1 Relativistic momentum

The momentum of a body in an inertial frame should depend only on its rest mass and velocity in this frame. This, of course, is not enough to define $\vec{p}$ uniquely, so we add the following axioms:

1. A restriction on direction of $\vec{p}$: it must be the same as the direction of $\vec{v}$.

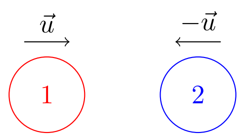
2. Due to isotropy of space, the absolute value of $\vec{p}$ should only depend on the mass and the absolute value of the velocity.

3. Consider two objects having the same velocity and location and masses m_1 and m_2. We would very much like our physics to be linear, so the momentum of the system of these two objects should equal the sum of the two momenta, so we postulate $\vec{p}$ to be proportional to m.

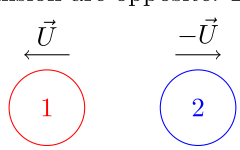
We thus infer that $\vec{p}(m, \vec{v})$ is of form $m\vec{v} \cdot \phi(v)$ (notice that the fact that the momentum of a body at rest is zero is a theorem).

Additionally, conservation of net momentum of a closed system should apply. We will not apply any particular mechanics for this restriction; instead, we will require a conditional: if momentum of a system is conserved in one inertial frame, it is also conserved in any other inertial frame.

Now, what form does $\phi(v)$ take? To figure that out, consider two balls of mass m with velocities $\vec{u}$ and $-\vec{u}$ (in K) respectively colliding elastically in plane Oxy:



Due to symmetry (I am not saying "due to conservation of net momentum", because we don't know if this conservation exists yet), the velocities after collision are opposite. Let them be $\vec{U}$ and $-\vec{U}$:



Now, what is the resulting velocity $\vec{U}$? This is the first problem we run into. The problem is we don't know if there is such a thing as conservation of energy, or that (in)elastic collisions are possible for high velocities.

What we can do is derive the formula for ϕ in assumption that the balls collide in such a way that $U_x = u_x$ and $U_y = -u_y$ (as if they "bounced off" each other), and then show that this formula also works for other $\vec{U}$s.

For an inertial frame K' moving with the speed $v = u_x$ along the X axis wrt. K, we expect the conservation of momentum to hold as well.

The formulae for the net momentum in K' before and after collision are:

$$\vec{p'} = m(\vec{u'_1}\phi(u'_1) + \vec{u'_2}\phi(u'_2)),$$
$$\vec{P'} = m(\vec{U'_1}\phi(U'_1) + \vec{U'_2}\phi(U'_2)).$$

If $\vec{p'}$ is to be conserved, then $p'_x = P'_x$ and $p'_y = P'_y$. It is easy to check that the former holds for all ϕ, and after substituting velocities to the latter and using some symmetries we get:

$$\frac{\phi(u'_1)}{\phi(u'_2)} = \frac{u'_{2y}}{u'_{1y}}.$$

The ratio of u'_y can be computed using the velocity addition formula:

$$u'_{1y} = \sqrt{1 - \frac{v^2}{c^2}} \cdot \frac{u_y}{1 - vu_x/c^2} = \sqrt{1 - \frac{u_x^2}{c^2}} \cdot \frac{u_y}{1 - u_x^2/c^2} = \gamma(u_y) \cdot u_y,$$
$$u'_{2y} = \sqrt{1 - \frac{v^2}{c^2}} \cdot \frac{-u_y}{1 + vu_x/c^2} = \sqrt{1 - \frac{u_x^2}{c^2}} \cdot \frac{-u_y}{1 + u_x^2/c^2}.$$

u_x can be calculated from u_{2x} as follows:

$$u'_{2x} = \frac{-u_x - v}{1 + vu_x/c^2} = \frac{-2u_x}{1 + u_x^2/c^2} \implies$$
$$u_x = \frac{-u'_{2x}}{\sqrt{1 - (u'_{2x})^2/c^2 + 1}}.$$

