CHAPTER 15

Estimating structural models of unemployment and job duration

Dale T. Mortensen and George R. Neumann

Available theoretical models of unemployment and job duration are based on a dynamic formulation of an individual worker's job search and job-matching problems. In general versions of the theory, the individual anticipates future arrival of an uncertain sequence of employment opportunities, whether currently employed or not. The worker controls the transition process between employment states and jobs by choosing search and acceptance strategies that maximize expected wealth given current information. In other words, the strategy is the solution to a well-defined dynamic programming problem.

Given a model of this type, the properties of the probability distributions of both the length of time spent looking for an acceptable job while unemployed and, once employed, the length of the specific job spell are endogenously determined by the optimal strategy and the structure of the decision problem. Hence, observations on the completed unemployment and job spell lengths experienced by a sample of workers provide information about the problem's structure. The purpose of this chapter is to develop methods suggested by the theory that an econometrician might find useful for the purpose of estimating structural parameters from available observations on realized unemployment and job spell lengths and to test their potential usefulness using Monte Carlo techniques.

Although structural models of unemployment and job spell duration have been available for some time, there are few attempts to estimate them in the literature. Instead, ad hoc specifications of the duration hazards borrowed from the statistical literature on survival and reliability analysis are estimated.[1] This state of affairs is not surprising given the problems

The research reported in this chapter was funded by a collaborative National Science Foundation grant.

[1] The empirical literature on unemployment duration that apply a search-theoretic framework include Ehrenberg and Oaxaca (1976), Kiefer and Neumann (1979a, 1981), Flinn and

posed by any dynamic optimization theory. The stumbling block is a consequence of the "forward-looking" nature of the solution to any intertemporal optimization problem. Given this nature, no general method for representing the optimal decision rules as closed-form functions of the structural parameters and regressors of the model is available. Hence, one does not know the form of the likelihood function or any other criterion for estimation.

In this chapter, the model considered is a generalization of Burdett's (1978) job search model.[2] Workers choose to search, whether employed or not, at endogenous intensities in the model studied. Hence, the worker is presumed to determine both an offer acceptance strategy and a search intensity strategy that maximize expected wealth. The optimal search strategy, given the structure of the decision problem, determines the hazard rates associated with the implied probability distribution over the length of both an unemployment spell and the subsequent job spell. Given a particular parameterization of the structure of the worker's decision problem, these spell length distributions and appropriate data on unemployment and job spells experienced by a sample of workers allow one to define a sample likelihood function of the parameters.

However, one cannot derive the likelihood function associated with any reasonable parameterization of the structure, at least not in closed form. In this chapter, several related methods for computing the value function are suggested that permit a numerical representation of the optimal search strategies and the associated spell duration hazard rates that appear in the likelihood function.

The principal method is based on the mathematical structure of discounted dynamic programming, the fact that the value function is the unique fixed point of a contraction map. The contraction map itself provides an algorithm for computing its own fixed point. However, in the

Footnote 1 *(cont.)*
Heckman (1983), Nickell (1979), Lancaster and Nickell (1980), and Burdett et al. (1984). Theoretical and empirical studies of job spell duration that emphasize job search and matching include Burdett (1978), Jovanovic (1979a, b, 1984), Mincer and Jovanovic (1981), Flinn (1986), Miller (1984), Mortensen (1984), and Topel and Ward (1985). Econometric methodology is discussed in many of these papers and in Flinn and Heckman (1982) and Heckman and Singer (1982). Among the empirical papers in this collection, only Miller attempts to estimate the structure of the underlying decision problem.

2 The original theoretical formulation of the search on the job problem can be found in Burdett (1978) although the model abstracts from the search effort choice. Burdett and Mortensen (1978) consider a generalization which allows for search intensity choice as well as search while employed. Benhabib and Bull (1983) represents a more recent contribution to the theory of search intensity choice. Finally, Jovanovic (1984) and Mortensen (1984) study generalizations of the model that allow for learning about match specific quality.

case under study, the modulus of the contraction is close to unity for reasonable parameter values. Consequently, a sufficiently close approximation can be expected to require considerable computing time as well as a special programming effort. The second method suggested is based on the fact that the optimal search and acceptance strategies can be represented as solutions to ordinary first-order differential equations. In this case, a numerical solution to the equation can be computed relatively easily using generally available software. Finally, for the model under study, the closed-form optimal strategy and the associated likelihood function can be derived for the limiting case in which the modulus of the contraction converges to unity. The estimates obtained using the functional form associated with this limiting case are of interest in their own right but also provide natural starting values for an iterative estimation procedure that uses either of the other two methods.[3]

The suggested computation methods are used here to estimate a special case of the model, that of constant search intensities, using synthetic data generated with the true model. That the computations are feasible and that the method of locating maximum-likelihood estimates (MLE) using them has satisfactory properties is established using Monte Carlo techniques.

