Varifolds and Surface Approximation

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joint work with
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Why varifolds ?

- **flexible**: you can endow both discrete and continuous objects with a varifold structure.
- encode **order** 1 information (tangent bundle): **unoriented objects**.
- provide **weak notion of curvatures**.
- natural distances to compare varifolds.

Plan

A simple example

What is a varifold ?

Generalized curvature of a varifold

Approximate curvature

Numerical illustrations

References

Second fundamental form

We start with

- $\Gamma \subset \mathbb{R}^2$ a C^2 closed curve,
- $\gamma : [0, L] \to \mathbb{R}^2$ an injective $(0 \sim L)$ **arc length** parametrization of Γ.
- **unit tangent vector** τ: for $x = \gamma(t) \in \Gamma$, $\tau(x) = \gamma'(t)$ and $\theta(x)$ the angle between $\tau(x)$ and the horizontal.
- **curvature vector** κ: for $x = \gamma(t) \in \Gamma$, $\kappa(x) = \gamma''(t)$.

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Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be C^1:

$$\int_0^L \frac{d}{dt}\varphi(\gamma(t))\gamma'(t)\,dt$$
$$\underbrace{=}_{\text{by parts}} \underbrace{\left[\varphi(\gamma(t))\gamma'(t)\right]_{t=0}^{L}}_{=0} - \int_0^L \varphi(\gamma(t))\,\underbrace{\gamma''(t)}_{\kappa(\gamma(t))}\,dt$$

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$$\begin{aligned}\int_0^L \frac{d}{dt}\varphi(\gamma(t))\gamma'(t)\,dt &= \int_0^L \big(\nabla\varphi(\gamma(t)) \cdot \gamma'(t)\big)\,\gamma'(t)\,dt \\ &= \int_\Gamma \left(\nabla\varphi(x) \cdot \tau(x)\right)\tau = \int_\Gamma \Pi_{\theta(x)}\left(\nabla\varphi(x)\right)\end{aligned}$$

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- **weak formulation** of curvature,
- relies only on the knowledge of $\left\{ \int_\Gamma \psi(x, \theta(x)) \;\middle|\; \begin{array}{l} \psi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \textbf{ continuous} \\ \forall x \in \mathbb{R}^2,\ \omega \mapsto \psi(x, \omega) \text{ is } \pi\textbf{-periodic} \end{array} \right\}.$

Our first varifold

$$\mathcal{C} = \left\{ \begin{array}{llll} \psi : & \mathbb{R}^2 \times \mathbb{R} & \to & \mathbb{R} \\ & (x, \omega) & \mapsto & \psi(x, \omega) \end{array} \;\middle|\; \begin{array}{l} \psi \text{ continuous and } \pi\text{–periodic} \\ \text{w.r.t. } \omega \end{array} \right\}.$$

The continuous linear form

$$\left\{ \begin{array}{llll} V_\Gamma & : & \mathcal{C} & \to & \mathbb{R} \\ & & \psi & \mapsto & \displaystyle\int_\Gamma \psi(x, \theta(x)) \end{array} \right.$$

is the 1–**varifold** naturally associated with Γ.

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The continuous linear form on $\mathcal{C}$

$$\left\{ \begin{array}{rcccl} V_\Gamma & : & \mathcal{C} & \to & \mathbb{R} \\ & & \psi & \mapsto & \displaystyle\int_\Gamma \psi(x, \theta(x)) \end{array} \right.$$

is the 1–**varifold** naturally associated with Γ.

With $\psi(x, \omega) = \Pi_\omega \nabla \varphi(x)$, $\boxed{\displaystyle\int_\Gamma \varphi(x)\kappa(x) = -V_\Gamma(\psi)}$ and

- Knowing V_Γ is enough to recover the curvature κ.
- Conversely, it is possible to define a notion of **generalized curvature** for any continuous linear form on $\mathcal{C}$, that is **for ANY** 1–**varifold** in $\mathbb{R}^2$.

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Varifolds ?

Generalized **surface** : *couples a* **weighted spatial** *information and a* **non oriented direction** *information.*

Introduced by Almgren in the 60' : weak notion of surface allowing good compactness properties.

