

# Higher Several Variable Calculus

## Math2111 UNSW

Hussain Nawaz  
hussain.nwz000@gmail.com

2022T1

## Contents

<b>1</b>	<b>Analysis</b>	<b>3</b>
1.1	Assumed . . . . .	3
1.2	Limits . . . . .	3
1.3	Metrics . . . . .	3
1.4	Limits of Sequences . . . . .	4
1.5	Open and Closed Sets . . . . .	5
1.6	Pinching and IVT Theorem . . . . .	6
<b>2</b>	<b>Differentiation</b>	<b>6</b>
2.1	Differentiability, Derivatives and Affine Approximations . . . . .	6
2.2	Gradients, Affine Approximations and Matrices . . . . .	7
2.3	Gradients, Tangent Planes and Affine Approximations . . . . .	8
2.4	Chain Rule, Directional Derivatives and Tangent Planes . . . . .	8
2.5	Taylor Series and Theorem . . . . .	9
2.6	Hessian Matrix and Stationary Points . . . . .	10
2.7	Lagrange Multipliers, Implicit and Inverse Function Theorems . . . . .	11
<b>3</b>	<b>Integration</b>	<b>11</b>
3.1	Riemann and Fubini . . . . .	11
3.2	Uniform Continuity, Leibniz . . . . .	12
3.3	Alternate Coordinates . . . . .	13
3.4	Mass, Centre of Mass, Centroid . . . . .	13
<b>4</b>	<b>Fourier Series</b>	<b>13</b>
4.1	Inner Products . . . . .	14
4.2	Norms . . . . .	14
4.3	Fourier Coefficient and Series . . . . .	15
4.4	Convergence of Fourier Series . . . . .	15
4.5	Convergence of Sequences . . . . .	16

<b>5</b>	<b>Vector Fields</b>	<b>17</b>
5.1	Vector Fields and Flows, Divergence and Curl . . . . .	17
5.2	Vector Identities . . . . .	18
<b>6</b>	<b>Path Integrals</b>	<b>19</b>
6.1	Path Integrals . . . . .	19
6.2	Applications Of Path Integrals . . . . .	19
<b>7</b>	<b>Vector Line Integrals</b>	<b>20</b>
7.1	Vector Line Integrals . . . . .	20
7.2	Fundamental Theorem of Line Integrals . . . . .	21
7.3	Green's Theorem . . . . .	22

# 1 Analysis

## 1.1 Assumed

### Assumed Concepts from Real Single-Variable Calculus

- limits
- continuity
- differentiability
- integrability

### Assumed Theorems

- Min/ Max Theorem
- Intermediate Value Theorem
- Mean Value Theorem

## 1.2 Limits

Recall that  $\lim_{x \rightarrow a} f(x) = L$  requires that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $|x - a| < \delta$  then

$$|f(x) - L| < \delta.$$

## 1.3 Metrics

We have metrics (distance functions) as

$$m : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

satisfying the following 3 axioms.

- **Positive Definite** such that for all  $x, y \in \mathbb{R}^n$ ,  $m(x, y) > 0$  and,  $m(x, y) = 0 \Leftrightarrow x = y$ .
- **Symmetric**  $m(x, y) = m(y, x)$ .
- **Triangle Inequality** such that for all  $x, y, z \in \mathbb{R}^n$ ,  $m(x, y) + m(y, z) \leq m(x, z)$ .

**Euclidian Distance** We allow the Euclidian distance to be defined as

$$d_n(x, y) := \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

We often allow  $d$  to be  $d_2$ .

**Norms** Norms will be revisited in the Fourier Series section. They can be thought of as the length of an element in vectors space.

**Equivalent Metrics** Two metrics  $d$  and  $\delta$  are considered equal if there exists constants  $0 < c < C < \infty$  such that

$$c\delta(x, y) \leq d(x, y) \leq C\delta(x, y).$$

## 1.4 Limits of Sequences

**Balls** A ball around  $\vec{a} \in \mathbb{R}$  is of radius  $\epsilon$  is the set

$$B(\vec{a}, \epsilon) = \{x \in \mathbb{R} : d(\vec{a}, x) < \epsilon\}.$$

**Limit in Sequence** For a sequence  $\{x_i\}$  of points in  $\mathbb{R}^n$ ,  $x$  is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \leq \epsilon.$$

Equivalently,

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(x, x_n) \in B(x, \epsilon).$$

### Theorems with Limits of Sequences

A sequence  $x_k$  converges to a limit  $x$

$$\begin{aligned} &\Leftrightarrow \text{The components of } x_k \\ &\quad \text{converge to the components of } x \\ &\Leftrightarrow d(x_k, x) \rightarrow 0. \end{aligned}$$

**Limits and Equivalent Metrics** Suppose that  $d$  and  $\delta$  are two equivalent metrics. That is,  $cd(x, y) \leq \delta(x, y) \leq Cd(x, y)$  for  $c, C > 0$ .

Considering  $d$  as the metric, suppose that

$$x_k \rightarrow x \quad \text{for } x_k, x \in \mathbb{R}^n.$$

That is,

$$\forall \epsilon > 0, \exists K : k \geq K \implies d(x_k, x) < \epsilon.$$

Using  $\delta$ , we may make an equivalent statement, choosing  $\epsilon > 0$  such that  $\epsilon' = C\epsilon$ . Considering that  $\epsilon > 0 \implies \exists K : \forall k \geq K \implies d(x_k, x) < \epsilon$  then,

$$\delta(x_k, x) \leq Cd(x_k, x) < C\epsilon = \epsilon'.$$

That is,  $\delta(x_k, x) < \epsilon'$ . Hence  $x_k \rightarrow x$  using an equivalent metric  $\delta$ .

