Distortions of lower probabilities as a tool for avoiding conflict

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Summary

We consider a number of imprecise probability models in conflict, meaning that their sets of compatible probabilities are disjoint.

Since the conjunction rule of aggregation is not applicable in this case, we propose to enlarge the models until the conflict is partially removed.

We analyse the properties of this procedure as an aggregation rule and compare it with a number of alternatives.

Outline

1. Preliminary concepts.
2. Aggregation of distorted models.
3. Non-uniform distortions.
4. Conclusions.

Imprecise probability models

Let $\mathcal{X}$ be a finite possibility space, and $\mathbb{P}(\mathcal{X})$ be the set of probability measures on $\mathcal{X}$.

A convex and closed set $\mathcal{M} \subseteq \mathbb{P}(\mathcal{X})$ is called a credal set.

It determines a coherent lower prevision on the set of gambles $\mathcal{L}(\mathcal{X}) := \{f : \mathcal{X} \to \mathbb{R}\}$ by $\underline{P}(f) := \min_{P\in\mathcal{M}} P(f)$ for all $f \in \mathcal{L}(\mathcal{X})$, and a coherent upper prevision by $\overline{P}(f) := \max_{P\in\mathcal{M}} P(f)$.

The restriction of $\underline{P}, \overline{P}$ to indicators of events are called coherent lower and upper probabilities.

Conversely, given a coherent lower prevision $\underline{P}$, we denote

$$\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}(\mathcal{X}) : P(f) \geq \underline{P}(f)\ \forall f\}.$$

A particular case of coherent lower probabilities are the 2-monotone ones, that satisfy

$$\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B) - \underline{P}(A \cap B) \quad \forall A, B \subseteq \mathcal{X}.$$

Aggregation rules

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

Consider a number of coherent lower probabilities $\underline{P}_1, \ldots, \underline{P}_n$ on $\mathcal{X}$ that model the opinions of a number of experts. Our goal is to aggregate them into a global model $\underline{Q}$.

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Conjunction: $\underline{Q}^C_{\text{agg}}$ is the lower envelope of the intersection $\cap_{i=1}^n \mathcal{M}(\underline{P}_i)$.

Disjunction: $\underline{Q}^D_{\text{agg}}(f) := \min_i \underline{P}_i(f)\ \forall f \in \mathcal{L}(\mathcal{X})$. It is the lower envelope of $\cup_{i=1}^n \mathcal{M}(\underline{P}_i)$.

Pareto: $\underline{Q}^P_{\text{agg}}(f) := \min\{\max_i \underline{P}_i(f), \min_i \overline{P}_i(f)\}\ \forall f \in \mathcal{L}(\mathcal{X})$.

Mixture: $\underline{Q}^M_{\text{agg}}(f) := \sum_{i=1}^n \alpha_i \underline{P}_i(f)\ \forall f \in \mathcal{L}(\mathcal{X})$ for some fixed $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$.

Aggregation under conflict

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

The conjunction rule is not applicable when $\cap_{i=1}^{n} \mathcal{M}(\underline{P}_i)$ is empty, i.e., in a situation of *conflict*.

On the other hand, disjunction may be too imprecise, and the Pareto rule may not preserve coherence.

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Distances between coherent lower probabilities

Consider the total variation distance between probability measures:

$$d_{\mathrm{TV}}(P_1, P_2) := \max_{A \subseteq \mathcal{X}} |P_1(A) - P_2(A)| = \frac{1}{2} \sum_{x \in \mathcal{X}} |P_1(\{x\}) - P_2(\{x\})|.$$

It can be generalised to coherent lower probabilities by

$$d_{\mathrm{TV}}^{\min}(\underline{P}_1, \underline{P}_2) := \min_{\substack{P_1 \in \mathcal{M}(\underline{P}_1) \\ P_2 \in \mathcal{M}(\underline{P}_2)}} d_{\mathrm{TV}}(P_1, P_2) = \min_{\substack{P_1 \in \mathcal{M}(\underline{P}_1) \\ P_2 \in \mathcal{M}(\underline{P}_2)}} \max_{A \subseteq \mathcal{X}} \big(P_1(A) - P_2(A)\big),$$

and

$$d'_{\mathrm{TV}}(\underline{P}_1, \underline{P}_2) := \max_{A \subseteq \mathcal{X}} \min_{\substack{P_1 \in \mathcal{M}(\underline{P}_1) \\ P_2 \in \mathcal{M}(\underline{P}_2)}} (P_1(A) - P_2(A)) = \max_{A \subseteq \mathcal{X}} \big(\underline{P}_1(A) - \overline{P}_2(A)\big).$$

