1 Governing Equations

Consider a Boussinesq, perfectly conducting fluid of density ρ_0, kinematic viscosity ν, and magnetic diffusivity η. Suppose the fluid is confined to a spherical shell of thickness H, rotating with an angular velocity $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$. The concentric spheres bounding the fluid have radii, respectively, of $r_i = R - H$, and $r_{\mathrm{CMB}} = R$, where R is the radius of the core. The equations of momentum conservation, induction, mass conservation, and Gauss's law for magnetism for the total velocity and magnetic fields $\mathbf{V}$ and $\mathbf{B}$ are respectively

$$\rho_0(\partial_t\mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{V} + 2\mathbf{\Omega}\times\mathbf{V}) = -\nabla P + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} + \Delta\rho\mathbf{g} + \rho_0\nu\nabla^2\mathbf{V} \tag{1.0.1a}$$

$$\partial_t\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{B} = \nabla\times(\mathbf{V}\times\mathbf{B}) + \eta\nabla^2\mathbf{B} \tag{1.0.1b}$$

$$\nabla\cdot\mathbf{V} = 0 \tag{1.0.1c}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{1.0.1d}$$

where ρ_0 is the ambient fluid density, P is the pressure, μ_0 is the magnetic permeability, $\Delta\rho$ is the density variation, and $\mathbf{g} = -g\hat{\mathbf{r}}$ is the acceleration due to gravity.

We consider small perturbations in velocity, $\mathbf{v}$, and magnetic field, $\mathbf{b}$, about background fields $\mathbf{V_0}$, and $\mathbf{B_0}$, which are assumed steady over the period of the waves. We assume $\mathbf{V_0} = 0$, and retain only the radial component, $B_0\hat{\mathbf{r}}$, of the background field $\mathbf{B_0}$ [1]. Assuming the stratified layer is thin, we may write the density perturbation as

$$\Delta\rho = -\mathbf{u}\cdot\nabla\rho_0 \simeq -u_r\frac{\partial\rho_0}{\partial r} \tag{1.0.2}$$

where $\mathbf{u}$ is the fluid displacement. For small amplitude waves, we have

$$\partial_t\mathbf{u} = \mathbf{v}. \tag{1.0.3}$$

We can conveniently write the buoyancy term $\Delta\rho\mathbf{g}$ in eq. (1.0.1a) in terms of the buoyancy frequency N, defined as

$$N^2 = -\frac{g}{\rho_0}\frac{\partial\rho_0}{\partial r}. \tag{1.0.4}$$

The system formed by eqs. (1.0.1a) to (1.0.1d) is written in non-dimensional form with the radius of the core, R, as the length scale, Ω^{-1} as the time scale, and $\sqrt{\Omega\rho_0\mu_0\eta}$ as the magnetic field scale. Additionally substituting eq. (1.0.2) and eq. (1.0.4) into eq. (1.0.1a), the resulting linear and dimensionless momentum and induction equations are

$$\partial_t\mathbf{v} + 2\hat{\mathbf{z}}\times\mathbf{v} = -\nabla P + E_\eta(\nabla\times\mathbf{b})\times\mathbf{B_0} - \tilde{N}^2u_r\hat{\mathbf{r}} + E\nabla^2\mathbf{v} \tag{1.0.5a}$$

$$\partial_t\mathbf{b} = \nabla\times(\mathbf{v}\times\mathbf{B_0}) + E_\eta\nabla^2\mathbf{b}, \tag{1.0.5b}$$

where the non-dimensional parameters

$$E = \frac{\nu}{\Omega R^2}, \quad E_\eta = \frac{\eta}{\Omega R^2}, \quad \tilde{N}^2 = \frac{N^2}{\Omega^2} \tag{1.0.6}$$

are respectively the Ekman number, magnetic Ekman number, and squared dimensionless buoyancy frequency.

[1] Specifically, we assume the magnetic field is randomly distributed over the core mantle boundary, and approximate the root mean square value as constant. We take B_0 to be the constant rms value.

2 Vector Spherical Harmonic Decomposition

As the velocity and magnetic perturbation fields $\mathbf{v}$ and $\mathbf{b}$ are solenoidal, they can be decomposed into poloidal and toroidal scalars W, Z, S and T as

$$\mathbf{v} = \nabla \times \nabla \times (W\hat{\mathbf{r}}) + \nabla \times (Z\hat{\mathbf{r}}) \tag{2.0.1a}$$
$$\mathbf{b} = \nabla \times \nabla \times (S\hat{\mathbf{r}}) + \nabla \times (T\hat{\mathbf{r}}). \tag{2.0.1b}$$

