Ex: If X_1 and X_2 are independent binomial r.v.s such that

$$\boxed{X_i \sim B(n_i, p)}, \quad i = 1, 2.$$

then,

$$X_1 + X_2 \sim B(n_1 + n_2, p).$$

$X = X_1 + X_2$.

$$M_X(t) = E[e^{tX}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1} e^{tX_2}] = E[e^{tX_1}]\, E[e^{tX_2}]$$

\# $E[e^{tX}] = \sum_{x=0}^{n} e^{tx}\, {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$

$$= \sum_{x=0}^{n} {}^{n}C_{x} (pe^{t})^{x} (1-p)^{n-x}$$

$$= [pe^{t} + 1 - p]^{n}$$

Note: m.g.f. identifies distribution fun^{n} uniquely.

Poisson Random Variable

$\mathbb{N} \cup \{0\}$

A r.v. X, taking on one of the values $0, 1, 2, 3, \ldots$, is said to be a Poisson r.v. with parameter $\lambda > 0$, we write $X \sim P(\lambda)$, if its p.m.f is given by

$$P[X=i] = \begin{cases} e^{-\lambda} \dfrac{\lambda^i}{i!}, & i = 0, 1, 2, 3, \ldots \\ 0 & \text{ow.} \end{cases}$$

(i) $\because \lambda > 0$, $p_X(i) \geq 0$. $\forall i \in \mathbb{N} \cup \{0\}$.

(ii) $$\sum_{i=0}^{\infty} p_X(i) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} e^{\lambda}$$

$$= 1.$$

Ex: If $X \sim P(\lambda)$, then

$$\boxed{E[X] = Var(X) = \lambda.}$$

Sol: $E[X] = \sum_{x=0}^{\infty} x\, p_X(x) = \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} \times x$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

$$E[X^2] = \sum_{x=0}^{\infty} x^2 p_X(x) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x x^2}{x!}$$

$$[x-1+1]$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{x \lambda^x}{(x-1)!}$$

$$= \lambda e^{-\lambda} \left[\sum_{x=2}^{\infty} \frac{(x-1)\lambda^{x-1}}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right]$$

$$= \lambda e^{-\lambda} \left[\lambda e^{\lambda} + e^{\lambda} \right] = \lambda^2 + \lambda$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$= (\cancel{\lambda^2} + \lambda) - \cancel{\lambda^2}$$

$$= \lambda$$

Poisson as an approximation of Binomial

- Let $X \sim B(n, p)$.
- Assume that n is large and p is small such that $\boxed{np < \infty}$.
- Define $\lambda = np > 0$.

$$\Rightarrow \boxed{p = \frac{\lambda}{n}}$$

$$P[X=i] = {}^{n}C_{i}\, p^{i}(1-p)^{n-i}$$

$$= \frac{n!}{i!\,(n-i)!}\left(\frac{\lambda}{n}\right)^{i}\left[1-\frac{\lambda}{n}\right]^{n-i}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{n^{i}}\,\frac{\lambda^{i}}{i!}\left[1-\frac{\lambda}{n}\right]^{n}\left[1-\frac{\lambda}{n}\right]^{-i}$$

$$= \frac{\cancel{n^{i}}\left[1-\frac{1}{n}\right]\left[1-\frac{2}{n}\right]\cdots\left[1-\frac{i-1}{n}\right]}{\cancel{n^{i}}}\,\frac{\lambda^{i}}{i!}\left[1-\frac{\lambda}{n}\right]^{n}\left[1-\frac{\lambda}{n}\right]^{-i}$$

$$\longrightarrow 1 \times \frac{\lambda^{i}}{i!} \times e^{-\lambda} \times 1 \quad \rightsquigarrow \quad n \to \infty$$

Ex: Suppose that the average number of accidents occuring weekly on a particular stretch of a higway equals 3.

Calculate the probability that there is at least one accident this week.

Let X = no. of accidents occurring weekly on a particular stretch of a higway.

$X \in \{0, 1, 2, 3, \ldots\}$

$X \sim P(\lambda)$, $\lambda = E[X]$ = Average/mean = 3.

$$p_X(x) = \begin{cases} e^{-3} \dfrac{3^i}{i!} & i = 0, 1, 2, 3, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$P[X \geq 1] = 1 - P[X=0]$$

$$= 1 - \frac{e^{-3}\,3^{0}}{0!}$$

$$= 1 - e^{-3} = 0.9502.$$

Ex:

If the average number of claims handled daily by an insurance company is 5, what proportion of days have less than 3 claims?

Sol:

X = no. of claims handled daily by an insurance company.

