Higher Several Variable Calculus
Math2111 UNSW

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2022T1

Contents

1 Curves and Surfaces

1.1 Curves

Curves A curve in $\mathbb{R}^n$ is a vector function

$$\mathbf{c} : I \to \mathbb{R}^n,$$

where I is an interval in $\mathbb{R}$.

Forms / Notations Curves may be defined in the following ways:

- **Parametrically** by $c(t) = (x_1(t), x_2(t), \ldots, x_n(t))$
- **Cartesian** by eliminating the t variable to get y in terms of x
- **Implicitly** As $F(x, y) = 0$.

2 Analysis

2.1 Assumed

Assumed Concepts from Real Single-Variable Calculus

- limits
- continuity
- differentiability
- integrability

Assumed Theorems

- Min/ Max Theorem
- Intermediate Value Theorem
- Mean Value Theorem

2.2 Limits

Recall that $\lim_{x\to a} f(x) = L$ requires that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x-a| < \delta$ then

$$|f(x) - L| < \delta.$$

2.3 Metrics

We have metrics (distance functions) as

$$m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

satisfying the following 3 axioms.

- **Positive Definite** such that for all $x, y \in \mathbb{R}^n$, $m(x,y) > 0$ and, $m(x,y) = 0 \iff x = y$.
- **Symmetric** $m(x,y) = m(y,x)$.z
- **Triangle Inequality** such that for all $x, y, z \in \mathbb{R}^n$, $m(x,y) + m(y,z) \leq m(x,z)$.

Euclidian Distance We allow the Euclidian distance to be defined as

$$d_n(x,y) := ||x - y|| = \sqrt{\sum_{i=1}^{n} (x_i - y_y)^2}$$

We often allow d to be d_2.

Norms Norms will be revisited in the Fourier Series section. They can be thought of as the length of an element in vectors space.

Equivalent Metrics Two metrics d and δ are considered equal if there exists constants $0 < c < C < \infty$ such that

$$c\delta(x, y) \leq d(x, y) \leq C\delta(x, y).$$

2.4 Limits of Sequences

Balls A ball around $\vec{a} \in \mathbb{R}$ is of radius ϵ is the set

$$B(\vec{a}, \epsilon), = \{x \in \mathbb{R} : d(\vec{a}, x) < \epsilon\}.$$

Limit in Sequence For a sequence $\{x_i\}$ of points in $\mathbb{R}^n$, x is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \leq \epsilon.$$

Equivalently,

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \in B(x, \epsilon).$$

Theorems with Limits of Sequences

$$\begin{aligned} \text{A sequence } x_k \text{ converges to a limit } x \\ &\Leftrightarrow \text{The components of } x_k \\ &\quad \text{converge to the componetents of } x \\ &\Leftrightarrow d(x_k, x) \to 0. \end{aligned}$$

Limits and Equivalent Metrics Suppose that d and δ are two equivalent metrics. That is, $cd(x, y) \leq \delta(x, y) \leq Cd(x, y)$ for $c, C > 0$.

Considering d as the metric, suppose that

$$x_k \to x \quad \text{ for } x_k, x \in \mathbb{R}^n.$$

That is,

$$\forall \epsilon > 0, \exists K : k \geq K \implies d(x_k, x) < \epsilon.$$

Using δ, we may make an equivalent statement, choosing $\epsilon > 0$ such that $\epsilon' = C\epsilon$. Considering that $e > 0 \implies \exists K : \forall k \geq K \implies d(x_k, x) < \epsilon$ then,

$$\delta(x_k, x) \leq Cd(x_k, x) < C\epsilon = \epsilon'.$$

That is, $\delta(x_k, x) < \epsilon'$. Hence $x_k \to x$ using an equivalent metric δ.

Cauchy Sequences A sequence $\{x_K\} \in \mathbb{R}$ is a Cauchy sequence if

$$\exists \epsilon > 0 \text{ such that } k, I > K \implies d(x_k, x_l) < \epsilon.$$

Cauchy Sequences and Convergence The following are equivalent:

$$\text{A sequence } \{x_k\} \text{ converges in } \mathbb{R}^2 \iff \{x_k\} \text{ is a Cauchy Sequence.}$$

2.5 Open and Closed Sets

Definitions Consider x_k

- $x_0 \in \Omega$ is an interior points of Ω if there is a ball around x completely contained in Ω. That is, there exists a $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq \Omega$.
- Ω is open if every point of Ω is an interior point.
- Ω is closed if its complement is open.
- $x_0 \in \Omega$ is a boundary point of Ω if every ball around x_0 contains points in Ω and points not in Ω.

