# Higher Several Variable Calculus
# Math2111 UNSW

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## Contents

# 1 Analysis

## 1.1 Assumed

**Assumed Concepts from Real Single-Variable Calculus**

- limits
- continuity
- differentiability
- integrability

**Assumed Theorems**

- Min/ Max Theorem
- Intermediate Value Theorem
- Mean Value Theorem

## 1.2 Limits

Recall that $\lim_{x \to a} f(x) = L$ requires that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$ then

$$|f(x) - L| < \delta.$$

## 1.3 Metrics

We have metrics (distance functions) as

$$m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

satisfying the following 3 axioms.

- **Positive Definite** such that for all $x, y \in \mathbb{R}^n$, $m(x, y) > 0$ and, $m(x, y) = 0 \iff x = y$.
- **Symmetric** $m(x, y) = m(y, x)$.z
- **Triangle Inequality** such that for all $x, y, z \in \mathbb{R}^n$, $m(x, y) + m(y, z) \leq m(x, z)$.

**Euclidian Distance** We allow the Euclidian distance to be defined as

$$d_n(x, y) := ||x - y|| = \sqrt{\sum_{i=1}^{n} (x_i - y_y)^2}$$

We often allow $d$ to be $d_2$.

**Norms** Norms will be revisited in the Fourier Series section. They can be thought of as the length of an element in vectors space.

**Equivalent Metrics** Two metrics $d$ and $\delta$ are considered equal if there exists constants $0 < c < C < \infty$ such that

$$c\delta(x, y) \leq d(x, y) \leq C\delta(x, y).$$

## 1.4 Limits of Sequences

**Balls** A ball around $\vec{a} \in \mathbb{R}$ is of radius $\epsilon$ is the set

$$B(\vec{a}, \epsilon), = \{x \in \mathbb{R} : d(\vec{a}, x) < \epsilon\}.$$

**Limit in Sequence** For a sequence $\{x_i\}$ of points in $\mathbb{R}^n$, $x$ is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \leq \epsilon.$$

Equivalently,

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \in B(x, \epsilon).$$

**Theorems with Limits of Sequences**

$$\begin{aligned} \text{A sequence } x_k \text{ converges to a limit } x & \\ & \Leftrightarrow \text{The components of } x_k \\ & \quad \text{converge to the componetents of } x \\ & \Leftrightarrow d(x_k, x) \to 0. \end{aligned}$$

**Limits and Equivalent Metrics** Suppose that $d$ and $\delta$ are two equivalent metrics. That is, $cd(x, y) \leq \delta(x, y) \leq Cd(x, y)$ for $c, C > 0$.

Considering $d$ as the metric, suppose that

$$x_k \to x \qquad \text{for } x_k, x \in \mathbb{R}^n.$$

That is,

$$\forall \epsilon > 0, \exists K : k \geq K \implies d(x_k, x) < \epsilon.$$

Using $\delta$, we may make an equivalent statement, choosing $\epsilon > 0$ such that $\epsilon' = C\epsilon$. Considering that $e > 0 \implies \exists K : \forall k \geq K \implies d(x_k, x) < \epsilon$ then,

$$\delta(x_k, x) \leq Cd(x_k, x) < C\epsilon = \epsilon'.$$

That is, $\delta(x_k, x) < \epsilon'$. Hence $x_k \to x$ using an equivalent metric $\delta$.

**Cauchy Sequences** A sequence $\{x_K\} \in \mathbb{R}$ is a Cauchy sequence if

$$\exists \epsilon > 0 \text{ such that } k, I > K \implies d(x_k, x_l) < \epsilon.$$

**Cauchy Sequences and Convergence** The following are equivalent:

$$\text{A sequence } \{x_k\} \text{ converges in } \mathbb{R}^2 \iff \{x_k\} \text{ is a Cauchy Sequence.}$$

## 1.5 Open and Closed Sets

**Definitions** Consider $x_k$

- $x_0 \in \Omega$ is an interior points of $\Omega$ if there is a ball around $x$ completely contained in $\Omega$. That is, there exists a $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq \Omega$.
- $\Omega$ is open if every point of $\Omega$ is an interior point.
- $\Omega$ is closed if its complement is open.
- $x_0 \in \Omega$ is a boundary point of $\Omega$ if every ball around $x_0$ contains points in $\Omega$ and points not in $\Omega$.