A d–**varifold** is a **Radon measure** in $\mathbb{R}^n \times G_{d,n}$.

The Grassmannian

Grassmannian of d–planes :

$$G_{d,n} = \{d\text{–vector sub-spaces of } \mathbb{R}^n\}.$$

$\rightsquigarrow$ **non-oriented** d–planes.

We identify $P \in G_{d,n}$ with the **orthogonal projector** Π_P onto P, so that $G_{d,n}$ can be seen as a **compact** subset of $M_n(\mathbb{R})$:

$$G_{d,n} \simeq \left\{ A \in M_n(R) \;\middle|\; \begin{array}{l} A^2 = A \\ A^T = A \\ \mathrm{Trace}(A) = d \end{array} \right\}$$

Distance on $G_{d,n}$: $d(P,Q) = \|\Pi_P - \Pi_Q\|$.

Radon measure

$X = \mathbb{R}^n$, $X = \mathbb{R}^n \times G_{d,n}$.

A **Radon measure** in X will equivalently be (thanks to Riesz theorem) :

- a **Borel mesure** on X that takes **finite** values on **compact** sets.
- a **positive linear form** on $\mathrm{C_c(X)}$.

Weak star convergence:

$$\mu_i \overset{*}{\rightharpoonup} \mu \quad \Leftrightarrow \quad \forall \varphi \in \mathrm{C_c(X)},\ \int_{\mathrm{X}} \varphi \, \mathrm{d}\mu_{\mathrm{i}} \to \int_{\mathrm{X}} \varphi \, \mathrm{d}\mu \, .$$

locally metrized for instance by the **flat distance** :

$$\Delta(\mu, \nu) = \sup \left\{ \int_X \varphi \, d\mu - \int_X \varphi \, d\nu \;\middle|\; \begin{array}{l} \varphi \text{ is 1–Lipschitz} \\ \sup_X |\varphi| \leq 1 \end{array} \right\}$$

About Δ

- **Condition** $\sup|\varphi| \leq 1$ **:** for $\varepsilon > 0$ and $\mu = (1+\varepsilon)\delta_0$, $\nu = \delta_0$,

$$\left|\int \varphi\, d\mu - \int \varphi\, d\nu\right| = \varepsilon\,|\varphi(0)| \xrightarrow[\varphi(0)\to+\infty]{} +\infty\,.$$

- **Condition** φ 1**–Lipschitz:** for $\varepsilon > 0$ and $\mu = \delta_\varepsilon$, $\nu = \delta_0$,

$$\left|\int \varphi\, d\mu - \int \varphi\, d\nu\right| = |\varphi(\varepsilon) - \varphi(0)| = 2\,.$$

with $\varphi(\varepsilon) = 1$ and $\varphi(0) = -1$.

- **Localized version :** $B \subset \mathbb{R}^n$

$$\boxed{\Delta_{\mathbf{B}}(\mu,\nu) = \sup\left\{ \int_X \varphi\, d\mu - \int_X \varphi\, d\nu \;\middle|\; \begin{array}{l} \varphi \text{ is } 1\text{–Lipschitz} \\ \sup_X |\varphi| \leq 1 \\ \operatorname{spt}\varphi \subset \mathbf{B} \end{array} \right\}}$$

First examples

1–**Varifold** associated with

- a segment $S \subset \mathbb{R}^n$ whose direction is $P \in G_{1,n}$:

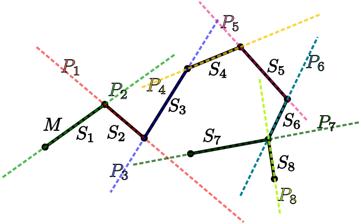
$$V = \mathcal{H}^1_{|S} \otimes \delta_P ,$$

- a union of segments

$$M = \cup_{i=1}^8 S_i \ ,$$

S_i of direction $P_i \in G_{1,n}$:

$$V = \sum_{i=1}^{8} \mathcal{H}^1_{|S_i} \otimes \delta_{P_i} .$$



2–**Varifold** associated with a triangulated surface $M = \bigcup_{T \in \mathcal{T}} T$, where T has direction $P_T \in G_{2,n}$:

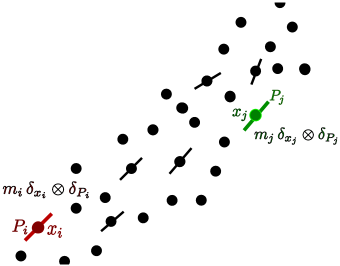
$$V = \sum_{T \in \mathcal{T}} \mathcal{L}^2_{|T} \otimes \delta_{P_T} \ .$$

Point cloud varifolds

d–**Varifold** associated with a point cloud in $\mathbb{R}^n$, that is

- a finite set of **points** $\{x_i\}_{i=1}^N \subset \mathbb{R}^n$,
- weighted by **masses** $\{m_i\}_{i=1}^N \subset \mathbb{R}_+^*$,
- provided with **directions** $\{P_i\}_{i=1}^N \subset G_{d,n}$.

$$V = \sum_{i=1}^{N} m_i \delta_{x_i} \otimes \delta_{P_i}.$$



Regular varifolds

When $M \subset \mathbb{R}^n$ is a d–sub-manifold (or a d–rectifiable set) :

1. $\boldsymbol{\mu}$ measure in $\mathbb{R}^{\boldsymbol{n}}$ supported in $\boldsymbol{M}$: $\boldsymbol{\mu = \mathcal{H}^d_{|M}}$.
2. a family $\boldsymbol{(\nu^x)_{x \in M}}$ **of probabilities in** $\boldsymbol{G_{d,n}}$: $\boldsymbol{\nu^x = \delta_{T_x M}}$.

Then define $\boldsymbol{V = \mu \otimes \nu^x}$ Radon measure in $\mathbb{R}^{\mathbf{n}} \times \boldsymbol{G_{d,n}}$

in the sense: for $\psi \in_x C_c(\mathbb{R}^n \times G_{d,n})$,

$$V(\psi) = \int \psi \, dV = \int_{x \in \mathbb{R}^n} \int_{P \in G_{d,n}} \psi(x, P) \, d\nu^x(P) \, d\mu(x)$$
$$= \int_M \psi(x, T_x M) \, d\mathcal{H}^d(x)$$

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Remember, for Γ*:* $V_\Gamma(\psi) = \displaystyle\int_\Gamma \psi(x, \theta(x))$.

Disintegration

Mass of a varifold V : it's the Radon measure $\|V\|$ in $\mathbb{R}^n$ defined as

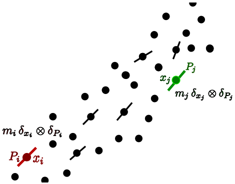
$$\|V\|(A) = V(A \times G_{d,n}) .$$

Disintegration : a d–varifold V can be decomposed as

$$\boldsymbol{V} = \boldsymbol{\mu} \otimes \boldsymbol{\nu_x} \quad \text{with} \quad \boldsymbol{\mu} = \|\boldsymbol{V}\|$$

where for $\|V\|$–a.e. x, ν_x is a probability measure in $G_{d,n}$.

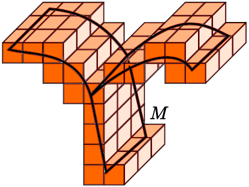
More varifolds ...



Point cloud

$$\sum_j m_j \delta_{x_j} \otimes \delta_{P_j}$$

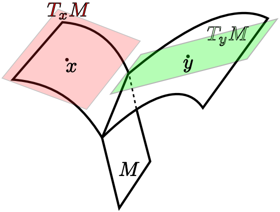
$$\|V\| = \sum_j m_j \delta_{x_j}$$



Volumic approx

$$\sum_{K \in \mathcal{K}} m_K \mathcal{L}^n_{|K} \otimes \delta_{P_K}$$

$$\|V\| = \sum_{K \in \mathcal{K}} m_K \mathcal{L}^n_{|K}$$



Rectifiable

$$\theta(x)\mathcal{H}^d_{|M} \otimes \delta_{T_xM}$$

$$\|V\| = \theta(x)\mathcal{H}^d_{|M}$$

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Divergence theorem

- $X \in \mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$,
- For $P \in G_{d,n}$, $\mathrm{div}_{\mathbf{P}} X = \sum_{k=1}^{n} (\Pi_{\mathbf{P}} \nabla X_k) \cdot e_k$,
- $M \subset \mathbb{R}^n$ closed d–sub-manifold C^2 with **mean curvature vector** H,

