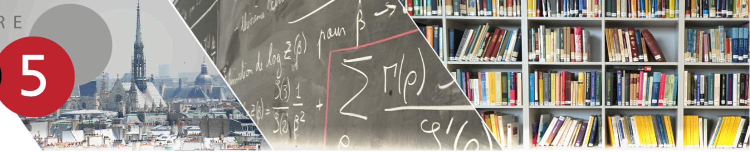
LABORATOIRE MAP5



The Shannon Total Variation

Rémy Abergel,
joint work with Lionel Moisan.

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UNIVERSITÉ PARIS DESCARTES

LABORATOIRE MAP5

Conference on variational methods and optimization in imaging,
The Mathematics of Imaging, IHP, February 6, 2019.

Given an image $U : \Omega_c \subset \mathbb{R}^2 \to \mathbb{R}$ that belongs to the Sobolev space $\mathcal{W}^{1,1}(\Omega_c)$, we note

$$\mathrm{TV}(U) = \int_{\Omega_c} |\mathrm{D}U(x,y)| \, dx \, dy \, .$$

This definition can be extended to the space $\mathrm{BV}(\Omega_c)$ of functions with bounded variation that are non-necessarily differentiable.

Definition (discrete total variation)

The total variation of a discrete image $u : \Omega \subset \mathbb{Z}^2 \to \mathbb{R}$ is generally defined by

$$\mathrm{TV}^{\mathrm{d}}(u) = \sum_{(x,y)\in\Omega} |\nabla u(x,y)| \, ,$$

where ∇ denotes a finite-differences scheme.

The total variation

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where ∇ denotes a finite-differences scheme.

The use of the total variation in image processing has become popular with the work of Rudin, Osher et Fatemi[1] (ROF), who proposed an image denoising model based on the minimization of the energy

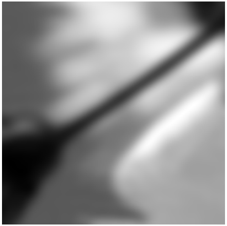
$$\forall u \in \mathbb{R}^{\Omega}, \quad E_{\mathrm{ROF}}(u) = \underbrace{\|u - u_0\|_2^2}_{\text{data fidelity}} + \lambda \underbrace{\mathrm{TV}^{\mathrm{d}}(u)}_{\substack{\text{regularity} \\ \text{(promoting sparsity)}}} \,,$$

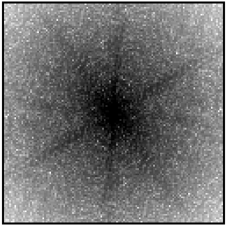
where u_0 represents the noisy image and $\lambda \in \mathbb{R}_+$ a regularity parameter that must be set by the user.

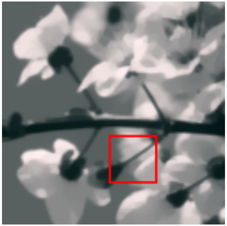
[1] **L. I. Rudin, S. Osher, and E. Fatemi**. "Nonlinear total variation based noise removal algorithms". *Physica D: Nonlinear Phenomena*, 1992.

We denoise an image u_0 by computing the minimizer of the ROF energy:

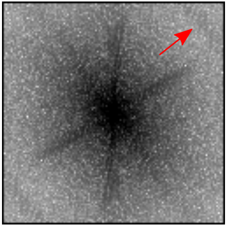

(a) input image u_0


Shannon zooming of (a)

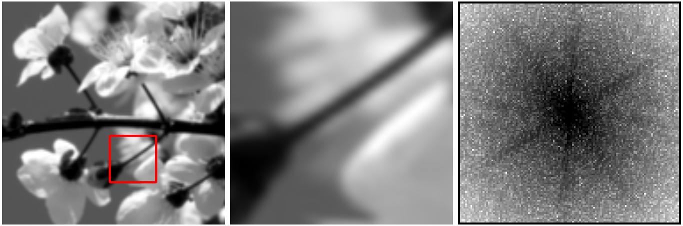

spectrum of (a)


(b) denoised image


Shannon zooming of (b)


spectrum of (b)

We denoise an image u_0 by computing the minimizer of the ROF energy:



(a) input image u_0 | Bicubic zooming of (a) | spectrum of (a)

(b) denoised image | Bicubic zooming of (b) | spectrum of (b)

This Theorem states that a *band-limited* function can be recovered exactly from an infinite set of samples.

