SC 220: Groups and Linear Algebra: Autumn 2020. B.Tech Sem-III

Gautam Dutta

December 6, 2020

EIgenvalues and Eigenvectors
Lecture 22

Eigenvalues and Eigenvectors

- *Eg.*1

  Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Eigenvalues and Eigenvectors

- *Eg.*1

  Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

$$A \begin{pmatrix} x \\ 0 \end{pmatrix} = 2 \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 0 \\ y \end{pmatrix} = 3 \begin{pmatrix} 0 \\ y \end{pmatrix}$$

Eigenvalues and Eigenvectors

- *Eg.*1

  Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

$$A \begin{pmatrix} x \\ 0 \end{pmatrix} = 2 \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 0 \\ y \end{pmatrix} = 3 \begin{pmatrix} 0 \\ y \end{pmatrix}$$

- We say that the action of A on the vectors $\begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ y \end{pmatrix}$ is simple. It just gives a vector parallel to the given vector.

Eigenvalues and Eigenvectors

- *Eg.*1

  Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

$$A \begin{pmatrix} x \\ 0 \end{pmatrix} = 2 \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 0 \\ y \end{pmatrix} = 3 \begin{pmatrix} 0 \\ y \end{pmatrix}$$

- We say that the action of A on the vectors $\begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ y \end{pmatrix}$ is simple. It just gives a vector parallel to the given vector.

  All other vectors get rotated under the action of A.

- *Eg.2*
  Now consider
  $$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

- *Eg.2*
  Now consider

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Let $v_1 = \begin{pmatrix} x \\ x \end{pmatrix}$,

- *Eg.2*
  Now consider
$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$
  Let $v_1 = \begin{pmatrix} x \\ x \end{pmatrix}, \quad v_2 = \begin{pmatrix} x \\ -x \end{pmatrix}$.

- *Eg.2*
  Now consider
  $$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$
  Let $v_1 = \begin{pmatrix} x \\ x \end{pmatrix}$, $v_2 = \begin{pmatrix} x \\ -x \end{pmatrix}$.
  Then $Av_1 = 4v_1$ and $Av_2 = 2v_2$.
  For any other vector v, Av is not parallel to v.

- *Eg.*2
  Now consider
  $$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$
  Let $v_1 = \begin{pmatrix} x \\ x \end{pmatrix}, \quad v_2 = \begin{pmatrix} x \\ -x \end{pmatrix}$.
  Then $Av_1 = 4v_1$ and $Av_2 = 2v_2$.
  For any other vector v, Av is not parallel to v.
- *Eg.*3
  $$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{Then} \quad A\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

- *Eg.*2
  Now consider
  $$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$
  Let $v_1 = \begin{pmatrix} x \\ x \end{pmatrix}, \quad v_2 = \begin{pmatrix} x \\ -x \end{pmatrix}$.
  Then $Av_1 = 4v_1$ and $Av_2 = 2v_2$.
  For any other vector v, Av is not parallel to v.
- *Eg.*3
  $$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{Then} \quad A\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
  For any other vector, $v, \ \ Av$ is not parallel to v.

*Eg.*4

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

*Eg.*4

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Under the action of A every vector in $\mathbb{R}^2$ gets rotated by an angle θ. We can't find any vector in $\mathbb{R}^2$ such that $Av = \lambda v$ for some number λ.

- *Def.* Eigenvector and Eigenvalue.
  Let V be a n-dimensional vector space over a field F.

- *Def.* Eigenvector and Eigenvalue.
  Let V be a n-dimensional vector space over a field F. Let T be a linear operator on V.

- *Def.* Eigenvector and Eigenvalue.
  Let V be a n-dimensional vector space over a field F. Let T be a linear operator on V.
  A non-zero vector $v \in V$ is called an eigenvector of T if $Tv = \lambda v, \;\; \lambda \in F$.

- *Def.* Eigenvector and Eigenvalue.
  Let V be a n-dimensional vector space over a field F. Let T be a linear operator on V.
  A non-zero vector $v \in V$ is called an eigenvector of T if $Tv = \lambda v, \;\; \lambda \in F$.
  λ is called the eigenvalue of T corresponding to the eigenvector v.

- How do we find the eigenvalues and eigenvector of T?

- How do we find the eigenvalues and eigenvector of T? Let A be the matrix representation of T and X be the matrix representation of the vector v in the basis B of V.