Distortions of credal sets

- $d_{\mathrm{TV}}^{\min} \geq d'_{\mathrm{TV}}$ and the equality holds whenever $\underline{P}_1, \underline{P}_2$ are 2-monotone.
- Given $\underline{P}_0 \in \underline{\mathbb{P}}(\mathcal{X})$ coherent and $d \in \{d_{\mathrm{TV}}^{\min}, d'_{\mathrm{TV}}\}$, the neighbourhood

$$B_d^{\delta}(\underline{P}_0) := \{Q \in \mathbb{P}(\mathcal{X}) \mid d(Q, \underline{P}_0) \leq \delta\}$$

determines the coherent lower probability

$$\underline{Q}(A) := \max\{\underline{P}_0(A) - \delta, 0\} \quad \forall A \neq \mathcal{X}, \underline{Q}(\mathcal{X}) := 1.$$

- However, $B^{\delta}_{d_{\mathrm{TV}}^{\min}}(\underline{P}_0)$ and $B^{\delta}_{d'_{\mathrm{TV}}}(\underline{P}_0)$ do not necessarily coincide: they determine different coherent lower previsions.

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Distortion-Conjunction aggregation rule

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

Let $\{\underline{P}_i\}_{i=1}^{n} \subseteq \underline{\mathbb{P}}(\mathcal{X})$ be coherent and such that $\cap_{i=1}^{n} \mathcal{M}(\underline{P}_i) = \emptyset$ (i.e. in global conflict).

Distortion-Conjunction aggregation rule

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

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$$\delta^* := \min\{\delta \geq 0 \mid \cap_{i=1}^n B_d^\delta(\underline{P}_i) \neq \emptyset\},$$

Distortion-Conjunction aggregation rule

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

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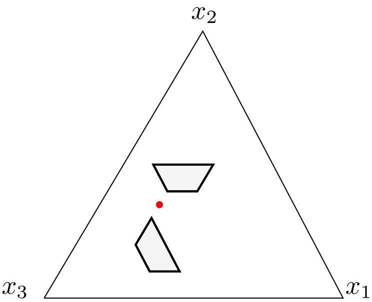
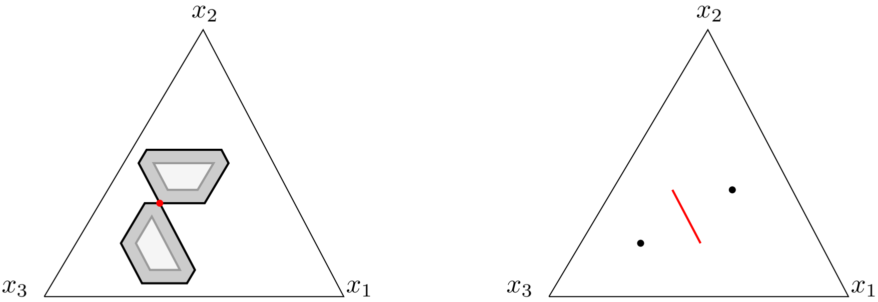
and, from it:

$$\underline{Q}_{\mathrm{agg}}(f) := \min\{P(f) \mid P \in \cap_{i=1}^n B_d^{\delta^*}(\underline{P}_i)\}, \quad \forall f \in \mathcal{L}(\mathcal{X})$$

Example

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Interpretation as a measure of discrepancy

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

Let us define the conflict between $P \in \mathbb{P}(\mathcal{X})$ and a coherent lower prevision $\underline{P}_i$ by

$$C_i(P) := \min_{P_i \in \mathcal{M}(\underline{P}_i)} d_{\mathrm{TV}}(P, P_i) = d^{\min}_{\mathrm{TV}}(P, \underline{P}_i).$$

and the maximal conflict between P and $\{\underline{P}_i\}_{i=1,\dots,n}$ as

$$C^{\max}_{\mathrm{group}}(P) := \max_i C_i(P) = \max_i d^{\min}_{\mathrm{TV}}(P, \underline{P}_i).$$

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Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

