Lipschitz-Killing curvatures of excursion sets for 2D random fields

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Collaboration

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Outlines

Motivation : Shot noise random fields

A **(Poisson) shot noise random field** is a random function $X : \mathbb{R}^d \to \mathbb{R}$ given by

$$\forall x \in \mathbb{R}^d, \quad X(x) = \sum_{i \in I} g_{m_i}(x - x_i), \text{ where}$$

- $\{x_i\}_{i \in I}$ is a Poisson point process of intensity $\lambda > 0$ in $\mathbb{R}^d$,
- $\{m_i\}_{i \in I}$ are independent « marks » with distribution $F(dm)$ on $\mathbb{R}^k$, and independent of $\{x_i\}_{i \in I}$.
- The functions g_m are real-valued deterministic functions, called **spot functions**, such that

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^d} |g_m(y)| \, dy \, F(dm) < +\infty.$$

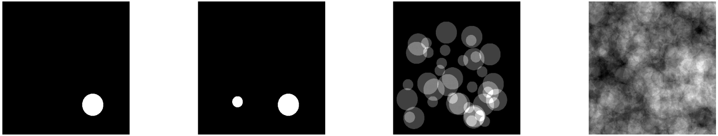
Here we consider $d = 2$ and, for sake of simplicity, $k \leq 2$ with a single $L^1(\mathbb{R}^2)$ function g randomly weighted and dilated : $(W, R) \sim F$ is a probability measure on $\mathbb{R} \times (0, +\infty)$ and for $m = (w, r)$

$$g_m(x) = wg(x/r).$$

Example 1 : disk with random radius

Let $d = 2$, T a bounded closed rectangle of $\mathbb{R}^2$ and $g = \mathbf{1}_D$. Consider random disk of radius $r = r_1$ or $r = r_2$ with $0 < r_1 < r_2$ (each with probability $1/2$), same weights $W = 1$ a.s. and intensity $\lambda > 0$

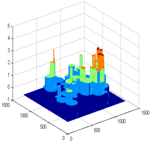
- The number n of centers in T is a Poisson random variable of parameter $\lambda|T|$
- The centers $x_1, \ldots, x_n$ are thrown uniformly, independently on T
- The radius $R_1, \ldots, R_n$ are attached to each center by flipping a coin to choose between r_1 or r_2.



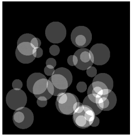
Excursion set

We consider the excursion set or the level set of level $u \in \mathbb{R}$ of X in T defined by

$$E_X(u) \cap T := \{x \in T; X(x) \geq u\} \text{ with } E_X(u) = \{X \geq u\}.$$



view 3D



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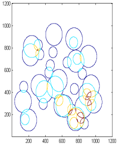
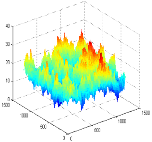
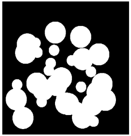
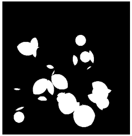


some level lines



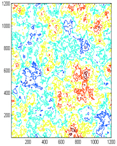


$u = 0.5$



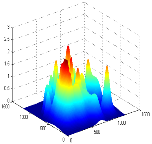
$u = 1.5$

$u = 2.5$

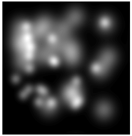


Example 2 : Gaussian kernel

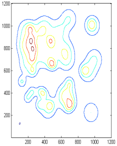
Let us choose $g(x) = e^{-\frac{\|x\|^2}{2}}$ instead of $\mathbf{1}_D$.



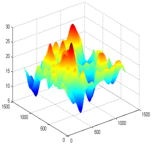
view 3D

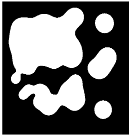


view 2D



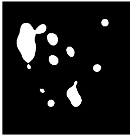
some level lines



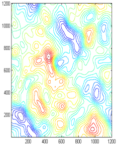


$u = 0.5$

$u = 1$



$u = 1.5$



Main questions

What can be said about "mean" geometry of excursion sets ? Area ? Perimeter ? Euler Characteristic=# connected components – # holes ?

Known results for

- Boolean model : Mecke (2001), Mecke, Wagner (1991)
- Smooth Gaussian and related random fields : Adler (2000), Adler, Taylor (2007), Azaïs, Wschebor (2009), ...
- High levels for some infinitely divisible fields : Adler, Samorodnitsky, Taylor (2010,2013),...

