

Higher Several Variable Calculus

Math2111 UNSW

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1 Curves and Surfaces

1.1 Curves

Curves A curve in \mathbb{R}^n is a vector function

$$\mathbf{c} : I \rightarrow \mathbb{R}^n,$$

where I is an interval in \mathbb{R} .

Forms / Notations Curves may be defined in the following ways:

- **Parametrically** by $c(t) = (x_1(t), x_2(t), \dots, x_n(t))$
- **Cartesian** by eliminating the t variable to get y in terms of x
- **Implicitly** As $F(x, y) = 0$.

2 Analysis

2.1 Assumed

Assumed Concepts from Real Single-Variable Calculus

- limits
- continuity
- differentiability
- integrability

Assumed Theorems

- Min/ Max Theorem
- Intermediate Value Theorem
- Mean Value Theorem

2.2 Limits

Recall that $\lim_{x \rightarrow a} f(x) = L$ requires that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$ then

$$|f(x) - L| < \delta.$$

2.3 Metrics

We have metrics (distance functions) as

$$m : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

satisfying the following 3 axioms.

- **Positive Definite** such that for all $x, y \in \mathbb{R}^n$, $m(x, y) > 0$ and, $m(x, y) = 0 \Leftrightarrow x = y$.
- **Symmetric** $m(x, y) = m(y, x)$.
- **Triangle Inequality** such that for all $x, y, z \in \mathbb{R}^n$, $m(x, y) + m(y, z) \leq m(x, z)$.

Euclidian Distance We allow the Euclidian distance to be defined as

$$d_n(x, y) := \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

We often allow d to be d_2 .

Norms Norms will be revisited in the Fourier Series section. They can be thought of as the length of an element in vectors space.

Equivalent Metrics Two metrics d and δ are considered equal if there exists constants $0 < c < C < \infty$ such that

$$c\delta(x, y) \leq d(x, y) \leq C\delta(x, y).$$

2.4 Limits of Sequences

Balls A ball around $\vec{a} \in \mathbb{R}$ is of radius ϵ is the set

$$B(\vec{a}, \epsilon) = \{x \in \mathbb{R} : d(\vec{a}, x) < \epsilon\}.$$

Limit in Sequence For a sequence $\{x_i\}$ of points in \mathbb{R}^n , x is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \leq \epsilon.$$

Equivalently,

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \in B(x, \epsilon).$$

Theorems with Limits of Sequences

A sequence x_k converges to a limit x

$$\begin{aligned} &\Leftrightarrow \text{The components of } x_k \\ &\quad \text{converge to the components of } x \\ &\Leftrightarrow d(x_k, x) \rightarrow 0. \end{aligned}$$

Limits and Equivalent Metrics Suppose that d and δ are two equivalent metrics. That is, $cd(x, y) \leq \delta(x, y) \leq Cd(x, y)$ for $c, C > 0$.

Considering d as the metric, suppose that

$$x_k \rightarrow x \quad \text{for } x_k, x \in \mathbb{R}^n.$$

That is,

$$\forall \epsilon > 0, \exists K : k \geq K \implies d(x_k, x) < \epsilon.$$

Using δ , we may make an equivalent statement, choosing $\epsilon > 0$ such that $\epsilon' = C\epsilon$. Considering that $\epsilon > 0 \implies \exists K : \forall k \geq K \implies d(x_k, x) < \epsilon$ then,

$$\delta(x_k, x) \leq Cd(x_k, x) < C\epsilon = \epsilon'.$$

That is, $\delta(x_k, x) < \epsilon'$. Hence $x_k \rightarrow x$ using an equivalent metric δ .

Cauchy Sequences A sequence $\{x_k\} \in \mathbb{R}$ is a Cauchy sequence if

$$\exists \epsilon > 0 \text{ such that } k, l > K \implies d(x_k, x_l) < \epsilon.$$

Cauchy Sequences and Convergence The following are equivalent:

A sequence $\{x_k\}$ converges in $\mathbb{R}^2 \iff \{x_k\}$ is a Cauchy Sequence.