Theorem (Shannon Sampling Theorem)

Let $U : \mathbb{R}^d \to \mathbb{R}$ be an absolutely integrable function whose Fourier transform

$$\forall \xi \in \mathbb{R}^d, \quad \widehat{U}(\xi) = \int_{\mathbb{R}^d} U(x) e^{-i\langle \xi, x \rangle} \, dx \,,$$

satisfies $\widehat{U}(\xi) = 0$ si $\xi \notin [-\pi, \pi]^d$. Then, U is uniquely determined by its values on $\mathbb{Z}^d$ since we have

$$\forall x \in \mathbb{R}^d, \quad U(x) = \sum_{k \in \mathbb{Z}^d} U(k) \, sinc(x - k) \,,$$

where we have set $sinc(x_1, \ldots, x_d) = \prod_{j=1}^{d} \frac{\sin(\pi x_j)}{\pi x_j}$ and $\frac{\sin(0)}{0} = 1$.

Definition (discrete Shannon interpolation (2D))

Given a discrete domain $\Omega = \{0, \dots, M-1\} \times \{0, \dots, N-1\}$, and an image $u : \Omega \to \mathbb{R}$, we call discrete Shannon interpolation of u the (M, N)-periodical function $U : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\forall (x, y) \in \mathbb{R}^2, \quad U(x, y) = \sum_{(k,\ell) \in \Omega} u(k, \ell)\, sincd_M(x-k)\, sincd_N(y-\ell)\,,$$

where

$$sincd_M(x) = \begin{cases} \dfrac{\sin(\pi x)}{M \sin\left(\frac{\pi x}{M}\right)} & \text{if M is odd,} \\[2ex] \dfrac{\sin(\pi x)}{M \tan\left(\frac{\pi x}{M}\right)} & \text{if M is even.} \end{cases}$$

We can show that the Shannon interpolate of a discrete image can be evaluated in the Fourier domain.

Proposition

The Shannon interpolate of a discrete image $u : \Omega \to \mathbb{R}$ satisfies

$$U(x,y) = \frac{1}{MN} \sum_{\substack{\alpha,\beta\in\mathbb{Z} \\ -\frac{M}{2}\leq\alpha\leq\frac{M}{2} \\ -\frac{N}{2}\leq\beta\leq\frac{N}{2}}} \varepsilon_M(\alpha)\varepsilon_N(\beta)\, \widehat{u}(\alpha,\beta)\, e^{2i\pi\left(\frac{\alpha x}{M}+\frac{\beta y}{N}\right)}\,,$$

where ε_M et ε_N are given[2] by

$$\varepsilon_M(\alpha) = \begin{cases} 1 & \text{si } |\alpha| < M/2 \\ 1/2 & \text{si } |\alpha| = M/2 \end{cases} \qquad \varepsilon_N(\beta) = \begin{cases} 1 & \text{si } |\beta| < N/2 \\ 1/2 & \text{si } |\beta| = N/2\,. \end{cases}$$

This interpolation formula is useful to apply precise subpixellic geometric transforms (rotations, translations, zoom) to discrete images.

[2] **R. Abergel and L. Moisan**: “The Shannon Total Variation”, Journal of Mathematical Imaging and Vision, 2017.

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We call Shannon total variation (STV) of the discrete image u the continuous totale variation of U.

Definition (STV_∞)

$$\mathrm{STV}_\infty(u) := \mathrm{TV}(U) = \int_{[0,M]\times[0,N]} |\mathrm{D}U(x,y)|\, dxdy\,.$$

For practical implementations, we can estimage $\mathrm{STV}_\infty(u)$ using a Riemann sum (involving an oversampling factor $n = 2$ or 3 for the integration domain).

Definition (STV_n)

For any integer $n \geq 1$, set

$$\mathrm{STV}_n(u) = \frac{1}{n^2} \sum_{(k,\ell)\in\Omega_n} \left|\mathrm{D}U\left(\tfrac{k}{n}, \tfrac{\ell}{n}\right)\right| = \frac{1}{n^2} \sum_{(k,\ell)\in\Omega_n} |\mathrm{D}_n u(k,\ell)|\,,$$

where $\mathrm{D}_n u(k,\ell) = \mathrm{D}U(\frac{k}{n}, \frac{\ell}{n})$, and $\Omega_n = \{0,\dots,nM-1\}\times\{0,\dots,nN-1\}$.

STV_n and TV^{d} share the same structure since $\mathrm{TV}^{\mathrm{d}}(u) = \sum_{(k,\ell)\in\Omega} |\nabla u(k,\ell)|$

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Related works

- **F. Malgouyres and F. Guichard**. “Edge direction preserving image zooming: a mathematical and numerical analysis.” SIAM Journal (2001).
- **L. Moisan**. “How to discretize the total variation of an image?” PAMM Proceedings (2007).
- **J. Preciozzi, P. Musé, A. Almansa, S. Durand, F. Cabot, Y. Kerr and B. Rougé**. “Sparsity-based restoration of SMOS images in the presence of outliers.” IGARSS (2012).
- **J. Preciozzi, P. Musé, A. Almansa, S. Durand, A. Khazaal and B. Rougé**. “SMOS images restoration from L1A data: A sparsity-based variational approach.” IGARSS Proceedings (2014).
- **D.C. Soncco, C. Barbanson, M. Nikolova, A. Almansa and Y. Ferrec**. “Fast and accurate multiplicative decomposition for fringe removal in interferometric images”. IEEE Transactions on Computational Imaging, (2017).
- **T. Briand and J. Vacher**. “How to apply a filter defined in the frequency domain by a continuous function”. Image Processing On Line, (2016).
- **R. Abergel and L. Moisan**. “The Shannon Total Variation”, Journal of Mathematical Imaging and Vision (2017).