Two different frameworks

1. Piecewize constant fields (elementary)
2. Smooth fields : at least C^2

Curvature measures

Let $E \subset \mathbb{R}^2$ be a "nice set". Its curvature measures $\Phi_j(E, \cdot)$, for $j = 0, 1, 2$, are defined for any Borel set $U \subset \mathbb{R}^2$ by

- $\Phi_2(E, U) = |E \cap U|$,
- $\Phi_1(E, U) = \frac{1}{2}\mathcal{H}^1(\partial E \cap U) = \frac{1}{2}\mathrm{Per}(E, U)$
- $\Phi_0(E, U) = \frac{1}{2\pi}\mathrm{TC}(\partial E, U)$,

where $\mathcal{H}^1(\partial E \cap U)$ is the lenght and $\mathrm{TC}(\partial E, U)$ the total curvature of the positively oriented curve ∂E in U.

Ref : Schneider, Weil, Stochastic and Integral Geometry

Piecewise regular curve

A Jordan curve $\Gamma \subset \mathbb{R}^2$ is piecewise regular if $\Gamma = \mathcal{R}_\Gamma \cup \mathcal{C}_\Gamma$ with $\#\mathcal{C}_\Gamma < +\infty$

- for $x \in \mathcal{R}_\Gamma$ one has $x = \gamma(s)$ for some $s \in (0, \varepsilon)$ with $\gamma : (0, \varepsilon) \to \Gamma$ C^2, arc length parametrized. Then, $|\gamma(0, \varepsilon)|_1 = \varepsilon$.
  The **signed curvature** $\kappa_\Gamma(x)$ of Γ at x is
  $$\kappa_\Gamma(x) = \langle \gamma''(s), \gamma'(s)^\perp \rangle.$$
- for $x \in \mathcal{C}_\Gamma$ one has $x = \gamma(0)$ with $\gamma : (-\varepsilon, \varepsilon) \to \Gamma$ continuous and C^2 on $(-\varepsilon, \varepsilon) \setminus \{0\}$ s.t. γ' admits limits $\gamma'(0^-) \in S^1$ and $\gamma'(0^+) \in S^1$ at 0. Then, $|\gamma(-\varepsilon, \varepsilon)|_1 = 2\varepsilon$.
  The **turning angle** at a corner point $x = \gamma(0) \in \mathcal{C}_\Gamma$ is the angle $\alpha_\Gamma(x) \in (-\pi, \pi)$ between the tangent "before" and the one "after" x
  $$\alpha_\Gamma(x) = \operatorname{Arg} \gamma'(0^+) - \operatorname{Arg} \gamma'(0^-) \qquad \in (-\pi, \pi).$$

Total curvature and Euler characteristic

The **total curvature** of Γ in U is defined as

$$\mathrm{TC}(\Gamma, U) := \int_{\mathcal{R}_\Gamma \cap U} \kappa_\Gamma(x)\mathcal{H}^1(dx) + \sum_{x \in \mathcal{C}_\Gamma \cap U} \alpha_\Gamma(x).$$

Gauss-Bonnet Theorem : Let $E \subset U$ be a regular region ie $E = \overline{\overset{o}{E}}$ such that ∂E is formed by n piecewise regular positively oriented disjoint Jordan curves $\Gamma_1, \ldots, \Gamma_n$ then

$$\mathrm{TC}(\partial E, U) := \sum_{i=1}^{n} \mathrm{TC}(\Gamma_i, U) = 2\pi\chi(E),$$

where $\chi(E)$ is the **Euler characteristic** of E,

$$\chi(E) = \#\text{connected components} - \#\text{ holes}.$$

It follows that $\Phi_0(E, U) = \chi(E)$.

Geometry of excursion sets

Let $X = (X(x))_{x \in \mathbb{R}^2}$ be a stationary "nice" random field and T a bounded closed rectangle with $\overset{o}{T} \neq \emptyset$. For $u \in \mathbb{R}$, we consider the excursion set of level u in T

$$E_X(u) \cap T := \{x \in T; X(x) \geq u\}.$$

the **LK curvatures** of the excursion set $E_X(u)$ within T are

$$C_j(X, u, T) := \Phi_j(E_X(u) \cap T, T), \quad \text{for } j = 0, 1, 2.$$

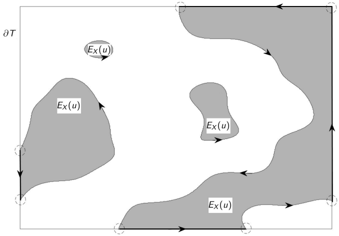
and, assuming the limits exist, the associated **LK densities** are

