Infimal-convolution-type regularization for inverse problems in imaging

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Motivation

Goals:

- Find flexible models for stable solution of inverse problems in function spaces
- Establish meaningful combined models

In particular:

- Combine multiple smoothness orders
- Separate image characteristics (e.g. cartoon/texture)
- Apply broadly to imaging problems (denoising, inpainting, inverse problems/reconstruction)

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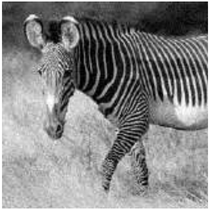

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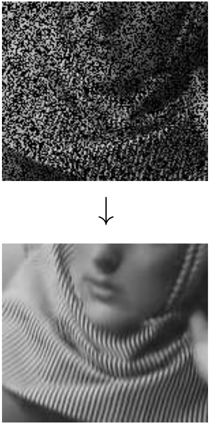

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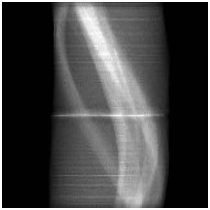

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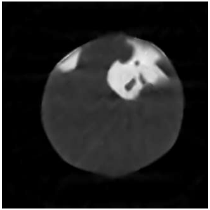

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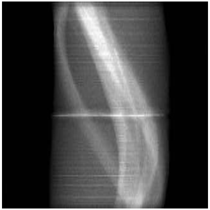

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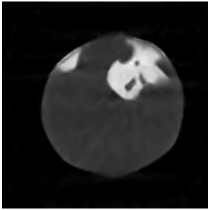

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Here: Infimal-convolution regularizers

Outline

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The infimal convolution

Definition:

- $\Phi_1, \Phi_2 : X \to \,]-\infty, \infty]$ proper, convex, l.s.c.

$$(\Phi_1 \Box \Phi_2)(u) = \inf_{u_1 + u_2 = u} \Phi_1(u_1) + \Phi_2(u_2)$$

- Infimal convolution of Φ_1 and Φ_2
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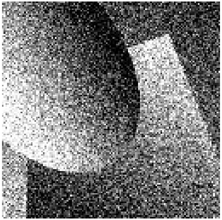
- Optimal balancing of m competing regularization functionals

Basic variational models



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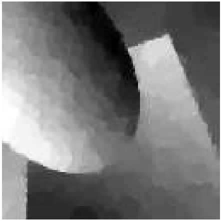


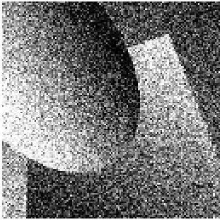
Smoothness-based:

- Total variation $\Phi(u) = \int_\Omega \mathrm{d}|\nabla u|$

[Rudin/Osher/Fatemi '92]

- Accounts for edges
- Unaware of higher-order smoothness
  $\rightsquigarrow$ staircasing effect

Basic variational models



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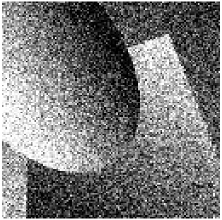
- Higher-order TV $\Phi(u) = \int_\Omega \mathrm{d}|\nabla^2 u|$

[Lysaker/Lundervold/Tai '03]
[Hinterberger/Scherzer '04]

- Favors smooth solutions
- Edges are not preserved

Infimal-convolution models



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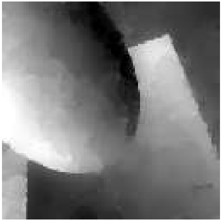

Smoothness-based:

- TV-TV^2 infimal convolution

$$\Phi(u) = \min_{u=u_1+u_2} \int_\Omega \mathrm{d}|\nabla u_1| + \beta \int_\Omega \mathrm{d}|\nabla^2 u_2|$$

[Chambolle/Lions '97]

- Models piecewise smooth images
- Staircase effect dominates solutions

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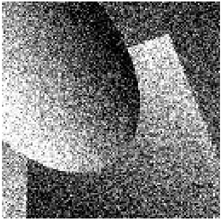

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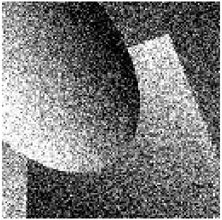
[Chan/Esedoglu/Park '05]

- Denoising similar to TV-TV^2 infimal convolution

Infimal-convolution-type models



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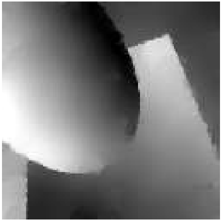

Variations:

- Total generalized variation of order 2

$$\Phi(u) = \mathrm{TGV}_\alpha^2(u) = \min_{\nabla u = u_1 + u_2} \alpha_1 \int_\Omega \mathrm{d}|u_1| + \alpha_0 \int_\Omega \mathrm{d}|\mathcal{E} u_2|$$

[B./Kunisch/Pock '10]

- Models piecewise smooth images
- Favors smoothness where given
- Preserves relevant edges

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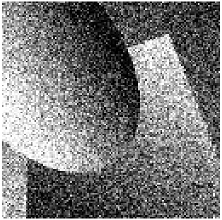

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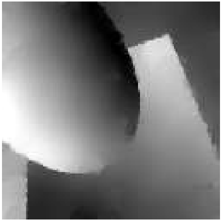

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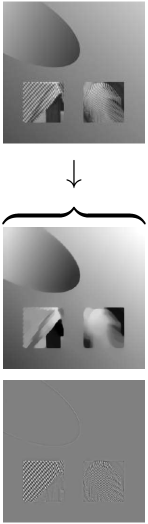
- Other variants by

[Setzer/Steidl/Teuber '11]

Cartoon/texture decomposition



Infimal convolution models:

- TV-G-norm decomposition

$$\Phi(u) = \min_{u_1+u_2=u} \int_\Omega \mathrm{d}|\nabla u_1| + \beta \|u_2\|_{\mathrm{TV}^*}$$

[Osher/Vese '03], [Aujol et al. '05]

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[Osher/Sole/Vese '03]

- Separates piecewise smooth and oscillatory parts
- G-norm/H^{-1}-norm part also captures non-oscillations

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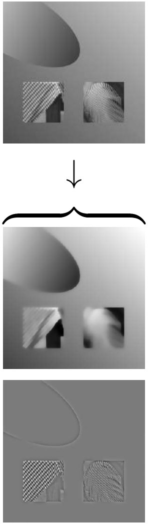

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Infimal convolution for inverse problems

Tikhonov regularization:

Solve $$\min_{u \in X} \; S(Ku, f) + \Phi(u)$$

- S discrepancy for $Ku = f$, Φ regularization functional

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Two viewpoints:

1. Monolithic regularization

$$\min_{u \in X} S(Ku, f) + \Phi(u), \qquad \Phi = \Phi_1 \square \ldots \square \Phi_m$$

2. Vector-valued regularization

$$\min_{(u_1,\ldots,u_m) \in X^m} S\bigl(K(u_1 + \ldots + u_m), f\bigr) + \Phi_1(u_1) + \ldots \Phi_m(u_m)$$