As in the discrete setting, a dual formulation of STV_n can be easily derived.

Proposition (dual formulation of STV_n)

$$\mathrm{STV}_n(u) = \max_{p:\Omega_n \to \mathbb{R}^2} \; \langle \tfrac{1}{n^2} \mathrm{D}_n u, p \rangle - \delta_{\mathscr{B}_*}(p)$$

where

$$\delta_{\mathscr{B}_*}(p) = \begin{cases} 0 & \text{if } \displaystyle\max_{(k,\ell)\in\Omega_n} |p(k,\ell)| \leq 1\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Sketch of proof.

1. The Legendre-Fenchel transform of $\|\cdot\|_{1,2}$ is $\|\cdot\|_{1,2}^\star = \delta_{\mathscr{B}_*}$,
2. thus, $\mathrm{STV}_n(u) = \|\frac{1}{n^2}\mathrm{D}_n u\|_{1,2} = \|\frac{1}{n^2}\mathrm{D}_n u\|_{1,2}^{\star\star} = \delta_{\mathscr{B}_*}^\star(\frac{1}{n^2}\mathrm{D}_n u)$,
3. besides, the supremum involved in $\delta_{\mathscr{B}_*}^\star$ is a maximum.

In order to evaluate

$$\mathrm{STV}_n(u) = \frac{1}{n^2} \sum_{(k,\ell)\in\Omega_n} |\mathrm{D}_n u(k,\ell)| \,,$$

we need to compute $\mathrm{D}_n u$ over Ω_n. The following proposition explain how $\mathrm{D}_n u$ can be computed efficiently in the Fourier domain.

Proposition (Fast evaluation of $\mathrm{D}_n u$)

Let $n > 1$ and $\widehat{\Omega}_n := \left[-\frac{nM}{2}, \frac{nM}{2}\right) \times \left[-\frac{nN}{2}, \frac{nN}{2}\right) \cap \mathbb{Z}^2$ the cannonical frequency domain associated to Ω_n. For all $(\alpha,\beta) \in \widehat{\Omega}_n$, we have

$$\widehat{\mathrm{D}_n u}(\alpha,\beta) = n^2\, \varepsilon_M(\alpha)\varepsilon_N(\beta)\, Z_n\widehat{u}(\alpha,\beta)\, 2i\pi \begin{pmatrix} \alpha/M \\ \beta/N \end{pmatrix} ,$$

where

$$Z_n\widehat{u}(\alpha,\beta) = \begin{cases} \widehat{u}(\alpha,\beta) & \text{si } |\alpha| \le \frac{M}{2} \text{ et } |\beta| \le \frac{N}{2} \\ 0 & \text{sinon.} \end{cases}$$

Besides, we have the upper-bound $|||\mathrm{D}_n||| \le \pi n \sqrt{2}$.

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Numerical evaluation of $\mathrm{STV}_n(u)$

In order to evaluate

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Besides, we have the upper-bound $|||\mathrm{D}_n||| \leq \pi n\sqrt{2}\,.$

Example of Matlab implementation

Practical computations of the gradient field D_n and its adjoint $D_n^* = -\mathrm{div}_n$ do not raise particular difficulties.