Each scalar is respectively decomposed into fully normalized scalar spherical harmonics Y_ℓ^m. For example, the poloidal scalar W is written as

$$W(r, \theta, \phi, t) = \sum_{m=-\ell}^{\ell} \sum_{l=0}^{\infty} W_\ell^m(r, t) Y_l^m(\theta, \phi), \tag{2.0.2}$$

where

$$Y_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_\ell^m(\cos(\theta)) e^{im\phi}, \tag{2.0.3}$$

and likewise for Z, S and T. Time dependence is represented for all coefficients by, for example, $W_\ell^m(r, t) = \tilde{W}_\ell^m(r) e^{i\omega t}$, where ω is the frequency of the wave. This decomposition serves to separate variables, so we may solve a system of PDE's in only one spatial variable, r. However, first we apply the operations of $\hat{\mathbf{r}} \cdot \nabla \times$ and $\hat{\mathbf{r}} \cdot \nabla \times \nabla \times$ to eq. (1.0.5a), and $\hat{\mathbf{r}} \cdot$ and $\hat{\mathbf{r}} \cdot \nabla \times$ to eq. (1.0.5b). This will cause the pressure gradient term in eq. (1.0.5a) to vanish, and through the Coriolis force result in coupling of the governing equations with directly neighboring spherical harmonics. If we had instead used orthogonality of the Y_ℓ^m first, the Coriolis force would cause the governing equations to couple across many spherical harmonics, complicating the system and increasing the number of equations to solve. We present expressions for the necessary curls and components of $\mathbf{v}, \mathbf{b}$ in terms of the scalars $W_\ell^m, Z_\ell^m, S_\ell^m$ and T_ℓ^m. Then, we separately treat the Coriolis and Lorentz force terms of the momentum equation, and the induction term of the magnetic induction equation.

2.1 Components of v and b

For notational simplicity, we derive expressions for the components only considering a single term in the scalar spherical harmonic expansions. For example, we will denote this by

$$(\hat{\mathbf{r}} \cdot v)_\ell^m \equiv \hat{\mathbf{r}} \cdot \nabla \times \nabla \times (W_\ell^m Y_\ell^m \hat{\mathbf{r}}) + \hat{\mathbf{r}} \cdot \nabla \times (Z_\ell^m Y_\ell^m \hat{\mathbf{r}}), \tag{2.1.1}$$

thus $\hat{\mathbf{r}} \cdot \mathbf{v} = \sum_{\ell,m} (\hat{\mathbf{r}} \cdot \mathbf{v})_\ell^m$. We use the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ to rewrite the poloidal part of $\mathbf{v}$ and $\mathbf{b}$. Vector derivatives in spherical coordinates are then evaluated, and simplified using the relation

$$L^2 Y_\ell^m = \ell(\ell + 1) Y_\ell^m, \tag{2.1.2}$$

where

$$L^2 = -\left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial\varphi^2}\right) \tag{2.1.3}$$

is the angular momentum operator, so that

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 - \frac{L^2}{r^2}. \tag{2.1.4}$$

The complete set of components, including those of $\mathbf{b}$, which follow directly from the $\mathbf{v}$ components, is

$$(\hat{\mathbf{r}} \cdot \mathbf{v})_\ell^m = \frac{\ell(\ell+1)}{r^2} W_\ell^m Y_\ell^m \tag{2.1.5a}$$

$$(\hat{\theta} \cdot \mathbf{v})_\ell^m = \frac{1}{r}\partial_r W_\ell^m \partial_\theta Y_\ell^m + \frac{1}{r\sin\theta} Z_\ell^m \partial_\phi Y_\ell^m \tag{2.1.5b}$$

$$(\hat{\phi} \cdot \mathbf{v})_\ell^m = \frac{1}{r\sin\theta}\partial_r W_\ell^m \partial_\phi Y_\ell^m - \frac{1}{r} Z_\ell^m \partial_\theta Y_\ell^m \tag{2.1.5c}$$

$$(\hat{\mathbf{r}} \cdot \mathbf{b})_\ell^m = \frac{\ell(\ell+1)}{r^2} S_\ell^m Y_\ell^m \tag{2.1.5d}$$

$$(\hat{\theta} \cdot \mathbf{b})_\ell^m = \frac{1}{r}\partial_r S_\ell^m \partial_\theta Y_\ell^m + \frac{1}{r\sin\theta} T_\ell^m \partial_\phi Y_\ell^m \tag{2.1.5e}$$

$$(\hat{\phi} \cdot \mathbf{b})_\ell^m = \frac{1}{r\sin\theta}\partial_r S_\ell^m \partial_\phi Y_\ell^m - \frac{1}{r} T_\ell^m \partial_\theta Y_\ell^m. \tag{2.1.5f}$$

