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A comparison between two different methods for solving KdV–Burgers equation

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Abstract

This paper presents two methods for finding the soliton solutions to the nonlinear dispersive and dissipative KdV–Burgers equation. The first method is a numerical one, namely the finite differences with variable mesh. The stability of the numerical scheme is discussed. The second method is the semi-analytic Adomian decomposition method. Test example is given. A comparison between the two methods is carried out to illustrate the pertinent feature of the proposed algorithm.

1. Introduction

This paper is concerned with the initial-boundary value problem associated with the nonlinear dispersive and dissipative wave which was formulated by Korteweg, de Vries and Burgers in the form:

$$u_t + \mu_1 u u_x + \mu_2 u_{xx} + \mu_3 u_{xxx} = 0, \quad x \in R, \tag{1}$$

where μ_1, μ_2 and μ_3 are constant coefficients, with the initial and boundary conditions:

$$u(x,0) = f(x),$$
$$u(0,t) = g(t),$$

where $u = u(x,t)$ is sufficiently smooth function, and $f(x)$ is bounded. In addition we shall assume that the solution $u(x,t)$, along with its derivatives, tends to zero as $|x| \to \infty$.

It is well known that many physical phenomena can be described by the Korteweg-de Vries–Burgers (KdVB) equation. Typical examples are provided by the behaviour of long waves in shallow water and waves in plasmas. Eq. (1) can serve as a nonlinear wave model of a fluid in an elastic tube [9], of a liquid with small bubbles [8] and turbulence [13,14]. The coefficients μ_2 and μ_3 in Eq. (1) represent the damping and the dispersion coefficients, respectively. We note that Eq. (1) is nonintegrable.

Soliton solutions of the KdV equation are known since long time [22]. Many problems, however, involve not only dispersion but also dissipation, and these are not governed by the KdV equation.

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More complicated problems are the flow of liquids containing gas bubbles [8], and the propagation of waves in an elastic tube filled with a viscous fluid [9]. Other cases regarded the governing evolution equation can be shown to be the so-called Korteweg-de Vries–Burgers' (KdVB) equation. A number of theoretical issues concerning the KdVB equation have received considerable attention. In particular, the travelling wave solution to the KdVB equation has been studied extensively. Johnson [10], Demiray [11] and Antar and Demiray [12] derived KdVB equation as the governing evolution equation for waves propagating in fluid-filled elastic or viscoelastic tubes in which the effects of dispersion, dissipation and nonlinearity are present.

The KdV–Burgers equation is a one-dimension generalization of the model description of the density and velocity fields that takes into account pressure forces as well as the viscosity and the dispersion. It may be a more flexible tool for physicists than the Burgers' equation.

Several studies in the literature, employing a large variety of methods, have been conducted to derive explicit solutions for KdV–Burgers equation (1). Grad and Hu [15] used a steady-state version of (1) to describe a weak shock profile in plasmas. They studied the same problem using a similar method to that used by Johnson [10], and a related problem was studied by Jeffrey [16]. A numerical investigation of the problem was carried out by Canosa and Gaxdag [17]. Bona and Schonbek [18] studied the existence and uniqueness of bounded travelling wave solution to (1) which tend to constant states at plus and minus infinity. More recently, Jeffrey and Xu [19] introduced a transformation which reduced the KdVB equation (1) to a quadratic form involving a new dependent variable and its partial derivatives. They also obtained exact solutions of the KdVB equation by solving this one in terms of a series of exponentials. A comprehensive account of the travelling wave solution to the KdVB equation can also be found in the review paper by Jeffrey and Kakutani [20]. For other theoretical issues and more details about these investigations concerning the KdVB equation, the reader is kindly referred to Jian-Jun [21] and the references therein.

It sounds more impressive when we realize that the only explicit solution were found by Euler (1765) and Lagrange (1772).

This paper is devoted to the study the KdV–Burgers equation. Our work here stems mainly from Adomian decomposition method [2–5], that has been widely used in applied sciences. The Adomian decomposition method provides the solution in a rapidly convergent series with components that are elegantly computed. Moreover, the obtained series solution is used to provide closed form solution. The main advantage of the method is that it can be applied directly to all types of differential equations without any need for restrictive assumptions. Another important advantage is that this method is capable of greatly reducing the size of computational work.