**Cauchy Sequences** A sequence  $\{x_k\} \in \mathbb{R}$  is a Cauchy sequence if

$$\exists \epsilon > 0 \text{ such that } k, l > K \implies d(x_k, x_l) < \epsilon.$$

**Cauchy Sequences and Convergence** The following are equivalent:

A sequence  $\{x_k\}$  converges in  $\mathbb{R}^2 \iff \{x_k\}$  is a Cauchy Sequence.

## 1.5 Open and Closed Sets

**Definitions** Consider  $x_k$

- $x_0 \in \Omega$  is an interior point of  $\Omega$  if there is a ball around  $x$  completely contained in  $\Omega$ . That is, there exists a  $\epsilon > 0$  such that  $B(x_0, \epsilon) \subseteq \Omega$ .
- $\Omega$  is open if every point of  $\Omega$  is an interior point.
- $\Omega$  is closed if its complement is open.
- $x_0 \in \Omega$  is a boundary point of  $\Omega$  if every ball around  $x_0$  contains points in  $\Omega$  and points not in  $\Omega$ .

**Closed Sets** A set  $\Omega \subset \mathbb{R}$  is closed iff and only if it contains all of its boundary points.

**Limit Points and Sets**  $x_0$  is a limit point of  $\Omega$  if there is a sequence  $\{x_i\}$  in  $\Omega$  with limit  $x_0$  and  $x_i \neq x$ .

- Every interior points of  $\Omega$  is a limit point of  $\Omega$ .
- $x_0$  is not necessarily in  $\Omega$
- A set is closed  $\Leftrightarrow$  it contains all of its limit points.

**Variations of a Set** Consider the set  $\Omega \in \mathbb{R}^n$ .

- The interior of  $\Omega$  is the set of all its interior points.
- The boundary  $\partial\Omega$  of  $\Omega$  is the set of all its boundary points.
- The closure of  $\Omega$ :  $\bar{\Omega} = \Omega \cup \partial\Omega$ .

The interior is the largest open subset and the closure is the smallest closed set containing  $\Omega$ .

**Limit of a Function at a Point** For  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $\lim_{x \rightarrow x_0}$  means that

$$\forall \epsilon \exists \delta > 0 \text{ such that for } x \in \Omega : \\ 0 < d(x, x_0) < \delta \implies d(f(x), b) < \epsilon.$$

Alternatively,

$$x \in B(x_0, \delta) \setminus \{x_0\} \implies f(x) \in B(b, \epsilon).$$

It is sufficient to consider the limits of the components of a function.

**Limits and sequences** The limit  $\lim_{x \rightarrow a} f(x) = b$  exists if and only if,  $\lim_{k \rightarrow \infty} f(x_k) = b$  for all sequences  $x_k$  such that  $x_k$  is an element of  $\Omega$  and,  $\lim_{k \rightarrow \infty} x_k = a$ .

This is very helpful for showing that a limit does not exist.

## 1.6 Pinching and IVT Theorem

### Pinching Theorem

IVT see 1141

## 2 Differentiation

### 2.1 Differentiability, Derivatives and Affine Approximations

**Differentiability in  $\mathbb{R}$**  A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  being differentiable at some  $a \in \mathbb{R}$  implies that there exists a *good* straight-line approximation to  $f$  at  $a$  called a *tangent line*. This function may be found as

$$T(x) = f(a) + f'(a)(x - a) = f(a) - f'(a)a + f'(a)x = y_0 + L(x)$$

where for all  $a$ ,  $y = f(a) - f'(a)a$  and  $L : \mathbb{R} \rightarrow \mathbb{R} = f'(a)x$ .

Recall that

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

**Affine Maps** A function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  being affine means that there exists a  $y_0$  such that for all  $x \in \mathbb{R}^n$

$$T(x) = y_0 + L(x)$$

In  $T : \mathbb{R} \rightarrow \mathbb{R}$  this is of the form  $y = mx + b$ .

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable if there is a good affine approximation to  $f$  of the form

$$T(x) = f(a) - f'(a)a + f'(a)x.$$

In this context good implies that  $f'(x)$  is defined in the usual manner and exists.

**Differentiability in  $\mathbb{R}^n \rightarrow \mathbb{R}^m$**  A function  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable for some  $a \in \Omega$  if there exists a linear map  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\lim_{x \rightarrow a} \frac{\|f(x) - f(a) - L(x - a)\|}{\|L(x - a)\|} = 0.$$

Notation: the matrix of the linear map  $L$ , the derivative of  $f$  at  $a$  is denoted by  $D_a f$ .

**Delta Epsilon Definition of Differentiability** A function  $f : \Omega \subset \mathbb{R} \rightarrow \mathbb{R}^m$  is differentiable on  $a \in \Omega$  if there is a linear map  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that  $\forall \epsilon > 0 \exists \delta > 0$  such that for all  $x \in \Omega$

$$\|x - a\| < \delta \Rightarrow \|f(x) - f(a) - L(x - a)\| < \epsilon \|x - a\|.$$

**Clairaut's Theorem / Mixed Derivative Theorem** Suppose  $f, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^2 f}{\partial x_j \partial x_i}$  all exist and are continuous on an open set around  $a$  then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

That is, the partial derivatives commute.

