Distorting lower probabilities using common distortion models

David Nieto-Barba
nietodavid@uniovi.es

Ignacio Montes
imontes@uniovi.es

Enrique Miranda
mirandaenrique@uniovi.es

UNIVERSIDAD OVIEDO

14th International Symposium on Imprecise Probabilities: Theories and Applications - ISIPTA'25
Bielefeld, Germany

Lower previsions and probabilities

Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathbb{P}(\mathcal{X})$ be the set of probabilities over $\mathcal{P}(\mathcal{X})$.

Definition (Lower previsions and probabilities)

*A **lower prevision** is a real-valued functional on $\mathcal{K} \subseteq \mathcal{L}(\mathcal{X}) := \{f : \mathcal{X} \to \mathbb{R}\}$ i.e. $\underline{P} : \mathcal{K} \to \mathbb{R}$. If $\mathcal{K} = \{I_A \mid A \subseteq \mathcal{X}\}$, $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$ is called **lower probability**.*

Definition (Credal set)

*The **credal set** of a lower prevision $\underline{P}$ on $\mathcal{K}$ is given by:*

$$\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}(\mathcal{X}) \mid P(f) \geq \underline{P}(f) \quad \forall f \in \mathcal{K}\}.$$

Definition (ASL, coherent and k-monotone lower previsions)

1. *$\underline{P}$ **avoids sure loss** if $\mathcal{M}(\underline{P}) \neq \emptyset$.*
2. *is **coherent** if $\underline{P}$ is the lower envelope of $\mathcal{M}(\underline{P})$.*
3. *is **k-monotone** if $\underline{P}\left(\vee_{i=1}^{p} f_i \right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,p\}} (-1)^{|I|+1} \underline{P}\left(\wedge_{i \in I}^{p} f_i \right)$.*

Classical distortion models

Definition (Classical neighbourhood model)

Let $P_0 \in \mathbb{P}(\mathcal{X})$, $\delta \geq 0$ *and* $d : \mathbb{P}(\mathcal{X}) \times \mathbb{P}(\mathcal{X}) \to [0, +\infty)$,

$$B_d^\delta(P_0) := \{P \in \mathbb{P}(\mathcal{X}) \mid d(P, P_0) \leq \delta\}.$$

Definition (Transformation of precise probabilities)

Let $\phi : [0,1] \to [0,1]$ *be non-decreasing and s.t.* $\phi \leq Id$,

$$\underline{P}_\phi(A) := \phi(P_0(A)) \quad \forall A \subset \mathcal{X}, \text{ and } \underline{P}(\mathcal{X}) := 1.$$

We will extrapolate families of common classical transformating functions to initial lower probabilities; specifically, **Vertical Barrier Models**.

Procedure and extrapolation of common models

Definition (Transformation procedure of lower probabilities)

Let $\{\phi_\lambda : [0,1] \to [0,1]\}_{\lambda\in\Lambda}$ *be a family of non-decreasing functions and s.t.* $\phi_\lambda \le Id$ *for every* $\lambda \in \Lambda$. *Given* $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$ *and* $\lambda \in \Lambda$, *we define* $\underline{Q}_\lambda[\underline{P}] : \mathcal{P}(\mathcal{X}) \to [0,1]$ *as:*

$$\underline{Q}_\lambda[\underline{P}](A) := (\phi_\lambda \circ \underline{P})(A) = \phi_\lambda\big(\underline{P}(A)\big) \quad \forall A \subset \mathcal{X},$$

and $\underline{Q}_\lambda[\underline{P}](\mathcal{X}) := 1$.

Definition

- **IVBM**: $\Lambda = \{(a,b) \,|\, a \le 0 \le b, a+b \le 1\}$, $\phi_{(a,b)}(A) = \max\{bt+a, 0\}$.
- **ITVM**: $\Lambda = [0,1)$, $\phi_\delta(t) = \max\{t-\delta, 0\}$.
- **ILVM**: $\Lambda = [0,1)$, $\phi_\delta(t) = \max\{(1-\delta)t, 0\}$.
- **IPMM**: $\Lambda = [0,+\infty)$, $\phi_\delta(t) = \max\{(1+\delta)t-\delta, 0\}$.

Desirable properties

Given $\{\underline{Q}_\lambda[\cdot]\}_{\lambda\in\Lambda}$ and $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$:

❶ (Expansion) $(\forall\lambda_1,\lambda_2 \in \Lambda)\ \lambda_1 \preceq \lambda_2 \Rightarrow \underline{Q}_{\lambda_2}[\underline{P}] \leq \underline{Q}_{\lambda_1}[\underline{P}]$.

