Variational Methods and Optimization in Imaging
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Stable Models and Algorithms for Backward Diffusion Evolutions

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joint work with

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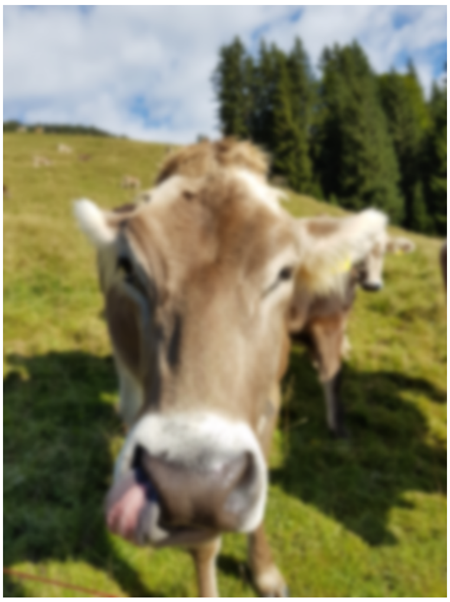
Introduction

- Forward diffusion equations blur or smooth images.
  $\Longrightarrow$ attempts to invert these evolutions for deblurring or sharpening images



$\leftrightarrow$

Introduction (2)

Problems

- Backward diffusion is typically regarded as ill-posed:
  - Solution does not exist for non-smooth initial data.
  - If it exists, it is highly sensitive w.r.t. perturbations.
- Thus, many researchers refrain from using backward diffusion.

Goals

- show how these problems can be
  - handled by sophisticated numerics
  - or circumvented by smart modelling
- demonstate these principles with two prototypical applications:
  - advanced numerics for the FAB diffusion of Gilboa et al. 2002
  - novel convex model for backward diffusion (Bergerhoff et al. 2018)

Outline

Outline

- FAB Diffusion
  - Continuous Model
  - Explicit Scheme
  - Efficient Numerics
  - Experiments
- Backward Diffusion with Convex Energy
  - Model and Theory
  - Numerical Algorithm
  - Experiment
- Conclusions

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FAB Diffusion: Continuous Model

The Perona–Malik Filter (1990)

- Consider open image domain $\Omega \subset \mathbb{R}^2$ and some bounded image $f : \Omega \to \mathbb{R}$.
- Create family of filtered versions $u(\boldsymbol{x}, t)$ of $f(\boldsymbol{x})$ as solution of

$$\begin{aligned} \partial_t u &= \operatorname{div}\left(g(|\boldsymbol{\nabla} u|^2)\, \boldsymbol{\nabla} u\right) && \text{on } \Omega \times (0,\infty), \\ u(\boldsymbol{x},0) &= f(\boldsymbol{x}) && \text{on } \Omega, \\ \boldsymbol{n}^\top \boldsymbol{\nabla} u &= 0 && \text{on } \partial\Omega \times (0,\infty), \end{aligned}$$

  where $\boldsymbol{n}$ denotes the outer normal vector to the image boundary $\partial\Omega$.
- diffusivity g is monotonically decreasing *positive* function of $|\boldsymbol{\nabla} u|^2$
- smoothes within flat regions and enhances edges between them
- gradient descent of a possibly nonconvex but monotone energy

$$E(u) = \int_\Omega \Psi(|\boldsymbol{\nabla} u|^2)\, d\boldsymbol{x}$$

  where the penaliser (potential) $\Psi(|\boldsymbol{\nabla} u|^2)$ satisfies $\Psi'(|\boldsymbol{\nabla} u|^2) = g(|\boldsymbol{\nabla} u|^2)$.

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FAB Diffusion: Continuous Model (2)

Forward-and-Backward (FAB) Diffusion

(Gilboa / Sochen / Zeevi 2002)

- goal: stronger sharpening than classical Perona-Malik filters
- equip Perona-Malik diffusion

$$\partial_t u \;=\; \operatorname{div}\left(g(|\boldsymbol{\nabla} u|^2)\,\boldsymbol{\nabla} u\right)$$

  with a diffusivity that takes positive and *negative* values.
- fairly mild assumptions in this talk:

$$g \in \mathrm{C}^1[0,\infty), \quad g(0) = c_1 > 0, \quad g(\,.\,) \ge -c_2 \quad \text{with} \quad c_1 > c_2 \ge 0.$$

- corresponds to *nonconvex and nonmonotone* potential $\Psi(|\boldsymbol{\nabla} u|^2)$

