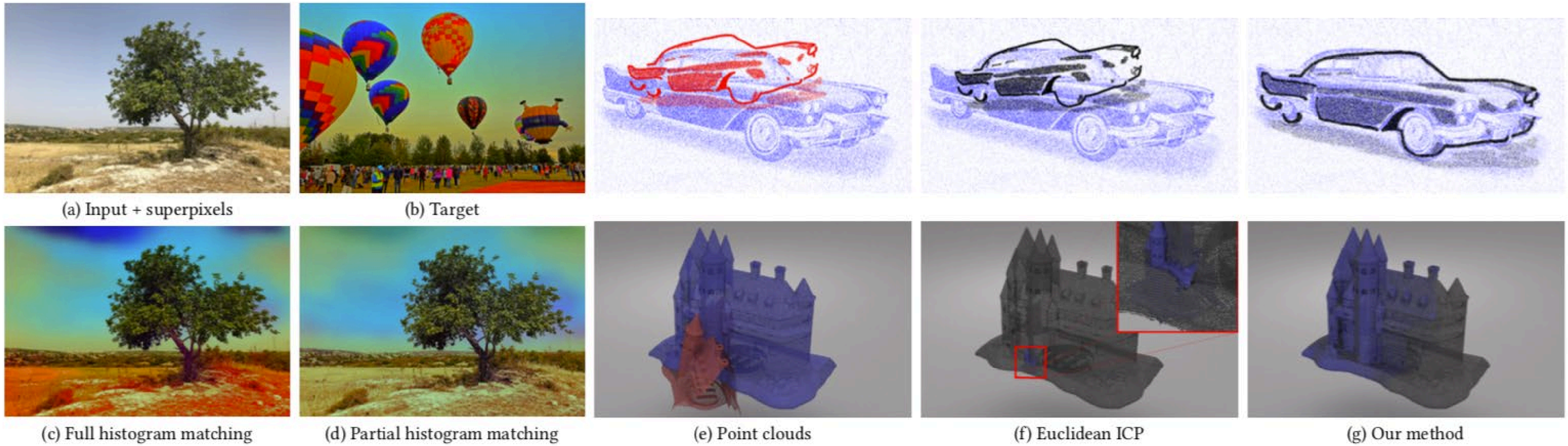
Sliced Partial Optimal Transport

Nicolas Bonneel*, David Coeurjolly*



(a) Input + superpixels (b) Target

(c) Full histogram matching (d) Partial histogram matching (e) Point clouds (f) Euclidean ICP (g) Our method

***CNRS, Univ. Lyon.**

Matching points

- Linear Assignment Problem

$$\min_{\text{T bijective}} \sum_i c(x_i, y_{T(i)})$$

- Optimal transport

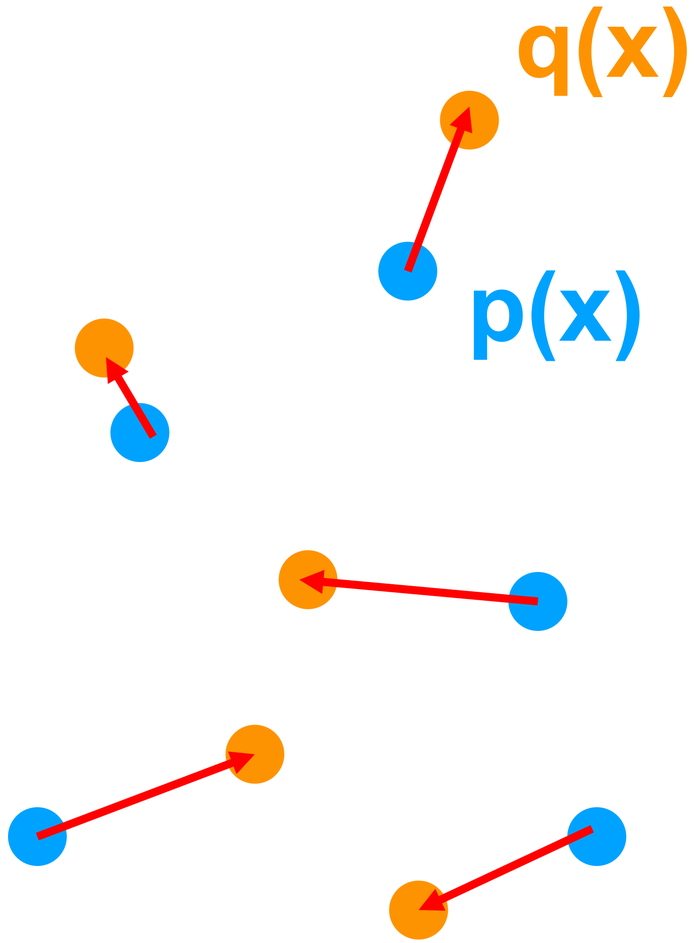
$$W(f,g) = \min \sum_{i,j} c(x_i, y_j)\pi_{i,j}$$

s.t.

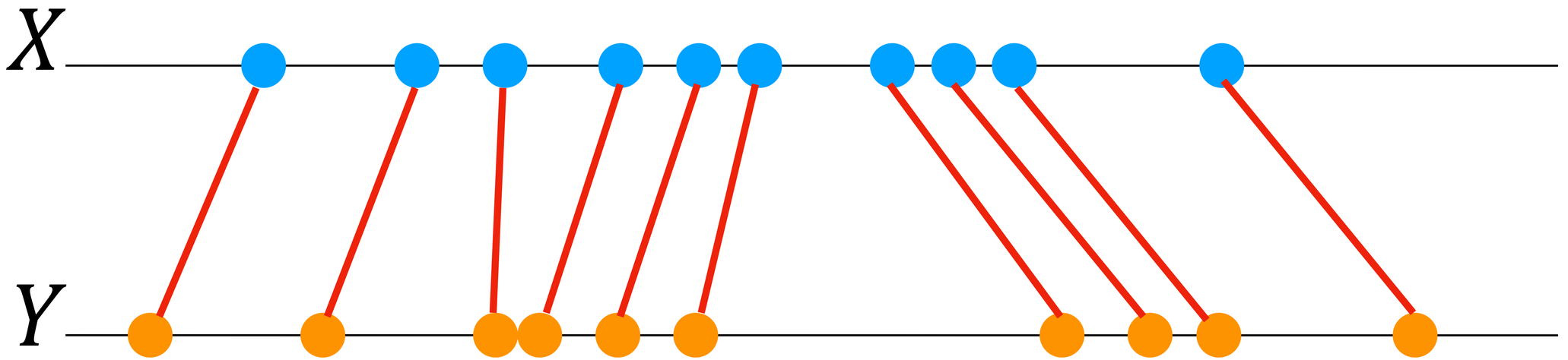
$$\sum_j \pi_{i,j} = 1$$

$$\sum_i \pi_{i,j} = 1$$

$$\pi_{i,j} \geq 0$$

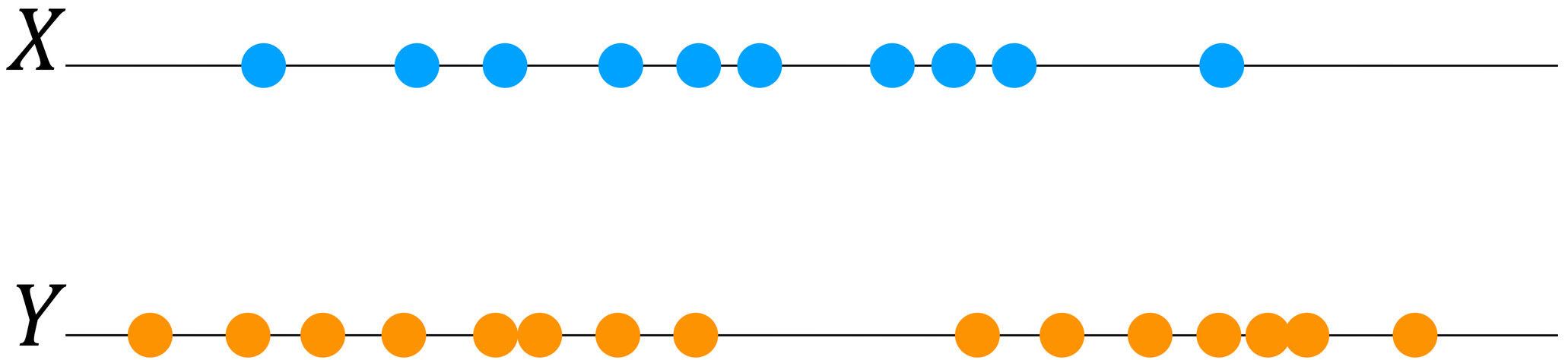


1-d Linear Assignment Problem is trivial*



*assuming the cost c is a convex function of |x-y|

Partial optimal assignment ?



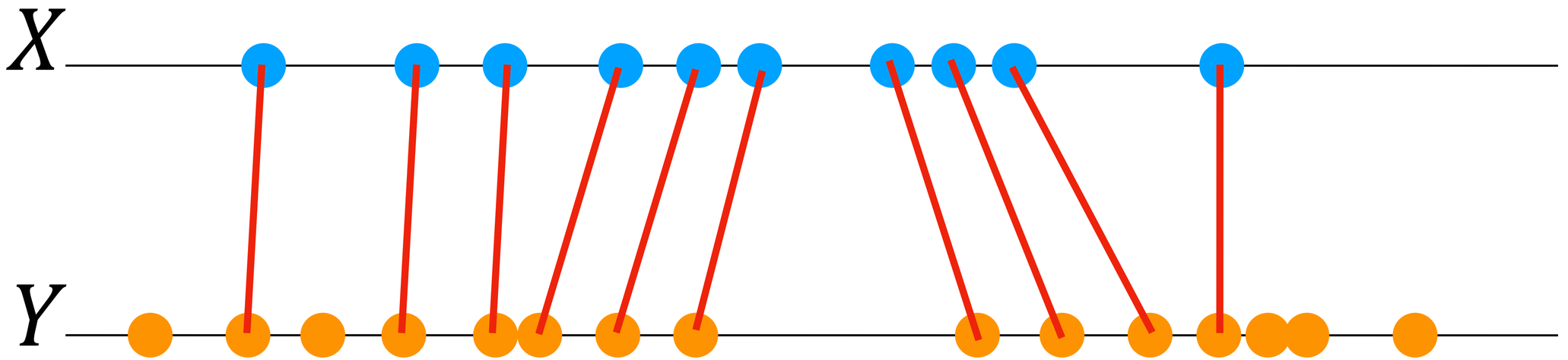
$$W(f,g) = \min \sum_{i,j} c(x_i, y_j)\pi_{i,j} \quad \textbf{s.t.} \quad \sum_j \pi_{i,j} = 1$$

$$\sum_i \pi_{i,j} \leq 1$$

$$\pi_{i,j} \geq 0$$

$$\min_{T \text{ injective}} \sum_i c(x_i, y_{T(i)})$$

Partial optimal assignment ?

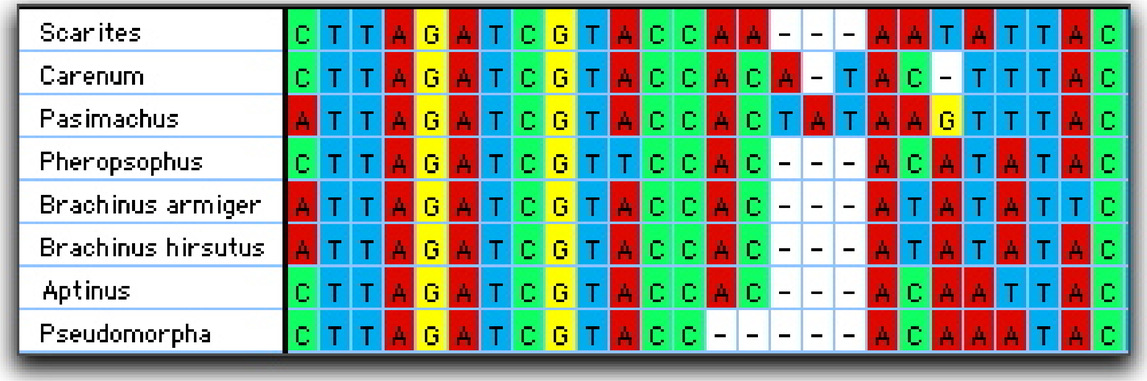


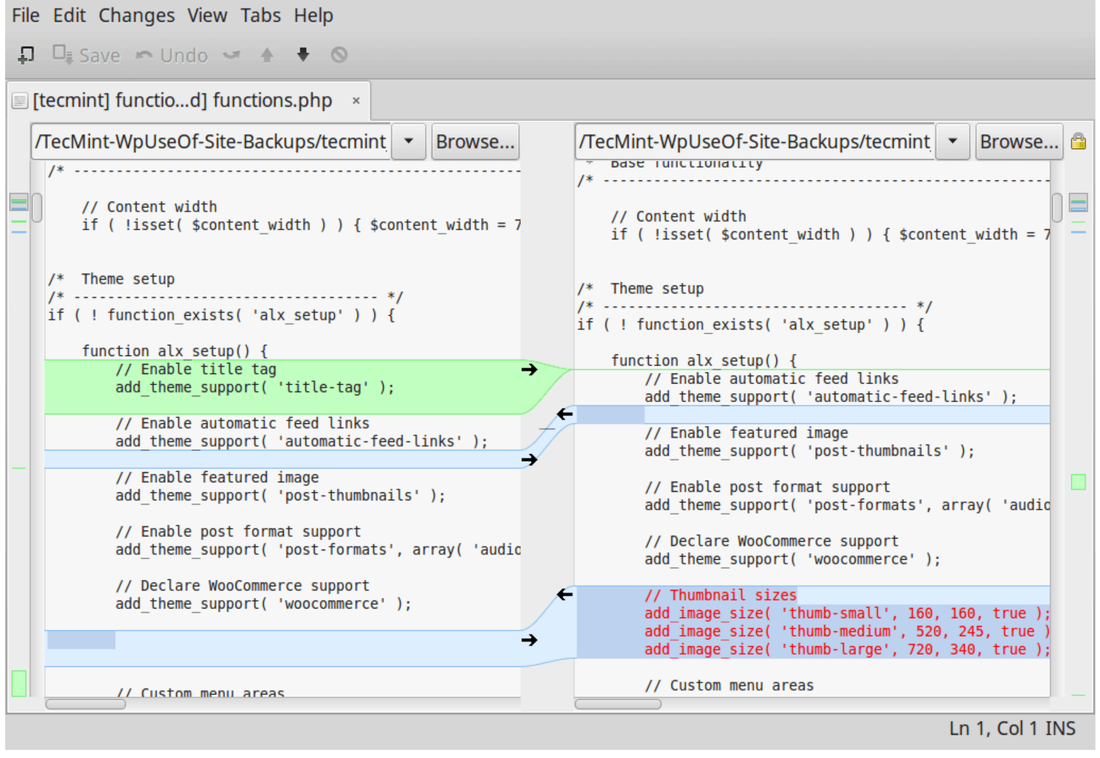
$$W(f,g) = \min \sum_{i,j} c_{i,j}\pi_{i,j} \quad \textbf{s.t.} \quad \sum_j \pi_{i,j} = 1 \qquad \sum_i \pi_{i,j} \leq 1 \qquad \pi_{i,j} \geq 0$$

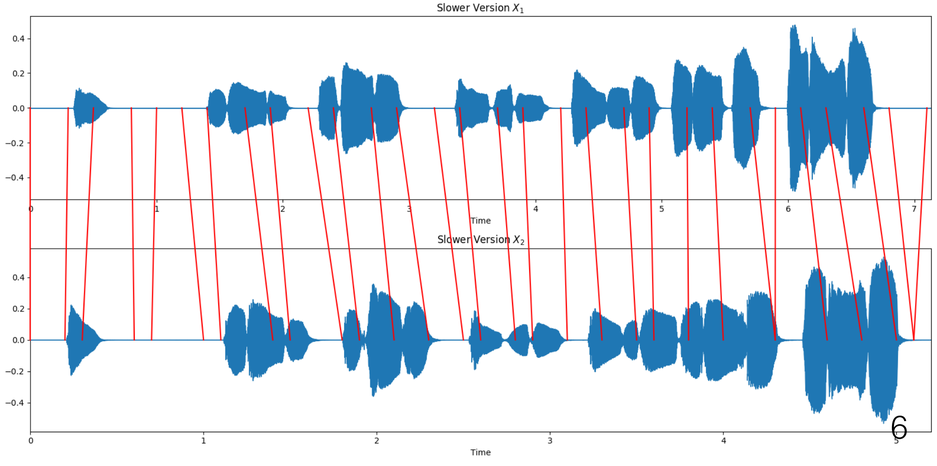
$$\min_{T \text{ injective}} \sum_i c(x_i, y_{T(i)})$$

Similar problems

- DNA sequence alignment
- Text alignment
  - Levenshtein distance
- Music synchronization
- …

