Higher Several Variable Calculus
Math2111 UNSW

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Contents

1 Analysis

1.1 Assumed

Assumed Concepts from Real Single-Variable Calculus

- limits
- continuity
- differentiability
- integrability

Assumed Theorems

- Min/ Max Theorem
- Intermediate Value Theorem
- Mean Value Theorem

1.2 Limits

Recall that $\lim_{x \to a} f(x) = L$ requires that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$ then

$$|f(x) - L| < \delta.$$

1.3 Metrics

We have metrics (distance functions) as

$$m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

satisfying the following 3 axioms.

- **Positive Definite** such that for all $x, y \in \mathbb{R}^n$, $m(x, y) > 0$ and, $m(x, y) = 0 \iff x = y$.
- **Symmetric** $m(x, y) = m(y, x)$.
- **Triangle Inequality** such that for all $x, y, z \in \mathbb{R}^n$, $m(x, y) + m(y, z) \leq m(x, z)$.

Euclidian Distance We allow the Euclidian distance to be defined as

$$d_n(x, y) := ||x - y|| = \sqrt{\sum_{i=1}^{n} (x_i - y_y)^2}$$

We often allow d to be d_2.

Norms Norms will be revisited in the Fourier Series section. They can be thought of as the length of an element in vectors space.

Equivalent Metrics Two metrics d and δ are considered equal if there exists constants $0 < c < C < \infty$ such that

$$c\delta(x, y) \leq d(x, y) \leq C\delta(x, y).$$

1.4 Limits of Sequences

Balls A ball around $\vec{a} \in \mathbb{R}$ is of radius ϵ is the set

$$B(\vec{a}, \epsilon), = \{x \in \mathbb{R} : d(\vec{a}, x) < \epsilon\}.$$

Limit in Sequence For a sequence $\{x_i\}$ of points in $\mathbb{R}^n$, x is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \leq \epsilon.$$

Equivalently,

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \in B(x, \epsilon).$$

Theorems with Limits of Sequences

$$\begin{aligned} \text{A sequence } x_k \text{ converges to a limit } x &\\ &\Leftrightarrow \text{The components of } x_k \\ &\quad \text{converge to the componetents of } x \\ &\Leftrightarrow d(x_k, x) \to 0. \end{aligned}$$

Limits and Equivalent Metrics Suppose that d and δ are two equivalent metrics. That is, $cd(x, y) \leq \delta(x, y) \leq Cd(x, y)$ for $c, C > 0$.

Considering d as the metric, suppose that

$$x_k \to x \qquad \text{for } x_k, x \in \mathbb{R}^n.$$

That is,

$$\forall \epsilon > 0, \exists K : k \geq K \implies d(x_k, x) < \epsilon.$$

Using δ, we may make an equivalent statement, choosing $\epsilon > 0$ such that $\epsilon' = C\epsilon$. Considering that $e > 0 \implies \exists K : \forall k \geq K \implies d(x_k, x) < \epsilon$ then,

$$\delta(x_k, x) \leq Cd(x_k, x) < C\epsilon = \epsilon'.$$

That is, $\delta(x_k, x) < \epsilon'$. Hence $x_k \to x$ using an equivalent metric δ.

Cauchy Sequences A sequence $\{x_K\} \in \mathbb{R}$ is a Cauchy sequence if

$$\exists \epsilon > 0 \text{ such that } k, I > K \implies d(x_k, x_l) < \epsilon.$$

Cauchy Sequences and Convergence The following are equivalent:

$$\text{A sequence } \{x_k\} \text{ converges in } \mathbb{R}^2 \iff \{x_k\} \text{ is a Cauchy Sequence.}$$

1.5 Open and Closed Sets

Definitions Consider x_k

- $x_0 \in \Omega$ is an interior points of Ω if there is a ball around x completely contained in Ω. That is, there exists a $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq \Omega$.
- Ω is open if every point of Ω is an interior point.
- Ω is closed if its complement is open.
- $x_0 \in \Omega$ is a boundary point of Ω if every ball around x_0 contains points in Ω and points not in Ω.

Closed Sets A set $\Omega \subset \mathbb{R}$ is closed iff and only if it contains all of its boundary points.

Limit Points and Sets x_0 is a limit point of Ω if there is a sequence $\{x_i\}$ in Ω with limit x_0 and $x_i \neq x$.

- Every interior points of Ω is a limit point of Ω.
- x_0 is not necessarily in Ω.
- A set is closed $\Leftrightarrow$ it contains all of its limit points.

Variations of a Set Consider the set $\Omega \in \mathbb{R}^n$.

- The interior of Ω is the set of all its interior points.
- The boundary $\partial\Omega$) of Ω is the set of all its boundary points.
- The closure of Ω: $\bar{\Omega} = \Omega \cup \partial\Omega$.

The interior is the largest open subset and the closure is the smallest closed set containing Ω.