$X \sim P(5)$, $\quad p_X(i) = e^{-5} \frac{5^i}{i!}$, $i = 0, 1, 2, 3, \ldots$

$$P[X<3]$$

$$= P[X=0] + P[X=1] + P[X=2]$$

$$= e^{-5} + e^{-5}\frac{5^1}{1!} + e^{-5}\frac{5^2}{2!}$$

Ex: If $X_1 \sim P(\lambda_1)$ & $X_2 \sim P(\lambda_2)$, then

$$X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$$

whenever X_1 & X_2 are independent r.v.

(**Sol:** Use moment generating function)

Ex: It has been established that the number of defective stereos produced daily at a certain plant is Poisson distributed with mean 4.

Over a 2-day span, what is the probability that the number of defective stereos does not exceed 3.

X_1 = no. of defective stereos produced during the 1st day

X_2 = 2nd day.

$$X_i \sim P(4), \quad i = 1, 2, \quad X_1 + X_2 \sim P(4+4=8)$$

$$P[X_1 + X_2 \le 3] = P$$

$$= P[X_1 + X_2 = 0] + P[X_1 + X_2 = 1] + P[X_1 + X_2 = 2] + P[X_1 + X_2 = 3].$$

Let $Y = X_1 + X_2 \sim P(8)$

$$p_Y(y) = \begin{cases} \dfrac{e^{-8}\, 8^{y}}{y!} & y = 0, 1, 2, 3, \ldots \\ 0 & \text{ow.} \end{cases}$$

$$\Rightarrow P[Y \leq 3] = \sum_{i=0}^{3} \frac{e^{-8}\, 8^{i}}{i!}$$

Geometric Random Variable

Independent trials are performed until a success occurs.

Each trial having probability of success is p, $0 < p < 1$.

Let X = no. of trials required.

$\in \{1, 2, 3, 4, \ldots\}$

$$P[X=i] = \underbrace{(1-p)\cdots(1-p)}_{(i-1)\text{-times}} \times p$$

$$= p(1-p)^{i-1} \quad , \quad i = 1, 2, 3, \ldots$$

$$\sum_{i=1}^{\infty} p(1-p)^{i-1} = p\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \ldots\right]$$

$$= p \times \frac{1}{1-(1-p)} = \underline{1}$$

Def^n:

Any r.v. X whose p.m.f is given by

$$p_X(i) = \begin{cases} p(1-p)^{i-1} & , i = 1, 2, 3, \ldots \\ 0 & \text{ow} \end{cases}$$

is called a Geometric r.v. and we write $X \sim Geo(p)$.

Ex: An urn contains N white and M black balls. Balls are randomly selected, one at a time, until a black one is obtained. If we assume that each ball is selected is replaced before the next one is drawn, what is the probability that

(a) exactly 'n' draws are needed?

(b) at least 'k' draws are needed?

Let X = no. of draws needed to select a black ball.

then $X \sim Geo\left(p = \frac{M}{M+N}\right)$

(a) $P[X = n] = (1-p)^{n-1} p = \frac{MN^{n-1}}{(M+N)^n}$

(b) $P[X \geq k] = (1-p)^{k-1} = 1 - P[X < k] = 1 - \sum_{i=0}^{k-1} P[X = i]$

Ex: If $X \sim Geo(p)$, then

$$E[X] = \frac{1}{p} \quad \& \quad Var(X) = \frac{1-p}{p^2}.$$

Sol:

$$E[X] = \sum_{x=1}^{\infty} x\, p_X(x)$$

$$= \sum_{x=1}^{\infty} p\, x (1-p)^{x-1}$$

$$= p \sum_{x=1}^{\infty} x (1-p)^{x-1} = pS$$

$$S = \sum_{n=1}^{\infty} n(1-p)^{n-1}$$

$$= 1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \cdots$$

$$-\quad (1-p)S = (1-p) + 2(1-p)^2 + 3(1-p)^3 + 4(1-p)^4 + \cdots$$

$$pS = 1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots$$

$$= \frac{1}{1-(1-p)} = \frac{1}{p} \implies S = \frac{1}{p^2}.$$

$$\Rightarrow E[X] = p \times \frac{1}{p^2} = \frac{1}{p}.$$

$$Var(X) = E[X^2] - (E[X])^2$$

M.G.F. of Bernoulli R.V.

$$X = \begin{cases} 1 & \text{if "success"} \\ 0 & \text{if "failure"} \end{cases}$$

$$P[X=1] = p, \quad P[X=0] = 1-p.$$

$$M_X(t) = E[e^{tX}] = \sum_{x \in \{0,1\}} e^{tx} p_X(x) = [(1-p) + e^t p]$$