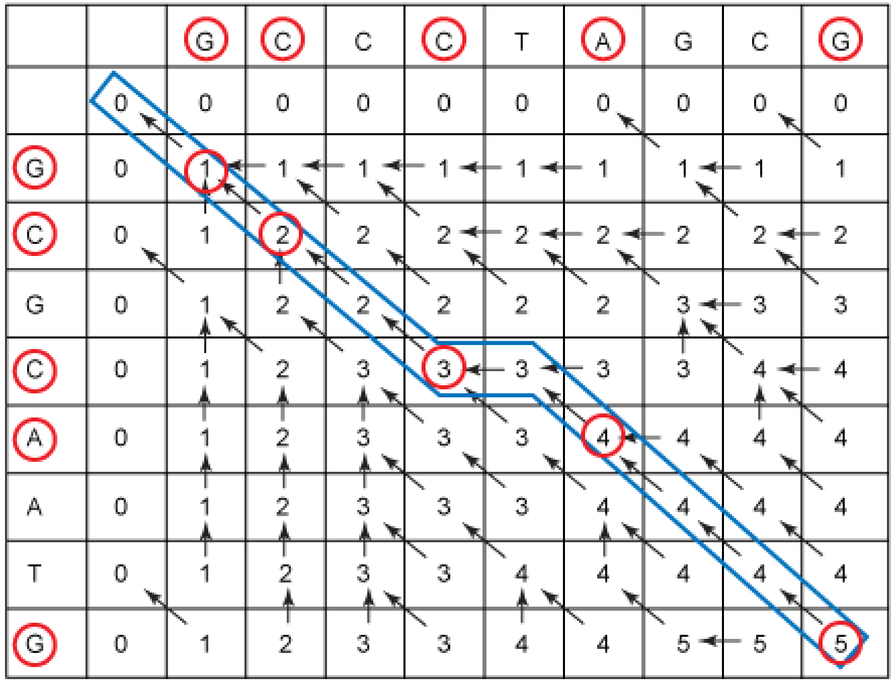






Existing Solutions

- Dynamic Time Warping
  - Solves a dynamic programming problem
  - Smith–Waterman algorithm, Needleman–Wunsch algorithm O(N^2) space and time
  - Hirschberg's algorithm O(N^2) time, O(N) space
- All end up doing variants of
  - $A_{i,j} = \min(A_{i-1,j-1} + cost, A_{i-1,j} + cost', A_{i,j-1} + cost'')$



Quadratic time complexity algorithm (linear space)

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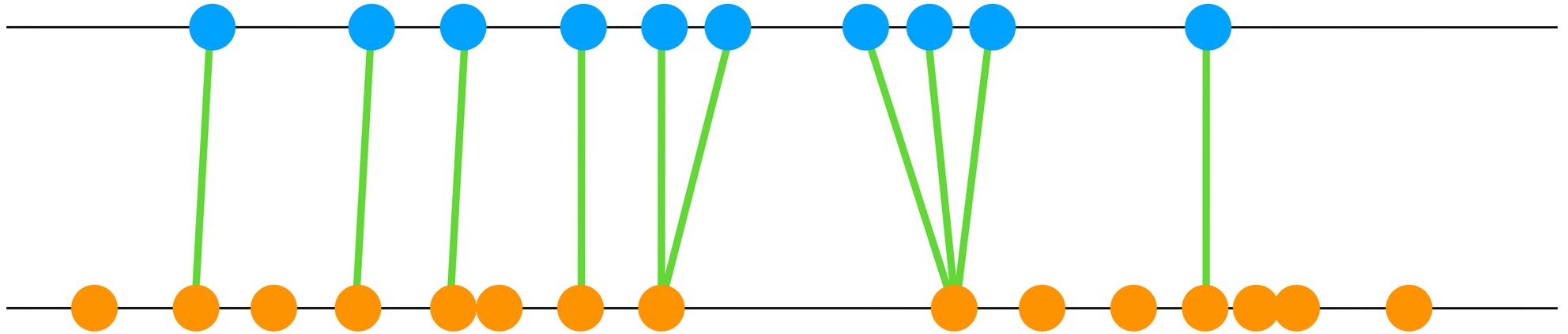
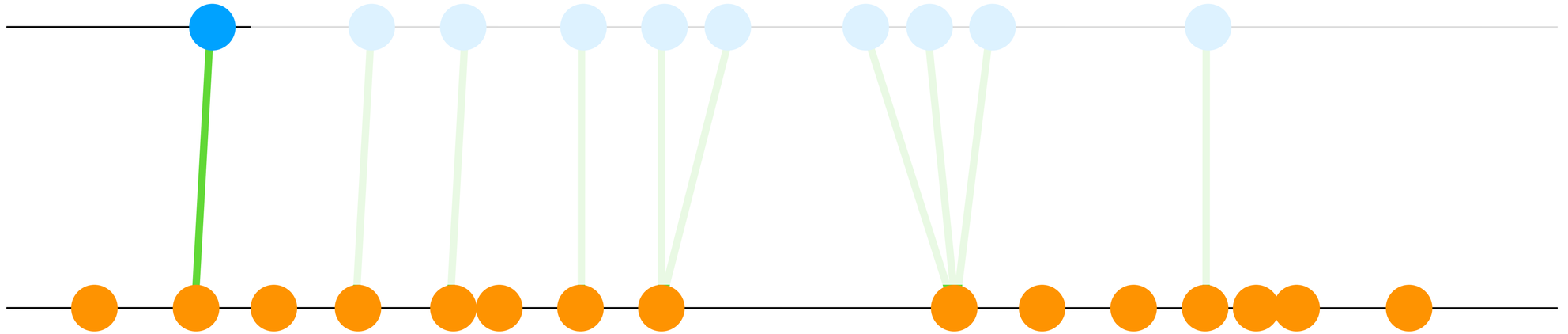


Euclidean Nearest Neighbor assignment

Quadratic time complexity algorithm (linear space)



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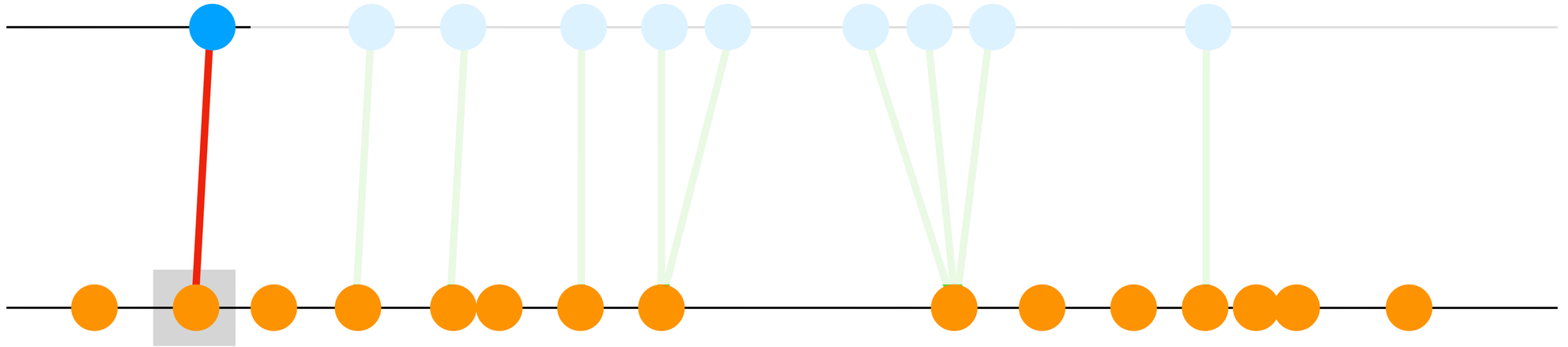

Euclidean Nearest Neighbor assignment

Optimal Transport assignment

Intervals of bijective assignments

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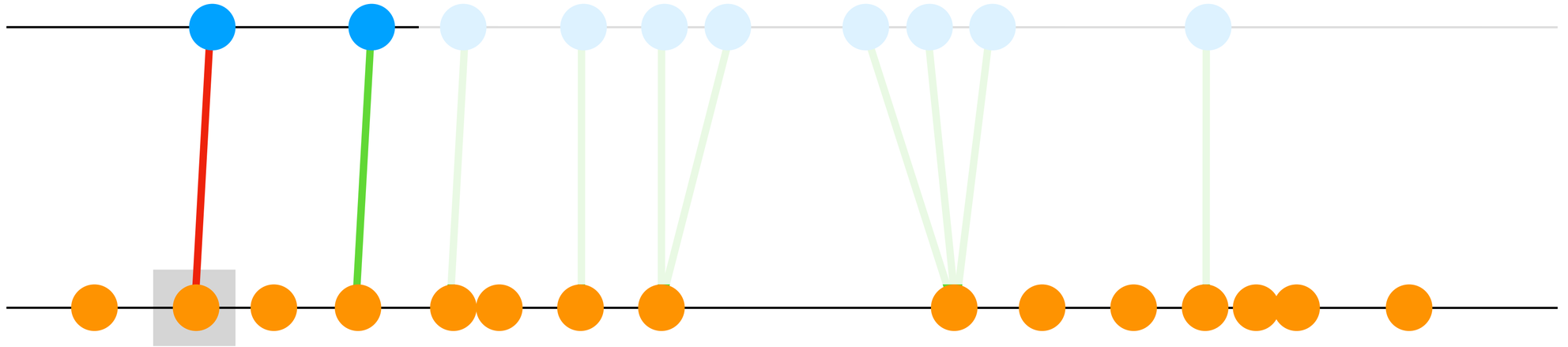

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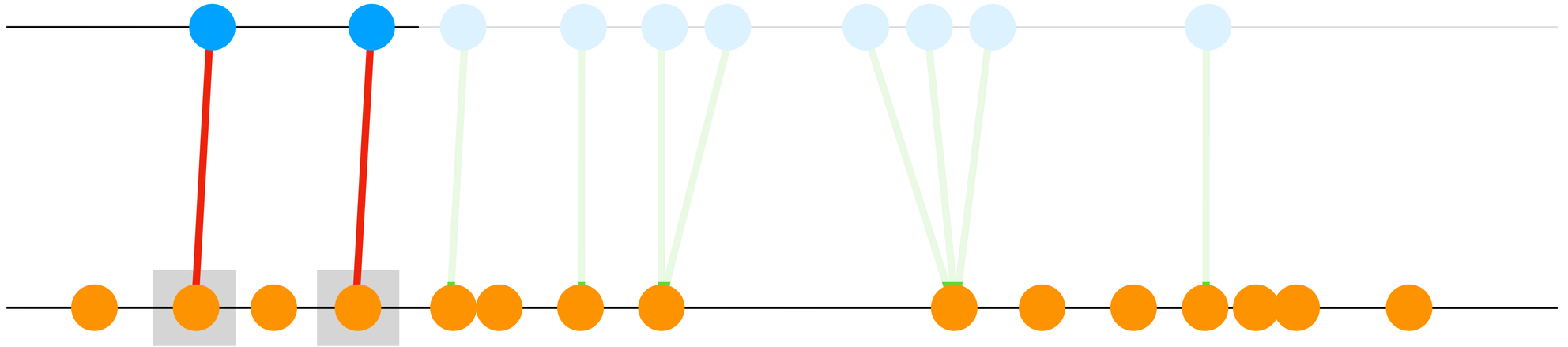

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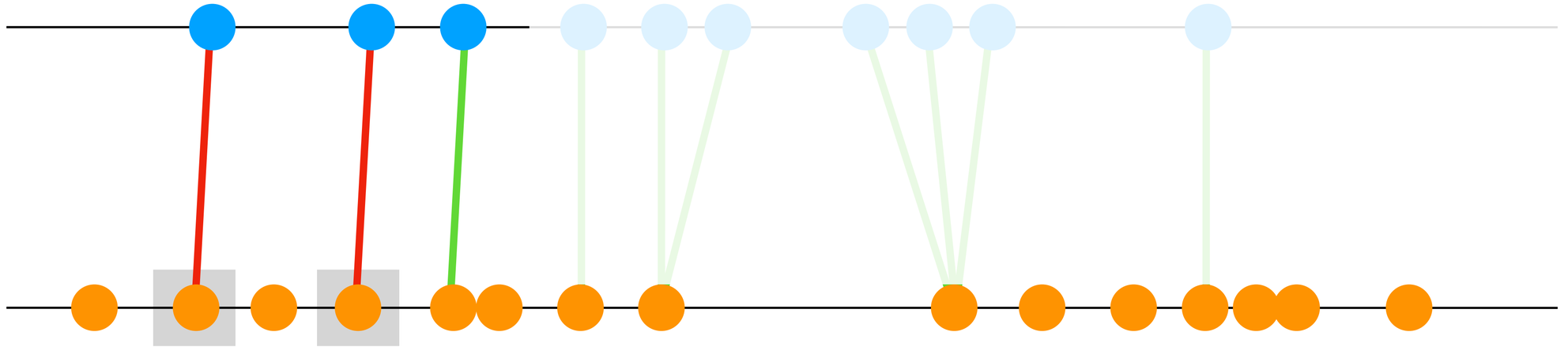

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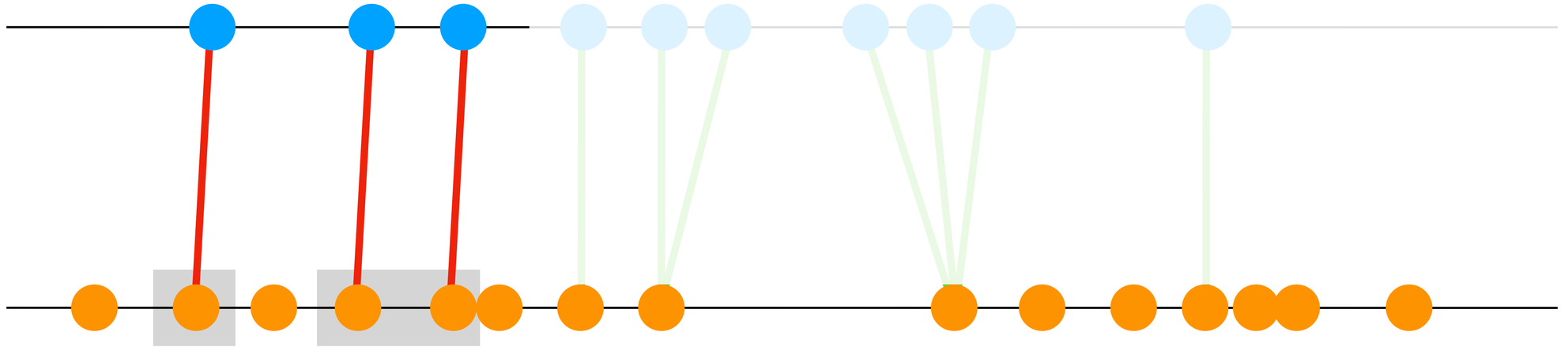

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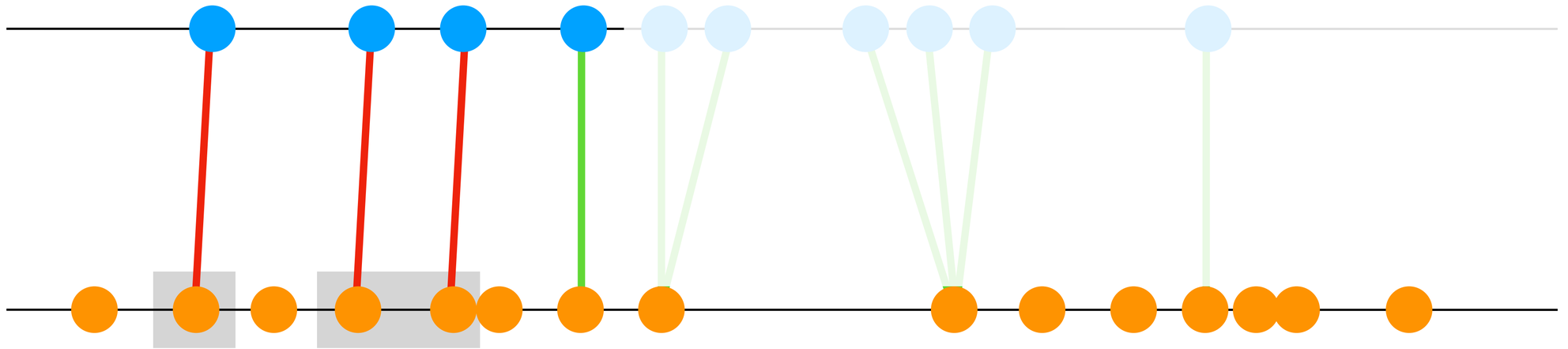

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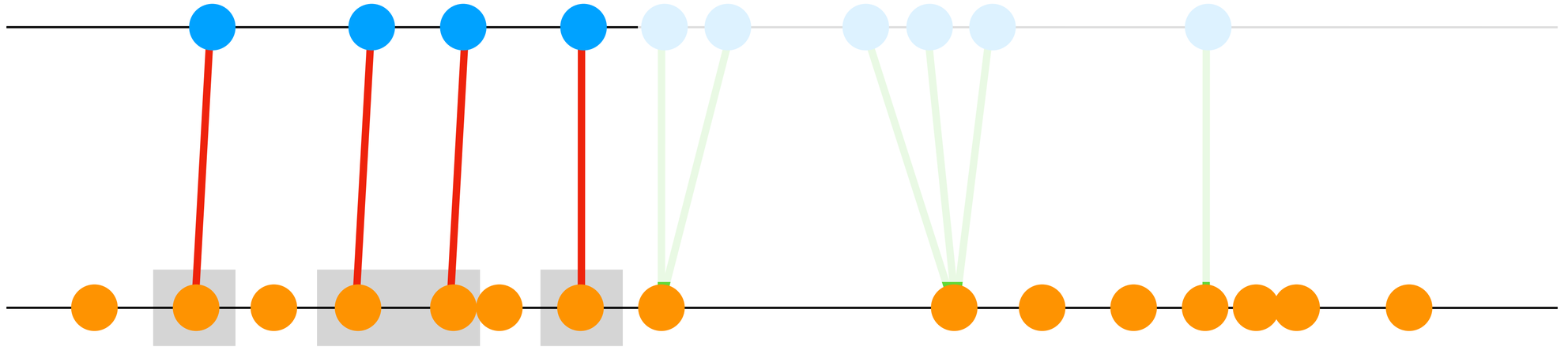

