Distortion of imprecise probabilities

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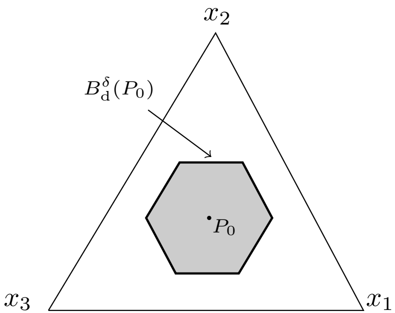
Motivation



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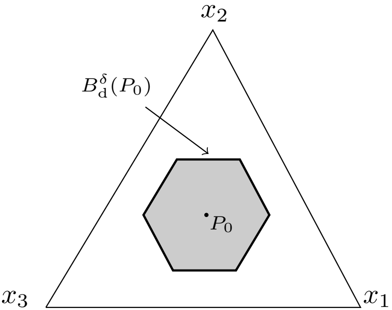

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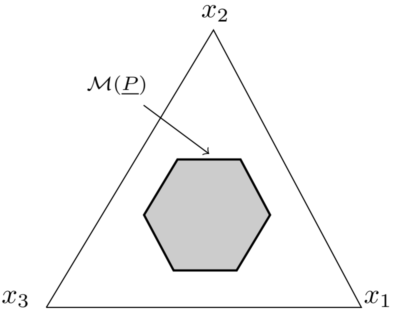

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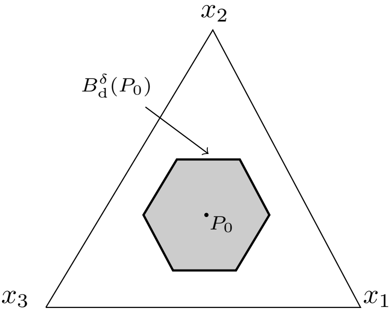

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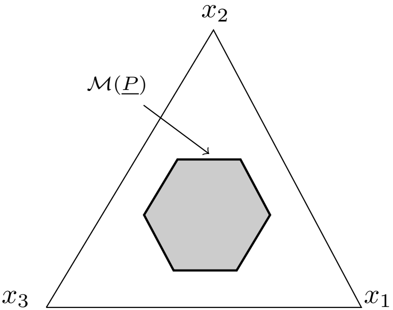

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1. Preliminaries
2. Distortion of imprecise models
3. Imprecise total variation
4. Conclusions

Preliminaries: lower probabilities

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Some particular properties that a coherent lower probability may satisfy are:

- ***2-monotonicity***. For every $A, B \subseteq \mathcal{X}$:

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2. ***k-monotonicity***. For every $A_1, \ldots, A_p \subseteq \mathcal{X}$ and $1 \leq p \leq k$:

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In this case, lower probabilities $\leftrightarrow$ lower previsions (Choquet integral).

Distortion procedures

	$P_0 \in \mathbb{P}(\mathcal{X})$	$\underline{P}$
(a)	$B_d^{\delta}(P_0) = \{Q \in \mathbb{P}(\mathcal{X}) : d(Q, P_0) \leq \delta\}$ lower $\downarrow$ envelope $\underline{Q}(A) = \inf\{P(A) : P \in B_d^{\delta}(P_0)\}$	$\bigcup_{P \in \mathcal{M}(\underline{P})} B_d^{\delta}(P)$ lower $\downarrow$ envelope $\underline{Q}(A)$

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(b)	$f_\delta : [0,1] \to [0,1]$ increasing, $f_\delta(0) = 0,\ f_\delta(1) = 1$ $\downarrow$ $\underline{Q}_\delta(A) := f_\delta(P_0(A))$	$f_\delta : [0,1] \to [0,1]$ increasing, $f_\delta(0) = 0,\ f_\delta(1) = 1$ $\downarrow$ $\underline{Q}_\delta(A) := f_\delta(\underline{P}(A))$

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- ***Expansion***. Given $\delta_1 > \delta_2 > 0$:

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(iv) ***Commutativity***. If $\underline{P}$ is coherent, then for any $\delta > 0$:

$$\mathcal{M}\big(\underline{Q}_{\delta}(\underline{P})\big) = \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}\big(\underline{Q}_{\delta}(P)\big).$$

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2. Distortion of imprecise models