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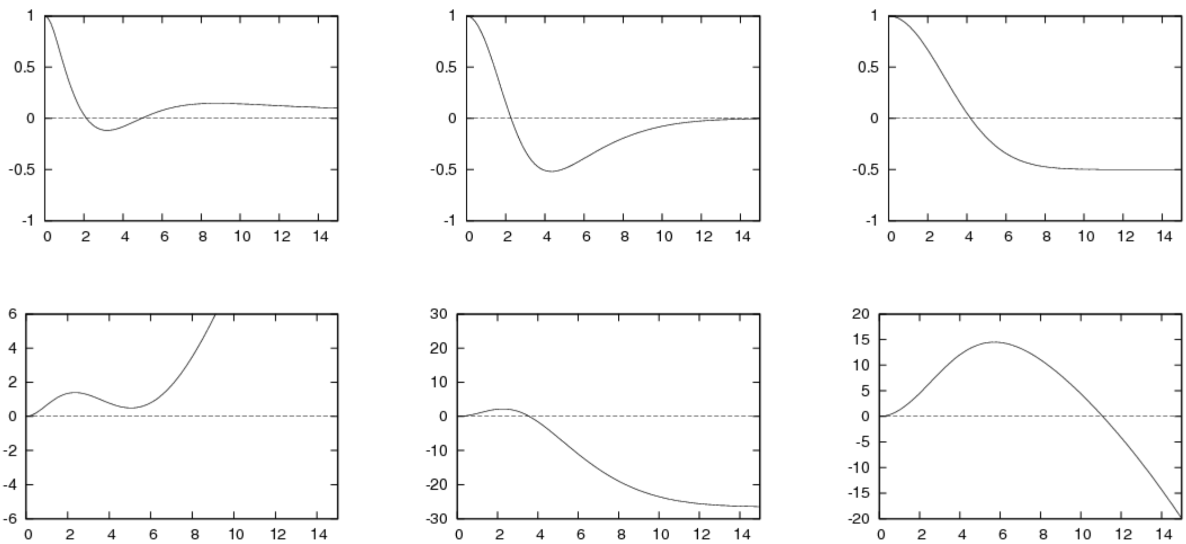
FAB Diffusion: Continuous Model (3)

How Unpleasant can this Become ?

Diffusivity $g(s^2)$ Diffusivity $g(s^2)$ Diffusivity $g(s^2)$



Potential $\Psi(s^2)$ Potential $\Psi(s^2)$ Potential $\Psi(s^2)$

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Theoretical Results so Far

- cannot be covered by standard theory for diffusion filters (W. 1998)
- no continuous well-posedness theory
- Gilboa / Sochen / Zeevi (IEEE TIP 2002):
  standard implementations violate extremum principle
- Gilboa / Sochen / Zeevi (JMIV 2004):
  experimental stabilisation with a fidelity term and biharmonic regularisation

Can we establish a fully discrete theory ?

Does this lead to practical algorithms for images?

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41

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FAB Diffusion: Explicit Scheme

Goal

- establish comprehensive theory for an explicit discretisation of FAB diffusion

Explicit Scheme

Explicit finite difference discretisation of diffusion equation

$$\partial_t u \;=\; \partial_x \Big(g(|\boldsymbol{\nabla} u|^2)\, \partial_x u \Big) \;+\; \partial_y \Big(g(|\boldsymbol{\nabla} u|^2)\, \partial_y u \Big)$$

in some inner pixel (i,j) at time level k yields the scheme

$$\begin{aligned}
\frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\tau} \;=\;& \frac{1}{h_1} \left(\frac{g_{i+1,j}^k + g_{i,j}^k}{2}\, \frac{u_{i+1,j}^k - u_{i,j}^k}{h_1} - \frac{g_{i,j}^k + g_{i-1,j}^k}{2}\, \frac{u_{i,j}^k - u_{i-1,j}^k}{h_1} \right) \\
&+ \frac{1}{h_2} \left(\frac{g_{i,j+1}^k + g_{i,j}^k}{2}\, \frac{u_{i,j+1}^k - u_{i,j}^k}{h_2} - \frac{g_{i,j}^k + g_{i,j-1}^k}{2}\, \frac{u_{i,j}^k - u_{i,j-1}^k}{h_2} \right)
\end{aligned}$$

with grid sizes h_1, h_2 and time step size τ.

Where Do Problems Arise ?

