Distorting lower probabilities using common distortion models

David Nieto-Barba nietobarba@uniovi.es
Ignacio Montes imontes@uniovi.es
Enrique Miranda mirandaenrique@uniovi.es

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Context

How can we robustify a probability measure P_0?

1. Distortion of probabilities

Neighbourhood models

d: distorting function comparing probabilities
δ: distorting factor

$$B_d^\delta(P_0) = \{P \mid d(P, P_0) \leq \delta\}$$

Increasing Transformations

$g : [0,1] \to [0,1]$ increasing
$g(t) \leq t$

Increasing transformations can be expressed as neighbourhood models

$$\underline{P}(A) = \begin{cases} g\big(P_0(A)\big) & A \neq \mathcal{X} \\ 1 & A = \mathcal{X} \end{cases}$$

2. Particular models

Vertical Barrier: $\underline{P}_{\mathrm{VBM}}(A) = \max\{bP_0(A) + a, 0\}$
$a \leq 0,\ b \geq 0,\ a + b \leq 1$
$A \neq \mathcal{X}$

Linear Vacuous: $\underline{P}_{\mathrm{LV}}(A) = (1-\delta)P_0(A)$
$\delta \in [0,1],\ A \neq \mathcal{X}$

Pari Mutuel: $\underline{P}_{\mathrm{PMM}}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\}$
$\delta \geq 0$

Total Variation: $\underline{P}_{\mathrm{TV}}(A) = \max\{P_0(A) - \delta, 0\}$
$\delta \geq 0,\ A \neq \mathcal{X}$

Proposal

And... what about starting with $\underline{P}$?

3. Distortion of lower probabilities

Distortion procedure

Δ: family of distortion parameters

$\{\phi_\lambda\}_{\lambda\in\Delta}$: family of transforming functions
$\phi_\lambda : [0,1] \to [0,1]$
ϕ_λ increasing
$\phi_\lambda(t) \leq t$

$$\underline{Q}_\lambda[\underline{P}](A) = \begin{cases} \phi_\lambda\big(\underline{P}(A)\big) & A \neq \mathcal{X} \\ 1 & A = \mathcal{X} \end{cases}$$

4. Particular (imprecise) models

Imprecise VBM: $\underline{Q}_{(a,b)}[\underline{P}](A) = \max\{b\underline{P}(A) + a, 0\}$

Imprecise LV: $\underline{Q}_\delta^{LV}[\underline{P}](A) = (1-\delta)\underline{P}(A)$

Imprecise PMM: $\underline{Q}_\delta^{PMM}[\underline{P}](A) = \max\{(1+\delta)\underline{P}(A) - \delta, 0\}$

Imprecise TV: $\underline{Q}_\delta^{TV}[\underline{P}](A) = \max\{\underline{P}(A) - \delta, 0\}$

Which are the desirable properties for a distortion procedure?

5. Desirable properties

Basic properties

Expansion (P1):
$\underline{Q}_{\lambda_1}[\underline{P}] \leq \underline{Q}_{\lambda_2}[\underline{P}]$ if $\lambda_1 \succeq \lambda_2$

Semigroup (a) (P2a):
$\underline{Q}_{\lambda_0}[\underline{P}] = \underline{P}$ for some $\lambda_0 \preceq \lambda$ for any $\lambda \in \Lambda$

Semigroup (b) (P2b):
$\underline{Q}_{\lambda_2+\lambda_1}[\underline{P}] = \underline{Q}_{\lambda_2}\big[\underline{Q}_{\lambda_1}[\underline{P}]\big]$

Structure preservation (P3):
$\underline{Q}_\lambda[\underline{P}]$ preserves the properties (ASL, coherence, ...) of $\underline{P}$

Reversibility (P4):
$\underline{P}(A) = \varphi_\lambda\big(\underline{Q}_\lambda[\underline{P}](A)\big)$ for some φ_λ

Distortion of credal sets

Expression as a neighbourhood (P8):
$\mathcal{M}(\underline{Q}_\lambda[\underline{P}]) = \{Q \mid d(Q, \underline{P}) \leq \mu\}$
for some function d comparing probabilities and lower probabilities and μ depending on λ

Extreme points commutativity (P9):

$$\underline{Q}_\lambda[\underline{P}](A) = \inf\left\{Q(A) \mid Q \in \bigcup_{P\in\mathrm{ext}(\mathcal{M}(\underline{P}))} \mathcal{M}(\underline{Q}_\lambda[P])\right\}$$