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and the maximal conflict between P and $\{\underline{P}_i\}_{i=1,\dots,n}$ as

$$C_{\mathrm{group}}^{\max}(P) := \max_i C_i(P) = \max_i d_{\mathrm{TV}}^{\min}(P, \underline{P}_i).$$

- $\cap_{i=1}^{n} \mathcal{M}(\underline{P}_i) = \emptyset \Leftrightarrow \forall P \in \mathbb{P}(\mathcal{X}) \quad C_{\mathrm{group}}^{\max}(P) \neq 0;$
- $\mathcal{M}\left(\underline{Q}_{\mathrm{agg}}\right) = \left\{Q \in \mathbb{P}(\mathcal{X}) \mid C_{\mathrm{group}}^{\max}(Q) = \min_{P \in \mathbb{P}(\mathcal{X})} C_{\mathrm{group}}^{\max}(P) = \delta^*\right\}.$

Properties of the distortion factor

Consider the case of $n = 2$, and assume the distortion function d is either $d_{\mathrm{TV}}^{\mathrm{min}}$ or d'_{TV}.

- $\delta^* = \frac{1}{2} d(\underline{P}_1, \underline{P}_2)$.
- $B_d^{\delta^*}(\underline{P}_1) \cap B_d^{\delta^*}(\underline{P}_2) = \partial B_d^{\delta^*}(\underline{P}_1) \cap \partial B_d^{\delta^*}(\underline{P}_2)$, where ∂ denotes the boundary of the set.

Properties as an aggregation rule

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

Coherence: $\underline{Q}_{\text{agg}}$ is coherent.

Symmetry: Permuting $\{\underline{P}_i\}_{i=1}^n$ does not affect $\underline{Q}_{\text{agg}}$.

Marginalisation: $\forall A \subseteq \mathcal{X}$, $\underline{Q}_{\text{agg}}(A)$ depends on $\{\underline{P}_i(A)\}_{i=1}^n$.

Precise preservation: $\{\underline{P}_i\}_{i=1}^n \subseteq \mathbb{P}(\mathcal{X})$ implies $\underline{Q}_{\text{agg}} \in \mathbb{P}(\mathcal{X})$.

Monotonicity: If certain $\underline{P}_i$ is replaced by $\underline{P}' \leq \underline{P}_i \Rightarrow \underline{Q}'_{\text{agg}} \leq \underline{Q}_{\text{agg}}$.

Total reconciliation: $\underline{Q}_{\text{agg}} \leq \min_i \underline{P}_i$.

Strong Pareto: $\underline{Q}_{\text{agg}}(f) \geq \min\{\max_i \underline{P}_i(f), \min_i \overline{P}_i(f)\}\ \forall f \in \mathcal{L}(\mathcal{X})$.

Unanimity: $\underline{Q}_{\text{agg}} \geq \min_i \underline{P}_i$.

Associativity: $\underline{Q}_{\text{agg}}$ coincides with the result of aggregating any $\underline{P}_i$ with the aggregation of the rest of the individuals.

Properties as an aggregation rule

The distortion-conjunction aggregation rule for $d \in \{d'_{\mathrm{TV}}, d^{\min}_{\mathrm{TV}}\}$ satisfies:

- **Coherence:** $\underline{Q}_{\mathrm{agg}}$ is coherent. (✓)
- **Symmetry:** Permuting $\{\underline{P}_i\}_{i=1}^n$ does not affect $\underline{Q}_{\mathrm{agg}}$.(✓)
- **Marginalisation:** $\forall A \subseteq \mathcal{X}$, $\underline{Q}_{\mathrm{agg}}(A)$ depends on $\{\underline{P}_i(A)\}_{i=1}^n$. (✗)
- **Precise preservation:** $\{\underline{P}_i\}_{i=1}^n \subseteq \mathbb{P}(\mathcal{X})$ implies $\underline{Q}_{\mathrm{agg}} \in \mathbb{P}(\mathcal{X})$. (✗)
- **Monotonicity:** If certain $\underline{P}_i$ is replaced by $\underline{P}' \leq \underline{P}_i \Rightarrow \underline{Q}'_{\mathrm{agg}} \leq \underline{Q}_{\mathrm{agg}}$. (✗)
- **Total reconciliation:** $\underline{Q}_{\mathrm{agg}} \leq \min_i \underline{P}_i$. (✗)
- **Strong Pareto:** $\underline{Q}_{\mathrm{agg}}(f) \geq \min\{\max_i \underline{P}_i(f), \min_i \overline{P}_i(f)\}\ \forall f \in \mathcal{L}(\mathcal{X})$. (✗)
- **Unanimity:** $\underline{Q}_{\mathrm{agg}} \geq \min_i \underline{P}_i$. (✗)
- **Associativity:** $\underline{Q}_{\mathrm{agg}}$ coincides with the result of aggregating any $\underline{P}_i$ with the aggregation of the rest of the individuals. (✗)