- The *standard discretisation* of $g(|\boldsymbol{\nabla} u|^2)$ is given by

$$g_{i,j}^k \;:=\; g \Bigg(\underbrace{\left(\frac{u_{i+1,j}^k - u_{i-1,j}^k}{2\,h_1} \right)^2 + \left(\frac{u_{i,j+1}^k - u_{i,j-1}^k}{2\,h_2} \right)^2}_{\text{can be positive at extrema}} \Bigg).$$

- It can create negative diffusivities in extrema, which give rise to instabilities.

Is There a Remedy ?

- A *nonstandard discretisation* produces a vanishing gradient in extrema:

$$\begin{aligned}
g_{i,j}^k \;:=\; g \Bigg(& \max \left(\frac{u_{i+1,j}^k - u_{i,j}^k}{h_1} \cdot \frac{u_{i,j}^k - u_{i-1,j}^k}{h_1},\; 0 \right) \\
& + \max \left(\frac{u_{i,j+1}^k - u_{i,j}^k}{h_2} \cdot \frac{u_{i,j}^k - u_{i,j-1}^k}{h_2},\; 0 \right) \Bigg).
\end{aligned}$$

- same quadratic consistency order as standard discretisation

FAB Diffusion: Explicit Scheme (3)

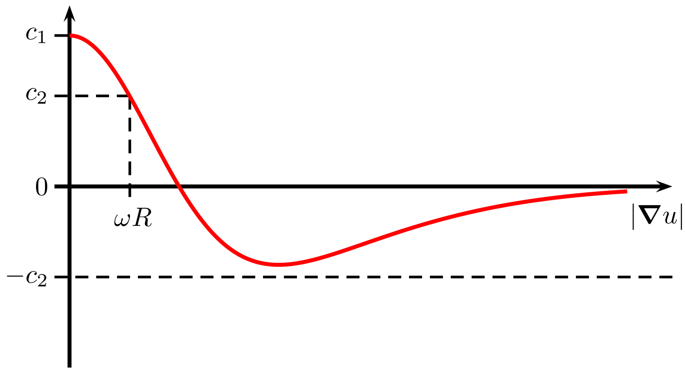
Two Technical Definitions

- The grey values of $\boldsymbol{f} = (f_i) \in \mathbb{R}^N$ are restricted to a finite interval of length

$$R \;:=\; \max_i f_i - \min_i f_i .$$

- Since g is continuous and $c_1 > c_2 \geq 0$, there exists a constant $\omega > 0$ such that

$$g(s^2) > c_2 \qquad \forall\, s \in (0,\, \omega R).$$



FAB Diffusion: Explicit Scheme (4)

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Theorem [Theory for the Explicit FAB Scheme]

With the preceding assumptions and definitions, consider the explicit scheme for FAB diffusion with nonstandard discretisation.

If the time step size τ satisfies

$$\tau \;\leq\; \frac{\omega^2\, h_1^4\, h_2^4}{2\, c_1 \cdot \left(h_1^2 + h_2^2\right) \cdot \left(\omega^2\, h_1^2\, h_2^2 + h_1^2 + h_2^2\right)}\,,$$

then this scheme has the following properties:

- **Well-Posedness**

  For every $k \in \mathbb{N}_0$, the solution $\boldsymbol{u}^{k+1}$ depends in a continuous way on perturbations of the initial image $\boldsymbol{f}$.

- **Average Grey Value Invariance**

$$\frac{1}{N}\sum_{j=1}^{N} u_j^k \;=\; \frac{1}{N}\sum_{j=1}^{N} f_j \;=:\; \mu \qquad \forall\, k \in \mathbb{N}_0 .$$

- **Maximum-Minimum Principle**

$$\min_j f_j \;\le\; u_i^k \;\le\; \max_j f_j \qquad \forall\, i,\ \forall\, k \in \mathbb{N}_0.$$

- **Lyapunov Sequence**

$$V^k \;:=\; \max_j\, u_j^k \;-\; \min_j\, u_j^k$$

is a Lyapunov sequence: decreasing in k and bounded from below.

- **Convergence to a Constant Steady State**

$$\lim_{k\to\infty} u_i^k \;=\; \mu \qquad \forall\, i.$$

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M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41

FAB Diffusion: Efficient Numerics

- Our explicit FAB scheme with nonstandard discretisation has a clean theory.
- However, it must satisfy a severe time step size restriction:
  This can lead to impractically small time steps: $10^{-6},\ldots,10^{-5}$
- **Reason:**
  - based on worst case *a priori* estimates
  - restrictions are not needed everywhere and at all time steps
- **Remedy:**
  - Replace pessimistic *a priori* estimates by realistic *a posteriori* estimates.
  - Act as locally/adaptive as possible in space and time.
  - Realisation: two-pixel interactions.