Closed Sets A set $\Omega \subset \mathbb{R}$ is closed iff and only if it contains all of its boundary points.

Limit Points and Sets x_0 is a limit point of Ω if there is a sequence $\{x_i\}$ in Ω with limit x_0 and $x_i \neq x$.

- Every interior points of Ω is a limit point of Ω.
- x_0 is not necessarily in $Omega$
- A set is closed $\Leftrightarrow$ it contains all of its limit points.

Variations of a Set Consider the set $\Omega \in \mathbb{R}^n$.

- The interior of Ω is the set of all its interior points.
- The boundary $\partial\Omega$) of Ω is the set of all its boundary points.
- The closure of Ω: $\bar{\Omega} = \Omega \cup \partial\Omega$.

The interior is the largest open subset and the closure is the smallest closed set containing Ω.

Limit of a Function at a Point For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $\lim_{x \to x_0}$ means that

$$\forall \epsilon \exists \delta > 0 \text{ such that for } x \in \Omega :$$
$$0 < d(x, x_0) < \delta \implies d(f(x), b) < \epsilon.$$

Alternatively,

$$x \in B(x_0, \delta) \backslash \{x_0\} \implies f(x) \in B(b, \epsilon).$$

It is sufficient to consider the limits of the components of a function.

Limits and sequences The limit $\lim_{x \to a} f(x) = b$ exists if and only if, $\lim_{k \to \infty} f(x_k) = b$ for all sequences x_k such that x_k is an element of Ω and, $\lim_{k \to \infty} x_k = a$.

This is very helpful for showing that a limit does not exists.

2.6 Pinching and IVT Theorem

Pinching Theorem

IVT see 1141

3 Analysis

4 Differentiation

5 Integration

6 Fourier Series

Fourier Series A Fourier series is the approximation of simple periodic functions by the sum of period functions of the form $\sin(x), \cos(x)$. Note that unlike Taylor series, a function f may be discontinuous. However, any lack of continuity leads to an infinite sum in the Fourier series.

6.1 Inner Products

Inner Products Let V be a real vector space. An inner product on V is a map that assigns each $f, g \in V$ a real number $\langle f, g \rangle$ such that the following properties hold for all $f, g, h \in V$ and $\lambda, \mu \in \mathbb{R}$:

- $\langle f, f \rangle \geq 0$,
- $\langle f, f \rangle = 0$ if and only if f is zero,
- $\langle \lambda f + \mu g, h \rangle, = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$,
- $\langle g, f \rangle = \langle f, g \rangle$.

Usual Inner Products

- The vector space R^n admits the following inner product

$$\langle u, v \rangle = u \cdot v = \sum_{i=1}^{n} u_i v_i.$$

- The vector space $C[a,b]$ consisting of all continuous function on the interval $[a,b]$ admits the following inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx.$$

Inner Product and Orthogonality We say functions are orthogonal if $\langle f, g\rangle = 0$.

6.2 Norms

A norm on V is a map that assigns each $f \in V$ a real number $||f||$ such that $\forall f \in V, \lambda \in \mathbb{R}$

- $||f|| > 0$,
- $||f|| = 0$ if and only if $f = 0$,
- $||\lambda f|| = \lambda ||f||$,
- $||f + g|| \leq ||f|| + ||g||$; that is, the triangle inequality holds.

Usual Norms

- The Euclidian norm (L^2-norm): is a norm on $C[a,b]$:

$$||f||_2 = \sqrt{\int_a^b f(x)^2 dx)}$$

- The max norm is a norm on $C[a,b]$:

$$||f||_\infty = \max_{a \leq x \leq b} \{|f(x)|\}$$

6.3 Fourier Coefficient and Series

Fourier Series Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is $2L$-periodic, - that is, $f(x) = f(x + 2L)$ - and is square integrable - that is, $\int_{-L}^{L} f(x)^2 dx < \infty$. Then, f may be represented by a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left[a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right)\right] \quad \forall x \in [-\pi, \pi].$$

This series converges to f as $n \to \infty$.

Fourier Coefficients

- $a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)$
- $b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)$

6.4 Convergence of Fourier Series

Continuity Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$. Suppose that the one-sided limits $f(c^+)$ and $f(c^-)$ exist.