For instance, in the case of odd dimensions,

```matlab
function [gx,gy] = sgrad(u,n)

  %% compute frequency grids
  [N,M] = size(u);
  nM = n*M; nN = n*N;
  a = ifftshift(-floor(M/2)+(0:M-1));
  b = ifftshift(-floor(N/2)+(0:N-1));
  [A,B] = meshgrid(a,b);

  %% compute the gradient field in the Fourier domain
  dft_u = fft2(u);
  dft_gx = zeros(nN,nM);
  dft_gy = zeros(nN,nM);

  dft_gx(1+mod(nM+a,nM),1+mod(nN+b,nN)) = n^2*2*1i*pi*(A/M) .* dft_u;
  dft_gy(1+mod(nM+a,nM),1+mod(nN+b,nN)) = n^2*2*1i*pi*(B/N) .* dft_u;

  %% get back to the spatial domain
  gx = ifft2(dft_gx);
  gy = ifft2(dft_gy);

end
```

```matlab
function div = sdiv(px,py,n)

  %% compute frequency grids
  [nN,nM] = size(px);
  M = nM/n; N = nN/n;
  a = ifftshift(-floor(M/2) + (0:M-1));
  b = ifftshift(-floor(N/2) + (0:N-1));
  [A,B] = meshgrid(a,b);

  %% compute the divergence in the Fourier domain
  dft_div = zeros(N,M);
  dft_px = fft2(px);
  dft_py = fft2(py);

  dft_div(1+mod(N+b,N),1+mod(M+a,M)) = 2*1i*pi*(dft_px(1+mod(nN+b,nN),1+mod(nM+a,nM)).*(A/M) + ...
                                                dft_py(1+mod(nN+b,nN),1+mod(nM+a,nM)).*(B/N));


  %% get back to the spatial domain
  div = ifft2(dft_div);

end
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More general implementations of D_n and its adjoint $D_n^* = -\mathrm{div}_n$ are available online in Matlab or C language.

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  %% compute the divergence in the Fourier domain
  dft_div = zeros(N,M);
  dft_px = fft2(px);
  dft_py = fft2(py);

  dft_div(1+mod(N+b,N),1+mod(M+a,M)) = 2*1i*pi*(dft_px(1+mod(nN+b,nN),1+mod(nM+a,nM)).*(A/M) + ...
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  %% get back to the spatial domain
  div = ifft2(dft_div);

end
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More general implementations of D_n and its adjoint $D_n^* = -\mathrm{div}_n$ are available online in Matlab or C language.

Given a noisy image u_0, we consider the STV_n variant of the ROF model

$$\operatorname*{argmin}_{u:\Omega\to\mathbb{R}} \|u - u_0\|_2^2 + \lambda\, \mathrm{STV}_n(u)\,.$$

We have the primal-dual reformulation

