On the closure of aggregation rules for imprecise probabilities

Enrique Miranda mirandaenrique@uniovi.es
Ignacio Montes imontes@uniovi.es

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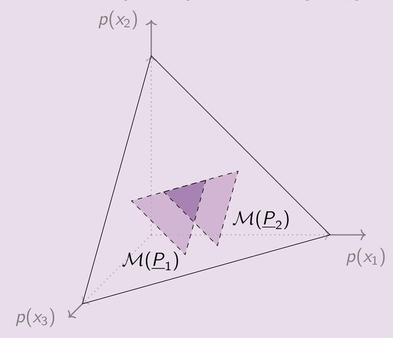
Aggregation of lower probabilities

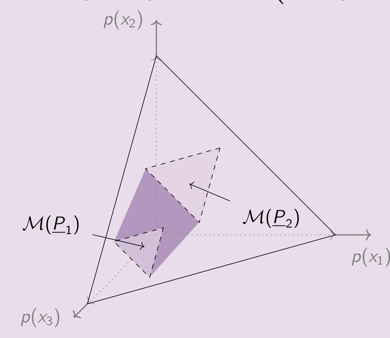
Conjunction ($\mathcal{A}_C$)

$\mathcal{A}_C$ is the natural extension of $\max\{\underline{P}_1, \underline{P}_2\}$
Equivalently: $\mathcal{M}(\mathcal{A}_C) = \mathcal{M}(\underline{P}_1) \cap \mathcal{M}(\underline{P}_2)$



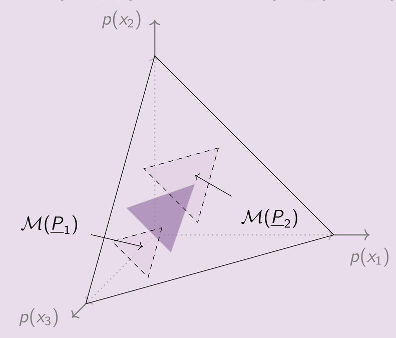
Disjunction ($\mathcal{A}_D$)

$\mathcal{A}_D = \min\{\underline{P}_1, \underline{P}_2\}$
Equivalently: $\mathcal{M}(\mathcal{A}_D) = \mathcal{CH}(\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2))$



Mixture ($\mathcal{A}_M$)

$\mathcal{A}_M = \alpha \underline{P}_1 + (1-\alpha)\underline{P}_2$
Equivalently: $\mathcal{M}(\mathcal{A}_M) = \alpha\mathcal{M}(\underline{P}_1) + (1-\alpha)\mathcal{M}(\underline{P}_2)$



Pareto rule ($\mathcal{A}_P$)

$\mathcal{A}_P(f) = \min\{\max_i \underline{P}_i(f), \min_i \overline{P}_i(f)\}$

Philosophy [4]: A gamble is desirable when it is desirable for at least one member of the group and it is not undesirable for any other member

If $\max_i \underline{P}$ avoids sure loss [2], then
$\mathcal{A}_P = \max_i \underline{P}_i$

Aggregation of particular models

Problem: Given $\underline{P}_1, \underline{P}_2$ two lower previsions in a family $\mathcal{H}$, does their aggregation $\mathcal{A}(\underline{P}_1, \underline{P}_2)$ belong to the same family $\mathcal{H}$?