$$C_j^*(X, u) := \lim_{T \nearrow \mathbb{R}^2} \frac{\mathbb{E}[C_j(X, u, T)]}{|T|}, \quad \text{for } j = 0, 1, 2,$$

where $\lim\limits_{T \nearrow \mathbb{R}^2}$ stands for the limit along any sequence of bounded rectangles that grows to $\mathbb{R}^2$. Note that

$$C_j^*(\sigma X + m, u) = C_j^*(X, (u - m)/\sigma).$$

LK densities



Note that

$$C_j(X,T,u) = \Phi_j(E_X(u)\cap T, T) = \Phi_j(E_X(u), \mathring{T}) + \Phi_j(T\cap E_X(u), \partial T).$$

Actually, $C_j^*(X,u) = \frac{\mathbb{E}[\Phi_j(E_X(u),\mathring{T})]}{|T|}$. Moreover, by stationarity

$$C_2^*(X,u) = \mathbb{P}(X(0)\geq u).$$

What for smooth deterministic functions ?

Assume that $f : W \to \mathbb{R}$ is C^2 with W open s.t. $T \subset W$ and note that $\partial E_f(u) \cap \mathring{T} = \{x \in \mathring{T}; f(x) = u\}$.

1. By Morse-Sard theorem, the image by f of the set of critical values of f has measure 0 in $\mathbb{R}$.
2. Let u be such a non-critical value. For a curve γ given by an implicit form $f(\gamma(s)) = u$, we have $\gamma'(s)^{\perp} = \nabla f(\gamma(s))/||\nabla f(\gamma(s))||$ and therefore the curvature at $x = \gamma(s)$ is given by

$$\kappa_f(x) = -\frac{D^2 f(x).(\nabla f^{\perp}(x), \nabla f^{\perp}(x))}{||\nabla f(x)||^3}.$$

3. The **coarea formula** for Lipschitz mappings states that, for any $\mathcal{L}$-integrable function g,

$$\int_{\mathbb{R}} \int_{\partial E_f(u) \cap \mathring{T}} g(x) \mathcal{H}^1(dx)\, du = \int_{\mathring{T}} g(x) \|\nabla f(x)\|\, dx.$$

Weak formula for Φ_1 and Φ_0

Let us choose $h : \mathbb{R} \to \mathbb{R}$ a bounded continuous function (test function).
Coarea formula with $g(x) = h(f(x))$:

$$\int_{\mathbb{R}} h(u)\Phi_1(E_f(u), \mathring{T})du = \frac{1}{2}\int_{\mathring{T}} h(f(x))\|\nabla f(x)\|\, dx.$$

Coarea formula with $g(x) = h(f(x))\kappa_f(x)$ for

$$\kappa_f(x) = -\frac{D^2 f(x).(\nabla f(x)^\perp, \nabla f(x)^\perp)}{\|\nabla f(x)\|^3}\mathbf{1}_{\|\nabla f(x)\|>0},$$

$$\int_{\mathbb{R}} h(u)\Phi_0(E_f(u), \mathring{T})du$$

$$= -\frac{1}{2\pi}\int_{\mathring{T}} h(f(x))\frac{D^2 f(x).(\nabla f(x)^\perp, \nabla f(x)^\perp)}{\|\nabla f(x)\|^2}\mathbf{1}_{\|\nabla f(x)\|>0}\, dx.$$

Expectation under stationarity

When X is a stationary field a.s. C^2 with $X(0)$, $\nabla X(0)$ and $D^2X(0)$ L^1

$$\int_{\mathbb{R}} h(u) C_1^*(X,u)du = \frac{1}{2}\mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right)$$

$$\int_{\mathbb{R}} h(u) C_0^*(X,u)du = \frac{-1}{2\pi}\mathbb{E}\left(h(X(0))\frac{D^2X(0).(\nabla X(0)^{\perp},\nabla X(0)^{\perp})}{\|\nabla X(0)\|^2}\mathbf{1}_{\|\nabla X(0)\|>0}\right)$$

➥ Allows information for a.e $u \in \mathbb{R}$

We also obtain for $h = 1$

- $TV^*(X) = 2\int_{\mathbb{R}} C_1^*(X,u)du$;
- $LTC^*(X) = 2\pi\int_{\mathbb{R}} C_0^*(X,u)du$.

