SDCA-Powered Inexact Dual Augmented Lagrangian Method for Fast CRF Learning

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Joint work with Shell Xu Hu

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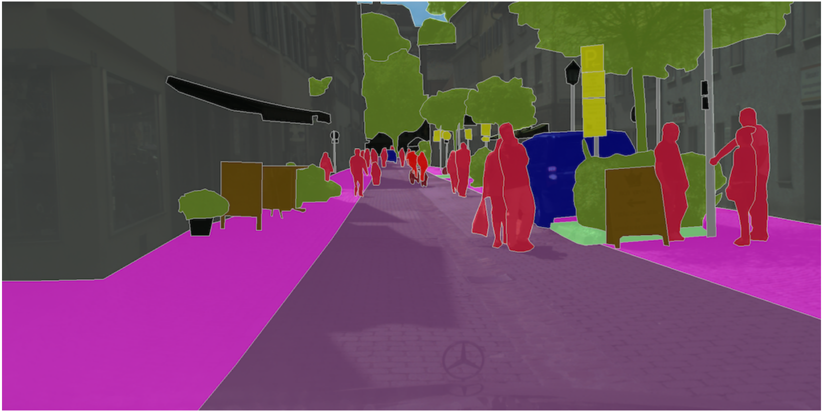
Outline

A motivating example: semantic segmentation



Cityscapes dataset (Cordts et al., 2016)

Recent fast algorithms for large sums of functions

$$\min_w F(w) + \frac{\lambda}{2}\|w\|_2^2 \qquad \text{with} \quad F(w) = \sum_{s=1}^{n} F_s(w)$$

and typically $F_s(w) = f_s(w^\intercal \varphi(x_s)) = \ell(w^\intercal \varphi(x_s), y_s)$

Stochastic gradient methods with variance reduction

Iterate: pick s at random and update $w^{t+1} = w^t - \eta g^t$ with

$$\begin{array}{lcllll}
\text{(SVRG)} & g^t = & \nabla F_s(w^t) - \nabla F_s(\widetilde{w}) & + \frac{1}{n} \nabla F(\widetilde{w}) & \text{and} & \widetilde{w} = \widetilde{w}_{\text{epoch}} \\
\text{(SAG)} & g^t = & \nabla F_s(w^t) - g_s^{t-1} & + g^{t-1} & \text{and} & g_s^t = \nabla F_s(w^t) \\
\text{(SAGA)} & g^t = & \nabla F_s(w^t) - g_s^{t-1} & + \frac{1}{n} g^{t-1} & \text{and} & g_s^t = \nabla F_s(w^t)
\end{array}$$

Stochastic Dual Coordinate Ascent (Implicit Variance reduction)

$$\max_{\alpha_1,\ldots,\alpha_n} \sum_{s=1}^{n} f_s^*(\alpha_s) + \frac{1}{2\lambda} \Big\| \sum_{s=1}^{n} \varphi(x_s)\alpha_s \Big\|_2^2$$

Iterate $\quad \alpha_s^{t+1} \leftarrow \text{Prox}_{\frac{\lambda}{L_s} f_s^*}\left(\alpha_s^t - \frac{1}{L_s}\varphi(x_s)^\intercal w^t\right), \qquad \alpha_i^{t+1} \leftarrow \alpha_i^t, \ \forall i \neq s.$

Variance reduction techniques yield improved rates

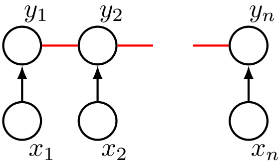
- κ : condition number
- d : ambient dimension

Running times to have $\mathrm{Obj}(w) - \mathrm{Obj}(w^*) \leq \varepsilon$

Stochastic GD	$d\,\kappa\,\frac{1}{\varepsilon}$
GD	$d\,n\kappa\,\log\frac{1}{\varepsilon}$
Accelerated GD	$d\,n\sqrt{\kappa}\,\log\frac{1}{\varepsilon}$
SAG(A), SVRG, SDCA, MISO	$d\,(n+\kappa)\,\log\frac{1}{\varepsilon}$
Accelerated variants	$d\,(n+\sqrt{n\kappa})\,\log\frac{1}{\varepsilon}$

- Exploiting sum structure yields faster algorithms...

