PRACTICAL LINEAR AND EXPONENTIAL FREQUENCY MODULATION FOR DIGITAL MUSIC SYNTHESIS

Kasper Nielsen

KarmaFX
Denmark
karmafx@karmafx.net

ABSTRACT

This paper explores Frequency Modulation (FM) for use in music synthesis. We take an in-depth look at Linear FM, Linear-Through-Zero FM, Phase Modulation (PM) and Exponential FM, and discuss their pros and cons for sound synthesis in a digital system. In the process we derive some useful formulas and discuss their implementation details. In particular we derive analytic expressions for DC correcting Exponential FM, and make it match the modulation depth of Linear FM. Finally, we review practical antialiasing solutions.

1. INTRODUCTION

FM synthesis was first introduced by John Chowning in his famous paper from 1973 [1], where he showed how natural sounding sounds could be synthesized by dynamically controlled Frequency Modulation. The technique was licensed by Yamaha for use in their DX7 synthesizer. Before that, FM was known from its use in radio transmission, originally designed as a replacement for Amplitude Modulation (AM) radio, since it is less susceptible to noise.

Chowning showed that a simple setup of a carrier waveform, modulated by a modulator waveform, could produce musical sounds by controlling the level of modulation using a Modulation Index *I*.

Expressed using just sinusoids, Chowning wrote the instantaneous FM modulated time-domain signal as:

$$y(t) = \sin(\omega_c t + I\sin(\omega_m t)) \tag{1}$$

where ω_c is the carrier frequency and ω_m is the modulator frequency [1]. Here, ω denotes the *angular frequency*, which for a linear frequency f is defined as $\omega=2\pi f$.

When I is zero, there is no modulation, but when I is increased, the modulation introduces sidebands in the resulting sound spectrum at frequency intervals of the modulating frequency [1]. See Figure 1.

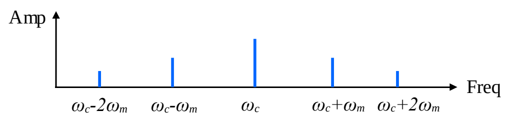

Figure 1: *FM spectrum*

Chowning's insight was that by controlling the Modulation Index and the ratio between the carrier and the modulating frequency, could add and remove sound harmonics, thereby controlling timbre.

For synthesizing harmonic spectra, the frequency ratio r should be rational [1]:

$$r = \frac{\omega_c}{\omega_m} = \frac{N_1}{N_2} \tag{2}$$

That is, N_1 and N_2 should be integers. Furthermore when disregarding common factors, the fundamental frequency of the resulting sound will be [1]:

$$\omega_0 = \frac{\omega_c}{N_1} = \frac{\omega_m}{N_2} \tag{3}$$

The sidebands of an FM signal and its associated magnitude coefficients can be determined using *Bessel functions* of the first kind, J_n [1]:

$$y(t) = \sum_{n=-\infty}^{\infty} J_n(I)\sin((\omega_c + n\omega_m)t) \tag{4}$$

Now, let us take a step back, and look at Chownings formula in equation (1). So the main frequency ω_c of the carrier signal is kept static and the modulating signal added as a phase offset: $I\sin(\omega_m t)$. So he is in fact modulating the phase. So why call it Frequency Modulation? Simply due to the fact that Frequency and Phase Modulation are closely related. Since frequency is the derivative of phase, modulating the signals phase will also modulate its frequency. FM and PM are therefore used interchangeably and sometimes simply referred to as *angle modulations* [2]. However, there are some key differences that we will look at in the following.

2. LINEAR FREQUENCY MODULATION

In principle, linear FM is simple: For a given carrier frequency, we allow the frequency to sweep up and down by a frequency deviation *Δω*. The important distinction being that it linearly sweeps up the same amount of Hz as it sweeps down. Mathematically we can write the modulated frequency as:

$$\omega_{lin}(t) = \omega_c + \Delta\omega \sin(\omega_m t) \tag{5}$$

and the resulting time-domain signal as:

$$y(t) = \sin((\omega_c + \Delta\omega \sin(\omega_m t))t) \tag{6}$$

This is *direct* Frequency Modulation and not the same as equation (1), as phase is left untouched.

The Modulation Index is defined as [2][3]:

$$I = \frac{\Delta\omega}{\omega_m} \tag{7}$$

This means that given a Modulation Index, the frequency deviation is calculated as [3]:

$$\Delta\omega = I\omega_m \tag{8}$$

That is, the peak frequency deviation is a multiple *I* of the modulating frequency.

2.1. Linear FM DC and Tuning

The link between the Modulation Index and frequency deviation means that slowly changing the modulation index, can make the resulting sound momentarily drift out of tune [2].