Expectation under stationarity and isotropy

When X is also **isotropic**, we get

$$\mathbb{E}\left(h(X(0))\|\nabla X(0)\|\right) = \frac{\pi}{2}\mathbb{E}\left(h(X(0))|X_1(0)|\right).$$

Moreover,

$$\mathbb{E}\left(h(X(0))\frac{D^2X(0).(\nabla X(0)^\perp, \nabla X(0)^\perp)}{\|\nabla X(0)\|^2}\mathbf{1}_{\|\nabla X(0)\|>0}\right) = \alpha_0(h) + 2\alpha_2(h),$$

with $\alpha_0(h) = \mathbb{E}\left(h(X(0))X_{11}(0)\right)$, and

$$\alpha_2(h) = -2\mathbb{E}\left(h(X(0))X_{12}(0)\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}\mathbf{1}_{\|\nabla X\|>0}\right).$$

Therefore

$$TV^*(X) = \frac{\pi}{2}\mathbb{E}\left(|X_1(0)|\right) \text{ and } LTC^*(X) = 4\mathbb{E}\left(X_{12}(0)\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}\mathbf{1}_{\|\nabla X\|>0}\right).$$

Gaussian case

For X a stationary standard isotropic C^2 Gaussian random field we note $\rho(x) = \mathrm{Cov}(X(x), X(0))$, and the second spectral moment

$$\lambda_2 = -\partial_k^2 \rho(0) = -\mathrm{Cov}(X(0), X_{kk}(0)) = \mathrm{Var}(X_k(0)).$$

By stationarity $\mathrm{Cov}(X(0), X_k(0)) = \mathrm{Cov}(X_k(0), X_{12}(0)) = 0$ and

$$\mathbb{E}\left(h(X(0))|X_1(0)|\right) = \mathbb{E}\left(h(X(0))\right)\mathbb{E}\left(|X_1(0)|\right) = \sqrt{\frac{2\lambda_2}{\pi}}\mathbb{E}\left(h(X(0))\right);$$

$$\begin{aligned}
\alpha_0(h) &= \mathbb{E}\left(h(X(0))X_{11}(0)\right) = \mathbb{E}\left(h(X(0))\mathbb{E}\left(X_{11}(0)|X(0)\right)\right) \\
&= -\lambda_2 \mathbb{E}\left(h(X(0))X(0)\right);
\end{aligned}$$

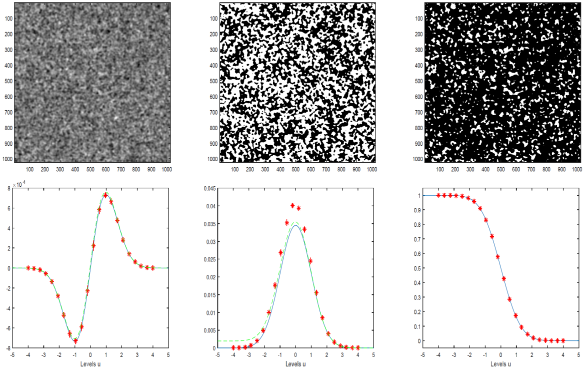
$$\begin{aligned}
\alpha_2(h) &= -2\mathbb{E}\left(h(X(0))X_{12}(0)\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}\mathbf{1}_{\|\nabla X\|>0}\right) \\
&= -2\mathbb{E}\left(h(X(0))X_{12}(0)\right)\mathbb{E}\left(\frac{X_1(0)X_2(0)}{\|\nabla X(0)\|^2}\mathbf{1}_{\|\nabla X\|>0}\right) = 0.
\end{aligned}$$

Hence $TV^*(X) = \sqrt{\frac{\pi\lambda_2}{2}}$ and $LTC^*(X) = 0$.

Gaussian case

For $X(0) \sim \mathcal{N}(0,1)$, this yields to for a.e. $u \in \mathbb{R}$

$$C_0^*(X,u) = \frac{1}{(2\pi)^{3/2}} \lambda_2 \, u \, e^{-\frac{u^2}{2}} \text{ and } C_1^*(X,u) = \frac{1}{4} \lambda_2^{1/2} \, e^{-\frac{u^2}{2}}.$$



$\rho(x) = e^{-\kappa^2 \|x\|^2}$, for $\kappa = 100/2^{10}$ in a domain of size $2^{10} \times 2^{10}$ pixels.