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Motivation for Two-Pixel Interactions

- most local diffusion interaction that respects a conservation law.

$$\boxed{u_{i,j}} \longleftrightarrow \boxed{u_{i+1,j}} \qquad \text{or} \qquad \begin{array}{c} \boxed{u_{i,j}} \\ \updownarrow \\ \boxed{u_{i,j+1}} \end{array}$$

- The explicit scheme performs four two-pixel interactions simultaneously:

$$u_{i,j}^{k+1} = u_{i,j}^{k} + \tau \left(\frac{g_{i+1,j}^{k} + g_{i,j}^{k}}{2} \cdot \frac{u_{i+1,j}^{k} - u_{i,j}^{k}}{h_1^2} - \frac{g_{i,j}^{k} + g_{i-1,j}^{k}}{2} \cdot \frac{u_{i,j}^{k} - u_{i-1,j}^{k}}{h_1^2} \right.$$
$$\left. + \frac{g_{i,j+1}^{k} + g_{i,j}^{k}}{2} \cdot \frac{u_{i,j+1}^{k} - u_{i,j}^{k}}{h_2^2} - \frac{g_{i,j}^{k} + g_{i,j-1}^{k}}{2} \cdot \frac{u_{i,j}^{k} - u_{i,j-1}^{k}}{h_2^2} \right)$$

Basic Idea behind Two-Pixel Scheme

- Decouple explicit scheme into *sequential (asynchroneous) two-pixel interactions.* Then stability follows trivially from the stability of each interaction.
- In each interaction, choose the largest time step ensuring two *stability criteria:*
  - For a positive diffusivity, the order of grey values must not be flipped.
  - For a negative diffusivity, we have a non-extremal pixel. It must not become larger/smaller than its largest/smallest neighbour.
- This gives highly localised time step size restrictions in space and time.
- To avoid directional bias, *randomise the order* of two-pixel interactions.
- Introduce *sync times* at which each all pixels reach the same time level. Use e.g. the stability bounds of an explicit forward diffusion scheme.
- selection probability for two-pixel interaction: proportional to remaining time until synchronisation.

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M I A
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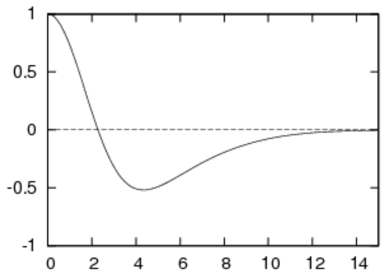
FAB Diffusion: Experiments

Technical Details



- We use the FAB diffusivity

$$g(s^2) \;=\; 2\,\exp\left(-\frac{\kappa^2 \ln 2}{\kappa^2-1}\cdot\frac{s^2}{\lambda^2}\right) \;-\; \exp\left(-\frac{\ln 2}{\kappa^2-1}\cdot\frac{s^2}{\lambda^2}\right)$$

  with contrast parameter $\lambda > 0$ and stretching parameter $\kappa > 1$.

- Run times refer to a C implementation on a single core.
  Hardware: Intel Core i5–5200U CPU at 2.20 GHz.

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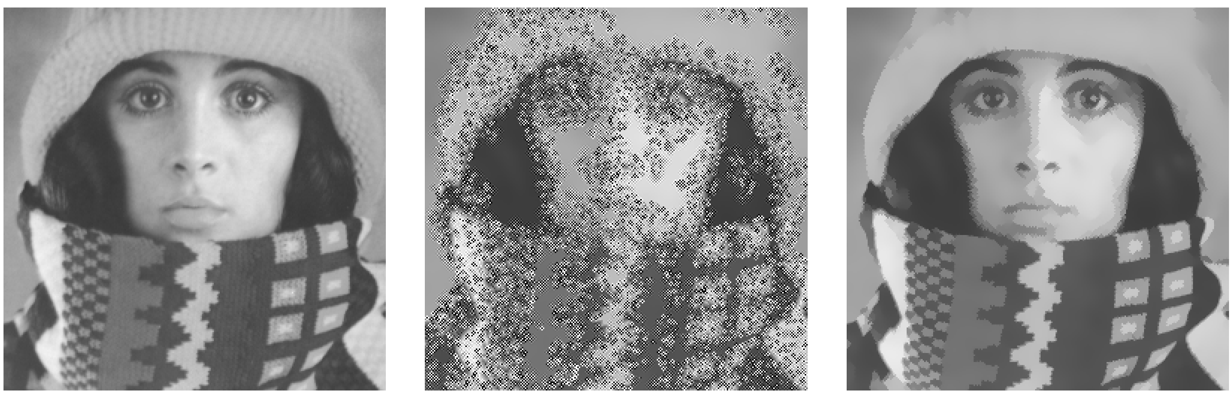
Stability: Standard versus Nonstandard Discretisation