For simplicity, we expressed the carrier and modulator using sinusoids. However, there is nothing stopping us from using other waveforms, such as more harmonically rich *Saw, Square and Triangle* waves. Yet, what happens if we use a modulating waveform that has a DC offset? In that case we will be adding this DC offset scaled by the frequency deviation *Δω* to the carrier frequency causing the resulting sound to go out of tune. Not a desirable behavior [2]. So for Linear FM, a DC offset in the modulating waveform should be avoided.

2.2. Linear-Through-Zero FM

An interesting aspect of Linear FM is what happens at 0 Hz. Since we are subtracting *Δω* from ω_c, the frequency can reach and even cross the 0 Hz line. We can choose to disallow this and simply stop frequencies at 0 Hz, or we can choose to allow it and let frequencies become negative. While both acceptable, the latter is more correct and known as *Linear-Through-Zero FM* (*LTZFM*) [4]. It is more correct because stopping at 0 Hz will in itself introduce a DC offset [2][3]. Also, the negative frequencies contribute to the sound. LTZFM simply sounds "sweeter" [4].

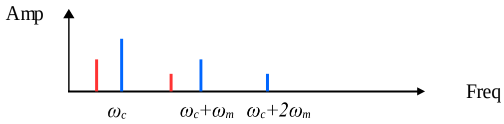

Figure 2: *Negative frequencies.*

So what happens below 0 Hz? Just like a wheel starting to spin backwards, negative frequencies generate sound but in reverse phase [3][4]: sin(-*x*) = -sin(*x*). That is, the negative frequency sidebands reflect back above 0 Hz, spilling into the positive frequencies with their phase inverted. See Figure 2.

3. PHASE MODULATION

Since frequency is the derivative of phase, modulating the phase will also modulate the frequency. Phase Modulation is therefore equivalent to Linear Frequency Modulation, and is in fact what Chowning describes in his paper [1]. Hence, it is also referred to as *Chowning-style FM* or *indirect FM*.

The frequency can be seen as the angular *velocity* of the phase angle in time. With FM, we change the frequency while keeping the phase constant. In PM, we change the phase while keeping the frequency constant. However, changing the phase also alters the instantaneous frequency, where the frequency and phase relationship can be written as [2]:

$$\omega(t) = \frac{d\theta}{dt} \tag{9}$$

$$\theta(t) = \int \omega(t)dt \tag{10}$$

That is, we get the frequency by differentiating the phase angle. And get the phase, by integrating the frequency.

Again, using just sinusoids, we can express the modulating phase function as:

$$\theta(t) = \Delta\theta \sin(\omega_m t) \tag{11}$$

where *Δθ* is the phase deviation. The resulting time-domain signal becomes:

$$y(t) = \sin(\omega_c t + \Delta\theta \sin(\omega_m t)) \tag{12}$$

The PM Modulation Index is simply defined as [2]:

$$I = \Delta\theta \tag{13}$$

Looking at the time-domain formula in equation (12), we see that DC in the modulating function will not change the tuning frequency of the carrier, but instead introduce a phase offset. Thus, PM is said to "fix" the DC tuning problem of Linear FM [2].

Since the phase modulation in itself can make the carrier signal run backwards, PM actually has built-in Linear-Through-Zero behaviour. Also, unlike direct Linear FM, altering the Modulation Index does not impose any momentary detuning [2].

Hence, for those reasons, PM is in general favoured over "naïve", direct Linear FM, and also why, when referring to FM, most papers and applications actually mean and apply PM.

Note that, due to the use of angular frequencies with trigonometric functions, the Modulation Index, for both FM and PM, may or may not have an implied scaling by 2π.

Aside from the above differences, Phase and frequency modulation give the same results. However, since frequency is the derivative of phase, there will always be a phase difference between FM and PM using the same modulation signal.

So we can safely say that Linear Phase Modulation *is* Linear Frequency Modulation. In fact, it is its more stable brother. Yet, the same cannot be said for Exponential FM.

4. EXPONENTIAL FM

Exponential FM is a form of frequency modulation where the modulation range follows the musical spacing of notes and octaves. For instance, given a frequency ω_c, going up one octave means doubling the frequency ($2\omega_c$), and going down one octave means halving the frequency ($\omega_c/2$).

Mathematically we can write the exponentially modulated frequency as [5][6]:

$$\omega_{\exp}(t) = \omega_c 2^{V\sin(\omega_m t)} \tag{14}$$

where V is the amplitude of the modulation. The sinusoidal time-domain signal then becomes:

$$y(t) = \sin(\omega_c 2^{V\sin(\omega_m t)} t) \tag{15}$$

Since trivial control over the tuning frequency is available in most analog synthesizers, Exponential FM is in a sense the simplest form of FM. At low rates, Exponential FM is simply *vibrato*.

With Linear FM/PM, the sidebands are equally spaced around the carrier frequency. However, with Exponential FM, the spacing of the sidebands is asymmetrical around the carrier, creating a different type of sound [5].

As we cannot halve a frequency to cross the 0 Hz line, exponential FM does not have Through-Zero behavior, but instead stops at 0 Hz.