- If $f^{c^+} = f^{c^-} = f(c)$ then f is continuous at c,
- If $f^{c^+} = f^{c^-} \neq f(c)$ then f has a removable discontinuity at c,
- If $f(c^+) \neq f(c^-)$ then, f has a jump discontinuity at at c.

Piecewise Continuity A function is piecewise continuous on $[a, b]$ if and only if

- $f(x^+)$ exists $\forall x \in [a, b]$,
- $f(x^-)$ exists $\forall x \in [a, b]$,
- f is continuous on (a, b) except at most a finite number of points.

Note that if f is only piecewise continuous then the partial sum of the Fourier series does not necessarily converge to f for all x.

Piecewise differentiability A function f is differentiable on c if and only if $f(c^+) = f(c^-) = f(c)$ and $D^+f(c) = D^-f(c)$

Note: $D^+f(c)$ is not necessarily the same as $\lim_{x \to c^+} f'(x)$.

A function is piecewise differentiable on $[a, b]$ if and only if

- $D^+f(x^)$ exists $\forall x \in [a, b)$,
- $D^-f(x^)$ exists $\forall x \in (a, b]$,
- f is differentiable on (a, b) except at most a finite number of points.

Pointwise convergence Let $c \in \mathbb{R}$. Suppose that a function has the following properties

- f is $2L$ periodic,
- f is piecewise continuous on $[-L, L]$,
- $D^+f(c), D^-f(c)$ exist.

Then,

$$S_f(c) = \frac{1}{2}[f(c^+) + f(c^-)].$$

Observe that if f is continuous at c then $S_f(c) = f(c)$.

Odd and Evenness Recall that odd and even functions are defined by the conditions $f(-x) = -f(x)$ and $f(x) = f(-x)$ respectively.

The following elementary properties hold:

- Odd $\times$ Even = Even,
- Odd $\times$ Odd = Even,
- Even $\times$ Even = Even,
- $\int_{-L}^{L} Odd = 0$.

6.5 Convergence of Sequences

Pointwise convergence Let $f_k : \mathbb{R} \to \mathbb{R}$. f_k converges to f on $[a, b]$ pointwisely iff and only if for all $x \in [a, b]$, $f_k(x) \to f(x)$ as $k \to infty$.

Epsilon Delta Definition Pointwise Convergence For all $x \in [a, b]$, $\epsilon > 0, \exists K$ (which will depend on ϵ, x such that

$$|f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

Uniform Convergence Let $f_k : \mathbb{R} \to \mathbb{R}$. f_k converges to f on [a, b] uniformly if and only if for all $\epsilon > 0, \exists K$ (depending on ϵ only) such that

$$\sup_{x \in [a,b]} |f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

Weierstrass test Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of a function f defined on $[a, b]$. Suppose that there exists a sequence of numbers c_k such that

$$|f_k(x)| \leq c_k \quad \forall x \in [a, b]$$

where $\sum_{k=1}^{\infty} c_k$ converges to a real number. Then $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f on $[a, b]$.

Note that this test also holds for function $f : \mathbb{R}^n \to \mathbb{R}$ for $x \in \Omega$ where Ω is a closed bounded set in $\mathbb{R}^n$.

Norm Convergence Using the supremum norm, the definition of uniform convergence can be equivalently written as: for all $\epsilon > 0, \exists K$ such that

$$||f_k - f|| \leq \epsilon \quad \forall k \geq K.$$

Equivalently,

$$\lim_{k \to \infty} ||f_k - f|| = 0.$$

We may extend this to define norm-convergence for any norm.

Extending Norm Convergence to L-2 Recall from the previous paragraph that norm-convergence is defined as follows:

$$\lim_{k\to\infty} ||f_k - f|| = 0.$$

As such, L^2 norm convergence, also knows as mean square convergence is equivalent to the following

$$\lim_{k\to\infty} \int_a^b [f_k(x) - f(x)]^2 dx = 0.$$

Parseval Theorem Let f be a 2π periodic and bounded function where $\int_{-pi}^{\pi} f(x)^2 dx < +\infty$. Then, the Fourier series of f converges to f in the mean square sense. Moreover, the Parseval's identity holds

$$\int_{-pi}^{\pi} f(x)^2 = ||f||_2^2 = \frac{\pi}{2} a_0 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This identity continues to hold for $2L$ periodic functions integrated over $[-L, L]$.

7 Path Integrals

8 Line Integrals

9 Surface Integrals

10 Integral Theorems

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