$Var(X+X) \neq 2Var(X)$.

Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ be two r.v.. The covariance of X and Y will be defined as

$$\boxed{Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]}$$

$[Y = X;\ Cov(X,X) = Var(X)]$

Def^n: (Standard Deviation)

$$S.D. = \sigma_X = \sqrt{Var(X)}$$

$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$$= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y]$$

$$= E[XY] - \mu_X \mu_Y - \cancel{\mu_Y \mu_X} + \cancel{\mu_X \mu_Y}$$

$$= E[XY] - E[X]E[Y]$$

(i) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

(ii) $\mathrm{Cov}(aX,Y) = E[(aX)Y] - E[aX]\,E[Y]$

$= aE[XY] - a\,E[X]\,E[Y]$

$= a\,\mathrm{Cov}(X,Y)$

(iii) $\mathrm{Cov}(X+Y,Z) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$

(iv) $\mathrm{Cov}\left(\sum_{i=1}^{n} X_i , Y\right) = \sum_{i=1}^{n} \mathrm{Cov}(X_i, Y)$

(v) $$\operatorname{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} \operatorname{Cov}\left(X_i, \sum_j Y_j\right)$$

$$= \sum_i \operatorname{Cov}\left(\sum_j Y_j, X_i\right)$$

$$= \sum_i \sum_j \operatorname{Cov}(X_i, Y_j)$$

(vi) $$\operatorname{Var}(X+Y) = \operatorname{Cov}(X+Y, X+Y)$$

$$= \operatorname{Cov}(X,X) + \operatorname{Cov}(Y,Y) + \operatorname{Cov}(X,Y) + \operatorname{Cov}(Y,X)$$

$$\Rightarrow \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$$

(vii) $$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i} \mathrm{Var}(X_i) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{Cov}(X_i, X_j)$$ [verify].

(viii) If X and Y are independent r.vs. then,

$$\operatorname{Cov}(X,Y) = 0$$

Sol:

$$E[XY] = \sum_i \sum_j x_i y_j p_{X,Y}(x_i, y_j)$$

$$= \sum_i \sum_j x_i y_j p_X(x_i) p_Y(y_j)$$

$$= \left(\sum_i x_i p_X(x_i)\right)\left(\sum_j y_j p_Y(y_j)\right)$$

$$\Rightarrow \quad E[XY] = E[X]\,E[Y]$$

$$\Rightarrow \quad \mathrm{Cov}(X,Y) = 0 .$$

Ex: let $X \sim U(-1,1)$, $\quad f_X(x) = \begin{cases} \frac{1}{2} & \text{if } -1 \leq x \leq 1 \\ 0 & \text{ow} \end{cases}$

Define $Y = X^2$.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-1}^{1} \frac{x}{2}\,dx = 0$$

$$E[Y] = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{-1}^{1} \frac{x^2}{2}\,dx = \frac{1}{3}$$

$$E[XY] = E[X^3] = \int_{-\infty}^{\infty} x^3 f_X(x)\,dx = \int_{-1}^{1} \frac{x^3}{2}\,dx = 0$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= 0 - 0 \times \frac{1}{3} = 0.$$

Ex Compute the variance of the sum obtained when 10 independent rolls of a fair dice are made.

X_i : outcome of ith roll of a fair dice

$X_i \in \{1, 2, \ldots, 6\}$, $i = 1, 2, \ldots, 10$.

$Var(X_i) = \frac{35}{12}$,

$Cov(X_i, X_j) = 0$ if $i \neq j$.

$$\mathrm{Var}\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} \mathrm{Var}(X_i) + \sum_{i=1}^{10} \sum_{\substack{j=1 \\ j \neq i}}^{10} \mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{10} \frac{35}{12} = \cancel{10}^{5} \times \frac{35}{\cancel{12}_{6}}$$

$$= \frac{175}{6}.$$

Ex: Compute the variance of the number of heads resulting from 10 independent tosses of a fair coin.