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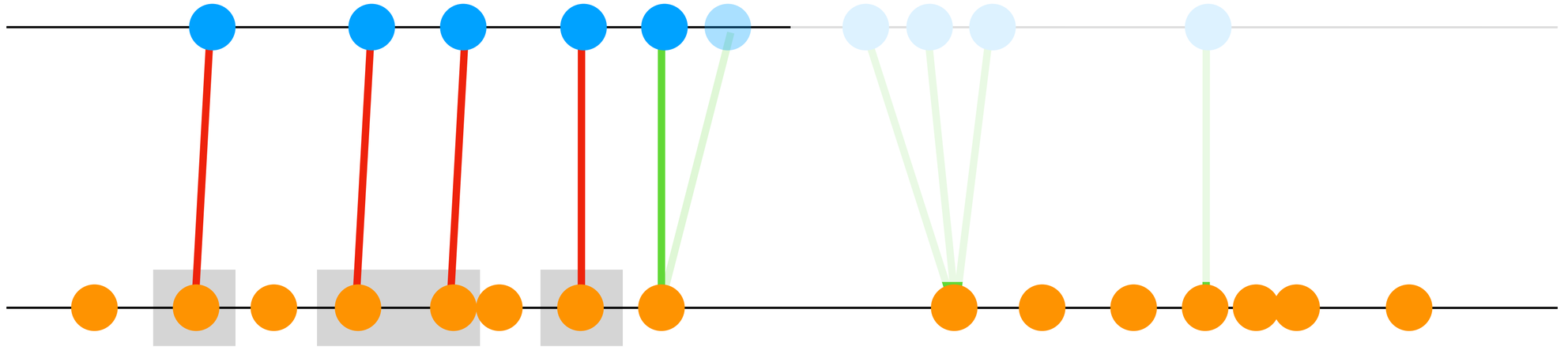

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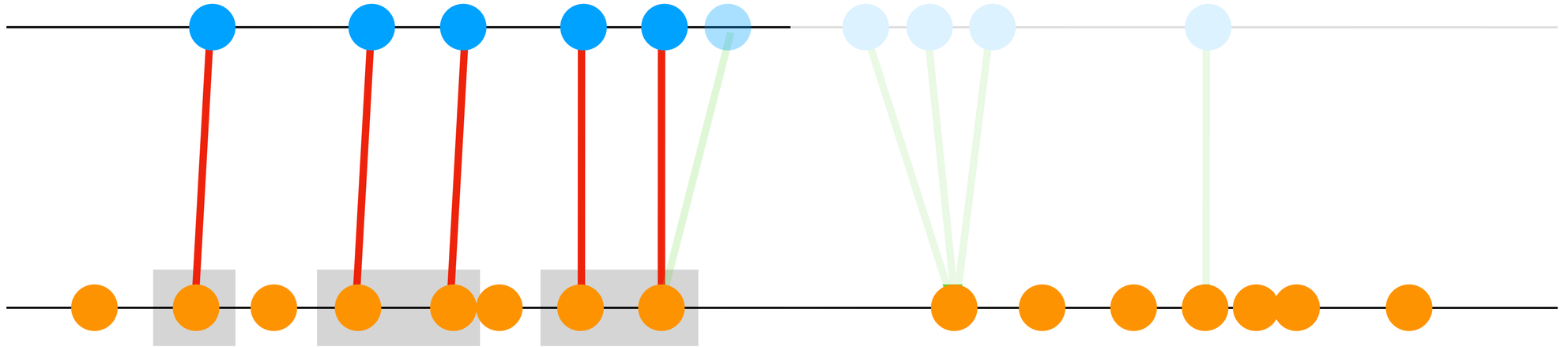

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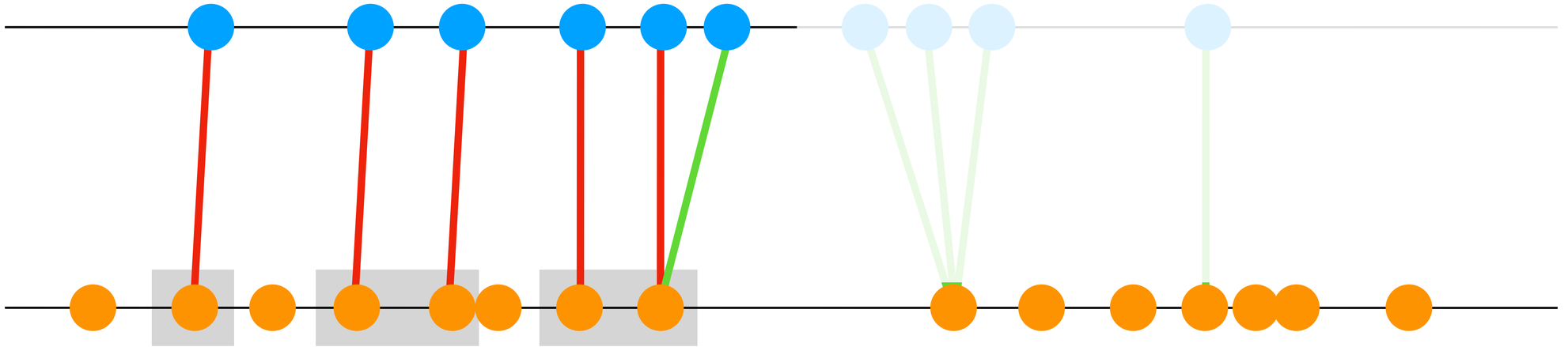

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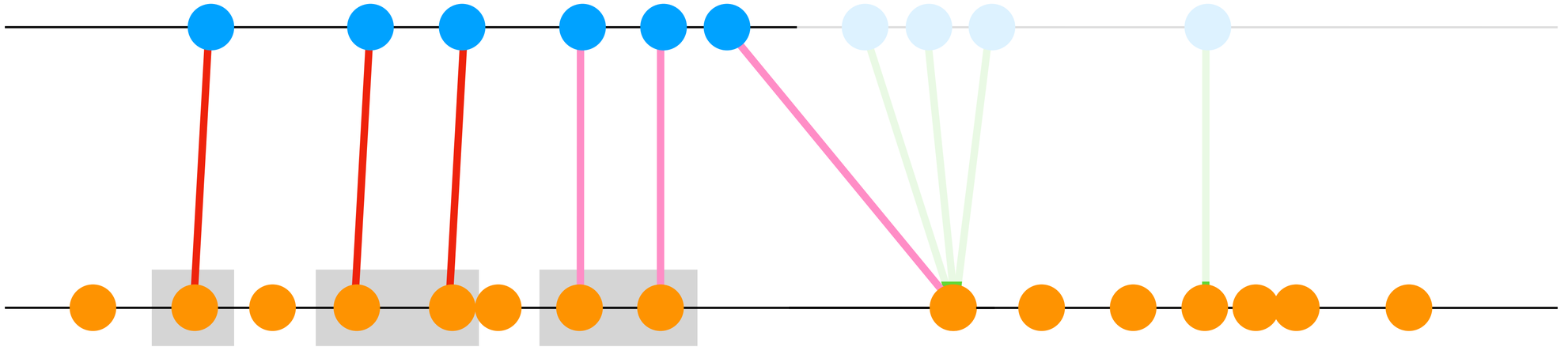

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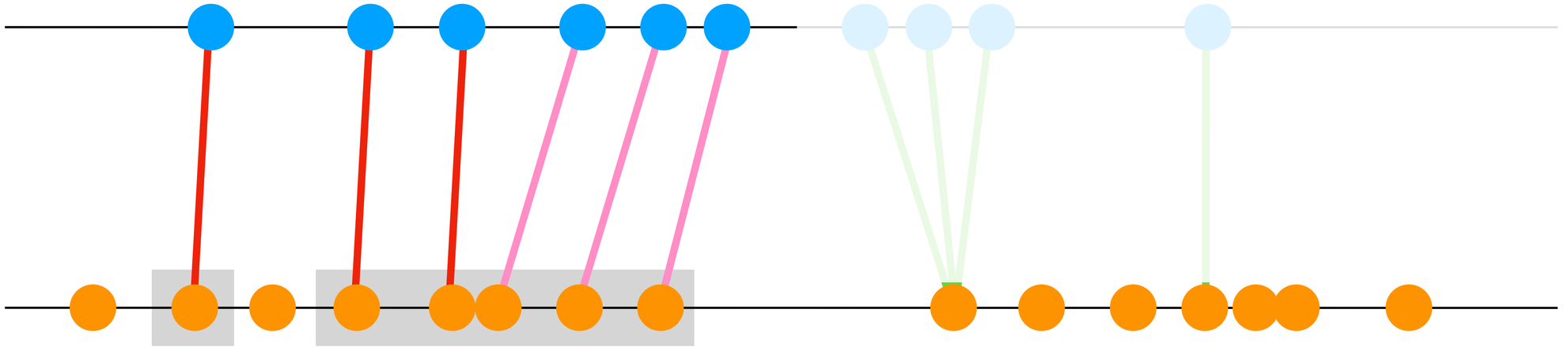

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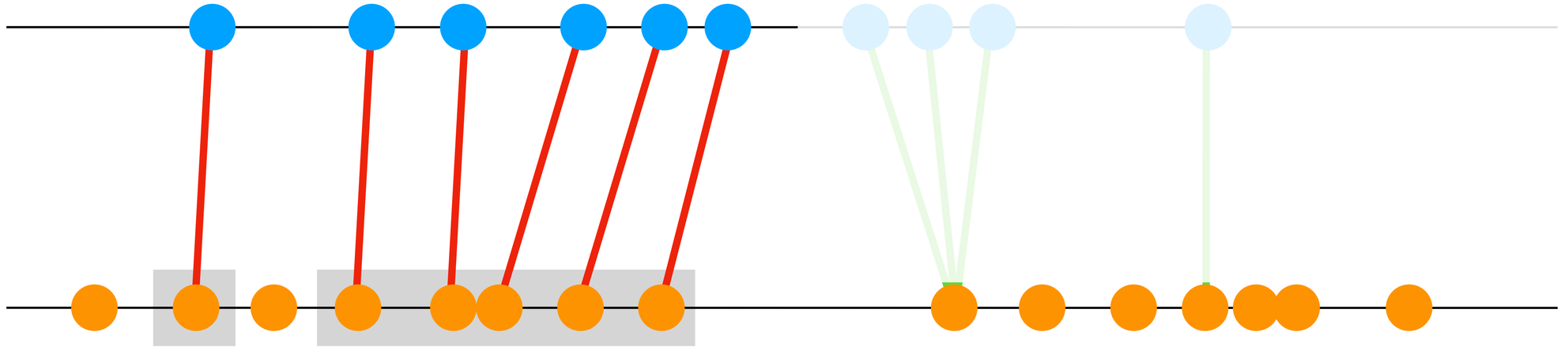

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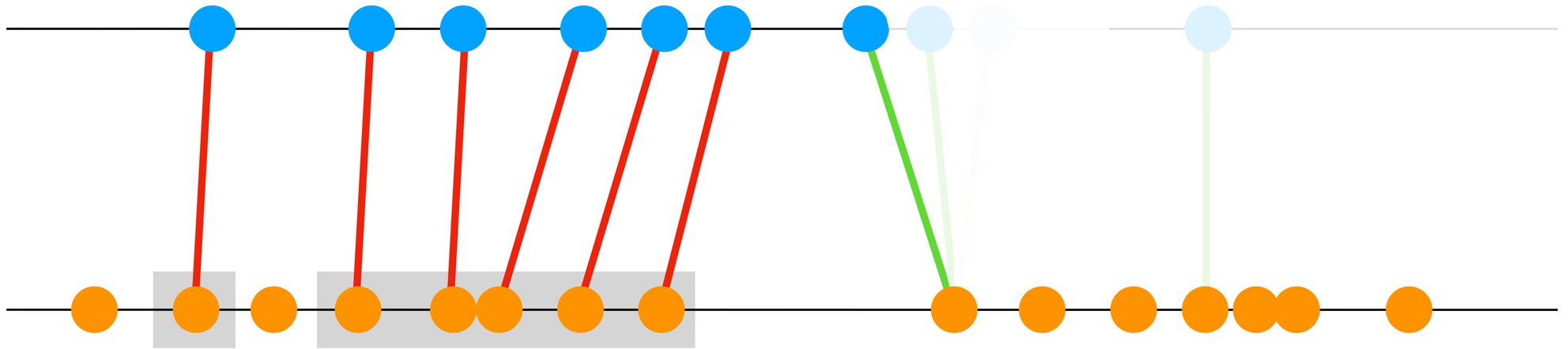

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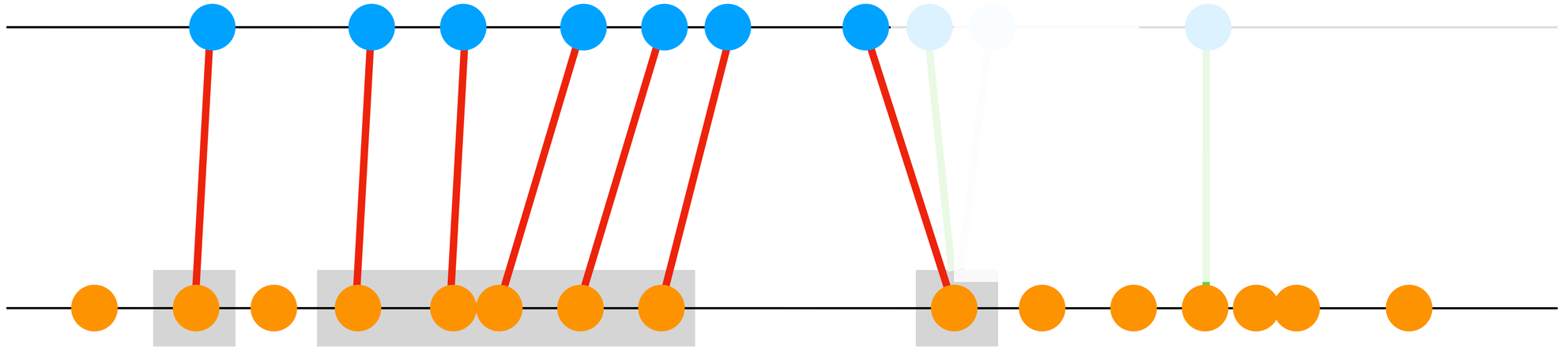

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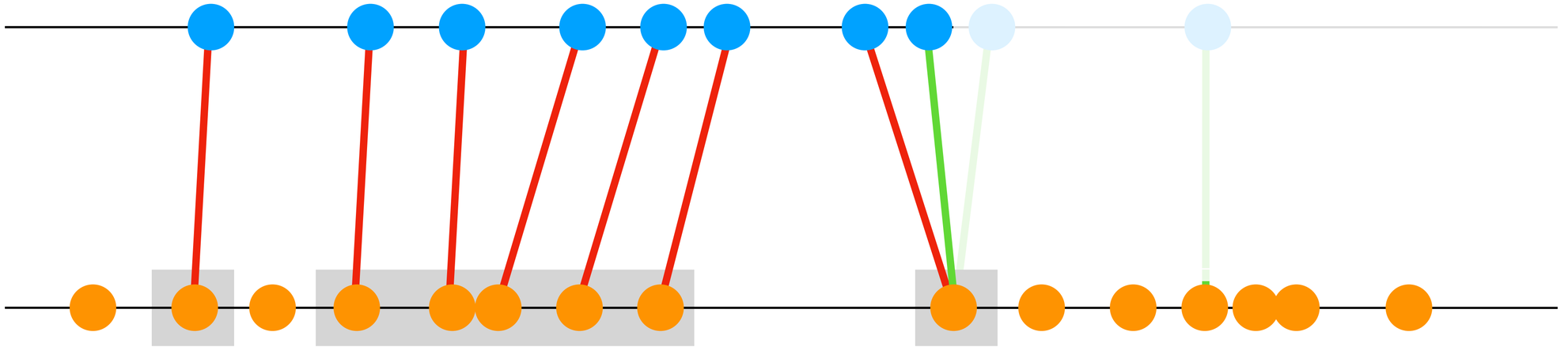

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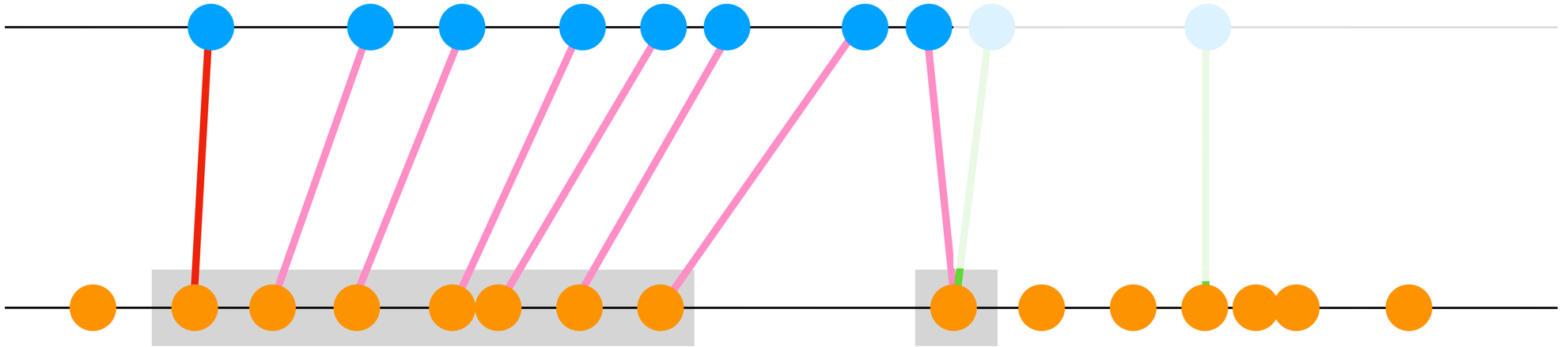