The text that follows is composed of four principal sections. In the first, the formal optimization problem that a worker faces is formulated and the optimal search strategy is characterized. The computation methods are developed in the second section. The third section reports on Monte Carlo estimation experiments applying the methods. This effort and directions for future research are summarized in the final section.

1 Theoretical model of unemployment and job duration

A job is characterized by a stationary wage which generally differs across employers for a given worker. Whether the worker is employed or not, the arrival of wage offers to the worker is generated by a Poisson process. The arrival rate depends on the worker's search intensity which is chosen subject to a cost that may depend on the worker's employment status.

[3] The only related literature on the estimation problem other than Miller (1984) of which we are aware are papers by Rust (1984) and Pakes (1986). The optimization problem that Rust considers does not include the model studied here as a special case; consequently, his theorems and method do not apply. Pakes, on the other hand, does estimate an optimal stopping model similar to ours using a maximum-likelihood procedure augmented by a computational algorithm that computes the optimal stopping rule at each iteration of the procedure. However, his data, duration distributions of patent lives, have a quite different structure than that available for job and unemployment spells. Hence, the specifics of his procedure are not applicable.

Let $c_i(s)$, $s \in \mathbb{R}_+$, denote search cost per period at intensity s, where $i=0$ and $i=1$ denote not employed and employed respectively. A wage offer, $w \in W$, that is a random draw from the offer distribution $F: W \to [0,1]$ is accepted or rejected on arrival. The worker is assumed to know the wage offer distribution and to make each acceptance decision and search intensity choice so as to maximize the expected present value of future income, expected wealth. The discount rate, denoted as r, is stationary and the working life is of indefinite length. When not employed, the worker receives a benefit b that is foregone once employed. Hence, the structure of the decision model is a specification of the interest rate, the unemployment benefit, the wage offer distribution, and the two cost-of-search functions.

Let $V_0(b)$ represent maximal expected wealth when unemployed and the benefit received is b, and let $V_1(w)$ denote expected wealth when employed at wage w. By applying Bellman's dynamic optimization principle, one can represent the two value functions as the solutions to the two functional equations represented in what follows. Because (1) the time required for the arrival of an alternative offer when unemployment, denoted as T, is exponentially distributed with hazard rate equal to the offer arrival rate as a consequence of the Poisson arrival assumption, (2) the offer arrival rate can be regarded as the worker's search intensity s without loss of generality, (3) the worker can choose between the wealth associated with continued unemployment and that offered by employment at the realized wage X given an arrival, and (4) the worker's income per period is b until the arrival of an offer, maximal expected wealth when unemployed, $V_0(b)$, solves the equation

$$\begin{aligned} V_0(b) &= \max_{s \geq 0} E\left\{\int_0^T [b - c_0(s)]e^{-rt}\,dt + e^{-rT}\max[V_0(b), V_1(X)]\right\} \\ &= \max_{s \geq 0}\left[\frac{b - c_0(s)}{r+s} + \frac{s}{r+s}\int \max[V_0(b), V_1(x)]\,dF(x)\right]. \end{aligned} \tag{1.a}$$

The first term on the right is the expected present value of benefits received net of the cost of search paid during the waiting time required until the next offer, and the second term is the present value of the worker's best choice at the time an offer arrives. Analogous reasoning for the case of employment at wage w implies that

$$\begin{aligned} V_1(w) &= \max_{s \geq 0} E\left\{\int_0^T [w - c_1(s)]e^{-rt}\,dt + e^{-rT}\max[V_1(w), V_1(X)]\right\} \\ &= \max_{s \geq 0}\left[\frac{w - c_1(s)}{r+s} + \frac{s}{r+s}\int \max[V_1(w), V_1(x)]\,dF(x)\right]. \end{aligned} \tag{1.b}$$