Thus

$$\frac{\phi(u'_1)}{\phi(u'_2)} = \frac{1 - u_x^2/c^2}{1 + u_x^2/c^2} = -1 + \frac{2}{1 + u_x^2/c^2} = -1 - \frac{u'_{2x}}{u_x} = \sqrt{1 - \frac{(u'_{2x})^2}{c^2}}.$$

Now, the right-hand side does not depend on the vertical velocity of the balls. This implies that the left-hand side does not depend on u_y. In particular, for infinitesimal u_y: u'_{1y} and u'_{2y} approach zero, and $u'_{1x} = 0$, hence $u'_1 = 0$ and $u'_2 = u'_{2x}$ and thus:

$$\frac{\phi(u'_1)}{\phi(u'_2)} = \frac{\gamma(u'_1)}{\gamma(u'_2)}.$$

Now, this statement is coordinate-independent and thus holds for all collisions similar to the one used. Also notice that this formula uses u'_1 and u'_2 instead of u_1 and u_2, and while the latter two are opposite velocities, there is no requirement on the former. It is in fact possible to construct a collision for any pair of u_1 and u_2, and thus this statement holds for absolutely arbitrary velocities.

Thus, $\phi(v)$ is of form $\alpha\gamma(v)$, where α is a universal constant. There is no way to narrow down this any further, because all functions of such form satisfy any reasonable dimensionless requirements. However, it seems reasonable for the well-known Newtonian momentum to approximate relativistic momentum for small velocities, and this yields $\alpha = 1$.

The relativistic momentum of a body of mass m and velocity $\vec{v}$ is:

$$\vec{p} = \gamma(v) \cdot m\vec{v}.$$

It can be shown algebraically that if net momentum of a system of bodies is conserved in any one inertial frame, it is also conserved in other inertial frames.

Whether this is indeed the only possible well-defined formula for ϕ is up in the air and apparently depends on the mechanics used.

2 Relativistic energy

Let $\vec{F}(\vec{q}, \vec{v}, t)$ be a vector field indicating the force applied to a particular object at coordinate $\vec{q}$ having velocity $\vec{v}$ at time t in inertial frame K.

The force satisfies $\vec{F}(\vec{q}, \vec{v}, t) = \frac{d\vec{p}}{dt}$.

We define the kinetic energy of a body as

$$\Delta K = \int_A^B \vec{F} d\vec{q}.$$

That is, the increase in the kinetic energy equals the work on the body. This requirement allows infinitely many definitions of kinetic energy, only differing in the integration constant. Hence we also require the kinetic energy of a body at rest to be zero.

The kinetic energy of a body of mass m is thus

$$\begin{aligned} K &= \int \vec{F} \cdot d\vec{q} = \int \frac{d\vec{p}}{dt} \cdot d\vec{q} = \int d\vec{p} \cdot \frac{d\vec{q}}{dt} = \int \vec{v} \cdot d\vec{p} = \vec{v} \cdot \vec{p} - \int \vec{p} \cdot d\vec{v} = \\ &= \gamma m v^2 - m \int \gamma \vec{v} \cdot d\vec{v} = \gamma m v^2 - \frac{1}{2} m \int \gamma \cdot d((\vec{v})^2) = \\ &= \gamma m v^2 + \frac{1}{2} m c^2 \int \gamma \cdot d\left(1 - \frac{v^2}{c^2}\right) = \gamma m v^2 + \frac{mc^2}{\gamma} + C = \gamma m c^2 + C. \end{aligned}$$

Where C is the constant of integration. For $\vec{v} = 0$, we want $K = 0 = mc^2 + C$, hence $C = -mc^2$ and

$$K = (\gamma - 1)mc^2$$

is the kinetic energy of a body.