$$\boxed{\int_{\mathbf{M}} \mathrm{div}_{\mathbf{T_x M}} X \, \mathbf{d}\mathcal{H}^{\mathbf{d}} = - \int_{\mathbf{M}} \mathbf{H} \cdot X \, \mathbf{d}\mathcal{H}^{\mathbf{d}}.}$$

For $\boldsymbol{V} = \boldsymbol{\mathcal{H}}^{\boldsymbol{d}}_{|\boldsymbol{M}} \otimes \boldsymbol{\delta_{T_x M}}$ the d–varifold associated with M :

$$\int_{\mathbb{R}^{\mathbf{n}} \times \mathbf{G_{d,n}}} \mathrm{div}_{\mathbf{P}} X(x) \, \mathbf{dV}(x, \mathbf{P}) = - \int_{\mathbb{R}^n} \mathbf{H} \cdot X \, \mathbf{d}\|\mathbf{V}\| \, .$$

$\longrightarrow$ **distributional** definition of mean curvature.

First variation of a varifold

First variation of V

$$\delta V : X \in \mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n) \longmapsto \int_{\mathbb{R}^n \times G_{d,n}} \operatorname{div}_P X(x)\, dV(x,P)\,.$$

δV is a distribution of order 1.

$V = \mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ associated with M (C^2 closed):

$$\delta V(X) = -\int_{\mathbf{M}} X \cdot \mathbf{H}\, \mathbf{d}\mathcal{H}^{\mathbf{d}}$$

thus $\boxed{\delta V = -H\, \mathcal{H}^d_{|M} = -H \|V\|}$ **order 0**.

$\rightsquigarrow$ **When δV is of order 0,**

- Riesz : δV is a vector **mesure de Radon**.
- Radon Nikodym : we decompose δV with respect to $\|V\|$:

$$\boxed{\delta V = -H\, \|V\| + (\delta V)_{sing},}$$

$H \in \mathrm{L}^1(\|V\|)$ **generalized curvature**.

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Regularization of the first variation

$\rho \in \mathrm{C}_c^\infty(\mathrm{B}_1(0))$ radial ≥ 0, $\int \rho = 1$ and $\rho_\varepsilon(x) = \dfrac{1}{\varepsilon^n}\rho\left(\dfrac{x}{\varepsilon}\right)$.

Regularized first variation

$$\delta V * \rho_\varepsilon(x) = \frac{1}{\varepsilon^{n+1}} \int_{B_\varepsilon(x) \times G_{d,n}} \nabla_S \, \rho \left(\frac{y-x}{\varepsilon} \right) dV(y,S) \, .$$

Well-defined for **any** varifold.

Case of a **point cloud** : $V = \sum_{i=1}^N m_i \delta_{(x_i, P_i)}$,

$$\boxed{\frac{1}{\varepsilon} \sum_{i=1}^N m_i \rho' \left(\frac{|x_i - x|}{\varepsilon} \right) \frac{\Pi_{P_i}(x_i - x)}{|x_i - x|}} .$$

$\longrightarrow$ **explicit** expression “easy” to implement numerically.

Approximate curvature

Radon-Nikodym derivative of $\delta V * \rho_\varepsilon$ with respect to $\|V\| * \xi_\varepsilon$:

$\delta V * \rho_\varepsilon = -H_\varepsilon(x,V)\,\|V\| * \xi_\varepsilon$ with

$$\boxed{H_\varepsilon(x,V) = -\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \xi_\varepsilon(x)}}.$$

• **Choice of ρ, ξ.** For V associated with M smooth, the leading term in the expansion of $|C\,H_\varepsilon - H|$ around a point is proportional to

$$\int_0^1 (s\rho'(s) + dC\xi(s))s^{d-1}\,ds \underbrace{= 0}_{\text{by PI}} \rightsquigarrow \boxed{\xi(s) = -\frac{s\rho'(s)}{dC} = -\frac{s\rho'(s)}{n}}.$$

Convergence

Let $V = \theta \mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ be a **rectifiable** d–varifold s.t. $\delta V = -\mathbf{H}\|V\| + (\delta V)_{sing}$. is a measure.