Limit of a Function at a Point For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $\lim_{x \to x_0}$ means that

$$\forall \epsilon \exists \delta > 0 \text{ such that for } x \in \Omega :$$
$$0 < d(x, x_0) < \delta \implies d(f(x), b) < \epsilon.$$

Alternatively,

$$x \in B(x_0, \delta) \backslash \{x_0\} \implies f(x) \in B(b, \epsilon).$$

It is sufficient to consider the limits of the components of a function.

Limits and sequences The limit $\lim_{x\to a} f(x) = b$ exists if and only if, $\lim_{k\to\infty} f(x_k) = b$ for all sequences x_k such that x_k is an element of Ω and, $\lim_{k\to\infty} x_k = a$.

This is very helpful for showing that a limit does not exists.

1.6 Pinching and IVT Theorem

Pinching Theorem Let $\Omega \subset \mathbb{R}^n$, let $\mathbf{a}$ be a limit point of Ω and let $f, g, h : \Omega \to \mathbb{R}$ be functions such that there exists $\epsilon > 0$ such that

$$g(\mathbf{x}) \leq f(\mathbf{x}) \leq h(\mathbf{x}) \qquad \forall \mathbf{x} \in B(\mathbf{a}, \epsilon) \cap \Omega.$$

Then

$$\lim_{x\to\mathbf{a}} g(\mathbf{x}) = \mathbf{b} = \lim_{x\to\mathbf{a}} h(\mathbf{x}) \implies \lim_{x\to\mathbf{a}} f(\mathbf{x}) = \mathbf{b}.$$

IVT Suppose that f is continuous on the closed interval $I = [a, b]$. Then, let $c \in [a, b]$. Suppose that z is some number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then,

$$f(c) = z, \quad \text{where} \quad c \in (a, b)..$$

2 Differentiation

2.1 Differentiability, Derivatives and Affine Approximations

Differentiability in $\mathbb{R}$ A function $f : \mathbb{R} \to \mathbb{R}$ being differentiable at some $a \in \mathbb{R}$ implies that there exists a *good* straight-line approximation to f at a called a *tangent line*. This function may be found as

$$T(x) = f(a) + f'(a)(x - a) = f(a) - f'(a)a + f'(a)x = y_0 + L(x)$$

where for all a, $y = f(a) - f'(a)a$ and $L : \mathbb{R} \to \mathbb{R} = f'(a)x$.

Recall that

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Affine Maps A function $T : \mathbb{R}^n \to \mathbb{R}^m$ being affine means that there exists a y_0 such that for all $x \in \mathbb{R}^n$

$$T(x) = y_0 + L(x).$$

In $T : \mathbb{R} \to \mathbb{R}$ this sis of the form $y = mx + b$.

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if there is a good affine approximation to f of the form

$$T(x) = f(a) - f'(a)a + f'(a)x.$$

In this context good implies that $f'(x)$ is defined in the usual manner and exists.

Differentiability in $\mathbb{R}^n \to \mathbb{R}^n$ A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable for some $a \in \Omega$ if there exists a linear map $L : \mathbb{R}n \to \mathbb{R}^m$ such that

$$\lim_{x \to a} \frac{||f(x) - f(a) - L(x-a)||}{||L(x-a)||} = 0.$$

Notation: the matrix of the linear map L, the derivative of f at a is denoted by $D_a f$.

Delta Epsilon Definition of Differentiability A function $f : \Omega \subset \mathbb{R} \to \mathbb{R}^m$ is differentiable on $a \in \Omega$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $\forall \epsilon > 0 \exists \delta > 0$ such that for all $x \in \Omega$

$$||x-a|| < \delta \Rightarrow ||f(x) - f(a) - L(x-a)|| < \epsilon \, ||x-a|| \, .$$

Clairaut's Theorem / Mixed Derivative Theorem Suppose $f, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^2 f}{\partial x_j \partial x_i}$ all exist and are continuous on an open set around a then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

That is, the partial derivatives commute.

Differentiability and Continuity Differentiability implies continuity. However, continuity does not imply differentiability. The proof of this is contingent on the fact that for $x \in \mathbb{R}^n$ and a $m \times n$ matrix L

$$\lim x \to 0 \, ||Lx|| = 0.$$

Partial Derivatives and Differentiability Suppose that $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$. If all partial derivatives $\dfrac{\partial f_j}{\partial x_i}$ exist for integers $i \in [1, n]$, $j \in [1, m]$ then f is differentiable on Ω.

2.2 Gradients, Affine Approximations and Matrices

Jacobian Matrices Suppose that all partial derivatives of $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ exist for some $a \in \Omega$. Then, the Jacobian matrix of f

$$J_a f = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}$$

may be evaluated at a point a. Where f is differentiable, its derivative is given by the Jacobian matrix.

Note however, that the Jacobian Matrix may exist even where f is not differentiable.

