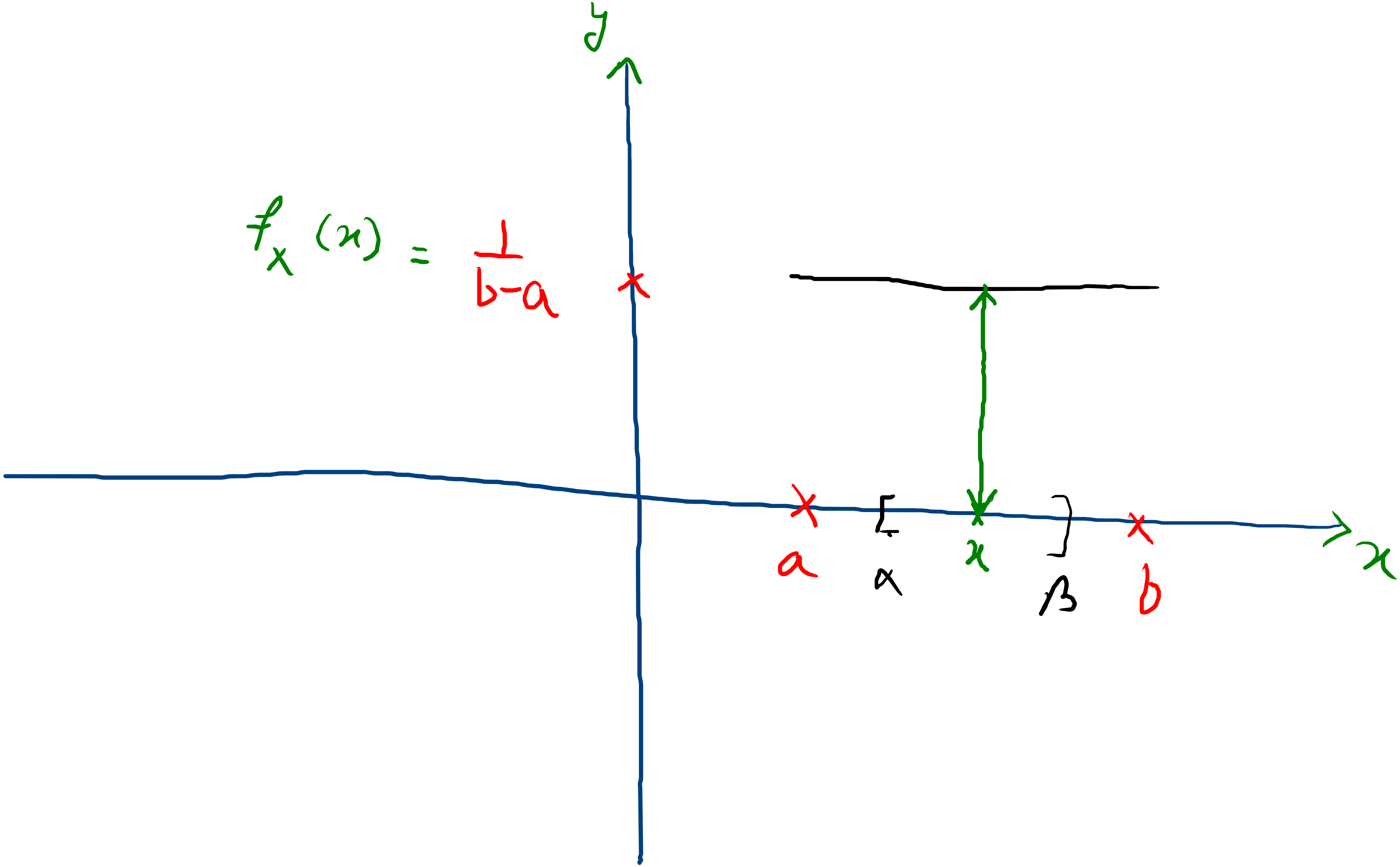
Uniform r.v.

A r.v. X is said to be uniformly distributed over the interval $[a,b]$ if its p.d.f is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

* We write $X \sim U[a,b]$.


y
$f_X(x) = \frac{1}{b-a}$
a
α
x
β
b
x

For $[\alpha, \beta] \subseteq [a, b]$,

$$P[\alpha < X < \beta] = \int_{\alpha}^{\beta} f_X(x)\, dx$$

$$= \boxed{\frac{\beta - \alpha}{b - a}} \leq 1$$

Ex: Buses arrive at a specified shop at 15-minute intervals starting at 7 A.M.

That is, they arrive at 7, 7:15, 7:30, and so on.