$$\operatorname*{argmin}_{u:\Omega\to\mathbb{R}} \max_{p:\Omega_n\to\mathbb{R}^2} \|u - u_0\|_2^2 + \langle \tfrac{\lambda}{n^2}\mathrm{D}_n u, p\rangle - \delta_{\mathscr{B}_*}(p)\,,$$

which can be numerically computed using, for instance, the Chambolle-Pock algorithm[3].

[3] **A. Chambolle, T. Pock**: "A first-order primal-dual algorithm for convex problems with applications to imaging", *Journal of Mathematical Imaging and Vision*, 2011.

Image denoising

Given a noisy image u_0, we consider the STV_n variant of the ROF model

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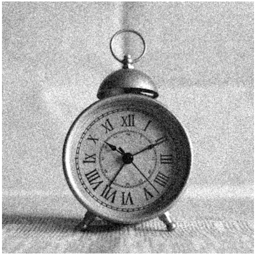
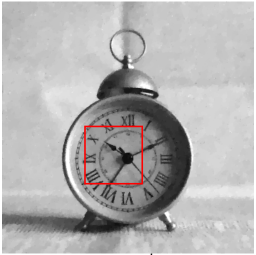
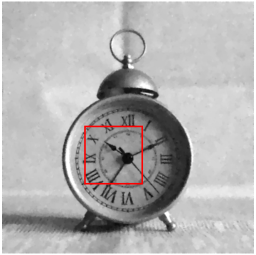


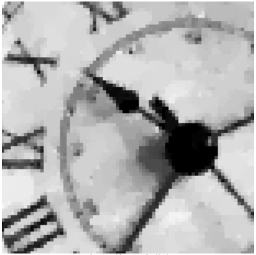
(a) noisy image u_0 ($\sigma = 20$)



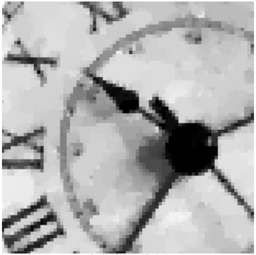
(b) TV^{d}



(c) STV_2

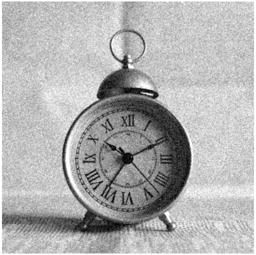


details of (b)

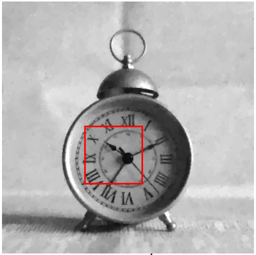


details of (c)

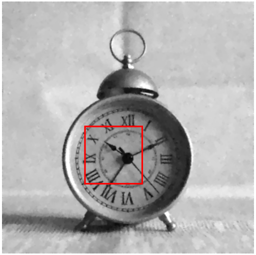
Image denoising



(a) noisy image u_0 ($\sigma = 20$)



(b) TV^{d}

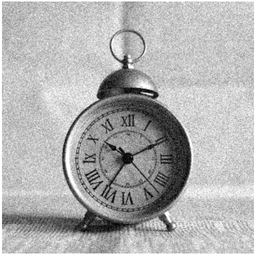


(c) STV_2

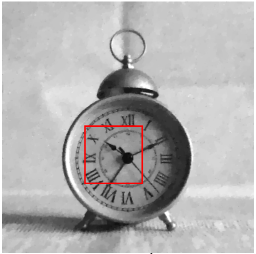
Bicubic zooming of (b)

Bicubic zooming of (c)

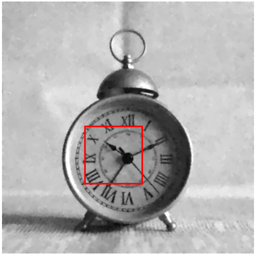
Image denoising



(a) noisy image u_0 ($\sigma = 20$)



(b) TV^{d}



(c) STV_2

Shannon zooming of (b)

Shannon zooming of (c)

We can also use STV_n as a regularizer for inverse problems. Given a linear operator $A : \mathbb{R}^\Omega \to \mathbb{R}^\omega$, and $u_0 : \omega \to \mathbb{R}$, consider

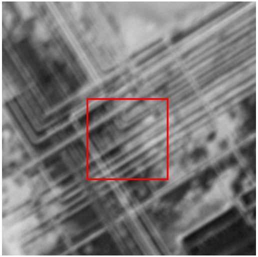
$$\operatorname*{argmin}_{u:\Omega\to\mathbb{R}} \quad \underbrace{\|Au - u_0\|_2^2}_{f(Au)} + \lambda \mathrm{STV}_n(u)\,,$$

with primal-dual reformulation (use $f(Au) = f^{\star\star}(Au)$)

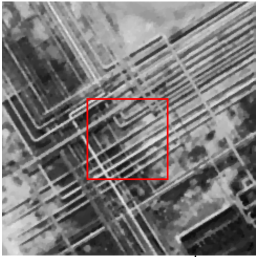
$$\operatorname*{argmin}_{u:\Omega\to\mathbb{R}} \quad \max_{\substack{p:\Omega_n\to\mathbb{R}^2\\ q:\omega\to\mathbb{R}}} \quad \left\langle \left(\tfrac{\lambda}{n^2}\mathrm{D}_n u, Au\right), (p,q)\right\rangle - \left(\delta_{\mathscr{B}_*}(p) + \|\tfrac{q}{2} + u_0\|_2^2\right).$$

and the Chambolle-Pock Algorithm can be used again.

