

Higher Theory of Statistics

Math2901 UNSW

Hussain Nawaz
hussain.nwz000@gmail.com

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Contents

1	Introduction	2
1.1	Experiments, Sample Space and Events	2
1.2	Sigma Algebra	2
1.3	Conditional Probability and Independence	3
1.4	Descriptive Statistics and R	4
2	Random Variables	4
2.1	Random Variables	4
2.2	Expectation and Variance	6
2.3	Moment Generating Functions	7
3	Common Distributions	9
3.1	Common Discrete Distributions	9
3.2	Continuous Distributions	10
4	Bivariate Distributions	11
4.1	Bivariate Transformations	14
5	Sums of Variables	14
6	Central Limit Theorem and Convergence of Random Variables	15
6.1	Central Limit Theorem	15
6.2	Convergence of Random Variables	16
7	Estimators	17
7.1	Data and Models	17
7.2	Estimators	17
7.3	Errors	18

1 Introduction

1.1 Experiments, Sample Space and Events

Experiments An experiment is any process that leads to a recorded observation.

Outcome and Sample Space An outcome is possible result of the experiment. The set of all possible outcomes is called the sample space. The sample space is often denoted by Ω .

Observe that not all sample spaces are countable. An uncountable example would be the set of all real number between 0 and 1.

Events An event is a set of outcomes that is, a subset of the sample space Ω .

Mutual Exclusion Events A, B are mutually exclusive (disjoint) if they have no outcomes in common. That is, $A \cap B = \emptyset$.

Set Operation Revision If you have trouble recalling the following laws, for associativity and distributivity, you may replace \cap with \times and \cup with $+$.

TODO: Associative and Distributive Law

1.2 Sigma Algebra

The σ algebra must be defined for rigorously working with probability. The formalization of this, is beyond the scope of this course.

The σ -algebra can be thought of as the family of all possible subsets or events in a sample space. Analogously, this may be conceptualised as the power-set of the sample space.

Probability The probability is a set function, often denoted by \mathcal{P} that maps events from the σ -algebra to $[0, 1]$ and satisfies certain properties.

Probability Space The triplet $\Omega, \mathcal{A}, \mathbb{P}$ is the probability space where

- Ω is the sample space,
- \mathcal{A} is the σ -algebra,
- \mathbb{P} is the probability function.

Properties of Probability Given the probability/sample space $\Omega, \mathcal{A}, \mathbb{P}$, the probability function \mathbb{P} must satisfy

- For all set $A \in \mathcal{A}$, $\mathbb{P}(A) \geq 0$
- $\mathbb{P}(\Omega) = 1$
- Countable additive. Suppose that the family of set A_i

Theorem: Continuity from below Given an increasing sequence of events $A_1 \subset A_2 \subset \dots \subset A_n$ then,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$$

Theorem: Continuity from above Given a decreasing sequence of events $A_1 \supset A_2 \supset \dots \supset A_n$ then,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$$

More Probability Lemmas

- $\mathbb{P}(\emptyset) = 0$,
- For any $A \in \mathcal{A}$, $\mathbb{P}(A) \leq 1$ and $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$,
- Suppose $A, B \in \mathcal{A}$ and $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

1.3 Conditional Probability and Independence

Conditional Probability The conditional probability that an event A occurs given that the event B has already occurred is denoted by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Independence The events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

A lemma on independence Given two events A, B , then

$$\mathbb{P}(A|B) = \mathbb{P}(A) \quad \text{if and only if} \quad \mathbb{P}(B|A) = \mathbb{P}(B)$$

Pairwise Independence of Sequences A countable sequence of events $A_{i \in \mathbb{N}}$ is pairwise independent if

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j) \quad \forall i \neq j.$$

Independence of Sequences A countable sequence of events $A_{i \in \mathbb{N}}$ is independent if for any sub-collection A_{i_1}, \dots, A_{i_n} we have

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_n}) = \prod_{j=1}^n \mathbb{P}(A_{i_j}).$$

Multiplicative Law Given A, B are events, then,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$$

This is equivalent to the multiplication down a decision tree.

Additive Law Let A, B be events. Then,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

This is analogous to the inclusion-exclusion principle from set theory.