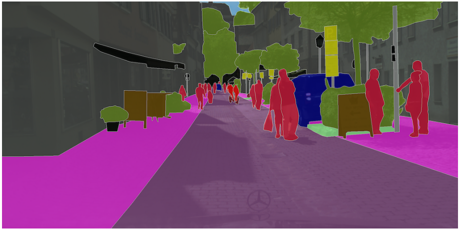
$$\min_w \sum_{s=1}^{n} \ell(w^\intercal \phi(x_s), y_s) + \frac{\lambda}{2}\|w\|_2^2$$



✓

Conditional Random Fields

- Input image x
- Features at pixel s: $\varphi_s(x)$
- Encoding of class at pixel s: $y_s = (y_{s1}, \dots, y_{sK})$ with
  - $y_{sk} = 1$ if in class k
  - $y_{sk} = 0$ else.



Options:

1. predict each pixel class individually: ***multiclass logistic regression***

$$p(y_s|x) \propto \exp\left(\sum_{k=1}^{K} y_{sk}\, w_k{}^\mathsf{T}\, \varphi_s(x)\right)$$

2. View image as a grid graph with vertices $\mathcal{V}$ and edges $\mathcal{E}$, and predict all pixels classes jointly while accounting for dependencies: ***CRF***

$$p(y_1, \dots, y_S|x) \propto \exp\left(\sum_{s\in\mathcal{V}} \sum_{k=1}^{K} y_{sk} w_{\tau_1,k}{}^\mathsf{T} \varphi_s(x) + \sum_{\{s,t\}\in\mathcal{E}} \sum_{k,l=1}^{K} w_{\tau_2,kl}\, y_{sk}\, y_{tl}\right)$$

Trick: log-likelihood as log-partition

$$
\begin{aligned}
-\log p(y^o|x^o) &= -\langle w, \phi(y^o, x^o)\rangle + A_{x^o}(w) \\
&= -\langle w, \phi(y^o, x^o)\rangle + \log \sum_{y} \exp\langle w, \phi(y, x^o)\rangle \\
&= \log \sum_{y} \exp\langle w, \phi(y, x^o) - \phi(y^o, x^o)\rangle \\
&= \log \sum_{y} \exp\Big(\sum_{c\in\mathcal{C}} \langle w_{\tau_c}, \phi_c(y_c, x^o) - \phi(y^o_c, x^o)\rangle\Big) \\
&= \log \sum_{y} \exp\Big(\sum_{c\in\mathcal{C}} \langle y_c, \theta_{(c)}\rangle\Big) \\
&\text{with} \quad \theta_{(c)} = \big(\langle w_{\tau_c}, \underbrace{\phi_c(y'_c, x^o) - \phi(y^o_c, x^o)}_{\psi_c(y'_c)}\rangle\big)_{y'_c\in\mathcal{Y}_c}.
\end{aligned}
$$

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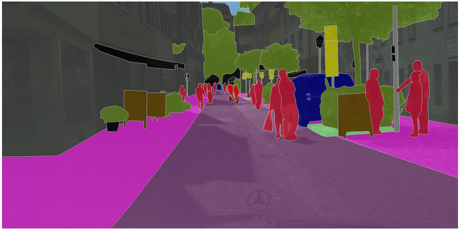

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Abstract CRF model

$$p(y^o|x^o) \propto \exp\left(\sum_{s\in\mathcal{V}}\sum_{k=1}^{K} y^o_{sk}\, w_{\tau_1,k}{}^{\intercal}\, \varphi_s(x^o) + \sum_{\{s,t\}\in\mathcal{E}}\sum_{k,l=1}^{K} w_{\tau_2,kl}\, y^o_{sk}\, y^o_{tl}\right)$$

$$p(y^o|x^o) \propto \exp\left(\sum_{s\in\mathcal{V}}\langle w_{\tau_1}, \phi_s(y^o_s, x^o)\rangle + \sum_{\{s,t\}\in\mathcal{E}}\langle w_{\tau_2}, \phi_{st}(y^o_s, y^o_t, x^o)\rangle\right)$$