In order to make numerical comparison, we use the finite-difference technique and in order to solve Eq. (1) a network of grid points is first established throughout the region occupied by the independent variables.

In the last section, an illustrative numerical examples are given and numerical comparison between both methods is presented.

2. Exact travelling wave solution

The well-known KdV–Burgers equation that involves both dispersion term u_{xxx}, and dissipation term u_{xx}, is

$$u_t + 2(u^3)_x - u_{xxx} + u_{xx} = 0.$$

This nonlinear partial differential equation has an exact special solution. We replace the independent parameters x and t by one composed parameter $(x - \frac{2}{9}t)$.

This transformation leads to the travelling wave solution

$$u(x,t) = \frac{1}{6}\left(1 + \tanh\left(\frac{1}{6}\left(x - \frac{2}{9}t\right)\right)\right).$$

It is worth noting that this exact solution is a special one.

3. The finite-difference method

There exist several different versions in finite-difference methods. In this study, we use Crank–Nicolson method. In order to solve Eq. (1) numerically, both space and time are divided into a large numbers of grid points and derivatives are replaced by finite-difference approximations.

Let h and k be the step sizes in coordinate and time grids and n the number of grid points in coordinate grid. The grid points (x_i, t_j) in this situation are given as $x_i = ih$ for $i = 0, 1, \ldots, n$ and $t_j = jk$, for $j \geqslant 0$.

The method is based on replacing u_x, u_{xx}, u_{xxx} by the mean of there finite-difference representations on the $(j+1)$th and jth time rows and approximates Eq. (1) by

$$\begin{aligned}
&-ru_{i+2,j+1}+\left(6u_{i,j}^2rh^2+2r+2rh\right)u_{i+1,j+1}+(4-4rh)u_{i,j+1}+\left(-6u_{i,j}^2rh^2+2rh-2r\right)u_{i-1,j+1}+ru_{i-2,j+1}\\
&\quad=ru_{i+2,j}-\left(6u_{i,j}^2rh^2+2r+2rh\right)u_{i+1,j}+(4+4rh)u_{i,j}+\left(6u_{i,j}^2rh^2-2rh+2r\right)u_{i-1,j}-ru_{i-2,j},
\end{aligned} \tag{2}$$

where $r=\frac{k}{h^3},\ i=1,2,\ldots,n$, and $j\geqslant 1$.

3.1. Stability analysis

Applying the well-known Von Neumann stability method in Eq. (2), one can prove that this numerical technique is stable under the following condition:

$$-1\leqslant\frac{k}{h^2}\leqslant 1.$$

This leads to the confirmation that, the numerical error will not grow for subsequent marching step in t, and the numerical solution will proceed in a stable manner.

3.2. Consistency

The difference equation is consistent with the differential equation if the limiting value of the local truncation error tends to zero as $h\to 0$, $k\to 0$.

To discuss the consistency of this scheme with the partial differential equation, we expand the terms

$$u_{i+2,j+1},u_{i+1,j+1},u_{i,j+1},u_{i-1,j+1},u_{i-2,j+1},u_{i+2,j},u_{i+1,j},u_{i-1,j},u_{i-2,j}$$

about the point (ih,ik) by Taylor's series, this leads to the following truncation error:

$$\begin{aligned}
T_{i,j}&=k^2\left(12u^2u_{xt}+2u_{tt}+2u_{xxt}-2u_{xxxxt}\right)+k^3\left(6u^2u_{xtt}+\frac{2}{3}u_{t^3}+u_{xxtt}\right)+k^4\left(2u^2u_{xt^3}+\frac{1}{6}u_{t^4}\right)+kh^2\left(4u^2u_{x^3}+\frac{1}{3}u_{x^4}\right)\\
&\quad+h^2k^2\left(2u^2u_{x^3t}\right)+\cdots,
\end{aligned} \tag{3}$$

which tends to zero as h and k tends to zero.

Hence the finite-difference equation is consistent with the differential equation.