Like with Linear FM, it would be advantageous to use phase modulation to get an Exponential FM response. However, in practice this is difficult. Getting the instantaneous phase function would require integration of the above frequency function, but this does not have a closed-form solution [7].

4.1. Exponential FM DC and Tuning

If we imagine a one octave modulation of a carrier frequency of 440 Hz, the peak maximum and minimum frequencies become 880 Hz and 220 Hz, which are not of equal distance to 440 Hz. So this implies that Exponential FM introduces a DC offset, which detunes the resulting sound [4].

Even modulation with a fully symmetric function, the "stretch" imposed by the exponential function will make the result non-symmetric, which means that the DC offset is dependent on the modulating waveform.

This may explain why musicians constantly have to retune a patch using Exponential FM [4]. Whenever the Modulation Index or the modulating waveform is altered, the patch will go out of tune.

To get an idea of what the DC offset is, we define an exponential factor k as the integral over one period of the modulating waveform:

$$k_{DC} = \int_{period} 2^{Vf(t)} \tag{16}$$

such that:

$$\omega_{\exp,noDC}(t) = \omega_c (2^{Vf(t)} - (k_{DC} - 1)) \tag{17}$$

Integration over *Sine* does not give a closed-form solution. Instead the exponential must be expanded into:

$$2^{V\sin(\omega t)} = I_0(V\ln(2)) + 2\sum_{k=1}^{\infty} I_k(V\ln(2))\cos(k\omega t) \tag{18}$$

Where I_n are the modified Bessel functions, equivalent to the regular Bessel functions of the first kind evaluated for purely imaginary arguments: $I_n(z)=(-i^n)J_n(iz)$ [6]. Since we are interested in the DC offset, we disregard the infinite sum, and write:

$$k_{DC|Sine} = \int_0^1 I_0(V\ln(2))dx = I_0(V\ln(2)) \tag{19}$$

The Modified Bessel function I_0 is not trivial to evaluate. However, good approximations are described in [8] and [9]. Integrating for ideal *Saw*, *Square and Triangle* waveforms yield nice closed-form analytic expressions:

$$k_{DC|Saw} = \int_0^1 2^{V\cdot Saw(x)} dx = \frac{\sinh(V\ln(2))}{V\ln(2)} \tag{20}$$

$$k_{DC|Square} = \int_0^1 2^{V\cdot Square(x)} dx = \cosh(V\ln(2)) \tag{21}$$

$$k_{DC|Triangle} = \int_0^1 2^{V\cdot Triangle(x)} dx = \frac{\sinh(V\ln(2))}{V\ln(2)} \tag{22}$$

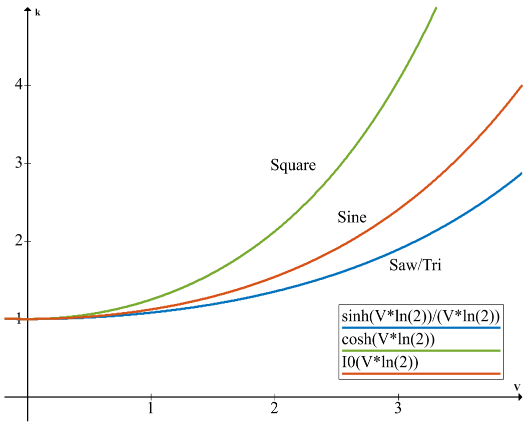

Figure 3: *Analytic DC corrections.*

We can use these expressions to compensate for the DC offset for (bipolar) Exponential FM with ideal modulating waveforms. See Figure 3.

To verify, we can, e.g., set V=4 and integrate over one period of a corrected modulating sinewave:

$$\int_0^1 \left(2^{4\sin(2\pi x)} - I_0(4\ln(2))\right)dx = 0 \tag{23}$$

which means that our average $\omega_{\exp,noDC}$ will equal ω_c. The integral over this corrected function is graphed in Figure 4.

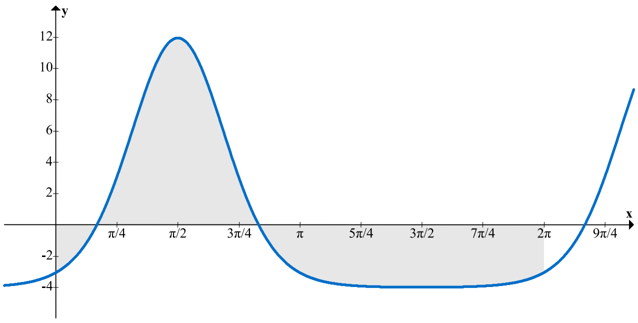

Figure 4: *DC corrected modulation.*

Care should be taken, as this correction may bring the instantaneous frequency below 0 Hz, thus again requiring a Through-Zero carrier oscillator.

In addition, since this correction is equivalent to altering the frequency by a ω_c scaled carrier offset, this skews the ratio, as ω_m is kept constant, resulting in a different sound.