2.5 Open and Closed Sets

Definitions Consider x_k

- $x_0 \in \Omega$ is an interior point of Ω if there is a ball around x completely contained in Ω . That is, there exists a $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq \Omega$.
- Ω is open if every point of Ω is an interior point.
- Ω is closed if its complement is open.
- $x_0 \in \Omega$ is a boundary point of Ω if every ball around x_0 contains points in Ω and points not in Ω .

Closed Sets A set $\Omega \subset \mathbb{R}$ is closed iff and only if it contains all of its boundary points.

Limit Points and Sets x_0 is a limit point of Ω if there is a sequence $\{x_i\}$ in Ω with limit x_0 and $x_i \neq x$.

- Every interior points of Ω is a limit point of Ω .
- x_0 is not necessarily in *Omega*
- A set is closed \Leftrightarrow it contains all of its limit points.

Variations of a Set Consider the set $\Omega \in \mathbb{R}^n$.

- The interior of Ω is the set of all its interior points.
- The boundary $\partial\Omega$ of Ω is the set of all its boundary points.
- The closure of Ω : $\bar{\Omega} = \Omega \cup \partial\Omega$.

The interior is the largest open subset and the closure is the smallest closed set containing Ω .

Limit of a Function at a Point For $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\lim_{x \rightarrow x_0}$ means that

$$\forall \epsilon \exists \delta > 0 \text{ such that for } x \in \Omega : \\ 0 < d(x, x_0) < \delta \implies d(f(x), b) < \epsilon.$$

Alternatively,

$$x \in B(x_0, \delta) \setminus \{x_0\} \implies f(x) \in B(b, \epsilon).$$

It is sufficient to consider the limits of the components of a function.

Limits and sequences The limit $\lim_{x \rightarrow a} f(x) = b$ exists if and only if, $\lim_{k \rightarrow \infty} f(x_k) = b$ for all sequences x_k such that x_k is an element of Ω and, $\lim_{k \rightarrow \infty} x_k = a$.

This is very helpful for showing that a limit does not exist.

2.6 Pinching and IVT Theorem

Pinching Theorem

IVT see 1141

3 Analysis

4 Differentiation

5 Integration

6 Fourier Series

Fourier Series A Fourier series is the approximation of simple periodic functions by the sum of period functions of the form $\sin(x)$, $\cos(x)$. Note that unlike Taylor series, a function f may be discontinuous. However, any lack of continuity leads to an infinite sum in the Fourier series.

6.1 Inner Products

Inner Products Let V be a real vector space. An inner product on V is a map that assigns each $f, g \in V$ a real number $\langle f, g \rangle$ such that the following properties hold for all $f, g, h \in V$ and $\lambda, \mu \in \mathbb{R}$:

- $\langle f, f \rangle \geq 0$,
- $\langle f, f \rangle = 0$ if and only if f is zero,
- $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$,
- $\langle g, f \rangle = \langle f, g \rangle$.

Usual Inner Products

- The vector space R^n admits the following inner product

$$\langle u, v \rangle = u \cdot v = \sum_{i=1}^n u_i v_i.$$

- The vector space $C[a, b]$ consisting of all continuous function on the interval $[a, b]$ admits the following inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

Inner Product and Orthogonality We say functions are orthogonal if $\langle f, g \rangle = 0$.

6.2 Norms

A norm on V is a map that assigns each $f \in V$ a real number $\|f\|$ such that $\forall f \in V, \lambda \in \mathbb{R}$

- $\|f\| > 0$,
- $\|f\| = 0$ if and only if $f = 0$,
- $\|\lambda f\| = \lambda\|f\|$,
- $\|f + g\| \leq \|f\| + \|g\|$; that is, the triangle inequality holds.

Usual Norms

- The Euclidian norm (L^2 -norm): is a norm on $C[a, b]$:

$$\|f\|_2 = \sqrt{\int_a^b f(x)^2 dx}$$

- The max norm is a norm on $C[a, b]$:

$$\|f\|_\infty = \max_{a \leq x \leq b} \{|f(x)|\}$$

6.3 Fourier Coefficient and Series

Fourier Series Suppose that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is $2L$ -periodic, - that is, $f(x) = f(x + 2L)$ - and is square integrable - that is, $\int_{-L}^L f(x)^2 dx < \infty$. Then, f may be represented by a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^n \left[a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right) \right] \quad \forall x \in [-\pi, \pi].$$

This series converges to f as $n \rightarrow \infty$.