Let $\mathcal{C} = \mathcal{V} \cup \mathcal{E}$, $\quad \log p_w(y^o|x^o) = \sum_{c\in\mathcal{C}}\langle w_{\tau_c}, \phi_c(x^o, y^o_c)\rangle - \log Z(x^o, w),$

with $y_{\{s,t\}} = y_s y_t^{\intercal}$ and $Z(x^o, w) = \sum_{y_1}\dots\sum_{y_S}\exp\left(\sum_{c\in\mathcal{C}}\langle w_{\tau_c}, \phi_c(x^o, y_c)\rangle\right)$

In fact $-\log p_w(y^o|x^o) = \log\sum_{y}\exp\left(\sum_{c\in\mathcal{C}}\langle w_{\tau_c}, \phi_c(x^o, y_c) - \phi_c(x^o, y^o_c)\rangle\right)$

$$= \log\sum_{y}\exp\sum_{c\in\mathcal{C}}\langle \Psi_{(c)}^{\intercal} w, y_c\rangle$$

$$=: f(\Psi^{\intercal} w) \quad \text{with} \quad f(\theta) = \log\sum_{y}\exp\sum_{c\in\mathcal{C}}\langle\theta_{(c)}, y_c\rangle.$$

Regularized maximum likelihood estimation

The regularized maximum likelihood estimation problem

$$\min_w - \log p_w(y^o|x^o) + \frac{\lambda}{2}\|w\|_2^2$$

is reformulated as

$$\min_w f(\Psi^\intercal w) + \frac{\lambda}{2}\|w\|_2^2 \qquad \text{with} \qquad f(\theta) = \log \sum_y \exp \sum_{c \in \mathcal{C}} \langle \theta_{(c)}, y_c \rangle,$$

f is essentially another way of writing the log-partition function A.

Major issue: NP-hardness of inference in graphical models

- f and its gradient are NP-difficult to compute.

$\Rightarrow$ the maximum likelihood estimator is intractable.

- f or ∇F can be estimated using MCMC methods to perform *approximate inference*.
- *Approximate inference* can also be solved as an optimization problem with *variational methods*.

Compare with the "disconnected graph" case

$$\min_w \sum_{s=1}^{S} \log p_w(y_s^o | x^o) + \frac{\lambda}{2} \|w\|_2^2$$

$$\min_w \sum_{s=1}^{S} f_s(\psi_s^\intercal w) + \frac{\lambda}{2} \|w\|_2^2 \quad \text{with} \quad f_s(\theta_{(s)}) := \log \sum_{y_s} \exp\langle \theta_{(s)}, y_s \rangle.$$

- f_s is easy to compute: the sum of K terms
- The objective is a sum of a large number of terms

$\Rightarrow$ Very fast randomized algorithms can be used to solve this problem

SAG Roux et al. (2012)
SVRG Johnson and Zhang (2013)
SAGA Defazio et al. (2014), etc
SDCA Shalev-Shwartz and Zhang (2016)

$$\max_{\alpha_1, \ldots, \alpha_S} \sum_{s=1}^{S} f_s^*(\alpha_s) + \frac{1}{2\lambda} \Big\| \sum_{s=1}^{S} \psi_s \alpha_s \Big\|_2^2$$

Could we do the same for CRFs? With SDCA?