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Monolithic regularization:

- Intrinsic decomposition & problem-adapted interpretation

General existence result

Theorem:

- X reflexive Banach space, Y normed space
- $K : X \to Y$ bounded linear
- $S(\cdot, f) : Y \to [0, \infty]$ proper, convex, l.s.c.
- $\Phi : X \in [0, \infty]$ proper, convex, l.s.c.
- Φ 1-homogeneous, $\dim \ker(\Phi) < \infty$, and coercive, i.e.,
$$\|u - Pu\|_X \leq C\Phi(u)$$
for $P : X \to \ker(\Phi)$ bounded linear projector
- Then: $$\min_{u \in X} \; S(Ku, f) + \alpha\Phi(u)$$
has a solution for each $\alpha > 0$

General regularization properties

$$\min_{u \in X} \; S(Ku, f) + \alpha \Phi(u)$$

- Let: Y Hilbert space, $S(v, f) = \frac{1}{2}\|v - f\|_Y^2$

Then:

- (Subsequential) stability for varying f
- Noise level $\delta \to 0$ and $\alpha \to 0$, $\delta^2/\alpha \to 0$
  $\Rightarrow$ (Subsequential) convergence
- Source condition $K^* w \in \partial\Phi(u^\dagger)$, $\alpha \sim \delta$
  $\Rightarrow$ $\mathcal{O}(\delta)$ for the Bregman distance w.r.t. Φ

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Difficulties:

- $\dim \ker(\Phi) = \infty \Rightarrow K$ has to be stably invertible on $\ker(\Phi)$
- Φ not coercive $\Rightarrow K$ has to be stably invertible on X

Infimal convolution regularization

Lemma:

- $\Phi_1, \Phi_2 : X \to [0, \infty]$ proper, convex, l.s.c.
- Φ_i 1-homogeneous, $\dim \ker(\Phi) < \infty$ for $i = 1, 2$
- Then: $\Phi_1 \Box \Phi_2 : X \to [0, \infty]$ is exact, proper, convex, l.s.c., 1-homogeneous, $\ker \Phi_1 \Box \Phi_2 = \ker \Phi_1 + \ker \Phi_2$

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Lemma:

- Additionally: $X \hookrightarrow Z$, Z Banach space and
$$\|u\|_X \leq C\big(\|u\|_Z + \Phi_i(u)\big)$$
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$\rightsquigarrow$ **sufficient conditions for a regularizer**

Examples

TV-TV^2 **infimal convolution:**

- TV and TV^2 have finite-dimensional kernels + embedding:

$$\|u\|_{\mathrm{BV}} \leq \|u\|_{\mathrm{BV}^2} \leq C\big(\|u\|_1 + \mathrm{TV}^2(u)\big)$$

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TV-$L^1 \circ \Delta$ **infimal convolution:**

- $\ker(\mathrm{TV} \,\square\, \|\cdot\|_{\mathcal{M}} \circ \Delta) = \{u \in \mathrm{BV} \mid u \text{ harmonic}\}$
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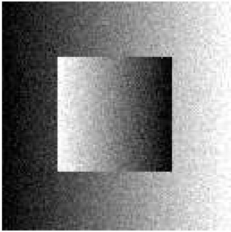
$\rightsquigarrow$ **Develop regularizing cartoon/texture models**
+ complex smoothness-based regularizers

Outline

Total generalized variation



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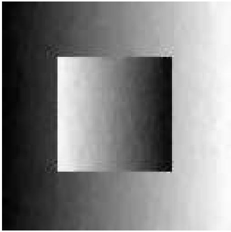

TV-regularization

Motivation:

- TV-based first-order regularization favors certain artifacts

Total generalized variation:

[B./Kunisch/Pock '10]

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- $\alpha = (\alpha_0, \ldots, \alpha_{k-1}) > 0$ weights

Second-order version:

$$\mathrm{TGV}_\alpha^2(u) = \min_{w \in \mathrm{BD}(\Omega)} \alpha_1 \|\nabla u - w\|_\mathcal{M} + \alpha_0 \|\mathcal{E} w\|_\mathcal{M}$$

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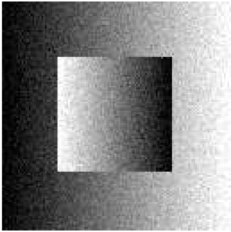

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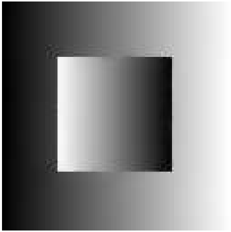

TGV_α^2-regularization

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Properties of TGV_α^2

Basic properties: *[B./Kunisch/Pock '10]*

- TGV_α^2 is proper, convex, lower semi-continuous
- TGV_α^2 is translation and rotation invariant
- $\mathrm{TGV}_\alpha^2 + \|\cdot\|_1$ gives the Banach space $\mathrm{BGV}_\alpha^2(\Omega)$
- $\ker(\mathrm{TGV}_\alpha^2) = \Pi^1$ affine functions
- TGV_α^2 measures piecewise affine only at the interfaces

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- $\mathrm{TGV}_\alpha^2 + \|\cdot\|_1$ gives the Banach space $\mathrm{BGV}_\alpha^2(\Omega)$
- $\ker(\mathrm{TGV}_\alpha^2) = \Pi^1$ affine functions
- TGV_α^2 measures piecewise affine only at the interfaces

Properties of TGV_α^2

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Advanced properties: *[B./Valkonen '11]*

- $\mathrm{BGV}_\alpha^2(\Omega) = \mathrm{BV}(\Omega)$ in the sense of equivalent norms
- TGV_α^2 is coercive in the sense

$$\|u - Pu\|_{\mathrm{BV}} \leq C\,\mathrm{TGV}_\alpha^2(u)$$

for $P : L^1(\Omega) \to \Pi^1$ continuous projection

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Existence and stability

Theorem: *[B./Valkonen '11]*

- $1 < p \leq d/(d-1)$
- $K : L^p(\Omega) \to H$ linear and continuous, H Hilbert space
- K injective on Π^1