**Differentiability and Continuity** Differentiability implies continuity. However, continuity does not imply differentiability. The proof of this is contingent on the fact that for  $x \in \mathbb{R}^n$  and a  $m \times n$  matrix  $L$

$$\lim_{x \rightarrow 0} \|Lx\| = 0.$$

**Partial Derivatives and Differentiability** Suppose that  $\Omega \subset \mathbb{R}^n$  is open and  $f : \Omega \rightarrow \mathbb{R}^m$ . If all partial derivatives  $\frac{\partial f_j}{\partial x_i}$  exist for integers  $i \in [1, n], j \in [1, m]$  then  $f$  is differentiable on  $\Omega$ .

## 2.2 Gradients, Affine Approximations and Matrices

**Jacobian Matrices** Suppose that all partial derivatives of  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  exist for some  $a \in \Omega$ . Then, the Jacobian matrix of  $f$

$$J_a f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

may be evaluated at a point  $a$ . Where  $f$  is differentiable, its derivative is given by the Jacobian matrix.

Note however, that the Jacobian Matrix may exist even where  $f$  is not differentiable.

## 2.3 Gradients, Tangent Planes and Affine Approximations

**Gradient** For  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , if the Jacobian exists, then it is given by the  $1 \times n$  matrix

$$Jf = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

This is equivalent to the gradient of  $f$ . That is,

$$\text{grad}(f) = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

**Affine Approximations** Allow  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  to be a differentiable function at  $a \in \Omega$ . The best affine approximation to  $f$  at  $a$  may be written in terms of the gradient vector as

$$T(x) = f(a) + \nabla f(a) \cdot (x - a).$$

**Tangent Planes** The tangent plane to a function  $z = f(x, y)$  is given by

$$z = T(x, y).$$

## 2.4 Chain Rule, Directional Derivatives and Tangent Planes

**Chain Rule** Suppose that  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $g : \Omega' \subset \mathbb{R}^m \rightarrow \mathbb{R}^p$  where  $f(\Omega) = \Omega'$ . If  $f$  and  $g$  are both differentiable then, so is  $g \circ f : \Omega \rightarrow \mathbb{R}^p$  such that

$$D_a(g \circ f) = D(f(a))gD_a f.$$

Equivalently,

$$D(g \circ f)(a) = Dg(f(a))Df(a).$$

**Directional Derivative** The directional derivative of  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  in the direction of the unit vector  $u$  at a point  $a \in \Omega$  is

$$D_u f(a) = f'_u(a) = \lim_{t \rightarrow 0} \frac{f(a + tu) - f(a)}{t}.$$

Equivalently, if  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at  $a$  then for a unit vector  $u$

$$D_u f(a) = Df(a) \cdot u = \nabla f(a) \cdot u.$$

Alternatively, allowing  $\theta$  to be the angle between  $\nabla f(a)$  and  $u$ ,

$$D_u f(a) = |\nabla f(a)| \cdot |u| \cdot \cos \theta.$$



**Tangent Planes** Consider the surface in  $\mathbb{R}^3$  defined by  $\phi(x, y, z) = \lambda$ . where  $\lambda$  is constant and  $\phi$  is differentiable.

Let  $c(t) = (c_1(t), c_2(t), c_3(t))$  be a differentiable curve lying on the vector space with a tangent vector given by  $c'(t) = (c'_1(t), c'_2(t), c'_3(t))$ .

Since all points  $c(t)$  lie on the surface,  $\phi(c(t)) = \lambda$ . Thus,

$$D(\phi(c(t)))Dc(t) = 0 \Rightarrow \nabla\phi c'(t) = 0.$$

Therefore, all curves passing through a point  $P$  on the surface have tangent vector normal to  $\nabla\phi$ . Thus, they all lie in the tangent plane at  $P$ .

## 2.5 Taylor Series and Theorem

**Taylor's Theorem** For all continuous and differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) \approx P_k(a) = \sum_{n=0}^k \frac{1}{n!} f^{(n)}(a)(x-a)^n + R$$

where the remainder  $R$  is

$$R = \frac{1}{(k+1)!} f^{(k+1)}(z)(x-a)^{k+1}.$$

for some  $z$  between  $x$  and  $a$ .

$P_0, P_1, P_2, P_3$  are the best constant, affine, quadratic, cubic approximations.

**Generalizing Taylor's Theorem** Let  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be  $C^r$  on the open set  $\Omega$ . Let  $a \in \Omega$  be such that the line segment joining  $a$  and  $x$  lies entirely in  $\Omega$ . Then,

$$f(x) = P_{r,a}(x) + R_{r,a}(a)$$

where for some  $z$  on the line segment between  $a$  and  $x$

$$P_{r,a}(x) = f(a) + \sum_{k=1}^{r-1} \frac{1}{k!} D^k f(a) \cdot (x-a)^k, \quad R_{r,a}(a) = \frac{1}{r!} D^r f(z) \cdot (x-a)^r$$

Note that  $\cdot$  is not a dot product.

**Second Degree Taylor Series** Let  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be  $C^3$  on  $\Omega$ . Then, ignoring the remainder,

$$f(x) = f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2} ((x-a) \cdot (Hf(a) \cdot (x-a)))$$

where  $H$  is the Hessian Matrix.