2.3 Gradients, Tangent Planes and Affine Approximations

Gradient For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, if the Jacobian exists, then it is given by the $1 \times n$ matrix

$$Jf = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{pmatrix}.$$

This is equivalent to the gradient of f. That is,

$$\mathrm{grad}(f) = \boldsymbol{\nabla} f = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{pmatrix}.$$

Affine Approximations Allow $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ to be a differentiable function at $a \in \Omega$. The best affine approximation to f at a may be written in terms of the gradient vector as

$$T(x) = f(a) + \boldsymbol{\nabla} f(a) \cdot (x - a).$$

Tangent Planes The tangent plane to a function $z = f(x, y)$ is given by

$$z = T(x, y).$$

2.4 Chain Rule, Directional Derivatives and Tangent Planes

Chain Rule Suppose that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : \Omega' \subset \mathbb{R}^m \to \mathbb{R}^P$ where $f(\Omega) = \Omega'$. If f and g are both differentiable then, so is $g \circ f : \Omega \to \mathbb{R}^p$ such that

$$D_a(g \circ f) = D_{f(a)} g \cdot D_a f.$$

Equivalently,

$$D(g \circ f)(a) = Dg(f(a))Df(a).$$

Directional Derivative The directional derivative of $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ in the direction of the unit vector u at a point $a \in \Omega$ is

$$D_u f(a) = f'_u(a) = \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}.$$

Equivalently, if $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at a then for a unit vector u

$$D_u f(a) = Df(a) \cdot u = \boldsymbol{\nabla} f(a) \cdot u.$$

Alternatively, allowing θ to be the angle between $\boldsymbol{\nabla} f(a)$ and u,

$$D_u f(a) = |\boldsymbol{\nabla} f(a)| \cdot |u| \cdot \cos\theta.$$

Tangent Planes Consider the surface in $\mathbb{R}^3$ defined by $\phi(x, y, z) = \lambda$. where λ is constant and ϕ is differentiable.

Let $c(t) = (c_1(t), c_2(t), c_3(t))$ be a differentiable curve lying on the vector space with a tangent vector given by $c'(t) = (c_1'(t), c_2'(t), c_3'(t))$.

Since all points $c(t)$ lie on the surface, $\phi(c(t)) = \lambda$. Thus,

$$D(\phi(c(t)))Dc(t) = 0 \Rightarrow \boldsymbol{\nabla}\phi c'(t) = 0.$$

Therefore, all curves passing through a point P on the surface have tangent vector normal to $\boldsymbol{\nabla}\phi$. Thus, they all lie in the tangent plane at P.

2.5 Taylor Series and Theorem

Taylor's Theorem For all continuous and differentiable functions $f : \mathbb{R} \to \mathbb{R}$, the taylor series in the neighbourhood of the point a is:

$$f(a) \approx P_k(a) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(a)(x-a)^n + R,$$

where the remainder R is

$$R = \frac{1}{(k+1)!} f^{(k+1)}(z)(x-a)^{k+1},$$

for some z between x and a.

P_0, P_1, P_2, P_3 are the best constant, affine, quadratic, cubic approximations.

Generalizing Taylor's Theorem Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be C^r on the open set Ω. Let $a \in \Omega$ be such that the line segment joining a and x lies entirely in Ω. Then,

$$f(x) = P_{r,a}(x) + R_{r,a}(a)$$

where for some z on the line segment between a and x

$$P_{r,a}(x) = f(a) + \sum_{k=1}^{r-1} \frac{1}{k!} D^k f(a) \cdot (x-a)^k, \quad , R_{r,a}(a) = \frac{1}{r!} D^r f(z) \cdot (x-a)^r$$

Note that $\cdot$ is not a dot product.

Second Degree Taylor Series Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be C^3 on Ω. Then, ignoring the remainder,

$$f(x) = f(a) + \boldsymbol{\nabla} f(a) \cdot (x-a) + \frac{1}{2}\left((x-a) \cdot (Hf(a) \cdot (x-a).)\right)$$

where H is the Hessian Matrix.

2.6 Hessian Matrix and Stationary Points

Hessian Matrix For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, the Hessian matrix of f at a point a is the $n \times n$ matrix

$$H(f,a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}.$$

Revision: Trace and Determinant Recall that the trace is the sum of its diagonal values. The trace of a hessian matrix is also the sum of its eigenvalues. Also, the determinant is the product of the eigenvalues. The eigenvalues of a matrix A can be found by calculating solutions to

$$|A - \lambda I| = 0.$$

Definite and Semi Definite Matrices For a $n \times n$ symmetric matrix H,

- All eigenvalues are $> 0 \Leftrightarrow$ positive definite
- All eigenvalues are $< 0 \Leftrightarrow$ negative definite
- All eigenvalues are $\geq 0 \Leftrightarrow$ positive semidefinite
- All eigenvalues are $\leq 0 \Leftrightarrow$ negative semidefinite

Sylvester's Criterion (for the Definite Property) Allow H_k to be the upper left $k \times k$ sub-matrix of h and let $\Delta_k = \det H_k$. Then,

- positive definite $\quad \Leftrightarrow \Delta_k > 0 \forall k$
- positive semidefinite $\quad \Leftrightarrow \Delta_k \leq 0 \forall k$
- negative definite $\quad \Leftrightarrow \Delta_k < 0$ for all odd k and $\Delta > 0$ for all even k
- negative semidefinite $\quad \Leftrightarrow \Delta_k \leq 0$ for all odd k and $\Delta \geq 0$ for all even k

The Definite Property and Classification of Stationary Points Suppose that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is C^2 and $\boldsymbol{\nabla} f(a) = 0$ at an interior point a of Ω. Then

- $H(f,a)$ is a positive definite $\Rightarrow$ f has a local maximum at a
- $H(f,a)$ is a negative definite $\Rightarrow$ f has a local minimum at a
- f has a local minimum at $a \Rightarrow H(f,a)$ is a positive semidefinite
- f has a local maximum at $a \Rightarrow H(f,a)$ is a negative semidefinite

Observe carefully that the semidefinite cases can also be saddle points.

