Two prominent examples of penalty-based aggregation of circular data

Raúl Pérez-Fernández and Bernard De Baets

Departamento de Estadística, I.O. y D.M., Universidad de Oviedo
KERMIT, Department of Data Analysis and Mathematical Modelling, Ghent University, Belgium

20th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems
Lisbon, Portugal, 22–26 July 2024

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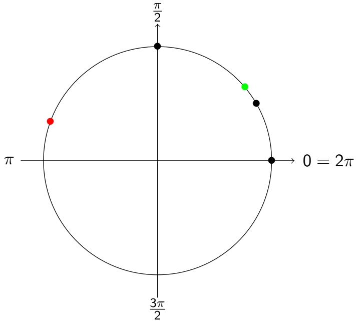
A real-life example of aggregation of circular data

$$\text{Coffee break} = A\,(\text{Start time}\,,\text{End time})$$

No arithmetic mean for circular data



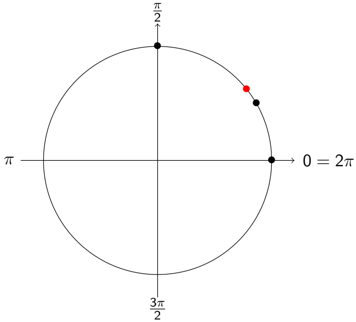
Let $\mathcal{D} = [0, 2\pi[$ denote the set of circular data

The arithmetic mean is not well-defined on $\mathcal{D}$:

$$A\left(0, \frac{\pi}{6}, \frac{\pi}{2}\right) = \frac{0 + \frac{\pi}{6} + \frac{\pi}{2}}{3} = \frac{2\pi}{9}$$

$$A\left(2\pi, \frac{\pi}{6}, \frac{\pi}{2}\right) = \frac{2\pi + \frac{\pi}{6} + \frac{\pi}{2}}{3} = \frac{8\pi}{9}$$

Aggregation of circular data: The circular mean



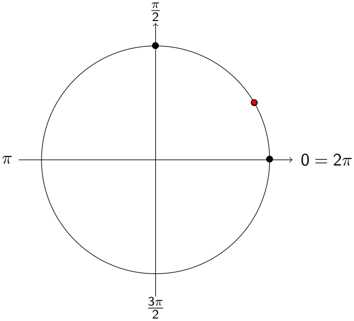
The circular mean $\overline{\mathbf{x}}$ of a list of angles $\mathbf{x}$ is defined as:

$$\overline{\mathbf{x}} = \begin{cases} \arctan(S/C)\,, & \text{if } C > 0\,, \\ \arctan(S/C) + \pi\,, & \text{if } C < 0\,, \\ \frac{\pi}{2}\,, & \text{if } C = 0 \text{ and } S > 0\,, \\ -\frac{\pi}{2}\,, & \text{if } C = 0 \text{ and } S < 0\,, \\ \text{undefined}\,, & \text{if } C = 0 = S\,, \end{cases}$$

where $C = \frac{1}{n}\sum_{i=1}^{n}\cos(x_i)$ and $S = \frac{1}{n}\sum_{i=1}^{n}\sin(x_i)$

$$\overline{\left(0, \frac{\pi}{6}, \frac{\pi}{2}\right)} = \arctan\left(\frac{3}{2+\sqrt{3}}\right) \approx \color{red}{0.677}$$

Aggregation of circular data: The circular median



If n is odd, then the circular median $\tilde{\mathbf{x}}$ of a list of angles $\mathbf{x}$ is defined as the angle in $\mathbf{x}$ that:

(i) $\lceil \frac{n}{2} \rceil$ of the angles in $\mathbf{x}$ lie on the arc $[\tilde{\mathbf{x}}, \tilde{\mathbf{x}} + \pi[$

(ii) the majority of the angles in $\mathbf{x}$ are closer to $\tilde{\mathbf{x}}$ than to $\tilde{\mathbf{x}} + \pi$

More involved definition if n is even

$$\widetilde{\left(0, \frac{\pi}{6}, \frac{\pi}{2}\right)} = \frac{\pi}{6}$$

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Aggregation functions in theory and practice

Definition:

Consider a bounded poset $(P, \leq, \mathbf{0}, \mathbf{1})$ and $n \in \mathbb{N}$. A function $A : P^n \to P$ is called an n-ary **aggregation function** on $(P, \leq)$ if