Law of Total Probability Suppose that $(A_i)_{i=1,\dots,k}$ are mutually exclusive and exhaustive of Ω . That is,

$$\bigcup_{i=1}^k A_i = \Omega.$$

Then for any event B , we have

$$\mathbb{P}(B) = \sum_{i=1}^k \mathbb{P}(B|A_i)\mathbb{P}(A_i).$$

1.4 Descriptive Statistics and R

Sample Variance and Mean Suppose that we are given observations x such that $x = (x_1, x_2, \dots, x_n)$.

Then, the **sample mean** is given by

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i.$$

The **sample variance** is given by

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2.$$

-

$$\frac{sdf}{ds} \frac{sdf}{5} \frac{4}{3} a + \frac{b}{7} \frac{a+b+(ddf)}{7} X = a_1 + a + X = a_1 + a_2 x = a_{22} - b_{22} +$$

This is the n -th This is x cringe.

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2 Random Variables

2.1 Random Variables

Definition: Random Variables Suppose that we work on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. And the outcomes in Ω are denoted by ω .

Then, a random variable (r.v) X is a function from Ω to \mathbb{R} such that $\forall x \in \mathbb{R}$, the set $A_x = \{\omega \in \Omega, X(\omega) \leq x\}$. That is, a random variable is a function that maps *Omega* to some space.

Convention on Random Variables Random variables are often denoted by capital letters while, the outcomes are denoted by the lower-case equivalent of the random variable.

Cumulative Distributive The cumulative distribution of a r.v X is defined by

$$F_X(x) = \mathbb{P}(\{\omega : X(\omega) \leq x\}) = \mathbb{P}(X \leq x).$$

Cumulative Distribution Theorems Suppose that F_X is cumulative distribution function of X . Then,

- It is bounded between zero and one and

$$\lim_{x \downarrow -\infty} = 0 \quad \text{and} \quad \lim_{x \uparrow \infty} = 1.$$

- It is non-decreasing. That is, if $x \leq y$ then, $F_X(x) \leq F_X(y)$.
- For any $x \leq y$,

$$\mathbb{P}(x < X < y) = \mathbb{P}(X \leq y) - \mathbb{P}(X \leq x) = F_X(y) - F_X(x).$$

- It is right continuous. That is,

$$\lim_{x \uparrow \infty} F_X \left(x + \frac{1}{n} \right) = F_X(x).$$

- it has finite left-hand limit and

$$\mathbb{P}(X < x) = \lim_{n \rightarrow \infty} F_x \left(x - \frac{1}{n} \right),$$

denoted by $F_X(x-)$. It is useful to observe that,

$$\mathbb{P}(X = x) = F_X(X) - F_X(x-) := F_X(x).$$

Discrete Random Variables A r.v. is said to be discrete if the image of X consists of countable many values x where $\mathbb{P}(X = x) > 0$. The probability function is $\Delta F_X(x) = \mathbb{P}(X = x)$ and satisfies

$$\sum_{\text{all } x} \mathbb{P}(X = x) = 1.$$

Continuous Random Variables and Probability Density Functions A r.v is continuous if the image of X takes a continuum of values.

The probability density function of a r.v is a real-valued function f_x on \mathbb{R} with the property that

$$\mathbb{P}(X \in A) = \int_A f_x(y) dy,$$

for any *Borel* subset of \mathbb{R}

Required Properties of a Density Function Valid density functions $f : \mathbb{R} \rightarrow \mathbb{R}$ must satisfy the following properties:

- $f(x) \geq 0, \forall x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f(x)dx = 1.$

Useful Properties of a Continuous Random Variable For all continuous random variables X , with density f_x ,

1. If $A = (-\infty, x]$ and creating a cumulative distribution function F_x such that $F_X(x) = \mathbb{P}(X \in A) = \mathbb{P}(X \leq x)$ then,

$$F_X(x) = \int_{-\infty}^x f_x(y)dy.$$

2. For all $a < b$,

$$\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)dx.$$

3. By the fundamental theorem of calculus and property 1,

$$F'_X(x) = \frac{d}{dx} \int_{-\infty}^x f_x(y)dy = f_X(x).$$

2.2 Expectation and Variance

Expectation The expectation of a r.v X , denoted by $\mathbb{E}(X)$ may be computed depending on when X is discrete or continuous.