For later use in writing out the final set of equations, we will treat the buoyancy term. Note that the term vanishes under the operation of taking one curl, and in taking two curls it follows from eq. (2.1.5a) that

$$\hat{\mathbf{r}} \cdot \nabla \times \nabla \times (\tilde{N}^2 u_r \hat{\mathbf{r}}) = \frac{\ell(\ell+1)}{r^2} \tilde{N}^2 u_r. \tag{2.1.6}$$

We define the scalar $D = \sum_{\ell,m} D_\ell^m Y_\ell^m$ such that

$$[u_r]_\ell^m = \frac{\ell(\ell+1)}{r^2} \tilde{N}^2 D_\ell^m Y_\ell^m, \tag{2.1.7}$$

hence $\partial_t D = W$. Thus, the buoyancy term becomes

$$[\hat{\mathbf{r}} \cdot \nabla \times \nabla \times (\tilde{N}^2 u_r \hat{\mathbf{r}})]_\ell^m = \frac{\ell^2(\ell+1)^2}{r^4} \tilde{N}^2 D_\ell^m Y_\ell^m. \tag{2.1.8}$$

2.2 Expressions for Curls of $\mathbf{v}, \mathbf{b}$

As the time derivative terms in eq. (1.0.5a) and eq. (1.0.5b) are directly proportional to $\mathbf{v}$ and $\mathbf{b}$, we also seek expressions for $\hat{\mathbf{r}} \cdot \nabla \times \mathbf{v}$, $\hat{\mathbf{r}} \cdot \nabla \times \mathbf{b}$ and $\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \mathbf{v}$. As $\mathbf{v}$, $\mathbf{b}$ are solenoidal, we can rewrite the diffusion terms as, for example,

$$\nabla^2 \mathbf{v} = -\nabla \times \nabla \times \mathbf{v}. \tag{2.2.1}$$

Thus, we seek expressions for $\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \mathbf{b}$, $\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \nabla \times \mathbf{b}$, $\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \nabla \times \mathbf{v}$ and $\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \nabla \times \nabla \times \mathbf{v}$. Following similar algebra as in section 2.1 we arrive at the following components of single curls of $\mathbf{v}$ and $\mathbf{b}$:

$$(\hat{\mathbf{r}} \cdot \nabla \times \mathbf{v})_\ell^m = \frac{\ell(\ell+1)}{r^2} Z_\ell^m Y_\ell^m \tag{2.2.2a}$$

$$(\hat{\theta} \cdot \nabla \times \mathbf{v})_\ell^m = \frac{1}{r\sin\theta}\left(\frac{\ell(\ell+1)}{r^2} W_\ell^m - \partial_{rr}^2 W_\ell^m\right)\partial_\phi Y_\ell^m + \frac{1}{r}\partial_r Z_\ell^m \partial_\theta Y_\ell^m \tag{2.2.2b}$$

$$(\hat{\phi} \cdot \nabla \times \mathbf{v})_\ell^m = -\frac{1}{r}\left(\frac{\ell(\ell+1)}{r^2} W_\ell^m - \partial_{rr}^2 W_\ell^m\right)\partial_\theta Y_\ell^m + \frac{1}{r\sin\theta}\partial_r Z_\ell^m \partial_\phi Y_\ell^m \tag{2.2.2c}$$

$$(\hat{\mathbf{r}} \cdot \nabla \times \mathbf{b})_\ell^m = \frac{\ell(\ell+1)}{r^2} T_\ell^m Y_\ell^m \tag{2.2.2d}$$

$$(\hat{\theta} \cdot \nabla \times \mathbf{b})_\ell^m = \frac{1}{r\sin\theta}\left(\frac{\ell(\ell+1)}{r^2} S_\ell^m - \partial_{rr}^2 S_\ell^m\right)\partial_\phi Y_\ell^m + \frac{1}{r}\partial_r T_\ell^m \partial_\theta Y_\ell^m \tag{2.2.2e}$$

$$(\hat{\phi} \cdot \nabla \times \mathbf{b})_\ell^m = -\frac{1}{r}\left(\frac{\ell(\ell+1)}{r^2} S_\ell^m - \partial_{rr}^2 S_\ell^m\right)\partial_\theta Y_\ell^m + \frac{1}{r\sin\theta}\partial_r T_\ell^m \partial_\phi Y_\ell^m, \tag{2.2.2f}$$

as well as double curls:

$$(\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \mathbf{v})_\ell^m = \frac{\ell(\ell+1)}{r^2}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) W_\ell^m Y_\ell^m \tag{2.2.3a}$$

$$\begin{aligned}(\hat{\theta} \cdot \nabla \times \nabla \times \mathbf{v})_\ell^m = {} & \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) W_\ell^m \partial_\theta Y_\ell^m \\ & + \frac{1}{r\sin\theta}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}2\right) Z_\ell^m \partial_\phi Y_\ell^m\end{aligned} \tag{2.2.3b}$$