2.7 Lagrange Multipliers, Implicit and Inverse Function Theorems

Lagrange Multipliers Consider two differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$. Lagrange multipliers are useful for finding local extrema of f under the constraint $S = \{x \in \mathbb{R}^n : g(x) = c, c \in \mathbb{R}\}$.

Then, if a local minimum or maximum of f occurs on $a \in S$ then, $\boldsymbol{\nabla} f()$ and $\boldsymbol{\nabla} g(a)$ are parallel. That is, when $\boldsymbol{\nabla} g(a) \neq 0$, there exists $\lambda \in \mathbb{R}$ such that

$$\boldsymbol{\nabla} f(a) = \lambda \boldsymbol{\nabla} g(a).$$

Note that this theorem will only provide possible candidates for minimum or maximum points. There is no guarantee that there exists minimum or maximums of f on S.

Inverse Function Theorem in $\mathbb{R} \to \mathbb{R}$ Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable on an interval $I \subset \mathbb{R}$ and $f'(x) \neq 0$ for all $x \in I$. Then, f is invertible on I and the inverse $(f^{-1})'(x)$

is differentiable such that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

That is, if $y = f(x)$ and f^{-1} exist and is differentiable with $x = f^{-1}(y)$ then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

Generalising the Inverse Function Theorem Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}^n$ be C^1. Suppose that $a \in \Omega$.

If the matrix $Df(a)$ is invertible, then f is invertible on an open set U containing a. That is, the inverse exists as

$$f^{-1} : f(U) \to U.$$

Further, f^{-1} is C^1 and for $x \in U$,

$$D_{f(x)} f^{-1} = (D_x f)^{-1}.$$

Consequently, f^{-1} has its best affine approximation at $f(a)$ as

$$f^{-1}(x) \approx a + (D_f)^{-1}(x - f(a)).$$

3 Integration

3.1 Riemann and Fubini

Riemann Integral For a bounded function $f : R \to \mathbb{R}$, if there exists a unique number I such that

$$\underline{S}_{\mathcal{P}_1,\mathcal{P}_2}(f) \leq I \leq \bar{S}_{\mathcal{P}_1,\mathcal{P}_2}(f)$$

for every pair of partitions $\mathcal{P}_1, \mathcal{P}_2$. Then, f is Riemann integrable on R and

$$I = \int\int_R f = \int\int_R f(x,y)dA.$$

I is called the Riemann integral of f over$R$.

Properties of the Riemann Integral The single variable interpretation of a Riemann integral is the (signed) area bound by the graph $y = f(x)$ and the x axis over the interval $[a,b]$. For two variables, $\int\int_R f$ is the (signed) volume bounded by the graph $z = f(x,y)$ and the xy-plane over the rectangle R.

If f, g are integrable on R,

- $\int\int_R \alpha f = \beta g = \alpha \int\int_r f + \beta \int\int_R g$
- If $f(x) \leq g(x), \forall x \in R$ then $\int\int_R \leq \int\int_R g$
- $|\int\int_R f| \leq \int\int_R |f|$
- If $R = R_1 \cup R_2$ and interior $R_1 \cap$ interior $R_2 = \emptyset$ then $\int\int_R f = \int\int_{R_1} f + \int\int_{R_2} f$

Fubini's Theorem Let $f : R \to \mathbb{R}$ be continuous on a rectangular domain $r = [a,b] \times [c,d]$. Then, $\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dy, dy$.

Fubini's Theorem - Discontinuous Let $f : R \to \mathbb{R}$ be bounded on a rectangular domain $R = [a,b] \times [c,d]$ with the discontinuities of f confined to a finite union of graphs of continuous functions. If $\int_c^d$ exists for each $x \int [a,b]$ then

$$\int\int_R = \int_a^b \left(\int_x^d f(x,y)dy \right) dx.$$

Since f is not continuous then there is no guarantee that these integrals exist.

3.2 Uniform Continuity, Leibniz

Uniform Continuity The function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous on Ω if for all $x, y \in \Omega$ and for all $\epsilon > 0$, there exists δ such that

$$d(x,y) < \delta \quad \Rightarrow \quad d(f(x), f(y)) < \epsilon.$$

δ may depend on x but, given an ϵ, the same δ must work for all x.

Continuity and Uniform Continuance A continuous function on a compact Ω is uniformly continuous on Ω.

