SC435
Introduction to Complex Networks

Groups of nodes

sub-graphs $\Rightarrow$ local structure of networks.

Group of nodes

- **clique:** is a maximal subset of vertices in an undirected network such that every member of the set is connected by an edge to any other vertex.
  - Cliques can overlap $\Rightarrow$ they can share vertices (one or more)
  - A clique in an otherwise sparse network indicates a highly cohesive subgroup.
- **k-plex:** of size n is a maximal subset of n vertices within a network such that each vertex is connected by an edge to atleast $n - k$ of the others.
  - k-plex can also be overlapping.
  - It is a useful concept in social network analysis but there is no solid rule what value k should take.

Group of nodes

Group of nodes

- **k-core:** is a *maximal* subset of vertices such that each is connected to atleast k-other in the subset.
- **k-core decomposition:** useful in obtaining the core-periphery structure of the network.
- degree of each node can be used to separate a network into distinct portions called shells.
  - periphery: low-degree outer shells
  - core: high degree (connectivity) inner shell

Algorithm

- start with $k = 0$ (singletons)
- Recursively remove all nodes of degree k, until there are no more left.
- The removed nodes make up the k-shell and the remaining nodes make up the $k + 1$-core
- If there are no more nodes left in the core, terminate else, increment k for the next iteration.

Can two k-cores overlap?

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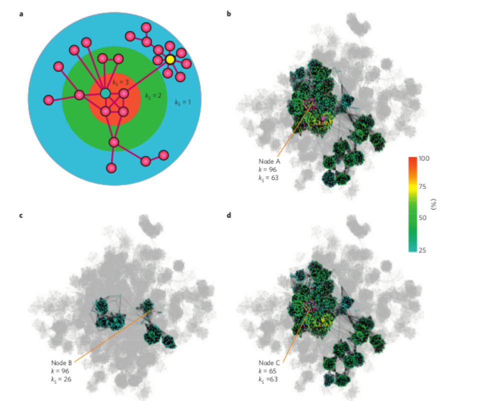
k-core



Hubs may not be important in spread.[1]

[1] Kitsak *et. al. Identification of influential spreaders in complex networks, Nature Physics* **6**, *2010.*

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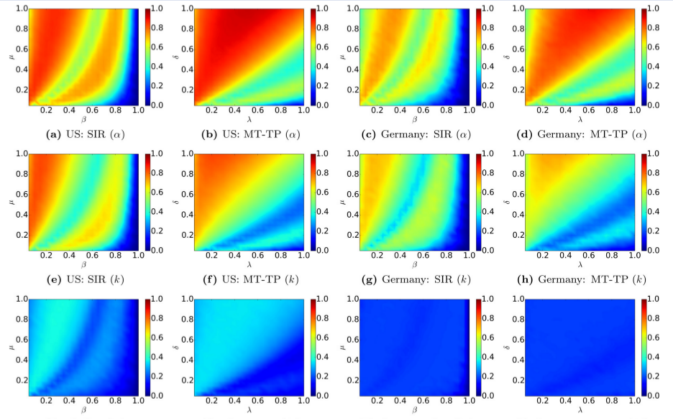

(a) US: SIR (α) (b) US: MT-TP (α) (c) Germany: SIR (α) (d) Germany: MT-TP (α)
(e) US: SIR (k) (f) US: MT-TP (k) (g) Germany: SIR (k) (h) Germany: MT-TP (k)
(i) US: SIR (k_c) (j) US: MT-TP (k_c) (k) Germany: SIR (k_c) (l) Germany: MT-TP (k_c)

Which is the most influential spreader Influential Spreader [2]

[2] Arruda *et. al.*, *Role of centrality for the identification of influential spreaders in complex networks*, Phys. Rev. E **90**, 032812 (2014)

Group of nodes

Group of nodes

- **k-clique:** is a *maximal* subset of vertices such that each is no more than a distance k away from any of the others via the edges of the network.
  - For $k = 1$: ordinary cliques.
  - not a very well-behaved one, since a k-clique by this definition need not be connected via paths that run within the subset
- **k-clan:**
- **k-club:**

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Components and k-component

- **Component:** A component in an undirected network is a maximal subset of vertices such that each is reachable by some path from each of the others.
- **k-component (k-connected component):** is a *maximal* subset of vertices such that each is reachable from each of the others by at least k *vertex-independent* paths.
  - **vertex-independent**: two paths are vertex-independent if they share none of the same vertices except the starting and the ending vertices.
  - number of vertex-independent paths between two vertices is equal to the size of the *vertex cut set* between the same two vertices.
  - k-component is connected with the idea of network robustness.