$$\begin{aligned}(\hat{\phi} \cdot \nabla \times \nabla \times \mathbf{v})_\ell^m = {} & \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) W_\ell^m \partial_\phi Y_\ell^m \\ & - \frac{1}{r}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) Z_\ell^m \partial_\theta Y_\ell^m\end{aligned} \tag{2.2.3c}$$

$$(\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \mathbf{b})_\ell^m = \frac{\ell(\ell+1)}{r^2}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) S_\ell^m Y_\ell^m \tag{2.2.3d}$$

$$\begin{aligned}(\hat{\theta} \cdot \nabla \times \nabla \times \mathbf{b})_\ell^m = {} & \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) S_\ell^m \partial_\theta Y_\ell^m \\ & + \frac{1}{r\sin\theta}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) T_\ell^m \partial_\phi Y_\ell^m\end{aligned} \tag{2.2.3e}$$

$$\begin{aligned}(\hat{\phi} \cdot \nabla \times \nabla \times \mathbf{b})_\ell^m = {} & \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) S_\ell^m \partial_\phi Y_\ell^m \\ & - \frac{1}{r}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) T_\ell^m \partial_\theta Y_\ell^m.\end{aligned} \tag{2.2.3f}$$

Lastly, using the components in eqs. (2.2.3a) to (2.2.3f) we arrive at expressions for the triple curl of $\mathbf{v}$ and $\mathbf{b}$ and the four-times curl of $\mathbf{v}$:

$$(\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \nabla \times \mathbf{v})_\ell^m = \frac{\ell(\ell+1)}{r^2}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) Z_\ell^m Y_\ell^m \tag{2.2.4a}$$

$$(\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \nabla \times \mathbf{b})_\ell^m = \frac{\ell(\ell+1)}{r^2}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right) T_\ell^m Y_\ell^m \tag{2.2.4b}$$

$$(\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \nabla \times \nabla \times \mathbf{v})_\ell^m = \frac{\ell(\ell+1)}{r^2}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right)^2 S_\ell^m Y_\ell^m. \tag{2.2.4c}$$

2.3 Treatment of Coriolis Term

From the components of $\mathbf{v}$ in eqs. (2.1.5a) to (2.1.5c), we can use the relation $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}$, and the cyclic property of the curl of the unit vectors to compute

$$[\hat{\mathbf{r}} \cdot (\hat{\mathbf{z}} \times \mathbf{v})]_\ell^m = \frac{\sin\theta}{r} Z_\ell^m \partial_\theta Y_\ell^m - \frac{1}{r}\partial_r W_\ell^m \partial_\phi Y_\ell^m \tag{2.3.1a}$$

$$[\hat{\theta} \cdot (\hat{\mathbf{z}} \times \mathbf{v})]_\ell^m = -\frac{\cos\theta}{r\sin\theta}\partial_r W_\ell^m \partial_\phi Y_\ell^m + \frac{\cos\theta}{r}\partial_\theta Z_\ell^m \tag{2.3.1b}$$

$$[\hat{\phi} \cdot (\hat{\mathbf{z}} \times \mathbf{v})]_\ell^m = \frac{\cos\theta}{r}\partial_r W_\ell^m \partial_\theta Y_\ell^m + \frac{\cos\theta}{r\sin\theta} Z_\ell^m \partial_\phi Y_\ell^m + \sin\theta\frac{\ell(\ell+1)}{r^2} W_\ell^m Y_\ell^m. \tag{2.3.1c}$$

The single and double curls of the Coriolis term are then found using the components eqs. (2.3.1a) to (2.3.1c):

$$[\hat{\mathbf{r}} \cdot \nabla \times (\hat{\mathbf{z}} \times \mathbf{v})]_\ell^m = \mathcal{A}_\ell^m W_\ell^m + \mathcal{B}_\ell^m \partial_r W_\ell^m + \mathcal{C}_\ell^m Z_\ell^m \tag{2.3.2a}$$

$$[\hat{\mathbf{r}} \cdot \nabla \times \nabla \times (\hat{\mathbf{z}} \times \mathbf{v})]_\ell^m = \mathcal{C}_\ell^m\left(\frac{\ell(\ell+1)}{r^2} W_\ell^m - \partial_{rr}^2 W_\ell^m\right) + \mathcal{A}_\ell^m \partial_r Z_\ell^m + \mathcal{B}_\ell^m \partial_r Z_\ell^m, \tag{2.3.2b}$$

where

$$\mathcal{A}_\ell^m = \frac{\ell(\ell+1)}{r^3}\left(\sin\theta\partial_\theta Y_\ell^m + 2\cos\theta Y_\ell^m\right) \tag{2.3.3a}$$