Now, most textbooks bring the constant of integration to attention and say that, in fact, if we take $C = 0$ (and what is more natural than a zero?), we get another formula for energy:

$$E = \gamma mc^2$$

Which is full energy, composed of kinetic energy and rest energy:

$$E_0 = mc^2$$
$$K = (\gamma - 1)mc^2$$
$$E = \gamma mc^2$$

The rest energy is said to be the energy stored within matter, that is, the binding energy.

This hand-waving brings up several questions:

1. The constant of integration is absolutely arbitrary, and so is rest energy. In fact, the only reasonable limitation on rest energy is its linearity with respect to mass; that is, $E_0 = \alpha mc^2$ where α is a universal dimensionless constant. Notice that this c^2 coefficient does not come from Maxwell equations or Lorentz transformations: it is just the simplest value of dimension $\frac{\text{energy}}{\text{mass}}$. Where does $\alpha = 1$ come from?

2. Energy is defined as the largest amount of work a body can exert on its surroundings. Can any body really exert exactly mc^2 of energy? Is there a body that can perform more work?

3. Mass having a role in rest energy seems somewhat arbitrary: Newton's second law for a body at rest does not include the mass of said body, so why wouldn't the rest energy depend on, say, electric charge as well?

Einstein tackles the first question as follows:

Consider a body emitting two symmetric electromagnetic plane waves of energy $L/2$ each. Due to symmetry, the velocity of the emitting body won't change, but due to law of conservation of energy, the internal energy of the body determined by mass will decrease.

In a frame K where the body is at rest, the full energies of the body before (E) and after (H) the emission are connected as follows:

$$E - H = L$$

In a frame K' moving with speed v perpendicular to the plane waves, the amount of energy emitted increases in a way similar to that of kinetic energy (look up on transformation of electromagnetic fields between inertial frames for more information), and so:

$$E' - H' = L' = \gamma L$$

Einstein notes that E and E' are the energies of the same body in different frames. The only difference between them is the velocity v of K' with respect to K. Since velocity only affects kinetic energy, we acquire:

$$E' - E = E'_K - E_K = (\gamma - 1)m_0c^2$$

Where m_0 is the original mass of the body. Similarly,

$$H' - H = H'_K - H_K = (\gamma - 1)m_1c^2$$

Where m_1 is the mass after the emission. Combining the four equations yields:

$$(m_0 - m_1)c^2 = L$$

Hence, a decrease of Δm in mess results in emission of Δmc^2 of energy, and vice versa. This implies that a mass-less body cannot emit energy, and thus

$$E_0 = mc^2$$

is the correct formula for energy at rest.

One oft arising question about this proof is that Einstein implicitly postulates a single body as seen from two inertial frames being identical to two bodies only differing in velocity, and not in mass, charge, or otherwise.

The logic in lack of difference in mass is apparent. We think of mass as a property intrinsic to matter. When we derived the expressions for momentum and energy, we were thinking of mass as a constant, not changing upon acceleration or switch of frames. So it's really no wonder rest energy, denoting an intrinsic property of matter, is somehow connected to mass.

The absence of any other influencing factors is due to the definition of matter. When we compute the rest energy of a body, we actually compute the rest energy of its matter, and not potential energy with regards to fields, be they electromagnetic or otherwise. Mass has a unique property of influencing the behavior of a particle in the now, and not via fields.

After seeing this derivation, other questions raise up:

1. Einstein requires velocity to be perpendicular to the wave front for L'/L to be a constant. What would happen for other angles?

2. This derivation assumes that a body can emit light at all. Hydrogen at room temperature cannot emit light–does this mean that hydrogen has zero rest energy for low temperatures?

3. Bodies do not actually emit light in plane waves. Would the results differ for other kinds of waves? And what if energy of a different kind was emitted?

The first question is easy to answer. Einstein used $\phi = 0$ purely for optimization. A more complete expression for transformation of energy gives

$$L' = \gamma L \left(1 - \frac{v}{c}\cos\phi\right).$$

When two plane waves are emitted at symmetric angles ϕ and $\pi + \phi$, the net energy is

$$L' = \gamma \frac{L}{2}\left(1 - \frac{v}{c}\cos\phi\right) + \gamma \frac{L}{2}\left(1 - \frac{v}{c}\cos(\pi + \phi)\right) = \gamma L,$$

which yields the same result.