**Closed Sets** A set $\Omega \subset \mathbb{R}$ is closed iff and only if it contains all of its boundary points.

**Limit Points and Sets** $x_0$ is a limit point of $\Omega$ if there is a sequence $\{x_i\}$ in $\Omega$ with limit $x_0$ and $x_i \neq x$.

- Every interior points of $\Omega$ is a limit point of $\Omega$.
- $x_0$ is not necessarily in $Omega$
- A set is closed $\Leftrightarrow$ it contains all of its limit points.

**Variations of a Set** Consider the set $\Omega \in \mathbb{R}^n$.

- The interior of $\Omega$ is the set of all its interior points.
- The boundary $\partial\Omega$) of $\Omega$ is the set of all its boundary points.
- The closure of $\Omega$: $\bar{\Omega} = \Omega \cup \partial\Omega$.

The interior is the largest open subset and the closure is the smallest closed set containing $\Omega$.

**Limit of a Function at a Point** For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $\lim_{x \to x_0}$ means that

$$\forall \epsilon \exists \delta > 0 \text{ such that for } x \in \Omega :$$
$$0 < d(x, x_0) < \delta \implies d(f(x), b) < \epsilon.$$

Alternatively,

$$x \in B(x_0, \delta) \backslash \{x_0\} \implies f(x) \in B(b, \epsilon).$$

It is sufficient to consider the limits of the components of a function.

**Limits and sequences** The limit $\lim_{x \to a} f(x) = b$ exists if and only if, $\lim_{k \to \infty} f(x_k) = b$ for all sequences $x_k$ such that $x_k$ is an element of $\Omega$ and, $\lim_{k \to \infty} x_k = a$.

This is very helpful for showing that a limit does not exists.

### 1.6 Pinching and IVT Theorem

**Pinching Theorem**

**IVT** see 1141

# 2 Differentiation

## 2.1 Differentiability, Derivatives and Affine Approximations

**Differentiability in $\mathbb{R}$** A function $f : \mathbb{R} \to \mathbb{R}$ being differentiable at some $a \in \mathbb{R}$ implies that there exists a *good* straight-line approximation to $f$ at $a$ called a *tangent line*. This function may be found as

$$T(x) = f(a) + f'(a)(x - a) = f(a) - f'(a)a + f'(a)x = y_0 + L(x)$$

where for all $a$, $y = f(a) - f'(a)a$ and $L : \mathbb{R} \to \mathbb{R} = f'(a)x$.

Recall that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

.

**Affine Maps** A function $T : \mathbb{R}^n \to \mathbb{R}^m$ being affine means that there exists a $y_0$ such that for all $x \in \mathbb{R}^n$

$$T(x) = y_0 + L(x)$$

.

In $T : \mathbb{R} \to \mathbb{R}$ this sis of the form $y = mx + b$.

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if there is a good affine approximation to $f$ of the form

$$T(x) = f(a) - f'(a)a + f'(a)x.$$

In this context good implies that $f'(x)$ is defined in the usual manner and exists.

**Differentiability in $\mathbb{R}^n \to \mathbb{R}^n$** A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable for some $a \in \Omega$ if there exists a linear map $L : \mathbb{R}n \to \mathbb{R}^m$ such that

$$\lim_{x \to a} \frac{||f(x) - f(a) - L(x - a)||}{||L(x - a)||} = 0.$$

Notation: the matrix of the linear map $L$, the derivative of $f$ at $a$ is denoted by $D_a f$.

**Delta Epsilon Definition of Differentiability** A function $f : \Omega \subset \mathbb{R} \to \mathbb{R}^m$ is differentiable on $a \in \Omega$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $\forall \epsilon > 0 \exists \delta > 0$ such that for all $x \in \Omega$

$$||x - a|| < \delta \Rightarrow ||f(x) - f(a) - L(x - a)|| < \epsilon \, ||x - a|| \, .$$

**Clairaut's Theorem / Mixed Derivative Theorem** Suppose $f, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^2 f}{\partial x_j \partial x_i}$ all exist and are continuous on an open set around $a$ then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

That is, the partial derivatives commute.