Expectation of Discrete Random Variables If X is a discrete random variable then,

$$\mathbb{E}(X) := \sum_{\text{all } x} x\mathbb{P}(X = x) = \sum_{\text{all } x} x\Delta F_x(x).$$

Expectation of continuous Random Variables If X is a continuous random variable then,

$$\mathbb{E}(X) := \int_{-\infty}^{\infty} xf_X(x)dx$$

Interpreting the Expectation Often $\mathbb{E}(x)$ is called the *mean* of X . Observe that mean and average are not necessarily the same. $\mathbb{E}(X)$ may be thought as the long-run average of the outcomes of X . That is, the average observation of X converges to $\mathbb{E}(X)$.

Where our density function represents a physical model, $\mathbb{E}(X)$ is equivalent to the center of mass.

Linearity of the Expectation We note that the expectation is linear. That is, for all constants $a, b \in \mathbb{R}$,

$$\mathbb{E}(aX + b) = a\mathbb{E}(X) + b.$$

Variance Let X be a r.v and set $\mu = \mathbb{E}(x)$. Then,

$$\text{Var}(X) := \mathbb{E}((X - \mu)^2).$$

The standard deviation is the square root of variance.

Properties of Variance Given a random variance X then, for any constants $a, b \in \mathbb{R}$,

1. $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$.,
2. $\text{Var}(aX) = a^2\text{Var}(X)$,
3. $\text{Var}(X + b) = \text{Var}(X)$,
4. $\text{Var}(b) = 0$.

Covariance Recall that variance is $\mathbb{E}((X - \mu)^2)$. Suppose that X, Y are random variables such that $\mathbb{E}(X) = \mu_X, \mathbb{E}(Y) = \mu_Y$. Then, the covariance is defined as

$$\text{Cov}(X, Y) = \mathbb{E}((X - \mu_X)(Y - \mu_Y)).$$

Observe that $\text{Cov}(X, X) = \text{Var}(X)$.

Variance and Linearity The variance is not linear. That is,

$$\text{Var}(X + Y) \neq \text{Var}(X) + \text{Var}(Y).$$

However,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y).$$

2.3 Moment Generating Functions

Moments A moment of the random variable is denoted by

$$\mathbb{E}[X^r], \quad r = 1, 2, \dots$$

Moments measure mean, variance, skewness, and kurtosis, all ways of looking at the shape of the distribution.

Suppose that $f(x)$ is a probability density function. Then,

$$\mathbb{E}[X^r] = \int_{-\infty}^{\infty} x^r f(x) dx$$

Kurtosis The kurtosis is the standard's 4th moment. It measures how *fat* the tail is. A positive kurtosis implies a thinner tail than negative kurtosis.

Moment Generating Function A moment generating function (MGF) is denoted as

$$M_X(u) = \mathbb{E}(e^{uX}) = \int_{\text{all } x} e^{ux} f_X(x) dx.$$

We say that the MGF of X exists if $M_X(u)$ is finite in some interval containing zero.

Using Moment Generating Function to Find Moments Suppose that the moment generating function exists. Then,

$$\mathbb{E}(X^r) = \lim_{u \rightarrow 0} M_X^{(r)}(u) =: \lim_{u \rightarrow 0} \frac{d^r}{du^r} M_X(u).$$

Equivalence of Moment Generation Functions Let X, Y be two random variables and suppose that $M_X(u) = M_Y(u)$ for all u in some interval containing 0. Then,

$$F_X(x) = F_Y(x), \forall x \in \mathbb{R}.$$

That is, a moment generating function (when it exists), uniquely characterises a cumulative distribution function of a random variable.

Existence of Moments and Moment Generating Functions If the moment generating function exists then all moments can be computed. However, the converse is not necessarily true. That is, if all the moments exist and are finite, this does not imply the moment generating function exists.

Useful Inequalities

Markov Inequality - Chebychev's First Inequality For all non-negative r.v X , for $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

Equivalently,

$$\int_a^\infty f(x) dx \leq \int_{-\infty}^\infty x f(x) dx.$$

Chebychev's Second Inequality Suppose that X is any r.v with $\mathbb{E}(X) = \mu$, $\text{Var}(x) = \sigma^2$ and $k > 0$. Then

$$\mathbb{P}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Convex and Concave Functions In probability, we may want to know if a function is concave but, cannot use the usual method of the second derivative as the function is not necessarily twice differentiable.