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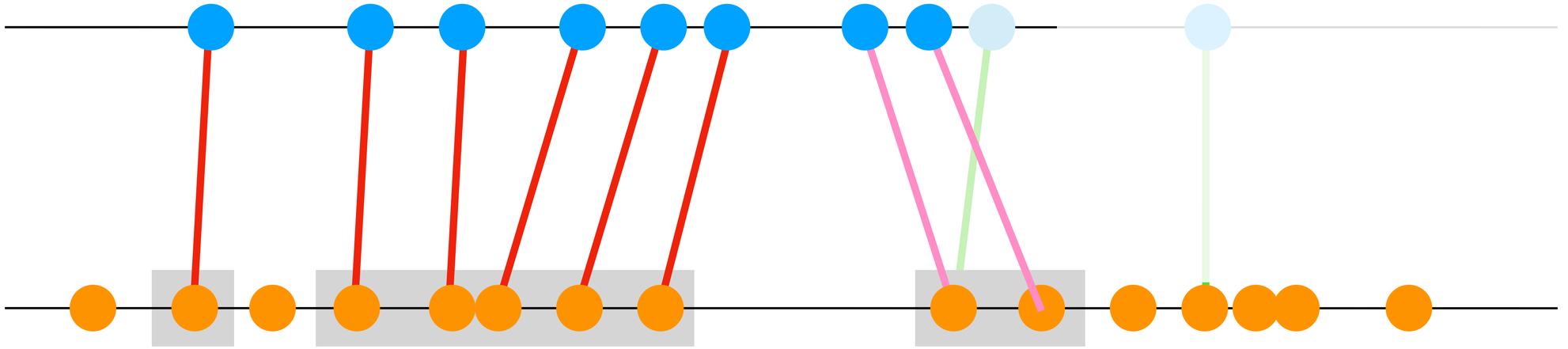

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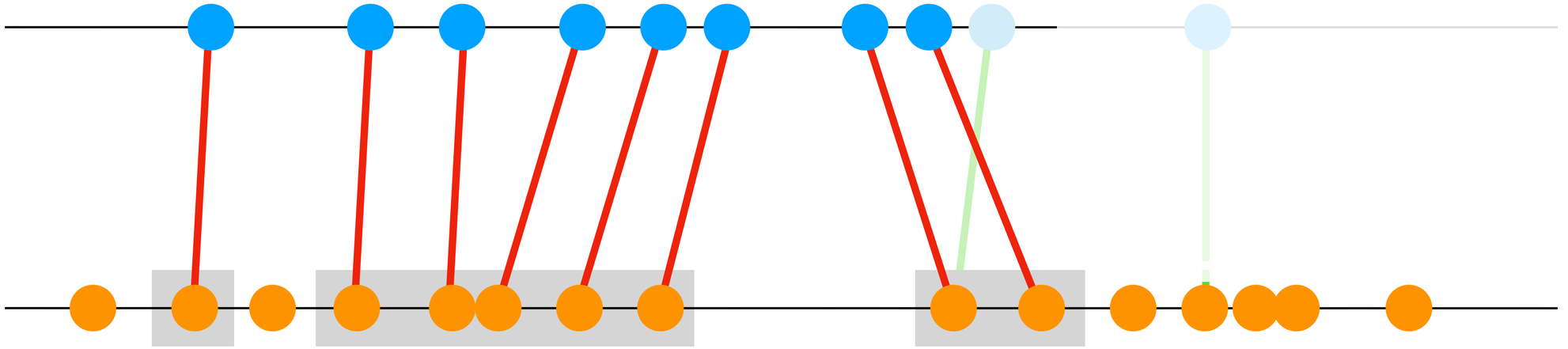

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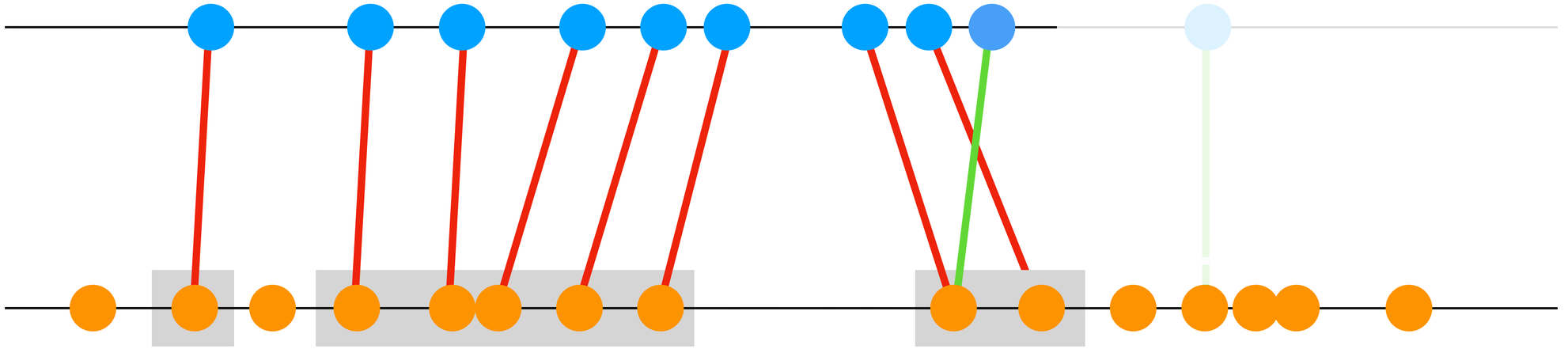

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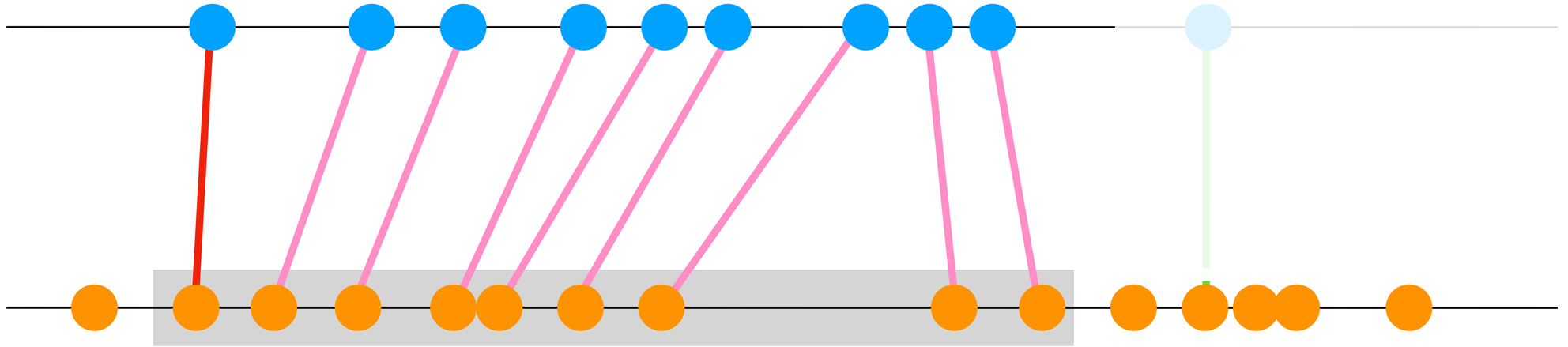

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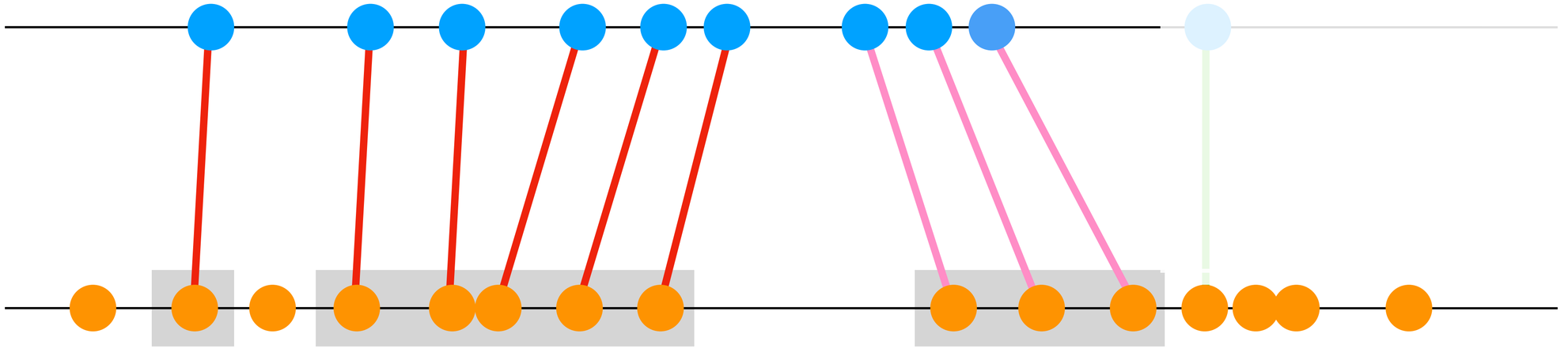

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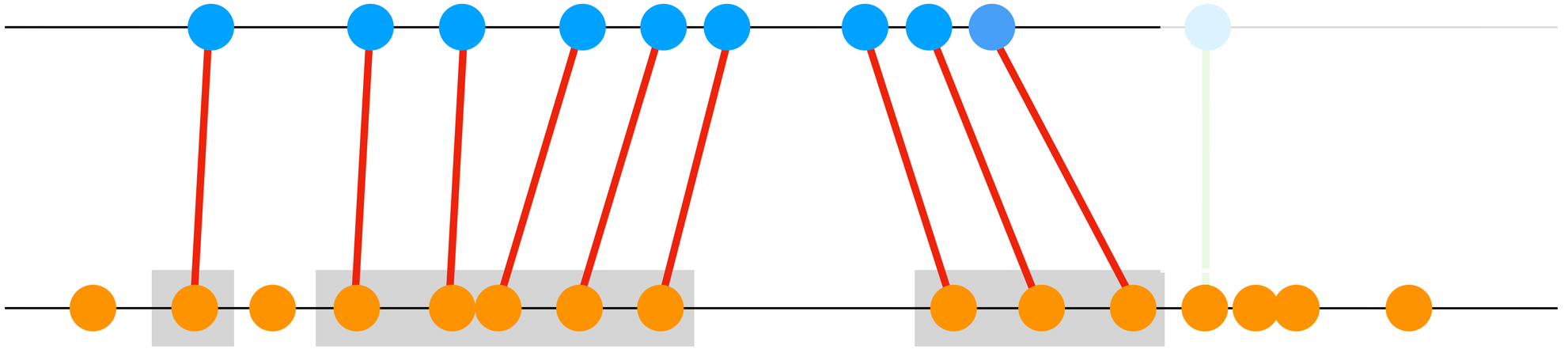

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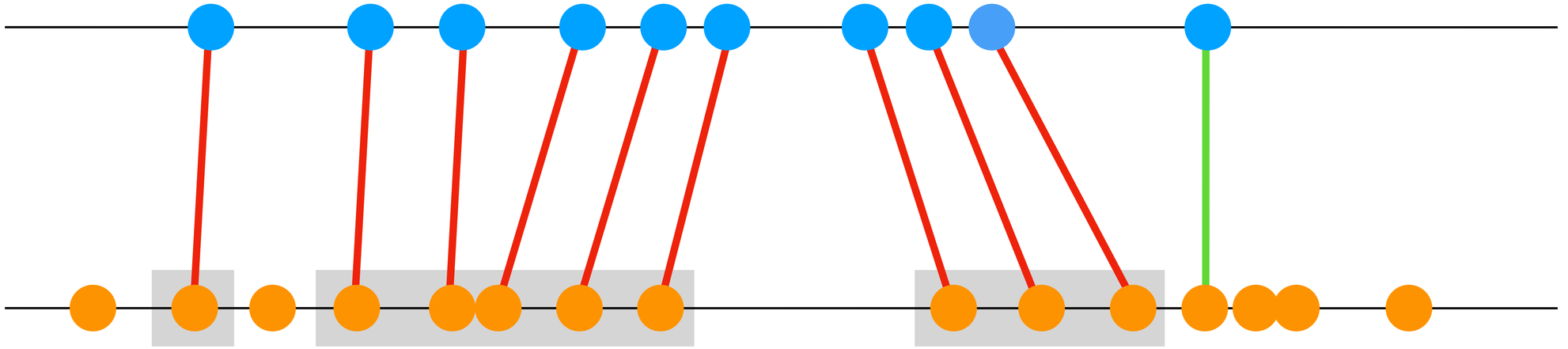

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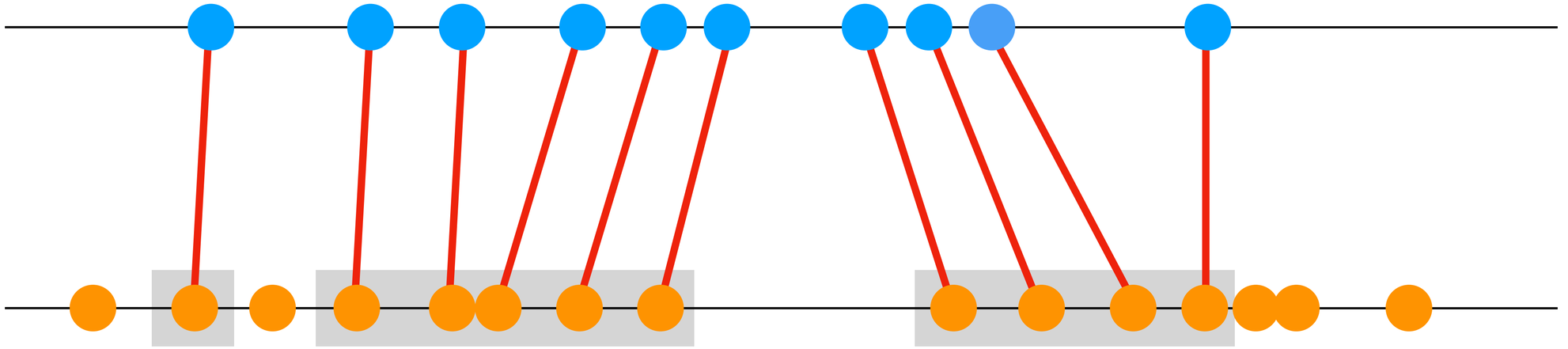

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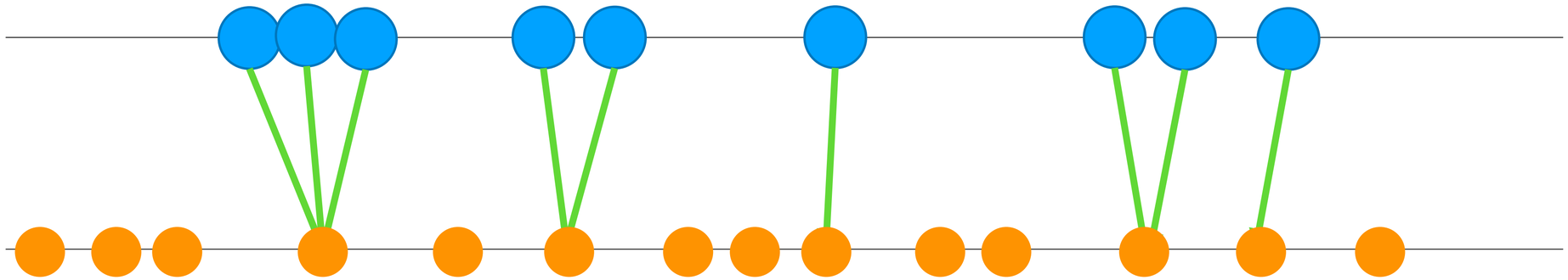
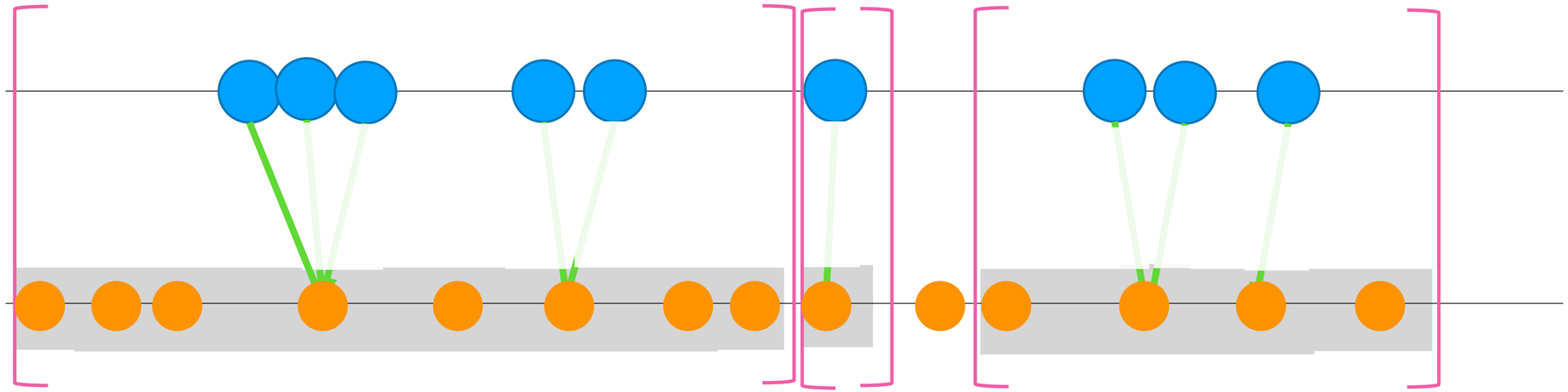
Intervals of bijective assignments