**Differentiability and Continuity** Differentiability implies continuity. However, continuity does not imply differentiability. The proof of this is contingent on the fact that for $x \in \mathbb{R}^n$ and a $m \times n$ matrix $L$

$$\lim x \to 0 \, ||Lx|| = 0.$$

**Partial Derivatives and Differentiability** Suppose that $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$. If all partial derivatives $\dfrac{\partial f_j}{\partial x_i}$ exist for integers $i \in [1, n]$, $j \in [1, m]$ then $f$ is differentiable on $\Omega$.

## 2.2 Gradients, Affine Approximations and Matrices

**Jacobian Matrices** Suppose that all partial derivatives of $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ exist for some $a \in \Omega$. Then, the Jacobian matrix of $f$

$$J_a f = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}$$

may be evaluated at a point $a$. Where $f$ is differentiable, its derivative is given by the Jacobian matrix.

Note however, that the Jacobian Matrix may exist even where $f$ is not differentiable.

## 2.3 Gradients, Tangent Planes and Affine Approximations

**Gradient** For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, if the Jacobian exists, then it is given by the $1 \times n$ matrix

$$Jf = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

This is equivalent to the gradient of $f$. That is,

$$\text{grad}(f) = \boldsymbol{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

**Affine Approximations** Allow $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ to be a differentiable function at $a \in \Omega$. The best affine approximation to $f$ at $a$ may be written in terms of the gradient vector as

$$T(x) = f(a) + \boldsymbol{\nabla} f(a) \cdot (x - a).$$

**Tangent Planes** The tangent plane to a function $z = f(x, y)$ is given by

$$z = T(x, y).$$

## 2.4 Chain Rule, Directional Derivatives and Tangent Planes

**Chain Rule** Suppose that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : \Omega' \subset \mathbb{R}^m \to \mathbb{R}^P$ where $f(\Omega) = \Omega'$. If $f$ and $g$ are both differentiable then, so is $g \circ f : \Omega \to \mathbb{R}^p$ such that

$$D_a(g \circ f) = D_(f_(a))gD_af.$$

Equivalently,

$$D(g \circ f)(a) = Dg(f(a))Df(a).$$

**Directional Derivative** The directional derivative of $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ in the direction of the unit vector $u$ at a point $a \in \Omega$ is

$$D_u f(a) = f'_u(a) = \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}.$$

Equivalently, if $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a$ then for a unit vector $u$

$$D_u f(a) = Df(a) \cdot u = \boldsymbol{\nabla} f(a) \cdot u.$$

Alternatively, allowing $\theta$ to be the angle between $\boldsymbol{\nabla} f(a)$ and $u$,

$$D_u f(a) = |\boldsymbol{\nabla} f(a)| \cdot |u| \cdot \cos \theta.$$

**Tangent Planes** Consider the surface in $\mathbb{R}^3$ defined by $\phi(x, y, z) = \lambda$. where $\lambda$ is constant and $\phi$ is differentiable.

Let $c(t) = (c_1(t), c_2(t), c_3(t))$ be a differentiable curve lying on the vector space with a tangent vector given by $c'(t) = (c_1'(t), c_2'(t), c_3'(t))$.

Since all points $c(t)$ lie on the surface, $\phi(c(t)) = \lambda$. Thus,

$$D(\phi(c(t)))Dc(t) = 0 \Rightarrow \boldsymbol{\nabla}\phi c'(t) = 0.$$

Therefore, all curves passing through a point $P$ on the surface have tangent vector normal to $\boldsymbol{\nabla}\phi$. Thus, they all lie in the tangent plane at $P$.

## 2.5 Taylor Series and Theorem

**Taylor's Theorem** For all continuous and differentiable functions $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) \approx P_k(a) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(x)(x-a)^n. + R$$

where the remainder $R$ is

$$R = \frac{1}{(k+1)!} f^{(k+1)}(z)(x-a)^k.$$

for some $z$ between $x$ and $a$.

$P_0, P_1, P_2, P_3$ are the best constant, affine, quadratic, cubic approximations.

**Generalizing Taylor's Theorem** Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be $C^r$ on the open set $\Omega$. Let $a \in \Omega$ be such that the line segment joining $a$ and $x$ lies entirely in $\Omega$. Then,

$$f(x) = P_{r,a}(x) + R_{r,a}(a)$$

where for some $z$ on the line segment between $a$ and $x$

$$P_{r,a}(x) = f(a) + \sum_{k=1}^{r-1} \frac{1}{k!} D^k f(a) \cdot (x-a)^k, \quad , R_{r,a}(a) = \frac{1}{r!} D^r f(z) \cdot (x-a)^r$$

Note that $\cdot$ is not a dot product.

