A correspondence between methods for ranking elements of a poset and stochastic orderings

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Overview

Basics on posets

Ranking elements in a poset

Stochastic orderings

Ranking elements in a poset VS Stochastic oderings

Conclusions

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Posets

Partially Ordered SET

P:	a set
$\leq_P$:	reflexive, antisymmetric and transitive relation
$(P, \leq_P)$:	poset

Related notions

Strict relation:	$x <_P y$ iff $x \leq_P y$ and $x \neq y$
Incomparability:	$x \parallel y$ iff $x \not\leq_P y$ and $y \not\leq_P x$
Maximal element:	x is maximal if $\nexists y \neq x$ such that $x \leq_P y$
Sub-poset:	$P' \subseteq P$ determines the poset $(P', \leq_{P'})$

Graphical representation

Covering relation

Covering: x is covered by y if $x < y$ and there is no z such that $x < z < y$

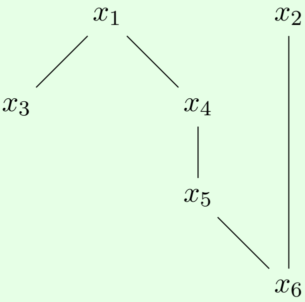
Notation: $x \lessdot y$

Hasse diagram

$$P = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

$$\lessdot = \{(x_3, x_1), (x_4, x_1), (x_5, x_4), (x_6, x_2), (x_6, x_5)\}$$

$$< = \{(x_3, x_1), (x_4, x_1), (x_5, x_4), (x_6, x_2), (x_6, x_5), (x_5, x_1), (x_6, x_1), (x_4, x_6)\}$$



Linear extension

Linear extension

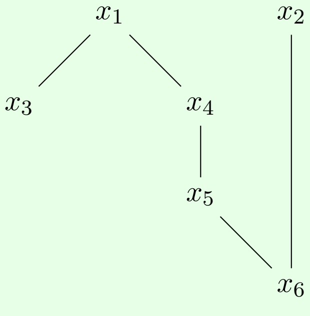
Extension:	$(P, \leq')$ such that $x \leq_P y$ implies $x \leq' y$
Linear extension:	an extension without incomparable elements
Notation:	$E_{\leq_P}$ denotes the set of linear extensions

Example

(Some) linear extensions

x_2	x_1	x_2
x_1	x_2	x_1
x_4	x_3	x_4
x_5	x_4	x_3
x_6	x_5	x_5
x_3	x_6	x_6

Poset



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Averaged rankings

Averaged ranking

$$\left.\begin{array}{l}(P, \leq_P) \\ e = (P, \leq_e) \in E_{\leq_P} \\ x \in P\end{array}\right\} \longrightarrow \begin{cases}\mathrm{Pos}_e(x) = |\{z \in P \mid x \leq_e z\}| \\ \mathrm{av}(x) = \frac{1}{|E_{\leq_P}|} \sum_{e \in E_{\leq_P}} \mathrm{Pos}_e(x)\end{cases}$$

Complete extension

Complete extension $(P, \precsim_{\mathrm{av}})$ given by:

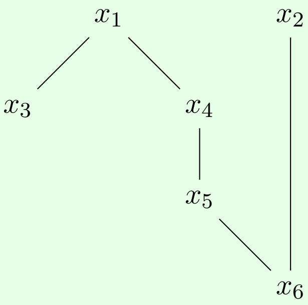
$$x \precsim_{\mathrm{av}} y \Leftrightarrow \mathrm{av}(x) \leq \mathrm{av}(y)$$

Averaged rankings: Example

$e \in E_{\leq_P}$

$$\begin{array}{c} x_2 \\ | \\ x_1 \\ | \\ x_4 \\ | \\ x_5 \\ | \\ x_6 \\ | \\ x_3 \end{array}$$