	Conjunction	Disjunction	Mixture	Pareto
Comparative Probabilities $\underline{P}$ is a comparative probability if there exists $\mathcal{L} \subseteq \mathcal{X} \times \mathcal{X}$ such that $\underline{P}(I_{x_i} - I_{x_j}) \geq 0$ for $(x_i, x_j) \in \mathcal{L}$	✓ The conjunction is closed!! ✓ If $\underline{P}_1$ and $\underline{P}_2$ are determined by $\mathcal{L}_1$ and $\mathcal{L}_2$, their conjunction is determined by $\mathcal{L}_1 \cup \mathcal{L}_2$	✗ The disjunction is not closed ✓ The disjunction gives a comparative probability if and only if it is determined by $\mathcal{L}_1 \cap \mathcal{L}_2$ ✓ Otherwise, the comparative probability determined by $\mathcal{L}_1 \cap \mathcal{L}_2$ is the unique undominated outer approximation	✗ The mixture is not closed ✓ The mixture gives a comparative probability if and only if $\underline{P}_1 = \underline{P}_2$	✗ The Pareto rule is not closed ✓ The Pareto rule gives a comparative probability if and only if it is determined by $\mathcal{L}_1 \cup \mathcal{L}_2$
2-monotone capacities $\underline{P}$ is 2-monotone if $\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B)$	✗ The conjunction is not closed ✓ If $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is convex, then the conjunction coincides with $\max\{\underline{P}_1, \underline{P}_2\}$ and it is 2-monotone ✗ The previous condition is only sufficient, but not necessary	✗ The disjunction is not closed ✓ If $\underline{P}_1$ is 2-monotone and $\underline{P}_2$ is vacuous on a fixed event $C \subseteq \mathcal{X}$, then the disjunction is 2-monotone	✓ The mixture is closed!!	✗ The Pareto rule is not closed... ✗ ...not even when $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss
Probability intervals $\underline{P}$ is a probability interval if there exists a family of intervals $\{[l_i, u_i]\}_{i=1,\dots,n}$ such that: $\mathcal{M}(\underline{P}) = \{P \mid P(\{x_i\}) \in [l_i, u_i]\}$	✗ The conjunction is not closed ✓ If the probability intervals satisfy $\sum_{i=1}^{n} \max\{l_i^1, l_i^2\} \leq 1 \leq \sum_{i=1}^{n} \min\{u_i^1, u_i^2\}$, then the conjunction is closed and ([1]): $\underline{\mathcal{A}}_C(\{x_i\}) = \max\left\{ l_i^1, l_i^2, 1 - \sum_{j \neq i]} \min\{u_i^1, u_i^2\} \right\}$ $\overline{\mathcal{A}}_C(\{x_i\}) = \min\left\{ u_i^1, u_i^2, 1 - \sum_{j \neq i]} \max\{l_i^1, l_i^2\} \right\}$	✗ The disjunction is not closed ✓ There is a unique undominated outer approximation determined by: $[\min\{l_i, l_i'\}, \max\{u_i, u_i'\}]$	✗ The mixture is not closed ✓ The probability interval determined by: $[\alpha l_i + (1-\alpha) l_i', \alpha u_i + (1-\alpha) u_i']$ is an undominated outer approximation of the mixture	✗ The Pareto rule is not closed ✗ When $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss, it holds that $\underline{\mathcal{A}}_P(\{x_i\}) = \max\{l_i^1, l_i^2\}$ $\overline{\mathcal{A}}_P(\{x_i\}) = \min\{u_i^1, u_i^2\}$ but $\mathcal{A}_P(\underline{P}_1, \underline{P}_2)$ may not be a probability interval
Belief functions $\underline{P}$ is a belief function if $m(A) = \sum_{B \subseteq A} (-1)^{\vert A \setminus B \vert} \underline{P}(B) \geq 0$ for any $A \subseteq \mathcal{X}$. A if a focal event if $m(A) > 0$. $\mathcal{F}$ denotes the set of focal events	✗ The conjunction is not closed... ✗ ...not even when $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss	✗ The disjunction is not closed ✓ If $\vert\mathcal{X}\vert = 3$ and $\underline{P}_2$ is minitive, $\mathcal{A}_D$ gives a belief function ✓ If $\underline{P}_2$ is a degenerate probability, the restriction to events of $\mathcal{A}_D(\underline{P}_1, \underline{P}_2)$ is a belief function	✓ The mixture is closed!! ✓ $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$	✗ The Pareto rule is not closed... ✗ ...not even when $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss
P-boxes $\underline{P}$ is a p-box determined by $(\underline{F}, \overline{F})$ if $\mathcal{M}(\underline{P}) = \{P \mid \underline{F} \leq F_P \leq \overline{F}\}$ for two ordered cdfs $\underline{F} \leq \overline{F}$.	✗ The conjunction is not closed... ✓ ...but it is when $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss ✓ In that case, $\mathcal{A}_C(\underline{P}_1, \underline{P}_2)$ is determined by $(\max\{\underline{F}_1, \underline{F}_2\}, \min\{\overline{F}_1, \overline{F}_2\})$	✗ The disjunction is not closed ✓ The p-box determined by $(\min\{\underline{F}_1, \underline{F}_2\}, \max\{\overline{F}_1, \overline{F}_2\})$ is the unique undominated outer approximation	✗ The mixture is not closed ✓ $\mathcal{A}_M(\underline{P}_1, \underline{P}_2)$ is a p-box if and only if $\mathcal{F}_1 \cup \mathcal{F}_2$ is ordered with respect to interval dominance	✗ The Pareto rule is not closed... ✗ ...not even when $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss
Minitive measures $\underline{P}$ is a minitive measure if $\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\}$. Its conjugate is maxitive: $\overline{P}(A \cap B) = \max\{\overline{P}(A), \overline{P}(B)\}$.	✗ The conjunction is not closed ✓ Letting $\pi_i(x) = \overline{P}_i(\{x\})$, $\overline{P} = \max\{\overline{P}_1, \overline{P}_2\}$ is maxitive iff [3]: $\min_{i=1,2}(\max_{j=1,2} \pi_i(x_j)) = \max_{j=1,2}(\min_{i=1,2} \pi_i(x_j))$	✗ The disjunction is not closed ✓ $\mathcal{A}_D(\underline{P}_1, \underline{P}_2)$ is minitive if and only if $\mathcal{F}_1 \cap \mathcal{F}_2 \subseteq \mathcal{F}_D$	✗ The mixture is not closed ✓ $\mathcal{A}_M(\underline{P}_1, \underline{P}_2)$ is minitive if and only if $\mathcal{F}_1 \cup \mathcal{F}_2$ is ordered by set inclusion	✗ The Pareto rule is not closed... ✗ ...not even when $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss

Summary

At a glance...

- $\mathcal{A}_C$, $\mathcal{A}_D$ and $\mathcal{A}_M$ preserve coherence [2], but they are not always closed for particular subfamilies of coherent lower previsions
- We have established some sufficient conditions for the aggregated model to be in the same family as the sources
- When the aggregated model does not belong to the same family, we may (a) look for inner/outer approximations; or (b) propose rules that are tailor-made for that specific family
- In particular: what about the Dempster-rule of combination for belief functions?

References

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[3] Miranda, Troffaes, Destercke. "A geometric and game-theoretic study of the conjunction of possibility measures". InfSci 2015
[4] Walley. "The elicitation and aggregation of beliefs". 1982