Linear time problem decomposition

Problem decomposition



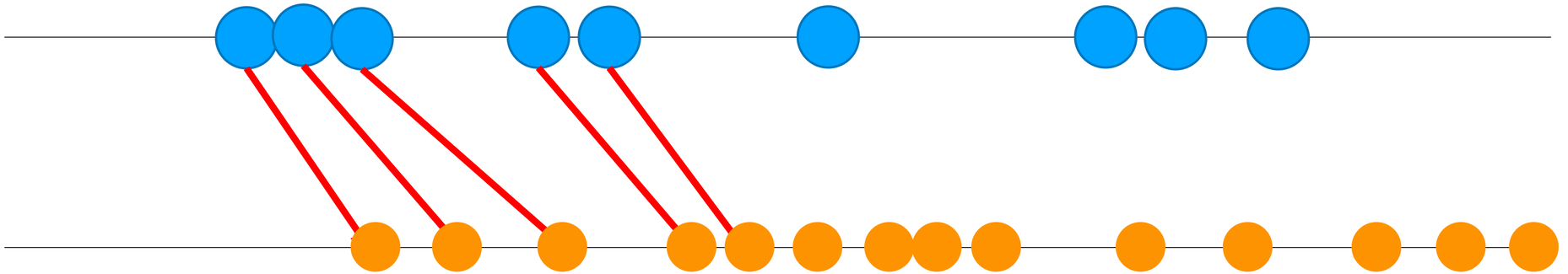
Problem decomposition



Simplifications

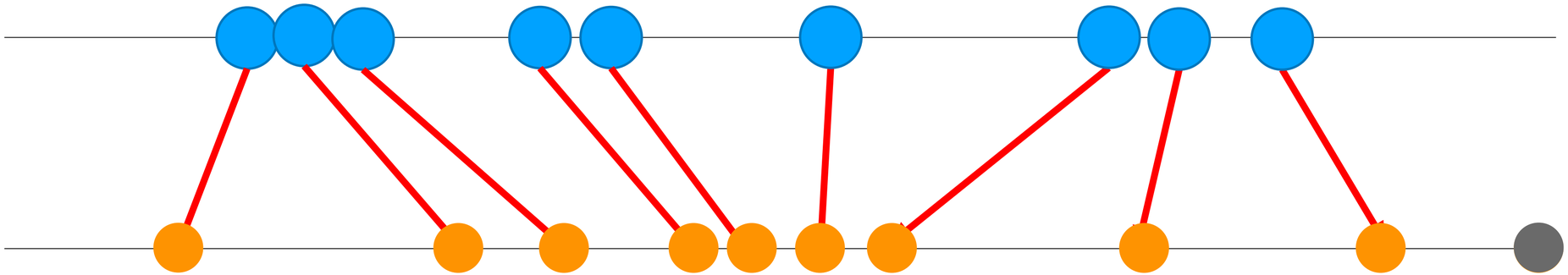
Simplification rules

- Matching N among N, or 1 among N: trivial
- Nearest neighbors injective: trivial
- Snap to the left (similarly to the right)



Simplification rules

- Matching N-1 among N
  - Idea from Hirschberg's algorithm
  - Minimum of matching together the K first + matching together N-K-1 lasts



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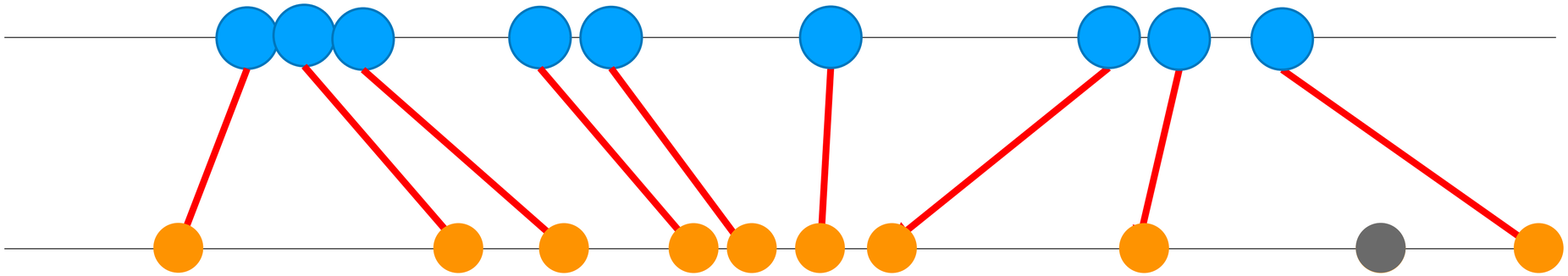

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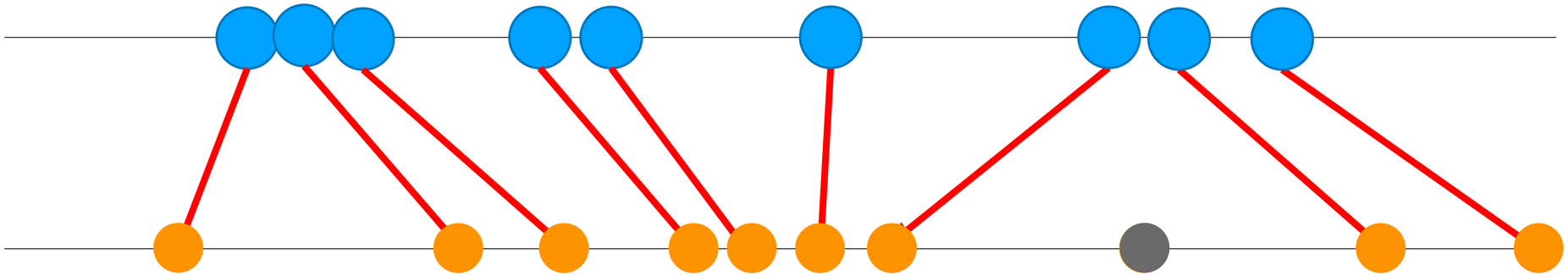

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  - Idea from Hirschberg's algorithm
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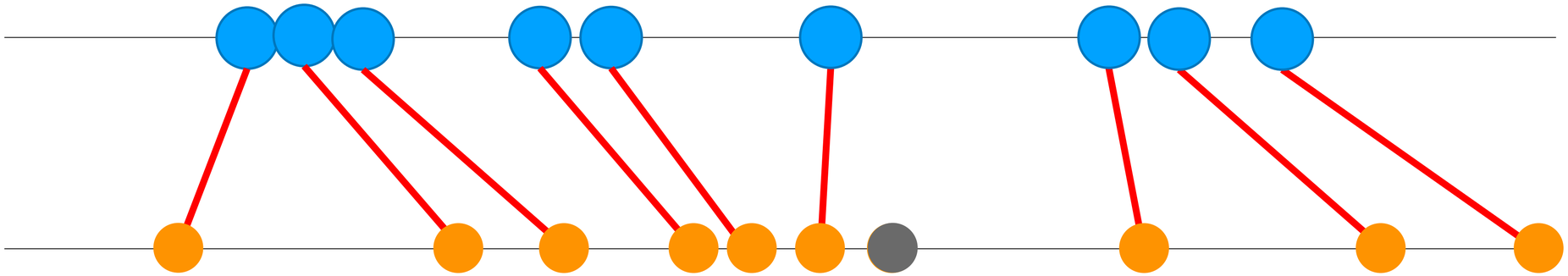
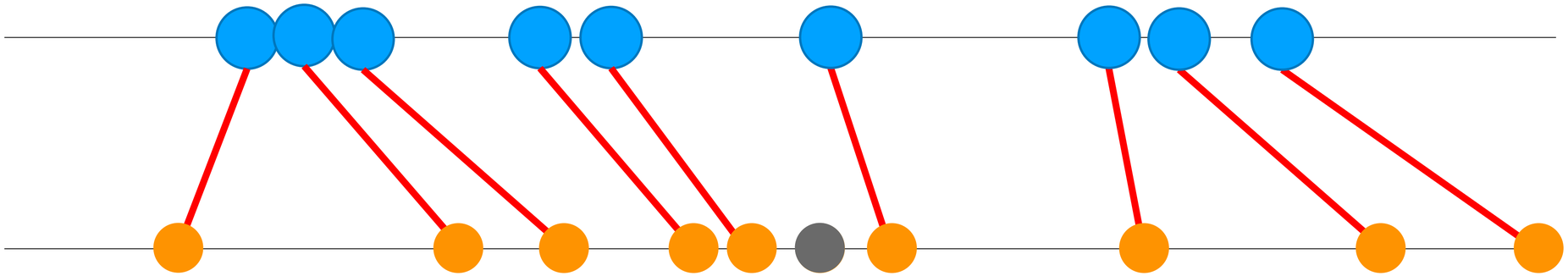


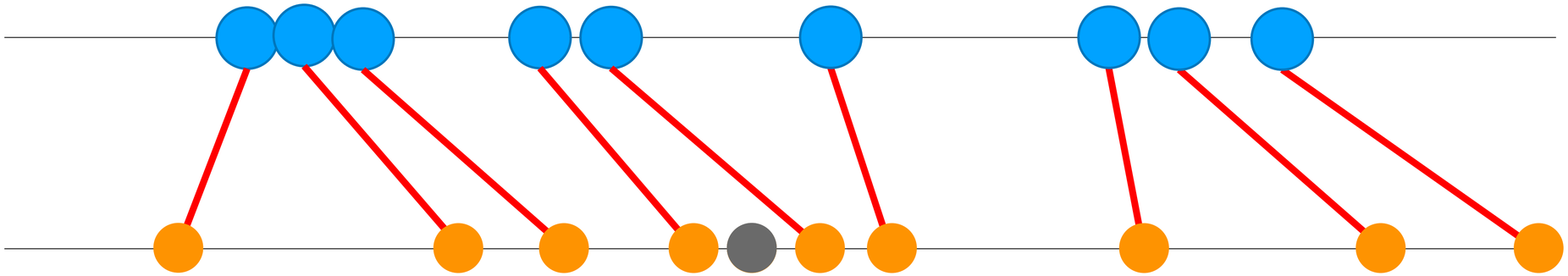
Simplification rules

- Matching N-1 among N
  - Idea from Hirschberg's algorithm
  - Minimum of matching together the K first + matching together N-K-1 lasts



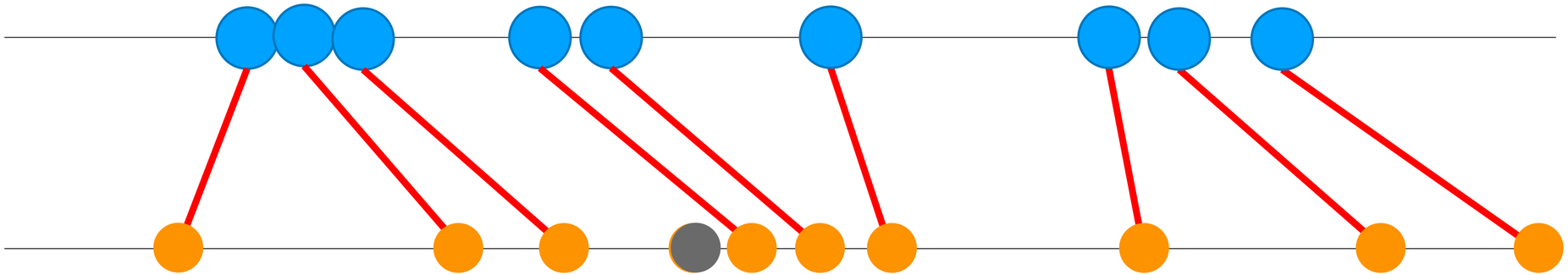
Simplification rules

- Matching N-1 among N
  - Idea from Hirschberg's algorithm
  - Minimum of matching together the K first + matching together N-K-1 lasts



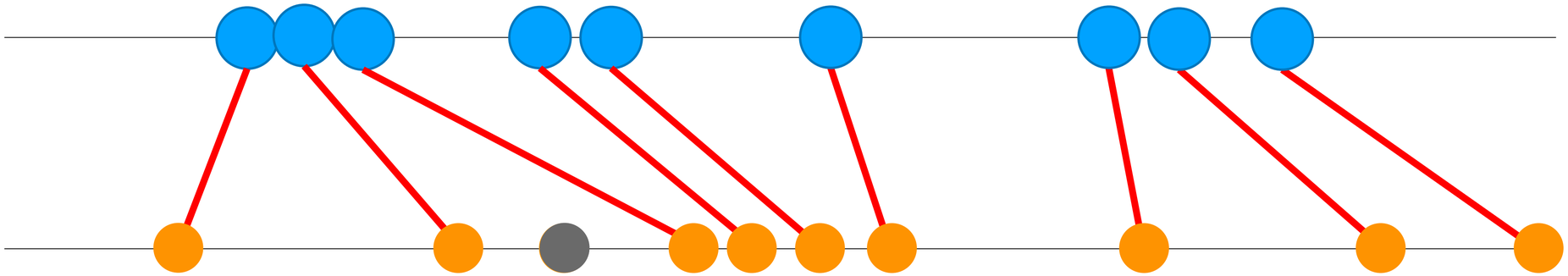
Simplification rules

- Matching N-1 among N
  - Idea from Hirschberg's algorithm
  - Minimum of matching together the K first + matching together N-K-1 lasts



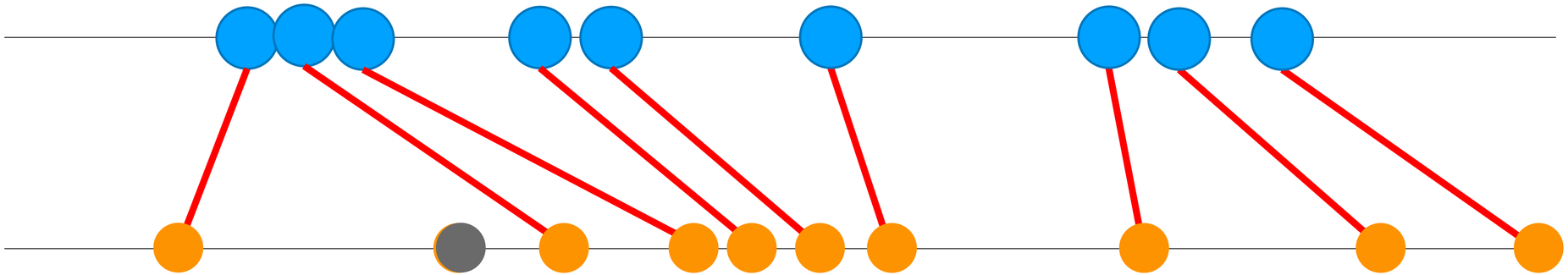
Simplification rules

- Matching N-1 among N
  - Idea from Hirschberg's algorithm
  - Minimum of matching together the K first + matching together N-K-1 lasts



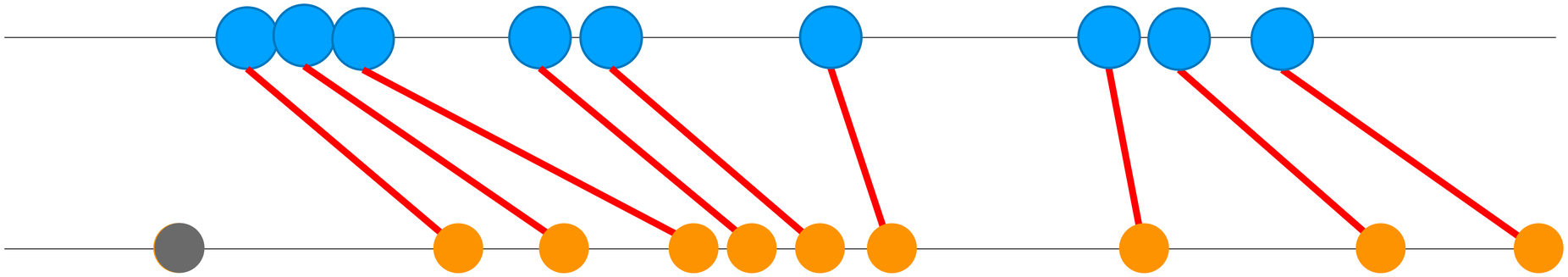
Simplification rules

- Matching N-1 among N
  - Idea from Hirschberg's algorithm
  - Minimum of matching together the K first + matching together N-K-1 lasts