## 2.6 Hessian Matrix and Stationary Points

**Hessian Matrix** For  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , the Hessian matrix of  $f$  at a point  $a$  is the  $n \times n$  matrix

$$H(f, a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}.$$

**Revision: Trace and Determinant** Recall that the trace is the sum of its diagonal values. The trace of a hessian matrix is also the sum of its eigenvalues. Also, the determinant is the product of the eigenvalues. The eigenvalues of a matrix  $A$  can be found by calculating solutions to

$$|A - \lambda I| = 0.$$

**Definite and Semi Definite Matrices** For a  $n \times n$  symmetric matrix  $H$ ,

- All eigenvalues are  $> 0 \Leftrightarrow$  positive definite
- All eigenvalues are  $< 0 \Leftrightarrow$  negative definite
- All eigenvalues are  $\geq 0 \Leftrightarrow$  positive semidefinite
- All eigenvalues are  $\leq 0 \Leftrightarrow$  negative semidefinite

**Sylvester's Criterion (for the Definite Property)** Allow  $H_k$  to be the upper left  $k \times k$  sub-matrix of  $h$  and let  $\Delta_k = \det H_k$ . Then,

- positive definite  $\Leftrightarrow \Delta_k > 0 \forall k$
- positive semidefinite  $\Leftrightarrow \Delta_k \leq 0 \forall k$
- negative definite  $\Leftrightarrow \Delta_k < 0$  for all odd  $k$  and  $\Delta > 0$  for all even  $k$
- negative semidefinite  $\Leftrightarrow \Delta_k \leq 0$  for all odd  $k$  and  $\Delta \geq 0$  for all even  $k$

**The Definite Property and Classification of Stationary Points** Suppose that  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^2$  and  $\nabla f(a) = 0$  at an interior point  $a$  of  $\Omega$ . Then

- $H(f, a)$  is a positive definite  $\Rightarrow f$  has a local maximum at  $a$
- $H(f, a)$  is a negative definite  $\Rightarrow f$  has a local minimum at  $a$
- $f$  has a local minimum at  $a \Rightarrow H(f, a)$  is a positive semidefinite
- $f$  has a local maximum at  $a \Rightarrow H(f, a)$  is a negative semidefinite

Observe carefully that the semidefinite cases can also be saddle points.

## 2.7 Lagrange Multipliers, Implicit and Inverse Function Theorems

**Lagrange Multipliers** Consider two differentiable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ . Lagrange multipliers are useful for finding local extrema of  $f$  under the constraint  $S = \{x \in \mathbb{R}^n : g(x) = c, c \in \mathbb{R}\}$ .

Then, if a local minimum or maximum of  $f$  occurs on  $a \in S$  then,  $\nabla f()$  and  $\nabla g(a)$  are parallel. That is, when  $\nabla g(a) \neq 0$ , there exists  $\lambda \in \mathbb{R}$  such that

$$\nabla f(a) = \lambda \nabla g(a).$$

Note that this theorem will only provide possible candidates for minimum or maximum points. There is no guarantee that there exists minimum or maximums of  $f$  on  $S$ .

**Inverse Function Theorem in  $\mathbb{R} \rightarrow \mathbb{R}$**  Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable on an interval  $I \subset \mathbb{R}$  and  $f'(x) \neq 0$  for all  $x \in I$ . Then,  $f$  is invertible on  $I$  and the inverse  $(f^{-1})'(x)$

is differentiable such that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

That is, if  $y = f(x)$  and  $f^{-1}$  exist and is differentiable with  $x = f^{-1}(y)$  then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

**Generalising the Inverse Function Theorem** Let  $\Omega \subset \mathbb{R}^n$  be open and  $f : \Omega \rightarrow \mathbb{R}^n$  be  $C^1$ . Suppose that  $a \in \Omega$ .

If the matrix  $Df(a)$  is invertible, then  $f$  is invertible on an open set  $U$  containing  $a$ . That is, the inverse exists as

$$f^{-1} : f(U) \rightarrow U.$$

Further,  $f^{-1}$  is  $C^1$  and for  $x \in U$ ,

$$D_{f(x)}f^{-1} = (D_x f)^{-1}.$$

Consequently,  $f^{-1}$  has its best affine approximation at  $f(a)$  as

$$f^{-1}(x) \approx a + (D_f)^{-1}(x - f(a)).$$

## 3 Integration

### 3.1 Riemann and Fubini

**Riemann Integral** For a bounded function  $f : R \rightarrow \mathbb{R}$ , if there exists a unique number  $I$  such that

$$\underline{S}_{\mathcal{P}_1, \mathcal{P}_2}(f) \leq I \leq \bar{S}_{\mathcal{P}_1, \mathcal{P}_2}(f)$$

for every pair of partitions  $\mathcal{P}_1, \mathcal{P}_2$ . Then,  $f$  is Riemann integrable on  $R$  and

$$I = \int \int_R f = \int \int_R f(x, y) dA.$$

$I$  is called the Riemann integral of  $f$  over  $R$ .

**Properties of the Riemann Integral** The single variable interpretation of a Riemann integral is the (signed) area bound by the graph  $y = f(x)$  and the  $x$  axis over the interval  $[a, b]$ . For two variables,  $\int \int_R f$  is the (signed) volume bounded by the graph  $z = f(x, y)$  and the  $xy$ -plane over the rectangle  $R$ .