Aggregation by event-dependent distortions

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

Our procedure makes a uniform distortion of the credal sets in conflict, even in those directions where conflict is not really present: it may be that $[\underline{P}_1(\{x\}), \overline{P}_1(\{x\})] = [\underline{P}_2(\{x\}), \overline{P}_2(\{x\})]$ and still we shall make the lower and upper probabilities of $\{x\}$ more imprecise.

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$\hookrightarrow$ Next we investigate if we can make a minimal (different) distortion in each direction so that conflict is avoided.

Aggregation by event-dependent distortions

Given $\underline{P}_1, \underline{P}_2 \in \mathbb{P}(\mathcal{X})$, let

$$\mathcal{A}^{=} = \{A \subseteq \mathcal{X} \mid [\underline{P}_1(A), \overline{P}_1(A)] \cap [\underline{P}_2(A), \overline{P}_2(A)] \neq \emptyset\},$$
$$\mathcal{A}^{1>2} = \{A \subseteq \mathcal{X} \mid \underline{P}_1(A) > \overline{P}_2(A)\},$$
$$\mathcal{A}^{2>1} = \{A \subseteq \mathcal{X} \mid \underline{P}_2(A) > \overline{P}_1(A)\}$$

and define the event-dependent distortion factor:

$$\delta_A = \begin{cases} \frac{1}{2}\left(\underline{P}_1(A) - \overline{P}_2(A)\right), & \text{if } A \in \mathcal{A}^{1>2}, \\ \frac{1}{2}\left(\underline{P}_2(A) - \overline{P}_1(A)\right), & \text{if } A \in \mathcal{A}^{2>1}. \end{cases}$$

We now define the distorted models $\underline{Q}_1, \underline{Q}_2$ by

$$\underline{Q}_i(A) = \begin{cases} \underline{P}_i(A), & \text{if } A \in \mathcal{A}^{=} \cup \mathcal{A}^{j>i}, \\ \underline{P}_i(A) - \delta_A, & \text{if } A \in \mathcal{A}^{i>j}. \end{cases}$$

Properties of the model

- $A \in \mathcal{A}^{1>2} \Leftrightarrow A^c \in \mathcal{A}^{2>1}$ and $A \in \mathcal{A}^{=} \Leftrightarrow A^c \in \mathcal{A}^{=}$.(✓)
- $\delta_A = \delta_{A^c}$ for every $A \subseteq \mathcal{X}$.(✓)
- $[\underline{Q}_1(A), \overline{Q}_1(A)] \cap [\underline{Q}_2(A), \overline{Q}_2(A)] \neq \emptyset$ for each $A \subseteq \mathcal{X}$. (✓)

Properties of the model

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- $\delta_A = \delta_{A^c}$ for every $A \subseteq \mathcal{X}$.(✓)
- $[\underline{Q}_1(A), \overline{Q}_1(A)] \cap [\underline{Q}_2(A), \overline{Q}_2(A)] \neq \emptyset$ for each $A \subseteq \mathcal{X}$. (✓)
- $\underline{Q}_1, \underline{Q}_2$ need not be coherent. (✗)
- $\mathcal{M}(\underline{Q}_1) \cap \mathcal{M}(\underline{Q}_2)$ may be empty. (✗)

$\hookrightarrow$ Removing the local conflict does not entail that the global conflict is resolved.

Conclusions

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

- Our procedure may not be more informative than the disjunction.
- While coherence and symmetry are satisfied, other properties of interest such as monotonicity and associativity are not.
- The event-dependent distortion should be refined to guarantee coherence.

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Future work:

- Comparison with other distortion measures.
- Analysis of the connection with axiomatic measures of conflict.
- Study of a pairwise approach to conflict removal.

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

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Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions

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Thank you for the attention...

Introduction
Aggregation under conflict
Event dependent aggregation
Conclusions