Higher Complex Analysis

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2021 T3

Contents

1 Assumed Knowledge

- Arithmetic of complex numbers

- Cartesian and Polar representation

- Argand Diagram

- de Moivre's Theorem

- Extracting n-th roots

2 Inequalities and Sets

2.1 Triangle Inequalities

The triangle inequality For all $w, z \in \mathbb{C}$,

$$|w + z| \leq |w| + |z|.$$

This can be generalised for an arbitrary amount of complex numbers such that

$$|z_0 + z_1 + z_2 + \ldots z_n| \leq |z_0| + |z_1| + |z_2| + \ldots |z_n|.$$

Reverse Triangle Inequality For all $w, z \in \mathbb{C}$,

$$||w| - |z|| \leq |w - z|.$$

Lemma For all real θ,
$$|e^{i\theta} - 1| \leq |\theta|.$$

2.2 Sets

Open Ball The open ball with center z_0 and radius ϵ, $\mathrm{B}(z_0, \epsilon$ is the set,

$$\{z \in \mathbb{C} : |z - z_0| < \epsilon\}.$$

Types of Points For any point z in $mathbbC$, there are 3 mutually exclusive and exhaustive possibilities:

- Interior Point,

- Boundary Point,

- Exterior Point.

Types of Sets

- A set is **open** if all of its points are interior points.

- A set is **closed** if it contains all of its boundary points or, its complement is open.

- The **closure** of a set S written as $\bar{S}$ is the set plus its boundary points.

- A set S is **bounded** if $S \subseteq B(0, R)$ for some real R.

- A set S is **compact** if it is both closed and bounded.

- A set is a **region** if it is an open set, with some, none, or all of its boundary points.

Paths   A polygonal path is a sequence of line segments such that the end point of one line is the start point of the next line. A simple, closed path is one that does not cross itself and, has the feature that its final point is the same as its initial point. The complement of a simple closed path has two pieces. A bounded interior and unbounded exterior.

- A set is **path connected** if and two points in the set can by joined by a path within that set.

- A set is **simply connected** if it is connected and the interior of every connected path remains entirely within the set. In other words, the set has no holes.

- The set is a **domain** if it is open and path connected.

3   Functions of a Complex Variable

Complex functions may be written as

$$f(z) = f(x, y) = u(x, y) + iv(x, y).$$

The Fundamental Theorem Of Algebra   Every non-constant complex polynomial of degree d factorizes uniquely. That is, there exist $\alpha_1, \alpha_2 \ldots alpha_d$ such that

$$p(z) = c \prod_{j=1}^{d} (z - \alpha_j).$$

4   Fractional Linear Transformations

4.1   Introduction to FLTs

Fractional Linear Transformations (FLTs) are functions of the form

$$f(z) \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \qquad ad - bc \neq 0.$$

They may be represented as a complex matrix M such that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If the determinant of this matrix is zero, then the function is constant.

Factorising Matrices Every 2×2 matrix may be written as a product of at most 3 matrices of the following two forms:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Inverse of FLT Suppose that

$$w = \frac{az + b}{cz + d}.$$

Then,

$$z = \frac{dw - b}{a - cw}.$$

4.2 Images of FLTs

Image Under a Line or Circle Let T_M be a fractional linear transformation. Then, the image of a line or a circle under T_M is either a line or a circle. If $\frac{-d}{c} \in \text{Domain}(T_M)$ then, the image of a FLT is a line. Otherwise, the image is bounded and is thus a circle.

5 Limits and Continuity

5.1 Limits

Most rules from real numbers apply.

Uniqueness of Limit Suppose that f is a complex function and that $z_0 \in \text{Domain}(f^{-1}$. If $\lim_{z \to z_0}$ exists, then it is unique.

5.2 Continuity

Definition Suppose that f is a complex function. f is continuous at the point z_0 if $f(z_0)$ is defined and $\lim_{z \to z_0} = f(z_0)$. The function if it is continuous on a set S is it is continuous at all points on its domain.

Some Continous Functions The following functions are continuous for all $z \in \mathbb{C}$ where $z \neq 0$.

- $z \mapsto z$

- $z \mapsto |z|$

- $z \mapsto \Re(z)$

- $z \mapsto \Im(z)$

Composition of Continuous Functions If $c \int \mathbb{C}$, f, g are continuous functions $S \mapsto \mathbb{C}$ then,

$$cd, f + g, |f|, f, \Re f, \Im f$$

are continuous on S as well as $\frac{f}{g}$ where $g(z) \neq 0$.