- How do we find the eigenvalues and eigenvector of T? Let A be the matrix representation of T and X be the matrix representation of the vector v in the basis B of V. Then

$$AX = \lambda X,$$

- How do we find the eigenvalues and eigenvector of T? Let A be the matrix representation of T and X be the matrix representation of the vector v in the basis B of V. Then

$$AX = \lambda X, \quad \Longrightarrow \quad (A - \lambda \mathbb{1})X = 0.$$

- How do we find the eigenvalues and eigenvector of T? Let A be the matrix representation of T and X be the matrix representation of the vector v in the basis B of V. Then

$$AX = \lambda X, \quad \implies \quad (A - \lambda \mathbb{1})X = 0.$$

- The equation has a non-trivial solution if $A - \lambda \mathbb{1}$ is a non-invertible matrix.i.e

- How do we find the eigenvalues and eigenvector of T? Let A be the matrix representation of T and X be the matrix representation of the vector v in the basis B of V. Then

$$AX = \lambda X, \quad \implies \quad (A - \lambda \mathbb{1})X = 0.$$

- The equation has a non-trivial solution if $A - \lambda \mathbb{1}$ is a non-invertible matrix.i.e

$$\det(A - \lambda \mathbb{1}) = 0 \tag{1}$$

- How do we find the eigenvalues and eigenvector of T? Let A be the matrix representation of T and X be the matrix representation of the vector v in the basis B of V. Then

$$AX = \lambda X, \quad \implies \quad (A - \lambda \mathbb{1})X = 0.$$

- The equation has a non-trivial solution if $A - \lambda \mathbb{1}$ is a non-invertible matrix.i.e

$$\det(A - \lambda \mathbb{1}) = 0 \tag{1}$$

This equation is an n^{th} degree polynomial in λ.

- How do we find the eigenvalues and eigenvector of T? Let A be the matrix representation of T and X be the matrix representation of the vector v in the basis B of V. Then

$$AX = \lambda X, \quad \implies \quad (A - \lambda \mathbb{1})X = 0.$$

- The equation has a non-trivial solution if $A - \lambda \mathbb{1}$ is a non-invertible matrix.i.e

$$\det(A - \lambda \mathbb{1}) = 0 \qquad (1)$$

This equation is an n^{th} degree polynomial in λ.
It is called the characteristic equation for A

- If the underlying vector space is over the complex field then the characteristic equation has n solutions, (possibly repeated)

- If the underlying vector space is over the complex field then the characteristic equation has n solutions, (possibly repeated)
  yielding n eigenvalues of A.

- If the underlying vector space is over the complex field then the characteristic equation has n solutions, (possibly repeated)
  yielding n eigenvalues of A.
- If the underlying field F is not $\mathbb{C}$ then the characteristic equation may yield $k \leq n$ eigenvalues in F.

- If the underlying vector space is over the complex field then the characteristic equation has n solutions, (possibly repeated)
  yielding n eigenvalues of A.
- If the underlying field F is not $\mathbb{C}$ then the characteristic equation may yield $k \leq n$ eigenvalues in F.
- The eigenvalues are also called the characteristic values of T.

- If the underlying vector space is over the complex field then the characteristic equation has n solutions, (possibly repeated)
  yielding n eigenvalues of A.
- If the underlying field F is not $\mathbb{C}$ then the characteristic equation may yield $k \leq n$ eigenvalues in F.
- The eigenvalues are also called the characteristic values of T.
- The eigenvectors v_i satisfying $Tv_i = \lambda_i v_i$ are also called the characteristic vectors of T.

*Eg.*1

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad AX = \lambda X$$

*Eg.*1

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad AX = \lambda X$$

To find a nontrivial X we must have $\det(A - \lambda \mathbb{1}) = 0$ i.e

*Eg.*1

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad AX = \lambda X$$

To find a nontrivial X we must have $\det(A - \lambda \mathbb{1}) = 0$ i.e

$$\det \begin{pmatrix} 4-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0$$

*Eg.*1

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad AX = \lambda X$$

To find a nontrivial X we must have $\det(A - \lambda \mathbb{1}) = 0$ i.e

$$\det \begin{pmatrix} 4-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0 \qquad (3-\lambda)[(4-\lambda)^2 - 4] = 0$$

*Eg.*1

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad AX = \lambda X$$

To find a nontrivial X we must have $\det(A - \lambda \mathbb{1}) = 0$ i.e

$$\det \begin{pmatrix} 4-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0 \qquad (3-\lambda)[(4-\lambda)^2 - 4] = 0$$

Eigenvalues are $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$.