4.2. Re-Tuning Exponential FM

Let us take a look at a practical example to see what this actually means. So given a sinusoidal oscillator playing the musical note C-3, we apply Exponential FM using a sinusoidal modulator at the same frequency, thus r=1, and say V=3.

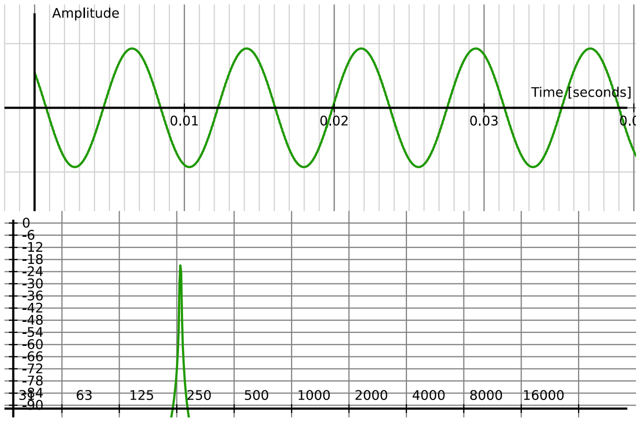

Figure 5: *Sinusoidal waveform and spectrum, without FM.*

For reference, we first observe the non-modulated waveform and spectrum shown in Figure 5. The fundamental frequency for the C-3 sinusoidal is shown in the spectrum, at exactly 130.81 Hz.

We then apply “naïve” exponential FM without any correction. See Figure 6. The result is an erratic waveform that sounds harsh, inharmonic and out-of-tune. From the spectrum we observe that a new fundamental frequency is introduced (~56 Hz), that has no harmonic relation to the fundamental.

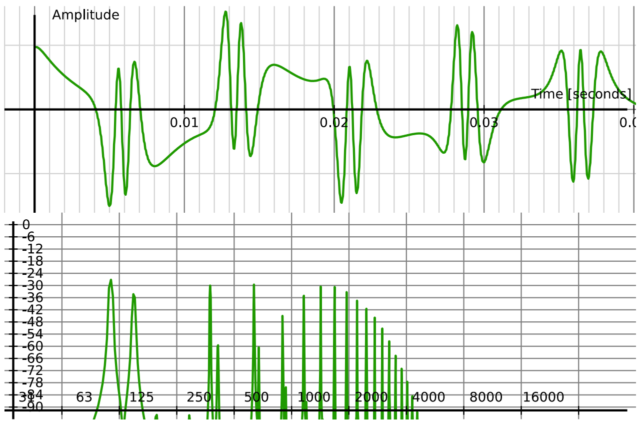

Figure 6: *Exponential FM without DC correction.*

Finally, we introduce the analytic DC correction. See Figure 7. The waveform becomes well-behaved, harmonic, and from the spectrum we observe that it is in-tune, since the fundamental frequency is kept intact, while the FM modulation only produces overtones.

Again, it is important to point out that this result is achieved using an oscillator capable of Through-Zero. If we apply this correction using an oscillator incapable of Through-Zero, we instead get the result shown in Figure 8, which in this case is just as inharmonic and out-of-tune as without correction.

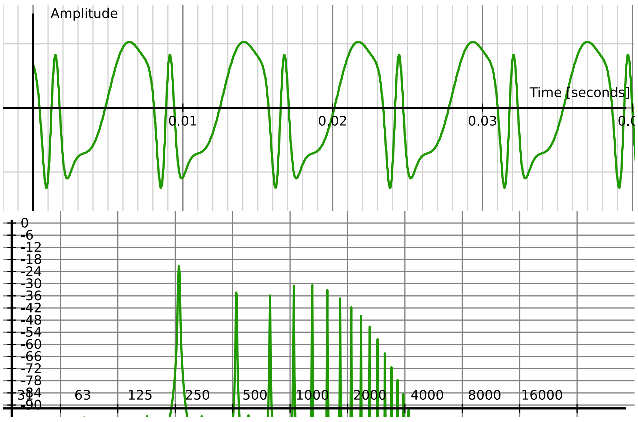

Figure 7: *Exponential FM with analytic DC correction.*

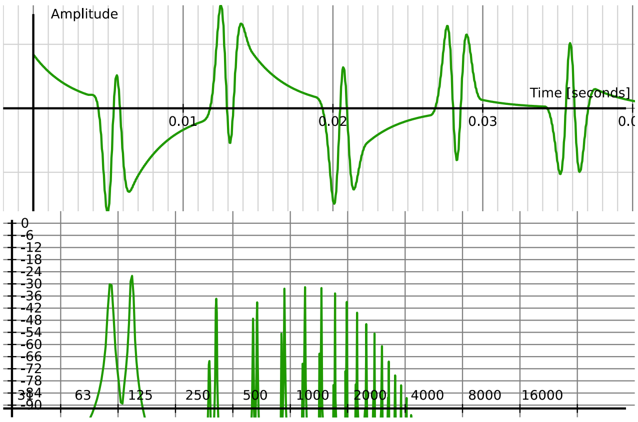

Figure 8: *Exponential FM, with DC correction but without Through-Zero.*

So are we not able to produce harmonic Exponential FM without Through-Zero? What happens if we instead alter the modulation frequency ω_m?