**Second Degree Taylor Series** Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be $C^3$ on $\Omega$. Then, ignoring the remainder,

$$f(x) = f(a) + \boldsymbol{\nabla} f(a) \cdot (x-a) + \frac{1}{2}\left((x-a) \cdot (Hf(a) \cdot (x-a).)\right)$$

where $H$ is the Hessian Matrix.

## 2.6 Hessian Matrix and Stationary Points

**Hessian Matrix** For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, the Hessian matrix of $f$ at a point $a$ is the $n \times n$ matrix

$$H(f,a) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_1} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_3} & \dfrac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_1} \end{pmatrix}.$$

**Revision: Trace and Determinant** Recall that the trace is the sum of its diagonal values. The trace of a hessian matrix is also the sum of its eigenvalues. Also, the determinant is the product of the eigenvalues. The eigenvalues of a matrix $A$ can be found by calculating solutions to

$$|A - \lambda I| = 0.$$

**Definite and Semi Definite Matrices** For a $n \times n$ symmetric matrix $H$,

- All eigenvalues are $> 0 \Leftrightarrow$ positive definite
- All eigenvalues are $< 0 \Leftrightarrow$ negative definite
- All eigenvalues are $\geq 0 \Leftrightarrow$ positive semidefinite
- All eigenvalues are $\leq 0 \Leftrightarrow$ negative semidefinite

**Sylvester's Criterion (for the Definite Property)** Allow $H_k$ to be the upper left $k \times k$ sub-matrix of $h$ and let $\Delta_k = \det H_k$. Then,

- positive definite $\quad \Leftrightarrow \Delta_k > 0 \forall k$
- positive semidefinite $\quad \Leftrightarrow \Delta_k \leq 0 \forall k$
- negative definite $\quad \Leftrightarrow \Delta_k < 0$ for all odd $k$ and $\Delta > 0$ for all even $k$
- negative semidefinite $\quad \Leftrightarrow \Delta_k \leq 0$ for all odd $k$ and $\Delta \geq 0$ for all even $k$

**The Definite Property and Classification of Stationary Points** Suppose that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is $C^2$ and $\boldsymbol{\nabla} f(a) = 0$ at an interior point $a$ of $\Omega$. Then

- $H(f,a)$ is a positive definite $\Rightarrow f$ has a local maximum at $a$
- $H(f,a)$ is a negative definite $\Rightarrow f$ has a local minimum at $a$
- $f$ has a local minimum at $a \Rightarrow H(f,a)$ is a positive semidefinite
- $f$ has a local maximum at $a \Rightarrow H(f,a)$ is a negative semidefinite

Observe carefully that the semidefinite cases can also be saddle points.

## 2.7 Lagrange Multipliers, Implicit and Inverse Function Theorems

**Lagrange Multipliers** Consider two differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$. Lagrange multipliers are useful for finding local extrema of $f$ under the constraint $S = \{x \in \mathbb{R}^n : g(x) = c, c \in \mathbb{R}\}$.

Then, if a local minimum or maximum of $f$ occurs on $a \in S$ then, $\boldsymbol{\nabla} f()$ and $\boldsymbol{\nabla} g(a)$ are parallel. That is, when $\boldsymbol{\nabla} g(a) \neq 0$, there exists $\lambda \in \mathbb{R}$ such that

$$\boldsymbol{\nabla} f(a) = \lambda \boldsymbol{\nabla} g(a).$$

Note that this theorem will only provide possible candidates for minimum or maximum points. There is no guarantee that there exists minimum or maximums of $f$ on $S$.

**Inverse Function Theorem in $\mathbb{R} \to \mathbb{R}$** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable on an interval $I \subset \mathbb{R}$ and $f'(x) \neq 0$ for all $x \in I$. Then, $f$ is invertible on $I$ and the inverse $(f^{-1})'(x)$

is differentiable such that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

That is, if $y = f(x)$ and $f^{-1}$ exist and is differentiable with $x = f^{-1}(y)$ then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

**Generalising the Inverse Function Theorem** Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}^n$ be $C^1$. Suppose that $a \in \Omega$.