*Eg.*1

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad AX = \lambda X$$

To find a nontrivial X we must have $\det(A - \lambda \mathbb{1}) = 0$ i.e

$$\det \begin{pmatrix} 4-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0 \qquad (3-\lambda)[(4-\lambda)^2 - 4] = 0$$

Eigenvalues are $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$.
Corresponding eigenvectors are

*Eg.*1

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad AX = \lambda X$$

To find a nontrivial X we must have $\det(A - \lambda \mathbb{1}) = 0$ i.e

$$\det \begin{pmatrix} 4-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0 \qquad (3-\lambda)[(4-\lambda)^2 - 4] = 0$$

Eigenvalues are $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$.
Corresponding eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Eg. 2:
If A is an upper(lower) triangular matrix, then eigenvalues are diagonal elements.

Eg. 2:

If A is an upper(lower) triangular matrix, then eigenvalues are diagonal elements.

Consider

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

Eg. 2:
If A is an upper(lower) triangular matrix, then eigenvalues are diagonal elements.
Consider

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & 3 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{pmatrix} = 0$$

Eg. 2:

If A is an upper(lower) triangular matrix, then eigenvalues are diagonal elements.

Consider

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & 3 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{pmatrix} = 0 \implies (2-\lambda)(4-\lambda)(5-\lambda) = 0$$

Eg. 2:
If A is an upper(lower) triangular matrix, then eigenvalues are diagonal elements.
Consider

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & 3 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{pmatrix} = 0 \implies (2-\lambda)(4-\lambda)(5-\lambda) = 0$$

So the eigenvalues are $\lambda_1 = 2, \lambda_2 = 4$ and $\lambda_3 = 5$.

Eg. 2:
If A is an upper(lower) triangular matrix, then eigenvalues are diagonal elements.
Consider

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & 3 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{pmatrix} = 0 \implies (2-\lambda)(4-\lambda)(5-\lambda) = 0$$

So the eigenvalues are $\lambda_1 = 2, \lambda_2 = 4$ and $\lambda_3 = 5$. The corresponding eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

0 Eigenvalues:

0 Eigenvalues:

- If 0 is one of the eigenvalues of A then

0 Eigenvalues:

- If 0 is one of the eigenvalues of A then

$$\det(A - 0 \cdot \mathbb{1}) = 0 \implies \det A = 0$$

0 Eigenvalues:

- If 0 is one of the eigenvalues of A then

$$\det(A - 0 \cdot \mathbb{1}) = 0 \implies \det A = 0$$

- $\implies$ A is not invertible.

0 Eigenvalues:

- If 0 is one of the eigenvalues of A then

$$\det(A - 0 \cdot \mathbb{1}) = 0 \implies \det A = 0$$

- $\implies$ A is not invertible.
- $\implies$ $AX = 0$ has non-trivial solutions.

0 Eigenvalues:

- If 0 is one of the eigenvalues of A then

$$\det(A - 0 \cdot \mathbb{1}) = 0 \implies \det A = 0$$

- $\implies$ A is not invertible.
- $\implies$ $AX = 0$ has non-trivial solutions.
- The solution space of this equation consists of eigenvectors of A corresponding to the 0-eigenvalue.

0 Eigenvalues:

- If 0 is one of the eigenvalues of A then

$$\det(A - 0 \cdot \mathbb{1}) = 0 \implies \det A = 0$$

- $\implies$ A is not invertible.
- $\implies$ $AX = 0$ has non-trivial solutions.
- The solution space of this equation consists of eigenvectors of A corresponding to the 0-eigenvalue. This indicates the following proposition.

0 Eigenvalues:

- If 0 is one of the eigenvalues of A then

$$\det(A - 0 \cdot \mathbb{1}) = 0 \implies \det A = 0$$

- $\implies$ A is not invertible.
- $\implies$ $AX = 0$ has non-trivial solutions.
- The solution space of this equation consists of eigenvectors of A corresponding to the 0-eigenvalue. This indicates the following proposition.
- *Proposition* 35
  An $n \times n$ matrix is invertible if and only if none of its eigenvalue is 0.

- *Proposition* 36
  Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  Proof:

- *Proposition* 36
  Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  Proof:
  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.