Leibniz' Rule Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on Ω and $\frac{\partial f}{\partial x}$ is uniformly continuous on Ω.

If,

$$F(x) = \int_a^b f(x,y)$$

then

$$F'(x) = \frac{d}{dx}\int_a^b f(x,y)dy = \int_a^b \frac{\partial f}{\partial x}(x,y)dy.$$

3.3 Alternate Coordinates

Change of Variable Suppose that $F : \Omega \in \mathbb{R}^n \to \mathbb{R}^n$ is C^1, $\det(J_x F \neq 0)$ for $x \in \Omega$ and F is one-to-one. Then, if f is integrable on $\Omega' = F(\Omega)$

$$\int_{\Omega'} f(x,y) = \int_{\Omega} (f \circ F)|\det JF|.$$

As an alternate notation consider,

$$\int_{\Omega'} f(x,y)dxdy = \int_{\Omega} f(x(u,v), y(u,v))|\det JF|dudv.$$

where

$$\det Jf = \det \left(\frac{\partial x}{\partial u} \quad \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \quad \frac{\partial y}{\partial v} \right).$$

Note that F is the function which maps the change of variable.

Change of Variable to Polar For a polar substitution, sub $x = r\cos\theta$ and $y = r\sin\theta$. In this case, the Jacobian determinant is r.

3.4 Mass, Centre of Mass, Centroid

For the following section, suppose that $\Omega \subset \mathbb{R}$ with a density function $p : \Omega to \mathbb{R}$.

Note that this can be generalised to n dimensions by the n-th integral rather than a double integral

Mass The total mass is

$$M = \int_\Omega p(x,y)dxdy.$$

Center of Mass The coordinates for centre of mass follow as such:

- $\bar{x} = \int\int_\Omega xp(x,y)dxdy$
- $\bar{y} = \int\int_\Omega yp(x,y)dxdy$

4 Fourier Series

Fourier Series A Fourier series is the approximation of simple periodic functions by the sum of period functions of the form $\sin(x), \cos(x)$. Note that unlike Taylor series, a function f may be discontinuous. However, any lack of continuity leads to an infinite sum in the Fourier series.

4.1 Inner Products

Inner Products Let V be a real vector space. An inner product on V is a map that assigns each $f, g \in V$ a real number $\langle f, g \rangle$ such that the following properties hold for all $f, g, h \in V$ and $\lambda, \mu \in \mathbb{R}$:

- $\langle f, f \rangle \geq 0$,
- $\langle f, f \rangle = 0$ if and only if f is zero,
- $\langle \lambda f + \mu g, h \rangle, = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$,
- $\langle g, f \rangle = \langle f, g \rangle$.

Usual Inner Products

- The vector space R^n admits the following inner product

$$\langle u, v \rangle = u \cdot v = \sum_{i=1}^{n} u_i v_i.$$

- The vector space $C[a,b]$ consisting of all continuous function on the interval $[a,b]$ admits the following inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

Inner Product and Orthogonality We say functions are orthogonal if $\langle f, g \rangle = 0$.

4.2 Norms

A norm on V is a map that assigns each $f \in V$ a real number $||f||$ such that $\forall f \in V, \lambda \in \mathbb{R}$

- $||f|| > 0$,
- $||f|| = 0$ if and only if $f = 0$,
- $||\lambda f|| = \lambda ||f||$,
- $||f + g|| \leq ||f|| + ||g||$; that is, the triangle inequality holds.

Usual Norms

- The Euclidian norm (L^2-norm): is a norm on $C[a, b]$:

$$||f||_2 = \sqrt{\int_a^b f(x)^2 dx)}$$

- The max norm is a norm on $C[a, b]$:

$$||f||_\infty = \max_{a \leq x \leq b} \{|f(x)|\}$$

4.3 Fourier Coefficient and Series

Fourier Series Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is $2L$-periodic, - that is, $f(x) = f(x + 2L)$ - and is square integrable - that is, $\int_{-L}^{L} f(x)^2 dx < \infty$. Then, f may be represented by a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left[a_k \cos\left(\frac{k\pi}{L} x\right) + b_k \sin\left(\frac{k\pi}{L} x\right) \right] \quad \forall x \in [-\pi, \pi].$$

This series converges to f as $n \to \infty$.

Fourier Coefficients

- $a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right)$
- $b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right)$

4.4 Convergence of Fourier Series

Continuity Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$. Suppose that the one-sided limits $f(c^+)$ and $f(c^-)$ exist.

- If $f^{c^+} = f^{c^-} = f(c)$ then f is continuous at c,
- If $f^{c^+} = f^{c^-} \neq f(c)$ then f has a removable discontinuity at c,
- If $f(c^+) \neq f(c^-)$ then, f has a jump discontinuity at at c.

Piecewise Continuity A function is piecewise continuous on $[a, b]$ if and only if

- $f(x^+)$ exists $\forall x \in [a, b]$,
- $f(x^-)$ exists $\forall x \in [a, b]$,
- f is continuous on (a, b) except at most a finite number of points.

