Higher Several Variable Calculus
Math2111 UNSW

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Contents

1 Curves and Surfaces

1.1 Curves

Curves A curve in $\mathbb{R}^n$ is a vector function

$$\mathbf{c} : I \to \mathbb{R}^n,$$

where I is an interval in $\mathbb{R}$.

Forms / Notations Curves may be defined in the following ways:

- **Parametrically** by $c(t) = (x_1(t), x_2(t), \ldots, x_n(t))$
- **Cartesian** by eliminating the t variable to get y in terms of x
- **Implicitly** As $F(x, y) = 0$.

2 Analysis

2.1 Assumed

Assumed Concepts from Real Single-Variable Calculus

- limits
- continuity
- differentiability
- integrability

Assumed Theorems

- Min/ Max Theorem
- Intermediate Value Theorem
- Mean Value Theorem

2.2 Limits

Recall that $\lim_{x \to a} f(x) = L$ requires that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$ then

$$|f(x) - L| < \delta.$$

2.3 Metrics

We have metrics (distance functions) as

$$m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

satisfying the following 3 axioms.

- **Positive Definite** such that for all $x, y \in \mathbb{R}^n$, $m(x, y) > 0$ and, $m(x, y) = 0 \iff x = y$.
- **Symmetric** $m(x, y) = m(y, x)$.z
- **Triangle Inequality** such that for all $x, y, z \in \mathbb{R}^n$, $m(x, y) + m(y, z) \leq m(x, z)$.

Euclidian Distance We allow the Euclidian distance to be defined as

$$d_n(x, y) := ||x - y|| = \sqrt{\sum_{i=1}^{n} (x_i - y_y)^2}$$

We often allow d to be d_2.

Norms Norms will be revisited in the Fourier Series section. They can be thought of as the length of an element in vectors space.

Equivalent Metrics Two metrics d and δ are considered equal if there exists constants $0 < c < C < \infty$ such that
$$c\delta(x, y) \leq d(x, y) \leq C\delta(x, y).$$

2.4 Limits of Sequences

Balls A ball around $\vec{a} \in \mathbb{R}$ is of radius ϵ is the set

$$B(\vec{a}, \epsilon), = \{x \in \mathbb{R} : d(\vec{a}, x) < \epsilon\}.$$

Limit in Sequence For a sequence $\{x_i\}$ of points in $\mathbb{R}^n$, x is the limit of the sequence if and only if
$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \leq \epsilon.$$
Equivalently,
$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \in B(x, \epsilon).$$

Theorems with Limits of Sequences

$$\begin{aligned} &\text{A sequence } x_k \text{ converges to a limit } x \\ &\qquad \Leftrightarrow \text{The components of } x_k \\ &\qquad\quad \text{converge to the componetents of } x \\ &\qquad \Leftrightarrow d(x_k, x) \to 0. \end{aligned}$$

Limits and Equivalent Metrics Suppose that d and δ are two equivalent metrics. That is, $cd(x, y) \leq \delta(x, y) \leq Cd(x, y)$ for $c, C > 0$.

Considering d as the metric, suppose that

$$x_k \to x \quad \text{for } x_k, x \in \mathbb{R}^n.$$

That is,
$$\forall \epsilon > 0, \exists K : k \geq K \implies d(x_k, x) < \epsilon.$$
Using δ, we may make an equivalent statement, choosing $\epsilon > 0$ such that $\epsilon' = C\epsilon$. Considering that $e > 0 \implies \exists K : \forall k \geq K \implies d(x_k, x) < \epsilon$ then,

$$\delta(x_k, x) \leq Cd(x_k, x) < C\epsilon = \epsilon'.$$

That is, $\delta(x_k, x) < \epsilon'$. Hence $x_k \to x$ using an equivalent metric δ.

Cauchy Sequences A sequence $\{x_K\} \in \mathbb{R}$ is a Cauchy sequence if

$$\exists \epsilon > 0 \text{ such that } k, I > K \implies d(x_k, x_l) < \epsilon.$$

Cauchy Sequences and Convergence The following are equivalent:

$$\text{A sequence } \{x_k\} \text{ converges in } \mathbb{R}^2 \iff \{x_k\} \text{ is a Cauchy Sequence.}$$

2.5 Open and Closed Sets

Definitions Consider x_k

- $x_0 \in \Omega$ is an interior points of Ω if there is a ball around x completely contained in Ω. That is, there exists a $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq \Omega$.
- Ω is open if every point of Ω is an interior point.
- Ω is closed if its complement is open.
- $x_0 \in \Omega$ is a boundary point of Ω if every ball around x_0 contains points in Ω and points not in Ω.

Closed Sets A set $\Omega \subset \mathbb{R}$ is closed iff and only if it contains all of its boundary points.

Limit Points and Sets x_0 is a limit point of Ω if there is a sequence $\{x_i\}$ in Ω with limit x_0 and $x_i \neq x$.

- Every interior points of Ω is a limit point of Ω.
- x_0 is not necessarily in $Omega$
- A set is closed $\Leftrightarrow$ it contains all of its limit points.

Variations of a Set Consider the set $\Omega \in \mathbb{R}^n$.

- The interior of Ω is the set of all its interior points.
- The boundary $\partial\Omega$) of Ω is the set of all its boundary points.
- The closure of Ω: $\bar{\Omega} = \Omega \cup \partial\Omega$.

The interior is the largest open subset and the closure is the smallest closed set containing Ω.

Limit of a Function at a Point For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $\lim_{x \to x_0}$ means that

$$\forall \epsilon \exists \delta > 0 \text{ such that for } x \in \Omega :$$
$$0 < d(x, x_0) < \delta \implies d(f(x), b) < \epsilon.$$

Alternatively,

$$x \in B(x_0, \delta) \backslash \{x_0\} \implies f(x) \in B(b, \epsilon).$$

It is sufficient to consider the limits of the components of a function.

Limits and sequences The limit $\lim_{x\to a} f(x) = b$ exists if and only if, $\lim_{k\to\infty} f(x_k) = b$ for all sequences x_k such that x_k is an element of Ω and, $\lim_{k\to\infty} x_k = a$.

2.6 Soon

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