- *Proposition* 36
  Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  Proof:
  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.
  Let $v_1, v_2, ..., v_k$ be the corresponding eigenvectors.
  Then

- *Proposition* 36
  Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  Proof:
  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.
  Let $v_1, v_2, ..., v_k$ be the corresponding eigenvectors.
  Then
  $$(T - \lambda_i \mathbb{1})v_i = 0, \quad i = 1, 2, ..., k$$

- *Proposition* 36
  Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  Proof:
  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.
  Let $v_1, v_2, ..., v_k$ be the corresponding eigenvectors. Then
  $$(T - \lambda_i \mathbb{1})v_i = 0, \quad i = 1, 2, ..., k$$
  Let $c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0$.

- *Proposition* 36

  Eigenvectors corresponding to distinct eigenvalues are linearly independent.

  Proof:

  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.

  Let $v_1, v_2, ..., v_k$ be the corresponding eigenvectors.

  Then

  $$(T - \lambda_i \mathbb{1})v_i = 0, \quad i = 1, 2, ..., k$$

  Let $c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0$.

  Consider the operator

  $$T_1 = (T - \lambda_2 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$$

- *Proposition* 36
  Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  Proof:
  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.
  Let $v_1, v_2, ..., v_k$ be the corresponding eigenvectors.
  Then
  $$(T - \lambda_i \mathbb{1})v_i = 0, \quad i = 1, 2, ..., k$$
  Let $c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0$.
  Consider the operator
  $$T_1 = (T - \lambda_2 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$$
  $$T_1(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = 0$$

- *Proposition* 36

  Eigenvectors corresponding to distinct eigenvalues are linearly independent.

  Proof:

  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.

  Let $v_1, v_2, ..., v_k$ be the corresponding eigenvectors.

  Then

  $$(T - \lambda_i \mathbb{1})v_i = 0, \quad i = 1, 2, ..., k$$

  Let $c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0$.

  Consider the operator

  $$T_1 = (T - \lambda_2 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$$

  $$T_1(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = 0$$

  $$\therefore \quad c_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)......(\lambda_1 - \lambda_k)v_1 = 0$$

- *Proposition* 36
  Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  Proof:
  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.
  Let $v_1, v_2, ..., v_k$ be the corresponding eigenvectors.
  Then
  $$(T - \lambda_i \mathbb{1})v_i = 0, \quad i = 1, 2, ..., k$$
  Let $c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0$.
  Consider the operator
  $$T_1 = (T - \lambda_2 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$$
  $T_1(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = 0$
  $\therefore \quad c_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)......(\lambda_1 - \lambda_k)v_1 = 0$
  Since $v_1 \neq 0$ and $\lambda_1 \neq \lambda_2, \lambda_3, ..., \lambda_k$,

- *Proposition* 36
  Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  Proof:
  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.
  Let $v_1, v_2, ..., v_k$ be the corresponding eigenvectors.
  Then
  $$(T - \lambda_i \mathbb{1})v_i = 0, \quad i = 1, 2, ..., k$$
  Let $c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0$.
  Consider the operator
  $$T_1 = (T - \lambda_2 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$$
  $T_1(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = 0$
  $\therefore \quad c_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)......(\lambda_1 - \lambda_k)v_1 = 0$
  Since $v_1 \neq 0$ and $\lambda_1 \neq \lambda_2, \lambda_3, ..., \lambda_k$,
  we have $c_1 = 0$.

- *Proposition* 36
  Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  Proof:
  Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k distinct eigenvalues of an operator T.
  Let $v_1, v_2, ..., v_k$ be the corresponding eigenvectors.
  Then
  $$(T - \lambda_i \mathbb{1})v_i = 0, \quad i = 1, 2, ..., k$$
  Let $c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0$.
  Consider the operator
  $$T_1 = (T - \lambda_2 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$$
  $T_1(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = 0$
  $\therefore \quad c_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)......(\lambda_1 - \lambda_k)v_1 = 0$
  Since $v_1 \neq 0$ and $\lambda_1 \neq \lambda_2, \lambda_3, ..., \lambda_k$,
  we have $c_1 = 0$.

.........contd.

...........contd.

...........contd.

Now considering the operator

$T_2 = (T - \lambda_1 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$

we can show that $c_2 = 0$.

...........contd.

Now considering the operator
$T_2 = (T - \lambda_1 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$
we can show that $c_2 = 0$.
Likewise $c_3 = ... = c_k = 0$.

...........contd.

Now considering the operator

$T_2 = (T - \lambda_1 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$

we can show that $c_2 = 0$.