If  $f, g$  are integrable on  $R$ ,

- $\int \int_R \alpha f = \beta g = \alpha \int \int_R f + \beta \int \int_R g$
- If  $f(x) \leq g(x), \forall x \in R$  then  $\int \int_R f \leq \int \int_R g$
- $|\int \int_R f| \leq \int \int_R |f|$
- If  $R = R_1 \cup R_2$  and interior  $R_1 \cap \text{interior } R_2 = \emptyset$  then  $\int \int_R f = \int \int_{R_1} f + \int \int_{R_2} f$

**Fubini's Theorem** Let  $f : R \rightarrow \mathbb{R}$  be continuous on a rectangular domain  $r = [a, b] \times [c, d]$ . Then,  $\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$ .

**Fubini's Theorem - Discontinuous** Let  $f : R \rightarrow \mathbb{R}$  be bounded on a rectangular domain  $R = [a, b] \times [c, d]$  with the discontinuities of  $f$  confined to a finite union of graphs of continuous functions. If  $\int_c^d$  exists for each  $x \in [a, b]$  then

$$\int \int_R = \int_a^b \left( \int_c^d f(x, y) dy \right) dx.$$

Since  $f$  is not continuous then there is no guarantee that these integrals exist.

## 3.2 Uniform Continuity, Leibniz

**Uniform Continuity** The function  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is uniformly continuous on  $\Omega$  if for all  $x, y \in \Omega$  and for all  $\epsilon > 0$ , there exists  $\delta$  such that

$$d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon.$$

$\delta$  may depend on  $x$  but, given an  $\epsilon$ , the same  $\delta$  must work for all  $x$ .

**Continuity and Uniform Continuance** A continuous function on a compact  $\Omega$  is uniformly continuous on  $\Omega$ .

**Leibniz' Rule** Consider a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  that is continuous on  $\Omega$  and  $\frac{\partial f}{\partial x}$  is uniformly continuous on  $\Omega$ .

If,

$$F(x) = \int_a^b f(x, y) dy$$

then

$$F'(x) = \frac{d}{dx} \int_a^b f(x, y) dy = \int_a^b \frac{\partial f}{\partial x}(x, y) dy.$$

### 3.3 Alternate Coordinates

**Change of Variable** Suppose that  $F : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $C^1$ ,  $\det(J_x F) \neq 0$  for  $x \in \Omega$  and  $F$  is one-to-one. Then, if  $f$  is integrable on  $\Omega' = F(\Omega)$

$$\int_{\Omega'} f(x, y) dx dy = \int_{\Omega} (f \circ F) |\det JF| du dv.$$

As an alternate notation consider,

$$\int_{\Omega'} f(x, y) dx dy = \int_{\Omega} f(x(u, v), y(u, v)) |\det JF| du dv.$$

where

$$\det Jf = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

Note that  $F$  is the function which maps the change of variable.

### 3.4 Mass, Centre of Mass, Centroid

For the following section, suppose that  $\Omega \subset \mathbb{R}^2$  with a density function  $p : \Omega \rightarrow \mathbb{R}$ .

Note that this can be generalised to  $n$  dimensions by the  $n$ -th integral rather than a double integral

**Mass** The total mass is

$$M = \int_{\Omega} p(x, y) dx dy.$$

**Center of Mass** The coordinates for centre of mass follow as such:

- $\bar{x} = \frac{1}{M} \int_{\Omega} x p(x, y) dx dy$
- $\bar{y} = \frac{1}{M} \int_{\Omega} y p(x, y) dx dy$

## 4 Fourier Series

**Fourier Series** A Fourier series is the approximation of simple periodic functions by the sum of period functions of the form  $\sin(x), \cos(x)$ . Note that unlike Taylor series, a function  $f$  may be discontinuous. However, any lack of continuity leads to an infinite sum in the Fourier series.

## 4.1 Inner Products

**Inner Products** Let  $V$  be a real vector space. An inner product on  $V$  is a map that assigns each  $f, g \in V$  a real number  $\langle f, g \rangle$  such that the following properties hold for all  $f, g, h \in V$  and  $\lambda, \mu \in \mathbb{R}$ :

- $\langle f, f \rangle \geq 0$ ,
- $\langle f, f \rangle = 0$  if and only if  $f$  is zero,
- $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$ ,
- $\langle g, f \rangle = \langle f, g \rangle$ .

### Usual Inner Products

- The vector space  $\mathbb{R}^n$  admits the following inner product

$$\langle u, v \rangle = u \cdot v = \sum_{i=1}^n u_i v_i.$$

- The vector space  $C[a, b]$  consisting of all continuous function on the interval  $[a, b]$  admits the following inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

**Inner Product and Orthogonality** We say functions are orthogonal if  $\langle f, g \rangle = 0$ .

## 4.2 Norms

A norm on  $V$  is a map that assigns each  $f \in V$  a real number  $\|f\|$  such that  $\forall f \in V, \lambda \in \mathbb{R}$

- $\|f\| \geq 0$ ,
- $\|f\| = 0$  if and only if  $f = 0$ ,
- $\|\lambda f\| = |\lambda| \|f\|$ ,
- $\|f + g\| \leq \|f\| + \|g\|$ ; that is, the triangle inequality holds.

