MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2 | **Mathematics 2000** | FALL 2004

SOLUTIONS

1. Let $f(x) = \frac{x}{2x^2+1}$. Clearly, $f(x)$ is continuous and positive for $n \geq 1$. To see that it is decreasing, note that

$$f'(x) = \frac{(2x^2+1) - x(4x)}{(2x^2+1)^2} = \frac{1-2x^2}{(2x^2+1)^2} < 0 \quad \text{for } x \geq 1.$$

So we can use the Integral Test. To carry out the integration, we use u-substitution with $u = 2x^2+1$ so $du = 4x\,dx$ and $\frac{1}{4}\,du = x\,dx$ (so that when $x = 1$, $u = 3$ and when $x = t$, $u = 2t^2+1$). We get

$$\begin{aligned} \int_1^\infty f(x)\,dx &= \lim_{t\to\infty} \int_1^t \frac{x}{2x^2+1}\,dx = \lim_{t\to\infty}\left[\frac{1}{4}\int_3^{2t^2+1}\frac{du}{u}\right] \\ &= \lim_{t\to\infty}\left[\frac{1}{4}\ln|u|\right]_3^{2t^2+1} = \lim_{t\to\infty}\left[\frac{1}{4}\ln(2t^2+1) - \frac{1}{4}\ln(3)\right] = \infty. \end{aligned}$$

2. (a) We use the Ratio Test with

$$a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} \quad \text{so} \quad a_{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{(n+1)!}.$$

Then

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{(n+1)!}\cdot\frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)} \\ &= \lim_{n\to\infty}\frac{2n+1}{n+1} = 2 = L. \end{aligned}$$

Since $L > 1$, the given series is divergent.

(b) Note that

$$\lim_{n\to\infty}\frac{n-7}{5n+3} = \frac{1}{5},$$

so the given series diverges by the Divergence Criterion.

(c) We use the Direct Comparison Test with the convergent geometric series $\displaystyle\sum_{n=0}^\infty\left(\frac{4}{5}\right)^n = \sum_{n=0}^\infty \frac{4^n}{5^n}$. Observe that

$$\begin{aligned} 3^n + 5^n &\geq 5^n \\ \frac{1}{3^n+5^n} &\leq \frac{1}{5^n} \\ \frac{4^n}{3^n+5^n} &\leq \frac{4^n}{5^n} \end{aligned}$$

(d) We use the Root Test, letting $a_n = (-1)^{n+1}\dfrac{n}{[\arctan(n)]^n}$. Then

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|(-1)^{n+1}\frac{n}{[\arctan(n)]^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{\arctan(n)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} = L < 1.$$

So the given series converges.

(e) We use the Limit Comparison Test with the (divergent) harmonic series. Then

$$\lim_{n\to\infty} \frac{\frac{3n-1}{\sqrt{n^4+n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^2-n}{\sqrt{n^4+n}} = \frac{3-\frac{1}{n}}{\sqrt{1+\frac{1}{n^3}}} = 3.$$

So the given series diverges as well.

3. First we consider the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$. To see if it converges, try the Limit Comparison Test with the divergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Then we have

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

so the absolute series is also divergent; hence the given series cannot be absolutely convergent. So now we must check the convergence of the given series; we use the Alternating Series Test. Observe that

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1+\frac{1}{n}} = 0.$$

Also, let $f(x) = \frac{\sqrt{x}}{x+1}$ so then

$$f'(x) = \frac{1-x}{2\sqrt{x}(x+1)^2} \leq 0 \quad \text{for } x \geq 1,$$

so $\left\{\frac{\sqrt{n}}{n+1}\right\}$ is decreasing. Hence, by the Alternating Series Test, the given series is convergent, and so it is conditionally convergent.

4. To determine the radius of convergence, we use the Ratio Test. Let

$$c_n = \frac{1}{(n+3)6^n} \quad \text{so} \quad c_{n+1} = \frac{1}{(n+4)6^{n+1}}.$$

Then

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+4)6^{n+1}} \cdot (n+3)6^n\right| = \lim_{n\to\infty} \frac{n+3}{6(n+4)} = \frac{1}{6} = \rho.$$

Thus the radius of convergence is $R = \frac{1}{\rho} = 6$ and the series is convergent for $|x-2| < 6$, that is, for $-6 < x-2 < 6$ or $-4 < x < 8$. We must check the endpoints $x = -4$ and $x = 8$. At $x = -4$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-6)^n}{(n+3)6^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3},$$