Fourier Coefficients

- $a_k = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{k\pi x}{L}\right)$
- $b_k = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{k\pi x}{L}\right)$

6.4 Convergence of Fourier Series

Continuity Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a point $c \in \mathbb{R}$. Suppose that the one-sided limits $f(c^+)$ and $f(c^-)$ exist.

- If $f^{c^+} = f^{c^-} = f(c)$ then f is continuous at c ,
- If $f^{c^+} = f^{c^-} \neq f(c)$ then f has a removable discontinuity at c ,
- If $f(c^+) \neq f(c^-)$ then, f has a jump discontinuity at c .

Piecewise Continuity A function is piecewise continuous on $[a, b]$ if and only if

- $f(x^+)$ exists $\forall x \in [a, b]$,
- $f(x^-)$ exists $\forall x \in [a, b]$,
- f is continuous on (a, b) except at most a finite number of points.

Note that if f is only piecewise continuous then the partial sum of the Fourier series does not necessarily converge to f for all x .

Piecewise differentiability A function f is differentiable on c if and only if $f(c^+) = f(c^-) = f(c)$ and $D^+f(c) = D^-f(c)$

Note: $D^+f(c)$ is not necessarily the same as $\lim_{x \rightarrow c^+} f'(x)$.

A function is piecewise differentiable on $[a, b]$ if and only if

- $D^+f(x)$ exists $\forall x \in [a, b]$,
- $D^-f(x)$ exists $\forall x \in (a, b]$,
- f is differentiable on (a, b) except at most a finite number of points.

Pointwise convergence Let $c \in \mathbb{R}$. Suppose that a function has the following properties

- f is $2L$ periodic,
- f is piecewise continuous on $[-L, L]$,
- $D^+f(c), D^-f(c)$ exist.

Then,

$$S_f(c) = \frac{1}{2}[f(c^+) + f(c^-)].$$

Observe that if f is continuous at c then $S_f(c) = f(c)$.

Odd and Evenness Recall that odd and even functions are defined by the conditions $f(-x) = -f(x)$ and $f(x) = f(-x)$ respectively.

The following elementary properties hold:

- Odd \times Even = Even,
- Odd \times Odd = Even,
- Even \times Even = Even,
- $\int_{-L}^L \text{Odd} = 0$.

6.5 Convergence of Sequences

Pointwise convergence Let $f_k : \mathbb{R} \rightarrow \mathbb{R}$. f_k converges to f on $[a, b]$ pointwisely iff and only if for all $x \in [a, b]$, $f_k(x) \rightarrow f(x)$ as $k \rightarrow \infty$.

Epsilon Delta Definition Pointwise Convergence For all $x \in [a, b]$, $\epsilon > 0, \exists K$ (which will depend on ϵ, x such that

$$|f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

Uniform Convergence Let $f_k : \mathbb{R} \rightarrow \mathbb{R}$. f_k converges to f on $[a, b]$ uniformly if and only if for all $\epsilon > 0, \exists K$ (depending on ϵ only) such that

$$\sup_{x \in [a, b]} |f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

Weierstrass test Let $f_k : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of a function f defined on $[a, b]$. Suppose that there exists a sequence of numbers c_k such that

$$|f_k(x)| \leq c_k \quad \forall x \in [a, b]$$

where $\sum_{k=1}^{\infty} c_k$ converges to a real number. Then $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f on $[a, b]$.

Note that this test also holds for function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ for $x \in \Omega$ where Ω is a closed bounded set in \mathbb{R}^n .

Norm Convergence Using the supremum norm, the definition of uniform convergence can be equivalently written as: for all $\epsilon > 0, \exists K$ such that

$$\|f_k - f\| \leq \epsilon \quad \forall k \geq K.$$

Equivalently,

$$\lim_{k \rightarrow \infty} \|f_k - f\| = 0.$$

We may extend this to define norm-convergence for any norm.