If the matrix $Df(a)$ is invertible, then $f$ is invertible on an open set $U$ containing $a$. That is, the inverse exists as

$$f^{-1} : f(U) \to U.$$

Further, $f^{-1}$ is $C^1$ and for $x \in U$,

$$D_{f(x)} f^{-1} = (D_x f)^{-1}.$$

Consequently, $f^{-1}$ has its best affine approximation at $f(a)$ as

$$f^{-1}(x) \approx a + (D_f)^{-1}(x - f(a)).$$

# 3 Integration

# 4 Fourier Series

**Fourier Series** A Fourier series is the approximation of simple periodic functions by the sum of period functions of the form $\sin(x), \cos(x)$. Note that unlike Taylor series, a function $f$ may be discontinuous. However, any lack of continuity leads to an infinite sum in the Fourier series.

## 4.1 Inner Products

**Inner Products** Let $V$ be a real vector space. An inner product on $V$ is a map that assigns each $f, g \in V$ a real number $\langle f, g \rangle$ such that the following properties hold for all $f, g, h \in V$ and $\lambda, \mu \in \mathbb{R}$:

- $\langle f, f \rangle \geq 0$,
- $\langle f, f \rangle = 0$ if and only if $f$ is zero,
- $\langle \lambda f + \mu g, h \rangle, = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$,
- $\langle g, f \rangle = \langle f, g \rangle$.

**Usual Inner Products**

- The vector space $R^n$ admits the following inner product

$$\langle u, v \rangle = u \cdot v = \sum_{i=1}^{n} u_i v_i.$$

- The vector space $C[a, b]$ consisting of all continuous function on the interval $[a, b]$ admits the following inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx.$$

**Inner Product and Orthogonality** We say functions are orthogonal if $\langle f, g \rangle = 0$.

## 4.2 Norms

A norm on $V$ is a map that assigns each $f \in V$ a real number $||f||$ such that $\forall f \in V, \lambda \in \mathbb{R}$

- $||f|| > 0$,
- $||f|| = 0$ if and only if $f = 0$,
- $||\lambda f|| = \lambda ||f||$,
- $||f + g|| \leq ||f|| + ||g||$; that is, the triangle inequality holds.

**Usual Norms**

- The Euclidian norm ($L^2$-norm): is a norm on $C[a,b]$:

$$||f||_2 = \sqrt{\int_a^b f(x)^2 dx)}$$

- The max norm is a norm on $C[a,b]$:

$$||f||_\infty = \max_{a\leq x\leq b}\{|f(x)|\}$$

## 4.3 Fourier Coefficient and Series

**Fourier Series** Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is $2L$-periodic, - that is, $f(x) = f(x+2L)$ - and is square integrable - that is, $\int_{-L}^{L} f(x)^2 dx < \infty$. Then, $f$ may be represented by a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left[a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right)\right] \quad \forall x \in [-\pi, \pi].$$

This series converges to $f$ as $n \to \infty$.

**Fourier Coefficients**

- $a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)$
- $b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)$

## 4.4 Convergence of Fourier Series

**Continuity** Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$. Suppose that the one-sided limits $f(c^+)$ and $f(c^-)$ exist.

- If $f^{c^+} = f^{c^-} = f(c)$ then $f$ is continuous at $c$,
- If $f^{c^+} = f^{c^-} \neq f(c)$ then $f$ has a removable discontinuity at $c$,
- If $f(c^+) \neq f(c^-)$ then, $f$ has a jump discontinuity at at c.

**Piecewise Continuity** A function is piecewise continuous on $[a,b]$ if and only if

- $f(x^+)$ exists $\forall x \in [a,b]$,
- $f(x^-)$ exists $\forall x \in [a,b]$,
- $f$ is continuous on $(a,b)$ except at most a finite number of points.

Note that if $f$ is only piecewise continuous then the partial sum of the Fourier series does not necessarily converge to $f$ for all $x$.

**Piecewise differentiability** A function $f$ is differentiable on $c$ if and only if $f(c^+) = f(c^-) = f(c)$ and $D^+f(c) = D^-f(c)$

Note: $D^+f(c)$ is not necessarily the same as $\lim_{x \to c^+} f'(x)$.

A function is piecewise differentiable on $[a, b]$ if and only if

- $D^+f(x^)$ exists $\forall x \in [a, b)$,
- $D^-f(x^)$ exists $\forall x \in (a, b]$,
- $f$ is differentiable on $(a, b)$ except at most a finite number of points.