❷ (Semigroup)

- ⓐ $(\exists\lambda_0 \in \Lambda)\ \underline{Q}_{\lambda_0}[\underline{P}] = \underline{P}$ and $\lambda_0 \preceq \lambda \quad \forall\lambda \in \Lambda$,
- ⓑ $(\forall\Lambda \ni \lambda_1,\lambda_2 \succeq \lambda_0$ s.t. $\lambda_1+\lambda_2 \in \Lambda)\ \underline{Q}_{\lambda_2+\lambda_1}[\underline{P}] = \underline{Q}_{\lambda_2}\left[\underline{Q}_{\lambda_1}[\underline{P}]\right]$.

❸ (Structure preservation) $(\forall\lambda \in \Lambda)\ \underline{P} \in \mathcal{H} \subseteq \underline{\mathbb{P}}(\mathcal{X}) \Rightarrow \underline{Q}_\lambda[\underline{P}] \in \mathcal{H}$.

Prop.	❶	❷	❸
IVBM	N.A.	N.A.	✓^{*1}
ITVM	✓✓	✓✓	✓^{*1}
ILVM	✓✓	✓✓ⓐ | ✗ⓑ	✓✓
IPMM	✓✓	✓✓ⓐ | ✗ⓑ	✓$^{*1/2}$

- ^{*1}: ASL, coherent, 2-monotone (✓✓); k-monotone $(k \geq 3)$ (✗).
- $^{*1/2}$: idem (✓✓); k-monotone $(k \geq 3)$ (✓ if $\underline{Q}_\lambda[\underline{P}] \in \underline{\mathbb{P}}^*(\mathcal{X})$).

Desirable properties

❹ (Reversibility)
$(\forall \lambda \in \Lambda)$, $\exists \varphi_\lambda : [0,1] \to [0,1]$ s.t. $\underline{P}(A) = \varphi_\lambda(\underline{Q}_\lambda[\underline{P}](A))\ \forall A \neq \emptyset, \mathcal{X}$.

❺ (Invariance) under permutations, ❻ marginalisations, ❼ conditioning.

$$\begin{tikzcd}
\underline{P} \arrow[r, "(\cdot)^\sigma"] \arrow[d, "\underline{Q}_\lambda[\cdot]"'] & \underline{P}^\sigma \arrow[d, "\underline{Q}_\lambda[\cdot]"'] \\
\underline{Q}_\lambda[\underline{P}] \arrow[r, "(\cdot)^\sigma"] & \underline{Q}_\lambda[\underline{P}^\sigma]
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P} \arrow[r, "(\cdot)^\Pi"] \arrow[d, "\underline{Q}_\lambda[\cdot]"'] & \underline{P}^\Pi \arrow[d, "\underline{Q}_\lambda[\cdot]"'] \\
\underline{Q}_\lambda[\underline{P}] \arrow[r, "(\cdot)^\Pi"] & \underline{Q}_\lambda[\underline{P}^\Pi]
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P} \arrow[r, "(\cdot)_B"] \arrow[d, "\underline{Q}_\lambda[\cdot]"'] & \underline{P}_B \arrow[d, "\underline{Q}_{\lambda^*}[\cdot]"'] \\
\underline{Q}_\lambda[\underline{P}] \arrow[r, "(\cdot)_B"] & \underline{Q}_{\lambda^*}[\underline{P}_B]
\end{tikzcd}$$

Prop.	❹	❺	❻	❼
IVBM	✓^{*_2}	✓✓	✓✓	✗
ITVM	✓^{*_2}	✓✓	✓✓	✗
ILVM	✓✓	✓✓	✓✓	✗
IPMM	✓^{*_2}	✓✓	✓✓	✗

- ^{*_2}: $\min_{x \in \mathcal{X}} \underline{P}(\{x\}) \geq {-a}/{b}$.

