Assignment 5 Spring 2014 Solutions

1. The Ratio Test. Assume $L = \lim\limits_{n\to\infty} |a_{n+1}/a_n|$.

   It $L < 1$ then $\sum a_n$ converges absolutely.

   If $L > 1$ the series diverges.

   If $L = 1$ the test is inconclusive.

   The Root Test. Assume $L = \lim\limits_{n\to\infty} |a_n|^{1/n}$.

   It $L < 1$ then $\sum a_n$ converges absolutely.

   If $L > 1$ the series diverges.

   If $L = 1$ the test is inconclusive.

2. Determine whether the series converge or diverge, using the Ratio Test or Root Test. If the test in inconclusive, use some other method.

   (a) $\sum\limits_{n=1}^{\infty} \frac{(1.5)^n}{n^5}$

   Use the Root Test. That is,

$$\lim_{n\to\infty}\left|\frac{(1.5)^n}{n^5}\right|^{1/n} = \lim_{n\to\infty}\frac{1.5}{n^{5/n}} = 1.5\lim_{n\to\infty}(n^{1/n})^{-5} = 1.5\left(\lim_{n\to\infty} n^{1/n}\right)^{-5} = 1.5(1)^{-5} = 1.5 > 1.$$

   Therefore, the given series diverges by the Root Test.

   (b) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n n}{(n+1)(n+2)}$

   Use the Ratio Test. That is,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)}{(n+2)(n+3)}\cdot\frac{(n+1)(n+2)}{(-1)^n n}\right| = \lim_{n\to\infty}\frac{(n+1)(n+1)}{n(n+3)} = \lim_{n\to\infty}\frac{n^2+2n+1}{n^2+3n} = 1.$$

   Therefore, the Ratio Test is inconclusive. Now, use the Alternating Series Test.
   Let $f(x) = \frac{x}{(x+1)(x+2)} = \frac{x}{x^2+3x+2}$. The given series has the form $\sum_{n=2}^{\infty}(-1)^n a_n$, where $a_n = f(n)$. Observe that

   i. $f(x) > 0$
   ii. $\lim\limits_{x\to\infty} \frac{x}{x^2+3x+2} = 0.$
   iii. $f'(x) = \frac{1(x^2+3x+2)-(2x+3)(x)}{(x^2+3x+2)^2} = \frac{x^2+3x+2-2x^2-3x}{(x^2+3x+2)^2} = \frac{2-x^2}{(x^2+2x+3)^2} < 0$, for $x \geq 2$. So, $f(x)$ is decreasing.

   Therefore the given series converges by the Alternating Series Test.

(c) $\sum_{n=1}^{\infty} 8^{-n^2}$

Use the Root Test. That is,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|8^{-n^2}\right|^{1/n} = \lim_{n\to\infty} 8^{-n} = \lim_{n\to\infty} \frac{1}{8^n} = 0 < 1.$$

Therefore, the given series converges by the Root Test.

(d) $\sum_{n=1}^{\infty} (-1)^n n^2 (0.8)^n$

Use the Root Test. That is,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |(-1)^n n^2 (0.8)^n|^{1/n} = \lim_{n\to\infty} 0.8 n^{2/n} = 0.8 \lim_{n\to\infty} n^{2/n} = 0.8 \cdot 1 = 0.8 < 1.$$

Therefore, the given series converges by the Root Test.

(e) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{3}{2} - \sqrt[n]{n}\right)^n$

Use the Root Test. That is,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|(-1)^{n+1}\left(\frac{3}{2} - \sqrt[n]{n}\right)^n\right|^{1/n} = \lim_{n\to\infty} \left(\frac{3}{2} - \sqrt[n]{n}\right) = \frac{3}{2} - 1 = \frac{1}{2} < 1.$$

Therefore, the given series converges by the Root Test.

(f) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Use the Ratio Test. That is,

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| \\
&= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} (n+1) \cdot \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\
&= e^{-1} \\
&< 1.
\end{aligned}$$

Therefore, the given series converges by the Ratio Test.

(g) $\sum_{n=1}^{\infty} (n^{1/n}-1)^n$

Use the Root Test. That is,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |(n^{1/n}-1)^n|^{1/n} = \lim_{n\to\infty} (n^{1/n}-1) = 1-1 = 0 < 1.$$

Therefore, the given series converges by the Root Test.

(h) $\sum_{n=1}^{\infty} \left(\frac{4n}{1-3n}\right)^{5n}$

Use the Root Test. That is,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\left(\frac{4n}{1-3n}\right)^{5n}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{4n}{3n-1}\right)^5 = \left(\lim_{n\to\infty} \frac{4n}{3n-1}\right)^5 = \left(\frac{4}{3}\right)^5 = \frac{1024}{243} > 1.$$

Therefore, the given series diverges by the Root Test.