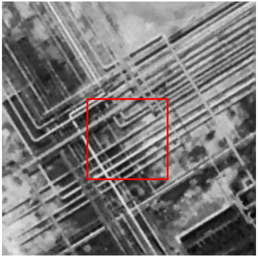
Consider that $Au = k * u$ is the convolution between u and a given motion blur kernel k.



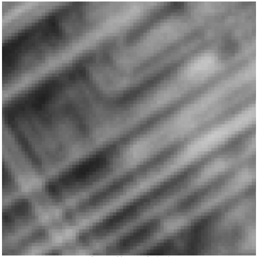
(a) blurry & noisy image u_0 ($\sigma = 2$)



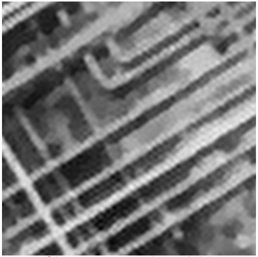
(b) discrete TV (TV^{d})



(c) Shannon TV (STV_2)



details of (a)



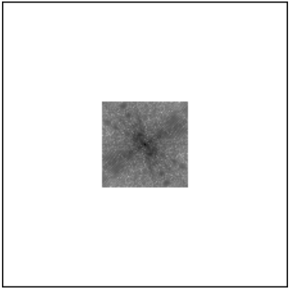
Shannon zooming of (b)

Shannon zooming of (c)

Now, A is a frequency masking operator of the type du type

$$\widehat{Au}(\alpha, \beta) = \begin{cases} \widehat{u}(\alpha, \beta) & \text{if } (\alpha, \beta) \in \widehat{\omega_0}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

image u_0

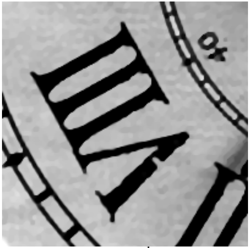


spectrum of u_0

[4] **F. Guichard and F. Malgouyres**: "Total variation based interpolation". In proceedings of the 9th European Signal Processing Conference (EUSIPCO), 1998

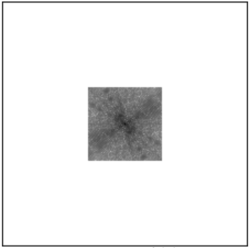
Spectrum extrapolation

(a) image u_0

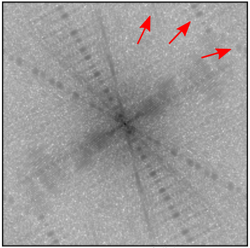


(b) TV^{d}

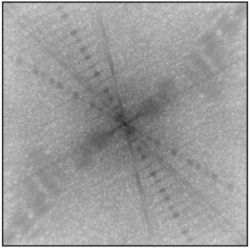
(c) STV_2



spectrum of (a)



spectrum of (b)

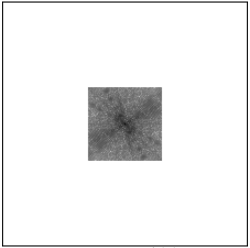


spectrum of (c)

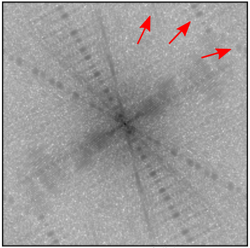
(a) image u_0

(b) TV^{d} (Shannon zooming)

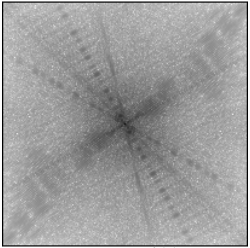
(c) STV_2 (Shannon zooming)



spectrum of (a)



spectrum of (b)



spectrum of (c)

Given an input image $u_0 : \Omega \to \mathbb{R}$ and a weight mapping $\gamma : \Omega \to \mathbb{R}_+$, we consider

$$\operatorname*{argmin}_{u:\Omega\to\mathbb{R}} \quad \|\widehat{u} - \widehat{u_0}\|_\gamma^2 + \lambda \, \mathrm{STV}_n(u)\,,$$

where

$$\|\widehat{u} - \widehat{u_0}\|_\gamma^2 = \frac{1}{|\widehat{\Omega}|} \sum_{(\alpha,\beta)\in\widehat{\Omega}} \gamma(\alpha,\beta)\cdot|\widehat{u}(\alpha,\beta) - \widehat{u_0}(\alpha,\beta)|^2\,,$$

is a weighted ℓ^2 square distance between $\widehat{u}$ et $\widehat{u_0}$, which makes the regularization adaptative with respect to the frequency.

A simple and interesting example of weighting:

$$\forall(\alpha,\beta)\in\widehat{\Omega}, \quad \gamma(\alpha,\beta) = e^{-2\pi^2\sigma^2\left(\frac{\alpha^2}{M^2}+\frac{\beta^2}{N^2}\right)}$$