Comments

- If one knows that $u \mapsto C_1^*(X,u)$ or $u \mapsto C_0^*(X,u)$ are continuous then a.e. is enough ! In Berzin, Latour, Leon (2017) general assumptions to ensure that $u \mapsto C_1^*(X,u)$ is continuous ;
- For isotropic stationary C^3 Gaussian field the formulas hold for all level (weakest assumptions cf Adler, Taylor (2007)) ; Moreover by kinematic formula

$$\mathbb{E}[C_0(X,T,u)] = C_0^*(X,u)|T| + \frac{1}{\pi} C_1^*(X,u)\mathcal{H}^1(\partial T) + C_2^*(X,u),$$
$$\mathbb{E}[C_1(X,T,u)] = C_1^*(X,u)|T| + \frac{1}{2} C_2^*(X,u)\mathcal{H}^1(\partial T).$$

- Due to Adler and Taylor, using Gaussian kinematic and Tube formulas, computations for fields of Gaussian type : $X = F(\mathbf{G})$ where $F : \mathbb{R}^k \mapsto \mathbb{R}$ C^2 and $\mathbf{G} = (G_1, \ldots, G_k)$ with $G_1, \ldots, G_k$ iid C^3 homogeneous Gaussian rf.

Chi2 case

For $k \geq 1$, $Z_k = G_1^2 + \ldots + G_k^2$ and normalized field

$$\widetilde{Z}_k(t) := \frac{1}{\sqrt{2k}}(Z_k(t) - k), \;\; t \in \mathbb{R}^2.$$

Then, for all $u \in \mathbb{R}$,

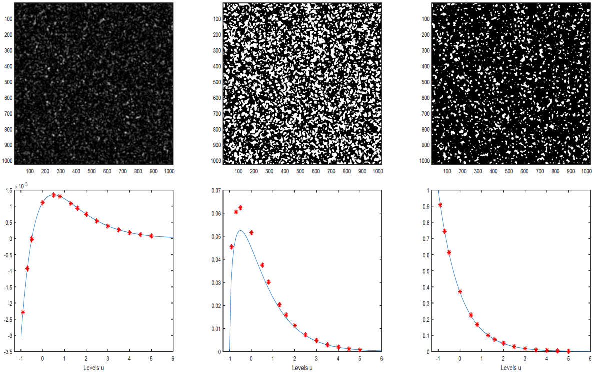
$$C_0^*(\widetilde{Z}_k, u) = \frac{\lambda_2}{\pi 2^{k/2}\Gamma(k/2)} \left(k + u\sqrt{2k}\right)^{(k-2)/2} \left(u\sqrt{2k} + 1\right) \exp\left(-\frac{k + u\sqrt{2k}}{2}\right)$$

$$C_1^*(\widetilde{Z}_k, u) = \frac{\sqrt{\pi\lambda_2}}{2^{(k+1)/2}\Gamma(k/2)} \left(k + u\sqrt{2k}\right)^{(k-1)/2} \exp\left(-\frac{k + u\sqrt{2k}}{2}\right),$$

$$C_2^*(\widetilde{Z}_k, u) = \mathbb{P}\left(\chi_k^2 \geq k + u\sqrt{2k}\right).$$

Hence $TV^*(\widetilde{Z}_k) = \sqrt{\frac{2\pi\lambda_2}{k}} \frac{\Gamma((k+1)/2)}{\Gamma(k/2)}$ and $LTC^*(\widetilde{Z}_k) = \sqrt{\frac{2}{k}}\lambda_2$.

Chi2 case



$\widetilde{Z}_k$ for $k = 2$ and for iid standard $G_1, \ldots, G_k$ with covariance function $\rho(x) = e^{-\kappa^2 \|x\|^2}$, for $\kappa = 100/2^{10}$ in a domain of size $2^{10} \times 2^{10}$ pixels.