1. A satisfies the boundary conditions:

$$A(\mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0} \quad \text{and} \quad A(\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1}$$

2. A is monotone increasing:

$$\mathbf{x} \leq \mathbf{y} \Rightarrow A(\mathbf{x}) \leq A(\mathbf{y})$$

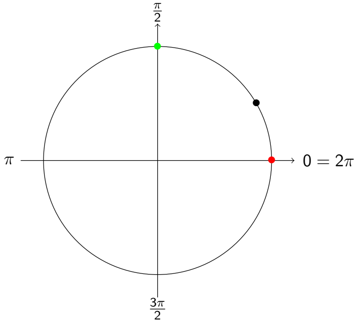
Practice:

This definition is considered a **standard** and works out for real numbers, ordinal (linguistic) scales, intervals, etc.

Examples:

Arithmetic mean, weighted arithmetic means, OWAs, t-norms, t-conorms, uninorms, Choquet integrals, Sugeno integrals...

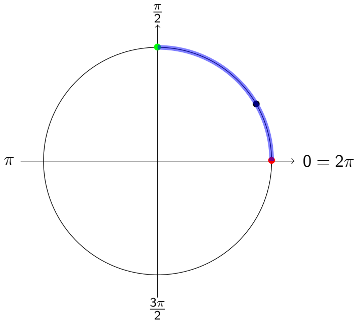
Problem: There is no order for circular data



Is $0 \leq \frac{\pi}{6} \leq \frac{\pi}{2}$? Is $2\pi \leq \frac{\pi}{6} \leq \frac{\pi}{2}$?

Unfortunately, there exists no order for circular data!

The set of circular data has an interesting structure



$\frac{\pi}{6}$ is in between 0 and $\frac{\pi}{2}$

A natural betweenness relation on $\mathcal{D}$ is defined as $(x, y, z) \in B_{\mathcal{D}}$ if

$$\left(\sin(z-x)\cdot\sin(y-x)\geq 0\right)\wedge\left(\cos(z-x)\leq\cos(y-x)\right)\}$$

The set of circular data is a bounded beset $(\mathcal{D}, B_{\mathcal{D}}, \mathcal{D})$
(in which all elements are bounds)

Aggregation functions on besets

Pérez-Fernández and De Baets, 2021

Definition:
Consider a bounded beset (X, B, S) and $n \in \mathbb{N}$. A function $A : X^n \to X$ is called an n-ary **aggregation function** on (X, B, S) if

1. A satisfies the boundary conditions:

$$A(o, \ldots, o) = o \text{ for any } o \in S$$

2. A is betweenness-preserving:

$$(o, x_i, y_i) \in B \text{ for any } i \in \{1, \ldots, n\}$$

implies

$$(o, A(x_1, \ldots, x_n), A(y_1, \ldots, y_n)) \in B$$

for any $o \in S$

Relation with classical aggregation

Theorem (Pérez-Fernández and De Baets, 2021)

Let $(X, \leq, 0, 1)$ *be a bounded poset and* $(X, B_{\leq}, \{0, 1\})$ *be the bounded beset where* $B_{\leq}$ *is induced by* $\leq$*. For a function* $A : X^n \to X$*, the following two conditions are equivalent:*

- A *is an aggregation function on* $(X, \leq, 0, 1)$
- A *is an aggregation function on* $(X, B_{\leq}, \{0, 1\})$

Extension of classical aggregation theory

This definition has been used successfully for formalizing aggregation processes for **rankings**, **compositional data**, **strings**

Sadly, both examples do not fit the current understanding of an aggregation function

Proposition

Consider the bounded beset $(\mathcal{D}, B_{\mathcal{D}}, \mathcal{D})$ and $n \in \mathbb{N}$. The circular mean $\bar{\cdot} : \mathcal{D}^n \to \mathcal{D}$ is not an aggregation function on $(\mathcal{D}, B_{\mathcal{D}}, \mathcal{D})$.

Proposition

Consider the bounded beset $(\mathcal{D}, B_{\mathcal{D}}, \mathcal{D})$ and $n \in \mathbb{N}$. The circular median $\tilde{\cdot} : \mathcal{D}^n \to \mathcal{P}(\mathcal{D})$ is not an aggregation function on $(\mathcal{D}, B_{\mathcal{D}}, \mathcal{D})$.

