

Higher Several Variable Calculus

Math2111 UNSW

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1 Curves and Surfaces

1.1 Curves

Curves A curve in \mathbb{R}^n is a vector function

$$\mathbf{c} : I \rightarrow \mathbb{R}^n,$$

where I is an interval in \mathbb{R} .

Forms / Notations Curves may be defined in the following ways:

- **Parametrically** by $c(t) = (x_1(t), x_2(t), \dots, x_n(t))$
- **Cartesian** by eliminating the t variable to get y in terms of x
- **Implicitly** As $F(x, y) = 0$.

2 Analysis

2.1 Assumed

Assumed Concepts from Real Single-Variable Calculus

- limits
- continuity
- differentiability
- integrability

Assumed Theorems

- Min/ Max Theorem
- Intermediate Value Theorem
- Mean Value Theorem

2.2 Limits

Recall that $\lim_{x \rightarrow a} f(x) = L$ requires that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$ then

$$|f(x) - L| < \delta.$$

2.3 Metrics

We have metrics (distance functions) as

$$m : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

satisfying the following 3 axioms.

- **Positive Definite** such that for all $x, y \in \mathbb{R}^n$, $m(x, y) > 0$ and, $m(x, y) = 0 \Leftrightarrow x = y$.
- **Symmetric** $m(x, y) = m(y, x)$.
- **Triangle Inequality** such that for all $x, y, z \in \mathbb{R}^n$, $m(x, y) + m(y, z) \leq m(x, z)$.

Euclidian Distance We allow the Euclidian distance to be defined as

$$d_n(x, y) := \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

We often allow d to be d_2 .

Norms Norms will be revisited in the Fourier Series section. They can be thought of as the length of an element in vectors space.

Equivalent Metrics Two metrics d and δ are considered equal if there exists constants $0 < c < C < \infty$ such that

$$c\delta(x, y) \leq d(x, y) \leq C\delta(x, y).$$

2.4 Limits of Sequences

Balls A ball around $\vec{a} \in \mathbb{R}$ is of radius ϵ is the set

$$B(\vec{a}, \epsilon) = \{x \in \mathbb{R} : d(\vec{a}, x) < \epsilon\}.$$

Limit in Sequence For a sequence $\{x_i\}$ of points in \mathbb{R}^n , x is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \leq \epsilon.$$

Equivalently,

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \in B(x, \epsilon).$$

Theorems with Limits of Sequences

A sequence x_k converges to a limit x

$$\begin{aligned} &\Leftrightarrow \text{The components of } x_k \\ &\quad \text{converge to the components of } x \\ &\Leftrightarrow d(x_k, x) \rightarrow 0. \end{aligned}$$

Limits and Equivalent Metrics Suppose that d and δ are two equivalent metrics. That is, $cd(x, y) \leq \delta(x, y) \leq Cd(x, y)$ for $c, C > 0$.

Considering d as the metric, suppose that

$$x_k \rightarrow x \quad \text{for } x_k, x \in \mathbb{R}^n.$$

That is,

$$\forall \epsilon > 0, \exists K : k \geq K \implies d(x_k, x) < \epsilon.$$

Using δ , we may make an equivalent statement, choosing $\epsilon > 0$ such that $\epsilon' = C\epsilon$. Considering that $\epsilon > 0 \implies \exists K : \forall k \geq K \implies d(x_k, x) < \epsilon$ then,

$$\delta(x_k, x) \leq Cd(x_k, x) < C\epsilon = \epsilon'.$$

That is, $\delta(x_k, x) < \epsilon'$. Hence $x_k \rightarrow x$ using an equivalent metric δ .

Cauchy Sequences A sequence $\{x_k\} \in \mathbb{R}$ is a Cauchy sequence if

$$\exists \epsilon > 0 \text{ such that } k, l > K \implies d(x_k, x_l) < \epsilon.$$

Cauchy Sequences and Convergence The following are equivalent:

A sequence $\{x_k\}$ converges in \mathbb{R}^2 \iff $\{x_k\}$ is a Cauchy Sequence.

2.5 Open and Closed Sets

Definitions Consider x_k

- $x_0 \in \Omega$ is an interior points of Ω if there is a ball around x completely contained in Ω . That is, there exists a $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq \Omega$.
- Ω is open if every point of Ω is an interior point.
- Ω is closed if its complement is open.
- $x_0 \in \Omega$ is a boundary point of Ω if every ball around x_0 contains points in Ω and points not in Ω .

2.6 Soon

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