Fenchel conjugate of the log-partition function

$$f(\theta) := \log \sum_{y} \exp \sum_{c \in \mathcal{C}} \langle \theta_{(c)}, y_c \rangle = \max_{\mu \in \mathcal{M}} \langle \mu, \theta \rangle + H_{\text{Shannon}}(\mu),$$

- The marginal polytope $\mathcal{M}$ is the set of all realizable moments vectors

$$\mathcal{M} := \Big\{ \mu = (\mu_c)_{c \in \mathcal{C}} \mid \exists Y \quad \text{s.t.} \quad \forall c \in \mathcal{C},\ \mu_c = \mathbb{E}[Y_c] \Big\}.$$

- H_{Shannon} is the Shannon entropy of the maximum entropy distribution with moments μ.

$$P^{\#}(w) := f\big(\Psi^{\intercal} w\big) + \frac{\lambda}{2} \|w\|_2^2$$

$$D^{\#}(\mu) := H_{\text{Shannon}}(\mu) - \iota_{\mathcal{M}}(\mu) - \frac{1}{2\lambda} \|\Psi \mu\|_2^2$$

$$\min_{w} P^{\#}(w) \qquad \text{and} \qquad \max_{\mu} D^{\#}(\mu)$$

form a pair of primal and dual optimization problems.

Both H_{Shannon} and $\mathcal{M}$ are intractable $\rightarrow$ NP-hard problem in general

Relaxing the marginal into the local polytope.

A classical relaxation for $\mathcal{M}$: the local polytope $\mathcal{L}$

For $\mathcal{C} = \mathcal{E} \cup \mathcal{V}$

Node and edge simplex constraints:

$$\forall s \in \mathcal{V}, \quad \boxed{\triangle_s := \left\{\mu_s \in \mathbb{R}_+^k \mid \mu_s^\intercal 1 = 1\right\}}$$

$$\forall \{s,t\} \in \mathcal{E}, \quad \boxed{\triangle_{\{s,t\}} := \left\{\mu_{st} \in \mathbb{R}_+^{k \times k} \mid 1^\intercal \mu_{st}^\intercal 1 = 1\right\}}.$$

$$\boxed{\mathcal{I} := \Big\{\mu = (\mu_c)_{c \in \mathcal{C}} \mid \forall c \in \mathcal{C}, \quad \mu_c \in \triangle_c\Big\}}$$

$$\mathcal{L} := \Big\{\mu \in \boxed{\mathcal{I}} \mid \boxed{\forall \{s,t\} \in \mathcal{E}, \quad \mu_{st}\,\mathbf{1} = \mu_s, \quad \mu_{st}^\intercal\,\mathbf{1} = \mu_t}\Big\}$$

$$\mathcal{L} = \boxed{\mathcal{I}} \cap \boxed{\{\mu \mid A\mu = 0\}}$$

for an appropriate definition of A...

Surrogates for the entropy

Various entropy surrogates exist, e.g.:

- Bethe entropy (nonconvex),
- Tree-reweighted entropy (TRW) (convex on $\mathcal{L}$ but not on $\mathcal{I}$)

Separable surrogates H_{approx}

We consider surrogates of the form $H_{\text{approx}}(\mu) = \sum_{c \in \mathcal{C}} h_c(\mu_c)$, such that

- each function h_c is **smooth**[a] and **convex on** $\triangle_c$ and
- H_{approx} is **strongly convex on** $\mathcal{L}$

In particular we propose to use

- the Gini entropy: $h_c(\mu_c) = 1 - \|\mu_c\|_F^2$
- a quadratic counterpart of the *oriented tree-reweighted entropy*:

[a] i.e. has Lipschitz gradients

Relaxed dual problem

$$\mathcal{M} \xrightarrow{\text{relax to}} \mathcal{L} = \mathcal{I} \cap \{\mu \mid A\mu = 0\}$$

$$H_{\text{Shannon}} \xrightarrow{\text{relax to}} H_{\text{approx}}(\mu) := \sum_{c \in \mathcal{C}} h_c(\mu_c).$$