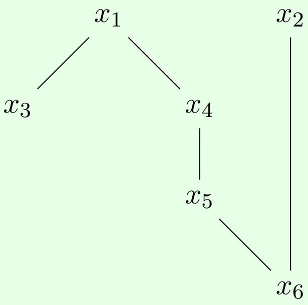
Poset



Averaged rankings: Example

	$e \in E_{\leq_P}$	$\mathrm{Pos}_e(x_i)$
1st	x_2	$\mathrm{Pos}_e(x_2) = 1$
2nd	x_1	$\mathrm{Pos}_e(x_1) = 2$
3rd	x_4	$\mathrm{Pos}_e(x_4) = 3$
4th	x_5	$\mathrm{Pos}_e(x_5) = 4$
5th	x_6	$\mathrm{Pos}_e(x_6) = 5$
6th	x_3	$\mathrm{Pos}_e(x_3) = 6$

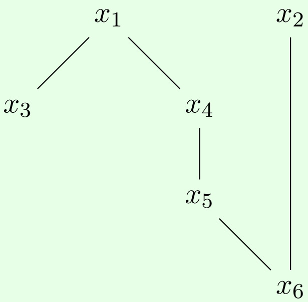
Poset



Averaged rankings: Example

Pos	1st	2nd	3rd	4th	5th	6th	av(x_i)
x_1	15	4	0	0	0	0	1.21
x_2	4	4	4	4	3	0	2.89
x_3	0	3	4	4	4	4	5.11
x_4	0	8	8	3	0	0	3.74
x_5	0	0	3	8	8	0	5.26
x_6	0	0	0	0	4	15	6.79

Poset



Averaged rankings: Example

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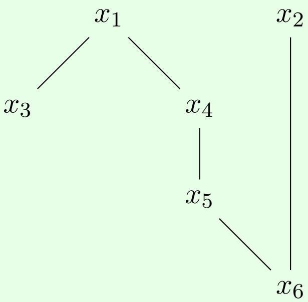


Complete extension

$$x_6 \prec_{\text{av}} x_5 \prec_{\text{av}} x_3 \prec_{\text{av}} x_4 \prec_{\text{av}} x_2 \prec_{\text{av}} x_1$$

Mutual rank probabilities

Mutual rank probability

$$\left.\begin{array}{l}(P,\leq_P)\\ x,y\in P\end{array}\right\} \quad \longrightarrow \quad p_{y<x}=\frac{|\{e\in E_{\leq_P}|y\leq_e x\}|}{|E_{\leq_P}|}$$

Complete extension

Complete extension $(P,\precsim_{\mathrm{mrp}})$ given by:

$$x\precsim_{\mathrm{mrp}} y \Longleftrightarrow p_{y<x}\geq \frac{1}{2}$$

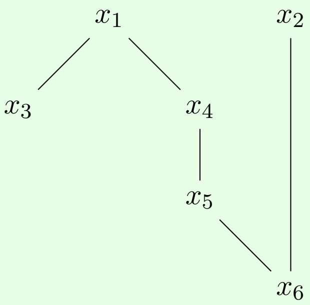
$\precsim_{\mathrm{mrp}}$ may not be transitive... take its transitive clousure $\precsim'_{\mathrm{mrp}}$.

Mutual rank probabilities: Example

Mutual rank probabilities

$p_{y<x}$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	$\frac{15}{19}$	1	1	1	1
x_2	$\frac{4}{19}$	0	$\frac{13}{19}$	$\frac{9}{19}$	$\frac{14}{19}$	1
x_3	0	$\frac{6}{19}$	0	$\frac{5}{19}$	$\frac{10}{19}$	$\frac{15}{19}$
x_4	0	$\frac{10}{19}$	$\frac{14}{19}$	0	1	1
x_5	0	$\frac{5}{19}$	$\frac{9}{19}$	0	0	1
x_6	0	0	$\frac{4}{19}$	0	0	0

Poset



Complete extension

$$x_6 \prec_{\mathrm{mrp}} x_5 \prec_{\mathrm{mrp}} x_3 \prec_{\mathrm{mrp}} x_2 \prec_{\mathrm{mrp}} x_4 \prec_{\mathrm{mrp}} x_1$$