4. The decomposition method

To begin with, Eq. (1) may be written in an operator form:

$$\begin{aligned}
&Lu=u_{xxx}-u_{xx}-2(u^3)_x,\\
&u(x,0)=f(x),
\end{aligned} \tag{4}$$

where the differential operator L is

$$L=\frac{\partial}{\partial t}. \tag{5}$$

It is assumed that the inverse operator L^{-1} is an integral operator given by

$$L^{-1}(\cdot)=\int_0^t(\cdot)\,\mathrm{d}t. \tag{6}$$

The Adomian decomposition method assumes that the unknown function $u(x,t)$ can be expressed as an infinite series of the form

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{7}$$

and the nonlinear operator $F(u)=(u^3)_x$ can be decomposed in an infinite series of polynomials given by

$$F(u)=\sum_{n=0}^{\infty}A_n, \tag{8}$$

where the components $u_n(x,t)$ will be determined recurrently, and A_n are the so-called Adomian polynomials of $u_0, u_1, \ldots, u_n$ defined by

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[F\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots \tag{9}$$

It is now well known that these polynomials can be constructed for all classes of nonlinearity according to algorithms set by Adomian [1,2] and recently developed by different alternative approaches [3–7].

Operating with the integral operator L^{-1} on both sides of (4) and using the initial condition, we find

$$u(x,t) = f(x) + L^{-1}(u_{xxx} - u_{xx} - 2(u^3)_x). \tag{10}$$

Substituting (7) and (8) into the functional equation (10) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L^{-1}\left(\left(\sum_{n=0}^{\infty} u_n\right)_{xxx} - \left(\sum_{n=0}^{\infty} u_n\right)_{xx} - 2\left(\sum_{n=0}^{\infty} A_n\right)\right). \tag{11}$$

Identify the zeroth component $u_0(x,t)$ by all terms that arise from the initial condition, and as a result, the remaining components $u_n(x,t)$, $n \geqslant 1$ can be determined by using the recurrence relation:

$$\begin{aligned} &u_0(x,t) = f(x),\\ &u_{k+1}(x,t) = L^{-1}((u_k)_{xxx} - (u_k)_{xx} - 2A_k), \quad k \geqslant 0, \end{aligned} \tag{12}$$

where A_k are Adomian polynomials that represent the nonlinear term $(u^3)_x$ and given by

$$\begin{aligned} A_0 &= 3u_0^2 u_{0x},\\ A_1 &= 6u_0 u_{0x} u_1 + 3u_0^2 u_{1x},\\ A_2 &= 3u_{0x} u_1^2 + 6u_0 u_{0x} u_2 + 6u_0 u_1 u_{1x} + 3u_0^2 u_{2x},\\ A_3 &= 6u_0 u_{0x} u_3 + 6u_1 u_2 u_{0x} + 3u_1^2 u_{1x} + 6u_0 u_2 u_{1x} + 6u_0 u_1 u_{2x} + 3u_0^2 u_{3x},\\ A_4 &= 6u_0 u_4 u_{0x} + 6u_1 u_3 u_{0x} + 3u_2^2 u_{0x} + 6u_0 u_3 u_{1x} + 6u_1 u_2 u_{1x} + 3u_1^2 u_{2x} + 6u_0 u_2 u_{2x} + 6u_0 u_1 u_{3x} + 3u_0^2 u_{4x}. \end{aligned} \tag{13}$$

Other polynomials can be generated in a like manner.

The first few components of $u_n(x,t)$ follow immediately upon setting:

$$\begin{aligned} u_0(x,t) &= f(x),\\ u_1(x,t) &= L^{-1}((u_0)_{xxx} - (u_0)_{xx} - 2A_0),\\ u_2(x,t) &= L^{-1}((u_1)_{xxx} - (u_1)_{xx} - 2A_1),\\ u_3(x,t) &= L^{-1}((u_2)_{xxx} - (u_2)_{xx} - 2A_2),\\ u_4(x,t) &= L^{-1}((u_3)_{xxx} - (u_3)_{xx} - 2A_3). \end{aligned} \tag{14}$$

The scheme in (14) can easily determine the components $u_n(x,t)$, $n \geqslant 0$. It is, in principle, possible to calculate more components in the decomposition series to enhance the approximation. Consequently, one can recursively determine every term of the series $\sum_{n=0}^{\infty} u_n(x,t)$, and hence the solution $u(x,t)$ is readily obtained in a series form.