Define the "permanent income" associated with unemployed and employed, respectively, as

$$v_0(b) \equiv rV_0(b) \quad \text{and} \quad v_1(w) \equiv rV_1(w). \tag{2}$$

Given (2), equations (1) can be rewritten as

$$v_0(b) = \max_{s \geq 0} \left[[1-\beta(s)][b - c_0(s)] + \beta(s) \int \max[v_0(b), v_1(x)]\, dF(x) \right] \tag{3.a}$$

and

$$v_1(w) = \max_{s \geq 0} \left[[1-\beta(s)][w - c_1(s)] + \beta(s) \int \max[v_1(w), v_1(x)]\, dF(x) \right], \tag{3.b}$$

where

$$\beta(s) \equiv s/(r+s). \tag{4}$$

The following series of results establish the existence of unique solutions to equations (3) and characterize the optimal search strategy associated with them. Three reasonable economic conditions are sufficient for existence:

Assumption 1. The set of possible wage offers W and the set of unemployment benefits B are both bounded.

Assumption 2. The interest of discount rate r is positive.

Assumption 3. The cost-of-search function $c_i(s)$ is twice differentiable, $c_i(0) = c_i'(0) = 0$, and $c_i''(s) > 0$ for $i = 0, 1$.

In stating the results, which are formally proved in the appendix, we use the fact that W can be regarded as a subset of B by virtue of Assumption 1 without loss of generality. Let

$$\bar{w} = \sup W \quad \text{and} \quad \underline{w} = \inf W \tag{5}$$

represent the largest and smallest wage offer, respectively.

Lemma 1. *Any solution* $v_0 \colon B \to \mathbb{R}$ *to* (3.a) *or* $v_1 \colon W \to \mathbb{R}$ *to* (3.b) *is continuous and strictly increasing.*

Lemma 1 implies that the employment acceptance decision satisfies the *reservation property,* that is, a unique reservation wage, denoted as $R(b)$, generally exists for every b such that a wage offer is acceptable if and only if it is no less than the reservation wage. Given the unemployment benefit, the reservation wage equates the values of employment and unemployment:

$$v_0(b) = v_1(R(b)). \tag{6}$$

Because both functions are increasing and continuous, the reservation wage increases continuously with the unemployment benefit. Finally, the optimal acceptance decision when employed is trivial since the indifference condition $v_1(w) = v_1(x)$ and the fact that $v_1(x)$ is continuous and increasing in x implies that any offer greater than or equal to the worker's current wage w yields a higher expected wealth.

Given any solution (v_0, v_1) to equations (3), Assumption 3 implies that a unique search-intensity strategy exists both when employed and not. Let $\lambda_0(R(b))$ and $\lambda_1(w)$ represent these optimal search intensity choices (the offer arrival rates) contingent on the worker's reservation wage when not employed and the wage when employed. These are the solutions to the maximization problems defined on the right sides of equations (3.a) and (3.b), respectively. Consequently, the optimal intensity equates the marginal cost and expected marginal return-to-search effort:

$$rc_0'(\lambda_0(R(b))) = \int_{R(b)}^{\bar{w}} [v_1(x) - v_1(R(b))]\, dF(x) \tag{7.a}$$

in the case of unemployment and

$$rc_1'(\lambda_1(w)) = \int_{w}^{\bar{w}} [v_1(x) - v_1(w)]\, dF(x) \tag{7.b}$$

in the case of employment by virtue of equations (2)–(4). Under Assumption 3, one can show that the first-order condition is sufficient. Hence, equations (7) imply that the optimal offer arrival rate is continuous and decreasing in the unemployment benefit when not employed and the wage when employed.

Note that equations (3), (4), and (7) imply that

$$v_0(b) = b + \lambda_0(R(b))c_0(\lambda_0(R(b))) - c_0(\lambda_0(R(b))) \tag{8.a}$$

and

$$v_1(w) = b + \lambda_1(w)c_1(\lambda_1(w)) - c_1(\lambda_1(w)). \tag{8.b}$$

Hence, given the optimal intensity functions, the reservation wage function $R(b)$ can be obtained using equation (6) after appropriate substitution from equations (8).

To complete the existence argument, we need the following.

Lemma 2. *If the solution to* (3.b), $v_1: W \to \mathbb{R}$, *is unique, continuous, and bounded, then* (3.a) *has a unique solution,* $v_0: B \to \mathbb{R}$.