Likewise $c_3 = ... = c_k = 0$.

$\therefore \quad v_1, v_2, ..., v_k$ are linearly independent.

♦◊♦

...........contd.

Now considering the operator
$T_2 = (T - \lambda_1 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$
we can show that $c_2 = 0$.
Likewise $c_3 = ... = c_k = 0$.
$\therefore \quad v_1, v_2, ..., v_k$ are linearly independent.

♦◊♦

- If an eigenvalue repeats k times then we may not get k linearly independent eigenvectors.

...........contd.

Now considering the operator

$T_2 = (T - \lambda_1 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$

we can show that $c_2 = 0$.

Likewise $c_3 = ... = c_k = 0$.

$\therefore \quad v_1, v_2, ..., v_k$ are linearly independent.

♦♢♦

- If an eigenvalue repeats k times then we may not get k linearly independent eigenvectors.
- *Eg.*

...........contd.

Now considering the operator

$T_2 = (T - \lambda_1 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$

we can show that $c_2 = 0$.

Likewise $c_3 = ... = c_k = 0$.

$\therefore \quad v_1, v_2, ..., v_k$ are linearly independent.

♦◊♦

- If an eigenvalue repeats k times then we may not get k linearly independent eigenvectors.
- *Eg.* If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ Then the characteristic equation is

...........contd.

Now considering the operator
$T_2 = (T - \lambda_1 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$
we can show that $c_2 = 0$.
Likewise $c_3 = ... = c_k = 0$.
$\therefore \quad v_1, v_2, ..., v_k$ are linearly independent.

♦◊♦

- If an eigenvalue repeats k times then we may not get k linearly independent eigenvectors.
- *Eg.* If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ Then the characteristic equation is $(1 - \lambda)^2 = 0$. So the eigenvalue $\lambda = 1$ has multiplicity 2.

...........contd.

Now considering the operator

$T_2 = (T - \lambda_1 \mathbb{1})(T - \lambda_3 \mathbb{1}).....(T - \lambda_k \mathbb{1})$

we can show that $c_2 = 0$.

Likewise $c_3 = ... = c_k = 0$.

$\therefore \quad v_1, v_2, ..., v_k$ are linearly independent.

♦◊♦

- If an eigenvalue repeats k times then we may not get k linearly independent eigenvectors.
- *Eg.* If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ Then the characteristic equation is $(1 - \lambda)^2 = 0$. So the eigenvalue $\lambda = 1$ has multiplicity 2. But the solution space of the homogeneous equation $(A - \lambda \mathbb{1})X = 0$ is only one dimensional yielding only one linearly independent eigenvector viz. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Diagonalization

Diagonalization

Let A be a $n \times n$ matrix.

Diagonalization

Let A be a $n \times n$ matrix. Let $C_1, C_2, ..., C_n$ be n linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.

Diagonalization

Let A be a $n \times n$ matrix. Let $C_1, C_2, ..., C_n$ be n linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.
Construct the $n \times n$ matrix $P = (C_1, C_2, ..., C_n)$.

Diagonalization

Let A be a $n \times n$ matrix. Let $C_1, C_2, ..., C_n$ be n linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.

Construct the $n \times n$ matrix $P = (C_1, C_2, ..., C_n)$.

P is invertible since its columns C_i are linearly independent.

Diagonalization

Let A be a $n \times n$ matrix. Let $C_1, C_2, ..., C_n$ be n linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.

Construct the $n \times n$ matrix $P = (C_1, C_2, ..., C_n)$.

P is invertible since its columns C_i are linearly independent.

Let $R_1, R_2, ..., R_n$ be the rows of P^{-1}.

Diagonalization

Let A be a $n \times n$ matrix. Let $C_1, C_2, ..., C_n$ be n linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.

Construct the $n \times n$ matrix $P = (C_1, C_2, ..., C_n)$.

P is invertible since its columns C_i are linearly independent.

Let $R_1, R_2, ..., R_n$ be the rows of P^{-1}.

$P^{-1}P = \mathbb{1} \implies R_i C_j = \delta_{ij}$.

Diagonalization

Let A be a $n \times n$ matrix. Let $C_1, C_2, ..., C_n$ be n linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.

Construct the $n \times n$ matrix $P = (C_1, C_2, ..., C_n)$.

P is invertible since its columns C_i are linearly independent.

Let $R_1, R_2, ..., R_n$ be the rows of P^{-1}.