**Pointwise convergence** Let $c \in \mathbb{R}$. Suppose that a function has the following properties

- $f$ is $2L$ periodic,
- $f$ is piecewise continuous on $[-L, L]$,
- $D^+f(c), D^-f(c)$ exist.

Then,

$$S_f(c) = \frac{1}{2}[f(c^+) + f(c^-)].$$

Observe that if $f$ is continuous at $c$ then $S_f(c) = f(c)$.

**Odd and Evenness** Recall that odd and even functions are defined by the conditions $f(-x) = -f(x)$ and $f(x) = f(-x)$ respectively.

The following elementary properties hold:

- Odd $\times$ Even $=$ Even,
- Odd $\times$ Odd $=$ Even,
- Even $\times$ Even $=$ Even,
- $\int_{-L}^{L} Odd = 0$.

## 4.5 Convergence of Sequences

**Pointwise convergence** Let $f_k : \mathbb{R} \to \mathbb{R}$. $f_k$ converges to $f$ on $[a, b]$ pointwisely iff and only if for all $x \in [a, b]$, $f_k(x) \to f(x)$ as $k \to infty$.

**Epsilon Delta Definition Pointwise Convergence** For all $x \in [a, b]$, $\epsilon > 0, \exists K$ (which will depend on $\epsilon, x$ such that

$$|f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

**Uniform Convergence** Let $f_k : \mathbb{R} \to \mathbb{R}$. $f_k$ converges to $f$ on [a, b] uniformly if and only if for all $\epsilon > 0, \exists K$ (depending on $\epsilon$ only) such that

$$\sup_{x\in[a,b]} |f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

**Weierstrass test** Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of a function $f$ defined on $[a, b]$. Suppose that there exists a sequence of numbers $c_k$ such that

$$|f_k(x)| \leq c_k \quad \forall x \in [a, b]$$

where $\sum_{k=1}^{\infty} c_k$ converges to a real number. Then $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function $f$ on $[a, b]$.

Note that this test also holds for function $f : \mathbb{R}^n \to \mathbb{R}$ for $x \in \Omega$ where $\Omega$ is a closed bounded set in $\mathbb{R}^n$.

**Norm Convergence** Using the supremum norm, the definition of uniform convergence can be equivalently written as: for all $\epsilon > 0, \exists K$ such that

$$||f_k - f|| \leq \epsilon \quad \forall k \geq K.$$

Equivalently,

$$\lim_{k\to\infty} ||f_k - f|| = 0.$$

We may extend this to define norm-convergence for any norm.

**Extending Norm Convergence to L-2** Recall from the previous paragraph that norm-convergence is defined as follows:

$$\lim_{k\to\infty} ||f_k - f|| = 0.$$

As such, $L^2$ norm convergence, also knows as mean square convergence is equivalent to the following

$$\lim_{k\to\infty} \int_a^b [f_k(x) - f(x)]^2 dx = 0.$$

**Parseval Theorem** Let $f$ be a $2\pi$ periodic and bounded function where $\int_{-pi}^{\pi} f(x)^2 dx < +\infty$. Then, the Fourier series of $f$ converges to $f$ in the mean square sense. Moreover, the Parseval's identity holds

$$\int_{-pi}^{\pi} f(x)^2 = ||f||_2^2 = \frac{\pi}{2} a_0 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This identity continues to hold for $2L$ periodic functions integrated over $[-L, L]$.

# 5 Vector Fields

## 5.1 Vector Fields and Flows, Divergence and Curl

**Flow Lines** If $F$ is a vector field, a flow line for $F$ is a path $c(t)$ such that

$$c'(t) = F(c(t)).$$

That is, that $F$ yields the velocity field of the path $c(t)$.

**The Del $\nabla$ operator** The vector differential operator $\nabla$ may be considered a symbolic vector. The differential operator may be written as

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k.$$

**Divergence** Given a field $F = (f_1, f_2, \ldots, f_n)$, the divergence of $F$ is

$$\text{div} F = \nabla \cdot F = \sum_{i=1}^{n} \nabla f_i.$$

Divergence may be thought as a type of derivative that describes the measure at which a vector field *spreads away* from a certain point. If the divergence is positive, then there is a net outflow while there is net inflow if the divergence is negative.

Observe that the divergence of a vector field will be real-valued.