Credal set properties

❽ (Generalised neighbourhood model) $\exists d : \mathbb{P}(\mathcal{X}) \times \underline{\mathbb{P}}(\mathcal{X}) \to \mathbb{R}$ such that:

$$(\forall \lambda \in \Lambda)\ \mathcal{M}\big(\underline{Q}_\lambda[\underline{P}]\big) = \{Q \in \mathbb{P}(\mathcal{X}) \mid d(Q, \underline{P}) \leq \mu(\lambda)\} =: B_d^{\mu(\lambda)}(\underline{P}).$$

❾ (Weak extreme point commutativity) If $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$ is coherent:

$$(\forall \lambda \in \Lambda) \quad \underline{Q}_\lambda[\underline{P}](A) = \inf \left\{ Q(A) \mid Q \in \bigcup_{P \in \mathrm{ext}(\mathcal{M}(\underline{P}))} \mathcal{M}\big(\underline{Q}_\lambda[P]\big) \right\}.$$

❿ (Strong commutativity) If $\underline{P} \in \underline{\mathbb{P}}(\mathcal{X})$ is coherent:

$$(\forall \lambda \in \Lambda) \quad \mathcal{M}\big(\underline{Q}_\lambda[\underline{P}]\big) = \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}\big(\underline{Q}_\lambda[P]\big).$$

Credal set properties

$$\mathcal{M}\left(\underline{Q}_{(a,b)}[\underline{P}]\right) = B^1_{d_{\mathrm{IVBM}}}(\underline{P}) \;\leftrightsquigarrow\; d_{\mathrm{IVBM}}(Q,\underline{P}) := \max_{A\subseteq\mathcal{X}} \frac{\underline{P}(A) - Q(A)}{(1-b)\underline{P}(A) - a};$$

$$\mathcal{M}\left(\underline{Q}^{\mathrm{TV}}_{\delta}[\underline{P}]\right) = B^{\delta}_{d_{\mathrm{ITVM}}}(\underline{P}) \;\leftrightsquigarrow\; d_{\mathrm{ITVM}}(Q,\underline{P}) = \max_{A\subseteq\mathcal{X}}(\underline{P}(A) - Q(A));$$

$$\mathcal{M}\left(\underline{Q}^{\mathrm{LV}}_{\delta}[\underline{P}]\right) = B^{\delta}_{d_{\mathrm{ILVM}}}(\underline{P}) \;\leftrightsquigarrow\; d_{\mathrm{ILVM}}(Q,\underline{P}) = \max_{A\mid\underline{P}(A)>0} \frac{\underline{P}(A) - Q(A)}{\underline{P}(A)};$$

$$\mathcal{M}\left(\underline{Q}^{\mathrm{PMM}}_{\delta}[\underline{P}]\right) = B^{\delta}_{d_{\mathrm{IPMM}}}(\underline{P}) \;\overset{*3}{\leftrightsquigarrow}\; d_{\mathrm{IPMM}}(Q,\underline{P}) = \max_{A\subset\mathcal{X}} \frac{\underline{P}(A) - Q(A)}{1-\underline{P}(A)}.$$

Prop.	❽	❾	❿
IVBM	✓✓	✓✓	✓^{*4}
ITVM	✓✓	✓✓	✓^{*4}
ILVM	✓✓	✓✓	✓^{*4}
IPMM	✓^{*3}	✓✓	✓^{*4}

- ^{*3}: if $\underline{P}(A) > 0\ \forall A \neq \emptyset$.
- ^{*4}: if $\underline{P}$ is 2-monotone.

Conclusions and future lines

Conclusions:

- The IVBM satisfies most of the desirable properties, with additional properties for particular submodels.
- Invariance under conditioning does not hold for imprecise distortions, unlike precise ones.

Future lines:

- Extension to the comparison of $\underline{P}, \underline{Q}$.
- Applications to game theory.
- Distortions of lower previsions and sets of desirable gambles.

References

Bronevich, A.G.: On the clousure of families of fuzzy measures under eventwise aggregations. Fuzzy Sets and Systems **153**, 45–70 (2005)

Herron, T., Seidenfeld, T., Wasserman, L.: Divisive conditioning: further results on dilation. Philosophy of Science **64**, 411–444 (1997)

Miranda, E., Pelessoni, R., Vicig, P.: Evaluating uncertainty with vertical barrier models. International Journal of Approximate Reasoning **167**, 109132 (2024)

Montes, I., Miranda, E., Destercke, S.: Unifying neighbourhood and distortion models: Part I- New results on old models. International Journal of General Systems **49**(6), 602–635 (2020)

Pelessoni, R., Vicig, P.: Dilation properties of coherent Nearly-Linear models. International Journal of Approximate Reasoning **140**, 211–231 (2022)

Funding

Proyect **PID2022-140585NB-I00** funded by
MICIU/AEI/10.13039/501100011033 y por FEDER, UE

Distorting lower probabilities using common distortion models

David Nieto-Barba
nietodavid@uniovi.es

Ignacio Montes
imontes@uniovi.es

Enrique Miranda
mirandaenrique@uniovi.es

14th International Symposium on Imprecise Probabilities: Theories and Applications - ISIPTA'25
Bielefeld, Germany