expectation →

$$\langle k \rangle = \int_{k_{min}}^{k_{max}} k\, p(k)\, dk = c \int_{k_{min}}^{k_{max}} k\, k^{-\alpha}\, dk$$

$$= c \int_{k_{min}}^{k_{max}} k^{1-\alpha}\, dk$$

$$= c \frac{k^{2-\alpha}}{2-\alpha} \Bigg|_{k_{min}}^{k_{max}}$$

now let's say $\alpha > 2$. (diverges otherwise)

to calculate var →

$$\langle k^2 \rangle = c \int_{k_{min}}^{k_{max}} k^2 p(k)\, dk = c \frac{k^{3-\alpha}}{3-\alpha} \Bigg|_{k_{min}}^{k_{max}}$$

now α needs to be greater than 3.

However,
we usually find that α is between 2 & 3.
$\alpha \in [2, 3]$.

$$\langle k \rangle \pm \frac{\sigma_k}{\sqrt{n}}$$

$n \to$ no. of nodes.
$\sigma \to$ s.d.

↓
confidence interval.

to calculate σ, we need $\alpha > 3$. Hence we can't calculat σ when $\alpha \in [2, 3]$.
∴ we have $\langle k \rangle$, but we cannot calculate the confidence.
∴ we call such distribution to be a scale free distribution.

$$\langle k^h \rangle \approx c \frac{k^{h+1-\alpha}}{h+1-\alpha} \Bigg|_{k_{min}}^{k_{max}}$$

scale free distributions are usually self similar structures.

Clauset
Newman (2004)

~~Ro~~ PARETO's Power Law distribution
$\boxed{80:20}$ → wealth distribution

$\boxed{\text{FRACTALS}}$

$$p(ak) = a^{(\,)}k^{-\alpha}$$

multiplying by a to change the scale.

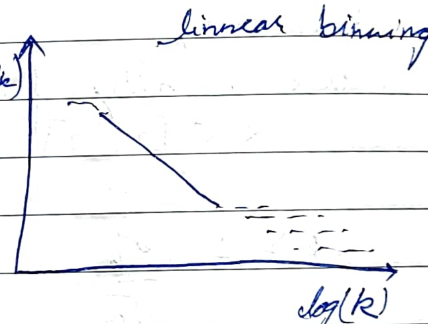
→ the distribution remains the same. Just multiplied by a constant.

80 : 20 → 20% of the nodes has 80% of the edges & 80% of the nodes has 20% of the edges.

Draw the distribution →

$k = 1, \ldots, k_{max}$

N_k
↳ no. of nodes with degree k



$$p_k = \frac{N_k}{N}$$ → total no. of nodes.

then make a log log plot & observe if it's linear or not.

logarithmic binning → degrees
$b_0 = 1$
$b_1 = 2, 3$
$b_2 = 4, 5, 6, 7$.

typically we don't do logarithmic binning till degree 10.

CDF → Rank - frequency method.
arrange degrees from highest to lowest.

10
9
8
7

calculate for exponential distribution.

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		r_i/n
10	1	1/7
9	2	2/7
8	3	3/7
7	4	4/7
6	5	
3	6	
2	7	

→ ~~this is not just for power law~~

$$\frac{\int_{x_{1/2}}^{\infty} p(x)\,dx}{\int_{x_{min}}^{\infty} p(x)\,dx} = \frac{1}{2}$$

$x_{1/2}$ → divides the distribution into 2 equal parts.

$$\frac{\int_{x_{1/2}}^{\infty} x\,p(x)\,dx}{\int_{x_{min}}^{\infty} x\,p(x)\,dx} = 2^{-\left(\frac{\alpha-2}{\alpha-1}\right)}$$

→ for a power law distribution

$k_{max} \sim f(N)$ → how does k_{max} depend on the no. of nodes.

$$\int_{k_{max}}^{\infty} p(k)\,dk = \frac{1}{N}$$

$c = (\alpha-1)\,k_{min}^{\alpha-1}$

$$c\int_{k_{max}}^{\infty} k^{-\alpha}\,dk = \frac{c\,k_{max}^{1-\alpha}}{\alpha-1} = \frac{1}{N}$$

$$k_{max} = k_{min} N^{-(1/1-\alpha)}$$

$$\boxed{k_{max} = k_{min} N^{(1/\alpha-1)}} \quad \alpha > 1.$$

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robust properties → properties that do not change on changing the network distribution size.

→ these properties give the insights on the distribution.

- hubs → power-law degree distribution
- small world → short avg. path length
- high clustering
- sparse networks.

Rule.

→ models.

Random Graphs (1959|1960)

Erdős - Rennys' model $G(n, m)$

Gilbert $G(n, p)$

n → no. of nodes
m → no. of links.
p → probability of connecting 2 nodes

in random graphs, one thinks of averages.