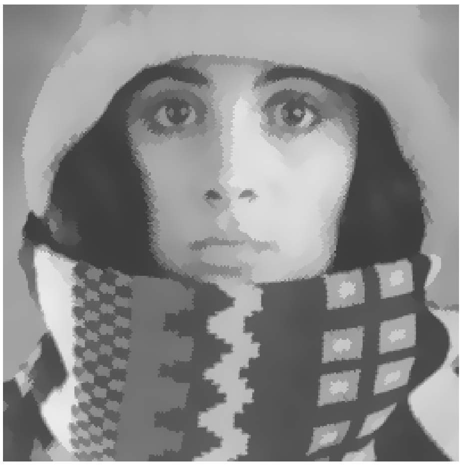
original image | standard discretisation | nonstandard discretisation



Left: Original image, 256×256 pixels. **Middle:** Using the standard discretisation within FAB diffusion creates instabilities. Diffusivity parameters $\lambda = 4$ and $\kappa = 2.5$, time step size $\tau = 10^{-5}$, and stopping time $t = 10$. Values outside $[0, 255]$ have been cropped in the visualisation. **Right:** The same experiment with nonstandard discretisation does not give rise to instabilities.

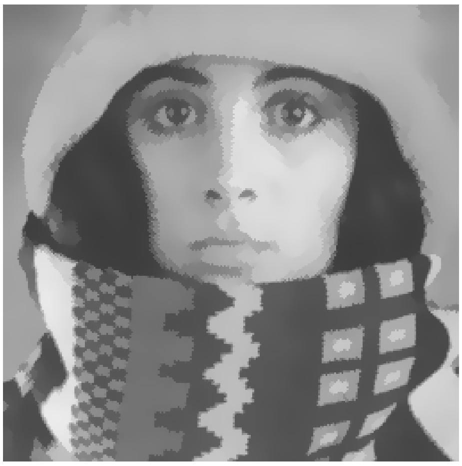
Efficiency: Explicit Scheme versus Two-Pixel Scheme

explicit scheme



10^6 iterations with $\tau = 10^{-5}$
run time: 66 min

two-pixel scheme



100 sync steps with $\tau_{\max} = 0.1$
average time step size: 0.0991
run time: 7.73 s

- Both schemes give results of comparable visual quality.
- The two-pixel scheme is 544 times faster.

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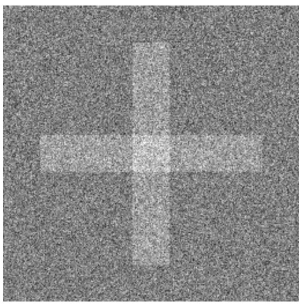
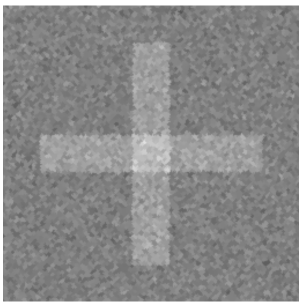
Scale-Space Behaviour

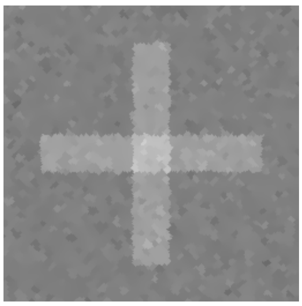
$t = 0$



$t = 6$



$t = 40$



$t = 200$

$t = 800$

$t = 3000$

Scale-space behaviour of FAB diffusion with $\lambda = 2$ and $\kappa = 2.5$. All computations use the two-pixel scheme with sync time step size $\tau_{\max} = 0.1$.