Student case

For $k \geq 3$, $T_k = G_{k+1}/\sqrt{Z_k/k}$ and normalized field

$$\widetilde{T}_k(t) := \sqrt{(k-2)/k}\,T_k(t),\ \ t \in \mathbb{R}^2.$$

Then,

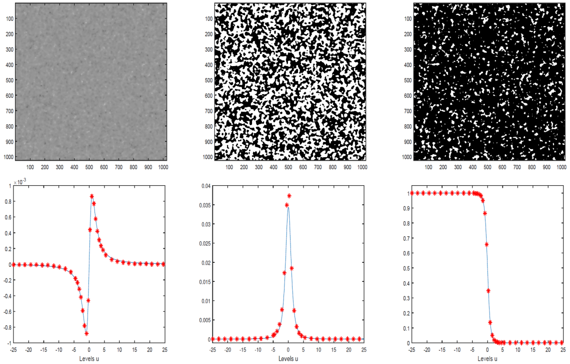
$$C_0^*(\widetilde{T}_k, u) = \frac{\lambda_2(k-1)}{4\pi^{\frac{3}{2}}} \frac{u}{\sqrt{k-2}} \frac{\Gamma\left(\frac{k-1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{u^2}{k-2}\right)^{\frac{1-k}{2}},$$

$$C_1^*(\widetilde{T}_k, u) = \frac{\sqrt{\lambda_2}}{4} \left(1 + \frac{u^2}{k-2}\right)^{\frac{1-k}{2}},$$

$$C_2^*(\widetilde{T}_k, u) = \mathbb{P}(Student(k) \geq u\sqrt{k/(k-2)}).$$

Hence $TV^*(\widetilde{T}_k) = \sqrt{(k-2)\pi\lambda_2}\frac{\Gamma((k-2)/2)}{2\Gamma((k-1)/2)}$ and $LTC^*(\widetilde{T}_k) = 0$.

Student case



$\widetilde{T}_k$ for $k = 4$ and and for iid standard $G_1, \dots, G_{k+1}$ with covariance function $\rho(x) = e^{-\kappa^2 \|x\|^2}$, for $\kappa = 100/2^{10}$ in a domain of size $2^{10} \times 2^{10}$ pixels.

Shot noise fields

The characteristic function is given by

$$\mathbb{E}\left(e^{itX(0)}\right) = \exp\left(\lambda \int_{\mathbb{R}^k \times \mathbb{R}^2} [e^{i[tg_m(x)]} - 1] F(dm)dx\right).$$

When g_m is smooth, we have also access to joint law of $(X(0), \nabla X(0), D^2X(0))$ via characteristic function and similar integral expression. In particular the joint characteristic function of X and $\partial_1 X$ is

$$\begin{aligned} \varphi(t,s) &= \mathbb{E}\left(e^{itX(0)+is\partial_1 X(0)}\right) \\ &= \exp\left(\lambda \iint [e^{itg_m(x)+is\partial_1 g_m(x)} - 1]\, F(dm)\, dx\right) \end{aligned}$$

Isotropic smooth Shot noise fields

The main idea is therefore to take $h_t(u) = e^{itu}$ to compute $\widehat{C_j^*}(t) = \int_{\mathbb{R}} e^{itu} C_j^*(X,u)du$. We obtain integral formulas :

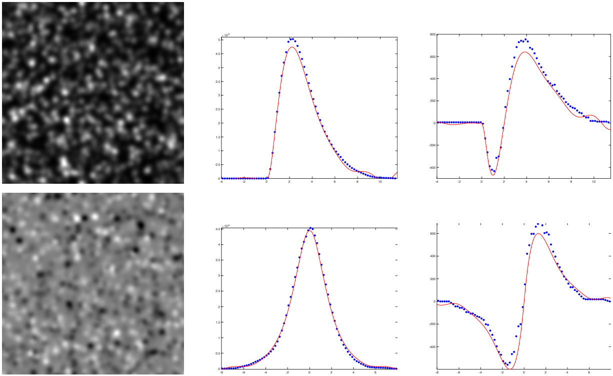
$$\widehat{C_1^*}(t) = \frac{1}{2}\int_0^{+\infty} \frac{\varphi(t,s)}{s} S_0(t,s)ds.$$

$$\widehat{C_0^*}(t) = S_1(t)\varphi(t,0) + \int_0^{+\infty} \frac{\varphi(t,s)}{s} S_2(t,s)ds,$$

with

$$\begin{aligned}
S_0(t) &= -i\lambda \int_{\mathbb{R}}\int_{\mathbb{R}^2} \partial_1 g_m(x) e^{i[tg_m(x)+s\partial_1 g_m(x)]}\, dx\, F(dm)\\
S_1(t) &= -\frac{\lambda}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \partial_1^2 g_m(x) e^{itg_m(x)}\, dx\, F(dm)\\
S_2(t,s) &= \frac{\lambda}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^2} [\partial_2^2 g_m(x) - \partial_1^2 g_m(x)] e^{i[tg_m(x)+s\partial_1 g_m(x)]}\, dx\, F(dm)
\end{aligned}$$

Shot noise Gaussian examples



We chose $g_m(x) = we^{-\frac{\|x/r\|^2}{2}}$ with $R = 1/a > 0$ a.s.