Note that if f is only piecewise continuous then the partial sum of the Fourier series does not necessarily converge to f for all x.

Piecewise differentiability A function f is differentiable on c if and only if $f(c^+) = f(c^-) = f(c)$ and $D^+f(c) = D^-f(c)$

Note: $D^+f(c)$ is not necessarily the same as $\lim_{x \to c^+} f'(x)$.

A function is piecewise differentiable on $[a, b]$ if and only if

- $D^+f(x^)$ exists $\forall x \in [a, b)$,
- $D^-f(x^)$ exists $\forall x \in (a, b]$,
- f is differentiable on (a, b) except at most a finite number of points.

Pointwise convergence Let $c \in \mathbb{R}$. Suppose that a function has the following properties

- f is $2L$ periodic,
- f is piecewise continuous on $[-L, L]$,
- $D^+f(c), D^-f(c)$ exist.

Then,

$$S_f(c) = \frac{1}{2}[f(c^+) + f(c^-)].$$

Observe that if f is continuous at c then $S_f(c) = f(c)$.

Odd and Evenness Recall that odd and even functions are defined by the conditions $f(-x) = -f(x)$ and $f(x) = f(-x)$ respectively.

The following elementary properties hold:

- Odd $\times$ Even = Even,
- Odd $\times$ Odd = Even,
- Even $\times$ Even = Even,
- $\int_{-L}^{L} Odd = 0$.

4.5 Convergence of Sequences

Pointwise convergence Let $f_k : \mathbb{R} \to \mathbb{R}$. f_k converges to f on $[a,b]$ pointwisely iff and only if for all $x \in [a,b]$, $f_k(x) \to f(x)$ as $k \to infty$.

Epsilon Delta Definition Pointwise Convergence For all $x \in [a,b]$, $\epsilon > 0, \exists K$ (which will depend on ϵ, x such that

$$|f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

Uniform Convergence Let $f_k : \mathbb{R} \to \mathbb{R}$. f_k converges to f on [a, b] uniformly if and only if for all $\epsilon > 0, \exists K$ (depending on ϵ only) such that

$$\sup_{x \in [a,b]} |f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

Weierstrass test Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of a function f defined on $[a,b]$. Suppose that there exists a sequence of numbers c_k such that

$$|f_k(x)| \leq c_k \quad \forall x \in [a,b]$$

where $\sum_{k=1}^{\infty} c_k$ converges to a real number. Then $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f on $[a,b]$.

Note that this test also holds for function $f : \mathbb{R}^n \to \mathbb{R}$ for $x \in \Omega$ where Ω is a closed bounded set in $\mathbb{R}^n$.

Norm Convergence Using the supremum norm, the definition of uniform convergence can be equivalently written as: for all $\epsilon > 0, \exists K$ such that

$$||f_k - f|| \leq \epsilon \quad \forall k \geq K.$$

Equivalently,

$$\lim_{k \to \infty} ||f_k - f|| = 0.$$

We may extend this to define norm-convergence for any norm.

Extending Norm Convergence to L-2 Recall from the previous paragraph that norm-convergence is defined as follows:

$$\lim_{k \to \infty} ||f_k - f|| = 0.$$

As such, L^2 norm convergence, also knows as mean square convergence is equivalent to the following

$$\lim_{k \to \infty} \int_a^b [f_k(x) - f(x)]^2 dx = 0.$$

Parseval Theorem Let f be a 2π periodic and bounded function where $\int_{-pi}^{\pi} f(x)^2 dx < +\infty$. Then, the Fourier series of f converges to f in the mean square sense. Moreover, the Parseval's identity holds

$$\int_{-pi}^{\pi} f(x)^2 = ||f||_2^2 = \frac{\pi}{2} a_0 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This identity continues to hold for $2L$ periodic functions integrated over $[-L, L]$.

5 Vector Fields

5.1 Vector Fields and Flows, Divergence and Curl

Flow Lines If F is a vector field, a flow line for F is a path $c(t)$ such that

$$c'(t) = F(c(t)).$$

That is, that F yields the velocity field of the path $c(t)$.

The Del ∇ operator The vector differential operator ∇ may be considered a symbolic vector. The differential operator may be written as

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k.$$

Divergence Given a field $F = (f_1, f_2, \ldots, f_n)$, the divergence of F is

$$\text{div} F = \nabla \cdot F = \sum_{i=1}^{n} \nabla f_i.$$

Divergence may be thought as a type of derivative that describes the measure at which a vector field *spreads away* from a certain point. If the divergence is positive, then there is a net outflow while there is net inflow if the divergence is negative.

Observe that the divergence of a vector field will be real-valued.

Curl If F is a vector field, then the curl may be defined as

$$\text{curl} F = \nabla \times F.$$

Curl is also analogous to a type of derivative for vector fields. The curl may be though as the measure at which the vector field *swirls* around a point. A positive swirl can be thought of as a counter clockwise rotation.

Observe that the curl of a vector field is also a vector field.