Strong commutativity (P10):

$$\mathcal{M}(\underline{Q}_\lambda[\underline{P}]) = \bigcup_{P\in\mathcal{M}(\underline{P})} \mathcal{M}(P)$$

Invariance properties

Permutations (P5)

σ: permutation
$A = \{x_{i_1}, \ldots, x_{i_k}\}$
$A^\sigma = \{x_{\sigma(i_1)}, \ldots, x_{\sigma(i_k)}\}$
$\underline{P}^\sigma(A) = \underline{P}(A^\sigma)$
$\underline{Q}_\lambda[\underline{P}]^\sigma(A) = \underline{Q}_\lambda[\underline{P}](A^\sigma)$

$$\begin{array}{ccc} \underline{P} & \longrightarrow & \underline{P}^\sigma \\ \downarrow & & \downarrow \\ \underline{Q}_\lambda[\underline{P}] & \longrightarrow & \big(\underline{Q}_\lambda[\underline{P}]\big)^\sigma = \underline{Q}_\lambda[\underline{P}^\sigma] \end{array}$$

Marginalisation (P6)

Π: partition of $\mathcal{X}$
$\underline{P}^\Pi$: restriction of $\underline{P}$ to $\mathcal{P}(\Pi)$
$\underline{Q}_\lambda[\underline{P}]^\Pi$: restriction of $\underline{Q}_\lambda[\underline{P}]$ to $\mathcal{P}(\Pi)$

$$\begin{array}{ccc} \underline{P} & \longrightarrow & \underline{P}^\Pi \\ \downarrow & & \downarrow \\ \underline{Q}_\lambda[\underline{P}] & \longrightarrow & \big(\underline{Q}_\lambda[\underline{P}]\big)^\Pi = \underline{Q}_\lambda[\underline{P}^\Pi] \end{array}$$

Conditioning (P7)

Regular extension
B such that
$\underline{Q}_\lambda[\underline{P}](B) > 0$
$\underline{P}_B = \underline{P}(\cdot|B)$
$\underline{Q}_\lambda[\underline{P}]_B = \underline{Q}_\lambda[\underline{P}](\cdot|B)$

$$\begin{array}{ccc} \underline{P} & \longrightarrow & \underline{P}_B \\ \downarrow & & \downarrow \\ \underline{Q}_\lambda[\underline{P}] & \longrightarrow & \big(\underline{Q}_\lambda[\underline{P}]\big)_B = \underline{Q}_{\lambda^*}[\underline{P}_B] \end{array}$$

Results

Which properties are satisfied by each model?

6. Results

Model	(P1)	(P2)		(P3)				(P4)	(P5)	(P6)	(P7)	(P8)	(P9)	(P10)
		(P2a)	(P2b)	ASL	Coh.	2-monot.	k-monot.							
IVBM	N.A.	N.A.	N.A.	✓✓	✓✓	✓✓	✗	✓ if $b \neq 0$ and $\min_x \underline{P}(\{x\}) \geq {-a}/{b}$	✓✓	✓✓	✗	✓✓	✓✓	✓ $\underline{P}$ 2-monot.
ITV	✓✓	✓✓	✓✓	✓✓	✓✓	✓✓	✗	✓ if $\delta \leq \min_x \underline{P}(\{x\})$	✓✓	✓✓	✗	✓✓	✓✓	✓ $\underline{P}$ 2-monot.
ILV	✓✓	✓✓	✗	✓✓	✓✓	✓✓	✓✓	✓✓	✓✓	✓✓	✗	✓✓	✓✓	✓ $\underline{P}$ 2-monot.
IPMM	✓✓	✓✓	✗	✓✓	✓✓	✓✓	✓ if $\underline{Q}[\underline{P}](A) > 0$ for $A \neq \emptyset$	✓ if $\min_x \underline{P}(\{x\}) \geq {\delta}/{1+\delta}$	✓✓	✓✓	✗	✓ if $\underline{P}(A) > 0$ for $A \neq \emptyset$	✓✓	✓ $\underline{P}$ 2-monot.

At a glance

7. Conclusions

- Some properties do not hold for all IVBM...
- ...but hold for specific models
- Remarkably, the ILV *always* preserves k-monotonicity and reversibility
- Under 2-monotonicity, distorting $\underline{P}$ is equivalent to distorting $\mathcal{M}(\underline{P})$

? is it possible to characterise the distortion through its associated set of almost desirable gambles?
? are the solutions of a coalitional game preserved by the distortion?

8. References

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