Groups of nodes

Transitivity

- Transitivity is most important in social networks
- Simplest relation: Connected by an edge
- Perfext transitivity only occurs in networks where each component is perfectly connected subgraph or clique.
- **Clustering coefficient**: Measure of partial transitivity

$$C = \frac{\text{number of closed paths of length 2}}{\text{number of paths of length 2}}$$

$$\boxed{C = \frac{6 \times \text{number of } \Delta's}{\text{number of paths of length 2}}}$$

$$\boxed{C = \frac{3 \times \text{number of } \Delta's}{\text{number of connected triplets}}}$$

Groups of nodes

Transitivity

- **Local Clustering**: Clustering coefficient for a single vertex $\Rightarrow$ Clustering coefficient of a node is the fraciton of the node's neighbor that are connected to each other.

$$\boxed{C(i) = \frac{\tau(i)}{\tau_{max}(i)} = \frac{\tau(i)}{\binom{k_i}{2}}}$$

- Average (global) clustering coefficient:

$$\boxed{C = \frac{\sum_{i:k_i>1} C(i)}{N_{k>1}}}$$

- Clustering coefficient in directed graphs: depends on the kind of triangles that are relevant to a specific case.
- High clustering coefficient $C >> p$, where

$$p = \frac{|E|}{\binom{n}{2}}$$

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Structural holes

- Local clustering coefficient is low for vertex with high degree.
- Local clustering coefficient is related with the concept of *structural holes*.
- While it is common for the neighbors of a vertex to be connected among themselves, it happens sometimes that these expected connections between neighbors are missing. These missing links are called structural holes.
- Structural holes can be a bad thing with regards to efficient spread of information.
- Can be good for vertex $i \Rightarrow$ high importance.

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Redundancy

- The redundancy R_i of vertex i is the mean number of connections from a neighbor of i to other neighbors of i.
- Redundancy of node i ranges from 0 to $k_i - 1$.
- $C(i) = \frac{R_i}{k_i - 1}$

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Reciprocity

- In directed networks there can be loops of size 2.
- Reciprocity measures the frequency of loops of size 2 in a directed network.

$$\boxed{r = \frac{1}{m}\sum_{ij} A_{ij}A_{ji} = \frac{1}{m}\mathrm{Tr}A^2}$$

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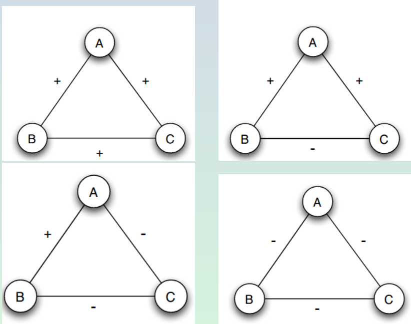
Signed networks

- In social networks, edges can be annotated as positive or negative to signify relationship
  - positive edge $\Rightarrow$ friendship
  - negative edge $\Rightarrow$ antagonism (people interact in a negative way)
  - connection between local and global network properties.

Groups of nodes

Structural Balance

- based on theories in social psychology (Heider) and extended to graphs (Cartwright & Hararay)
- In a group of 3 certain combinations of $+/-$ are socially more plausible.



Groups of nodes

Cartwright-Harary Theorem

If a labeled complete graph is balanced, then either all pairs of nodes are friends, or else the nodes can be divided into two groups, X and Y, such that every pair of people in X like each other, every pair of peoplein Y like each other, and everyone in X is the enemy of everyone in Y.