5.2 Vector Identities

Basic Vector Identities

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(\lambda f) = \lambda \nabla f$ where $\lambda \in \mathbb{R}$
3. $\nabla(fg) = g\nabla f + f\nabla g$. You may draw analogies to the product.
4. $\nabla \frac{f}{g} = \frac{f\nabla g - g\nabla f}{g^2}$ where $g \neq 0$. This is analogous to the quotient rule.
5. $\nabla \cdot (F+G) = \nabla \cdot F + \nabla \cdot G$
6. $\nabla \times (F+G) = \nabla \times F + \nabla \times G$
7. $\nabla \cdot (fF) = f\nabla \cdot F = F \cdot \nabla f$
8. $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$
9. $\nabla \cdot (\nabla \times F) = 0$
10. $\nabla \times (fF) = f\nabla \times F = \nabla f \times F$
11. $\nabla \times (\nabla f) = 0$
12. $\nabla^2(fg) = f\nabla^2 g + 2((\nabla f \cdot \nabla g)) + g\nabla^2 f$
13. $\nabla \cdot (\nabla \times \nabla g) = 0$
14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla f^2$

6 Path Integrals

6.1 Path Integrals

Path (Scalar Line) Integrals Suppose that a vector-valued function $c(t)$ parametrises a curve C for $t \in [a,b]$. The scalar line integral may be thought as the integral of along c.

Computing a Scalar Line Integral Let $c(t)$ parametrise a curve C for $t \in [a,b]$. Assume that $f(x,y,z)$ and $c(t)$ are continuous. Then,

$$\int_C f(x,y,z)ds = \int_a^b f\left(c\left(t\right)\right) \cdot ||c(t)||dt.$$

Elementary Properties of Path Integrals

- $\int_C f_1 ds + \int_C f_2 ds = \int_C (f_1 + f_2)ds$,
- $\int_C \lambda f ds = \lambda \int_C f ds, \quad \lambda \in \mathbb{R}$.

6.2 Applications Of Path Integrals

Suppose that $\delta = \delta(x, y, z)$ which is a density function.

Mass

$$M = \int_C \delta(x, y, z) ds.$$

First Moments About the Coordinate Plane

- $M_{yz} = \int_C x\delta ds$
- $M_{xz} = \int_C y\delta ds$
- $M_{xy} = \int_C z\delta ds$

Coordinates of Center of Mass

- $\bar{x} = \frac{M_{yz}}{M}$
- $\bar{y} = \frac{M_{xz}}{M}$
- $\bar{z} = \frac{M_{xy}}{M}$

Moments of Inertia About Axes

- $I_x = \int_C (y^2 + z^2)\delta ds$
- $I_x = \int_C (x^2 + z^2)\delta ds$
- $I_x = \int_C (x^2 + y^2)\delta ds$

7 Vector Line Integrals

7.1 Vector Line Integrals

Vector Line Integrals Vector line integrals are different from scalar line integrals in the sense that to define a vector line integral, we must specify a direction along the path or curve C.

Computing a Vector Line Integral Let $c(t)$ parametrise an oriented curve C for $t \in [a, b]$. Then,

$$\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t).$$

Link to Path Integrals Suppose that C is a smooth curve with a parametrisation $c(t)$ for $t \in [a, b]$ where $c(t)$ is continuously differentiable and $c'(t) \neq 0$ for all $t \in [a, b]$.

Then, $c'(t)$ is a non-zero tangent vector pointing in the forward direction and the unit tangent vector is

$$T(c(t)) = \frac{c'(t)}{||c'(t)||}.$$

Then,

$$\int_C F \cdot ds = \int_C F \cdot T ds.$$

Summing Paths Suppose that C is made of n finitely many paths C_i. Then, $C = \sum_i^n C_i$. Note that all the curves must be joined end to end. Then,

$$\int_C F \cdot ds = \sum_i^n \int_{C_i} F \cdot ds.$$

Work and Other Alternative Notations Suppose that $c(t) = (x(t), y(t), z(t))$ and $F = (M, N, P)$. Then, we denote work as any of the following notations

$$\begin{aligned} W &= \int_C F \cdot ds \\ &= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt \\ &= \int_C M ds + N ds + P ds. \end{aligned}$$

Properties fo Line Integrals

- Linearity
- Reversing Orient
- Additivity

Flow Integrals and Circulation Suppose that F represents a velocity field of a fluid flowing through a region in space. Then, the flow across a curve may be defined as the following

$$\text{Flow} = \int_a^b F \cdot \hat{T} ds.$$

This integral is called the flow integral. If the curve is a closed loop then this is called the *circulation* around the curve.

