1 Differentiation

1.1 Differentiability, Derivatives and Affine Approximations

Differentiability in $\mathbb{R}$ A function $f : \mathbb{R} \to \mathbb{R}$ being differentiable at some $a \in \mathbb{R}$ implies that there exists a *good* straight-line approximation to f at a called a *tangent line*. This function may be found as

$$T(x) = f(a) + f'(a)(x - a) = f(a) - f'(a)a + f'(a)x = y_0 + L(x)$$

where for all a, $y = f(a) - f'(a)a$ and $L : \mathbb{R} \to \mathbb{R} = f'(a)x$.

Recall that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

.

Affine Maps A function $T : \mathbb{R}^n \to \mathbb{R}^m$ being affine means that there exists a y_0 such that for all $x \in \mathbb{R}^n$

$$T(x) = y_0 + L(x)$$

.

In $T : \mathbb{R} \to \mathbb{R}$ this sis of the form $y = mx + b$.

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if there is a good affine approximation to f of the form

$$T(x) = f(a) - f'(a)a + f'(a)x.$$

In this context good implies that $f'(x)$ is defined in the usual manner and exists.

Differentiability in $\mathbb{R}^n \to \mathbb{R}^n$ A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable for some $a \in \nleq$ if there exists a linear map $L : \mathbb{R}n \to \mathbb{R}^m$ such that

$$\lim_{x \to a} \frac{||f(x) - f(a) - L(x - a)||}{||L(x - a)||} = 0.$$

Notation: the matrix of the linear map L, the derivative of f at a is denoted by $D_a f$.

Delta Epsilon Definition of Differentiability A function $f : \Omega \subset \mathbb{R} \to \mathbb{R}^m$ is differentiable on $a \in \Omega$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $\forall \epsilon > 0 \exists \delta > 0$ such that for all $x \in \Omega$

$$||x - a|| < \delta \Rightarrow ||f(x) - f(a) - L(x - a)|| < \epsilon \, ||x - a|| \, .$$

Clairaut's Theorem / Mixed Derivative Theorem Suppose $f, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^2 f}{\partial x_j \partial x_i}$ all exist and are continuous on an open set around a then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

That is, the partial derivatives commute.

Differentiability and Continuity Differentiability implies continuity. However, continuity does not imply differentiability.

1.2 Gradients, Affine Approximations and Matrices

Jacobian