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FAB Diffusion: Experiments (5)

Robustness under Noise

test image *barbara*, 512×512 pixels

Gaussian noise with $\sigma = 50$, truncated outside $[0, 255]$

FAB diffusion, 2-pixel scheme ($\lambda = 2,\ \kappa = 2.5,\ t = 120$)

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M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41

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Backward Diffusion with Convex Energy: Model and Theory

Problem

- common stabilisation constraints for backward diffusion:
  - forward or zero diffusion at extrema:
    $\Longrightarrow$ requires sophisticated numerical schemes
  - fidelity term:
    $\Longrightarrow$ dependence on fidelity weights and input data

Remedy: Smarter Modelling

- novel backward diffusion model with *globally negative diffusivities*
- results from gradient descent of a *convex (!) energy*
- stabilisation through reflecting boundary conditions in the *co-domain*
- will allow standard numerics

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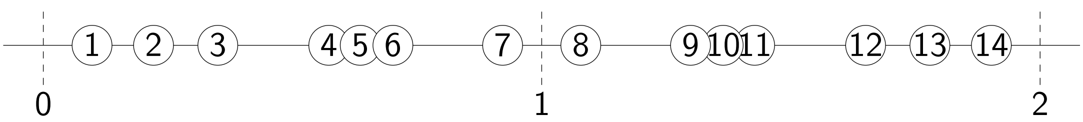
Model and Theory

- vector $\boldsymbol{v} = (v_1, \ldots, v_N)^\top \in (0,1)^N$ with N distinct 1D particle positions v_i
- extend $\boldsymbol{v}$ with additional particles $v_{N+1}, \ldots, v_{2N}$:
  mirror all positions $v_1, \ldots, v_N$ at the right domain boundary 1



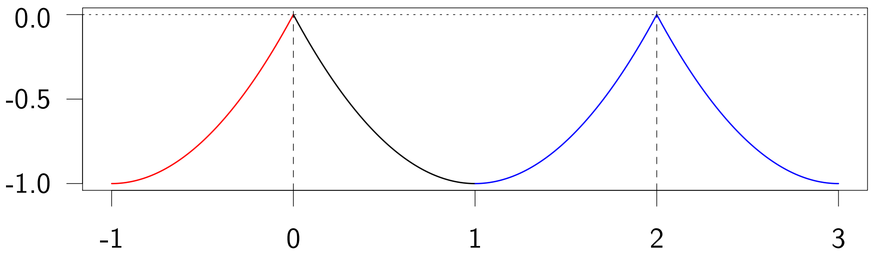
Exemplary setup for $N = 7$ particles.

- As our *baseline model*, we consider an energy with nonlocal interactions:

$$E(\boldsymbol{v}) \;=\; \frac{1}{2} \cdot \sum_{i=1}^{2N} \sum_{j=1}^{2N} \Psi\big((v_j - v_i)^2\big)\,,$$

where $\Psi : \mathbb{R}_0^+ \to \mathbb{R}$ is a specific repulsive penaliser function.

Which Repulsive Penaliser $\Psi(s^2)$ Do We Choose?



Penaliser $\Psi(s^2) = (s-1)^2 - 1$ for $s \in [0,1]$,
extended to $[-1,1]$ by symmetry and to $[-1,3]$ by periodicity.

- decreasing and *strictly convex* for $s \in [0,1]$
- extension to $[-1,1]$ by symmetry
- extension to $\mathbb{R}$ by periodicity
- differentiable everywhere except at even integers

What is the Gradient Descent Evolution of Our Model ?

- Our discrete energy function

$$E(\boldsymbol{v}) \;=\; \frac{1}{2} \cdot \sum_{i=1}^{2N} \sum_{j=1}^{2N} \Psi\big((v_j - v_i)^2\big)$$

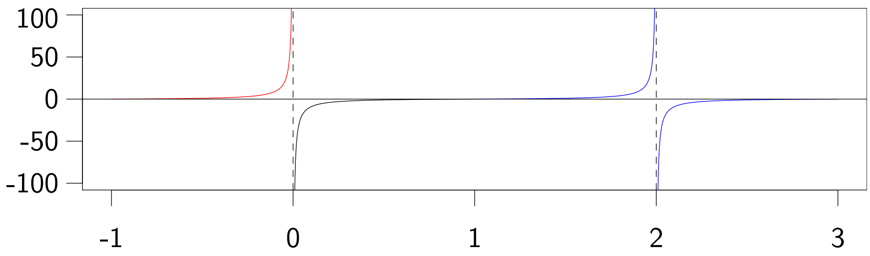
  yields the gradient descent evolution

$$\partial_t v_i \;=\; \sum_{\substack{j=1 \\ j \neq i}}^{2N} \Psi'\big((v_j - v_i)^2\big)\,(v_j - v_i) \;=:\; \sum_{\substack{j=1 \\ j \neq i}}^{2N} \Phi(v_j - v_i)\,.$$

- This is a *space-discrete nonlocal diffusion* model with
  - *diffusivity function* $\Psi'(s^2)$
  - *flux function* $\Phi(s) := \Psi'(s^2)\,s$