Flux in the Plane If C is a smooth closed curve in the domain of a continuous vector field $F = M(x,y)i + N(x,y)j + N(x,y)j$ and, n is the outward pointing unit-normal on C then, the flux of f across C is the following expression

$$\int_C F \cdot \hat{n} ds.$$

Calculating Flux Across a Smooth Closed Plane Curve Suppose that $F = Mi = Nj$. Let $G = -N, M$

$$\text{Flux of } F \text{ across } C = \oint_C Mdy - Ndx = \oint_C Gds$$

7.2 Fundamental Theorem of Line Integrals

Gradient Fields A vector field F is called a gradient vector field if there exists a real-valued function ϕ such that $F = \nabla\phi$. That is,

$$\begin{pmatrix} M \\ N \\ P \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{pmatrix}.$$

If such a function ϕ exists then ϕ is called the potential function of F where, F is *conservative.*

Fundamental Theorem for Gradient Vector Fields If $F = \nabla\phi$ on a domain $\mathcal{D}$. Then, for all oriented curves C in $\mathcal{D}$ with an initial point P and a terminal point Q,

$$\int_C F \cdot ds = \phi(Q) - \phi(P).$$

The integral is independent of the path.

Cross Partials of Gradient Vector Fields are Equal Let $F = (F_1, F_2, F_3)$ be a gradient vector field whose components have continuous partial derivatives. Then, the cross partials are equal. That is,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x},$$
$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y},$$
$$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

Equivalently,

$$\boldsymbol{\nabla} \times F = 0.$$

7.3 Green's Theorem

Green's Theorem connects double integrals and line integrals. It is very useful f or line integrals over complicated vector fields with simpler partial derivatives.

Green's Theorem: Flux Divergence or Normal Form Let D be a bounded simple region in $\mathbb{R}^2$ with a nonempty interior whose boundary consists of a finite number of smooth curves. Let Cbe the boundary of D with a positive (counter-clockwise) direction. Let $F = Mi + Nj$ be a continuously different boundary vector field on D.

Then, the outward flux of F across the curve C equals the double integral of divergence $\boldsymbol{\nabla} \cdot F$ over D. That is,

$$\oint_C (F \cdot \hat{n}) ds = \oint_C -N dx + M dy = \int\int_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial Y} \right).$$

Once again, note the assumptions:

- D is bounded and simple with a non-empty interior,
- The boundary C is oriented in the positive (counter-clockwise) direction, and is the finite union of smooth curves,
- The vector field F is continuously differentiable on D.

Green's Theorem: Circulation-Curl or Tangential Form Let D be a bounded simple region in $\mathbb{R}^2$ with a nonempty interior whose boundary consists of a finite number of smooth curves. Let Cbe the boundary of D with a positive (counter-clockwise) direction. Let $F = Mi + Nj$ be a continuously different boundary vector field on D.

Then, the counter-clockwise circulation of F around C equals the double integral of $(\boldsymbol{\nabla} \times F) \cdot k$ over D. That is,

$$\oint_C (F \cdot \hat{T}) ds = \oint_C M dx + N dy = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

Area of a Region Let D be a simple and bounded region with a non-empty interior and let C be the boundary of D which is the finite union of smooth curves. Then, the area of D can be calculated as such

$$\text{Area}(D) = \frac{1}{2} \oint_C (-y dx + x dy).$$

8 Surface Integrals

8.1 Parametrisations of Surfaces

Cone A cone $z^2 = x^2 + y^2$ can be parametrised as

$$\phi(u, v) = (u \cos v, u \sin v, u), \quad u \in \mathbb{R}, v \in [0, 2\pi].$$

Cylinder A cylinder of radius R, of form $x^2 + y^2 = R^2$ can be parametrised as

$$\phi(\theta, z) = (R\cos\theta, R\sin\theta, z) \quad \theta \in [0, 2\pi], x \in \mathbb{R}.$$

Sphere A sphere of radius R of equation $x^2 + y^2 + z^2$ is parametrised as

$$\phi(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$$

where $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$.

8.2 Area of a Surface

Surface Area Let $\Phi(u, v)$ be a parametrisation of a smooth surface S with parameter domain D.

The area of the surface is

$$\int\int_D ||T_u \times T_v|| \, .$$

Equivalently we have:

8.3 Surface Integrals of Scalar and Vector Valued Functions

9 Integral Theorems

9.1 Stokes' Theorem

Stokes' Theorem With Stokes' Theorem, we can compute line integrals over closed curves by using surface integrals. Suppose that

- S is a smooth oriented surface,
- ∂S is the boundary of S in the anti-clockwise direction when looking at S from the positive direction,
- F is a continuously differentiable vector field on S.

Then,

$$\iint_S (\nabla \times F) \cdot ds = \oint_{\partial S} F \cdot ds.$$

9.2 Gauss' Divergence Theorem

The divergence theorem allows for computation of surface integrals over close surfaces with a triple integral over the 3D solid bound by the surface. If

- The region $W \subset \mathbb{R}^3$ is a bounded, solid, simple region.
- S is the piecewise smooth boundary, oriented so that the normal vector points outwards

- F is a C^1 vector field on W.

Then,

$$\iint_S F \cdot dS = \iiint_W \nabla \cdot F \cdot dV.$$

This is very helpful for complex vector fields with simpler expressions for divergence.

Note that $\iiint_W dV$ is the volume of W. Also note that for a conversion from $x, y, z \to r, \theta, \phi$, the Jacobian determinant is $r^2 \sin \phi$.