Higher Theory of Statistics
Math2901 UNSW

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Contents

1 Introduction

1.1 Experiments, Sample Space and Events

Experiments An experiment is any process that leads to a recorded observation.

Outcome and Sample Space An outcome is possible result of the experiment. The set of all possible outcomes is called the sample space. The sample space is often denoted by Ω.

Observe that not all sample spaces are countable. An uncountable example would be the set of all real number between 0 and 1.

Events An event is a set of outcomes that is, a subset of the sample space Ω.

Mutual Exclusion Events A, B are mutually exclusive (disjoint) if they have no outcomes in common. That is, $A \cap B = \emptyset$.

Set Operation Revision If you have trouble recalling the following laws, for associativity and distributivity, you may replace $\cap$ with $\times$ and $\cup$ with $+$.

TODO: Associative and Distributive Law

1.2 Sigma Algebra

The σ algebra must be defined for rigorously working with probability. The formalization of this, is beyond the scope of this course.

The σ-algebra can be thought of as the family of all possible subsets or events in a sample space. Analogously, this may be conceptualised as the power-set of the sample space.

Probability The probability is a set function, often denoted by $\mathcal{P}$ that maps events from the σ-algebra to $[0, 1]$ and satisfies certain properties.

Probability Space The triplet $\Omega, \mathcal{A}, \mathbb{P}$ is the probability space where

- Ω is the sample space,
- $\mathcal{A}$ is the σ-algebra,
- $\mathbb{P}$ is the probability function.

Properties of Probability Given the probability/sample space $\Omega, \mathcal{A}, \mathbb{P}$, the probability function $\mathbb{P}$ must satisfy

- For all set $A \in \mathcal{A}$, $\mathbb{P}(A) \geq 0$
- $\mathbb{P}(\Omega) = 1$
- Countable additive. Suppose that the family of set A_i

Theorem: Continuity from below Given an increasing sequence of events $A_1 \subset A_2 \subset \cdots \subset A_n$ then,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} = \mathbb{P}(A_n)$$

Theorem: Continuity from below Given a decreasing sequence of events $A_1 \supset A_2 \supset \ldots A_n$ then,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} = \mathbb{P}(A_n)$$

More Probability Lemmas

- $\mathbb{P}(\emptyset) = 0$,
- For any $A \in \mathcal{A}$, $\mathbb{P}(A) \leq 1$ and $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$,
- Suppose $A, B \in \mathcal{A}$ and $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

1.3 Conditional Probability and Independence

Conditional Probability The conditional probability that an event A occurs given that the event B has already occurred is denoted by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Independence The events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

A lemma on independence Given two events A, B, then

$$\mathbb{P}(A|B) = A \quad \text{if and only if} \quad \mathbb{P}(B|A) = B$$

Pairwise Independence of Sequences A countable sequence of events $A_{i\in\mathbb{N}}$ is pairwise independent if

$$\mathbb{P}(A_i 0 \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j) \quad \forall i \neq j.$$

Independence of Sequences A countable sequence of events $A_{i\in\mathbb{N}}$ is independent is independent if for any sub-collection $A_{i_1}, \ldots, A_{i_n}$ we have

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}) \prod_{j=1}^{n} \mathbb{P}(A_{i_j}).$$

Multiplicative Law Given A, B are events, then,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$$

This is equivalent to the multiplication down a decision tree.

Additive Law Let A, B be events. Then,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

This is analogous to the inclusion-exclusion principle from set theory.

Law of Total Probability Suppose that $(A_i)_{i=1,\ldots,k}$ are mutually exclusive and exhaustive of Ω. That is,

$$\bigcup_{i=1}^{k} A_i = \Omega.$$

Then for any event B, we have

$$\mathbb{P}(B) = \sum_{i=1}^{k} \mathbb{P}(B|A_i)\mathbb{P}(A_i).$$