Extending Norm Convergence to L-2 Recall from the previous paragraph that norm-convergence is defined as follows:

$$\lim_{k \rightarrow \infty} \|f_k - f\| = 0.$$

As such, L^2 norm convergence, also known as mean square convergence is equivalent to the following

$$\lim_{k \rightarrow \infty} \int_a^b [f_k(x) - f(x)]^2 dx = 0.$$

Parseval Theorem Let f be a 2π periodic and bounded function where $\int_{-\pi}^{\pi} f(x)^2 dx < +\infty$. Then, the Fourier series of f converges to f in the mean square sense. Moreover, the Parseval's identity holds

$$\int_{-\pi}^{\pi} f(x)^2 dx = \|f\|_2^2 = \frac{\pi}{2} a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This identity continues to hold for $2L$ periodic functions integrated over $[-L, L]$.

7 Vector Fields

7.1 Vector Fields and Flows, Divergence and Curl

Flow Lines If F is a vector field, a flow line for F is a path $c(t)$ such that

$$c'(t) = F(c(t)).$$

That is, that F yields the velocity field of the path $c(t)$.

The Del ∇ operator The vector differential operator ∇ may be considered a symbolic vector. The differential operator may be written as

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k.$$

Divergence Simply put,

$$\text{div} F = \nabla \cdot F.$$

Divergence may be thought as a type of derivative that describes the measure at which a vector field *spreads away* from a certain point. If the divergence is positive, then there is a net outflow while there is net inflow if the divergence is negative.

Observe that the divergence of a vector field will be real-valued.

Curl If F is a vector field, then the curl may be defined as

$$\operatorname{curl} F = \nabla \times F.$$

Curl is also analogous to a type of derivative for vector fields. The curl may be thought of as the measure at which the vector field *swirls* around a point. A positive swirl can be thought of as a counter clockwise rotation.

Observe that the curl of a vector field is also a vector field.

8 Path Integrals

8.1 Path Integrals

Path (Scalar Line) Integrals Suppose that a vector-valued function $c(t)$ parametrises a curve C for $t \in [a, b]$. The scalar line integral may be thought of as the integral of along c .

Computing a Scalar Line Integral Let $c(t)$ parametrise a curve C for $t \in [a, b]$. Assume that $f(x, y, z)$ and $c(t)$ are continuous. Then,

$$\int_C f(x, y, z) ds = \int_a^b f(c(t)) \cdot \|c'(t)\| dt.$$

Elementary Properties of Path Integrals

- $\int_C f_1 ds + \int_C f_2 ds = \int_C (f_1 + f_2) ds,$
- $\int_C \lambda f ds = \lambda \int_C f ds, \quad \lambda \in \mathbb{R}.$

8.2 Applications Of Path Integrals

Suppose that $\delta = \delta(x, y, z)$ which is a density function.

Mass

$$M = \int_C \delta(x, y, z) ds.$$

First Moments About the Coordinate Plane

- $M_{yz} = \int_C x \delta ds$
- $M_{xz} = \int_C y \delta ds$
- $M_{xy} = \int_C z \delta ds$

Coordinates of Center of Mass

- $\bar{x} = \frac{M_{yz}}{M}$
- $\bar{y} = \frac{M_{xz}}{M}$
- $\bar{z} = \frac{M_{xy}}{M}$

Moments of Inertia About Axes

- $I_x = \int_C (y^2 + z^2) \delta ds$
- $I_x = \int_C (x^2 + z^2) \delta ds$
- $I_x = \int_C (x^2 + y^2) \delta ds$

9 Vector Line Integrals

Vector Line Integrals Vector line integrals are different from scalar line integrals in the sense that to define a vector line integral, we must specify a direction along the path or curve C .

Computing a Vector Line Integral Let $c(t)$ parametrise an oriented curve C for $t \in [a, b]$. Then,

$$\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t) dt.$$

Link to Path Integrals Suppose that C is a smooth curve with a parametrisation $c(t)$ for $t \in [a, b]$ where $c(t)$ is continuously differentiable and $c'(t) \neq 0$ for all $t \in [a, b]$.