According to Hutchins et al. [6] the condition for a harmonic spectrum with an Exponential FM pure sinusoidal oscillator is:

$$\frac{\omega_c}{\omega_m} = \frac{1}{I_0(V\ln(2))}\frac{N_1}{N_2} \tag{24}$$

This forces one of the sidebands to fall on zero frequency, so “the carrier, all normal sidebands, and any significant reflected sidebands will fall on positions that are multiples of a common fundamental” [6].

So this means that, unlike with Linear FM, the harmonic ratio may be non-rational, and we can express the harmonic modulation frequency as:

$$\omega_m = \frac{I_0(V\ln(2))}{r}\omega_c \tag{25}$$

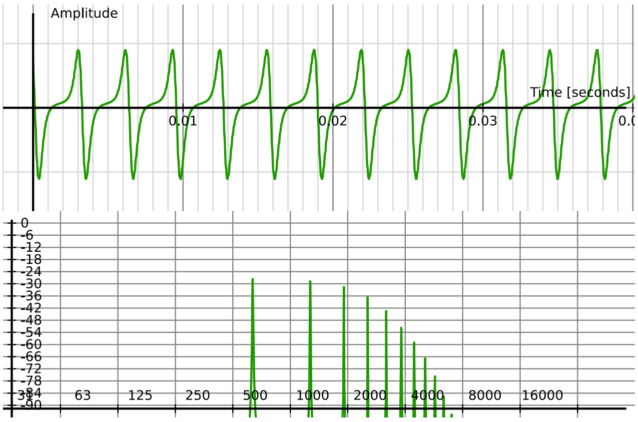

Figure 9: *Exponential FM with corrected modulation frequency.*

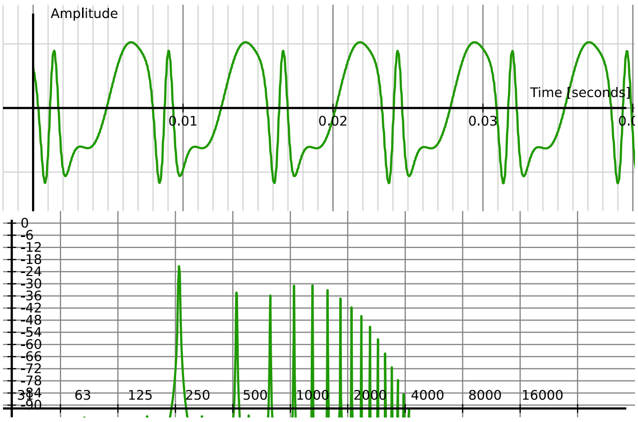

Figure 10: *Exponential FM with tracked DC correction.*

The numerator looks exactly like our DC offset. Let us explore this further. So we again perform “naïve” Exponential FM, except we now alter the modulation frequency according to equation (25). The result is shown in Figure 9. While the output is now harmonic and sounds good, it is not in-tune, as the fundamental has shifted to ~320 Hz. A shift that depends on V. Yet, given an oscillator incapable of Through-Zero, this correction does produce a harmonic sound. So if we know the modulating waveform, we can get the modulation frequency that will produce a harmonic spectrum by:

$$\omega_m = \frac{k_{DC}}{r}\omega_c \tag{26}$$

What if the modulating waveform is unknown? An alternative is to apply a DC filter to the frequency control signal, and use the estimated DC offset for correction. However, depending on the cutoff point, the filter will not work instantly, resulting in an inharmonic glide-effect when changing Modulation Index or switching notes. For completeness, Figure 10 shows the resulting signal when using such a tracked DC offset as our k_{DC}. The output matches our analytic results (Figure 7) when in a steady-state. A DC filter can also be applied in conjunction with an analytic correction in cases where the modulating waveform diverts from the analytic inputs. Finally, the integral over a known but non-ideal waveform can be precalculated into a look-up table, and used for correction when the analytic formulas do not apply.

4.3. Exponential Modulation Index

So far we have referred to the modulation amplitude as *V*. While *V* is linked to the Modulation Index, we have yet to establish this link. In [6] & [7] *V* is simply referred to as the *Modulation Depth*, while it is pointed out that the carrier frequency increases rapidly when *V* reaches values of 8 or more [7]. This is problem for digital synthesis, as high frequencies are prone to alias in a digital system.

Timoney et al.[7] suggests a “useful formula for low-aliasing digital implementations of Exponential FM” that estimates the frequency where sidebands fall below a threshold (See Section 5.1). The expression is limited to sinusoidal modulation.