$$\mathcal{B}_\ell^m = -\frac{\ell(\ell+1)}{r^2}\cos\theta Y_\ell^m - \frac{\sin\theta}{r^2}\partial_\theta Y_\ell^m \tag{2.3.3b}$$

$$\mathcal{C}_\ell^m = -\frac{1}{r^2}\partial_\phi Y_\ell^m. \tag{2.3.3c}$$

These coefficients can be simplified using relations for spherical harmonics and their derivatives. Defining the coefficient

$$c_\ell^m = \sqrt{\frac{(\ell-m)(\ell+m}{(2\ell-1)(2\ell+1)}}, \tag{2.3.4}$$

we have

$$\sin\theta\partial_\theta Y_\ell^m = \ell c_{\ell+1}^m Y_{\ell+1}^m - (\ell+1)c_\ell^m Y_{\ell-1}^m \tag{2.3.5a}$$

$$\cos\theta Y_\ell^m = c_{\ell+1}^m Y_{\ell+1}^m + c_\ell^m Y_{\ell-1}^m \tag{2.3.5b}$$

$$\partial_\phi Y_\ell^m = imY_\ell^m. \tag{2.3.5c}$$

The simplified coefficients are then given by

$$\mathcal{A}_\ell^m = \frac{\ell(\ell+1)}{r^3}\left((\ell+2)c_{\ell+1}^m Y_{\ell+1}^m - (\ell-1)c_\ell^m Y_{\ell-1}^m\right) \tag{2.3.6a}$$

$$\mathcal{B}_\ell^m = -\frac{1}{r^2}\left(\ell(\ell+2)c_{\ell+1}^m Y_{\ell+1}^m + (\ell-1)(\ell+1)c_\ell^m Y_{\ell-1}^m\right) \tag{2.3.6b}$$

$$\mathcal{C}_\ell^m = -\frac{im}{r^2}Y_\ell^m. \tag{2.3.6c}$$

2.4 Treatment of Lorentz Force Term

We can rewrite the Lorentz force term $(\nabla\times\mathbf{B})\times\mathbf{B}$ from eq. (1.0.1a) as

$$(\nabla\times\mathbf{B})\times\mathbf{B} = -\frac{1}{2}\nabla(\mathbf{B}\cdot\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{B}. \tag{2.4.1}$$

The first term can be included into the pressure, as it will vanish under the application of the curl operator. For the calculation of two curls, it proves easier to write the Lorentz force using the (dimensionless) current density $\mathbf{J}$, where $(\nabla\times\mathbf{B})\times\mathbf{B} = \mathbf{J}\times\mathbf{B}$. Using another vector identity, we have

$$\hat{\mathbf{r}}\cdot\nabla\times\nabla\times\mathbf{J}\times\mathbf{B} = \hat{\mathbf{r}}\cdot\nabla\times[(\mathbf{B}\cdot\nabla)\mathbf{J} - (\mathbf{J}\cdot\nabla)\mathbf{B}]. \tag{2.4.2}$$

Thus, we seek expressions for $\hat{\mathbf{r}}\times\nabla\times(\mathbf{B}\cdot\nabla)\mathbf{B}$, and $\hat{\mathbf{r}}\cdot\nabla\times[(\mathbf{B}\cdot\nabla)\mathbf{J} - (\mathbf{J}\cdot\nabla)\mathbf{B}]$.

Writing $\mathbf{B} = \mathbf{B_0} + \mathbf{b}$, with $\mathbf{B_0} = B_0\hat{\mathbf{r}}$, and neglecting terms that are nonlinear in $\mathbf{b}$, we calculate the necessary components of the directional derivative:

$$[\hat{\theta}\cdot(\mathbf{B}\cdot\nabla)\mathbf{B}]_\ell^m = \frac{B_0}{r}\partial_{rr}^2 S_\ell^m \partial_\theta Y_\ell^m + \frac{B_0}{r\sin\theta}\partial_r T_\ell^m \partial_\phi Y_\ell^m \tag{2.4.3a}$$

$$[\hat{\phi}\cdot(\mathbf{B}\cdot\nabla)\mathbf{B}]_\ell^m = \frac{B_0}{r\sin\theta}\partial_{rr}^2 S_\ell^m \partial_\phi Y_\ell^m - \frac{B_0}{r}\partial_r T_\ell^m \partial_\theta Y_\ell^m. \tag{2.4.3b}$$

Using eq. (2.4.3) we get the expression for one curl of the Lorentz force term:

$$[\hat{\mathbf{r}}\cdot\nabla\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]]_\ell^m = B_0\frac{\ell(\ell+1)}{r^2}\partial_r T_\ell^m Y_\ell^m. \tag{2.4.4}$$