$P^{-1}P = \mathbb{1} \implies R_i C_j = \delta_{ij}$.

$$\therefore P^{-1}AP = P^{-1}(AC_1, AC_2, ..., AC_n)$$

Diagonalization

Let A be a $n \times n$ matrix. Let $C_1, C_2, ..., C_n$ be n linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.
Construct the $n \times n$ matrix $P = (C_1, C_2, ..., C_n)$.
P is invertible since its columns C_i are linearly independent.
Let $R_1, R_2, ..., R_n$ be the rows of P^{-1}.
$P^{-1}P = \mathbb{1} \implies R_i C_j = \delta_{ij}$.

$$\begin{aligned} \therefore P^{-1}AP &= P^{-1}(AC_1, AC_2, ..., AC_n) \\ &= \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix} (\lambda_1 C_1 \ \ldots \ \lambda_n C_n) \end{aligned}$$

Diagonalization

Let A be a $n \times n$ matrix. Let $C_1, C_2, ..., C_n$ be n linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.
Construct the $n \times n$ matrix $P = (C_1, C_2, ..., C_n)$.
P is invertible since its columns C_i are linearly independent.
Let $R_1, R_2, ..., R_n$ be the rows of P^{-1}.
$P^{-1}P = \mathbb{1} \implies R_i C_j = \delta_{ij}$.

$$\begin{aligned}
\therefore P^{-1}AP &= P^{-1}(AC_1, AC_2, ..., AC_n) \\
&= \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix} (\lambda_1 C_1 \ \dots \ \lambda_n C_n) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \\
&= D
\end{aligned}$$

Diagonalization

Let A be a $n \times n$ matrix. Let $C_1, C_2, ..., C_n$ be n linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.
Construct the $n \times n$ matrix $P = (C_1, C_2, ..., C_n)$.
P is invertible since its columns C_i are linearly independent.
Let $R_1, R_2, ..., R_n$ be the rows of P^{-1}.
$P^{-1}P = \mathbb{1} \implies R_iC_j = \delta_{ij}$.

$$\begin{aligned}
\therefore P^{-1}AP &= P^{-1}(AC_1, AC_2, ..., AC_n) \\
&= \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix} (\lambda_1 C_1 \ \dots \ \lambda_n C_n) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \\
&= D
\end{aligned}$$

where D is a diagonal matrix with the diagonal elements as the eigenvalues of A.

*Eg.*1:

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

*Eg.*1:

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Characteristic equation : $\det \begin{pmatrix} 3-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} = 0$

*Eg.*1:

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Characteristic equation : $\det \begin{pmatrix} 3-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} = 0$

$\therefore (4-\lambda)[(3-\lambda)^2 - 1] = 0$

*Eg.*1:

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Characteristic equation : $\det \begin{pmatrix} 3-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} = 0$

$\therefore (4-\lambda)[(3-\lambda)^2 - 1] = 0 \implies \lambda = 4$ or $\lambda = 2$.

*Eg.*1:

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Characteristic equation : $\det \begin{pmatrix} 3-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} = 0$

$\therefore (4-\lambda)[(3-\lambda)^2 - 1] = 0 \implies \lambda = 4$ or $\lambda = 2$.
Eigenvalue 4 occurs with multiplicity 2.

*Eg.*1:

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Characteristic equation : $\det \begin{pmatrix} 3-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} = 0$

$\therefore (4-\lambda)[(3-\lambda)^2 - 1] = 0 \implies \lambda = 4 \text{ or } \lambda = 2.$
Eigenvalue 4 occurs with multiplicity 2.

Eigenvectors: For $\lambda = 2$,

*Eg.*1:

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Characteristic equation : $\det \begin{pmatrix} 3-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} = 0$

$\therefore (4-\lambda)[(3-\lambda)^2 - 1] = 0 \implies \lambda = 4$ or $\lambda = 2$.

Eigenvalue 4 occurs with multiplicity 2.

Eigenvectors: For $\lambda = 2$, $C_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda = 4$ we get two linearly independent eigenvectors.

*Eg.*1:

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Characteristic equation : $\det \begin{pmatrix} 3-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} = 0$

$\therefore (4-\lambda)[(3-\lambda)^2 - 1] = 0 \implies \lambda = 4$ or $\lambda = 2$.

Eigenvalue 4 occurs with multiplicity 2.

Eigenvectors: For $\lambda = 2$, $C_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda = 4$ we get two linearly independent eigenvectors.