To make Exponential FM easier to control, we will instead derive an expression for *V* in relation to the Linear Modulation Index. To keep *V* in check, we set the size of the full exponential frequency range equal to the Linear FM one and solve for *V*:

$$\omega_c\left(2^V - 2^{-V}\right) = 2I\omega_m \tag{27}$$

⇕

$$2\sinh(V\ln(2)) = \frac{2I\omega_m}{\omega_c} \tag{28}$$

⇕

$$V = \frac{\operatorname{asinh}\left(\frac{I}{r}\right)}{\ln(2)} \tag{29}$$

Given a Modulation Index, this makes Exponential FM span the same size frequency range as Linear FM, thus keeping *V* within a more sensible range. See Figure 11. This also makes it easier to compare the two, both visually and sound-wise. The Linear FM vs. the Exponential FM spectrum is shown in Figure 12.

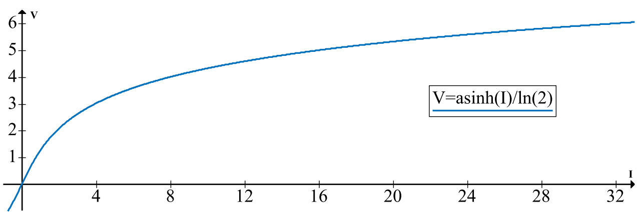

Figure 11: *Modulation Depth V vs. Modulation Index I, for ratio r*=1.

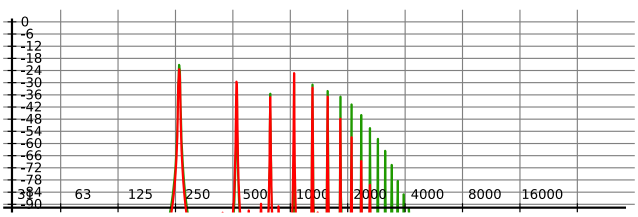

Figure 12: *Exponential FM*(*green*) *vs. Linear FM spectrum*(*red*).

4.4. Operators & Feedback FM

In classic FM, the term *operator* describes a waveform generator, who's amplitude is controlled by an envelope, and can function as either a carrier or a modulator [1][2]. Each operator can be routed to control the frequency of another operator. In this way it is possible to do complex FM routing, referred to as *algorithms*, where operators are connected in series or in parallel, to form complex evolving sounds. In part, this is what made the Yamaha DX7 so famous.

Now, this design also made it possible to feed the output from a chain of operators back into the input frequency of a chained operator, creating variable feedback by a factor *β*. See Figure 13.

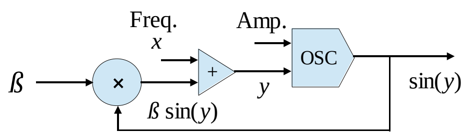

Figure 13*: Feedback FM.*

Feedback can breathe life into a static sound, but historically it also solved a problem. Normally, when the Modulation Index changes, the amplitude of the sidebands change unevenly as dictated by the Bessel functions. This results in a characteristic unnatural or “electronic” sound [5]. By introducing feedback, this unevenness is made more linear, which makes changes to the Modulation Index sound more natural. Roads et al. [5] describes this mathematically by:

$$y(t) = \sum_{n=1}^{\infty} \frac{2}{n\beta} J_n(n\beta)\sin((\omega_c + n\omega_m)t) \tag{30}$$

This reveals a scaling by $2/n\beta$, but also shows that the inputs to the Bessel functions are now scaled by their order *n*, which accounts for the even spread of the sidebands across a wider spectrum compared to non-feedback FM [5].

An inherent problem with feedback is that it tends to be unstable. Feeding a Sine back into its frequency might be okay. The sinewave actually warps into a sawtooth. But feeding a Saw into itself, will quickly result in noise. One solution is to lowpass filter the feedback, thus taming the high frequencies that cause the noise. Although the DX7 was Sine only, it had built-in lowpass feedback filtering.

5. ANTIALIASING

When implementing FM in a digital system we have to address digital *aliasing*, i.e., the fact that we cannot represent frequencies higher than the *Nyquist* frequency. Since one of the main points of FM synthesis is to generate sidebands, these can cross half the sampling rate, where they wrap around and reappear as distortion, or aliasing.

Interestingly, the corrections suggested in section 4 for Exponential FM, reduce the frequency range, and thus the aliasing risk. However, for both Linear FM, PM & Exponential FM, aliasing remains a concern, especially for non-sinusoidal carriers.