A function h is convex if for any $\lambda \in [0, 1]$ and x_1, x_2 in the domain of h ,

$$h(\lambda x_1 + (1 - \lambda)x_2) \leq (\geq) \lambda h(x_1) + (1 - \lambda)h(x_2).$$

Jensen's Inequality Let X be a random variable. Suppose that h is a convex function. Then

$$h(\mathbb{E}(X)) \leq \mathbb{E}(h(X)).$$

If h is concave then,

$$h(\mathbb{E}(X)) \geq \mathbb{E}(h(X)).$$

Applications of Jensen's Inequality Using Jensen's inequality, it can be shown that

$$\text{Arithmetic Mean} \geq \text{Geometric Mean} \geq \text{Harmonic Mean}.$$

3 Common Distributions

3.1 Common Discrete Distributions

Bernoulli Distributions A Bernoulli trial is an experiment with two outcomes; success and failure. A random variable X is defined with

$$X = \begin{cases} 1 & \text{if success,} \\ 0 & \text{if failure.} \end{cases}$$

Let $p \in [0, 1]$ be probability of success. Then, we denote $X \sim \text{Bernoulli}(p)$

1. $\mathbb{P}(X = 1) = p,$
2. $\mathbb{P}(X = 0) = 1 - p,$
3. $\mathbb{E}(X) = p,$
4. $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 1 \times p - p^2 = p(1 - p).$

Binomial Distribution When there are n independent bernoulli trials with a success rate of p , and $X :=$ total number of successes. Then, X is a Binomial r.v with parameter n and p such that we write $X \sim \text{Bin}(n, p)$.

Let $(Y_i)_{i=1, \dots, n}$ be a sequence of independent bernoulli trials with success rate p . Then

$$X := \sum_{i=1}^n Y_i \text{ is } \text{Bin}(n, p).$$

Expectation exists as

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^n Y_i\right) = \sum_{i=1}^n \mathbb{E}(Y_i) = np.$$

Alternatively, using combinatorics,

$$\mathbb{P}(X = k) = C_k^n p^k (1-p)^{n-k}.$$

Poisson Distribution A random variable X follows the poisson distribution with parameter λ if its probability function is

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Observe that

$$\mathbb{E}(X) = \lambda.$$

Use The poisson distribution is used to model count data. That is, counting the number of times an event occurs within a time period. The parameter λ represents the average number of times the event occurs in the time period of interest.

Hypergeometric Distribution A random variable has hypergeometric distribution with parameter N, m, n and is written as $X \sim \text{Hyp}(n, m, N)$ if

$$\mathbb{P}(X = k) = \frac{C_x^m \times C_{n-x}^{N-m}}{C_n^N}, \quad \text{where } x = 1, \dots, n.$$

Example of Hypergeometric Given a box of N balls, m are red and $N - m$ are black. Draw n balls at random and let X be the number of red balls drawn. Then, $X \sim \text{Hyp}(n, m, N)$.

Remark The “I feel like skipping this... discrete problems are not very interesting” was stated while covering this. Interpretation of this is left as an exercise to the reader.

3.2 Continuous Distributions

Gaussian - Normal Random Variable Given parameters μ, σ^2 has a probability density function as

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

The expectation and variance are μ, σ^2 respectively.

Linear Transforms Let X be a r.v with a probability density function f_X . Let $y = a + bX$ then for $b > 0$ and $a \in \mathbb{R}$, then

$$f_Y(x) = \frac{1}{b} f_X\left(\frac{x-a}{b}\right).$$

Linear Transformation of Normally Distributed Random Variable Suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$ and $a \in \mathbb{R}$ and $b > 0$. Then, the random variable $Y := a + bX$ is normally distributed as

$$Y \sim \mathcal{N}(a + b\mu, b^2\sigma^2).$$

That is, normally distributed random variable are closed under linear transformation.

Indicator Function An indicator function of a set A is defined by

$$I_A(X) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Commonly, we may see indicator functions over intervals that follow the notation $I_{[a,b]}$.