Maximal method

Maximal method

$M_1 = \{x \in P \mid x \text{ maximal in } (P, \leq_P)\} \to P_1 = P \setminus M_1$
$M_2 = \{x \in P \mid x \text{ maximal in } (P_1, \leq_{P_1})\} \to P_2 = P \setminus (M_1 \cup M_2)$
$\dots$
$M_1, M_2, \dots, M_k$ partition of P

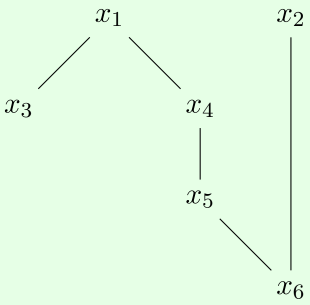
Complete extension

Complete extension $(P, \precsim_{\max})$ given by:

$$x \precsim_{\max} y \iff x \in M_i,\ y \in M_j \text{ with } i \leq j$$

Maximal method: Example

Poset

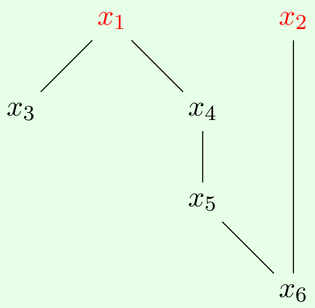


Maximal method: Example

Partition

$M_1 = \{x_1, x_2\}$

Poset



Maximal method: Example

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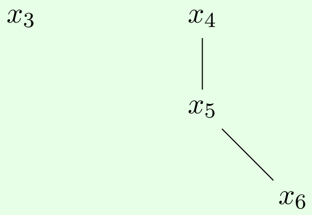


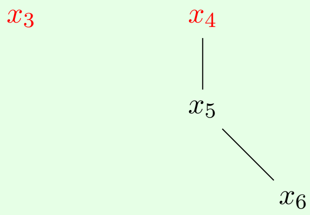
Maximal method: Example

Partition

$M_1 = \{x_1, x_2\}$

$M_2 = \{x_3, x_4\}$

Poset



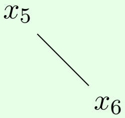
Maximal method: Example

Partition

$M_1 = \{x_1, x_2\}$

$M_2 = \{x_3, x_4\}$

Poset



Maximal method: Example

Partition

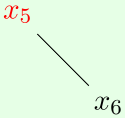
$M_1 = \{x_1, x_2\}$

$M_2 = \{x_3, x_4\}$

$M_3 = \{x_5\}$

$M_4 = \{x_6\}$

Poset



Maximal method: Example

Partition

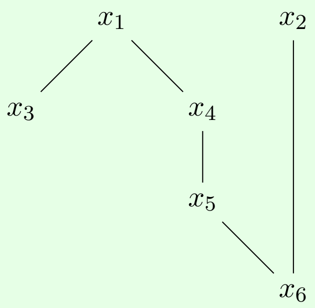
$M_1 = \{x_1, x_2\}$

$M_2 = \{x_3, x_4\}$

$M_3 = \{x_5\}$

$M_4 = \{x_6\}$

Poset



Complete extension

$$x_6 \prec_{\mathrm{mm}} x_5 \prec_{\mathrm{mm}} x_4 \sim_{\mathrm{mm}} x_3 \prec_{\mathrm{mm}} x_2 \sim_{\mathrm{mm}} x_1$$

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Expected value

Expected value

Let $\mathcal{A}$ be a set of random variables. Given $\mathbf{x}, \mathbf{y} \in \mathcal{A}$, $\mathbf{x}$ is said to be preferred to $\mathbf{y}$ with respect to expected value, denoted by $\mathbf{x} \succsim_{\text{EV}} \mathbf{y}$, if $E(\mathbf{x}) \geq E(\mathbf{y})$.