- Top : $W \sim \mathcal{E}(\mu)$, we find $\varphi(t) = \left(\frac{\mu}{\mu - it}\right)^{2\pi\lambda/a}$ and $X(0) \sim \gamma(\mu, 2\pi\lambda/a)$;
- Bottom : $W \sim \mathcal{L}(\mu)$, $\varphi(t) = \left(\frac{\mu^2}{\mu^2 + t^2}\right)^{\pi\lambda/a}$ and $X(0) \sim GS\mathcal{L}(\mu, \pi\lambda/a)$.

Shot noise disk examples

Considering $g_m(x) = \mathbf{1}_{rD}(x)$ we note

$$\bar{a} = \pi\mathbb{E}(R^2) \text{ and } \bar{p} = 2\pi\mathbb{E}(R)$$

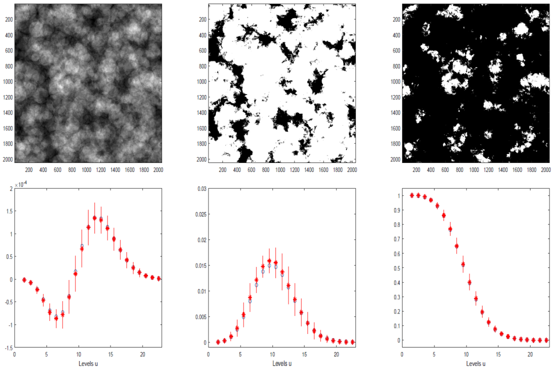
and get

$$X(0) \sim \mathcal{P}(\lambda\bar{a}).$$

Moreover, for $u \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, it holds that

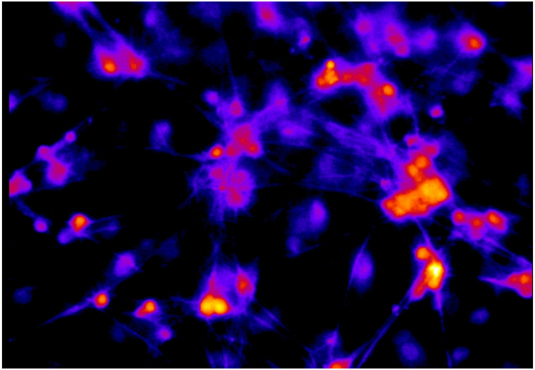
$$\begin{aligned}
C_0^*(u) &= e^{-\lambda\bar{a}} \frac{(\lambda\bar{a})^{\lfloor u \rfloor}}{\lfloor u \rfloor !} \lambda \left(1 - \lambda\frac{\bar{p}^2}{4\pi} + \lfloor u \rfloor \frac{\bar{p}^2}{4\pi\bar{a}}\right), \\
C_1^*(u) &= \frac{1}{2} e^{-\lambda\bar{a}} \frac{(\lambda\bar{a})^{\lfloor u \rfloor}}{\lfloor u \rfloor !} \lambda\bar{p} \\
C_2^*(u) &= e^{-\lambda\bar{a}} \sum_{k>u} \frac{(\lambda\bar{a})^k}{k!}.
\end{aligned}$$

Shot noise disk case



$R = 50$ or $R = 100$ each with probability $1/2$ and $\lambda = 5 \times 10^{-4}$ in a domain of size $2^{10} \times 2^{10}$ pixels.