**Curl** If $F$ is a vector field, then the curl may be defined as

$$\text{curl} F = \nabla \times F.$$

Curl is also analogous to a type of derivative for vector fields. The curl may be though as the measure at which the vector field *swirls* around a point. A positive swirl can be thought of as a counter clockwise rotation.

Observe that the curl of a vector field is also a vector field.

## 5.2 Vector Identities

### Basic Vector Identities

1. $\nabla(f + g) = \nabla f + \nabla g$
2. $\nabla(\lambda f) = \lambda \nabla f$ where $\lambda \in \mathbb{R}$
3. $\nabla(fg) = g\nabla f + f\nabla g$. You may draw analogies to the product.
4. $\nabla \dfrac{f}{g} = \dfrac{f\nabla g - g\nabla f}{g^2}$ where $g \neq 0$. This is analogous to the quotient rule.

5. $\nabla \cdot (F + G) = \nabla \cdot F + \nabla \cdot G$

6. $\nabla \times (F + G) = \nabla \times F + \nabla \times G$

7. $\nabla \cdot (fF) = f\nabla \cdot F = F \cdot \nabla f$

8. $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$

9. $\nabla \cdot (\nabla \times F) = 0$

10. $\nabla \times (fF) = f\nabla \times F = \nabla f \times F$

11. $\nabla \times (\nabla f) = 0$

12. $\nabla^2(fg) = f\nabla^2 g + 2((\nabla f \cdot \nabla g)) + g\nabla^2 f$

13. $\nabla \cdot (\nabla \times \nabla g) = 0$

14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla f^2$

# 6 Path Integrals

## 6.1 Path Integrals

**Path (Scalar Line) Integrals** Suppose that a vector-valued function $c(t)$ parametrises a curve $C$ for $t \in [a, b]$. The scalar line integral may be thought as the integral of along $c$.

**Computing a Scalar Line Integral** Let $c(t)$ parametrise a curve $C$ for $t \in [a, b]$. Assume that $f(x, y, z)$ and $c(t)$ are continuous. Then,

$$\int_C f(x, y, z)ds = \int_a^b f(c(t)) \cdot ||c(t)||dt.$$

**Elementary Properties of Path Integrals**

- $\int_C f_1 ds + \int_C f_2 ds = \int_C (f_1 + f_2)ds$,
- $\int_C \lambda f ds = \lambda \int_C f ds, \quad \lambda \in \mathbb{R}$.

## 6.2 Applications Of Path Integrals

Suppose that $\delta = \delta(x, y, z)$ which is a density function.

**Mass**

$$M = \int_C \delta(x, y, z)ds.$$

**First Moments About the Coordinate Plane**

- $M_{yz} = \int_C x\delta ds$
- $M_{xz} = \int_C y\delta ds$
- $M_{xy} = \int_C z\delta ds$

**Coordinates of Center of Mass**

- $\bar{x} = \frac{M_{yz}}{M}$
- $\bar{y} = \frac{M_{xz}}{M}$
- $\bar{z} = \frac{M_{xy}}{M}$

**Moments of Inertia About Axes**

- $I_x = \int_C (y^2 + z^2)\delta ds$
- $I_x = \int_C (x^2 + z^2)\delta ds$
- $I_x = \int_C (x^2 + y^2)\delta ds$

# 7 Vector Line Integrals

**Vector Line Integrals** Vector line integrals are different from scalar line integrals in the sense that to define a vector line integral, we must specify a direction along the path or curve $C$.

**Computing a Vector Line Integral** Let $c(t)$ parametrise an oriented curve $C$ for $t \in [a, b]$. Then,

$$\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t).$$

**Link to Path Integrals** Suppose that $C$ is a smooth curve with a parametrisation $c(t)$ for $t \in [a, b]$ where $c(t)$ is continuously differentiable and $c'(t) \neq 0$ for all $t \in [a, b]$.