$\Rightarrow$ Optimization problem

$$\min_{u \in L^p(\Omega)} \frac{1}{2}\|Ku - f\|^2 + \mathrm{TGV}_\alpha^2(u)$$

possesses a solution

UNI GRAZ

Existence and stability

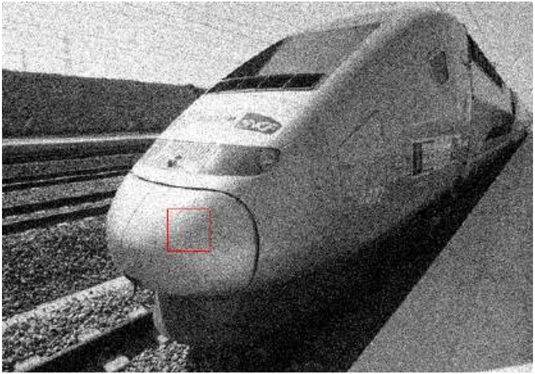
Theorem: *[B./Valkonen '11]*

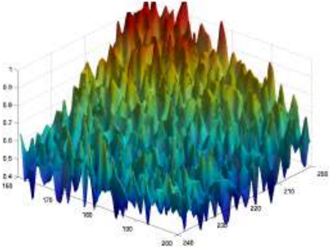
$$\left.\begin{array}{l} \blacksquare\ 1 < p \leq d/(d-1) \\ \blacksquare\ K : L^p(\Omega) \to H \\ \quad \text{linear and continuous,} \\ \quad H \text{ Hilbert space} \\ \blacksquare\ K \text{ injective on } \Pi^1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \text{Optimization problem} \\ \displaystyle\min_{u \in L^p(\Omega)} \frac{1}{2}\|Ku - f\|^2 \\ \qquad\qquad + \mathrm{TGV}_\alpha^2(u) \\ \text{possesses a solution} \end{array}\right.$$

Stability: $f^n \to f$ in $H \Rightarrow \begin{cases} u^n \rightharpoonup u \text{ in } L^p(\Omega) \text{ (subseq.)} \\ \mathrm{TGV}_\alpha^2(u^n) \to \mathrm{TGV}_\alpha^2(u) \end{cases}$

Example: Denoising a "cartoon" image

Solve: $$\min_{u \in L^2(\Omega)} \frac{\|u - f\|^2}{2} + \mathrm{TGV}_\alpha^2(u)$$





noisy image

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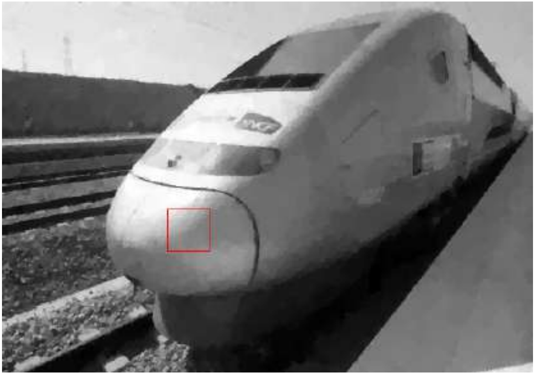

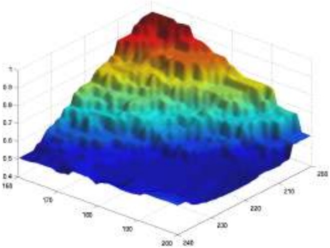

TV regularization

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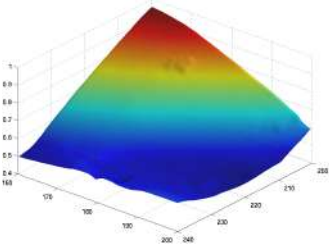
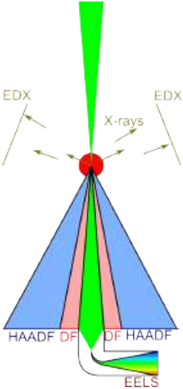


TGV_α^2 regularization

Electron tomography

Joint work with Georg Haberfehlner and Richard Huber

Scanning Transmission Electron Microscopy (STEM):



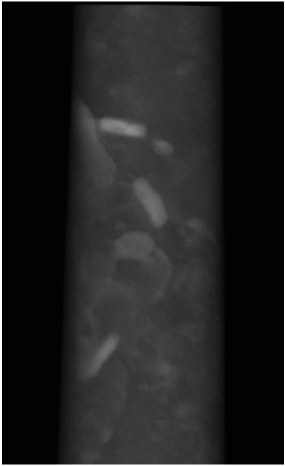
- Focused electron beam
- High Angle Annular Dark Field (HAADF)

$\rightsquigarrow$ non Bragg-scattered electrons

$\rightsquigarrow$ proportional to mass-thickness

$\rightsquigarrow$ **Reconstruction via Radon inversion**

Regularized reconstruction

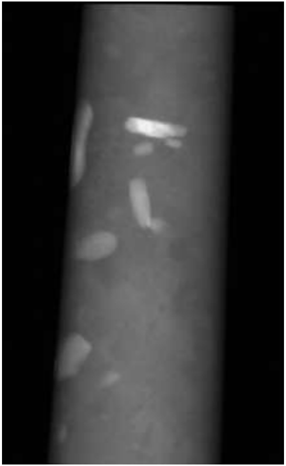


Projections

Reconstruction of one slice:

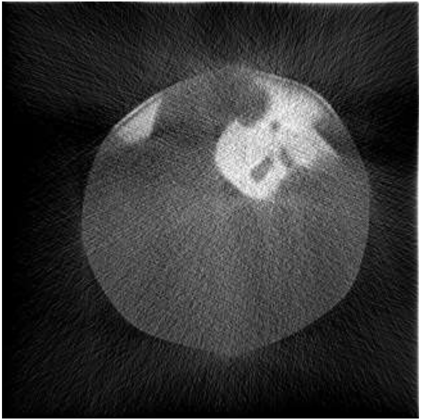
Sinogram

Regularized reconstruction



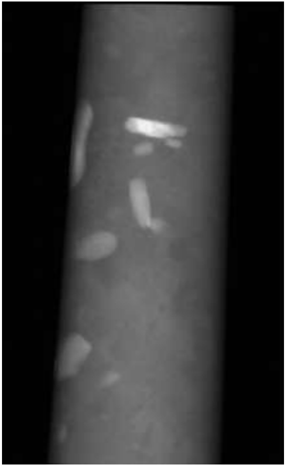
Projections

Reconstruction of one slice:



Filtered back-projection

Regularized reconstruction

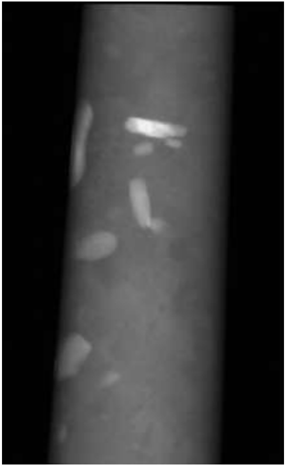


Projections

Reconstruction of one slice:

TV-regularization

Regularized reconstruction



Projections

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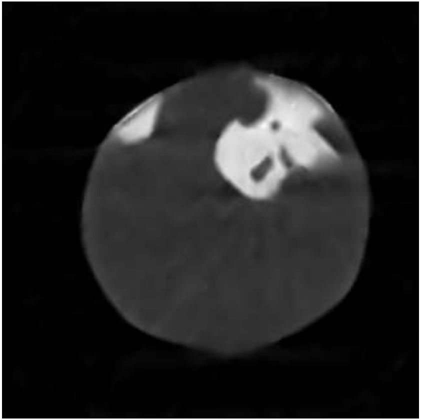