If a passenger arrives at the shop at a time that is uniformly distributed between 7 and 7:30, find the prob. that he waits

(a) less than 5 minutes for a bus;

(b) at least 12 minutes for a bus.

Sol: Let X = the time (in minutes) past 7 A.M. that the passenger arrives at the stop.

$$\Rightarrow \quad X \sim U[0,30]$$

(a) 7, 7:15, 7:30; $P[\{10 < X < 15\} \cup \{25 < X < 30\}]$

$$= P[10 < X < 15] + P[25 < X < 30]$$

$$= \frac{15-10}{30-0} + \frac{30-25}{30-0} = \frac{10}{30} = \frac{1}{3}.$$

(b) 7:00 to 7:03

7:15 to 7:18

$$P[\{0 < X < 3\} \cup \{15 < X < 18\}] = \frac{3}{30} + \frac{3}{30} = \frac{6}{30} = \frac{1}{5}.$$

Ex: Let $X \sim U[a,b]$. Then

$$E[X] = \frac{a+b}{2}$$

$$Var(X) = \frac{(b-a)^2}{12}$$

Sol: $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \frac{1}{b-a}\int_a^b x\, dx = \frac{b^2-a^2}{2(b-a)}$

$$= \frac{a+b}{2}.$$

$E[X^2] = ?$, $Var(X) = E[X^2] - (E[X])^2$.

Ex: The current in a semiconductor diode is often measured by the Shockley equation

$$\boxed{I = I_0 \left[e^{aV} - 1 \right]}$$

where V is the voltage across the diode; I_0 is the reverse current; a is a constant; and I is the resulting diode current.

Find $E[I]$ if $a=5$, $I_0 = 10^{-6}$

and $V \sim U[1,3]$.

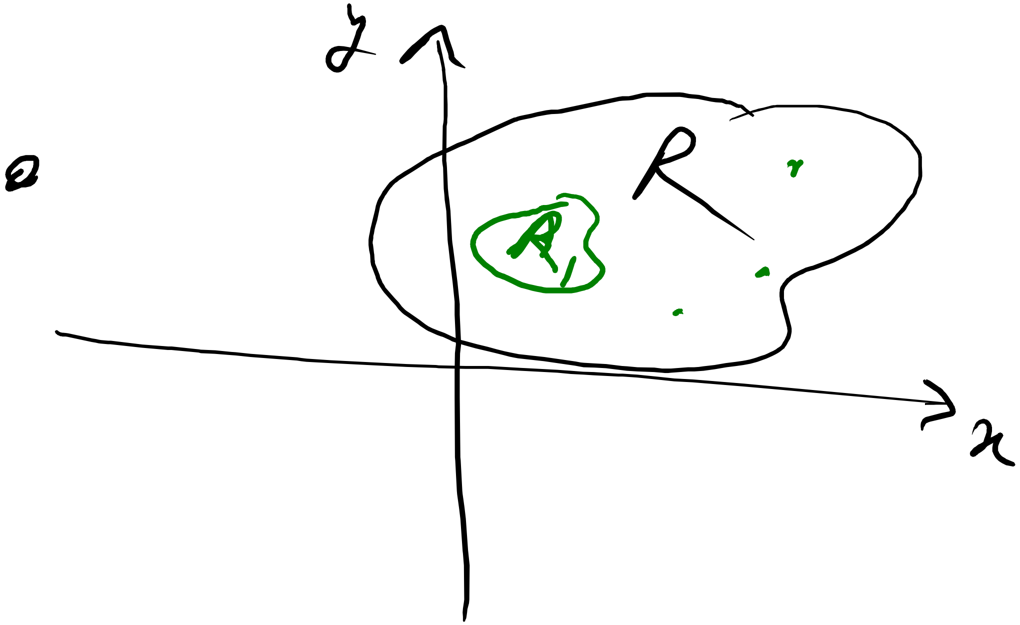
Sol:

$$I = I_0\left[e^{aV} - 1\right] = 10^{-6}\left[e^{5V} - 1\right]$$
$$= g(V)$$

$$E[I] = \int_{-\infty}^{\infty} g(v)\, f_V(v)\, dv$$

$$= \frac{1}{2}\int_1^3 \left[10^{-6}\left(e^{-5v}-1\right)\right] dv$$

$$\approx 0.3269\ldots$$



$$\frac{1}{\text{Area of } R}\iint_R dA$$

Def^n: The random vector (X, Y) is said to have a uniform distribution over the 2-D region R if its joint p.d.f is given by