3. **Imprecise total variation**

4. Conclusions

Total Variation distance

Definition (Total Variation distance)

$d_{TV} : \mathbb{P}(\mathcal{X}) \times \mathbb{P}(\mathcal{X}) \to [0, \infty)$ *given by:*

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Proposition (Herron et al. '97)

The lower envelope of $B^{\delta}_{d_{\mathrm{TV}}}(P)$ *is given by:*

$$\underline{Q}_P(A) = \max\{P(A) - \delta, 0\} \quad \forall A \subset \mathcal{X}, \quad \underline{Q}_P(\mathcal{X}) = 1.$$

Imprecise Total Variation (first approach)

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Let $\underline{P}$ *be a coherent lower probability and* $\delta > 0$*. The* **imprecise total variation model** *induced by* $(\underline{P}, \delta)$ *is the lower probability* $\underline{Q}$ *given by*

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This generalises the eponymous model for the precise case, and has also appeared as the *strong* δ*-core* in game theory (used to find a solution of a distorted game when $\mathcal{M}(\underline{P}) = \emptyset$).

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Let $\underline{P}$ be a lower probability, $\delta > 0$ and let $\underline{Q}$ be the total variation model they induce. If $\underline{P}$ has any of the following properties:

1. *avoiding sure loss (i.e. $\mathcal{M}(\underline{P}) \neq \emptyset$),*
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so does $\underline{Q}$.

However, k-monotonicity is not preserved in general.

Imprecise Total Variation (second approach)

What about conmmutativity?

Consider:

$$\underline{Q}'(A) = \min\left\{Q(A) : Q \in \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}(\underline{Q}_P)\right\}, \quad \forall A \subseteq \mathcal{X}. \qquad \text{(ITV2)}$$

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Proposition

Let $\underline{P}$ be a coherent lower probability, $\delta > 0$ and $\underline{Q}, \underline{Q}'$ the lower probabilities induced by Eqs. (ITV1), (ITV2). *Then $\underline{Q}' = \underline{Q}$.*

Consider the premetric:

$$d_{\mathrm{TV}}^{\min}(\underline{P},\underline{Q}) = \min_{\substack{P\in\mathcal{M}(\underline{P})\\ Q\in\mathcal{M}(\underline{Q})}} d_{\mathrm{TV}}(P,Q) = \min_{\substack{P\in\mathcal{M}(\underline{P})\\ Q\in\mathcal{M}(\underline{Q})}} \max_{A\subseteq\mathcal{X}} |P(A)-Q(A)|.$$

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It holds:

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Example

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$	$\{x_6\}$
$P_1(A)$	0.2	0.061	0.061	0.19	0.199	0.289
$P_2(A)$	0.161	0.1	0.1	0.238	0.141	0.26
$P_3(A)$	0.239	0.099	0.12	0.161	0.17	0.211
$P_4(A)$	0.161	0.177	0.13	0.161	0.102	0.269
$P_5(A)$	0.22	0.041	0.109	0.21	0.199	0.221
$P_6(A)$	0.22	0.041	0.041	0.23	0.199	0.269
$P_7(A)$	0.178	0.16	0.111	0.161	0.1	0.29
$P_8(A)$	0.161	0.157	0.14	0.181	0.073	0.288
$P_9(A)$	0.19	0.073	0.078	0.21	0.15	0.299

Example (Continuation)

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$	$\{x_6\}$
$P_{10}(A)$	0.161	0.158	0.159	0.161	0.112	0.249
$P_{11}(A)$	0.2	0.071	0.14	0.199	0.199	0.191
$P_{12}(A)$	0.161	0.177	0.138	0.161	0.073	0.29
$P_{13}(A)$	0.239	0.071	0.1	0.189	0.199	0.202
$P_{14}(A)$	0.22	0.041	0.081	0.238	0.199	0.221
$P_{15}(A)$	0.219	0.119	0.061	0.161	0.17	0.27
$P_{16}(A)$	0.218	0.043	0.062	0.238	0.168	0.271
$P_{17}(A)$	0.2	0.0705	0.1585	0.1805	0.199	0.1915

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$$Q := (0.2, 0.05, 0.05, 0.201, 0.199, 0.3) \in \mathcal{M}(\underline{Q}) \setminus B^{\delta}_{d^{\min}_{\mathrm{TV}}}(\underline{P}),$$

where $\underline{P}(A) = \min\{P_i(A) : i = 1, \ldots, 17\}$ *for any* $A \subseteq \mathcal{X}$, $\delta = 0.011$.