TGV-regularization

Outline

Regularization for image sequences

Motivation: Straightforward spatio-temporal regularization

$$\mathrm{TV}_{\epsilon_x}(u) = \int_{[0,T]\times\Omega} \sqrt{\epsilon|\partial_{x_1} u|^2 + \epsilon|\partial_{x_2} u|^2 + |\partial_t u|^2}\, \mathrm{d}t\, \mathrm{d}x$$

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Compressed

TV_{ϵ_x}

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$$\mathrm{TV}_{\epsilon_t}(u) = \int_{[0,T]\times\Omega} \sqrt{\ |\partial_{x_1} u|^2 + \ |\partial_{x_2} u|^2 + \epsilon |\partial_t u|^2}\ \mathrm{d}t\ \mathrm{d}x$$

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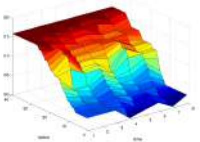

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Approach: Optimal balancing with infimal convolution (IC)

$$\mathrm{ICTV}_{\epsilon} = \mathrm{TV}_{\epsilon_x} \,\square\, \mathrm{TV}_{\epsilon_t}$$

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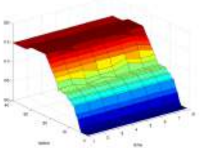

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u_1

u_2

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u_1

u_2

$\rightsquigarrow$ **Use infimal convolution of TGV**

Infimal convolution TGV

Definition: Anisotropic Total Generalized Variation

$$\mathrm{TGV}_\beta^k(u) = \sup \left\{ \int_\Omega u \,\mathrm{div}^k\, v \;\mathrm{d}x \;\middle|\; v \in \mathcal{C}_c^k(\Omega, \mathrm{Sym}^k(\mathbb{R}^d)), \right.$$
$$\left. \|\mathrm{div}^l\, v\|_{\infty,\beta_l^*} \leq 1,\ \mathrm{l}{=}0,\ldots,\mathrm{k}{-}1 \right\}$$

- $\beta = (|\cdot|_{\beta_0}, \ldots, |\cdot|_{\beta_{k-1}})$ tensor norms
- $\|v_l\|_{\infty,\beta_l^*} = \sup_{x\in\Omega} |v_l(x)|_{\beta_l^*}$, $|\cdot|_{\beta_l^*}$ dual norm

Infimal convolution TGV

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Infimal convolution TGV: *[Holler/Kunisch '14]*

$$\mathrm{ICTGV}_\beta^k = \mathrm{TGV}_{\beta^1}^{k_1} \,\square\, \dots \,\square\, \mathrm{TGV}_{\beta^m}^{k_m}$$

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Regularization properties:

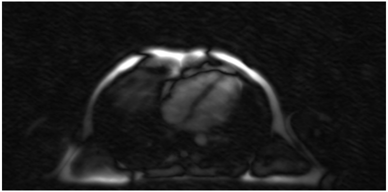
- Each $\mathrm{TGV}_{\beta^i}^{k_i}$ has finite-dimensional kernel + embedding

$$\|u\|_{\mathrm{BV}} \leq C\big(\|u\|_1 + \mathrm{TGV}_{\beta^i}^{k_i}(u)\big)$$

$\rightsquigarrow$ **ICTGV is a regularizer**

Accelerated dynamic MRI

Joint work with Martin Holler and Matthias Schlögl


sum of squares

Dynamic MRI:

- Acquire image sequences of moving objects
- Due to limited acquisition speed, only low-resolution images are obtainable

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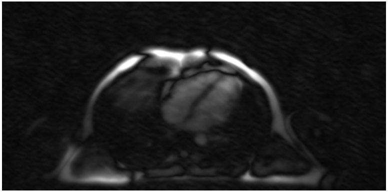

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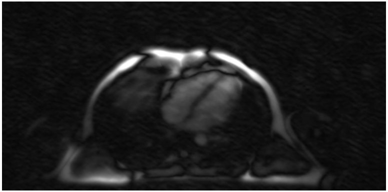

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Goal:

- Improve spatio-temporal resolution by undersampling reconstruction

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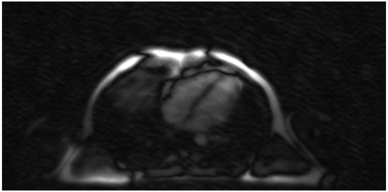

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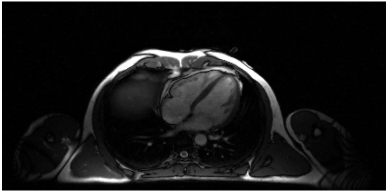

regularized

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- Due to limited acquisition speed, only low-resolution images are obtainable

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$\rightsquigarrow$ **Apply ICTGV regularization**

The variational model

Minimization problem:

$$\min_u \sum_{t,c} \frac{\lambda}{2} \|K_{t,c}(u_t) - d_{t,c}\|_2^2 + \mathrm{ICTGV}_\beta^2(u)$$

- $K_{t,c}(u_t) = M_t \mathcal{F}(u_t \sigma_c)$ masked Fourier transform (σ_c complex coil sensitivities)
- $\mathrm{ICTGV}_\beta^2 = \mathrm{TGV}_{\beta^1}^2 \,\square\, \mathrm{TGV}_{\beta^2}^2$

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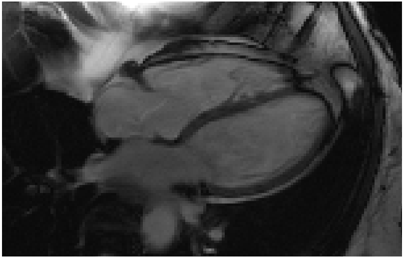
Primal-dual algorithm:

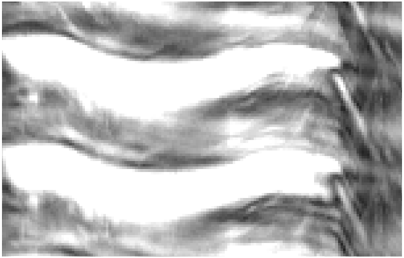
- Guaranteed convergence, duality-based stopping criterion
- GPU-optimized version:
  $\approx$ 160 seconds including coil-sensitivity estimation
  (NVidia GeForce GTX770 with AGILE library)

Numerical test

Acceleration factor 8:



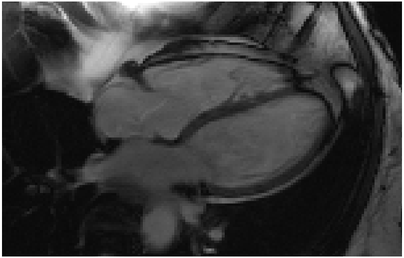
Reference data



Unregularized reconstruction

Numerical test

Acceleration factor 8:



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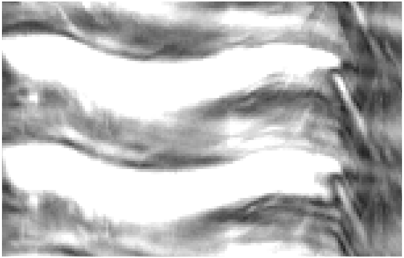

Unregularized reconstruction

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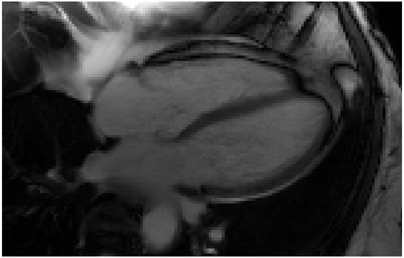
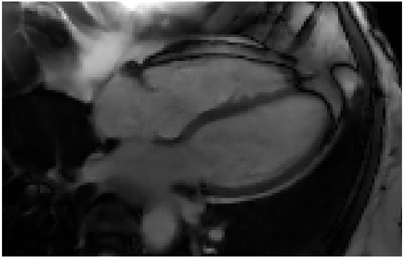


Low rank + sparse model

Difference

Numerical test

Acceleration factor 8:

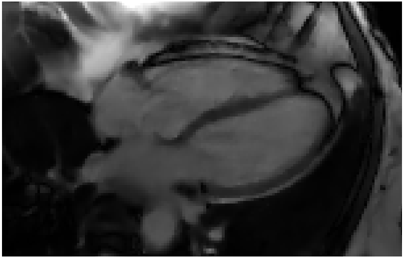


ICTGV model

Difference

Numerical test

Acceleration factor 16:

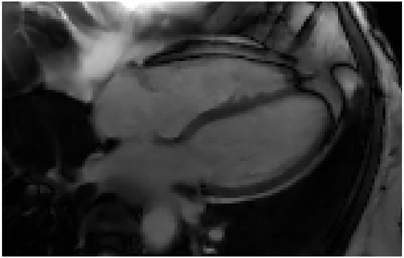


ICTGV model

Difference

Numerical test

Acceleration factor 8:

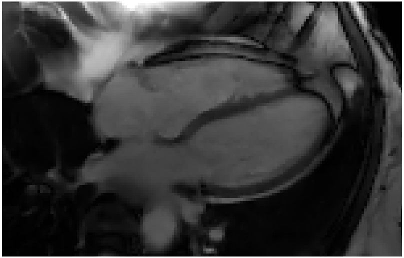


Slow component

Fast component

Numerical test

Acceleration factor 8:



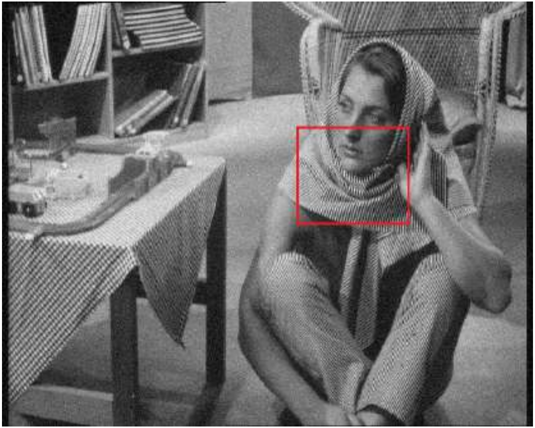
Slow component

Fast component

- Favorable quantitative comparison
- 2nd place at the ISMRM 2013 reconstruction challenge

Outline

Motivation: Denoising “barbara”



noisy image

Motivation: Denoising "barbara"

TGV_α^2 regularization

Motivation: Denoising “barbara”

TGV_α^2 regularization

$\rightsquigarrow$ **Capture oscillatory structures** $\rightsquigarrow$ $\mathrm{TGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}$

Oscillation TGV

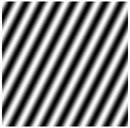
Joint work with Yiming Gao

Idea:

- Choose differential equation with oscillatory functions as solutions

Oscillation TGV

Joint work with Yiming Gao



Idea:

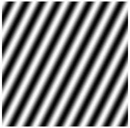
- Choose differential equation with oscillatory functions as solutions

$$\nabla^2 u + \mathbf{c} u = 0, \qquad \mathbf{c} = \omega \otimes \omega, \qquad \omega \in \mathbb{R}^d,\ \omega \neq 0$$

- Solutions: $u(x) = C_1 \cos(\omega \cdot x) + C_2 \sin(\omega \cdot x),\ C_1, C_2 \in \mathbb{R}$

Oscillation TGV

Joint work with Yiming Gao



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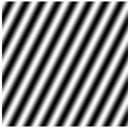
$$\nabla^2 u + \mathbf{c}u = 0, \qquad \mathbf{c} = \omega \otimes \omega, \qquad \omega \in \mathbb{R}^d,\ \omega \neq 0$$

- Solutions: $u(x) = C_1 \cos(\omega \cdot x) + C_2 \sin(\omega \cdot x),\ C_1, C_2 \in \mathbb{R}$
- Adapt second-order TGV to the corresponding operator

$$\nabla^2 u + \mathbf{c}u = 0$$

Oscillation TGV

Joint work with Yiming Gao



Idea:

- Choose differential equation with oscillatory functions as solutions

$$\nabla^2 u + \mathbf{c} u = 0, \qquad \mathbf{c} = \omega \otimes \omega, \qquad \omega \in \mathbb{R}^d,\ \omega \neq 0$$

- Solutions: $u(x) = C_1 \cos(\omega \cdot x) + C_2 \sin(\omega \cdot x), \ C_1, C_2 \in \mathbb{R}$
- Adapt second-order TGV to the corresponding operator

$$\nabla u - w = 0, \qquad \mathcal{E} w + \mathbf{c} u = 0$$

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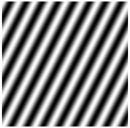

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Total generalized variation (second order):

$$\mathrm{TGV}^2_{\alpha,\beta}(u) = \min_{w \in \mathrm{BD}(\Omega)} \alpha \int_\Omega \mathrm{d}|\nabla u - w| + \beta \int_\Omega \mathrm{d}|\mathcal{E}w \quad |$$

- $\alpha > 0,\ \beta > 0$

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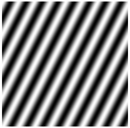

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Oscillation total generalized variation:

$$\mathrm{TGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}(u) = \min_{w \in \mathrm{BD}(\Omega)} \alpha \int_\Omega \mathrm{d}|\nabla u - w| + \beta \int_\Omega \mathrm{d}|\mathcal{E}w + \mathbf{c}u|$$

- $\alpha > 0,\ \beta > 0,\ \mathbf{c} = \omega \otimes \omega, \omega \in \mathbb{R}^d, \omega \neq 0$