Problem relaxation

$$D^{\#}(\mu) := H_{\text{Shannon}}(\mu) - \iota_{\mathcal{M}}(\mu) - \frac{1}{2\lambda}\|\Psi\mu\|_2^2$$

relax to $\downarrow$

$$D(\mu) := H_{\text{approx}}(\mu) - \iota_{\mathcal{I}}(\mu) - \iota_{\{A\mu=0\}} - \frac{1}{2\lambda}\|\Psi\mu\|_2^2$$

so that with

$$f_c^*(\mu_c) : -h_c(\mu_c) + \iota_{\triangle_c}(\mu_c) \quad \text{and} \quad g^*(\mu) = \frac{1}{2\lambda}\|\Psi\mu\|_2^2$$

we have $$D(\mu) = -\sum_{c \in \mathcal{C}} f_c^*(\mu_c) - g^*(\mu) - \iota_{\{A\mu=0\}}.$$

A dual augmented Lagrangian formulation

$$D(\mu) = -\sum_{c \in \mathcal{C}} f_c^*(\mu_c) - g^*(\mu) - \iota_{\{A\mu=0\}}$$

Idea: without the linear constraint, we could exploit the form of the objective to use a fast algorithm such as *stochastic dual coordinate ascent*.

$$D_\rho(\mu, \xi) = -\sum_{c \in \mathcal{C}} f_c^*(\mu_c) - g^*(\mu) - \langle \xi, A\mu \rangle - \frac{1}{2\rho}\|A\mu\|_2^2$$

By strong duality, we need to solve

$$\min_\xi d(\xi) \qquad \text{with} \qquad d(\xi) := \max_\mu D_\rho(\mu, \xi).$$

The algorithm

Need to solve

$$\min_{\xi} d(\xi) \qquad \text{with} \qquad d(\xi) := \max_{\mu} D_\rho(\mu, \xi).$$

with

$$D_\rho(\mu, \xi) = -\sum_{c \in \mathcal{C}} f_c^*(\mu_c) - g^*(\mu) - \langle \xi, A\mu \rangle - \frac{1}{2\rho}\|A\mu\|_2^2.$$

Note that we have $\nabla d(\xi) = A\mu_\xi$ with $\mu_\xi = \arg\min_{\xi} D_\rho(\mu, \xi)$.

Combining an *inexact dual Lagrangian method* with a subsolver $\mathcal{A}$

At epoch t:

- Maximize D_ρ partially w.r.t. μ using a fixed number of steps of a (stochastic) linearly convergent algorithm $\mathcal{A}$ to get $\hat{\mu}^t$ from the $\hat{\mu}^{t-1}$.
- Take an inexact gradient step on d with $\xi^{t+1} = \xi^t - \frac{1}{L} A\hat{\mu}^t$

Main technical lemma

- Let ξ^t (resp. $\hat{\mu}^t$) the value of ξ (resp. μ) at the end of epoch t
- Let $\hat{\Delta}_t := \max_{\mu} D_\rho(\mu, \xi^t) - D_\rho(\hat{\mu}^t, \xi^t)$ and $\Gamma_t := d(\xi^t) - d(\xi^*)$.
- Let $\Delta_t^0 := \max_{\mu} D_\rho(\mu, \xi^t) - D_\rho(\mu_0^t, \xi^t)$

If algorithm $\mathcal{A}$ used at epoch t to maximize $D_\rho(\mu, \xi)$ w.r.t. μ is such that

$$\exists \beta \in (0,1), \qquad \mathbb{E}[\hat{\Delta}_t] \leq \beta\, \mathbb{E}[\Delta_t^0] \quad ,$$

then $\exists\, \kappa \in (0,1)$ characterizing d and $\exists\, C > 0$ such that, if $\mu_0^t = \hat{\mu}^{t-1}$,

$$\left\| \begin{matrix} \mathbb{E}[\hat{\Delta}_{T_{\text{ex}}}] \\ \mathbb{E}[\Gamma_{T_{\text{ex}}}] \end{matrix} \right\| \leq C\, \lambda_{\max}(\beta)^{T_{\text{ex}}} \left\| \begin{matrix} \mathbb{E}[\hat{\Delta}_0] \\ \mathbb{E}[\Gamma_0] \end{matrix} \right\| \quad ,$$

where $\lambda_{\max}(\beta)$ is the largest eigenvalue of the matrix $M(\beta) = \begin{bmatrix} 6\beta & 3\beta \\ 1 & 1-\kappa \end{bmatrix}$

Main theoretical result: linear convergence in the dual

Let $\mathcal{A}$ be an *iterative* algorithm used to solve partially $\max_{\mu} D_\rho(\mu, \xi)$.