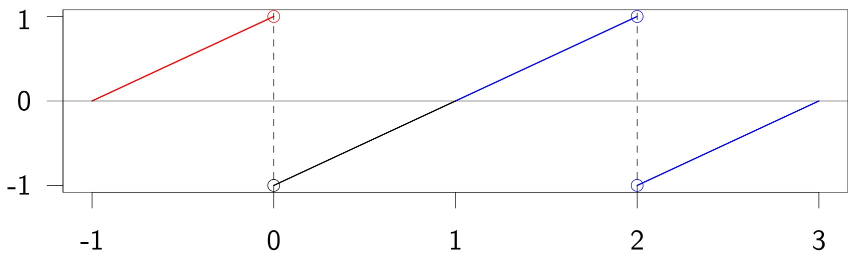
Diffusivity and Flux Functions for $\Psi(s^2) = (s-1)^2 - 1$

- The diffusivity function $\Psi'(s^2)$ is a *shifted backward TV* diffusivity in $(0, 1]$:



For $s \in (0, 1]$, the diffusivity satisfies $\Psi'(s^2) = 1 - \frac{1}{s}$.

- The flux function $\Phi(s) := \Psi'(s^2)\, s$ is negative everywhere in $(0, 1)$.



For $s \in (0, 2)$, the flux function is given by $\Phi(s) = s - 1$.

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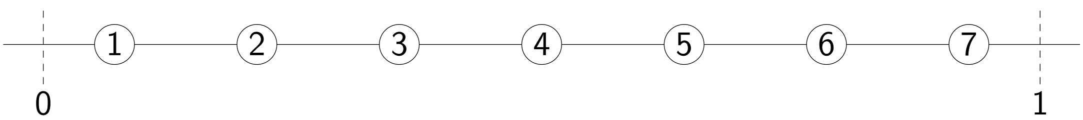
Which Properties can be Proven for the Baseline Model ?

- Well-posedness:
  The strictly convex energy has a unique minimiser.
  The gradient descent evolution depends continuously on the input data.
- Particles can never reach the domain boundaries.
- Particles cannot occupy the same position.
- Strict global minimum of $E(\boldsymbol{v})$:
  Convergence to equilibrium point $\boldsymbol{v}^*$ for $t \to \infty$
- Steady-state solution $\boldsymbol{v}^*$ explicitly known:
  Particles are distributed equidistantly in $(0, 1)$.



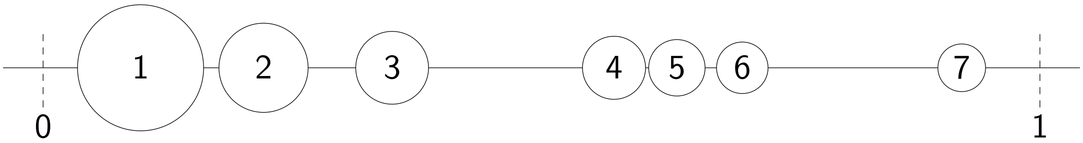
Steady state for $N = 7$ particles.

Can we change the model such that we get a more interesting steady state?

Generalised Model with Weights

- We can assign fixed nonnegative weights $w_1, \ldots, w_N$ to our N particles.
- They are mirrored and periodically extened like the particles.
- With $p_i := \sqrt{w_i} \cdot v_i$ the energy for the *generalised model* reads

$$E(\boldsymbol{p}, \boldsymbol{w}) \;=\; \frac{1}{2} \cdot \sum_{i=1}^{2N} \sum_{j=1}^{2N} w_i \cdot w_j \cdot \Psi\left(\left(\frac{p_j}{\sqrt{w_j}} - \frac{p_i}{\sqrt{w_i}}\right)^2\right).$$



Exemplary setup for $N = 7$ particles with $w_i = {}^1\!/_i$.

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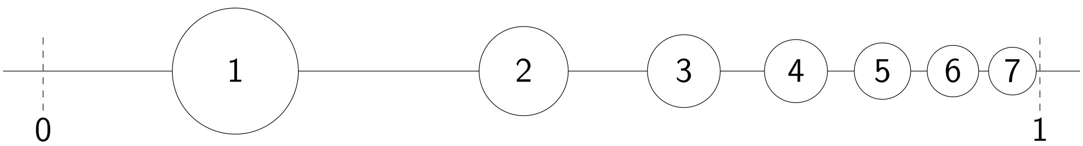
Which Properties can be Proven for the Generalised Model ?