Properties of $\mathrm{TGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}$

Basic properties: *[Gao/B. '17]*

- $\mathrm{TGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}$ is proper, convex, lower semi-continuous
- $\mathrm{TGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}$ is translation and rotation invariant
- $\mathrm{TGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}} + \|\cdot\|_1$ gives the Banach space $\mathrm{BGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}(\Omega)$
- $\ker(\mathrm{TGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}) = \mathrm{span}\{x \mapsto \cos(\omega \cdot x), x \mapsto \sin(\omega \cdot x)\}$
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Advanced properties:

- $\mathrm{BGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}(\Omega) = \mathrm{BV}(\Omega)$ in the sense of equivalent norms
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Infimal convolution of $\mathrm{TGV}^{\mathrm{osci}}$

Next steps:

- Separate cartoon components from oscillatory components
- Allow for multiple directions and frequencies

$\rightsquigarrow$ ***m*-fold infimal convolution**

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$$\mathrm{ICTGV}^{\mathrm{osci}}_{\vec{\alpha},\vec{\beta},\vec{\mathbf{c}}}(u) = (\mathrm{TGV}^{\mathrm{osci}}_{\alpha_1,\beta_1,\mathbf{c}_1} \,\square\, \ldots \,\square\, \mathrm{TGV}^{\mathrm{osci}}_{\alpha_m,\beta_m,\mathbf{c}_m})(u)$$

- $\alpha_i > 0$, $\beta_i > 0$, $\mathbf{c}_i = \omega_i \otimes \omega_i$, $\omega_i \in \mathbb{R}^d$

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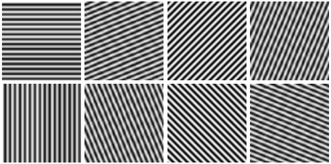
Choice of parameters:

- $\alpha_i, \beta_i \rightsquigarrow$ regularization parameters, similar to TGV^2
- $\omega_i \rightsquigarrow$ directions and frequencies, e.g., in 2D:

$$\omega_i = f \begin{pmatrix} \sin(\frac{i\pi}{k}) \\ \cos(\frac{i\pi}{k}) \end{pmatrix}, \quad k > 0$$

$f > 0$ frequency



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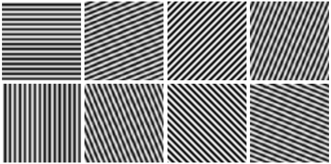

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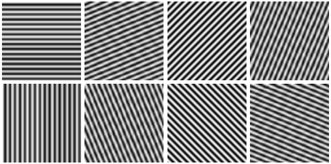

Remark:

- For $\omega = 0 \Rightarrow \mathrm{TGV}^{\mathrm{osci}} = \mathrm{TGV}^2 \rightsquigarrow$ cartoon component

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Infimal convolution of $\mathrm{TGV}^{\mathrm{osci}}$

Typical choice for cartoon/texture models:

- $\omega_1 = 0 \rightsquigarrow$ cartoon component
- $\omega_2, \ldots, \omega_9 \rightsquigarrow$ 8 texture directions, fixed frequency

Optionally:

- $\omega_{10}, \ldots, \omega_{17} \rightsquigarrow$ 8 texture directions, higher frequency

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Variant of $\mathrm{ICTGV}^{\mathrm{osci}}$:

$$\mathrm{ICTGV}^{\mathrm{osci}}_{\vec{\alpha},\vec{\beta},\vec{\mathbf{c}},\vec{\gamma}}(u) = \big((\gamma_1\|\cdot\|_1 + \mathrm{TGV}^{\mathrm{osci}}_{\alpha_1,\beta_1,\mathbf{c}_1})\,\square \\ \ldots\,\square(\gamma_m\|\cdot\|_1 + \mathrm{TGV}^{\mathrm{osci}}_{\alpha_m,\beta_m,\mathbf{c}_m})\big)(u)$$

- $\gamma_i \geq 0$ sparsifying parameter $\rightsquigarrow$ sparser textures

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Basic properties:

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Advanced properties:

- $\mathrm{ICTGV}^{\mathrm{osci}}_{\vec{\alpha},\vec{\beta},\vec{\mathbf{c}},\vec{\gamma}}$ is lower semi-continuous and exact, i.e., the infimum in the definition is attained
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Solution of imaging problems

Theorem:

$$\left.\begin{array}{l} \blacksquare\ 1 < p \leq d/(d-1) \\ \blacksquare\ K : L^p(\Omega) \to Y \\ \quad \text{linear and continuous,} \\ \quad Y \text{ Banach space} \\ \blacksquare\ f \in Y,\ 1 \leq q < \infty \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \text{Optimization problem} \\ \displaystyle\min_{u \in L^p(\Omega)} \ \frac{1}{q}\|Ku - f\|_Y^q \\ \qquad\qquad + \mathrm{ICTGV}^{\mathrm{osci}}_{\vec{\alpha},\vec{\beta},\vec{\mathbf{c}},\vec{\gamma}}(u) \\ \text{possesses a solution} \end{array}\right.$$

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Remarks:

- Ensures applicability to a variety of imaging problems (denoising, deblurring, image reconstruction (CT, MRI), . . .)
- Intrinsic cartoon/texture decomposition for these problems

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Numerical realization

Discretization in 2D and for $Y = L^2(\Omega')$:

- Discretize $\mathrm{TGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}$ with forward/backward differences
- Maintain kernel, i.e.,
  $\ker(\mathrm{TGV}^{\mathrm{osci}}_{\alpha,\beta,\mathbf{c}}) = \mathrm{span}\{x \mapsto \cos(\omega \cdot x), x \mapsto \sin(\omega \cdot x)\}$

$$\rightsquigarrow \mathbf{c} = \begin{bmatrix} 2 - 2\cos(\omega_1) & 1 + \cos(\omega_1 - \omega_2) - \cos(\omega_1) - \cos(\omega_2) \\ 1 + \cos(\omega_1 - \omega_2) - \cos(\omega_1) - \cos(\omega_2) & 2 - 2\cos(\omega_2) \end{bmatrix}$$

for $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2 \setminus \pi\mathbb{Z}^2$

- Solve non-smooth discrete optimization problems with first-order methods, such as a primal-dual iteration

[Chambolle/Pock '11]

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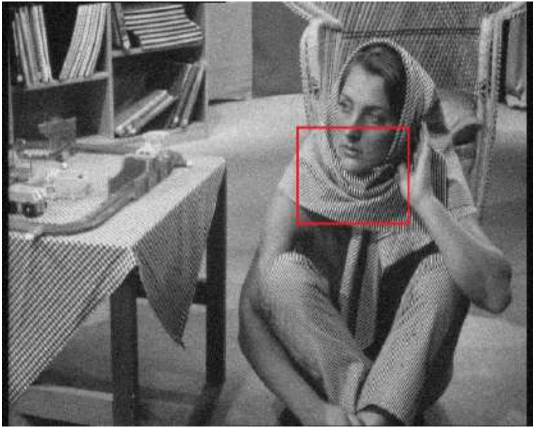
Optimization algorithm