Then, $c'(t)$ is a non-zero tangent vector pointing in the forward direction and the unit tangent vector is

$$T(c(t)) = \frac{c'(t)}{||c'(t)||}.$$

Then,

$$\int_C F \cdot ds = \int_C F \cdot T ds.$$

**Summing Paths** Suppose that $C$ is made of $n$ finitely many paths $C_i$. Then, $C = \sum_i^n C_i$. Note that all the curves must be joined end to end. Then,

$$\int_C F \cdot ds = \sum_i^n \int_{C_i} F \cdot ds.$$

**Work and Other Alternative Notations** Suppose that $c(t) = (x(t), y(t), z(t))$ and $F = (M, N, P)$. Then, we denote work as any of the following notations

$$\begin{aligned} W &= \int_C F \cdot ds \\ &= \int_a^b \left( M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt \\ &= \int_C Mds + Nds + Pds. \end{aligned}$$

**Properties fo Line Integrals**

- Linearity
- Reversing Orient
- Additivity

**Flow Integrals and Circulation** Suppose that $F$ represents a velocity field of a fluid flowing through a region in space. Then, the flow across a curve may be defined as the following

$$\text{Flow} = \int_a^b F \cdot \hat{T} ds.$$

This integral is called the flow integral. If the curve is a closed loop then this is called the *circulation* around the curve.

**Flux in the Plane** If $C$ is a smooth closed curve in the domain of a continuous vector field $F = M(x,y)i + N(x,y)j + N(x,y)j$ and, $n$ is the outward pointing unit-normal on $C$ then, the flux of $f$ across $C$ is the following expression

$$\int_C F \cdot \hat{n} ds.$$

**Calculating Flux Across a Smooth Closed Plane Curve** Suppose that $F = Mi = Nj$. Let $G = -N, M$

$$\text{Flux of } F \text{ across } C = \oint_C Mdy - Ndx = \oint_C Gds$$

## 7.1 Fundamental Theorem of Line Integrals

**Gradient Fields** A vector field $F$ is called a gradient vector field if there exists a real-valued function $\phi$ such that $F = \nabla\phi$. That is,

$$\begin{pmatrix} M \\ N \\ P \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \phi}{\partial z} \end{pmatrix}.$$

If such a function $\phi$ exists then $\phi$ is called the potential function of $F$ where, $F$ is *conservative.*

**Fundamental Theorem for Gradient Vector Fields** If $F = \nabla\phi$ on a domain $\mathcal{D}$. Then, for all oriented curves $C$ in $\mathcal{D}$ with an initial point $P$ and a terminal point $Q$,

$$\int_C F \cdot ds = \phi(Q) - \phi(P).$$

The integral is independent of the path.

**Cross Partials of Gradient Vector Fields are Equal** Let $F = (F_1, F_2, F_3)$ be a gradient vector field whose components have continuous partial derivatives. Then, the cross partials are equal. That is,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x},$$
$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y},$$
$$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

Equivalently,

$$\boldsymbol{\nabla} \times F = 0.$$

## 7.2 Green's Theorem

Green's Theorem connects double integrals and line integrals.

**Green's Theorem: Flux Divergence or Normal Form** Let $D$ be a bounded simple region in $\mathbb{R}^2$ with a nonempty interior whose boundary consists of a finite number of smooth curves. Let $C$be the boundary of $D$ with a positive (counter-clockwise) direction. Let $F = Mi + Nj$ be a continuously different boundary vector field on D.

Then, the outward flux of $F$ across the curve $C$ equals the double integral of divergence $\boldsymbol{\nabla} \cdot F$ over $D$. That is,

$$\oint_C (F \cdot \hat{n}) ds = \oint_C -N dx + M dy = \int\int_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial Y} \right).$$

Once again, note the assumptions:

- $D$ is bounded and simple with a non-empty interior,
- The boundary $C$ is oriented in the positive (counter-clockwise) direction, and is the finite union of smooth curves,
- The vector field $F$ is continuously differentiable on $D$.

**Green's Theorem: Circulation-Curl or Tangential Form** Let $D$ be a bounded simple region in $\mathbb{R}^2$ with a nonempty interior whose boundary consists of a finite number of smooth curves. Let $C$be the boundary of $D$ with a positive (counter-clockwise) direction. Let $F = Mi + Nj$ be a continuously different boundary vector field on D.

Then, the counter-clockwise circulation of $F$ around $C$ equals the double integral of $(\boldsymbol{\nabla} \times F) \cdot k$ over $D$. That is,

$$\oint_C (F \cdot \hat{T}) ds = \oint_C M dx + N dy = \int \int_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

**Area of a Region** Let $D$ be a simple and bounded region with a non-empty interior and let $C$ be the boundary of $D$ which is the finite union of smooth curves. Then, the area of $D$ can be calculated as such

$$\text{Area}(D) = \frac{1}{2} \oint_C (-y dx + x dy) .$$