It is interesting to note that we obtained the series solution by using the initial condition only. The obtained series may lead to the exact solution.

5. Application

In the following, we will examine the KdV–Burgers equation:

$$u_t + 2(u^3)_x - u_{xxx} + u_{xx} = 0, \tag{15}$$

subject to initial condition

$$u(x,0) = \frac{1}{6}\left(1 + \tanh\left(\frac{x}{6}\right)\right)$$

with $u = u(x,t)$ is a sufficiently smooth function, and $f(x)$ is bounded. As indicated before, we shall assume that the solution $u(x,t)$, along with its derivatives, tends to zero as $|x| \to \infty$.

Applying the inverse operator L^{-1} of (6) on both sides of (15) and using the decomposition series (7) and (8) yield

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{1}{6}\left(1+\tanh\left(\frac{x}{6}\right)\right) + L^{-1}\left(\left(\sum_{n=0}^{\infty} u_n\right)_{xxx} - \left(\sum_{n=0}^{\infty} u_n\right)_{xx} - 2\left(\sum_{n=0}^{\infty} A_n\right)\right). \tag{16}$$

Proceeding as before, Adomian decomposition method [1,2] gives the recurrence relation:

$$\begin{aligned} u_0(x,t) &= \frac{1}{6}\left(1+\tanh\left(\frac{x}{6}\right)\right), \\ u_{k+1}(x,t) &= L^{-1}((u_k)_{xxx} - (u_k)_{xx} - 2A_k), \quad k \geqslant 0. \end{aligned} \tag{17}$$

The resulting components are

$$\begin{aligned} u_0(x,t) &= \frac{1}{6}\left(1+\tanh\frac{x}{6}\right), \\ u_1(x,t) &= L^{-1}((u_0)_{xxx} - (u_0)_{xx} - 2A_0) = \frac{-1}{162} t\,\mathrm{sech}\left(\frac{x}{6}\right)^2, \\ u_2(x,t) &= L^{-1}((u_1)_{xxx} - (u_1)_{xx} - 2A_1) = \frac{-t^2\,\mathrm{sech}\left(\frac{x}{6}\right)^2 \tanh\frac{x}{6}}{4374}, \\ u_3(x,t) &= L^{-1}((u_2)_{xxx} - (u_2)_{xx} - 2A_2) = \frac{-t^3\left(-2+\cosh\left(\frac{x}{3}\right)\right)\mathrm{sech}\left(\frac{x}{6}\right)^4}{354,294}, \\ u_4(x,t) &= L^{-1}((u_3)_{xxx} - (u_3)_{xx} - 2A_3) = \frac{-t^4\,\mathrm{sech}\left(\frac{x}{6}\right)^5 \sinh\frac{x}{2}}{38,263,752} + \frac{11t^4\,\mathrm{sech}\left(\frac{x}{6}\right)^4 \tanh\frac{x}{6}}{38,263,752}. \end{aligned} \tag{18}$$

In view of (18), the solution in series form is

$$\begin{aligned} u(x,t) = \frac{1}{9,565,938}\Bigg[&1,594,323\left(1+\tanh\frac{x}{6}\right) + 3t^3\,\mathrm{sech}\left(\frac{x}{6}\right)^4\left(27+t\tanh\frac{x}{6}\right) - t\,\mathrm{sech}\left(\frac{x}{6}\right)^2 \\ &\times\left(59,049+54t^2+t(2187+t^2)\tanh\frac{x}{6}\right)\Bigg]. \end{aligned} \tag{19}$$

5.1. Numerical results and some illustrations

In this section, we present the following tables to describe the errors between the exact and numerical solutions. The tables illustrate the errors for both methods, the Adomian decomposition method and the finite-difference method compared with the exact solution, at different values of x and t.

A comparison between the numerical results obtained by using Crank Nicolson and the decomposition methods with those obtained by exact solution are given for $h = 0.1$, and different values of x in Tables 1 and 2. We can observe from the tables, that the decomposition method is more accurate as compared with the finite-difference method (Crank Nicolson). Figs. 1–3 illustrate the above results and methods.