### Usual Norms

- The Euclidian norm ( $L^2$ -norm): is a norm on  $C[a, b]$ :

$$\|f\|_2 = \sqrt{\int_a^b f(x)^2 dx}$$

- The max norm is a norm on  $C[a, b]$ :

$$\|f\|_\infty = \max_{a \leq x \leq b} \{|f(x)|\}$$

### 4.3 Fourier Coefficient and Series

**Fourier Series** Suppose that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is  $2L$ -periodic, - that is,  $f(x) = f(x + 2L)$  - and is square integrable - that is,  $\int_{-L}^L f(x)^2 dx < \infty$ . Then,  $f$  may be represented by a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^n \left[ a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right) \right] \quad \forall x \in [-\pi, \pi].$$

This series converges to  $f$  as  $n \rightarrow \infty$ .

#### Fourier Coefficients

- $a_k = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{k\pi x}{L}\right)$
- $b_k = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{k\pi x}{L}\right)$

### 4.4 Convergence of Fourier Series

**Continuity** Consider a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a point  $c \in \mathbb{R}$ . Suppose that the one-sided limits  $f(c^+)$  and  $f(c^-)$  exist.

- If  $f(c^+) = f(c^-) = f(c)$  then  $f$  is continuous at  $c$ ,
- If  $f(c^+) = f(c^-) \neq f(c)$  then  $f$  has a removable discontinuity at  $c$ ,
- If  $f(c^+) \neq f(c^-)$  then,  $f$  has a jump discontinuity at  $c$ .

**Piecewise Continuity** A function is piecewise continuous on  $[a, b]$  if and only if

- $f(x^+)$  exists  $\forall x \in [a, b]$ ,
- $f(x^-)$  exists  $\forall x \in [a, b]$ ,
- $f$  is continuous on  $(a, b)$  except at most a finite number of points.

Note that if  $f$  is only piecewise continuous then the partial sum of the Fourier series does not necessarily converge to  $f$  for all  $x$ .

**Piecewise differentiability** A function  $f$  is differentiable on  $c$  if and only if  $f(c^+) = f(c^-) = f(c)$  and  $D^+f(c) = D^-f(c)$

Note:  $D^+f(c)$  is not necessarily the same as  $\lim_{x \rightarrow c^+} f'(x)$ .

A function is piecewise differentiable on  $[a, b]$  if and only if

- $D^+f(x)$  exists  $\forall x \in [a, b]$ ,
- $D^-f(x)$  exists  $\forall x \in (a, b]$ ,
- $f$  is differentiable on  $(a, b)$  except at most a finite number of points.

**Pointwise convergence** Let  $c \in \mathbb{R}$ . Suppose that a function has the following properties

- $f$  is  $2L$  periodic,
- $f$  is piecewise continuous on  $[-L, L]$ ,
- $D^+f(c), D^-f(c)$  exist.

Then,

$$S_f(c) = \frac{1}{2}[f(c^+) + f(c^-)].$$

Observe that if  $f$  is continuous at  $c$  then  $S_f(c) = f(c)$ .

**Odd and Evenness** Recall that odd and even functions are defined by the conditions  $f(-x) = -f(x)$  and  $f(x) = f(-x)$  respectively.

The following elementary properties hold:

- Odd  $\times$  Even = Even,
- Odd  $\times$  Odd = Even,
- Even  $\times$  Even = Even,
- $\int_{-L}^L \text{Odd} = 0$ .

## 4.5 Convergence of Sequences

**Pointwise convergence** Let  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ .  $f_k$  converges to  $f$  on  $[a, b]$  pointwisely iff and only if for all  $x \in [a, b]$ ,  $f_k(x) \rightarrow f(x)$  as  $k \rightarrow \text{infy}$ .

**Epsilon Delta Definition Pointwise Convergence** For all  $x \in [a, b]$ ,  $\epsilon > 0, \exists K$  (which will depend on  $\epsilon, x$  such that

$$|f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

**Uniform Convergence** Let  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ .  $f_k$  converges to  $f$  on  $[a, b]$  uniformly if and only if for all  $\epsilon > 0, \exists K$  (depending on  $\epsilon$  only) such that

$$\sup_{x \in [a, b]} |f_k(x) - f(x)| \leq \epsilon \quad \forall k \geq K.$$

**Weierstrass test** Let  $f_k : \mathbb{R} \rightarrow \mathbb{R}$  be a sequence of a function  $f$  defined on  $[a, b]$ . Suppose that there exists a sequence of numbers  $c_k$  such that

$$|f_k(x)| \leq c_k \quad \forall x \in [a, b]$$

where  $\sum_{k=1}^{\infty} c_k$  converges to a real number. Then  $\sum_{k=1}^{\infty} f_k$  converges uniformly to a function  $f$  on  $[a, b]$ .

Note that this test also holds for function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $x \in \Omega$  where  $\Omega$  is a closed bounded set in  $\mathbb{R}^n$ .



**Norm Convergence** Using the supremum norm, the definition of uniform convergence can be equivalently written as: for all  $\epsilon > 0, \exists K$  such that

$$\|f_k - f\| \leq \epsilon \quad \forall k \geq K.$$

Equivalently,

$$\lim_{k \rightarrow \infty} \|f_k - f\| = 0.$$

We may extend this to define norm-convergence for any norm.