Iteration:

$$
\begin{cases}
\lambda^{n+1} = \big(\lambda^n + \sigma(K \sum_{i=1}^{m} \bar{u}_i^n - f)\big)/(1+\sigma) \\
\text{for } i = 1, \ldots, m \text{ do} \\
\qquad p_i^{n+1} = \mathcal{P}_{\alpha_i}\big(p_i^n + \sigma(\nabla \bar{u}_i^n - \bar{w}_i^n)\big) \\
\qquad q_i^{n+1} = \mathcal{P}_{\beta_i}\big(q_i^n + \sigma(\mathcal{E}\bar{w}_i^n + \mathbf{c}_i \bar{u}_i^n)\big) \\
\qquad \tilde{u}_i^{n+1} = u_i^n - \tau(K^*\lambda^{n+1} - \operatorname{div}_1 p_i^{n+1} + \mathbf{c}_i q_i^{n+1}) \\
\qquad u_i^{n+1} = \operatorname{Shrink}_{\tau\gamma_i}(\tilde{u}_i^{n+1}) \\
\qquad w_i^{n+1} = w_i^n + \tau(p_i^{n+1} + \operatorname{div}_2 q_i^{n+1}) \\
\qquad \bar{u}_i^{n+1} = 2u_i^{n+1} - u_i^n \\
\qquad \bar{w}_i^{n+1} = 2w_i^{n+1} - w_i^n \\
\text{end for}
\end{cases}
$$

Example: Denoising "barbara"



noisy image

Example: Denoising “barbara”

TGV^2 regularization

Example: Denoising “barbara”

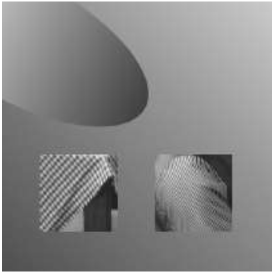
ICTGV$^{\text{osci}}$ regularization

Example: Denoising “barbara”

ICTGV$^{\text{osci}}$ regularization

$\rightsquigarrow$ **Better recovery of oscillatory structures**

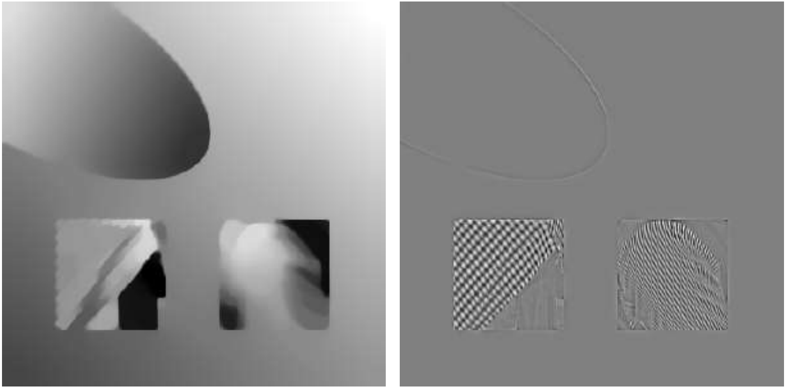
Cartoon/texture decomposition



Tested algorithms:

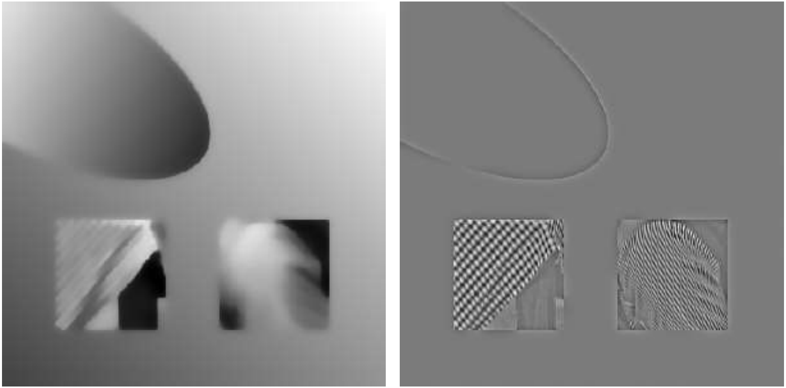
- Total variation/G-norm decomposition *[Aujol et al. '05]*
- Total variation/H^{-1}-norm decomposition *[Osher/Sole/Vese '03]*
- Framelet/Local discrete cosine transform model *[Cai/Osher/Shen '09]*
- Infimal convolution of oscillation TGV

Cartoon/texture decomposition



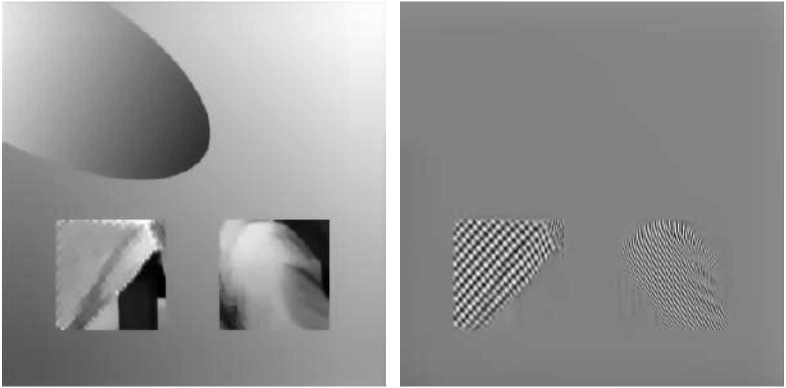
TV-G-norm decomposition

Cartoon/texture decomposition



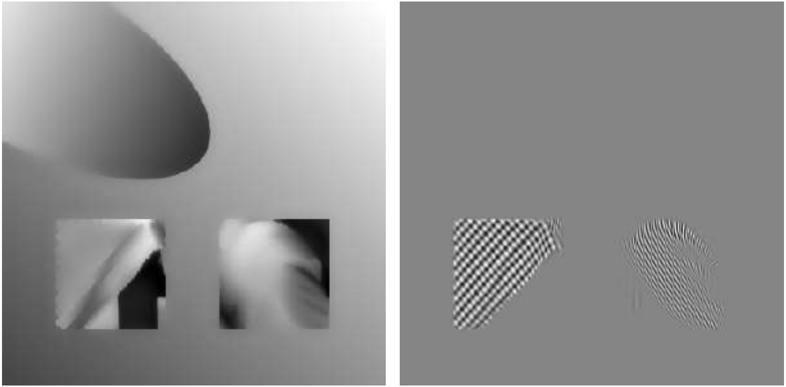
TV-H^{-1}-norm decomposition

Cartoon/texture decomposition



Framelet+LDCT model

Cartoon/texture decomposition



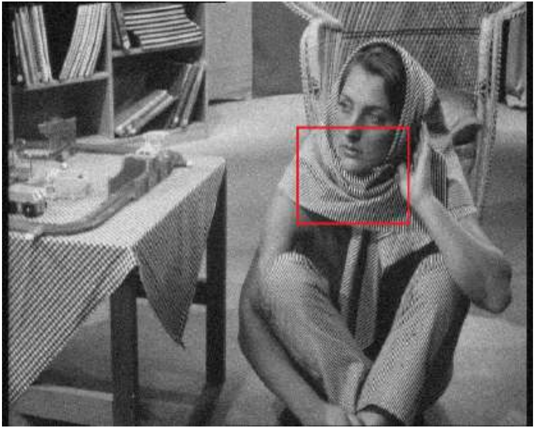
ICTGV$^{\mathrm{osci}}$ model

Example: Image denoising

Tested models/algorithms:

- Total generalized variation of second order *[B./Kunisch/Pock '10]*
- Nonlocal total variation *[Gilboa/Osher '08]*
- Infimal convolution of total generalized variation *[Holler/Kunisch '14]*
- Framelet + local discrete cosine transform *[Cai/Osher/Shen '09]*
- Infimal convolution of oscillation TGV
- Block matching and 3D filtering *[Dabov et al. '07]*

Example: Denoising "barbara"



noisy image

Example: Denoising "barbara"

TGV^2 regularization (27.41 dB)

Example: Denoising "barbara"

Nonlocal TV regularization (32.05 dB)

Example: Denoising "barbara"

ICTGV regularization (31.15 dB)

Example: Denoising "barbara"

Framelet + LDCT model (31.70 dB)

Example: Denoising “barbara”

ICTGV$^{\text{osci}}$ regularization (32.21 dB)

Example: Denoising "barbara"

BM3D (34.43 dB)

Example: Image inpainting

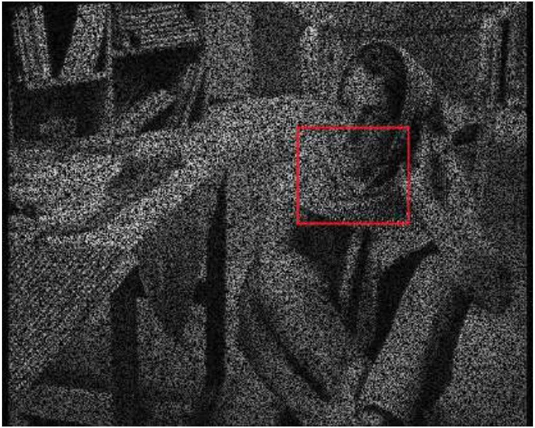
Setup:

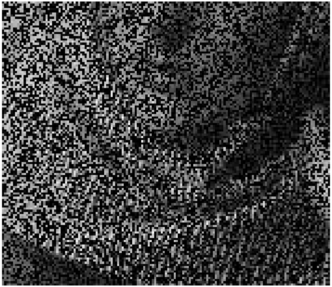
- Recover an image from 50% randomly deleted pixels

Tested algorithms:

- Total generalized variation *[B./Kunisch/Pock '10]*
- Framelet/Local discrete cosine transform model *[Cai/Osher/Shen '09]*
- Infimal convolution of oscillation TGV

Example: Inpainting "barbara"





corrupted image

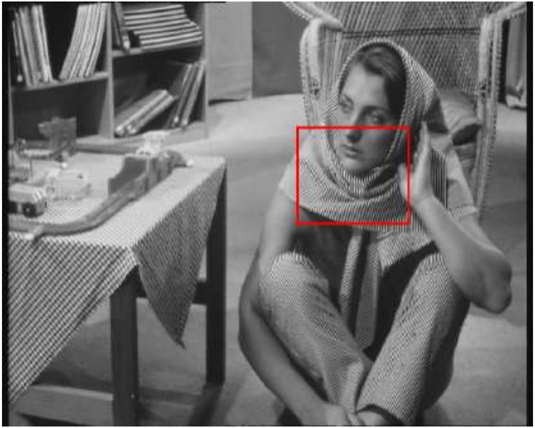
Example: Inpainting "barbara"

TGV^2 regularization (27.49 dB)

Example: Inpainting “barbara”

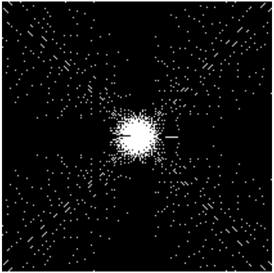
Framelet + LDCT model (32.75 dB)

Example: Inpainting “barbara”



$\mathrm{ICTGV}^{\mathrm{osci}}$ regularization (34.03 dB)

Example: Undersampled MRI



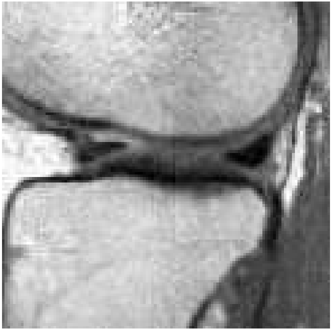
Setup:

- Recover an image from a few radially sampled Fourier coefficients

Tested algorithms:

- Total generalized variation + shearlet model *[Guo/Qin/Yin '14]*
- Infimal convolution of oscillation TGV

Example: MRI reconstruction



TGV + shearlet model (70 radial lines)

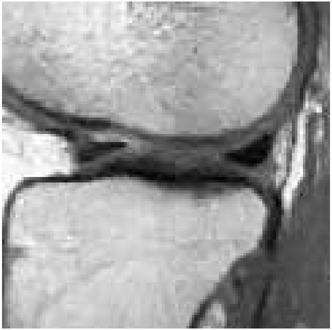
Example: MRI reconstruction

TGV + shearlet model (60 radial lines)

Example: MRI reconstruction

TGV + shearlet model (50 radial lines)

Example: MRI reconstruction



ICTGV$^{\text{osci}}$ model (70 radial lines)

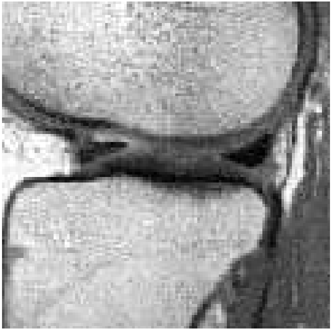
Example: MRI reconstruction

ICTGV$^{\text{osci}}$ model (60 radial lines)

Example: MRI reconstruction

ICTGV$^{\text{osci}}$ model (50 radial lines)

Example: MRI reconstruction



ground truth

Example: MRI reconstruction

Quantitative comparison:

Model	50 lines	60 lines	70 lines
TGV + shearlet	28.77 dB	29.59 dB	30.10 dB
$\mathrm{ICTGV}^{\mathrm{osci}}$	29.16 dB	29.98 dB	30.56 dB

Outline

Summary

- Infimal convolution is a flexible tool to combine variational models, covering several known approaches
- It provides a regularizer for linear inverse problems in many cases
- It allows to construct stabilizing and regularizing cartoon/texture decomposition models
- Numerical experiments show a promising efficiency

Yiming Gao and Kristian Bredies.
Infimal convolution of oscillation total generalized variation for the recovery of images with structured texture.
SIAM Journal on Imaging Sciences, 11(3):2021–2063, 2018.

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