(i) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

Use the Ratio Test. That is,

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{[(n+1)!]^2}{[2(n+1)]!} \cdot \frac{(2n)!}{(n!)^2} \\ &= \lim_{n\to\infty} \frac{[(n+1)!]^2}{(n!)^2} \cdot \frac{(2n)!}{(2n+2)!} \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} \\ &= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} \\ &= \frac{1}{4} \\ &< 1. \end{aligned}$$

Therefore, the given series is convergent by the Ratio Test.

(j) $\sum_{n=0}^{\infty} \frac{4^n}{n!}$

Use the Ratio Test. That is,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot \frac{4^n}{4^{n+1}} = \lim_{n\to\infty} \frac{4}{n+1} = 0 < 1.$$

Therefore, the given series is convergent by the Ratio Test.

(k) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$

Use the Root Test. That is,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \left(1-\frac{1}{n}\right)^n \right|^{1/n} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right) = 1.$$

Therefore, the Ratio Test is inconclusive. Using the Divergence tests yields

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \frac{1}{e} \neq 0$$

Therefore, the given series diverges by the Divergence Test.

(l) $\sum_{n=1}^{\infty} \left(\frac{2}{e^{-8n}-1}\right)^n$

Use the Root Test. That is,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \left(\frac{2}{e^{-8n}-1}\right)^n \right|^{1/n} = \lim_{n\to\infty} \left(\frac{2}{1-e^{-8n}}\right) = \frac{2}{1-0} = 2 > 1.$$

Therefore, the given series diverges by the Root Test.

(m) $\sum_{n=1}^{\infty} \frac{n!}{4\cdot 8\cdot 12\cdots(4n)}$

Use the Ratio Test. That is,

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)!}{4\cdot 8\cdot 12\cdots(4(n+1))} \cdot \frac{4\cdot 8\cdot 12\cdots(4n)}{n!} \\ &= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{4\cdot 8\cdot 12\cdots(4n+4)}{4\cdot 8\cdot 12\cdots(4n)} \\ &= \lim_{n\to\infty} \frac{n+1}{4n+4} \\ &= \frac{1}{4} \\ &< 1. \end{aligned}$$

Therefore, the given series converges by the Ratio Test.

3. Use the power series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x|<1$, to find a power series $\sum_{n=0}^{\infty} a_n x^n$ for each of the following functions. Identify the interval of convergence.

(a) $f(x) = \frac{8}{4x+7}$

$$\begin{aligned}
f(x) &= \frac{8}{4x+7} \\
&= 8 \cdot \frac{1}{7+4x} \\
&= \frac{8}{7} \cdot \frac{1}{1+\frac{4x}{7}} \\
&= \frac{8}{7} \cdot \frac{1}{1-\left(-\frac{4x}{7}\right)} \\
&= \frac{8}{7} \sum_{n=0}^{\infty} \left(-\frac{4x}{7}\right)^n \\
&= \frac{8}{7} \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{7^n} x^n \\
&= 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{4}{7}\right)^{n+1} x^n.
\end{aligned}$$

This series will converge for all $\left|-\frac{4x}{7}\right| < 1$. That is, $-1 < \frac{4x}{7} < 1$ or $-\frac{7}{4} < x < \frac{7}{4}$.

(b) $f(x) = \frac{-4x^3}{(1+x^4)^2}$

Since the denominator of the given series is squared, we must differentiate the base power series. That is,

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\left[\frac{1}{1-x}\right]' &= \left[\sum_{n=0}^{\infty} x^n\right]' \\
\frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} n x^{n-1} \\
\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} (n+1) x^n \\
\frac{1}{(1+x^4)^2} &= \sum_{n=0}^{\infty} (n+1)(-x^4)^n \\
\frac{1}{(1+x^4)^2} &= \sum_{n=0}^{\infty} (-1)^n (n+1) x^{4n} \\
\frac{-4x^3}{(1+x^4)^2} &= -4x^3 \sum_{n=0}^{\infty} (-1)^n (n+1) x^{4n} \\
\frac{-4x^3}{(1+x^4)^2} &= 4 \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^{4n+3}.
\end{aligned}$$

The possible interval of convergence is $|-x^4| < 1$. That is, $|x| < 1$ or $-1 < x < 1$. Checking the endpoints yields:

i. $\underline{x = -1}$: $4\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)(-1)^{4n+3} = 4\sum_{n=0}^{\infty}(-1)^{5n+4}(n+1)$, which diverges by the Divergence Test.

ii. $\underline{x = 1}$: $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)(1)^{4n+3} = \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)$, which diverges by the Divergence Test.