Again neglecting terms that are nonlinear in $\mathbf{b}$, we similarly find the necessary components of $[(\mathbf{B}\cdot\nabla)\mathbf{J} - (\mathbf{J}\cdot\nabla)\mathbf{B}]$:

$$\left[\hat{\theta}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{J} - (\mathbf{J}\cdot\nabla)\mathbf{B}]\right]_\ell^m = B_0 r\frac{\partial}{\partial r}\frac{J_{\theta,\phi}}{r} \tag{2.4.5a}$$

$$\left[\hat{\phi}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{J} - (\mathbf{J}\cdot\nabla)\mathbf{B}]\right]_\ell^m = B_0 r\frac{\partial}{\partial r}\frac{J_{\theta,\phi}}{r} \tag{2.4.5b}$$

where $J_{\theta,\phi} = [\nabla\times\mathbf{b}]_{\theta,\phi}$ is found in eq. (2.2.2e) and eq. (2.2.2f). Using eq. (2.4.5) we find an expression for two curls of the Lorentz force term:

$$[\hat{\mathbf{r}}\cdot\nabla\times\nabla\times\mathbf{J}\times\mathbf{B}]_\ell^m = B_0\frac{\partial}{\partial r}\left[\frac{\ell(\ell+1)}{r^2}\left(\frac{\ell(\ell+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right)\right]S_\ell^m Y_\ell^m. \tag{2.4.6}$$

2.5 Treatment of Induction Term

Lastly, we seek expressions for $\hat{\mathbf{r}} \times \nabla \times \mathbf{v} \times \mathbf{B_0}$, and $\hat{\mathbf{r}} \times \nabla \times \nabla \times \mathbf{v} \times \mathbf{B_0}$. As $\mathbf{B_0} = B_0\hat{\mathbf{r}}$, we have that

$$\hat{\mathbf{r}} \cdot \mathbf{v} \times \mathbf{B_0} = 0, \quad \hat{\theta} \cdot \mathbf{v} \times \mathbf{B_0} = B_0 v_\phi, \quad \hat{\phi} \cdot \mathbf{v} \times \mathbf{B_0} = -B_0 v_\theta, \tag{2.5.1}$$

with the components of $\mathbf{v}$ given by eq. (2.1.5b) and eq. (2.1.5c). Using eq. (2.5.1) we find the desired expression for the radial component:

$$[\hat{\mathbf{r}} \times \nabla \times \mathbf{v} \times \mathbf{B_0}]_\ell^m = B_0 \frac{\ell(\ell+1)}{r^2} \partial_r W_\ell^m Y_\ell^m. \tag{2.5.2}$$

Similarily, we find

$$\hat{\theta} \cdot \nabla \times \mathbf{v} \times \mathbf{B_0} = \frac{B_0}{r} \frac{\partial}{\partial r}(r v_\theta), \quad \hat{\phi} \cdot \nabla \times \mathbf{v} \times \mathbf{B_0} = \frac{B_0}{r} \frac{\partial}{\partial r}(r v_\phi), \tag{2.5.3}$$

from which we get our expression for one curl of the induction term:

$$[\hat{\mathbf{r}} \cdot \nabla \times \nabla \times \mathbf{v} \times \mathbf{B_0}]_\ell^m = B_0 \frac{\ell(\ell+1)}{r^2} \partial_r Z_\ell^m Y_\ell^m. \tag{2.5.4}$$

3 System of Equations

Using the results of section 2, we apply the operations of $\hat{\mathbf{r}} \cdot \nabla \times$ and $\hat{\mathbf{r}} \cdot \nabla \times \nabla \times$ to eq. (1.0.5a), and $\hat{\mathbf{r}} \cdot$ and $\hat{\mathbf{r}} \cdot \nabla \times$ to eq. (1.0.5b). For notational simplicity, we define the operator

$$\tilde{\nabla}^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}. \tag{3.0.1}$$

We introduce the variables $\zeta \equiv \tilde{\nabla}^2 W$, and $\beta \equiv \tilde{\nabla}^2 S$ to reduce the degree of the final equations. Then, we project eq. (1.0.5a) and eq. (1.0.5b) onto a single spherical harmonic degree by applying $\iint \mathrm{d}\Omega Y_\ell^m$ (where we integrate over the full surface of the sphere), use orthogonality of the scalar spherical harmonics. Note that the Coriolis coefficients $\mathcal{A}_\ell^m$ and $\mathcal{B}_\ell^m$, summed against some scalar coefficient which is a function of r only, $U_\ell^m(r)$, transform under this operation as

$$\iint \mathrm{d}\Omega Y_\ell^m \left(\sum_{\ell',m'} \mathcal{A}_{\ell'}^{m'} U_{\ell'}^{m'} \right) = \frac{\ell(\ell+1)}{r^3} \left((\ell-1) c_\ell^m U_{\ell-1}^m - (\ell+2) c_{\ell+1}^m U_{\ell+1}^m \right) \tag{3.0.2a}$$

$$\iint \mathrm{d}\Omega Y_\ell^m \left(\sum_{\ell',m'} \mathcal{B}_{\ell'}^{m'} U_{\ell'}^{m'} \right) = -\frac{1}{r^2} \left((\ell-1)(\ell+1) c_\ell^m U_{\ell-1}^m + \ell(\ell+2) c_{\ell+1}^m U_{\ell+1}^m \right). \tag{3.0.2b}$$