5.1. Modulation Index Limiting

The extent of the sidebands is known as the *bandwidth*. Chowning uses *Carson's* rule for the estimated bandwidth of sinusoidal Linear FM [1]:

$$B_{FM} \approx 2(\Delta f + f_m) = 2 f_m (I + 1) \tag{31}$$

Where Δf is the frequency deviation (in Hz), and f_m is the maximum modulation frequency. If we set $f_c+f_m(I_{max}+1)=F_S/2$, we can compute an approximate maximum Modulation Index:

$$I_{\max} = \frac{F_S/2 - f_c}{f_m} - 1 \tag{32}$$

Where F_S is the sampling-rate and f_c is the carrier frequency. Thus, keeping the Modulation Index below this limit will not cause significant aliasing for sinusoidal FM. For non-sinusoidal carrier signals, f_c should be replaced by their highest overtone frequency. We can use I_{max} to design a simple expression for a low-aliasing Modulation Index I':

$$I' = clamp\langle I, 0, I_{\max} \rangle \tag{33}$$

An adapted bandwidth rule may be chosen for Exponential FM, estimating where sidebands fall below -80 dB peak value [7]:

$$B_{EFM} \approx f_c e^{V_0 \ln(2)} 2^{V_m - 1} 2.771 + f_m (V_m + 4.303) \tag{34}$$

Where V_0 is the DC term and V_m the time-varying amplitude of the modulation signal [7]:

$$V(t) = V_0 + V_m f(\omega_m t) \tag{35}$$

5.2. Oversampling

Another way to reduce aliasing is to *oversample*, i.e., perform frequency modulation at an N times higher sample-rate, and then downsample the result, which involves lowpass filtering and decimation [10].

Ideally we want a lowpass filter that suppresses frequencies above Nyquist at the target sample-rate, leaves the pass-band flat, and has a short transition-band. One way to design such a filter is through a linear-phase FIR, whose output is given by the convolution:

$$y[n] = \sum_{k=0}^{M} h[k] x[n-k] \tag{36}$$

where $h[k]$ are the M+1 filter coefficients and $x[n]$ is the sampled input signal [10]. The symmetric coefficients for a Windowed-Sinc filter are:

$$h[i] = \frac{\sin(2\pi f_c (i - M/2))}{i - M/2} w[i] \tag{37}$$

where f_c is the normalized cutoff frequency, and $w[i]$ is a window weighting function, such as *Hamming* or *Blackman* [10]. For $i=M/2$, $h[i]=2\pi f_c w[i]$. The final filter kernel h should be normalized for unity gain.

The number of coefficients is typically limited, so it is important to consider the normalized bandwidth of the transition-band, which approximately is $BW=4/M$ [10]. So for an M of say 64 coefficients, at 16x oversampling, the transition-band becomes twice the size of the pass-band. Even if we adjust the cutoff frequency by $-BW/2$, the roll-off is still too slow and we get just 20 dB attenuation at Nyquist.

One option is of course to use more coefficients, but that makes the filter more costly. That aside, doing filtering before decimation is inherently ineffective, as we throw away N-1 samples that were just computed. A different approach is to decimate before filtering. This is made possible with a class of filters called *polyphase filters* [11].

For downsampling by a factor N, a polyphase filter consists of N parallel chains of allpass filters, that each have a different phase-response (Hence the term “polyphase”). The output is the sum of these allpass filters.

While the filters can be realized for any factor using both FIR and IIR prototype filters, we will focus on a polyphase IIR decimation by a factor of 2, as these are relatively cheap and can be cascaded to form any power of two decimation. The digital z-domain transfer function for such a polyphase *halfband* filter is [11][12]:

$$H(z) = \frac{1}{2}\left(\prod_{k=0}^{K/2-1} \frac{a_{2k+1} + z^{-2}}{1 + a_{2k+1} z^{-2}} + z^{-1} \prod_{k=0}^{(K-1)/2} \frac{a_{2k} + z^{-2}}{1 + a_{2k} z^{-2}} \right) \tag{38}$$

where n is the order and a the $K=(n-1)/2$ allpass coefficients. This formula tells us that we have two chains of cascaded 1st order allpass filters than process the input data interleaved. Hence the reason for using z^{-2} and the extra sample delay z^{-1}. For a specified transition bandwidth (ω_t) and stop-band attenuation (d_s), the calculation of n and the coefficients is shown in, e.g., [11] and [12].

For 16x oversampling, we can cascade $log_2(16)=4$ polyphase 2x decimation filters [12] and compute the normalized transition bandwidth f_t for each section backwards by $f_t[stage]=(f_t[stage-1]+0.5)/2$ [12]:

$$\begin{aligned} &2x \to 1x: && f_t[3] = 0.01 \\ &4x \to 2x: && f_t[2] = (0.01 + 0.5)/2 = 0.255 \\ &8x \to 4x: && f_t[1] = (0.255 + 0.5)/2 = 0.3775 \\ &16x \to 8x: && f_t[0] = (0.3775 + 0.5)/2 = 0.43875 \end{aligned} \tag{39}$$

While IIR filters have fewer coefficients, and thus lower runtime than FIR, they have a non-linear phase response and are more susceptible to quantization noise. Since the transition-band requirements are less critical for the first few sections, de Soras suggests using FIR filtering there and only polyphase for the last pass (2x→1x) [12].