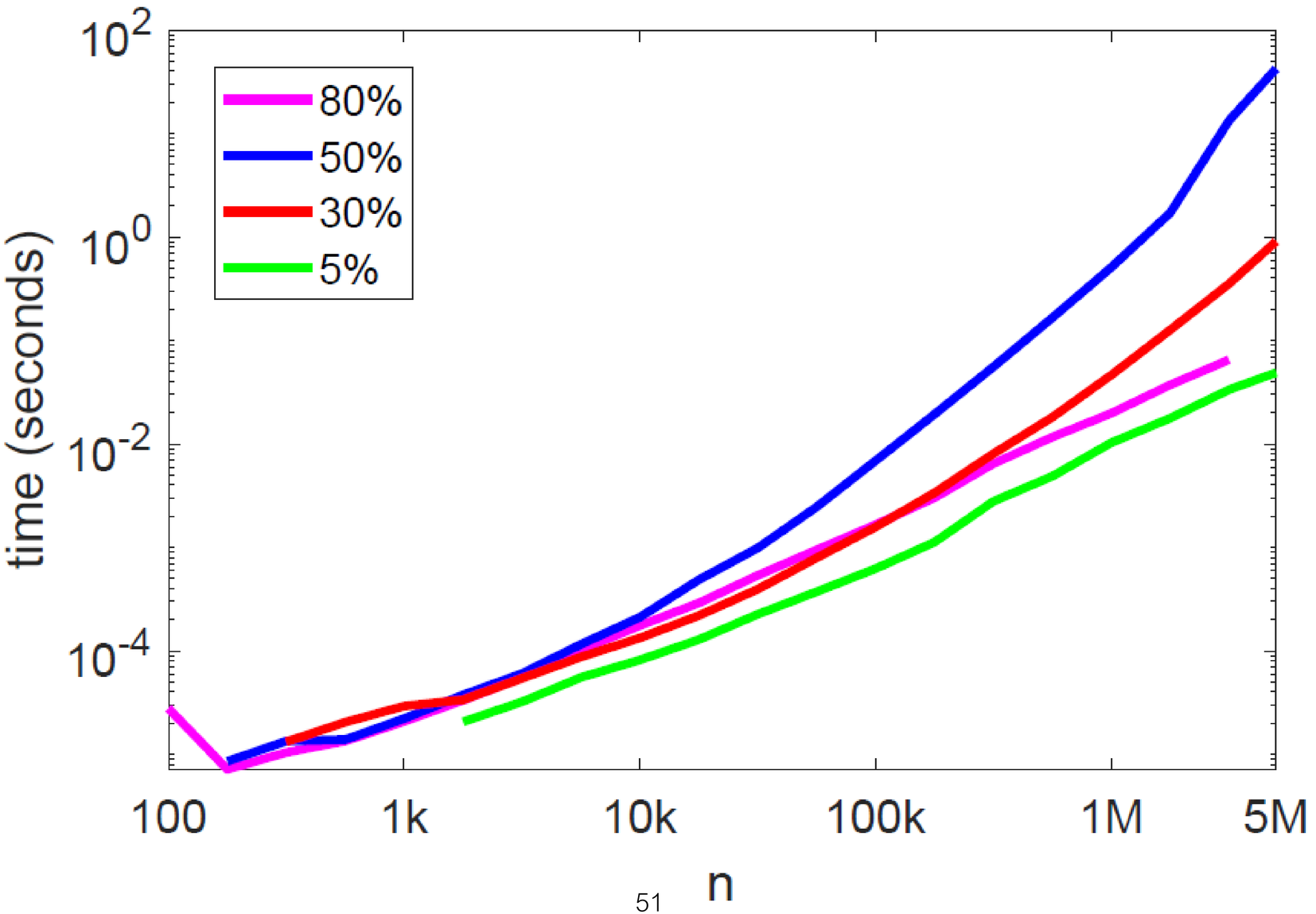


Simplification rules

- Matching N-1 among N
  - Idea from Hirschberg's algorithm
  - Minimum of matching together the K first + matching together N-K-1 lasts



Speed

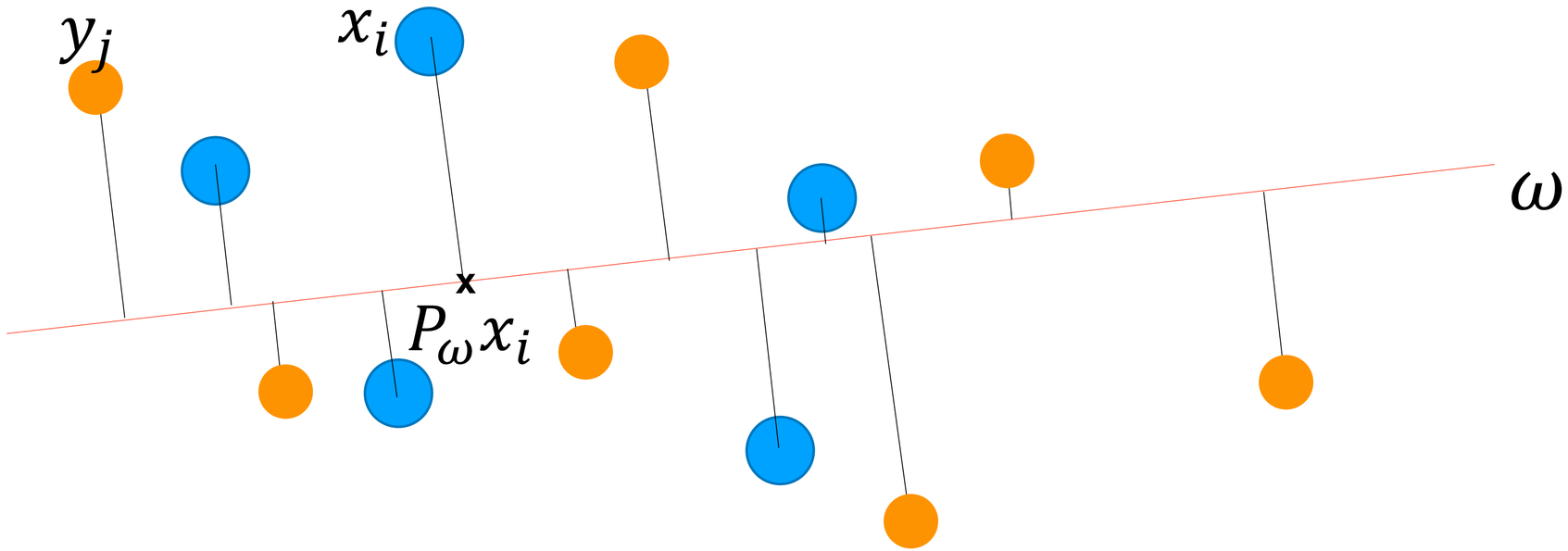


Sliced Partial Optimal Transport (SPOT)

Extension to d dimensions

- Sliced optimal transport cost :

$$E(\{x_i\}_i, \{y_i\}_i) = \int_{\mathbb{S}^{d-1}} \min_{T_\omega} \sum_i \left(P_\omega x_i - P_\omega y_{T_\omega(i)}\right)^2 d\omega$$

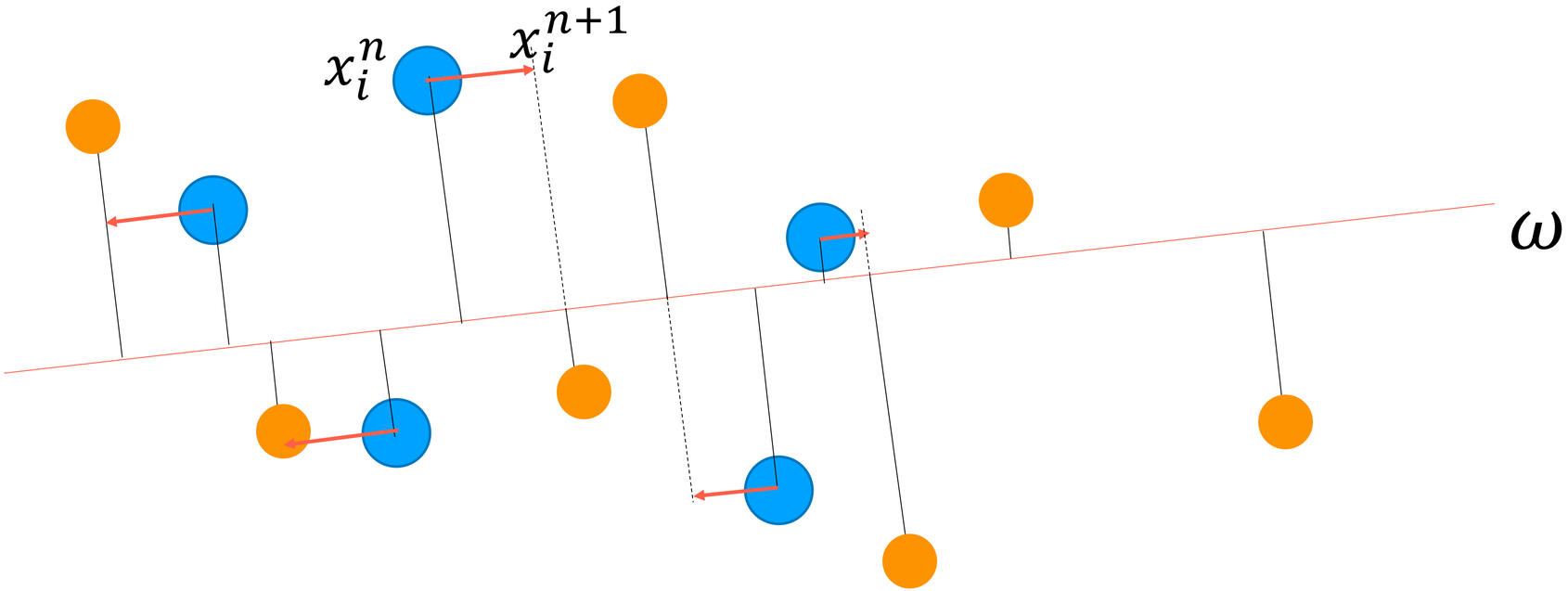


Extension to d dimensions

- Sliced optimal transport gradient flow :

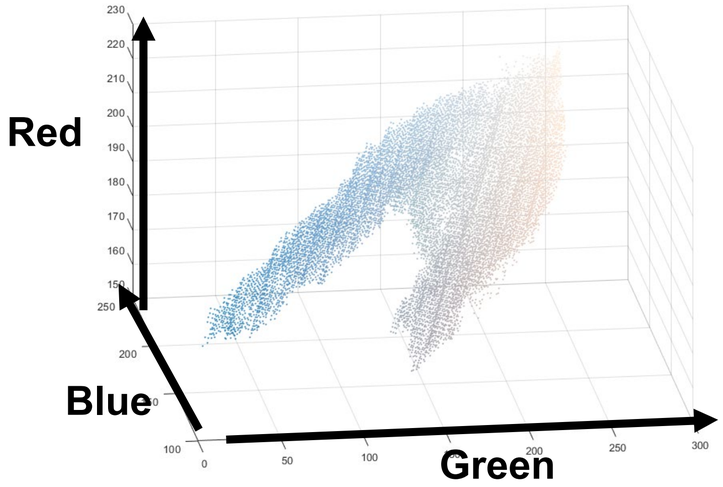
$$x_i^{n+1} = x_i^n - \alpha \nabla_x E$$

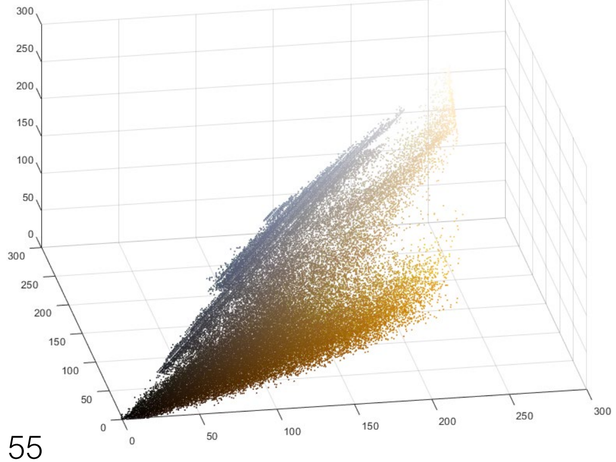
$$\Rightarrow x_i^{n+1} = x_i^n + \alpha \int_{\mathbb{S}^{d-1}} \left(P_\omega y_{T_\omega(i)} - P_\omega x_i \right) \omega \, d\omega$$



Color Transfer Application

- Images seen as point clouds in RGB space





Color Transfer Application

Full Transfer*

Target 20% larger

Target 40% larger

* [Bonneel et al. 2015] Sliced and Radon Wasserstein Barycenters of Measures.

Color Transfer Application

Full Transfer **Target 20% larger** **Target 40% larger**

Color Transfer Application

Full Transfer

Target 20% larger

Target 40% larger

1 superpixel | 2 superpixels | 6 superpixels | 13 superpixels | target

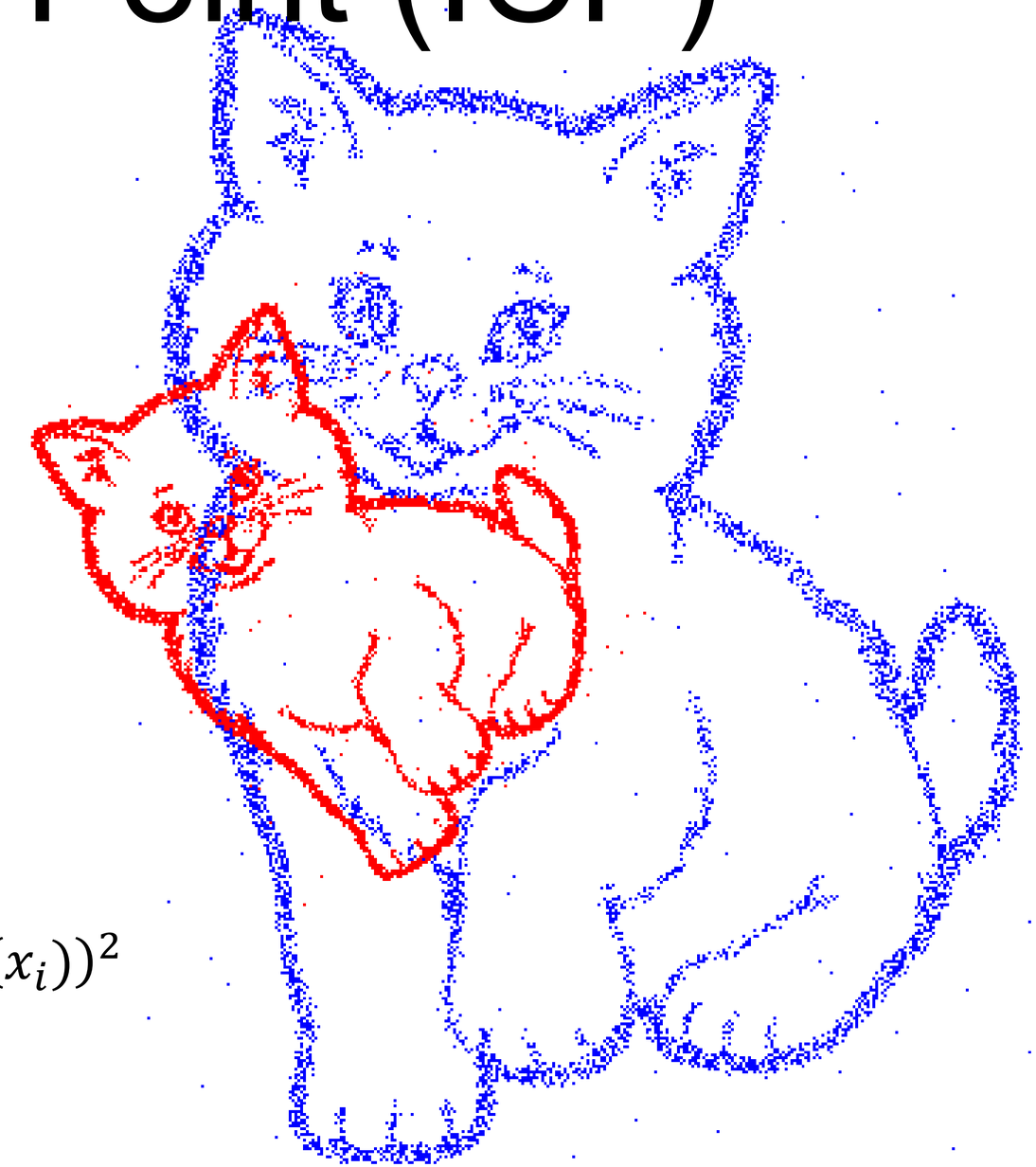
Fast Iterative Sliced Transport

Iterative Closest Point (ICP)