Therefore, the IOC $= (-1, 1)$.

(c) $f(x) = \frac{2}{(1-x)^3}$

Since the denominator of the given series is cubed, we must differentiate the base power series twice. That is,

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\left[\frac{1}{1-x}\right]' &= \left[\sum_{n=0}^{\infty} x^n\right]' \\
\frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^{n-1} \\
\frac{2}{(1-x)^3} &= \sum_{n=2}^{\infty} n(n-1)x^{n-2} \\
\frac{2}{(1-x)^3} &= \sum_{n=0}^{\infty}(n+1)(n+2)x^n.
\end{aligned}$$

The possible interval of convergence is $|x| < 1$, or $-1 < x < 1$. Checking the endpoints yields:

i. $\underline{x = -1}$: $\sum_{n=0}^{\infty}(n+1)(n+2)(-1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)$, which diverges by the Divergence Test.

ii. $\underline{x = 1}$: $\sum_{n=0}^{\infty}(n+1)(n+2)(1)^n = \sum_{n=0}^{\infty}(n+1)(n+2)$, which diverges by the Divergence Test.

Therefore, the IOC $= (-1, 1)$.

(d) $f(x) = \ln(5x+1)$

Note first that $f'(x) = \frac{1}{5x+1} \cdot 5 = \frac{5}{1+5x}$. Finding the power series of $f'(x)$ yields:

$$\begin{aligned}
\frac{5}{1+5x} &= 5 \cdot \frac{1}{1-(-5x)} \\
\frac{5}{1+5x} &= 5\sum_{n=0}^{\infty}(-5x)^n \\
\frac{5}{1+5x} &= 5\sum_{n=0}^{\infty}(-1)^n 5^n x^n \\
\frac{5}{1+5x} &= \sum_{n=0}^{\infty}(-1)^n 5^{n+1} x^n \\
\int \frac{5}{1+5x}dx &= \int \sum_{n=0}^{\infty}(-1)^n 5^{n+1} x^n dx \\
\ln(5x+1) &= \sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}}{n+1}x^{n+1} + C.
\end{aligned}$$

Substituting $x = 0$ yields:

$$\ln(1) = 0 = \sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}}{n+1}(0)^{n+1} + C.$$

Hence, $C = 0$. Therefore, the power series to the given function is

$$\sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}}{n+1}x^{n+1}.$$

We are guaranteed convergence for $|-5x| < 1$, or $-1 < 5x < 1$, or $-\frac{1}{5} < x < \frac{1}{5}$. Checking the end points yields:

i. $\underline{x = -\frac{1}{5}}$: $\sum\limits_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}}{n+1}\left(-\frac{1}{5}\right)^{n+1} = \sum\limits_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}}{n+1}(-1)^{n+1}\frac{1}{5^{n+1}} = \sum\limits_{n=0}^{\infty} -\frac{1}{n+1}$, which diverges by the Limit Comparison Test with $\sum t_n = \sum\limits_{n=1}^{\infty}\frac{1}{n}$, which is a divergent p-series.

ii. $\underline{x = \frac{1}{5}}$: $\sum\limits_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}}{n+1}\left(\frac{1}{5}\right)^{n+1} = \sum\limits_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}}{n+1}\cdot\frac{1}{5^{n+1}} = \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{n+1}$, which converges by the Alternating Series Test.

Therefore, the IOC $= \left(-\frac{1}{5}, \frac{1}{5}\right]$.

(e) $f(x) = \frac{2}{3-x}$

$$\begin{aligned}
f(x) &= \frac{2}{3-x} \\
&= 2 \cdot \frac{1}{3-x} \\
&= \frac{2}{3} \cdot \frac{1}{1-\frac{x}{3}} \\
&= \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n \\
&= \frac{2}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} x^n \\
&= 2 \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n.
\end{aligned}$$

This series will converge for all $\left|\frac{x}{3}\right| < 1$. That is, $-1 < \frac{x}{3} < 1$ or $-3 < x < 3$.

(f) $f(x) = \frac{x^2}{2x-4}$

$$\begin{aligned}
f(x) &= \frac{x^2}{2x-4} \\
&= x^2 \cdot \frac{1}{-4+2x} \\
&= \frac{x^2}{-4} \cdot \frac{1}{1-\frac{x}{2}} \\
&= -\frac{x^2}{4} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \\
&= -\frac{x^2}{2^2} \sum_{n=0}^{\infty} \frac{1}{2^n} x^n \\
&= -\sum_{n=0}^{\infty} \frac{1}{2^{n+2}} x^{n+2}.
\end{aligned}$$

This series will converge for all $\left|\frac{x}{2}\right| < 1$. That is, $-1 < \frac{x}{2} < 1$ or $-2 < x < 2$.