Example

$\mathcal{A} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ set of discrete, uniform and independent r.v. with supports:

$$D_1 = \{1, 3, 4, 15, 16, 17\} \longrightarrow E(\mathbf{x}_1) = {}^{28}\!/_3$$

$$D_2 = \{2, 10, 11, 12, 13, 14\} \longrightarrow E(\mathbf{x}_2) = {}^{31}\!/_3$$

$$D_3 = \{5, 6, 7, 8, 9, 18\} \longrightarrow E(\mathbf{x}_3) = {}^{53}\!/_6$$

$$\mathbf{x}_2 \succ_{\text{EV}} \mathbf{x}_1 \succ_{\text{EV}} \mathbf{x}_3$$

Statistical preference

Statistical preference

Let $\mathcal{A}$ be a set of random variables. Given $\mathbf{x}, \mathbf{y} \in \mathcal{A}$, the winning probability of $\mathbf{x}$ over $\mathbf{y}$ is given by:

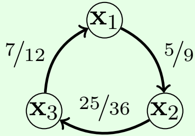
$$Q(\mathbf{x}, \mathbf{y}) = \mathsf{Prob}(\mathbf{x} > \mathbf{y}) + \frac{1}{2}\mathsf{Prob}(\mathbf{x} = \mathbf{y}).$$

$\mathbf{x}$ is said to be statistically preferred to $\mathbf{y}$, denoted by $\mathbf{x} \succsim_{\mathrm{SP}} \mathbf{y}$, if $Q(\mathbf{x}, \mathbf{y}) \geq \frac{1}{2}$.

Example

$\mathcal{A} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ set of discrete, uniform and independent r.v. with supports:

$$D_1 = \{1, 3, 4, 15, 16, 17\}$$
$$D_2 = \{2, 10, 11, 12, 13, 14\}$$
$$D_3 = \{5, 6, 7, 8, 9, 18\}$$



Probabilistic preference

Probabilistic preference

Let $\mathcal{A}$ be a set of random variables. Given $\mathbf{x} \in \mathcal{A}$, its multivariate winning probability is given by:

$$\Pi_{\mathcal{A}}(\mathbf{x}) = \sum_{\mathcal{Y} \subseteq \mathcal{A} \setminus \{\mathbf{x}\}} \frac{1}{1+|\mathcal{Y}|} \mathrm{Prob}\left((\forall \mathbf{z} \in \mathcal{Y}) (\forall \mathbf{w} \in \mathcal{A} \setminus (\{\mathbf{x}\} \cup \mathcal{Y})) (\mathbf{x} = \mathbf{z} > \mathbf{w}) \right).$$

If $P(\mathbf{x} = \mathbf{y}) = 0$ for any $\mathbf{x}, \mathbf{y} \in \mathcal{A}$: $\Pi_{\mathcal{A}}(\mathbf{x}) = \mathrm{Prob}\big(\mathbf{x} > \max_{\mathbf{y} \neq \mathbf{x}} \mathbf{y}\big)$.

Iterative procedure

$\mathcal{A}_1 = \{\mathbf{x} \in \mathcal{A} \mid \Pi_{\mathcal{A}}(\mathbf{x}) > 0\}$
$\mathcal{A}_2 = \{\mathbf{x} \in \mathcal{A} \setminus \mathcal{A}_1 \mid \Pi_{\mathcal{A} \setminus \mathcal{A}_1}(\mathbf{x}) > 0\}$
...
$\mathcal{A}_{j+1} = \{\mathbf{x} \in \mathcal{A} \setminus \{\mathcal{A}_1 \cup \mathcal{A}_j\} \mid \Pi_{\mathcal{A} \setminus \{\mathcal{A}_1 \cup \mathcal{A}_j\}}(\mathbf{x}) > 0\}$