6 Complex Differentiability

Suppose that $S \subseteq \mathbb{C}$ and that $f : S \rightarrow C$ is a complex function. Then we say that f is differentiable at the point z_0 in S if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is finite. If so, then it is the derivative of f at z_0.

6.1 Cauchy-Reimann Equations

If f is differentiable at every point of an open set, then the Cauchy-Reimann equations hold at every point of that set.

The equations
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The derivative in terms of u and v

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Holomorphic Functions Suppose that Ω is an open subset of $\mathbb{C}$ and $f : \Omega \rightarrow \mathbb{C}$ is a function. If f is differentiable at every point of Ω then, we say that f is **holomorphic** or **complex analytic** or **analytic** in Ω. If f is analytic on the entire complex plane then, it is **entire.**

7 Harmonic Functions

Definition   Suppose that $u : \Omega \to \mathbb{R}$ is a function where Ω is an open subset of $\mathbb{R}^2$ and, that u is twice continuously differentiable. u is harmonic if, it satisfies *Laplace's Equation*:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Harmonic and Holomorphic Functions   To find a harmonic function, it is sufficient to find a holomorphic function. Suppose that $f \int H(\Omega)$ where Ω is an open subset of $\mathbb{C}$, that f is twice continuously differentiable and that

$$f(x + iy) = u(x, y) + iv(x, y)$$

for all $x + iy \in \Omega$ where u and v are real-valued. Then u and v are harmonic functions.

Infinite Differentiability   For the above, the twice differentiable requirement is redundant since, if $f \in H(\Omega)then$, f is infinitely differentiable on Ω.

Harmonic Conjugate   If Ω is a simply connected domain, and $u : \Omega \to \mathbb{R}$ is harmonic then, there exists a harmonic function $v : \Omega \to \mathbb{R}$ such that f given by

$$f(x + iy) = u(x, y) + iv(x, y)$$

in Ω is holomorphic. In fact, any such functions v differ only by a complex constant. This function v is called the **harmonic conjugate** of u

Function From Harmonic Components   If u, v is found then, you may evaluate $f(z, 0)$ in place of $f(x, y)$ and allow that to be your $f(z)$ due to analytic continuity. See more about this later.

8 Power Series

8.1 Definition and Convergence

Definition   A complex power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n \left(z - z_0\right),$$

centered at z_0 where, a_n are all fixed complex numbers.

Ratio Test   The radius of convergence p, may be found as

$$p = \lim_{n \to \infty} \frac{|a_n|}{a_{n+1}}$$

as long as the limit exists or, is $+\infty$.

Root Test The radius of convergence p is given by

$$p = \lim_{n \to \infty} \frac{1}{|a_n|^{\frac{1}{n}}}$$

as long as the limit exists or, is $+\infty$.

In other to use the root test for coefficients with factorials, **Stirling'g formula** may be useful.

$$\lim_{n \to \infty} \frac{1}{\ln((2\pi)^1/2e^{-n}n^{n+1/2})} = 1 \text{ and } \lim_{n \to \infty} \frac{n!}{(2\pi)^{1/2}e^{-n}n^{n+1/2}} = 1.$$

8.2 Algebra and Calculus of Power Series

Suppose that both power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

converge in $B(z_0, p)$. Then, the following series also converge in $B(z_0, p)$:

1. $\sum ca_n(z - z_0)^n = c \sum a_n(z - z_0)^n$

2. $\sum(a_n + b_n)(z - z_0)^n = \sum a_n(z - z_0)^n + \sum b_n(z - z_0)^n$

3. $\sum_{n=0}^{\infty} a_n(z - z_0)^n \times \sum b_n(z - z_0)^n = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$

Derivatives of Power Series The derivative of the power series can be found by simply taking the sum of derivatives of each element. The derivative exists in the same radius of convergence as the original function. That is, for $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$,

$$f'(z) = \sum_{n=0}^{\infty} a_n n(z - z_0)^{n-1}.$$

An infinite power series of the form above may be differentiated infinitely.

9 Exponential, Hyperbolic and Trigonometric Function

9.1 The exponential function

Definition We may define the exponential function with the following power series:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbb{C}.$$

If a function satisfies $f(0) = 1$ and $f'(z) = f(z)$ then, that function must be the exponential function.