The indicator unifies expectation and probability since, the probability is the expectation of the indicator function. Therefore, it may be written that

$$\mathbb{P}(X \in A) = \int_A f_X(x)dx = \int_{-\infty}^{\infty} I_A(x)f_X(x)dx = \mathbb{E}(I_A(X)).$$

4 Bivariate Distributions

Definition: Bivariate Distribution The joint density function of two continuous random variables X, Y is given by a bivariate function $f_{X,Y}$ with the following properties

1. $f(x, y) \geq 0, \forall (x, y) \in \mathbb{R}^2$.

- 2.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx dy = 1.$$

3. For any (measurable) sets A, B ,

$$\mathbb{P}(X \in A, Y \in B) = \int_A \int_B f_{X,Y}(x, y)dx dy.$$

Min and Max Notation We may write $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

Revision: Double Integral Theorems

1. **Tonelli's Theorem:** Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}_+$. Then,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)dy dx.$$

2. **Fubini's Theorem:** Suppose that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. If

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x, y)| dx dy < \infty, \text{ or } \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x, y)| dy dx < \infty,$$

Then,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)dx dy < \infty, = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)dy dx < \infty, .$$

Lemma: Expected Value of Bounded Borel Function For any bounded Borel function $g : \mathbb{R} \rightarrow \mathbb{R}$ and random variables X, Y , if the following sums / integrals are finite then,

$$\mathbb{E}(g(X, Y)) = \begin{cases} \sum_{\forall x} \sum_{\forall y} g(x, y) \mathbb{P}(X = x, Y = y) & \text{discrete ,} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy. & \end{cases}$$

Marginal Probability Density This is the name of the individual density functions of a joint density function. That is, f_X, f_Y are the marginal densities of $f_{X,Y}$.

The marginal densities are given as

$$f_X(x) = \int_{\mathbb{R}} f_{x,y} dy..$$

Similarly, for the discrete case,

$$\mathbb{P}(X = x) = \sum_{\forall y} \mathbb{P}(X = x, Y = y)..$$

Independence Two random discrete variables X, Y are independent if

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \mathbb{P}(Y = y), \forall x, y.$$

Similarly, for the continuous case, with a joint probability of $f_{X,Y}$,

$$f_{X,Y} = f_X(x) f_Y(y), \forall x, y.$$

Independent Product of Expectation If X, Y are independent for bounded functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ then

$$\mathbb{E}(g(X) f(Y)) = \mathbb{E}(g(X)) \mathbb{E}(f(Y)).$$

Conditional Probability Conditional probability for functions extends the standard case of sets. That is, in the discrete case,

$$\mathbb{P}(X = x | Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}..$$

Similarly for the continuous case,

$$f_{x|y} := \frac{f_{x,y}(x, y)}{f_y(y)}.$$

Multivariate Gaussian A random vector $X = (X_1, X_2)$ is said to be Gaussian with $\mu_X = (\mu_{x_1}, \mu_{x_2})$ and covariance matrix V if

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^d |V|}} \exp \left(-\frac{1}{2} (X - \mu_X)^T V^{-1} (X - \mu_X) \right).$$

That is

$$f_X(x).$$

$d = 2$ (dimensions), V^{-1} is the matrix inverse of V and $|V|$ is the determinant of V .

Variance Matrix The variance matrix is a symmetric matrix with entries

$$V_{ij} = \text{Cov}(X_i, X_j), \text{ where } i, j \in 1, \dots, d.$$

If $X = (X_1, X_2)$ is multivariate Gaussian then (X_i) for $i = 1, 2$ must be one-dimensional Gaussian but, the converse is not true.

Conditional Variance and Expection Given any bound Borel function g , the conditional expection of $g(x)$ given the set $\{Y = y\}$ is

$$\mathbb{E}(g(x)|Y = y) = \begin{cases} \sum_x g(x)\mathbb{P}(X = x|Y = y) & \text{discrete,} \\ \int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)dx & \text{continuous.} \end{cases}.$$

Conditional variance follows such that when given $Y = y$,

$$\text{Var}(X|Y = y) = \mathbb{E}(X^2|Y = y) - (\mathbb{E}(X|Y = y))^2.$$

Independent Conditional Variance and Expection Let X, Y be independent random variables. Then,

$$\mathbb{E}(X|Y = y) = \mathbb{E}(X) \text{ and } \text{Var}(X|Y = y) = \text{Var}(X).$$