~~to~~ calculating those averages is a little difficult for a $G(n, m)$ model as compared to a $G(n, p)$ model.

for a $G(n, p)$ model, the no. of links aren't going to be the same for different configurations of the network.

$G(n, p) \rightarrow$ probability of connecting node pairs

average no. of links $\rightarrow \langle m \rangle$

$$\langle m \rangle = {}^{n}C_{2} \cdot p = \binom{n}{2} p$$

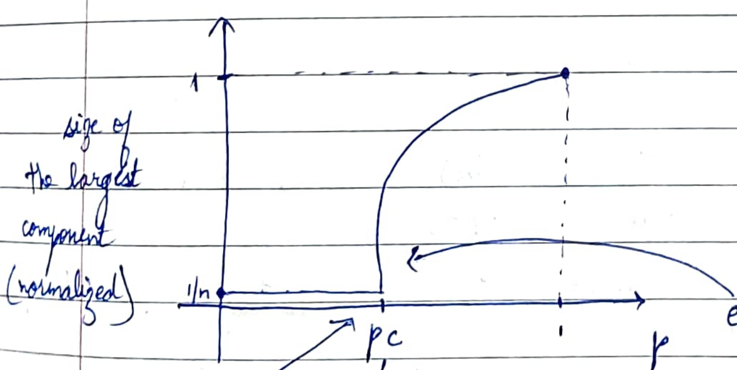
average degree $\rightarrow \langle k \rangle$

$$\langle k \rangle = \frac{2m}{n} = \frac{2\langle m \rangle}{n} = (n-1)p \simeq np$$

$\downarrow$ for large graphs.

Density $\rightarrow \rho$

$$\rho = p = \frac{\langle m \rangle}{\binom{n}{2}}$$



$1/n \simeq 0$ for large graphs.

emergence of a giant component.

$pc \rightarrow$ critical value of probability

phase transition
bifurcation

$\rightarrow$ a very sharp transition from a disconnected graph to a connected graph.

$\langle k \rangle = 1$ at pc.

if $\langle k \rangle < 1 \rightarrow$ disconnected graph (mostly)

if $\langle k \rangle > 1 \rightarrow$ connected graph (mostly)

sparse connected graphs → usually trees.
↳ avg. degree close to 1.

$\langle k \rangle \simeq 1$

$\langle k \rangle \simeq np$

$p = \frac{\langle k \rangle}{n}$

$p << 1$ → tree kind of a structure
↓
clustering coefficient close to 0.

for a random graph → clustering coefficient $\simeq p$.
i.e. for $p << 1$, such graphs have a low value of clustering coefficient.

$\boxed{C = p}$

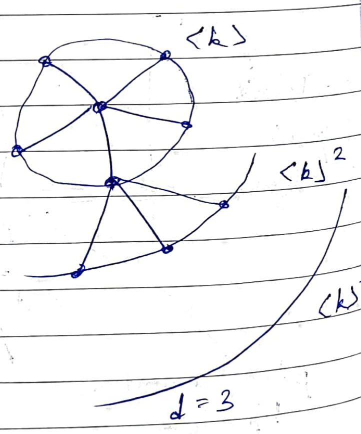
Average path length →

average degree $= \langle k \rangle$

no. of neighbours on 1^{st} level $= \langle k \rangle$

on 2^{nd} level $= \langle k \rangle^2$

3^{rd} level $= \langle k \rangle^3$



$\therefore \; n = 1 + \langle k \rangle + \langle k \rangle^2 + \dots + \langle k \rangle^d$

$= \frac{\langle k \rangle^{d+1} - 1}{\langle k \rangle - 1}$

$\simeq \langle k \rangle^d$

$d \simeq \frac{\ln n}{\ln \langle k \rangle}$

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Dunbar's no. = 75 → avg. no. of people a person knows in their lifetime.

let's work with 100.

$\therefore\ n = 100^{d}$

Degree distribution →

$$P(k) = p_k = {}^{n-1}C_k\, p^k (1-p)^{n-1-k}$$

k neighbours out of n-1 possible neighbours

(Binomial distribution)

~~P(k)~~ $p_k \quad \lim_{n \to \infty} \quad \to \langle k \rangle = np$

$$p_k \approx \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!} \to \text{poisson distribution}$$

mean $= \langle k \rangle$

variance $= \langle k \rangle$

the short avg. path length is obtained because of the randomness in a graph & not because of hubs.

clustering, degree distribution, power law behaviour isn't just because of randomness. It depicts the kind of graph we're working with.

if we try to make things regular, let's say increase the clustering coefficient, we might end up increasing the avg. path length.

∴ if we combine the idea of regular & random graphs, we might end up with a network having a high clustering coeff. & a short avg. path length.

small world phenomenon
Collective phenomenon
science
Waltz & Strogatz

make a graph with a lot of triangles, high clustering.
form a bridge
remove rest of the edges at random.