Table 1
Comparison between the numerical results obtained by using Crank-Nicolson and the ADM with those obtained by the exact solution for $h = 0.1$ at $t = 0.02$

$h = 0.1$	$t = 0.02$		$t = 0.001$	
x	$u_{exact} - u_{crank}$	$u_{exact} - u_{decomposition}$	$u_{exact} - u_{crank}$	$u_{exact} - u_{decomposition}$
0.1	0.0889801	−2.77556E−17	−0.00194939	0.0
0.3	0.123026	−2.77556E−17	0.0000544889	0.0
0.5	0.0399017	0.0	−0.000298439	−2.77556E−17
0.7	−0.0795711	−2.77556E−17	−0.00122232	0.0
0.9	0.0545362	−2.77556E−17	−0.0031547	−5.55112E−17

Table 2
Comparison between the numerical results obtained by using Crank-Nicolson and the ADM with those obtained by the exact solution for $h = 0.1$ at $t = 0.001$

$h = 0.1$	$t = 0.001$	
x	$u_{\text{exact}} - u_{\text{crank}}$	$u_{\text{exact}} - u_{\text{decomposition}}$
1	1.38595E−07	5.55112E−17
2	2.61206E−10	2.77556E−17
3	3.54754E−10	0
4	7.1927E−10	5.55112E−17
5	1.78176E−6	5.55112E−17

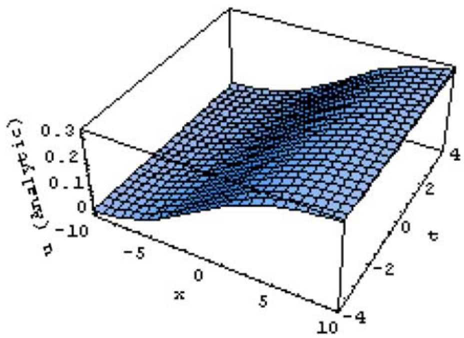

Fig. 1. The Adomian decomposition method results, for: x: $-10 \to 10$ and t: $-4 \to 4$.

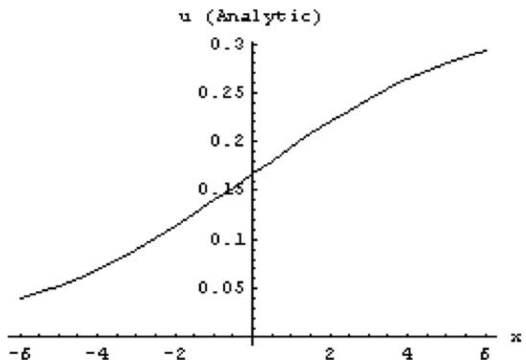

Fig. 2. The Adomian decomposition method results, for: x: $-6 \to 6$ and $t = 0.001$.

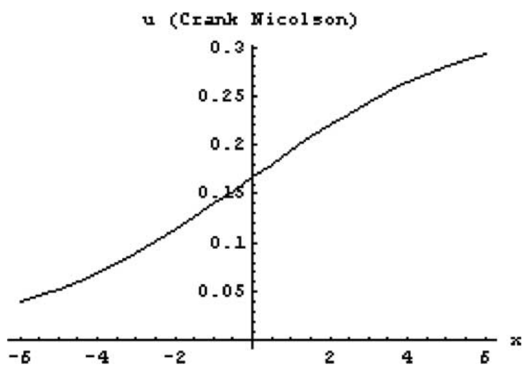

Fig. 3. The result of Crank Nicolson method, for: x: $-6 \to 6$ and $t = 0.001$.

6. Conclusion

In this paper, two different methods for solving the KdV–Burgers equation have been used. The Adomian decomposition method was successfully used to develop the solution. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. The Crank Nicolson finite-difference method had been applied as well. This finite-difference method requires great efforts and carefulness to study the stability and the consistency of the used scheme. In other cases, however, the numerical approach may be more adequate for finding the solution and comparing with those obtained by analytic or semi-analytic methods. Finally, we conclude that the nonlinear KdV–Burgers equation gives soliton solution, which represents an important application in Physics and physical problems.

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