- Registering point sets
  - Class of transformation known
  - Rigid, affine, similarity…
- ICP algorithm
  - Match points to their nearest neighbor
  - Compute transformation
    - Orthogonal Procrustes Problem
- Locally minimizes

$$\min_{\mathrm{M}} \sum_{i} (y_{nn(M(x_i))} - M(x_i))^2$$

$nn(M(x_i))$ Nearest neighbor of $M(x_i)$



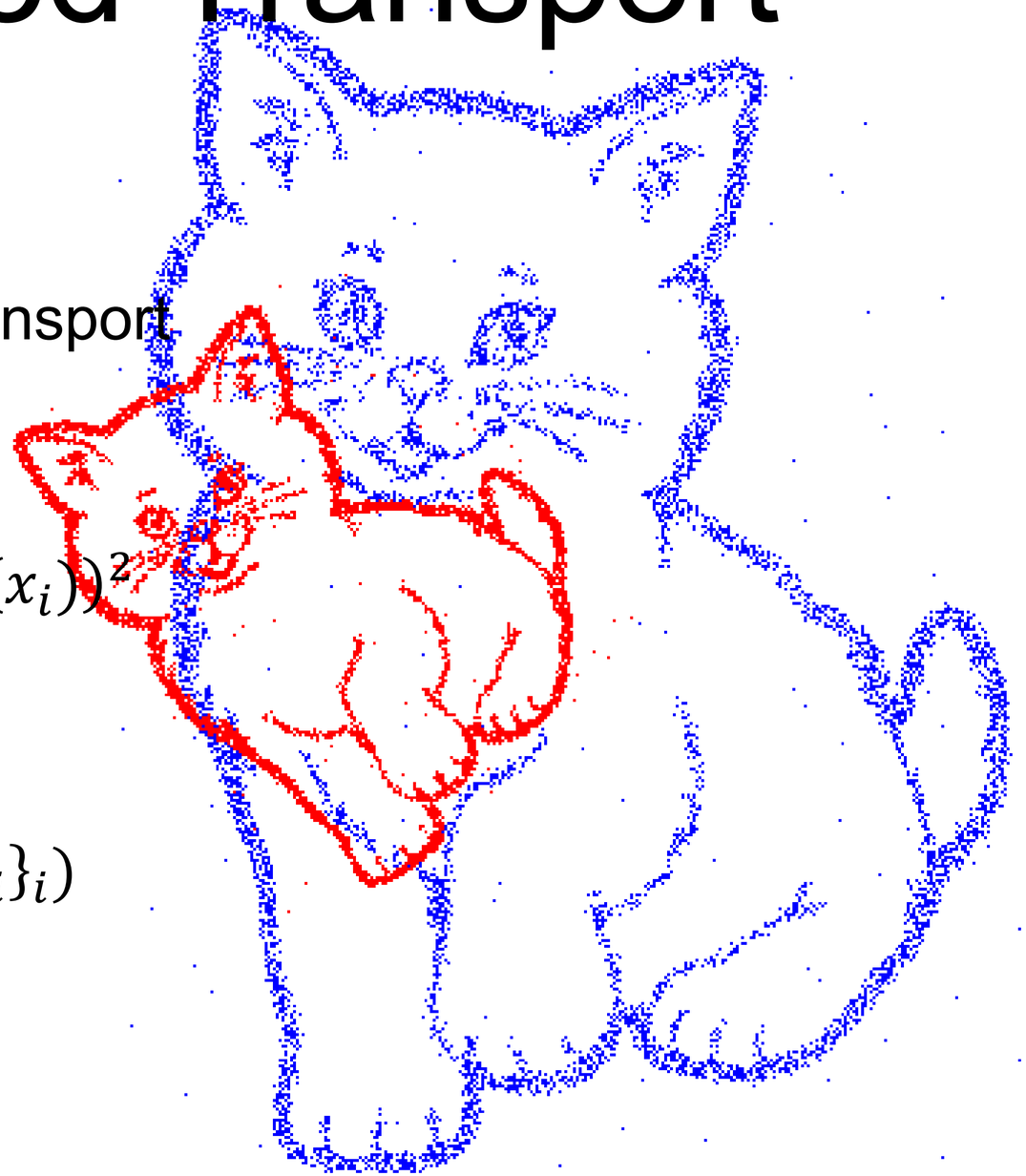
Fast Iterative Sliced Transport

- Replaces nearest neighbor with optimal transport
- Ideally, minimizes

$$\min_{\mathrm{M}} \sum_{i} (y_{OT(M(x_i))} - M(x_i))^2$$

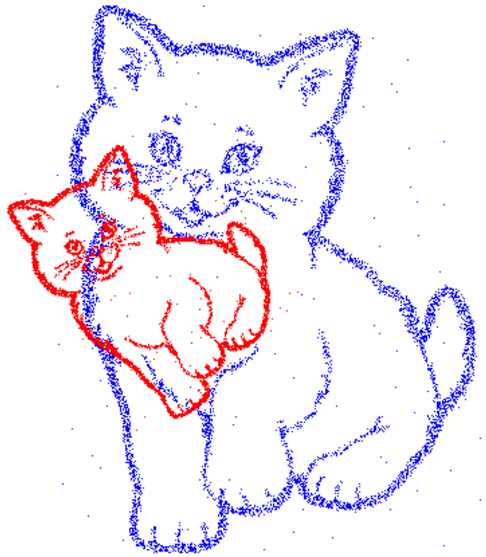
In practice:

$$\min_{\mathrm{M}} E(\{M(x_i)\}_i, \{y_i\}_i)$$



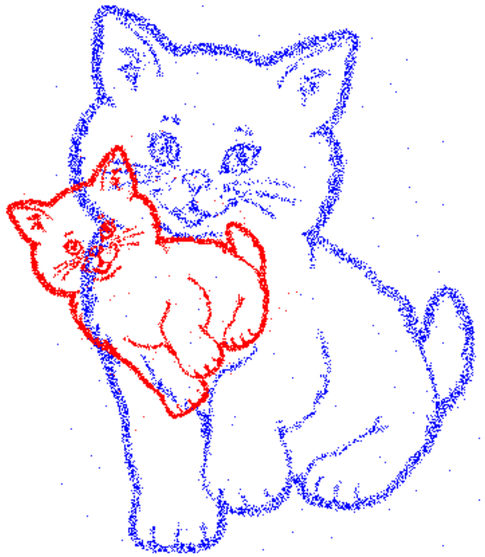
Source: 10k samples
Target: 10k samples

0



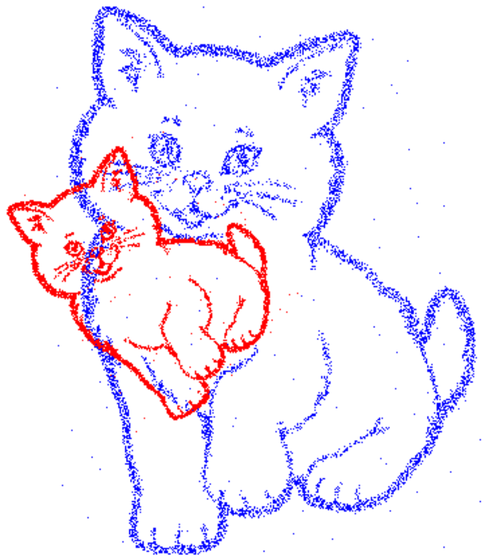
ICP
(0.006 s / iteration)

0



Iterative Transport
with network simplex
(155 s / iteration)

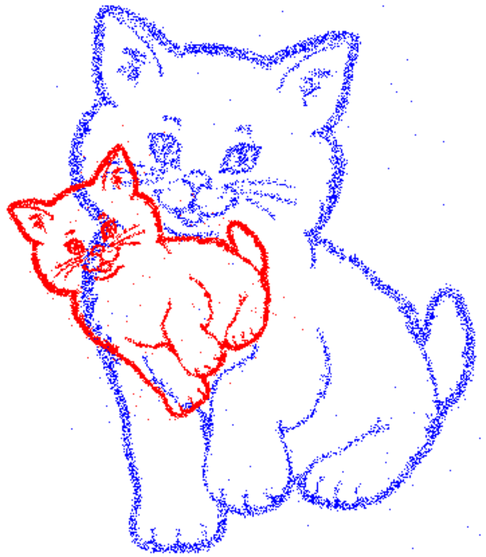
0



Our FIST algorithm
(0.04 s / iteration)

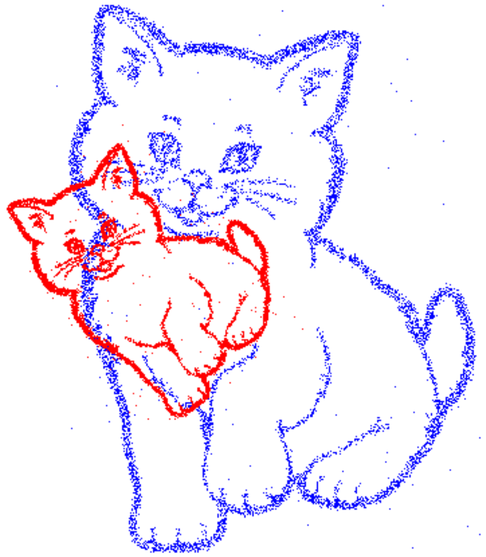
Source: 8k samples
Target: 10k samples

0



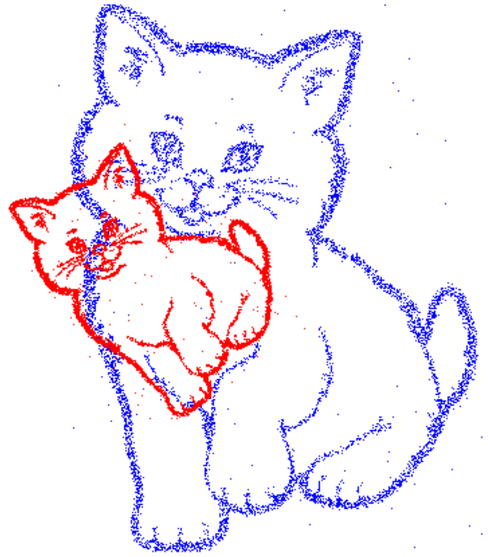
ICP
(0.005 s / iteration)

0



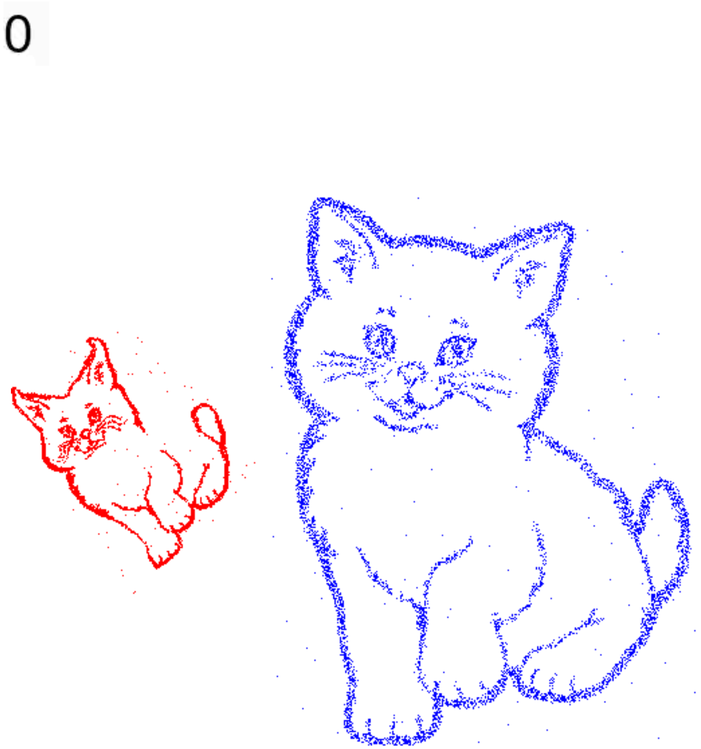
Iterative Transport
with network simplex
(40 s / iteration)

0

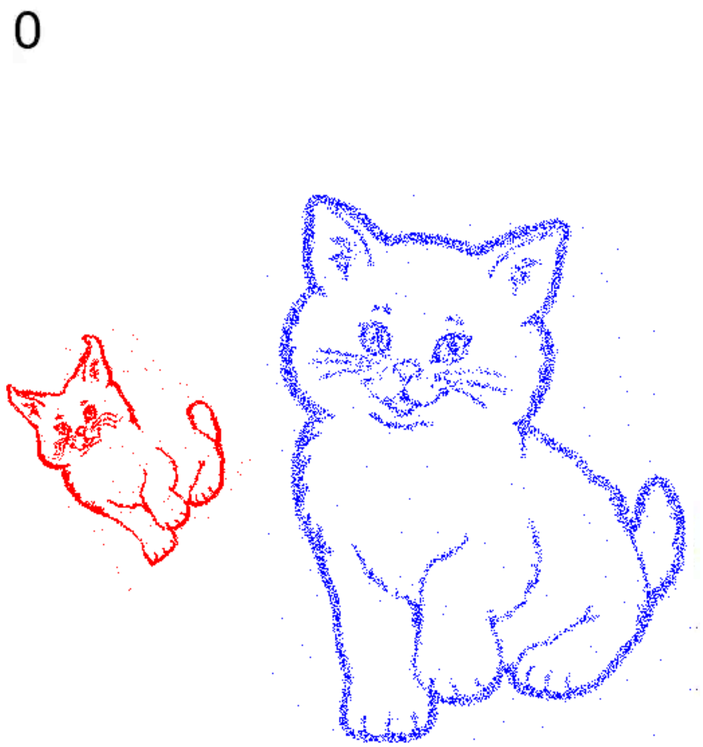


Our FIST algorithm
(0.04 s / iteration)

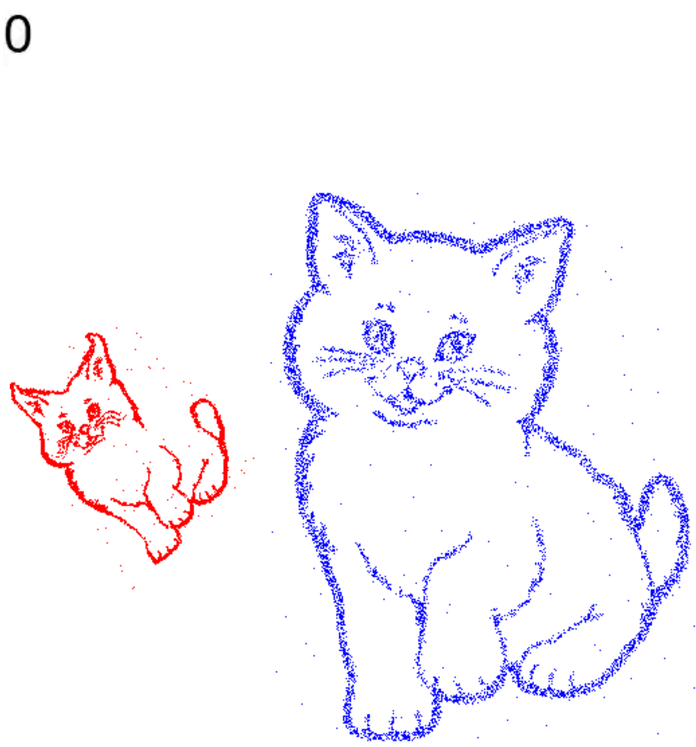
Source: 8k samples
Target: 10k samples



ICP
(0.005 s / iteration)



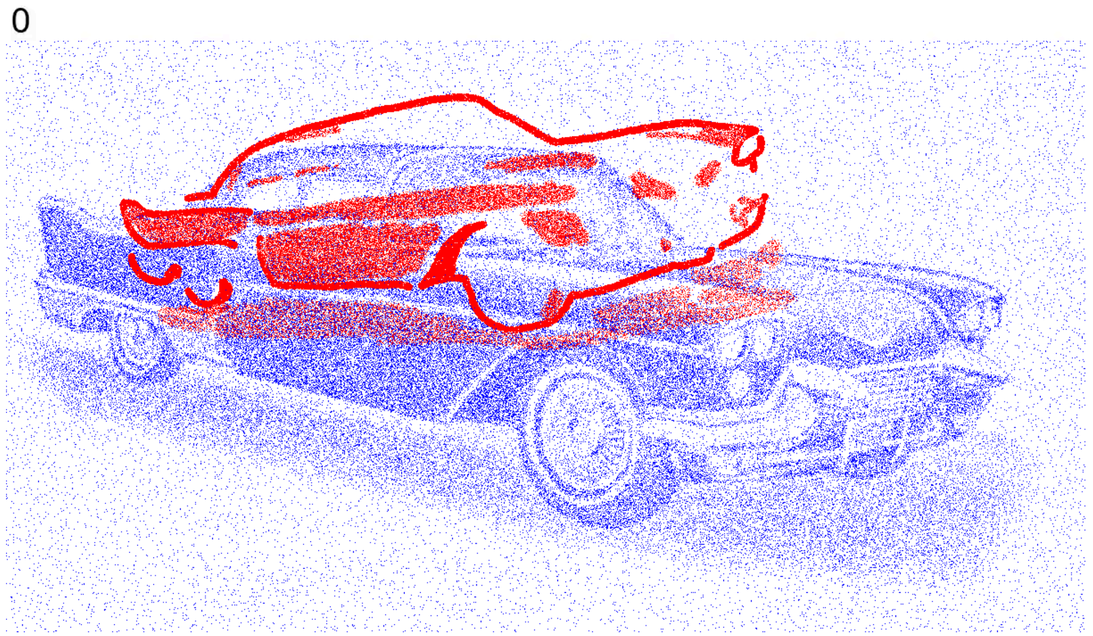
Iterative Transport
with network simplex
(40 s / iteration)



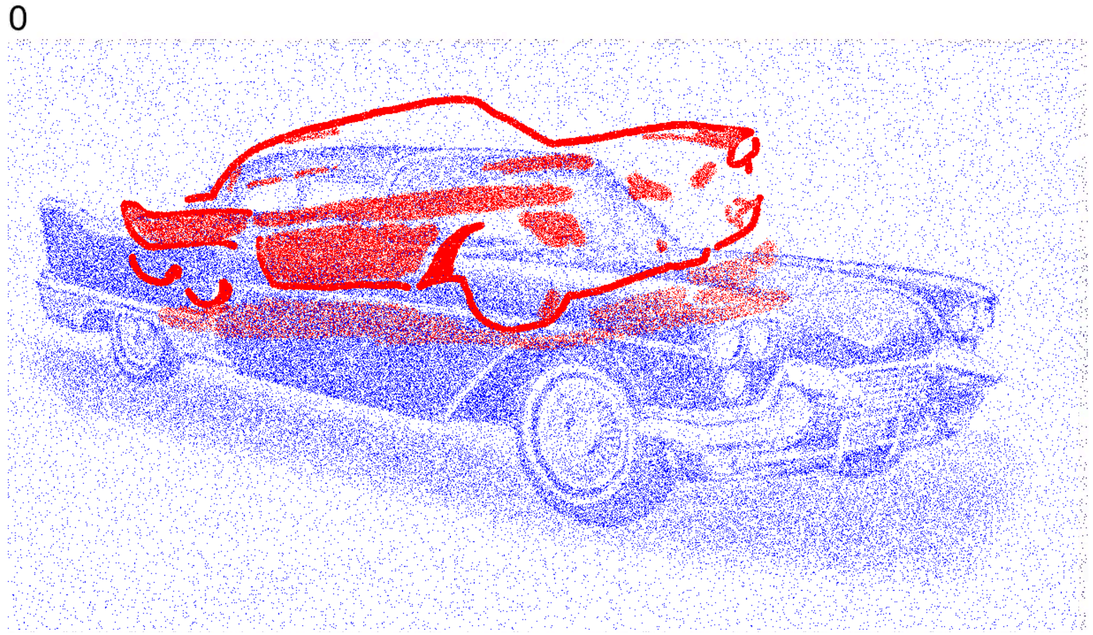
Our FIST algorithm
(0.04 s / iteration)