The answer to the second question is that we assume that full energy can be separated into two parts: rest energy that is a function of the internal state of a body and does not depend on velocity, and kinetic energy that only depends on velocity and mass, and not on the internal state. This is an arguable distinction: Planck has shown in 1907 that such a separation is impossible for black cavity radiation. If, however, we believe that this is possible for matter, then rest energy does not depend on temperature and therefore hydrogen indeed has the same rest energy at all temperatures.

As for the third question, this is an implication of law of conservation of energy. Consider body A emitting a plane wave of energy L which leads to a decrease of $\Delta m_1 = L/c^2$ in mass, and body B emitting a different kind of energy of the same magnitude that leads to a different decrease in mass $\Delta m_2 < \Delta m_1$. Then let the second body lose additional $\Delta m = \Delta m_1 - \Delta m_2$ mass as plane wave radiation. The masses of the two bodies are now identical, but the second body has emitted additional Δmc^2 amount of energy, which contradicts law of conservation of energy–that is, assuming rest energy is a function of mass, of course.

So yes, $E_0 = mc^2$ is in fact a conjecture or an implication of several laws we took as-is from Newtonian mechanics. That it holds in reality according to experiments is a nice bonus. Binding energy, for instance, almost sums up rest energy, and the behavior of mass-less and massive elementary particles upon conversion also confirms the theory.

There is also artificial significance in rest energy being exactly mc^2: it helps link energy to momentum.

If $E_0 = mc^2$ and $K = (\gamma - 1)mc^2$, then total energy $E = \gamma mc^2$. After performing some algebraic transformations we get

$$\begin{aligned} E^2 &= m^2c^4 + \left(\frac{Ev}{c}\right)^2 \implies \\ E^2 &= m^2c^4 + (\gamma mv)^2c^2 \implies \\ E^2 &= m^2c^4 + p^2c^2. \end{aligned}$$

This is one of the more well-known identities of special relativity. After some more reordering we get

$$(E_0/c)^2 = (E/c)^2 - p^2,$$

and this can be interpreted as E_0/c being the absolute value of a vector in 4-space, called 4-momentum:

$$\frac{E_0}{c} = \|\mathbf{p}\| = \left\| \left(\frac{E}{c}, p_x, p_y, p_z \right) \right\|,$$

where the metric signature is chosen to be $(+,-,-,-)$. As E_0/c is the same across inertial frames, it is Lorentz-invariant just like space-time interval.

Moreover, 4-momentum behaves the same way as 4-position under Lorentz transformation, that is, it is Lorentz-covariant.

The Lorentz transformation for 4-position (ct, x, y, z) is:

$$\begin{aligned}
ct' &= \gamma \left(ct - x\frac{v}{c} \right), \\
x' &= \gamma \left(x - ct\frac{v}{c} \right), \\
y' &= y, \\
z' &= z.
\end{aligned}$$

And the Lorentz transformation for 4-momentum $(E/c, p_x, p_y, p_z)$ is:

$$\begin{aligned}
\frac{E'}{c} &= \gamma \left(\frac{E}{c} - p_x\frac{v}{c} \right), \\
p_x' &= \gamma \left(p_x - \frac{E}{c}\frac{v}{c} \right), \\
p_y' &= p_y, \\
p_z' &= p_z.
\end{aligned}$$

This is incredibly obvious for a body that is at rest in frame K, that is, for $\vec{p} = \vec{0}$:

$$\begin{aligned}
\frac{E'}{c} &= \gamma\frac{E}{c} = \frac{\gamma mc^2}{c}, \\
p_x' &= -\gamma\frac{E}{c}\frac{v}{c} = -\gamma mv, \\
p_y' &= 0, \\
p_z' &= 0.
\end{aligned}$$