Proposition

Let $\underline{P}$ be a 2-monotone lower probability, $\delta > 0$ and let $\underline{Q}$ be the lower probability defined by (ITV1)*. Then, $\mathcal{M}(\underline{Q}) = B^{\delta}_{d_{\mathrm{TV}}^{\min}}(\underline{P})$.*

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Proof.

Let $\mathcal{H} \subseteq \mathcal{L}(\mathcal{X})$ be the set of gambles that take values in $[0,1]$. Define, for a given $Q \in \mathbb{P}(\mathcal{X})$, $f_Q(P,g) := P(g) - Q(g)$ over $\mathcal{M}(\underline{P}) \times \mathcal{H}$.

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Let $\mathcal{H} \subseteq \mathcal{L}(\mathcal{X})$ be the set of gambles that take values in $[0,1]$. Define, for a given $Q \in \mathbb{P}(\mathcal{X})$, $f_Q(P,g) := P(g) - Q(g)$ over $\mathcal{M}(\underline{P}) \times \mathcal{H}$.

$$\max_{\mathcal{H}} \min_{\mathcal{M}(\underline{P})} f_Q(P,g) \leq \delta \Leftrightarrow Q \in \mathcal{M}\big(\underline{Q}\big) \quad \text{and}$$

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Proposition

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Applying the min-max theorem,

$$\min_{\mathcal{M}(\underline{P})} \max_{\mathcal{H}} f_Q(P,g) = \max_{\mathcal{H}} \min_{\mathcal{M}(\underline{P})} f_Q(P,g). \quad \square$$

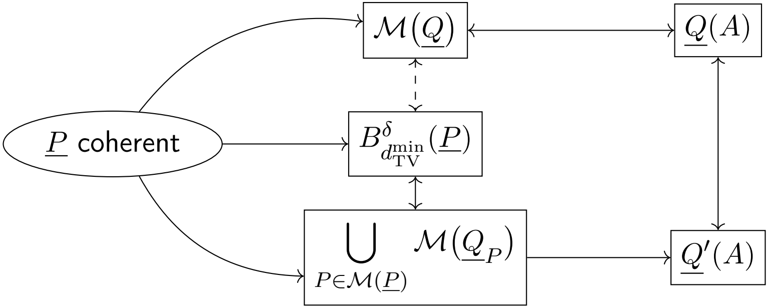
1. Preliminaries
2. Distortion of imprecise models
3. Imprecise total variation
4. **Conclusions**

Synthesis

Properties	Imprecise TV
Expansion	✓
Aggregation	✓
Structure Preservation	

Synthesis

Properties	Imprecise TV
Expansion	✓
Aggregation	✓
Structure Preservation (ASL, coherence, 2-monot., minitivity)	✓
Structure Preservation (k-monotonicity)	X
Commutativity	$\sim$ (✓ under 2-monot.)


$\underline{P}$ coherent
$\mathcal{M}(\underline{Q})$
$\underline{Q}(A)$
$B^{\delta}_{d^{\min}_{TV}}(\underline{P})$
$\bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}(\underline{Q}_P)$
$\underline{Q}'(A)$

$$\begin{tikzcd}
 & \mathcal{M}(\underline{Q}) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, dashed] & \underline{Q}(A) \arrow[dd, leftrightarrow] \\
\underline{P} \text{ coherent} \arrow[ur, bend left] \arrow[r] \arrow[dr, bend right] & B^{\delta}_{d_{\mathrm{TV}}^{\min}}(\underline{P}) \arrow[d, leftrightarrow] & \\
 & \bigcup_{P\in\mathcal{M}(\underline{P})} \mathcal{M}(\underline{Q}_P) \arrow[r] & \underline{Q}'(A)
\end{tikzcd}$$

The dashed line expresses the partial correspondence, in this case, under 2-monotonicity.

Future lines

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- Distortion of other IP models (e.g. Moral), and also through different premetrics (e.g. maximum instead of minimum).
- Connections with game theory (e.g. weak δ-core, through a penalised version of d_{TV}).

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