- low frequencies: $\gamma(\alpha,\beta)$ is high, we enforce $\widehat{u}(\alpha,\beta) \approx \widehat{u_0}(\alpha,\beta)$;
- high frequencies: $\gamma(\alpha,\beta)$ is low, the computed value $\widehat{u}(\alpha,\beta)$ is mostly driven by the regularity term STV_n.

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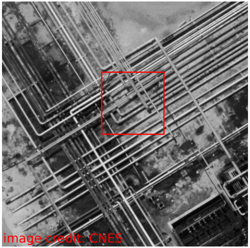
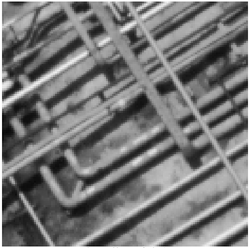
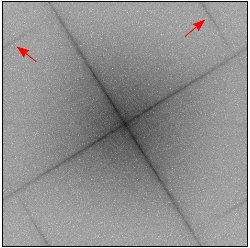
A new model: the image “Shannonizer”



(a) initial image u_0


details of (a)


spectrum of (a)

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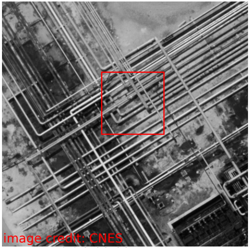

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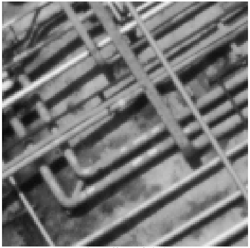
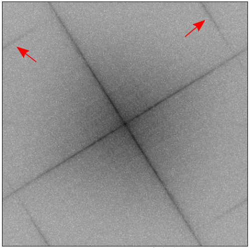
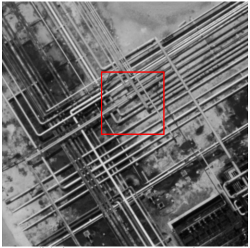


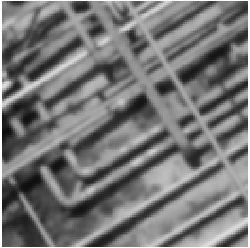
details of (a)



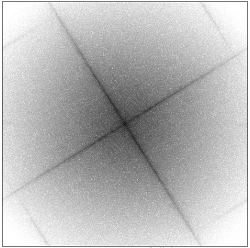
spectrum of (a)



(b) frequency attenuation

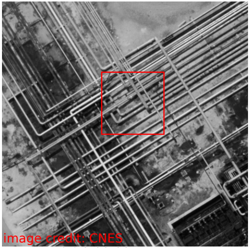


details of (b)

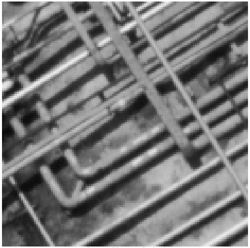


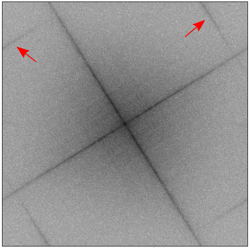
spectum of (b)

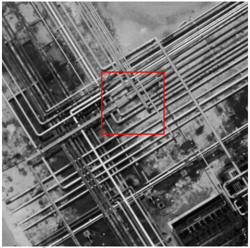
A new model: the image “Shannonizer”

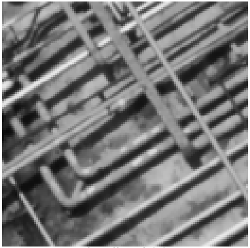


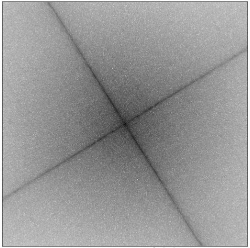
(a) initial image u_0


details of (a)

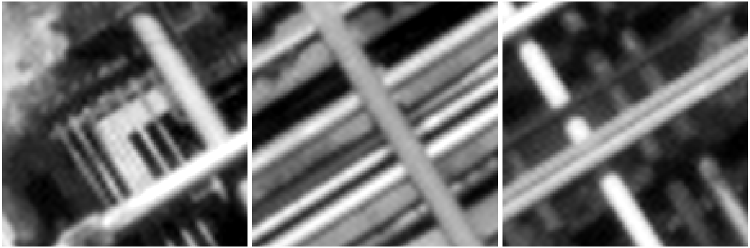

spectrum of (a)


(c) Shannonized image (STV_2)

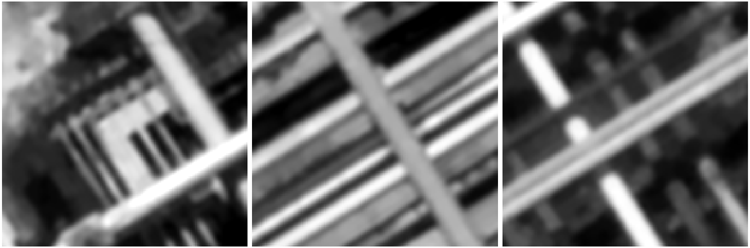

details of (c)


spectrum of (c)

A new model: the image “Shannonizer”

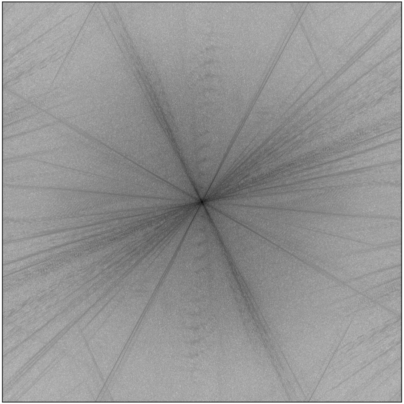


Shannon zoomings of the initial image

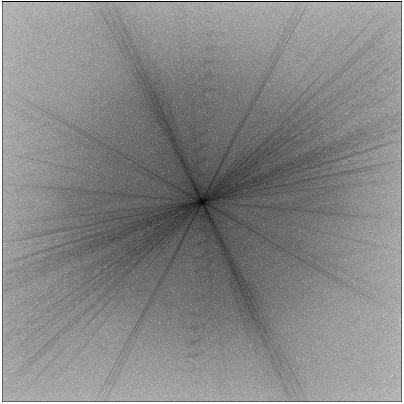


Shannon zoomings of the Shannonized image

(a) initial image

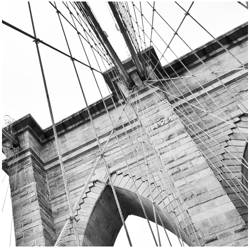

spectrum of (a)

(b) Shannonized image



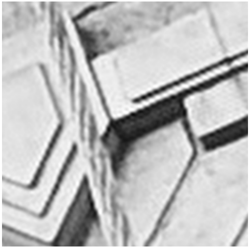
spectrum of (b)

A new model: the image “Shannonizer”

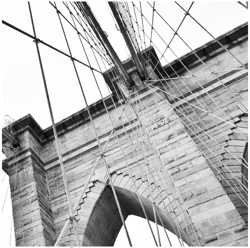


(a) initial image

details of (a)



Shannon zooming of (a)

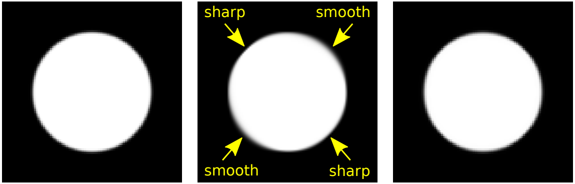


(b) Shannonized image

details of (b)

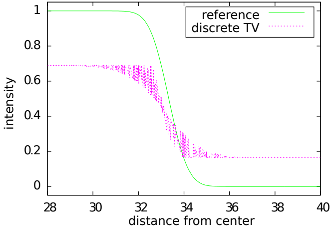
Shannon zooming of (b)

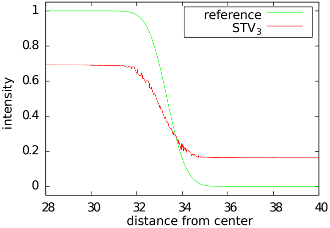