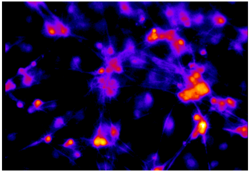
Biological movies

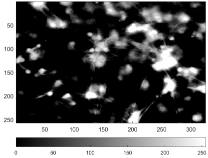


- Calcium imaging recordings for mouse nasal explants
- Context : study of pulsativity for Gonadotropin-Releasing Hormone-1 neurons and calcium event synchronization
- Collaboration : A. Duittoz (INRA Nouzilly) and C. Georgelin (IDP Tours)

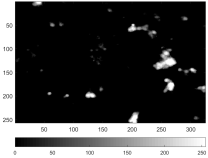
The data

$D = 700$ images of size 257×325 pixels in RGB coded in 8 bit :

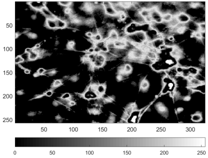




R



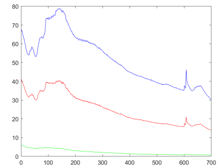
G



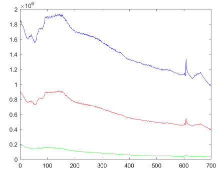
B

Statistics of the data

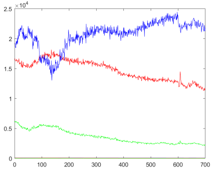
Evolution according to time



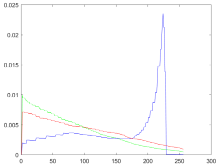
mean



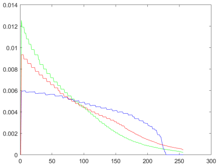
$TV^* \times |T|$



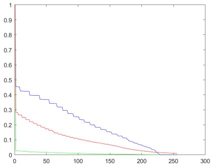
$LTC^* \times |T|$



$C_0^* / \int |C_0^*|$



C_1^* / TV^*

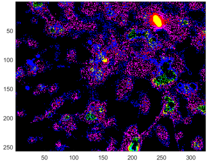


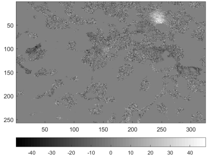
C_2^*

Mean over time / according to values

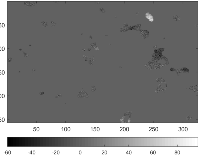
The time differences

Consider the difference vs time $\Delta X_t = X_{t+1} - X_t$

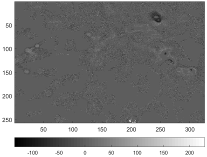




R



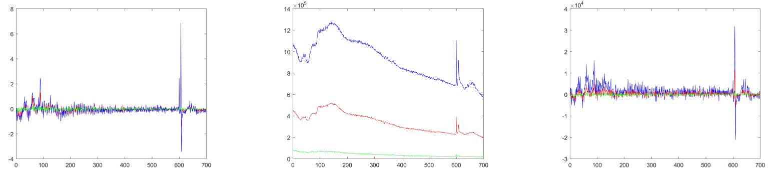
G



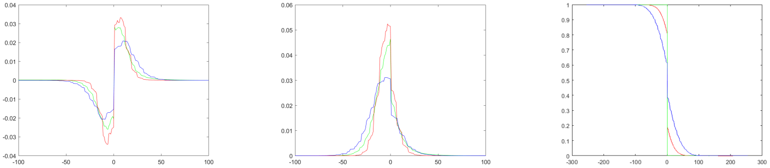
B

Statistics of time differences

Evolution according time



mean $\qquad TV^* \times |T| \qquad LTC^* \times |T|$



$C_0^* / \int |C_0^*| \qquad C_1^* / TV^* \qquad C_2^*$

Mean over time / according to values

Perspectives

- Effect of discretization of random fields
- Use for statistical inference
- Detect main topological changes in a sequence of images
- Beyond expectation : higher order moments (variance at least, and then CLT)
- Higher dimension

References

R. Adler, J. Taylor : Random fields and Geometry. *Springer, NY (2007).*

J.M. Azais, M. Wschebor : Level sets and extrema of random processes and fields. *Wiley (2009).*

C. Berzin, A. Latour, J. Leon : Kac-Rice formulas for random fields and their applications in : random geometry, roots of random polynomials and some engineering problems. *Instituto Venezolano de Investigaciones Científicas, (2017).*

H. Biermé, A. Desolneux : On the perimeter of excursion sets of shot noise random fields. *Annals of Probability,* **44***(1), 521-543, (2016).*

H. Biermé, A. Desolneux : Mean Geometry of 2D random fields : level perimeter and level total curvature integrals. *revised version, 2018.*

H. Biermé, E. Di Bernardino, C. Duval, A. Estrade : Lipschitz Killing curvatures of excursion sets for 2D random fields *to appear in EJS*