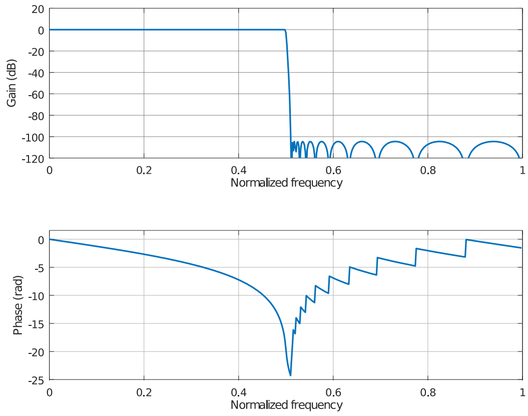

Figure 14: *Polyphase Halfband Filter Frequency/Phase Response.*

The Frequency and Phase response of a 2x→1x polyphase filter of order 25 with normalized transition-band of 0.01 and a stop-band attenuation of 96 dB is shown in Figure 14. We observe that it has the wanted cutoff, attenuation and fast transition, although with a non-linear phase response.

6. CONCLUSION

We have looked at FM synthesis in its many different forms. We have described the differences between Linear FM, Linear-Through-Zero FM, PM and Exponential FM and listed some practical formulas.

Most notably we have reviewed different techniques for correcting the DC offset and detuning of Exponential FM, and derived analytic expressions that produce harmonic and in-tune results: Equations (19, 20, 21, 22). If the modulation function is known, this makes it possible to synthesize the classic sound of Exponential FM, without the need for manual retuning, making it easier to implement and use in a digital system.

In addition we have derived an expression for the Exponential modulation depth *V*, equation (29), that makes it match the size of Linear FM, making it both easier to control and compare.

The differences between Linear FM and Phase Modulation have also been discussed, and why Through-Zero oscillators are important for both Linear and Exponential FM. We have also briefly touched on Feedback FM and its uses. Finally, we have addressed the aliasing problem, and reviewed ways to overcome it when implementing FM in a digital system.

Many sound synthesis textbooks only touch on Chowning-style FM. However, once we dig deeper it becomes clear that Frequency Modulation is a much broader subject, with lots of different variations that still makes this complex synthesis technique interesting, many years after its discovery. This paper has shed some light on these variations, their individual pros and cons, and practical ways to implement them in a digital system.

7. REFERENCES

[1] J.M. Chowning, “The Synthesis of Complex Audio Spectra by Means of Frequency Modulation”, in *J. Audio Eng. Soc.* 21, 7, 1973. https://ccrma.stanford.edu/sites/default/files/user/jc/fm_synthesispaper-2.pdf

[2] W. Pirkle, *Designing Software Synthesizer Plug-Ins in C++*, Focal Press, October 2014.

[3] J. Chowning and D. Bristow, “FM Theory and Applications: By Musicians for Musicians”, Yamaha, October 1, 1987.

[4] C. Meyer, "Understanding the Differences Between Exponential, Linear, and Through Zero FM", in *Learning Modular.com* Blog post, March 13, 2018. https://learningmodular.com/understanding-the-differences-between-exponential-linear-and-through-zero-fm/

[5] C. Roads and J. Strawn, *The Computer Music Tutorial*, MIT Press, 1996.

[6] Jr. Hutchins and A. Bernard, "The Frequency Modulation Spectrum of an Exponential Voltage-Controlled Oscillator", in *JAES* Volume 23 Issue 3 pp. 200-206; 1975. http://electronotes.netfirms.com/AES3.PDF

[7] J. Timoney and V. Lazzarini, "Exponential Frequency Modulation Bandwidth Criterion for Virtual Analog Applications", in *Proc. of the 14th Int. Conference on Digital Audio Effects* (*DAFx-11*), September 19-23, Paris, 2011. http://recherche.ircam.fr/pub/dafx11/Papers/60_e.pdf

[8] W.H. Press et al., *Numerical Recipes in C, The Art of Scientific Computing*, Cambridge University Press, 2 edition, 1992.

[9] J. Olivares et al., "A simple approximation for the modified Bessel function of zero order I0(x)", in *Journal of Physics: Conference Series*, Volume 1043, IOP Publishing Ltd, 2018. https://iopscience.iop.org/article/10.1088/1742-6596/1043/1/012003/pdf

[10] S.W. Smith, "The Scientist and Engineer's Guide to Digital Signal Processing", California Technical Pub, 1997. http://dspguide.com

[11] R.A. Valenzuela and A.G. Constantinides, "Digital signal processing schemes for efficient interpolation and decimation", in *Circuits, Devices and Systems*, IEE Proceedings G 130(6):225-235,1984. https://www.researchgate.net/publication/224336631

[12] L. De Soras, *readme.txt* in *hiir: An oversampling and Hilbert transform library in C++*, 2005. http://ldesoras.free.fr/