Source: 90k samples
Target: 100k samples

(input too large for iterative transport with network simplex)



ICP
(0.05 s / iteration)



Our FIST algorithm
(0.66 s / iteration)

Source: 120k samples
Target: 200k samples

(input too large for iterative transport with network simplex)

ICP
(0.07 s / iteration)

Our FIST algorithm
(1.72 s / iteration)

Source: 90k samples
Target: 100k samples

(input too large for iterative transport with network simplex)

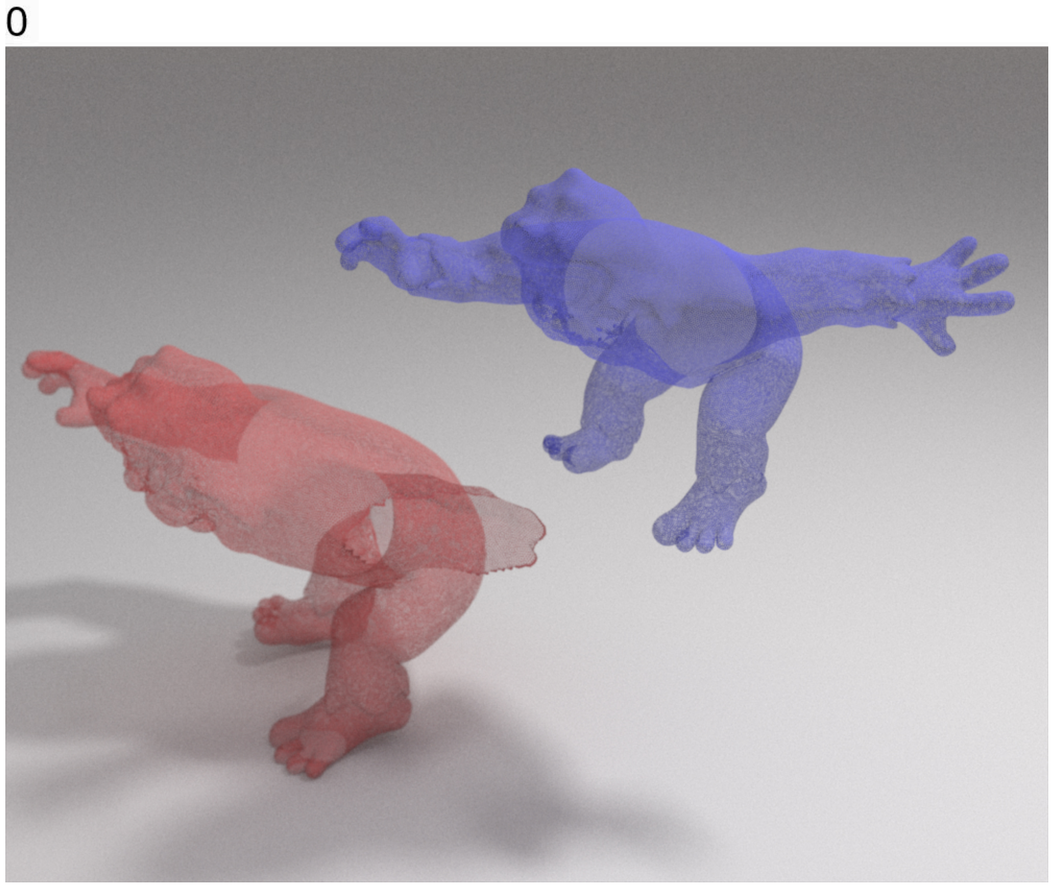
ICP
(0.05 s / iteration)

Our FIST algorithm
(0.69 s / iteration)

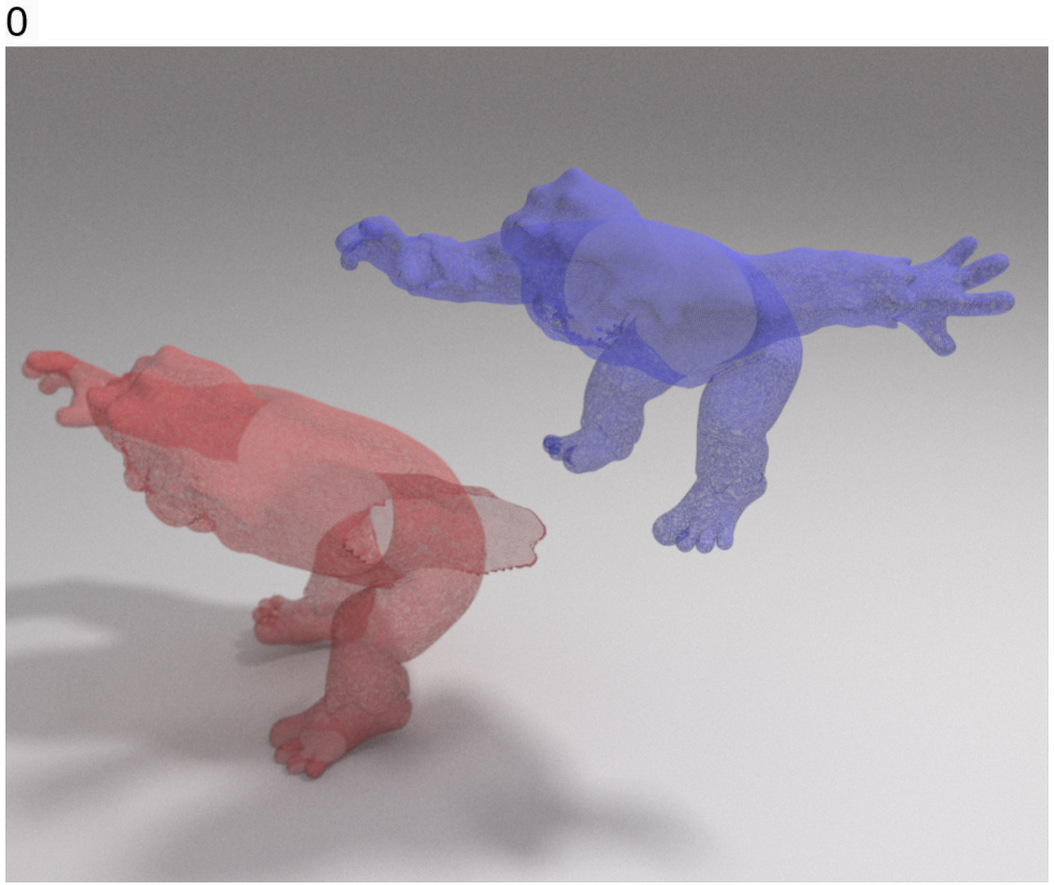
$$\min_{\mathrm{M}} \sum_{i} (y_{nn(\boldsymbol{M}(\boldsymbol{x_i}))} - M(x_i))^2 \;=\; 0\,!$$

Source: 80k samples
Target: 100k samples

(input too large for iterative t transport with network simplex)



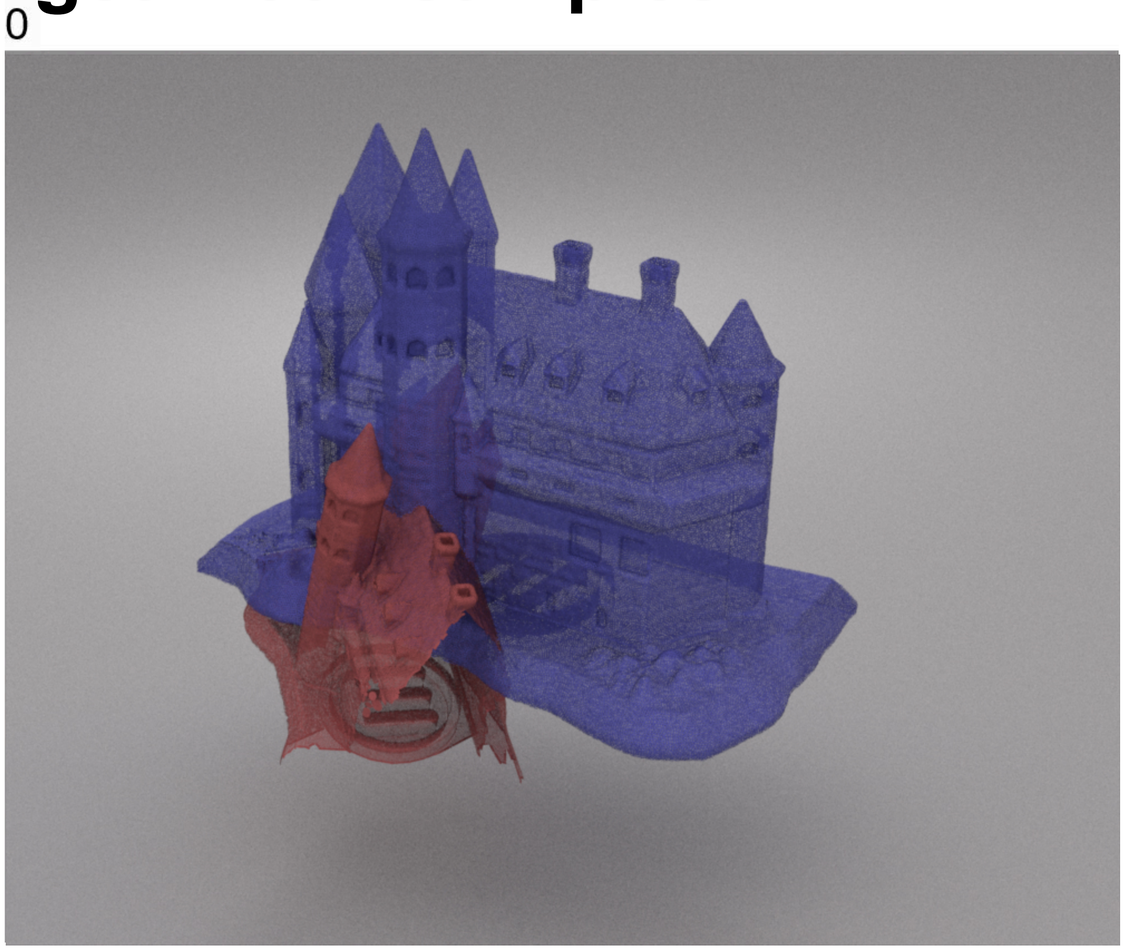
ICP
(0.05 s / iteration)



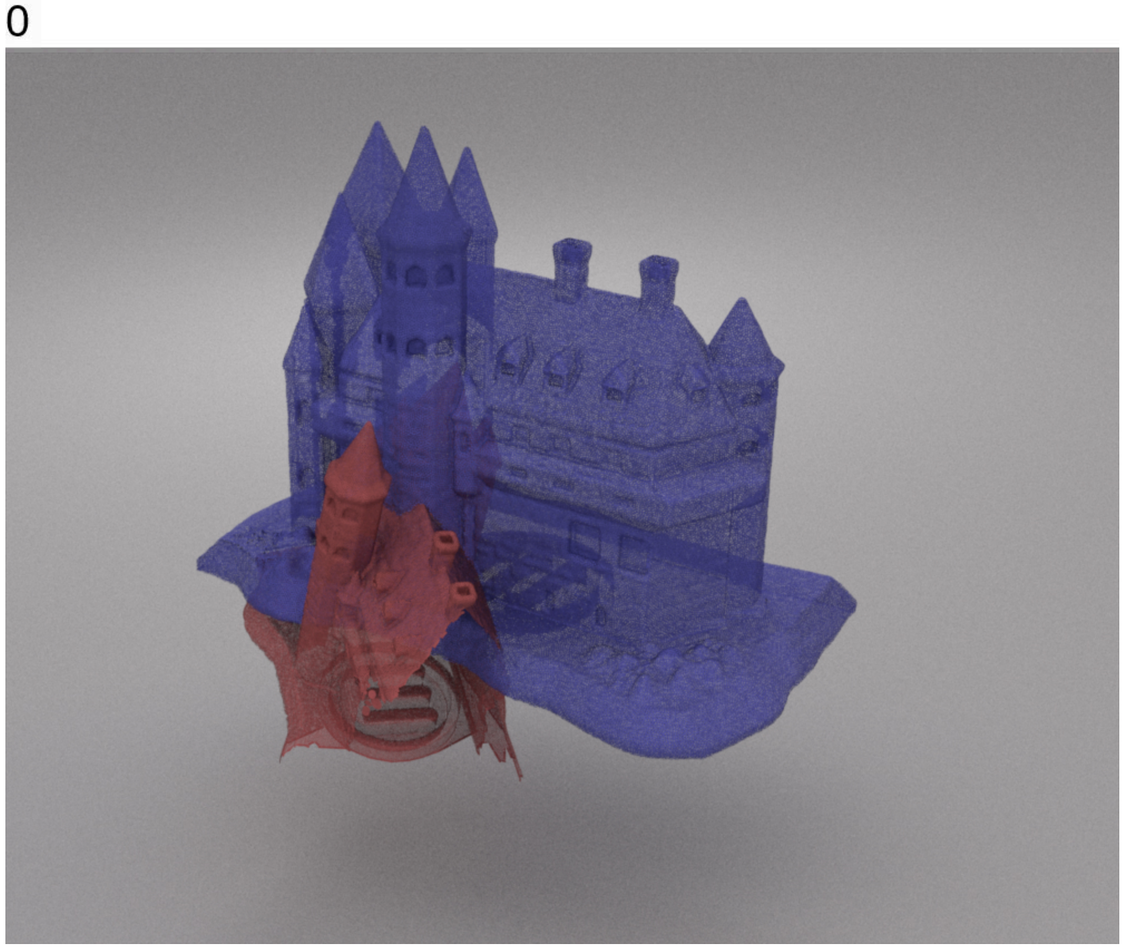
Our FIST algorithm
(2.29 s / iteration)

Source: 150k samples
Target: 200k samples

(input too large for iterative transport with network simplex)

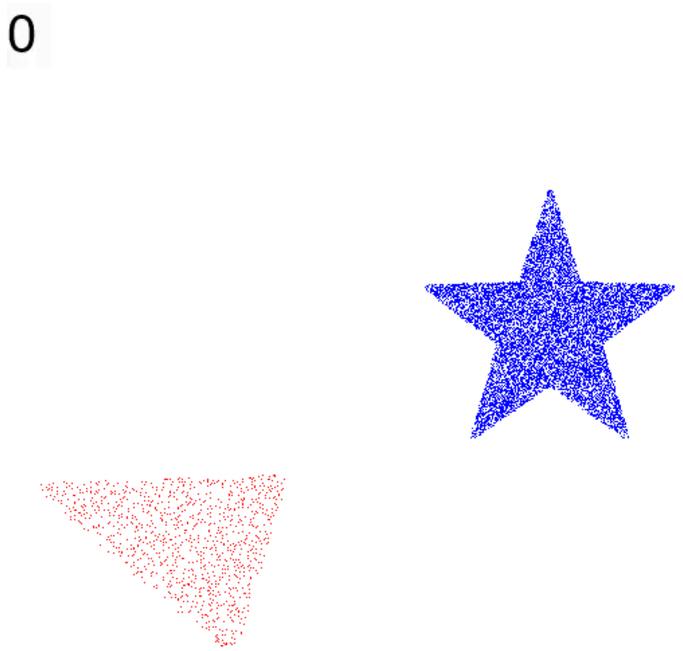


ICP
(0.09 s / iteration)

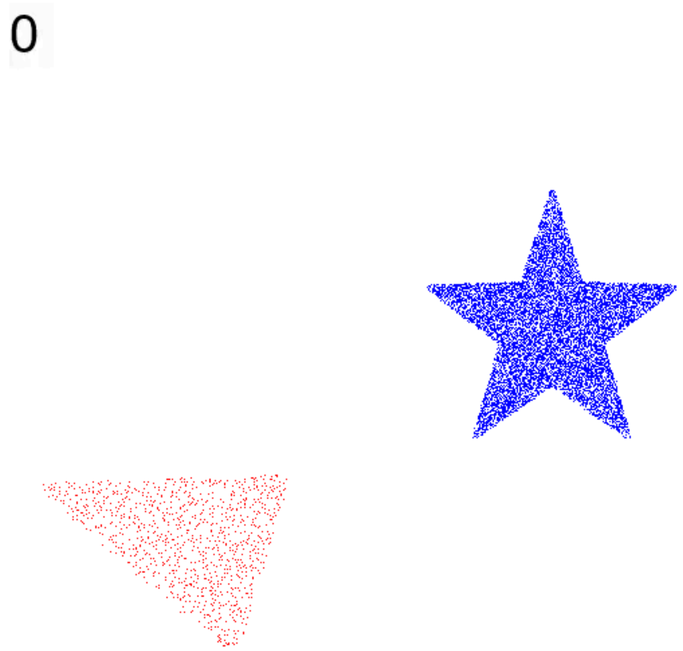


Our FIST algorithm
(2.18 s / iteration)

Failure case: the transport is optimal only on projections



Iterative Transport with Network Simplex



Our FIST algorithm

Conclusions

- Fast partial optimal transport in 1d
  - Quadratic-time algorithm (worst case)
  - Quasi-linear time decomposition

- Sliced Partial Optimal Transport
  - Same implementation for any dimension
  - Convergence in high dimension ?

- Fast Iterative Sliced Transport : variant of ICP

- Applications: point cloud registration, color matching
  - …. Machine learning ?