$C_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$; $C_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Using C_1, C_2, C_3 we construct

Using C_1, C_2, C_3 we construct $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

Using C_1, C_2, C_3 we construct $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

$P^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.

Using C_1, C_2, C_3 we construct $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

$P^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.

$P^{-1}AP = \mathrm{diag}(4, 4, 2)$

Eg.2: $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

$$Eg.2\text{: } A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$ Eigenvalues are 2, 2 and 3.

Eg.2: $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. Eigenvalues are 2, 2 and 3.

Eigenvector for $\lambda = 3$:

Eg.2: $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. Eigenvalues are 2, 2 and 3.

Eigenvector for $\lambda = 3$:

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

*Eg.*2: $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. Eigenvalues are 2, 2 and 3.

Eigenvector for $\lambda = 3$:

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{array}{l} y = 2z \\ x = 9z \end{array}$$

Eg.2: $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. Eigenvalues are 2, 2 and 3.

Eigenvector for $\lambda = 3$:

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{array}{l} y = 2z \\ x = 9z \end{array}$$

$$\therefore C_1 = \begin{pmatrix} 9 \\ 2 \\ 1 \end{pmatrix}.$$

$Eg.2$: $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. Eigenvalues are 2, 2 and 3.

Eigenvector for $\lambda = 3$:

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{array}{l} y = 2z \\ x = 9z \end{array}$$

$\therefore C_1 = \begin{pmatrix} 9 \\ 2 \\ 1 \end{pmatrix}$. For $\lambda = 2$ we get $C_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Eg.2: $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. Eigenvalues are 2, 2 and 3.

Eigenvector for $\lambda = 3$:

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{array}{l} y = 2z \\ x = 9z \end{array}$$

$\therefore C_1 = \begin{pmatrix} 9 \\ 2 \\ 1 \end{pmatrix}$. For $\lambda = 2$ we get $C_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Though this eigenvalue has multiplicity 2 there is only one corresponding eigenvector.

*Eg.*2: $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. Eigenvalues are 2, 2 and 3.

Eigenvector for $\lambda = 3$:

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{matrix} y = 2z \\ x = 9z \end{matrix}$$

$\therefore C_1 = \begin{pmatrix} 9 \\ 2 \\ 1 \end{pmatrix}$. For $\lambda = 2$ we get $C_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Though this eigenvalue has multiplicity 2 there is only one corresponding eigenvector. We can't construct the matrix P such that $P^{-1}AP = \mathrm{diag}(2, 2, 3)$.

So A is not diagonalizable.

Conclusion

Conclusion

- If all the eigenvalues of a matrix A are distinct, then it is diagonalizable.

Conclusion

- If all the eigenvalues of a matrix A are distinct, then it is diagonalizable.
- If an eigenvalue has multiplicity k, then A is diagonalizable only if the eigenspace for this eigenvalue is k-dimensional.

Conclusion

- If all the eigenvalues of a matrix A are distinct, then it is diagonalizable.
- If an eigenvalue has multiplicity k, then A is diagonalizable only if the eigenspace for this eigenvalue is k-dimensional.
- Let T be a linear operator on a vector space V. If the matrix representation of T is diagonalizable then the range of T is spanned by the eigenvectors of the nonzero eigenvalues of T.

Conclusion

- If all the eigenvalues of a matrix A are distinct, then it is diagonalizable.
- If an eigenvalue has multiplicity k, then A is diagonalizable only if the eigenspace for this eigenvalue is k-dimensional.
- Let T be a linear operator on a vector space V. If the matrix representation of T is diagonalizable then the range of T is spanned by the eigenvectors of the nonzero eigenvalues of T.
  kernel T is spanned by the eigenvectors of the 0-eigenvalue of T.

- *Def.*:**Similarity**
  A matrix A is similar to matrix B if $\exists$ an invertible matrix P such that $B = PAP^{-1}$.

- *Def.*:**Similarity**
  A matrix A is similar to matrix B if $\exists$ an invertible matrix P such that $B = PAP^{-1}$.
- Similarity is an equivalence relation on the set of matrices.

- *Def.*:**Similarity**
  A matrix A is similar to matrix B if $\exists$ an invertible matrix P such that $B = PAP^{-1}$.
- Similarity is an equivalence relation on the set of matrices.
- Similar matrices have the same characteristic polynomial. Hence they will have the same set of eigenvalues.