**Extending Norm Convergence to L-2** Recall from the previous paragraph that norm-convergence is defined as follows:

$$\lim_{k \rightarrow \infty} \|f_k - f\| = 0.$$

As such,  $L^2$  norm convergence, also known as mean square convergence is equivalent to the following

$$\lim_{k \rightarrow \infty} \int_a^b [f_k(x) - f(x)]^2 dx = 0.$$

**Parseval Theorem** Let  $f$  be a  $2\pi$  periodic and bounded function where  $\int_{-\pi}^{\pi} f(x)^2 dx < +\infty$ . Then, the Fourier series of  $f$  converges to  $f$  in the mean square sense. Moreover, the Parseval's identity holds

$$\int_{-\pi}^{\pi} f(x)^2 dx = \|f\|_2^2 = \frac{\pi}{2} a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This identity continues to hold for  $2L$  periodic functions integrated over  $[-L, L]$ .

## 5 Vector Fields

### 5.1 Vector Fields and Flows, Divergence and Curl

**Flow Lines** If  $F$  is a vector field, a flow line for  $F$  is a path  $c(t)$  such that

$$c'(t) = F(c(t)).$$

That is, that  $F$  yields the velocity field of the path  $c(t)$ .

**The Del  $\nabla$  operator** The vector differential operator  $\nabla$  may be considered a symbolic vector. The differential operator may be written as

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k.$$

**Divergence** Given a field  $F = (f_1, f_2, \dots, f_n)$ , the divergence of  $F$  is

$$\operatorname{div} F = \nabla \cdot F = \sum_{i=1}^n \nabla f_i.$$

Divergence may be thought as a type of derivative that describes the measure at which a vector field *spreads away* from a certain point. If the divergence is positive, then there is a net outflow while there is net inflow if the divergence is negative.

Observe that the divergence of a vector field will be real-valued.

**Curl** If  $F$  is a vector field, then the curl may be defined as

$$\operatorname{curl} F = \nabla \times F.$$

Curl is also analogous to a type of derivative for vector fields. The curl may be thought as the measure at which the vector field *swirls* around a point. A positive swirl can be thought of as a counter clockwise rotation.

Observe that the curl of a vector field is also a vector field.

## 5.2 Vector Identities

### Basic Vector Identities

1.  $\nabla(f + g) = \nabla f + \nabla g$
2.  $\nabla(\lambda f) = \lambda \nabla f$  where  $\lambda \in \mathbb{R}$
3.  $\nabla(fg) = g \nabla f + f \nabla g$ . You may draw analogies to the product.
4.  $\nabla \frac{f}{g} = \frac{f \nabla g - g \nabla f}{g^2}$  where  $g \neq 0$ . This is analogous to the quotient rule.
5.  $\nabla \cdot (F + G) = \nabla \cdot F + \nabla \cdot G$
6.  $\nabla \times (F + G) = \nabla \times F + \nabla \times G$
7.  $\nabla \cdot (fF) = f \nabla \cdot F = F \cdot \nabla f$
8.  $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$
9.  $\nabla \cdot (\nabla \times F) = 0$
10.  $\nabla \times (fF) = f \nabla \times F = \nabla f \times F$
11.  $\nabla \times (\nabla f) = 0$
12.  $\nabla^2(fg) = f \nabla^2 g + 2((\nabla f \cdot \nabla g)) + g \nabla^2 f$
13.  $\nabla \cdot (\nabla \times \nabla g) = 0$
14.  $\nabla \cdot (f \nabla g - g \nabla f) = f \nabla^2 g - g \nabla^2 f$

## 6 Path Integrals

### 6.1 Path Integrals

**Path (Scalar Line) Integrals** Suppose that a vector-valued function  $c(t)$  parametrises a curve  $C$  for  $t \in [a, b]$ . The scalar line integral may be thought as the integral of along  $c$ .

**Computing a Scalar Line Integral** Let  $c(t)$  parametrise a curve  $C$  for  $t \in [a, b]$ . Assume that  $f(x, y, z)$  and  $c(t)$  are continuous. Then,

$$\int_C f(x, y, z) ds = \int_a^b f(c(t)) \cdot \|c'(t)\| dt.$$

#### Elementary Properties of Path Integrals

- $\int_C f_1 ds + \int_C f_2 ds = \int_C (f_1 + f_2) ds,$
- $\int_C \lambda f ds = \lambda \int_C f ds, \quad \lambda \in \mathbb{R}.$

### 6.2 Applications Of Path Integrals

Suppose that  $\delta = \delta(x, y, z)$  which is a density function.

Mass

$$M = \int_C \delta(x, y, z) ds.$$

#### First Moments About the Coordinate Plane

- $M_{yz} = \int_C x \delta ds$
- $M_{xz} = \int_C y \delta ds$
- $M_{xy} = \int_C z \delta ds$

#### Coordinates of Center of Mass

- $\bar{x} = \frac{M_{yz}}{M}$
- $\bar{y} = \frac{M_{xz}}{M}$
- $\bar{z} = \frac{M_{xy}}{M}$

#### Moments of Inertia About Axes

- $I_x = \int_C (y^2 + z^2) \delta ds$
- $I_y = \int_C (x^2 + z^2) \delta ds$
- $I_z = \int_C (x^2 + y^2) \delta ds$

## 7 Vector Line Integrals

### 7.1 Vector Line Integrals

**Vector Line Integrals** Vector line integrals are different from scalar line integrals in the sense that to define a vector line integral, we must specify a direction along the path or curve  $C$ .

**Computing a Vector Line Integral** Let  $c(t)$  parametrise an oriented curve  $C$  for  $t \in [a, b]$ . Then,

$$\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t) dt.$$

**Link to Path Integrals** Suppose that  $C$  is a smooth curve with a parametrisation  $c(t)$  for  $t \in [a, b]$  where  $c(t)$  is continuously differentiable and  $c'(t) \neq 0$  for all  $t \in [a, b]$ .