Sol: let $I_j = \begin{cases} 1 & \text{if } j\text{th toss results heads} \\ 0 & \text{tails} \end{cases}$

Total no. of Heads $= \sum_{j=1}^{10} I_j$,

$Cov(I_i, I_j) = 0$ if $i \neq j$.

$$\Rightarrow \operatorname{Var}\left(\sum_{j=1}^{10} I_j\right) = \sum_{j=1}^{10} \operatorname{Var}(I_j)$$

$$= \sum_{j=1}^{10} \frac{1}{2}\left(1-\frac{1}{2}\right)$$

$$= 10 \times \frac{1}{4} = \frac{5}{2}.$$

$$\operatorname{Var}(I)$$

$$= P(A)[1-P(A)]$$

$X = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{if } \text{———} \text{ does not occur.} \end{cases}$

$$Y = \begin{cases} 1 & \text{if event } B \text{ occurs} \\ 0 & \text{if } \text{———} \text{ does not occur.} \end{cases}$$

$$XY = \begin{cases} 1 & \text{if } X=1, Y=1 \\ 0 & \text{o/w.} \end{cases}$$

$$E[XY] = P[X=1, Y=1]$$

$$E[X] = P[X=1]$$

$$E[Y] = P[Y=1]$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= P[X=1, Y=1] - P[X=1]P[Y=1].$$

Now,

$$\operatorname{Cov}(X,Y) \underset{(<)}{>} 0$$

$$\Leftrightarrow \quad P[X=1, Y=1] \underset{(<)}{>} P[X=1]\,P[Y=1]$$

$$\Leftrightarrow \quad \frac{P[X=1, Y=1]}{P[X=1]} \underset{(<)}{>} P[Y=1]$$

$$\Leftrightarrow \quad P[Y=1 \mid X=1] \underset{(<)}{>} P[Y=1].$$

In general,

$Cov(X,Y) > 0$ indicates that Y tends to increase as X does.

$Cov(X,Y) < 0$ indicates that Y tends to decrease as X increases.

Def: Correlation between X and Y.

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \, \sigma_Y}.$$

$$= \frac{Cov(X,Y)}{\sqrt{Var(X)\, Var(Y)}}$$

Ex: $\rho_{X,Y} \in [-1,1]$. [Hint: $E\left[\left((X-\mu_X)t+(Y-\mu_Y)\right)^2\right]$]

Markov's Inequality

If X is a random variable that takes only non-negative values, then for any value $a>0$,

$$P[X \geq a] \leq \frac{E[X]}{a}.$$

Proof:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{0}^{\infty} x f_X(x)\,dx$$

$$= \int_{0}^{a} x f_X(x)\,dx + \int_{a}^{\infty} x f_X(x)\,dx$$

$$\geq \int_{a}^{\infty} x f_X(x)\,dx \geq a\int_{a}^{\infty} f_X(x)\,dx$$

$$= a\,P[X \geq a]$$

$\forall\, a > 0,$

$$P[X \geq a] \leq \frac{E[X]}{a}.$$

Chebyshev's inequality

If X is a r.v. with mean μ_X and variance σ_X^2, then for any $k>0$,

$$P\left[|X-\mu_X| \geq k\right] \leq \frac{\sigma^2}{k^2}$$

Sol:

$$\{|X-\mu_X| \geq k\} \iff \{(X-\mu_X)^2 \geq k^2\}$$

Now,

$$P\left[(X-\mu_X)^2 \geq k^2\right] \leq \frac{E\left[(X-\mu_X)^2\right]}{k^2}$$

$$= \frac{\sigma_X^2}{k^2}$$

$$\Rightarrow P\left[|X-\mu_X| \geq k\right] \leq \frac{\sigma_X^2}{k^2}$$

The weak law of large numbers

Let $X_1, X_2, \ldots$ be a sequence of identically distributed independent (i.i.d.) r.v s each having mean $\mu = E[X]$.

Then, for any $\varepsilon > 0$,

$$P\left[\left|\underbrace{\frac{X_1+X_2+\cdots+X_n}{n}}_{\text{Sample mean}} - \mu\right| > \varepsilon\right] \to 0 \quad \text{as } n \to \infty.$$

Proof: $\sigma_X^2 < \infty.$

$$E\left[\frac{X_1+\cdots+X_n}{n}\right] = \frac{1}{n}\left[E[X_1]+\cdots+E[X_n]\right] = \mu$$

$$\mathrm{Var}\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n)$$

$$= \frac{1}{n^2}\sum_i \mathrm{Var}(X_i) = \frac{\cancel{n}\,\sigma_X^2}{n^{\cancel{2}}}$$

$$= \frac{\sigma_X^2}{\cancel{n}}$$

Chebyshev's inequality :

$$P\left[\,|X-\mu| \geq k\,\right] \leq \frac{\sigma_X^2}{k^2}$$

Replacing

$$X = \frac{X_1 + \cdots + X_n}{n}, \quad \mu = \mu, \quad \sigma_X^2 \text{ by } \frac{\sigma_X^2}{n}, \quad k = \varepsilon.$$

$$P\left[\,\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right] \leq \frac{\sigma_X^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty$$