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{Area of } R} & \text{if } (x,y) \in R \\ 0 & \text{ow} \end{cases}$$

In the above case,

$$P[(X,Y)\in A] = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

$$= \frac{\text{Area of } A}{\text{Area of } R}$$

Ex: Suppose that (X,Y) is uniformly distributed over the following rectangular region R:

$$R = \{ (x,y) : 0 \leq x \leq a , \ 0 \leq y \leq b \}$$

Then, X, Y are independent random variables.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{ab} & \text{if } (x,y) \in R \\ 0 & \text{o.w.} \end{cases}$$

Sol:

$$P[X \le x, Y \le y] = \frac{1}{ab}\int_0^x \int_0^y dy\, dx$$

$$\left[\int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\, dy\, dx \right]$$

$$= \frac{xy}{ab}.$$

$$F_X(x) = P[X \le x] = P[X \le x, Y \le b] = \frac{1}{ab}\int_0^x \left(\int_0^b dy \right) dx$$

$$= \frac{1}{a}\int_0^x dx = \frac{x}{a}.$$

$$\parallel^{ly}$$

$$F_Y(y) = \frac{y}{b}.$$

$$F_{X,Y}(x,y) = F_X(x)\,F_Y(y).$$

$\Rightarrow$ X & Y are independent.

Normal Random Variable

A r.v. X is said to be normally distributed with parameters μ and σ^2 and we write $\boxed{X \sim N(\mu, \sigma^2)}$, if its p.d.f. is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$-\infty < x < \infty$$