- **Consistency** $C = C_{\rho,\xi} > 0$ constant, for $\mathcal{H}^d$–a.e. $x \in M$,

$$C\, H_\varepsilon(x,V) \xrightarrow[\varepsilon\to 0]{} H(x) = -\frac{\delta V}{\|V\|}(x)$$

- **Stability** :
  - $x \in \operatorname{spt}\|V\|$ and $z_i \xrightarrow[i\to\infty]{} 0$
  - $(V_i)_i$ sequence of d–varifolds weak–$*$ converges to V with a localized flat distance around x controlled by $d_i \downarrow 0$. Then, for $\varepsilon_i \downarrow 0$ satisfying $\boxed{\dfrac{d_i + |z_i - x|}{\varepsilon_i^2} \xrightarrow[i\to\infty]{} 0}$,

$$|H_{\varepsilon_i}(z_i, V_i) - H_{\varepsilon_i}(x,V)| = O_{i\to\infty}\left(\frac{d_i + |z_i - x|}{\varepsilon_i^2}\right)$$

Case of a point cloud varifold

Let $V = \sum_{i=1}^{N} m_i \delta_{(x_i, P_i)}$,

$$\boxed{H_\varepsilon(x, V) = - \frac{\dfrac{1}{\varepsilon} \displaystyle\sum_{i=1}^{N} m_i \rho' \left(\frac{|x_i - x|}{\varepsilon} \right) \frac{\Pi_{P_i}(x_i - x)}{|x_i - x|}}{\displaystyle\sum_{i=1}^{N} m_i \xi \left(\frac{|x_i - x|}{\varepsilon} \right)} .}$$

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Mean curvature

Code C++ using `nanoflann` library.

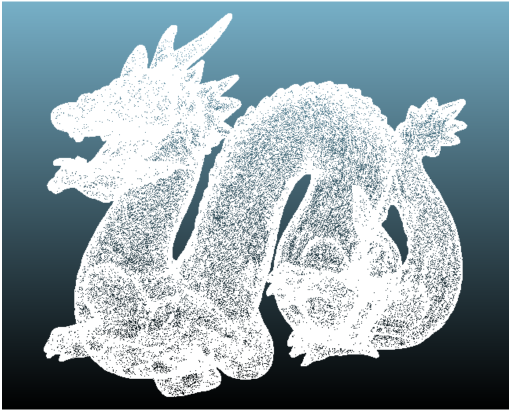

Figure: Intensity of mean curvature from blue (zero) to red through white $\varepsilon = 0.007$ *for a diameter* 1

Mean curvature

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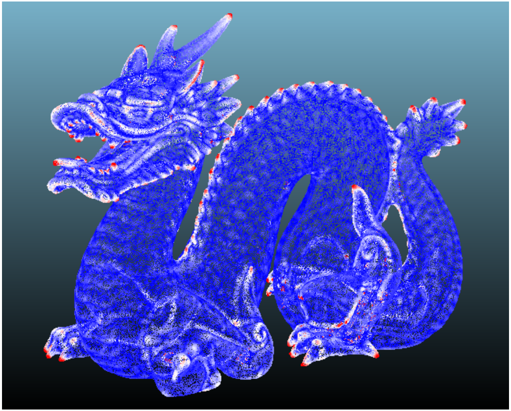


Figure: Intensity of mean curvature from blue (zero) to red through white $\varepsilon = 0.007$ *for a diameter* 1