9.2　The hyperbolic functions

We may define the hyperbolic functions with the following power series:

$$\cosh(z) = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

and

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.$$

Observe that

- $\cos(iz) = \cosh(z)$

- $\cosh(iz) = \cos(z)$

- $\sin(iz) = i\sinh(z)$

- $\sinh(iz) = i\sin(z)$

9.3　The trigonometric functions

We may define the trigonometric functions with the following power series:

$$\cos(z) = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

and,

$$\sin(z) = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}.$$

10　Logarithms and Roots

Square Root　We define the principle value of the square root as:

$$PV w^{\frac{1}{2}} = \begin{cases} |w|^{\frac{1}{2}} e^{i\mathrm{Arg}(w)/2} & \text{if } w \neq 0 \\ 0 & \text{if } w = 0. \end{cases}$$

n-th Roots　The principle value of the nth root is given by

$$PV z^{1/n} = \exp\left(\frac{\mathrm{Log}(z)}{n}\right) = |z|^{\frac{1}{n}} e^{\frac{i\mathrm{Arg}(z)}{n}}.$$

The function $PV z^{1/n}$ is differentiable in $\mathbb{C}\backslash(-\infty, 0]$.

11   Inverses of exponential and related function

Logarithm   We define the complex logarithm as

$$\log(z) = \ln(z) + \arg(z)i.$$

Complex Powers   We may define raising to a complex power α as the following

$$z^\alpha = \exp(\alpha \log(z).$$

That is,

$$z^\alpha = \exp(\alpha \ln|z| + i\mathrm{Arg}(z)\alpha + \alpha 2k\pi i,$$

where, k is an integer.

The function $z \mapsto PV z^\alpha$ is differentiable on $\mathbb{C}\backslash(-\infty, 0)$.

Inverse sinh   We define the principal inverse of sinh as

$$PV \sinh^{-1} w = \mathrm{Log}(w + PV(w^2 + 1)^{1/2}).$$

Inverse cosh   We define the principal inverse of sinh as

$$PV \cosh^{-1} w = \mathrm{Log}(w + PV(w + 1)^{1/2}PV(w - 1)^{1/2}).$$

12   Contour Integrals

Definition   A contour is an oriented range of a piece-wise smooth curve in the complex plane.

Contour Integrals   Given a piecewise smooth curve $\gamma(t) : [a, b] \to \mathbb{C}$ and a continuous (not necessarily differentiable) function f defined on the range of γ, we define the complex line integral of f over *gamma* as

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Properties

- The integral is linear

- The integral is independent of the parametrization

- The integral is additive for joins

- The integral is dependent of the orientation

- The size of the integral depends on the size of the curve and function. See the ML lemma

The ML Lemma

$$|\int_\gamma f(z)dz \le ML$$

where L is the length of γ and M is the maximizer. That is, M is some number such that, for all $z \in \text{range}(\gamma)$, $f(z) \le M$.

13 The Cauchy Goursat Theorem

The Cauchy Goursat Theorem Suppose that Ω is a simply connected domain such that $f \in H(\Omega)$ and that Γ is a closed contour in Ω. Then,

$$\int_\Gamma f(z)dz = 0.$$

Existence Of Primitives Suppose that Ω is a simply connected domain in $\mathbb{C}$ and that $f \in H(\Omega)$. Then, there exists a function F on Ω such that

$$\int_\Gamma f(z) = F(q) - F(p)$$

for all contours in Ω from p to q. Further, it is the case that F is differentiable and, $F' = f$.

14 Cauchy's Integral Formula

14.1 The integral formula

Suppose that Ω is a simply connected domain in $\mathbb{C}$, that f is holomorphic on Ω, that Γ is a simply connected domain and that $w \in \text{Int}(\Gamma)$. Then,

$$f(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}dz.$$

In other words,

$$\int_\Gamma \frac{f(z)}{z-w} = 2\pi i f(w).$$

Cauchy's Integral Formula with Power Series With the same conditions as above, where $\Omega = B(z_0, R)$.

$$f(w) = \sum_{n=0}^\infty c_n(z-z_0)^n$$

where,

$$c_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^n}dz$$

The radius of convergence is at least R.

We may use this, combined with the fact that $f^{(n)}(z_0) = n!c_n$ to get Cauchy's Generalised Integral Formula.

14.2 The Generalised Integral Formula

Cauchy's Generalised Integral Formula

$$f^{(n)}(z_0) = \frac{n!}{p\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}.$$

Liouville's Theorem Suppose that f is a bounded entire function. Then, f is constant.