- Let ξ^t (resp. $\hat{\mu}^t$) the value of ξ (resp. μ) at the end of epoch t
- Let $\hat{\Delta}_t := \max_{\mu} D_\rho(\mu, \xi^t) - D_\rho(\hat{\mu}^t, \xi^t)$ and $\Gamma_t := d(\xi^t) - d(\xi^*)$.

Proposition: If

- $\mathcal{A}$ is a linearly convergent algorithm
- at epoch t, $\mathcal{A}$ is initialized with $\hat{\mu}^{t-1}$ ($\rightarrow$ use of warm-starts)
- $\mathcal{A}$ is run for a fixed ahead T_{in} number of iteration at each epoch

then we have

- $\hat{\Delta}_t$, $\Gamma_t \xrightarrow{a.s.} 0$ linearly
- the residuals $\|A\hat{\mu}^t\|_2^2 \xrightarrow{a.s.} 0$ linearly
- the smooth part of the objective a.s. converges linearly

Global linear convergence in the primal

Let P be the relaxed primal objective

$$P(w) := F_{\mathcal{L}}\big(\Psi^\intercal w\big) + \frac{\lambda}{2}\|w\|_2^2, \quad \text{with} \quad F_{\mathcal{L}}(\theta) := \max_{\mu \in \mathcal{L}} \langle \theta, \mu \rangle + H_{\text{approx}}(\mu).$$

Corollary

Let $\quad \hat{w}^t = -\frac{1}{\lambda}\Psi\hat{\mu}^t$.

If

- $\mathcal{A}$ is a *linearly convergent* algorithm and
- the function $\mu \mapsto -H_{\text{approx}}(\mu) + \frac{1}{2\rho}\|A\mu\|_2^2$ is *strongly convex*,

then $\quad P(\hat{w}^t) - P(w^\star)$ converges to 0 linearly a.s.

Since a fixed nb of inner iterations are done at each epoch, the linear convergence is as a function of the total number of clique updates.

Related work

A lot of work on approximate inference for CRFs:

- Komodakis et al. (2007); Sontag et al. (2008); Savchynskyy et al. (2011)

Learning method going beyond saddle formulations:

- Meshi et al. (2010); Hazan and Urtasun (2010); Lacoste-Julien et al. (2013)

Learning in the dual for structured SVMs **with only clique-wise updates**:

- With relaxation + smoothing of the linear constraints Meshi et al. (2015) and using block coordinate Frank-Wolfe (BCFW) or block coordinate ascent.
- With multiplier and a greedy primal dual algorithm, Yen et al. (2016) show a global linear convergence result in the dual.

Convergence rates for approximate gradient descent

- Schmidt et al. (2011); Devolder et al. (2014); Lan and Monteiro (2016); Lin et al. (2017)

Related work on BCFW with linear constraints: Gidel et al. (2018)

Experiments: Algorithms

SoftBCFW	Stochastic block coordinate Frank-Wolfe + penalty method (Meshi et al., 2015)
SoftSDCA	Stochastic block coordinate prox ascent + penalty method
GDMM	Dual decomposed learning with factorwise oracle (Yen et al., 2016)
IDAL	Our algorithm

Datasets

Gaussian mixture Potts model

- 10×10 grid graph with 5 classes
- Gaussian features in $\mathbb{R}^{10}$
- $(w_{\tau_1} \in \mathbb{R}^{10\times 5}, w_{\tau_2} \in \mathbb{R}^{5\times 5})$
- 50 training grids