$\boxed{\sigma > 0}$

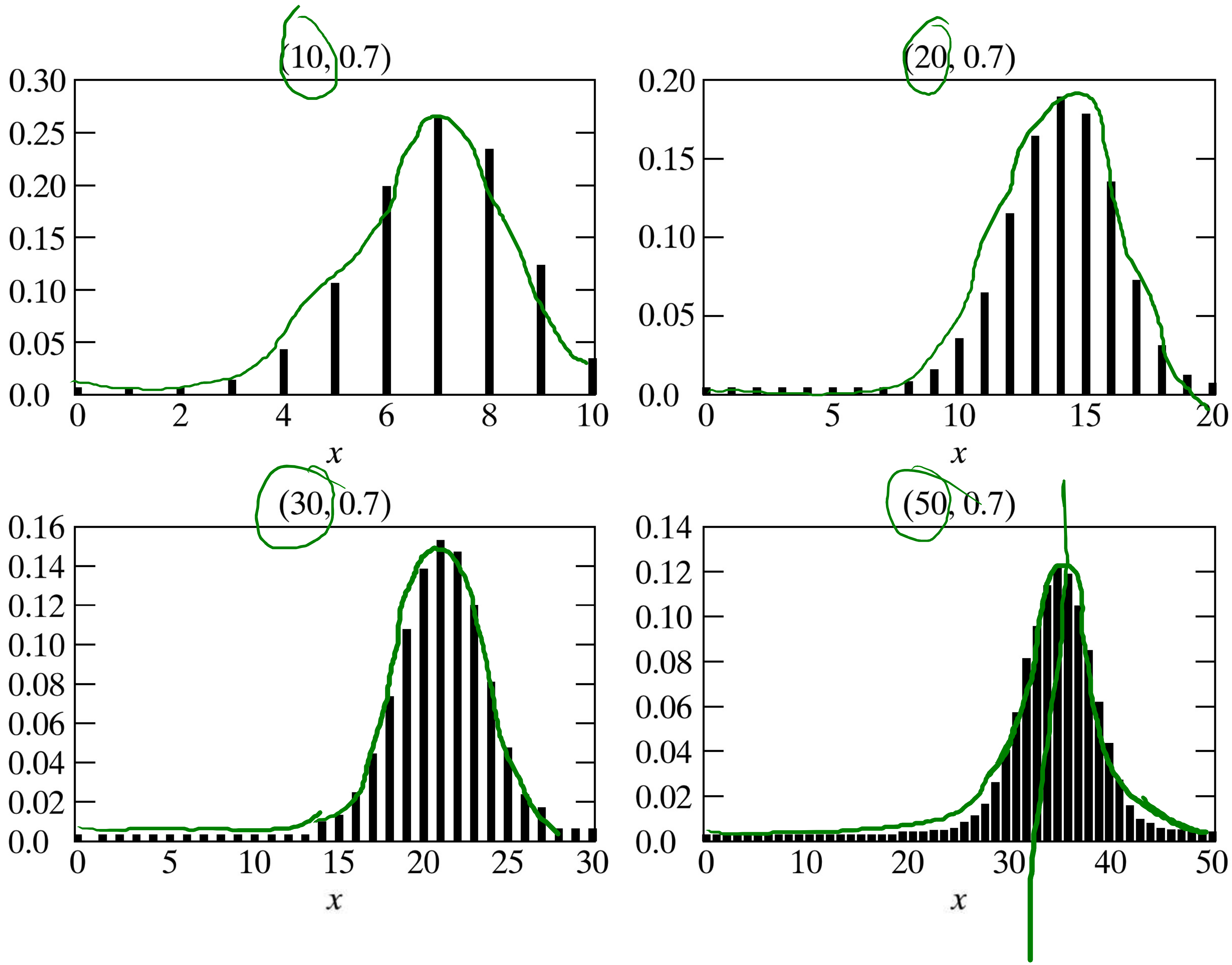


FIGURE 5.6: The probability mass function of a binomial (n, p) random variable becomes more and more "normal" as n becomes larger and larger.

Ex: Let $X \sim N(\mu, \sigma^2)$. Show that

$$E[X] = \mu$$

$$\& \quad Var(X) = \sigma^2.$$

Sol:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), \quad -\infty < x < \infty$$

$$E[X-\mu] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)\, \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

$$y = \frac{x-\mu}{\sigma} \Rightarrow dy = \frac{dx}{\sigma}$$

$$E[X-\mu] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma y)\, \exp\left(-\, y^2/2\right) dy$$

$$= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y\, e^{-y^2/2}\, dy \quad ; \quad y^2/2 = z \Rightarrow y\,dy = dz$$

$$= 0.$$

$$Var(X) = E[(X-\mu)^2]$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y \cdot y \, e^{-y^2/2} \, dy = \sigma^2.$$

$$\left[\int y^2 e^{-y^2/2} dy = -y e^{-y^2/2} + \int e^{-y^2/2} dy \right]$$

$$\boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1}$$

Ex: 1 Let $X \sim N(\mu, \sigma^2)$. Then, for any constants a and b, $b \neq 0$, the r.v.

$$Y = a + bX \sim N(a + b\mu, b^2\sigma^2).$$

Ex: 2 Let X be normal, then bX will also be normal $(b \neq 0)$

Ex: 3 Let X be normal, then $X + a$ will also be a normal.

① $X \sim N(\mu, \sigma^2)$

$$Y = a + bX, \quad b \neq 0.$$

$$E[Y] = a + bE[X] = a + b\mu$$

$$Var[Y] = Var(a+bX) = b^2 Var(X) = b^2 \sigma^2.$$

Now, $F_Y(y) = P[Y \leq y] = P[a + bX \leq y]$

$$= \begin{cases} P\left[X \leq \frac{y-a}{b}\right] & \text{if } b > 0 \\ P\left[X \geq \frac{y-a}{b}\right] & \text{if } b < 0 \end{cases}$$

$$\Rightarrow F_Y(y) = \begin{cases} F_X\left(\frac{y-a}{b}\right) & \text{if } b>0 \\ 1 - F_X\left(\frac{y-a}{b}\right) & \text{if } b<0 . \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{b} f_X\left(\frac{y-a}{b}\right) & \text{if } b>0 \\ -\frac{1}{b} f_X\left(\frac{y-a}{b}\right) & \text{if } b<0 \end{cases}$$

$$= \frac{1}{|b|} f_X\left(\frac{y-a}{b}\right) ..$$

$$\Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma|b|} \exp\left[-\frac{1}{2} \left\{ \frac{(y-a)/b - \mu}{\sigma} \right\}^2 \right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma|b|} \exp\left[-\frac{1}{2\sigma^2} \left(\frac{y-a-b\mu}{b} \right)^2 \right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma|b|} \exp\left[-\frac{1}{2\,\cancel{\sigma^2}} \left(\frac{y-a-b\mu}{\sigma|b|} \right)^2 \right]$$

$$\sim N\left(a+b\mu, (\sigma|b|)^2 \right) = N\left(a+b\mu, \sigma^2 b^2 \right).$$

Remark: If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1).$$

$$\boxed{\begin{aligned} a &= -\frac{\mu}{\sigma} \\ b &= \frac{1}{\sigma} \end{aligned}}$$