15 Morera's Theorem and Analytic Continuation

Morera's Theorem Suppose that Ω is a domain and that the function $f : \Omega \to \mathbb{C}$ is continuous and that

$$\int_\Gamma f(z)dz = 0$$

whenever Γ is a closed contour in Ω. Then, f is holomorphic in $Omega$.

Analytic Continuation Suppose that Υ is a nonempty, open subset of a domain Ω in $\mathbb{C}$ and that $f \in H(\Omega)$. If $f(z) = 0$ for all z in Υ then, $f(z) = 0$ for all z in Ω.

16 Taylor Series

A taylor series for the function f around z_0 is of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Algebra and Calculus of Taylor Series The same rules that apply for power series, also apply for Taylor series.

Differentiability Suppose that the taylor series for f converges in a ball centered at z_0, of radius r. Then f is differentiable on that same domain.

Lagrange inversion theorem Suppose that f is holomorphic on Ω and that $f(a) = b$ and $f'(a) \neq 0$. Then, there is a holomorphic function g defined on an open set Υ containing b such that $g \circ f(z) = z$ for all z near a and, $f \circ g(w) = w$ for all w near b. Further, $g(w) = a + \sum_{n=1}^{\infty} c_n(w - b)^n$ where

$$c_n = \frac{1}{n!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}}$$

when $n \geq 1$.

17   Laurent's Theorem

Suppose that A is the annulus $\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$ and $R_1 < r < R_2$. If f is holomorphic on A then,

$$f(w) = \sum_{n=-\infty}^{\infty} c_n(w - z_0)^n \quad \forall w \in A$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}}.$$

18   Singularities

Isolated Singularities   A function has an *isolated singularity* at z_0 in $\mathbb{C}$ if f is holomorphic in the punctured ball $B^\circ(z_0, R)$ for some $R \in \mathbb{R}^+$ and, f is not differentiable at z_0.

Types of Singularities and Zeroes   If $c_n = 0$ for all n in $\mathbb{Z}$, then, $f(z_0) = 0$. Otherwise, following examples are mutually exclusive and exhaustive

- There are infinitely many $n \in \mathbb{Z}^-$ such that $c_n \neq 0$. Then, z_0 is an **essential** singularity.

- There are no $n \in \mathbb{Z}$ such that $c_n \neq 0$. Then, z_0 is a removable singularity. Otherwise, if there exist $m \in \mathbb{Z}$ such that $c_m \neq 0$ and $c_n = 0$ for all $n < m$ then, f has a zero of order m at z_0.

- There is atleast one, but finitely many $n \in \mathbb{Z}$ such that $c_n \neq 0$. Then there exists $m \in \mathbb{Z}^-$ such that $c_m \neq 0$ and $c_n = 0$ for all $n < m$. Then, f has a pole of order $-M$

19   Residues and Residue Theorem

19.1   Residues

Definition   Suppose that the function f has an isolated singularity at z_0. Then, f is holomorphic on some punctured ball $B^\circ z_0$ and, has a Laurent series there of form $\sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$. The residue of f at z_0 is written as:

$$\text{Res}(f, z_0) = c_{-1}.$$

Laurent's Theorem Connection   From Laurent's theorem, if f is holomorphic in the area $B^\circ(z_0, 0)$ then,

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \int_\Gamma f(z) dz.$$

19.2 Cauchy's Residue Theorem

Residue Theorem Suppose that Γ is a simple, closed contour with the standard orientation in a domain Ω and $f \in H(\Omega)$ and, that $\text{Int}(\Gamma) \cap \Omega = \text{Int}(\Gamma) \backslash \{z_1, z_2, z_3, \ldots, z_K\}$. Then,

$$\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{K} \text{Res}(f, z_k).$$

19.3 Computing Singularities

Suppose that f has an isolated singularity at the point z_0. The, there are a few ways to computer the residue at z_0, depending on the type of singularity.

Removable Singularity The residue of removable singularities is zero.

Essential Singularities There is no formulaic method to compute residues at essential singularities. A possible approach may be to consider integration, or manipulation of series.

Poles of non-zero order Suppose that f has a pole of order N at z_0. Then,

$$\text{Res}(f, z_0) = \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{N-1}}{dz^{N-1}}(z - z_0)^N f(z).$$

The p/q' formula Suppose that $f(z) = \frac{p(z)}{q(z)}$ in Ω and that $p(z) \neq 0$, while $q(z) = 0$. If z_0 is a simple pole of order 1 then,

$$\text{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$