Note that any solution to equation (3.b) is a fixed point of the transformation T defined by the right side of the equation:

$$\begin{aligned}(Tv)(w) &= \max_{s \in \mathbb{R}_+} \Big[[1-\beta(s)][b-c_0(s)] \\ &\qquad + \beta(s) \int \max[v_0(b), v_1(x)]\, dF(x) \Big] \\ &= [1-\beta(\lambda_1(w))][w - c_1(\lambda_1(w))] \\ &\qquad + \beta(\lambda_1(w)) \int \max[v_1(w), v_1(x)]\, dF(x). \end{aligned} \tag{9}$$

By virtue of Lemma 1, the fixed point must be an element of the space of continuous and increasing real-valued functions. Furthermore, for any function in this space, equations (9) and (7.b) imply

$$w \le v_1(w) \le (Tv_1)(\bar{w}) = \bar{w} \quad \text{because} \ \lambda_1(\bar{w}) = 0. \tag{10}$$

Let $\mathfrak{F}$ represent the set of functions with these properties, that is,

$$\mathfrak{F} = \{v: W \to \mathbb{R} \mid v(w) \text{ is continuous and increasing and } w \le v(w) \le \bar{w}\}. \tag{11}$$

Lemma 3. *The transformation T* (a) *maps the function space* $\mathfrak{F}$ *into itself and* (b) *is a contraction on* $\mathfrak{F}$.

Because $\mathfrak{F}$ is a complete metric space under the supremum norm, Lemma 3 implies that T has a unique fixed point. [See Ross (1970, p. 192) for a proof.] Consequently, the three lemmas together imply existence and uniqueness of an optimal search strategy:

Theorem. *Unique, continuous, and strictly increasing value functions exist.*

As an implication of the model, a completed spell of unemployment given the unemployment benefit b is an exponentially distributed random variable with hazard rate equal to

$$\eta_0(R(b)) = \lambda_0(R(b))[1-F(R(b))]. \tag{12}$$

Similarly, the completed length of any subsequent job-paying wage w is exponentially distributed with hazard

$$\eta_1(w) = \lambda_1(w)[1 - F(w)]. \tag{13}$$

These facts provide the means for estimating the structure of the model: interest rate, cost-of-search functions, and wage offer distribution. Specifically, given a parameterization of the cost-of-search and wage offer distribution functions, one can use equations (12) and (13) to construct the likelihood function for a sample of unemployment and job spell durations experienced by identical workers. Ideal data for the purpose would include observations on the completed length of an unemployment spell and the unemployment benefit received during the spell together with the length of the subsequent postunemployment job spell (complete or not) and the wage earned for each worker in the sample. However, in order to characterize the likelihood function and compute its value for any point in the parameter space, one needs to solve the model for the optimal search intensity strategies and the reservation wage expressed as functions of both the "regressors" b or w and the model's structural parameters.

2 Computing optimal search strategy functions

In order to estimate the model's structure, one must solve for the optimal search strategy functions given specific parameterizations of the cost-of-search and the wage offer distribution functions. Closed-form solutions cannot be obtained in general even for very simple specifications. The problem is the forward-looking nature of the worker's dynamic decision problem. As a consequence of this nature, the current decision depends on the structure not only through its effect on the current decision criterion, the right sides of equations (3) given the value function. A given structural parameter also affects the criteria for future search intensity choices, and these effects feed back as determinants of the value function that enters the criterion for the current choice.

All the effects on future decisions of any one of the problem's structural parameters are embedded in the value function. Although closed-form solutions for the value of the problem and the associated optimal strategy cannot be obtained, both can be computed. Because the value-of-employment function is the unique fixed point of a contraction map defined on a known function space, strategy can be computed to any degree of accuracy desired.

Specifically, consider the sequence of functions generated by iterating the contraction T defined in equation (9):

$$\begin{aligned} v_1^{n+1} &= Tv_1^n \\ &\equiv \max_{s \geq 0} \left[[1-\beta(s)][w - c_1(s)] \right. \\ &\qquad \left. + \beta(s) \int \max[v_1^n(w), v_1^n(x)]\, dF(x) \right]. \end{aligned} \tag{14}$$

Because T is a contraction on $\mathfrak{F}$, any sequence of value functions that initiates in $\mathfrak{F}$ converges uniformly (in the supremum norm) to the unique fixed point of T. Hence, the computed function is an approximation of the true function for sufficiently large n. Given this approximation, one can compute the associated optimal search intensity functions and the reservation wage function using equations (6)–(8).