After canceling common factors, and replacing $\partial_t \to i\omega$, the full equation set is

$$\begin{aligned} i\omega\zeta_\ell^m = {} & \left(E\tilde{\nabla}^2 - \frac{2im}{\ell(\ell+1)}\right)\zeta_\ell^m + \tilde{N}^2\frac{\ell(\ell+1)}{r^2}D_\ell^m \\ & - 2\frac{\ell-1}{\ell}c_\ell^m\left(\frac{\partial}{\partial r} - \frac{\ell}{r}\right)Z_{\ell-1}^m - 2\frac{\ell+2}{\ell+1}c_{\ell+1}^m\left(\frac{\partial}{\partial r} + \frac{\ell+1}{r}\right)Z_{\ell+1}^m \\ & + B_0\frac{\partial}{\partial r}\beta_\ell^m \end{aligned} \tag{3.0.3a}$$

$$0 = \tilde{\nabla}^2 W_\ell^m - \zeta_\ell^m \tag{3.0.3b}$$

$$i\omega D_\ell^m = W_\ell^m \tag{3.0.3c}$$

$$\begin{aligned} i\omega Z_\ell^m = {} & \left(E\tilde{\nabla}^2 + \frac{2im}{\ell(\ell+1)}\right)Z_\ell^m \\ & + 2\frac{\ell-1}{\ell}c_\ell^m\left(\frac{\partial}{\partial r} - \frac{\ell}{r}\right)W_{\ell-1}^m + 2\frac{\ell+2}{\ell+1}c_{\ell+1}^m\left(\frac{\partial}{\partial r} + \frac{\ell+1}{r}\right)W_{\ell+1}^m \\ & + B_0\frac{\partial}{\partial r}T_\ell^m \end{aligned} \tag{3.0.3d}$$

$$i\omega T_\ell^m = E_\eta\tilde{\nabla}^2 T_\ell^m + B_0\frac{\partial}{\partial r}Z_\ell^m \tag{3.0.3e}$$

$$i\omega S_\ell^m = E_\eta\tilde{\nabla}^2 S_\ell^m + B_0\frac{\partial}{\partial r}W_\ell^m \tag{3.0.3f}$$

$$0 = \tilde{\nabla}^2 S_\ell^m - \beta_\ell^m, \tag{3.0.3g}$$

where eq. (3.0.3a) and eq. (3.0.3d) are respectively the poloidal and toroidal components of the momentum equation, and eq. (3.0.3e) and eq. (3.0.3f) are respectively the toroidal and poloidal parts of the magnetic induction equation. The remaining equations define relations on the introduced variables ζ, D and β.

We can simplify eqs. (3.0.3a) to (3.0.3g), as we may neglect the Toroidal part of the magnetic field perturbation, S, on the basis of scaling arguments. Additionally, we seek axisymmetric modes, so we set $m = 0$ in all equations. Thus, we will henceforth omit the superscript m from all variables, and assume the system to be axisymmetric, setting derivatives in ϕ to be zero. The resulting, simplified equations are the final set which is to be solved numerically:

$$
\begin{aligned}
i\omega\zeta_\ell = E\tilde{\nabla}^2\zeta_\ell + \tilde{N}^2\frac{\ell(\ell+1)}{r^2}D_\ell \\
- 2\frac{\ell-1}{\ell}c_\ell\left(\frac{\partial}{\partial r}-\frac{\ell}{r}\right)Z_{\ell-1} - 2\frac{\ell+2}{\ell+1}c_{\ell+1}\left(\frac{\partial}{\partial r}+\frac{\ell+1}{r}\right)Z_{\ell+1}
\end{aligned}
\tag{3.0.4a}
$$

$$0 = \tilde{\nabla}^2 W_\ell - \zeta_\ell \tag{3.0.4b}$$

$$i\omega D_\ell = W_\ell \tag{3.0.4c}$$

$$
\begin{aligned}
i\omega Z_\ell = E\tilde{\nabla}^2 Z_\ell \\
+ 2\frac{\ell-1}{\ell}c_\ell\left(\frac{\partial}{\partial r}-\frac{\ell}{r}\right)W_{\ell-1} + 2\frac{\ell+2}{\ell+1}c_{\ell+1}\left(\frac{\partial}{\partial r}+\frac{\ell+1}{r}\right)W_{\ell+1} \\
+ B_0\frac{\partial}{\partial r}T_\ell
\end{aligned}
\tag{3.0.4d}
$$

$$i\omega T_\ell = E_\eta\tilde{\nabla}^2 T_\ell + B_0\frac{\partial}{\partial r}Z_\ell. \tag{3.0.4e}$$