Gaussian curvature

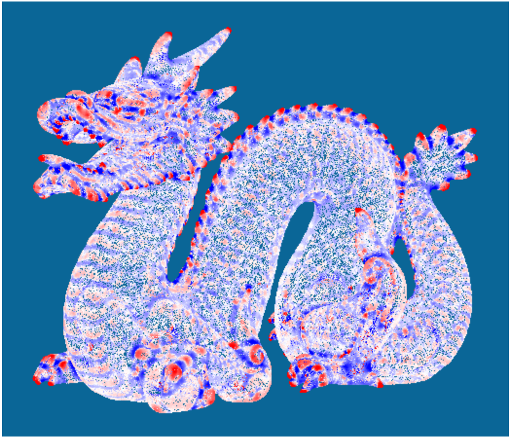


Figure: Gaussian curvature, negative (blue), zero (white), positive (red)

Sharp features

Figure: $|k_1| + |k_2|$ from blue (zero) to red (high) through white

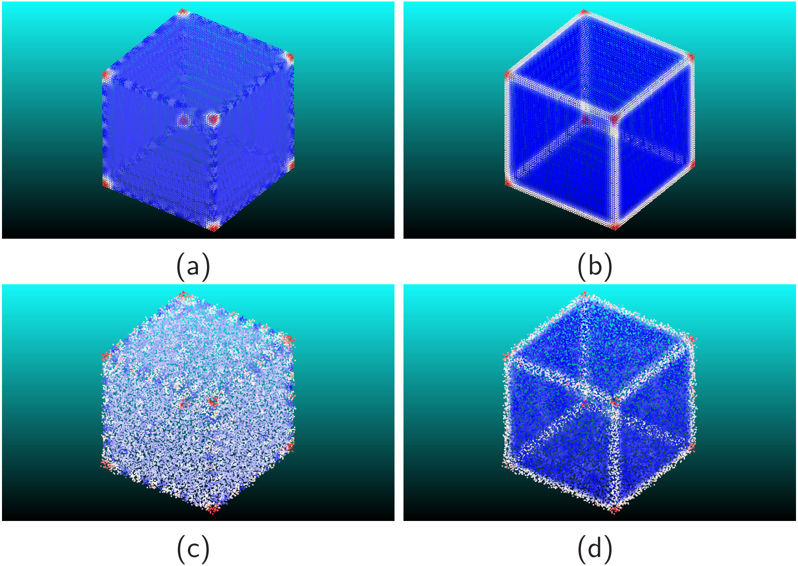
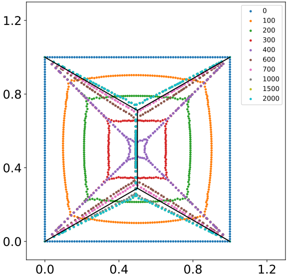
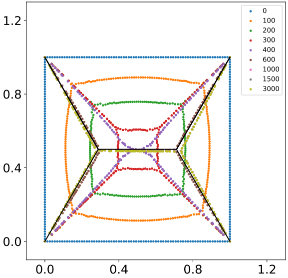


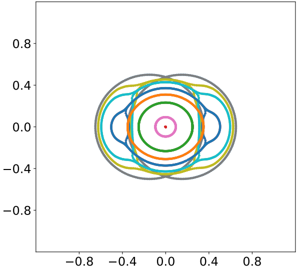
Figure: Left: Gaussian curvature, Right: $|k_1| + |k_2|$, Top: without noise, Bottom: with white noise


1.2
0.8
0.4
0.0
0.0
0.4
0.8
1.2
0
100
200
300
400
600
700
1000
1500
2000

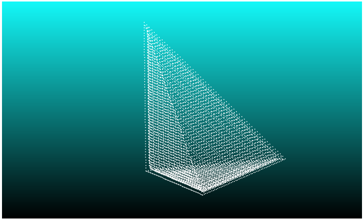
(a)


1.2
0.8
0.4
0.0
0.0
0.4
0.8
1.2
0
100
200
300
400
600
1000
1500
3000

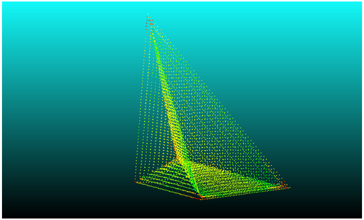
(b)