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Framework of penalty-based aggregation

Yager, Rybalov, Beliakov, Calvo, Bustince, ...

Definition: Consider $n \in \mathbb{N}$ and $I = [a, b] \subseteq \mathbb{R}$. A function $P : I^{n+1} \to \mathbb{R}$ is called a **penalty function** if:

1. $P(y; \mathbf{x}) \geq 0$
2. $P(y; \mathbf{x}) = 0$ if and only if $\mathbf{x} = (y, \ldots, y)$
3. for every $\mathbf{x}$, $P(\cdot; \mathbf{x})$ is quasi-convex and lower-semicontinuous

The function $f : I^n \to I$ defined by

$$\boxed{f(\mathbf{x}) = \frac{l(\mathbf{x}) + r(\mathbf{x})}{2},}$$

where $[l(\mathbf{x}), r(\mathbf{x})]$ is the set of minimizers of $P(\cdot; \mathbf{x})$, is called the **penalty-based (aggregation) function** associated with P

Penalty-based aggregation on besets

Pérez-Fernández and De Baets, 2019

Definition: Consider $n \in \mathbb{N}$, a set X and a betweenness relation B on X^n. A function $P : X \times X^n \to \mathbb{R}^+$ is called a **penalty function (compatible with** B**)** if the following four properties hold:

1. $P(y; \mathbf{x}) \geq 0$, for any $y \in X$ and any $\mathbf{x} \in X^n$
2. $P(y; \mathbf{x}) = 0$ if and only if $\mathbf{x} = (y, \ldots, y)$
3. The set of minimizers of $P(\cdot; \mathbf{x})$ is non-empty, for any $\mathbf{x} \in X^n$
4. $P(y; \mathbf{x}) \leq P(y; \mathbf{x}')$, for any $y \in X$ and any $\mathbf{x}, \mathbf{x}' \in X^n$ such that $((y, \ldots, y), \mathbf{x}, \mathbf{x}') \in B$

The function $f : X^n \to \mathcal{P}(X)$ defined by

$$f(\mathbf{x}) = \arg\min_{y \in X} P(y; \mathbf{x}),$$

for any $\mathbf{x} \in X^n$, is called the **penalty-based (aggregation) function** associated with P

The circular mean is a penalty-based function

Proposition

Consider $n \in \mathbb{N}$, $\mathcal{D}$ and the betweenness relation $(B_{\mathcal{D}})^{(n)}$ on $\mathcal{D}^n$. The function $P_1 : \mathcal{D} \times \mathcal{D}^n \to \mathbb{R}^+$ defined as

$$P_1(y; \mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} (1 - \cos(x_i - y))$$

is a penalty function compatible with $(B_{\mathcal{D}})^{(n)}$ and the circular mean $\bar{\cdot} : \mathcal{D}^n \to \mathcal{D}$ is a penalty-based (aggregation) function associated with P_1.

The circular median is a penalty-based function

Proposition

Consider $n \in \mathbb{N}$, $\mathcal{D}$ and the betweenness relation $(B_{\mathcal{D}})^{(n)}$ on $\mathcal{D}^n$. The function $P_2 : \mathcal{D} \times \mathcal{D}^n \to \mathbb{R}^+$ defined as

$$P_2(y; \mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} (\pi - |\pi - |x_i - y||) = \frac{1}{n} \sum_{i=1}^{n} \min(x_i - y, 2\pi - x_i + y)$$

is a penalty function compatible with $(B_{\mathcal{D}})^{(n)}$ and the circular median $\tilde{\cdot} : \mathcal{D}^n \to \mathcal{P}(\mathcal{D})$ is a penalty-based (aggregation) function associated with P_2.

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Conclusions

- It has been shown that **the circular mean and the circular median** can be accommodated within the framework of **penalty-based data aggregation on besets**
- Unfortunately, even though these two prominent functions for the aggregation of circular data satisfy the boundary conditions and are actually idempotent (as a result of property (P2) of a penalty function), it is shown that they **are not aggregation functions on the bounded beset of circular data**
- A future study subject is to explore weaker properties than monotonicity that could accommodate the circular mean and circular median

Funding

RPF received funding from grant PID2022-140585NB-I00 funded by MICIU/AEI/10.13039/501100011033 and "FEDER/UE".

BDB received funding from the Flemish Government under the "Onderzoeksprogramma Artificiële Intelligentie (AI) Vlaanderen" programme.

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