Covariance Let X, Y be random variables. Then,

$$\text{Cov}(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))).$$

Properties of Covariance For random variables X, Y ,

1. $\text{Cov}(X, X) = \text{Var}(X)$.
2. $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.
3. $\text{Cov}(X, Y) = 0$ if X, Y are independent. However a zero covariance does not imply independence.
4. The covariance is bilinear such that $\forall a, b \in \mathbb{R}$, and random variables X, Y, Z ,

$$\text{Cov}(aX + bY, Z) = a\text{Cov}(X, Z) + b\text{Cov}(Y, Z).$$

The same principle holds for $\text{Cov}(X, aY + bZ)$.

Correlation The correlation may be thought of as a measure of linear independence. Note that though independence implies variables are uncorrelated, uncorrelated variables are not necessarily independent.

The correlation can be thought of as a normalisation of covariance. That is,

$$\text{Corr}(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}.$$

Properties of Correlation Let X, Y be two random variables.

1. $|\text{Corr}(X, Y)| \leq 1$
2. $\text{Corr}(X, Y) = 1$ if and only if there exists an $a, b \in \mathbb{R}$ where $b < 0$ such that $\mathbb{P}(Y = a + bX) = 1$.
3. $\text{Corr}(X, Y) = -1$ if and only if there exists an $a, b \in \mathbb{R}$ where $b > 0$ such that $\mathbb{P}(Y = a + bX) = 1$.

Transformations - Continuous Random Variables A real valued function h over $A \subset \mathbb{R}$ is said to be monotone on A if h is strictly increasing or decreasing over A .

Note that h is invertible.

Calculating PDF of Monotone Transformations Suppose that X is a random variable with density f_x . If h is monotone over $\{x : f_X(x) > 0\}$ then, the probability density of $Y := h(X)$ is given by

$$f_Y(y) = f_X(x) \left| \frac{dy}{dx} \right| = f_X \circ h^{-1}(y) \left| \frac{dh^{-1}(y)}{dy} \right|.$$

4.1 Bivariate Transformations

Suppose that X, Y are random variables over \mathbb{R} with transforms U, V respectively. Then, the joint probability density is given by

$$f_{U,V} = f_{X,Y}(x, y) |\det(J)|.$$

Observe that $X = U^{-1}, Y = V^{-1}$. Recall from multivariate calculus that J is the Jacobian Matrix such that

$$J = \begin{pmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{pmatrix}.$$

5 Sums of Variables

Convolution Formula Suppose X, Y are independent, continuous r.v. with density f_X, f_Y . Let $Z = X + Y$. Then,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) dy.$$

In the discrete case,

$$\mathbb{P}(X + Y = z) = \sum_y \mathbb{P}(X = z - y) \mathbb{P}(Y = y).$$

Convolutions and Exponential / Gamma The method for convolutions can be use to show that if $X \sim \Gamma(\alpha, 1)$ and, $Y \sim \Gamma(\beta, 1)$, where X, Y are independent, then,

$$Z := X + Y \sim \Gamma(\alpha + \beta, 1)..$$

Observe $0 < y < z < \infty$.

Convolutions and Moment Generating Functions Suppose that X, Y are random variables with moment generating functions. Then,

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

Generally, if $\{X_i\}_{i=1}^n$ is an independent sequence of random variables then

$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t).$$

Useful Results about Sums Let $(X_i)_{i=1, \dots, n}$ be an independent sequence of random variables and set $Y := \sum_{i=1}^n X_i$. Then,

$$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2) \implies Y \sim \mathcal{N}\left(\sum \mu_i, \sum \sigma_i^2\right)$$

$$X_i \sim \exp(\lambda) \implies Y \sim \Gamma(n, \lambda)$$

$$X_i \sim \Gamma(1, \lambda) \implies Y \sim \Gamma(n, \lambda)$$

$$X_i \sim \Gamma(\alpha_i, \beta) \implies Y \sim \Gamma\left(\sum \alpha_i, \beta\right)$$

$$X_i \sim \text{Poisson}(\lambda_i) \implies Y \sim \text{Poisson}\left(\sum \lambda_i\right)$$

$$X_i \sim \text{Bernoulli}(p_i) \implies Y \sim \text{Bin}(n, p)$$

$$X_i \sim \text{Bin}(n_i, p) \implies Y \sim \text{Bin}\left(\sum n_i, p\right) ..$$