We call Z as a standard normal variate. Its c.d.f. is

$\Phi(z)$

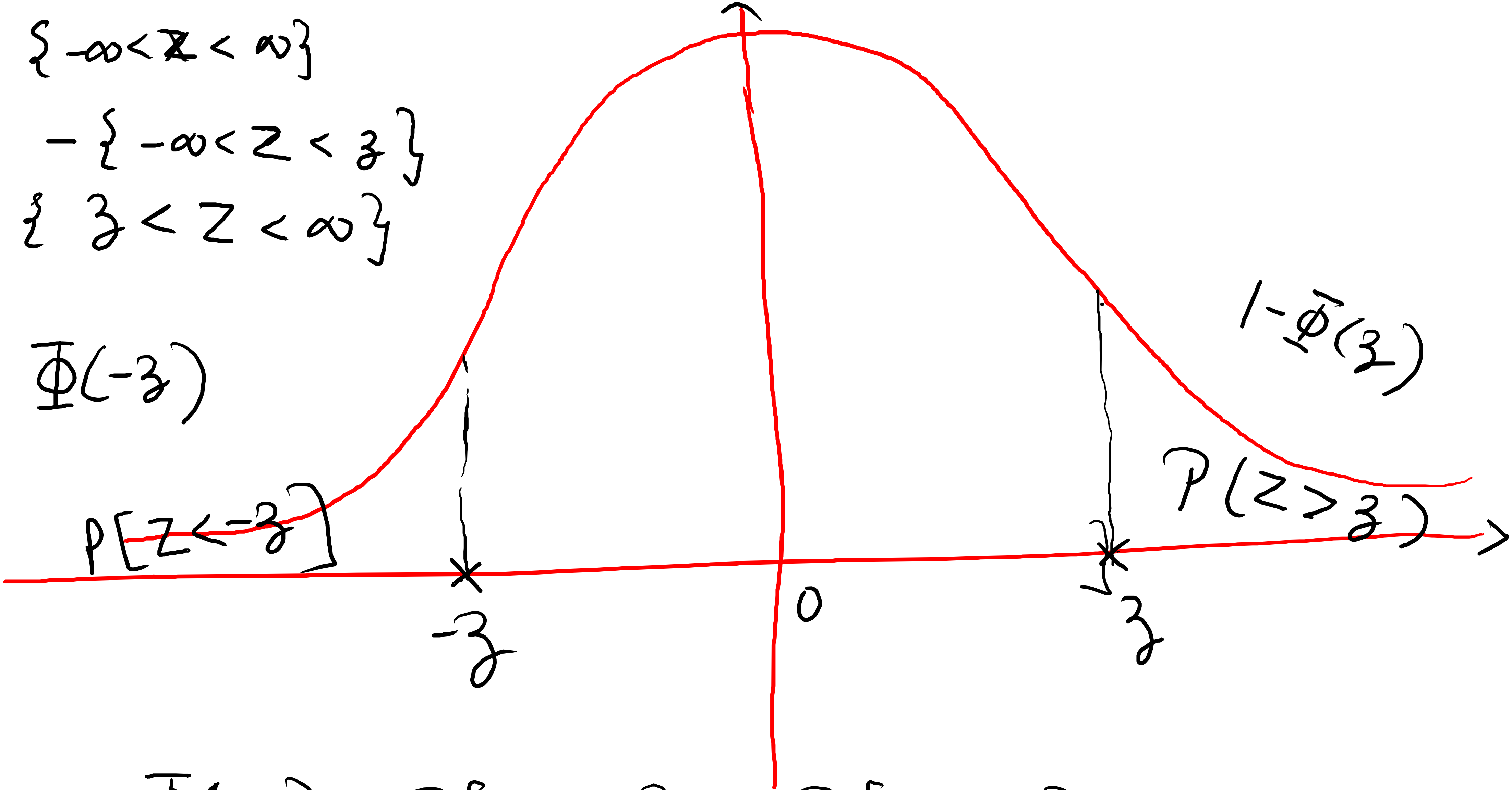
$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy, \quad -\infty < z < \infty$$

$$\Phi(-z) = 1 - \Phi(z) \quad \text{[Due to symmetry]}$$

$$\Phi(-z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-y^2/2}\, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy - \int_{-\infty}^{z} e^{-y^2/2}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-y^2/2}\, dy.$$

$$\{-\infty < Z < \infty\}$$
$$-\{-\infty < Z < 3\}$$
$$= \{3 < Z < \infty\}$$



$$\Phi(-3) = P[Z \le -3] = P[Z > 3] = 1 - \Phi(3)$$

$X \sim N(\mu, \sigma^2)$

(i) $P[X < b] = P\left[\frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right]$

$= P\left[Z < \frac{b-\mu}{\sigma}\right] = \Phi\left(\frac{b-\mu}{\sigma}\right)$