3.1 Boundary Conditions

Boundary conditions of no-penetration and free-stress are imposed at the top and bottom of the stratified layer, which are written respectively as

$$v_r|_{r_i, r_{\mathrm{CMB}}} = 0, \quad \left.\frac{\partial}{\partial r}\frac{v_{\theta,\phi}}{r}\right|_{r_i, r_{\mathrm{CMB}}} = 0. \tag{3.1.1}$$

We discretize the system in the radial direction, with grid points $r_0 = r_i, r_1, r_2, \cdots, r_{N-1}, r_N = r_{\mathrm{CMB}}$. We will denote by, for example, $W_{\ell,j} = W(r_j)$ the coefficient W_ℓ evaluated on the j-th grid point. Using the components of $\mathbf{v}$ from eqs. (2.1.5a) to (2.1.5c), the conditions in eq. (3.1.1) are translated to the discretized coefficients W_ℓ and Z_ℓ. No-penetration at the lower and upper boundary yield the boundary conditions on W_ℓ, where $W_{\ell,0} = W_{\ell,N} = 0$. From these conditions on W_ℓ we find the necessary boundary conditions on D_ℓ, which are $D_{\ell,0} = D_{\ell,N} = 0$. Free stress yields the following relations on W_ℓ and Z_ℓ:

$$0 = \left.\frac{\partial^2 W_\ell}{\partial r^2}\right|_{r_0, r_N} - \frac{2}{r}\left.\frac{\partial W}{\partial r}\right|_{r_0, r_N} \tag{3.1.2a}$$

$$0 = \left.\frac{\partial Z_\ell}{\partial r}\right|_{r_0, r_N} - \frac{2}{r}Z_{\ell,0}. \tag{3.1.2b}$$

Equation (3.1.2b) provide the boundary conditions for Z_ℓ. Equation (3.1.2a) is used to find conditions on the ghost-points $W_{\ell,-1}$ and $W_{\ell,N+1}$, which are used to find boundary conditions on ζ:

$$\zeta_{\ell,0} = \frac{W_{\ell,1}}{\Delta r^2}\left(1 + \frac{\Delta r - r_0}{\Delta r + r_0}\right) \tag{3.1.3a}$$

$$\zeta_{\ell,N} = \frac{W_{\ell,N-1}}{\Delta r^2}\left(1 + \frac{\Delta r + r_0}{\Delta r - r_0}\right), \tag{3.1.3b}$$

where Δr is the (possibly non-uniform) grid-spacing.

Pseudo-vacuum conditions can be applied on $\mathbf{b}$, where we set $b_\phi = 0$ at the top and bottom of the domain, and thus $T_{\ell,0} = T_{\ell,N} = 0$. Alternatively, we can impose a different condition on the bottom of the domain, where we require continuity of the magnetic field and the horizontal electric field. This condition can be found by integrating the induction equation over an infinitesimal radial distance across the lower boundary, and is written as

$$[E_\eta \partial_r b_\phi]_-^+ + B_0[v_\phi]_-^+ = -0, \tag{3.1.3c}$$

where $[\ ]_-^+$ denotes the difference in the enclosed quantity across the boundary. Assuming the flow in the region below the stratified layer vanishes, so that $\mathbf{v}_- = 0$, the magnetic field perturbation obeys a diffusion equation in this region. Solutions can thus be approximated by

$$b_\phi^-(\phi) = b_\phi^-(\phi)e^{(1-i)(r-r_i)/\delta}, \tag{3.1.3d}$$

where $\delta = \sqrt{2\eta/\omega}$ is the skin depth. Substituting eq. (3.1.3d) into eq. (3.1.3c) yields, in dimensionless form,

$$\partial_r b_\phi^+ - (1-i)\frac{b_\phi^+}{\tilde{\delta}} + \frac{B_0}{E_\eta} v_\phi^+ = 0, \tag{3.1.3e}$$

with $\tilde{\delta} = \delta/R$. Finally, substituting eq. (2.1.5c) and eq. (2.1.5f) into eq. (3.1.3e) yield the lower boundary condition on T_ℓ:

$$0 = \frac{1}{r}T_{\ell,0} - \left.\frac{\partial T_\ell}{\partial r}\right|_{r_0} + \frac{1-i}{\tilde{\delta}}T_{\ell,0} - \frac{B_0}{E_\eta}Z_{\ell,0} \tag{3.1.3f}$$