Probabilistic preference

Probabilistic preference

Given $\mathbf{x} \in \mathcal{A}_i$ and $\mathbf{y} \in \mathcal{A}_j$, $\mathbf{x}$ is said to be probabilistically preferred to $\mathbf{y}$, denoted by $\mathbf{x} \succsim_{\mathrm{PP}} \mathbf{y}$, if $i < j$ or $i = j$ and

$$\Pi_{\mathcal{A} \setminus \{\mathcal{A}_1 \cup \mathcal{A}_j\}}(\mathbf{x}) \geq \Pi_{\mathcal{A} \setminus \{\mathcal{A}_1 \cup \mathcal{A}_j\}}(\mathbf{y})$$

Example

$\mathcal{A} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ set of discrete, uniform and independent r.v. with supports:

$$D_1 = \{1, 3, 4, 15, 16, 17\} \longrightarrow \Pi_{\mathcal{A}}(\mathbf{x}_1) = 0.4167$$

$$D_2 = \{2, 10, 11, 12, 13, 14\} \longrightarrow \Pi_{\mathcal{A}}(\mathbf{x}_2) = 0.3472$$

$$D_3 = \{5, 6, 7, 8, 9, 18\} \longrightarrow \Pi_{\mathcal{A}}(\mathbf{x}_3) = 0.2361$$

$$\mathbf{x}_1 \succ_{\mathrm{PP}} \mathbf{x}_2 \succ_{\mathrm{PP}} \mathbf{x}_3$$

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Setting

Posets		**Random variables**
$(P, \leq_P)$	$\longrightarrow$	$(E_{\leq_P}, \mathcal{P}(E_{\leq_P}), \mathcal{U})$
$a \in P$	$\longrightarrow$	$\begin{array}{rccc} \mathbf{x}_a : & E_{\leq_P} & \longrightarrow & \{1, \ldots, \lvert P \rvert\} \\ & e & \mapsto & \mathbf{x}_a(e) = \mathsf{Pos}_e(a) \end{array}$
P	$\longrightarrow$	$\mathcal{A} = \{\mathbf{x}_a : a \in P\}$

Main results

Averaged ranking VS Expected value

$$a \precsim_{\mathrm{av}} b \iff \mathbf{x}_b \succeq_{\mathrm{EV}} \mathbf{x}_a$$

Mutual rank probabilities VS Statistical preference

$$a \precsim_{\mathrm{mrp}} b \iff \mathbf{x}_b \succeq_{\mathrm{SP}} \mathbf{x}_a$$

Maximal method VS Probabilistic preference

$$a \precsim_{\mathrm{mm}} b \implies \mathbf{x}_b \succeq_{\mathrm{PP}} \mathbf{x}_a$$

$$M_i \iff \mathcal{A}_i$$

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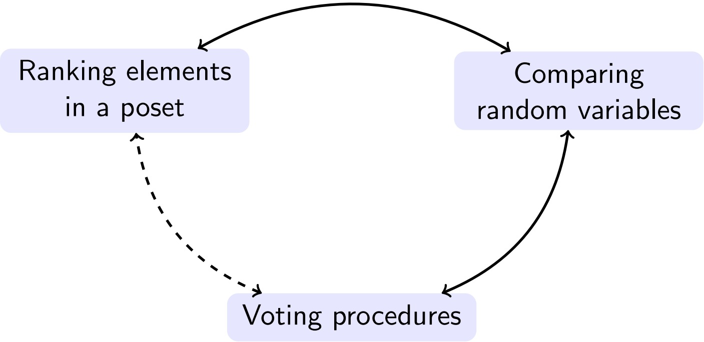
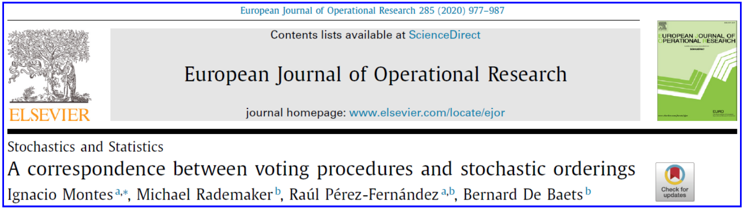
Ranking elements in a poset

Comparing random variables

Conclusions

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Stochastics and Statistics

A correspondence between voting procedures and stochastic orderings

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