6 Central Limit Theorem and Convergence of Random Variables

6.1 Central Limit Theorem

Definition: Central Limit Theorem Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables with common mean $\mu = \mathbb{E}(X_1)$ and variance $\sigma^2 = \text{Var}(X_1) < \infty$. Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ then,

$$\frac{\bar{x}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1).$$

Normal Approximation to the Binomial Distribution This is really just an extension of the central limit theorem but for binomial distributions. Suppose $X \sim \text{Bin}(n, p)$. Then,

$$\frac{X - np}{\sqrt{np(1-p)}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1).$$

6.2 Convergence of Random Variables

Convergence of Random Variables Let $(X_i)_{i \in \mathbb{N}}$ be a random sequence of random variables. X_n converges to X in terms of distribution if, for all x where $F_X(x)$ is continuous,

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x).$$

This is denoted as $X_N \xrightarrow{d} X$.

Convergence of Moment Generating Functions and Existence of CDF Suppose $(X_n)_{n \in \mathbb{N}}$ is a sequence of random variables with moment generating function $M_{X_n}(t)$. Suppose

$$M(t) = \lim_{n \rightarrow \infty} M_{X_n}(t)$$

exists. Then, there exists a unique valid cumulative distribution function F and random variable X such that $F_X = F$.

The moments are uniquely determined by M and for all points of continuity,

$$\lim_{n \rightarrow \infty} F_{X_n}(X)S = F(X) + F_X(x).$$

Convergence: Epsilon Definition for Probability Consider a sequence of variable $(X_n)_{n=1, \dots}$, converges to a random variable X if for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| > \epsilon) = 0.$$

Law of Large Numbers (Strong Version) Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables with mean μ , and finite variable σ^A . Set $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Then

$$\bar{X}_n \xrightarrow{a.s.} \mu.$$

(Weak Version): Same Thing but using *almost surely* probability for convergence.

Almost Surely Convergence Two random variables converge *almost surely* if for random variables X, Y $\mathbb{P}(Y = X) = 1$. The same can be said for sequences, written as $X \xrightarrow{a.s.} X$.

Slutsky's Theorem Suppose that $(X_n)_{n \in \mathbb{N}}$ is a sequence of independent random variables that converge to X in distribution and $(Y_i)_{i \in \mathbb{N}}$ is a sequence of independent random variables that converge in probability to a constant C .

Delta Method Recall that the central limit theorem informs us that sequences of random variables converge to a normal distribution. The delta-method is concerned with the square of this random variable.

Suppose

$$\frac{X_n - \theta}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1)$$

and g is a differentiable function in the neighbourhood of θ and $g'(\theta) \neq 0$. Then,

$$\sqrt{n} \left(g(X_n) - g(\theta) \xrightarrow{d} \mathcal{N}(0, \theta^2 [g'(\theta)]^2) \right).$$

Equivalently (and better),

$$\frac{g(X_n) - g(\theta)}{\sigma g'(\theta) / \sqrt{n}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1).$$

You should be thinking of this as

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\text{Var}(X_n)}}.$$

7 Estimators

7.1 Data and Models

Samples and Data A random sample is a collection of (random) observations (X_1, \dots, X_n) . The sample data is x_1, \dots, x_n . For the sake of laziness, this may be represented as $X[1, n]$ and $x[1, n]$ respectively. This is not standard notation.

Parametric Model A parametric model for a random sample $X[1, n]$ is a family of probability density functions $f(x; \theta)$ for $\theta \in \Theta$ where $\Theta \in \mathbb{R}^d$ is called the parameter space.

Data being modelled by a parametric family $f(x; \theta) : \theta \in \Theta$ can be written as

$$X[1, n] \sim \{f(x; \theta) : \theta \in \Theta\}, \text{ for } x \in \mathbb{R}.$$

7.2 Estimators

Estimators Suppose that $X[1, n] \sim \{f(x; \theta) : \theta \in \Theta\}$. An estimator of θ is denoted by $(\hat{\theta})_n$. That is,

$$\hat{\theta}_n = \hat{\theta}_n(X[1, n]) = g(X[1, n]),$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}^d$.