Then, $c'(t)$ is a non-zero tangent vector pointing in the forward direction and the unit tangent vector is

$$T(c(t)) = \frac{c'(t)}{\|c'(t)\|}.$$

Then,

$$\int_C F \cdot ds = \int_C F \cdot T ds.$$

Summing Paths Suppose that C is made of n finitely many paths C_i . Then, $C = \sum_i^n C_i$. Note that all the curves must be joined end to end. Then,

$$\int_C F \cdot ds = \sum_i^n \int_{C_i} F \cdot ds.$$

Work and Other Alternative Notations Suppose that $c(t) = (x(t), y(t), z(t))$ and $F = (M, N, P)$. Then, we denote work as any of the following notations

$$\begin{aligned} W &= \int_C F \cdot ds \\ &= \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt \\ &= \int_C M ds + N ds + P ds. \end{aligned}$$

Properties fo Line Integrals

- Linearity
- Reversing Orient
- Additivity

Flow Integrals and Circulation Suppose that F represents a velocity field of a fluid flowing through a region in space. Then, the flow across a curve may be defined as the following

$$\text{Flow} = \int_a^b F \cdot \hat{T} ds.$$

This integral is called the flow integral. If the curve is a closed loop then this is called the *circulation* around the curve.

Flux in the Plane If C is a smooth closed curve in the domain of a continuous vector field $F = M(x, y)i + N(x, y)j + N(x, y)j$ and, n is the outward pointing unit-normal on C then, the flux of f across C is the following expression

$$\int_C F \cdot \hat{n} ds.$$

Calculating Flux Across a Smooth Closed Plane Curve Suppose that $F = Mi = Nj$. Let $G = -N, M$

$$\text{Flux of } F \text{ across } C = \oint_C M dy - N dx = \oint_C G ds$$

9.1 Fundamental Theorem of Line Integrals

Gradient Fields A vector field F is called a gradient vector field if there exists a real-valued function ϕ such that $F = \nabla\phi$. That is,

$$\begin{pmatrix} M \\ N \\ P \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{pmatrix}.$$

If such a function ϕ exists then ϕ is called the potential function of F where, F is *conservative*.

Fundamental Theorem for Gradient Vector Fields If $F = \nabla\phi$ on a domain \mathcal{D} . Then, for all oriented curves C in \mathcal{D} with an initial point P and a terminal point Q ,

$$\int_C F \cdot ds = \phi(Q) - \phi(P).$$

The integral is independent of the path.

Cross Partial of Gradient Vector Fields are Equal Let $F = (F_1, F_2, F_3)$ be a gradient vector field whose components have continuous partial derivatives. Then, the cross partials are equal. That is,

$$\begin{aligned}\frac{\partial F_1}{\partial y} &= \frac{\partial F_2}{\partial x}, \\ \frac{\partial F_2}{\partial z} &= \frac{\partial F_3}{\partial y}, \\ \frac{\partial F_3}{\partial x} &= \frac{\partial F_1}{\partial z}.\end{aligned}$$

Equivalently,

$$\nabla \times F = 0.$$

9.2 Green's Theorem

Green's Theorem connects double integrals and line integrals.

Green's Theorem: Flux Divergence or Normal Form Let D be a bounded simple region in \mathbb{R}^2 with a nonempty interior whose boundary consists of a finite number of smooth curves. Let C be the boundary of D with a positive (counter-clockwise) direction. Let $F = Mi + Nj$ be a continuously differentiable boundary vector field on D .

Then, the outward flux of F across the curve C equals the double integral of divergence $\nabla \cdot F$ over D . That is,

$$\oint_C (F \cdot \hat{n}) ds = \oint_C -N dx + M dy = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right).$$

Once again, note the assumptions:

- D is bounded and simple with a non-empty interior,
- The boundary C is oriented in the positive (counter-clockwise) direction, and is the finite union of smooth curves,
- The vector field F is continuously differentiable on D .

Green's Theorem: Circulation-Curl or Tangential Form Let D be a bounded simple region in \mathbb{R}^2 with a nonempty interior whose boundary consists of a finite number of smooth curves. Let C be the boundary of D with a positive (counter-clockwise) direction. Let $F = Mi + Nj$ be a continuously differentiable boundary vector field on D .

Then, the counter-clockwise circulation of F around C equals the double integral of $(\nabla \times F) \cdot k$ over D . That is,

$$\oint_C (F \cdot \hat{T}) ds = \oint_C M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

Area of a Region Let D be a simple and bounded region with a non-empty interior and let C be the boundary of D which is the finite union of smooth curves. Then, the area of D can be calculated as such

$$\text{Area}(D) = \frac{1}{2} \oint_C (-y dx + x dy).$$

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