Let us denoise a rotationally invariant image using the ROF model and its STV variant.



(a) reference (b) discrete TV (c) Shannon TV

We can compare the denoising models in terms of isotropy by plotting the gray levels of the denoised image as a function of their distance to the disk center.



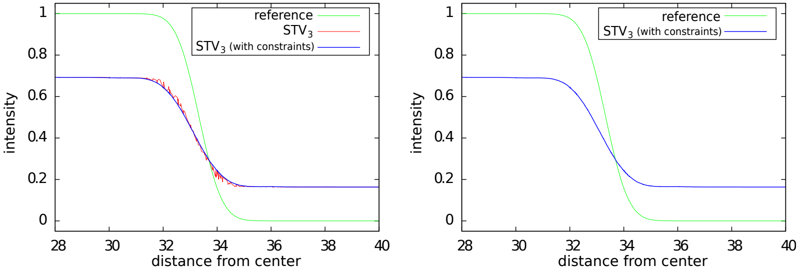


We can force the isotropy of the frequency support by adding the constraint $\mathrm{Supp}(\widehat{u}) \subset \mathcal{D}_{\widehat{\Omega}}$, where

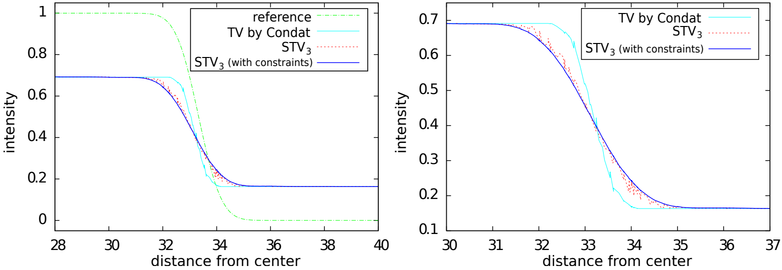
$$\mathcal{D}_{\widehat{\Omega}} = \left\{(\alpha,\beta) \in \widehat{\Omega},\ \left(\frac{\alpha}{M/2}\right)^2 + \left(\frac{\beta}{N/2}\right)^2 \leq 1\right\}.$$

We consider the constrained problem

$$\operatorname*{argmin}_{u:\Omega\to\mathbb{R}} \|u - u_0\|_2^2 + \lambda \mathrm{STV}_n(u) \quad \text{subject to} \quad \mathrm{Supp}(\widehat{u}) \subset \mathcal{D}_{\widehat{\Omega}}\,.$$



We can compare this result to that obtain using the isotropic variant of TV^{d} proposed by Condat[5].



[5] **L. Condat**: "Discrete total variation: New definition and minimization," SIAM Journal on Imaging Sciences, 2017.

Conclusion and perspectives

We studied a TV discretization scheme based on the Shannon interpolation called STV.

- STV reconciliates TV regularization with Shannon interpolation, and thus, with the Shannon Sampling Theory.

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Conclusion and perspectives

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Conclusion and perspectives

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- Preliminary results indicate an excellent level of isotropy offered by the STV model.
- We can imagine extensions of this approach to other TV models (such as TGV), but also to other functionals.

Open source implementations

Two implementations (one in langage C, the other in Matlab) of all presented algorithms are available on my webpage.

- Implementation in C language : contains independant modules for denoising, deconvolution, spectrum extrapolation and image Shannonization.
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Thank you for your attention.