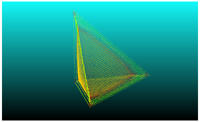
(c)



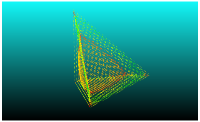
(a) Step 1



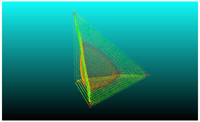
(b) Step 101



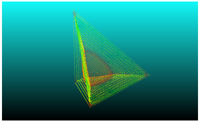
(c) Step 13



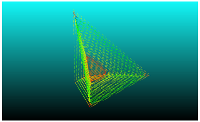
(d) Step 25



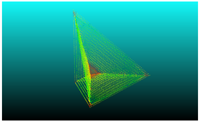
(e) Step 37



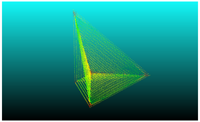
(f) Step 49



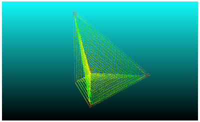
(g) Step 61



(h) Step 73

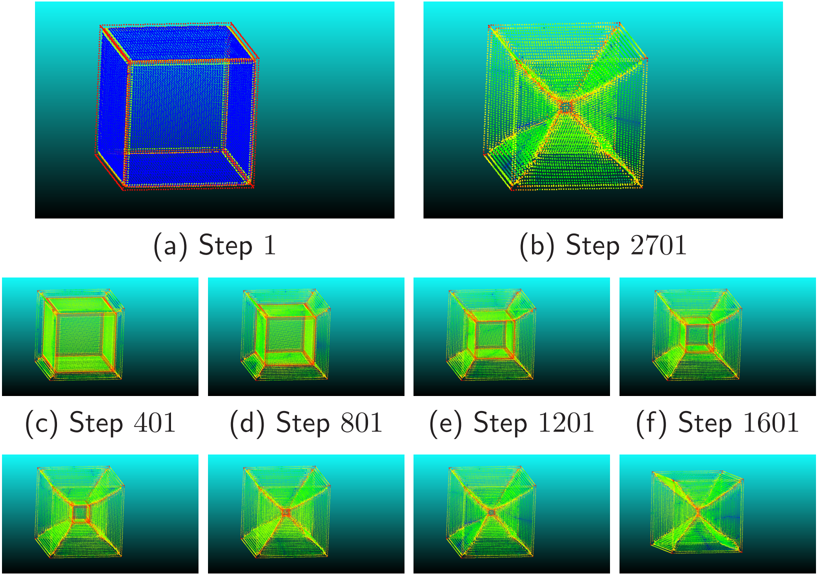


(i) Step 85



(j) Step 97

Figure: Evolution of a tetrahedron whose edges are fixed, discretized with $N = 6052$ points and for a time–step $\tau = 0.005$.



(a) Step 1 (b) Step 2701

(c) Step 401 (d) Step 801 (e) Step 1201 (f) Step 1601

(g) Step 2001 (h) Step 2401 (i) Step 2701 (j) Step 2701

Figure: Evolution of a cube whose edges are fixed, discretized with $N = 18600$ points and for a time–step $\tau = 0.01$.

Thanks for your attention !

Plan

A simple example

What is a varifold ?

Generalized curvature of a varifold

Approximate curvature

Numerical illustrations

References

Second fundamental form

À propos des varifolds

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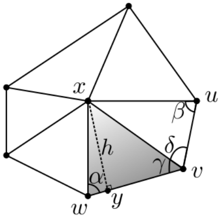
Second fundamental form

$$\frac{\frac{d}{n}\sum_{l=1}^{N} m_l \rho'\left(\frac{|x_{l_0}-x_l|}{\varepsilon}\right)\frac{P_l(x_{l_0}-x_l)}{|x_{l_0}-x_l|}\cdot\frac{1}{2}\left(\left(P_l-P_{l_0}\right)_{jk}e_i+\left(P_l-P_{l_0}\right)_{ik}e_j-\left(P_l-P_{l_0}\right)_{ij}e_k\right)}{\sum_{l=1}^{N} m_l \xi\left(\frac{|x_{l_0}-x_l|}{\varepsilon}\right)}.$$