- same as for the baseline model
- Only difference: The steady-state p_i^* is no longer equidistantly distributed:

$$p_i^* \;=\; \sqrt{w_i} \cdot \frac{\sum\limits_{j=1}^{i} w_j - \frac{1}{2} w_i}{\sum\limits_{j=1}^{N} w_j}, \qquad i = 1, \ldots, N$$



Steady state for $N = 7$ particles with $w_i = {}^1\!/_i$.

Outline

- FAB Diffusion
  - Continuous Model
  - Explicit Scheme
  - Efficient Numerics
  - Experiments
- Backward Diffusion with Convex Energy
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- Conclusions

Backward Diffusion with Convex Energy: Numerical Algorithm

- A simple *explicit* time discretisation of the N particle evolution works well !
- time step size restriction involves *Lipschitz constant* L_Φ of flux $\Phi(s)$, $s \in (0, 2)$:

$$0 < \tau < \frac{1}{2\, L_\Phi \sum\limits_{i=1}^{N} w_i}$$

- algorithm reproduces the stability properties of time-continuous evolution:
  - Particles cannot reach the domain boundary.
  - Particles do not change their order.
- no problems due to negative diffusivities

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Backward Diffusion with Convex Energy: Experiment

Goal

- enhance global contrast of a digital greyscale image
  $f: \{1, \ldots, n_x\} \times \{1, \ldots, n_y\} \to (0, 1)$

Algorithm Using our Generalised Model

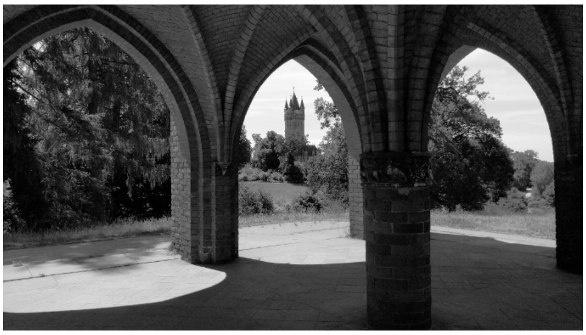
- If a grey value i appears n times, set its weight to $w_i := n$.
- Evolve the explicit scheme for some given time t,
  or use the known steady state solution for $t \to \infty$.
- Map the original grey values to the processed ones to get the enhanced image.

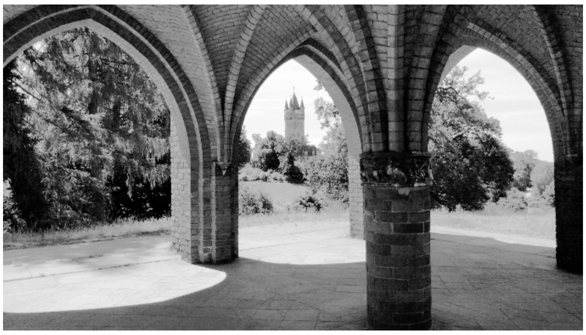
M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41

Backward Diffusion with Convex Energy: Experiment (2)

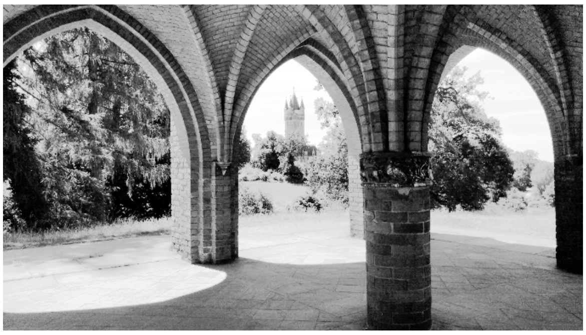
original image

small stopping time





large stopping time



steady state

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Outline

Outline

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Conclusions

- Backward diffusion can be tamed by sophisticated numerics or smart models.
- important components of sophisticated numerics:
  - nonstandard discretisations to preserve continuous qualities
  - two-pixel interactions to achieve highest locality
  - local time step size adaptations to increase efficiency
  - asynchronous splittings for simple stability guarantees
  - randomisation to avoid directional bias
- features of well-posed smart models:
  - gradient descent of strictly convex energies
  - stabilisation through range constraints
  - allow simple numerical schemes

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Download: `www.mia.uni-saarland.de/weickert/publications.shtml`

Postdoc Opening in Inpainting-Based Image Compression

- funded by ERC Advanced Grant
- required: expertise in optimisation, variational methods, or PDEs
- e-mail CV: `weickert@mia.uni-saarland.de`

Thank you !

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