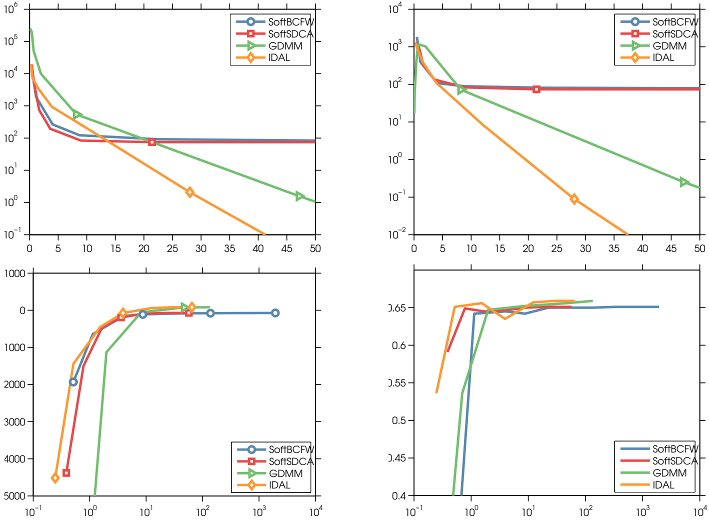
Semantic segmentation of images

- MSRC-21 dataset (Shotton et al., 2006)
- 21 classes
- 50 features $(w_{\tau_1} \in \mathbb{R}^{50\times 21}, w_{\tau_2} \in \mathbb{R}^{21\times 21})$
- 335 training images

Results for Gaussian mixture Potts model $(\lambda = 10, \rho = 1)$

Bound on duality gap

Gap on marginalization constraints

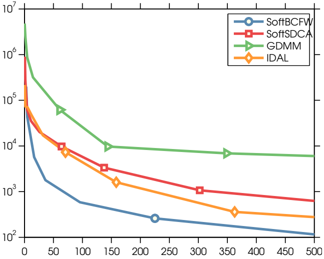


Dual objective

Accuracy on test data

Result on segmentation dataset, max margin variant $(\lambda = 1, \rho = 0.1)$

Bound on duality gap



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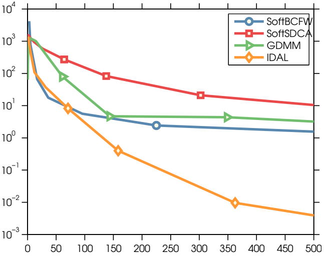

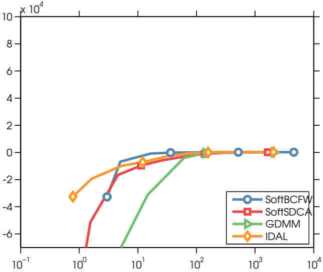

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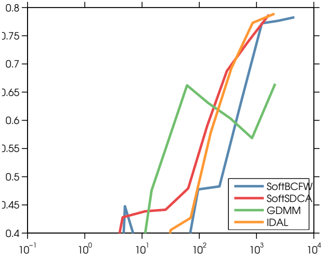

Accuracy on test data

Summary and conclusions

We proposed an algorithm combining SDCA and an inexact dual Lagrangian method that obtains

- Global linear convergences for the relaxed objective
  - In the primal and for the dual augmented Lagrangian formulation
  - Obtains good practical performance

Other contributions:

- Computable duality gaps to track convergence in the primal
- Representer theorem in the structured learning case "inside the graph"
- Unified derivation connecting formulations of previous work
- SDCA can accommodate linear constraints on the dual parameter.

Open questions:

- Use a better approx. for the entropy like OTRW ($\rightarrow$ non Lipschitz gradients)?
- Be stochastic on ξ as well?

Paper: SDCA-Powered Inexact Dual Augmented Lagrangian Method for Fast CRF Learning, X. Hu, G. Obozinski, AIStats, 2018.

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