Link with the Cotangent formula

- Let $\mathcal{T} = (\mathcal{F}, \mathcal{E}, \mathcal{V})$ be a triangulation in $\mathbb{R}^3$, where $\mathcal{V} \subset \mathbb{R}^3$ is the set of vertices, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges and $\mathcal{F}$ is the set of triangle faces.
- 2–varifold
$$V_{\mathcal{T}} = \sum_{F \in \mathcal{F}} \mathcal{H}^2_{|F} \otimes \delta_{P_F} \ ,$$
- The nodal function φ_v, $v \in \mathcal{V}$, associated with $\mathcal{T}$ is defined by $\varphi_v(v) = 1$, $\varphi_v(w) = 0$ for $w \in \mathcal{V}$, $w \neq v$ and φ_v affine on each face $F \in \mathcal{F}$.



$$\delta V_{\mathcal{T}}(\widehat{\varphi_x}) = -\frac{1}{2} \sum_{v \in \mathcal{V}(x)} (\cot \alpha_{xv} + \cot \beta_{xv})\,(v - x).$$

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Second fundamental form

Second fundamental form

Back to the divergence theorem:

- $M \subset \mathbb{R}^n$ C^2 closed;
- $P(x) = (P_{jk}(x))_{jk} \in M_n(\mathbb{R})$ orthogonal projection onto $T_x M$;
- $\varphi \in \mathrm{C}^1_\mathrm{c}(\Omega \times \mathrm{M_n}(\mathbb{R}))$,

$X(x) := \varphi(x, P(x))e_i$.

Divergence theorem $0 = \displaystyle\int_M \mathrm{div}_P(PX)$ leads to a weak formulation of the second fundamental form through

$$A_{ijk} = (P(x)\nabla P_{jk}(x))_i \ :$$

$$-\int_M (P(x)\nabla_x \varphi)_i \, d\mathcal{H}^d =$$

$$\int_M \Big(\sum_{j,k} \underbrace{(P(x)\nabla P_{jk}(x))_i}_{=:A_{ijk}} D_{jk}\varphi + \sum_q \underbrace{(P(x)\nabla P_{iq}(x))_q}_{A_{qiq}} \varphi \Big) \, d\mathcal{H}^d$$

Second fundamental form

Back to the divergence theorem:

- $M \subset \mathbb{R}^n$ C^2 closed;
- $P(x) = (P_{jk}(x))_{jk} \in M_n(\mathbb{R})$ orthogonal projection onto $T_x M$;
- $\varphi \in \mathrm{C}^1_\mathrm{c}(\Omega \times \mathrm{M_n}(\mathbb{R}))$,

$X(x) := \varphi(x) P_{jk}(x) e_i$.

Divergence theorem $0 = \int_M \mathrm{div}_P(PX)$ leads to a weak formulation of the second fundamental form through

$$A_{ijk} = (P(x)\nabla P_{jk}(x))_i \ :$$

$$-\int_M (P(x)\nabla\varphi)_i \, d\mathcal{H}^d = \int_M \left(A_{ijk}\varphi + P_{jk}(x) \sum_q A_{qiq}\varphi \right) d\mathcal{H}^d$$

It is then possible to define, for $i, j, k = 1 \dots n$,

$$\delta_{ijk} V : X \in \mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n) \longmapsto \int_{\mathbb{R}^n \times G_{d,n}} \mathbf{S_{jk}} \operatorname{div}_S X(x) \, dV(x, S) \cdot e_i \, .$$

When those distributions are **Radon measures**, we define β_{ijk} s.t.

$$\delta_{ijk} V = -\beta_{ijk} \|V\| + (\delta_{ijk} V)_{sing} \, .$$

And for $\|V\|$–a.e. x, we can define A_{ijk} as the pointwise solution of the linear system with n^3 equations :

$$\boxed{A_{ijk} + c_{jk} \sum_{q=1}^{n} A_{qiq} = \beta_{ijk}}$$

with $c_{jk}(x) = \int_{G_{d,n}} S \, d\nu_x(S)$ and $V = \|V\| \otimes \nu_x$.