Then,  $c'(t)$  is a non-zero tangent vector pointing in the forward direction and the unit tangent vector is

$$T(c(t)) = \frac{c'(t)}{\|c'(t)\|}.$$

Then,

$$\int_C F \cdot ds = \int_C F \cdot T ds.$$

**Summing Paths** Suppose that  $C$  is made of  $n$  finitely many paths  $C_i$ . Then,  $C = \sum_i^n C_i$ . Note that all the curves must be joined end to end. Then,

$$\int_C F \cdot ds = \sum_i^n \int_{C_i} F \cdot ds.$$

**Work and Other Alternative Notations** Suppose that  $c(t) = (x(t), y(t), z(t))$  and  $F = (M, N, P)$ . Then, we denote work as any of the following notations

$$\begin{aligned} W &= \int_C F \cdot ds \\ &= \int_a^b \left( M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt \\ &= \int_C M ds + N ds + P ds. \end{aligned}$$

#### Properties of Line Integrals

- Linearity
- Reversing Orient
- Additivity

**Flow Integrals and Circulation** Suppose that  $F$  represents a velocity field of a fluid flowing through a region in space. Then, the flow across a curve may be defined as the following

$$\text{Flow} = \int_a^b F \cdot \hat{T} ds.$$

This integral is called the flow integral. If the curve is a closed loop then this is called the *circulation* around the curve.

**Flux in the Plane** If  $C$  is a smooth closed curve in the domain of a continuous vector field  $F = M(x, y)i + N(x, y)j + N(x, y)j$  and,  $n$  is the outward pointing unit-normal on  $C$  then, the flux of  $f$  across  $C$  is the following expression

$$\int_C F \cdot \hat{n} ds.$$

**Calculating Flux Across a Smooth Closed Plane Curve** Suppose that  $F = Mi = Nj$ . Let  $G = -N, M$

$$\text{Flux of } F \text{ across } C = \oint_C Mdy - Ndx = \oint_C Gds$$

## 7.2 Fundamental Theorem of Line Integrals

**Gradient Fields** A vector field  $F$  is called a gradient vector field if there exists a real-valued function  $\phi$  such that  $F = \nabla\phi$ . That is,

$$\begin{pmatrix} M \\ N \\ P \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{pmatrix}.$$

If such a function  $\phi$  exists then  $\phi$  is called the potential function of  $F$  where,  $F$  is *conservative*.

**Fundamental Theorem for Gradient Vector Fields** If  $F = \nabla\phi$  on a domain  $\mathcal{D}$ . Then, for all oriented curves  $C$  in  $\mathcal{D}$  with an initial point  $P$  and a terminal point  $Q$ ,

$$\int_C F \cdot ds = \phi(Q) - \phi(P).$$

The integral is independent of the path.

**Cross Partial of Gradient Vector Fields are Equal** Let  $F = (F_1, F_2, F_3)$  be a gradient vector field whose components have continuous partial derivatives. Then, the cross partials are equal. That is,

$$\begin{aligned}\frac{\partial F_1}{\partial y} &= \frac{\partial F_2}{\partial x}, \\ \frac{\partial F_2}{\partial z} &= \frac{\partial F_3}{\partial y}, \\ \frac{\partial F_3}{\partial x} &= \frac{\partial F_1}{\partial z}.\end{aligned}$$

Equivalently,

$$\nabla \times F = 0.$$

### 7.3 Green's Theorem

Green's Theorem connects double integrals and line integrals.

**Green's Theorem: Flux Divergence or Normal Form** Let  $D$  be a bounded simple region in  $\mathbb{R}^2$  with a nonempty interior whose boundary consists of a finite number of smooth curves. Let  $C$  be the boundary of  $D$  with a positive (counter-clockwise) direction. Let  $F = Mi + Nj$  be a continuously differentiable boundary vector field on  $D$ .

Then, the outward flux of  $F$  across the curve  $C$  equals the double integral of divergence  $\nabla \cdot F$  over  $D$ . That is,

$$\oint_C (F \cdot \hat{n}) ds = \oint_C -N dx + M dy = \int \int_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy.$$

Once again, note the assumptions:

- $D$  is bounded and simple with a non-empty interior,
- The boundary  $C$  is oriented in the positive (counter-clockwise) direction, and is the finite union of smooth curves,
- The vector field  $F$  is continuously differentiable on  $D$ .

**Green's Theorem: Circulation-Curl or Tangential Form** Let  $D$  be a bounded simple region in  $\mathbb{R}^2$  with a nonempty interior whose boundary consists of a finite number of smooth curves. Let  $C$  be the boundary of  $D$  with a positive (counter-clockwise) direction. Let  $F = Mi + Nj$  be a continuously differentiable boundary vector field on  $D$ .

Then, the counter-clockwise circulation of  $F$  around  $C$  equals the double integral of  $(\nabla \times F) \cdot k$  over  $D$ . That is,

$$\oint_C (F \cdot \hat{T}) ds = \oint_C M dx + N dy = \int \int_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

**Area of a Region** Let  $D$  be a simple and bounded region with a non-empty interior and let  $C$  be the boundary of  $D$  which is the finite union of smooth curves. Then, the area of  $D$  can be calculated as such

$$\text{Area}(D) = \frac{1}{2} \oint_C (-ydx + xdy) .$$