Estimators of Mean of Normal Distribution Suppose that $X[1, n]$ are independent samples from a normal distribution $\mathcal{N}(\mu, \sigma^2)$.

Then, there are a few methods of calculating the mean:

- Average: $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n X_i$
- Median: $\hat{\mu}_n = X_{n/2}$
- $\hat{\mu}_n = \frac{X_1 + X_n}{2}$.

By the law of large numbers, the first estimator approaches μ and is the best estimator.

Bias The bias is a way to assess how good an estimator is. Is calculated as

$$\text{Bias}(\hat{\theta}) = \mathbb{E}(\theta) - \hat{\theta}.$$

If $\text{Bias}(\hat{\theta}) = 0$ then, θ is an unbiased estimator.

Variance Estimators From the definition of variance, we may infer that if $\hat{\mu}_n = \overline{X}_n$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \hat{\mu})^2.$$

As such $\text{Bias}(\hat{\sigma}^2) = \mathbb{E}(\text{sigma}^2) - \sigma^2$. Equivalently,

$$\text{Bias}(\hat{\sigma}^2) = \sigma^2 \left(1 - \frac{1}{n}\right) - \sigma^2 = \text{sigma}^2 \left(-\frac{1}{n}\right).$$

On average, sigma^2 will underestimate the variance but, the becomes unbiased for large n .

Student t -Distribution A random variable T has a t -distribution with a degree of freedom v if it has a probability density function of

$$f_T(x) = \frac{\Gamma(\nu/2)}{\Gamma(\nu/2)\Gamma(1/2)} v^{-\frac{1}{2}} \left(1 + \frac{x^2}{\nu}\right)^{(1-\nu)/2}.$$

This looks like a normal distribution. In fact, as $\nu \rightarrow \infty$, $T_\nu \rightarrow Z \sim \mathcal{N}(0, 1)$. The t -distribution has a fatter tail.

Normal and t -Distribution Let Y, Z be independent random variables with $Y \sim \mathcal{N}(0, 1)$ and $Z \sim \chi_\nu^2$.

Recall that χ_n^2 is equivalent to the distribution $\Gamma(n, \frac{1}{2})$.

7.3 Errors

Standard Error The standard error of a estimator $\hat{\sigma}$ is

$$\text{Se}(\hat{\theta}) = \frac{1}{\sqrt{n}} \sqrt{\text{Var}(\hat{\theta})}.$$

Similarly, the estimated standard error is

$$\text{Se}(\hat{\theta}) = \frac{1}{\sqrt{n}} \left[\sqrt{\text{Var}(\hat{\theta})} \right]_{\theta=\hat{\theta}}.$$

Mean Square Error Define Mean Square Error (MSE) of an estimator to be

$$\text{MSE}(\hat{\theta}) := \mathbb{E} \left(\left(\hat{\theta} - \theta \right)^2 \right).$$

Equivalently,

$$\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta}) + [\text{Bias}(\hat{\theta})]^2.$$

This is estimated as

$$\text{MSE}(\hat{\theta}) = \text{Bias}(\hat{\theta})^2 + \text{Se}(\hat{\theta})^2.$$

Variance Bias Tradeoff For a fixed MSE, there is a trade-off between variance and bias. In statistics, we care for reducing bias most of the time. However, in other applications such as machine learning, there is a need for low variance.

Comparing Estimators with MSE Suppose that $\hat{\theta}_1, \hat{\theta}_2$ are two estimators of θ . Then, $\hat{\theta}_1$ is better than $\hat{\theta}_2$ at θ_0 if

$$\text{MSE}_{\theta_0}(\hat{\theta}_1) < \text{MSE}(\hat{\theta}_2).$$

A estimator $\hat{\theta}_1$ is uniformly better than $\hat{\theta}_2$ if, $\hat{\theta}_1$ is better for all $\theta \in \Theta$.

Asymptotic Properties of the Estimator An estimator $\hat{\theta}$ is a consistent estimator of θ if as $n \rightarrow \infty$,

$$\hat{\theta} \xrightarrow{\mathbb{P}} \theta.$$

By the weak law of large numbers \overline{X} is a consistent estimator of μ .