Although the algorithm suggested by (14) is technically feasible, it can be very computer time intensive. An important factor determining the computer time required by the algorithm to find a sufficiently close approximate value function is the speed with which the contraction T converges. The slower the rate of convergence, the greater is the number of iterations required to obtain any allowable approximation error. Specifically, the fact that

$$\|v_1^{n+1}-v_1^n\| \le \beta \|v_1^n - v_1^{n-1}\|, \tag{15.a}$$

where $\|\cdot\|$ represents the supremum norm,

$$\beta = \hat{\lambda}/(r+\hat{\lambda}) \quad [\hat{\lambda} \text{ solves } (r+\hat{\lambda})c_1'(\hat{\lambda}) - c_1(\hat{\lambda}) = \bar{w} - \underline{w}], \tag{15.b}$$

is established in the proof to Lemma 3. Bounds on the number of iterations of the contraction required to guarantee a given approximation error can be determined using either of the following inequalities:

$$\|v_1 - v_1^n\| \le [\beta/(1-\beta)]\,\|v_1^n - v_1^{n-1}\| \le [\beta^n/(1-\beta)]\,|\bar{w}-\underline{w}|, \tag{15.c}$$

where the first inequality is an implication of (15.a) and the second is an implication of (15.a) and the fact that $\underline{w} \le v(w) \le \bar{w}$ for all $v \in \mathfrak{F}$.[4]

Later we will argue that β is likely to be close to unity in practice because available evidence suggests that the offer arrival rate when employed at the reservation wage is large relative to the interest rate. If so, equations (15) imply that the number of iterations required to guarantee a reasonable bound on approximation error is large. Consequently, alternative computation methods may prove useful as a means of significantly reducing computation costs. The remainder of the section considers a general and several special alternative approaches.

In some cases of interest, one does not need the value of employment function. For example, in the limiting case of costless search up to some upper bound on intensity, that is, $c_i(s) = 0$ for $s \le \lambda_i$ and $c_i(s) = \infty$ otherwise, equation (6) together with (3) and (4) imply

[4] The second inequality can be used to compute an absolute upper bound on the number of iterations of the contraction required to attain a given approximation error ex ante. The first inequality provides a stopping rule that guarantees a given approximation error. In practice, the actual number of iterations that are required by the stopping rule, the first inequality, should be significantly smaller than the absolute upper bound implied by the second. We thank Nicholas Kiefer for pointing out the first of these two inequalities.

$$v_1(R(b)) = R(b) + \frac{\lambda_1}{r} \int_{R(b)}^{\bar{w}} [v_1(x) - v_1(R(b))]\, dF(x)$$
$$= b + \frac{\lambda_0}{r} \int_{R(b)}^{\bar{w}} [v_1(x) - v_1(R(b))]\, dF(x)$$

since $s_i = \lambda_i$ is optimal.[5] If $F(x)$ is continuous, the value of employment is differentiable. Indeed,

$$v_1'(w) = \frac{r}{r + \lambda_1[1 - F(w)]}$$

by virtue of equations (3.b) and (4). Finally, because integration by parts implies

$$\int_{R(b)}^{\bar{w}} [v_1(x) - v_1(R(b))]\, dF(x) = \int_{R(b)}^{\bar{w}} v'(x)[1 - F(x)]\, dx,$$

$R(b)$ can be numerically computed as the unique solution to

$$R(b) = b + (\lambda_0 - \lambda_1) \int_{R(b)}^{\bar{w}} \frac{1 - F(x)}{r + \lambda_1[1 - F(x)]}\, dx. \tag{16}$$

Note that the reservation wage is greater than the unemployment benefit in this case if and only if the offer arrival rate when unemployed, λ_0, exceeds the offer arrival rate when employed, λ_1. This implication is a reflection of the fact that employment is less desirable in the short run to the extent that the waiting time required for the arrival of the next offer is greater when employed than when not.

Computing the value function by iterating the contraction can be avoided for the more general model as well because the optimal search intensity strategy when employed can be represented as the solution to an ordinary first-order differential equation, provided that the wage offer distribution is continuous and the cost-of-search function is twice differentiable. To establish this claim, note that equation (3.b) and the fact that the value-of-employment function is increasing in w imply

$$v_1'(w) = \frac{r}{r + \lambda_1(w)[1 - F(w)]} \tag{17}$$

by virtue of the envelope theorem. Consequently, by differentiating both sides of equation (7.b) and then substituting from equation (17), one obtains the ordinary first-order differential equation

$$\lambda_1'(w) c_1''(\lambda_1(w)) = \frac{-[1 - F(w)]}{r + \lambda_1(w)[1 - F(w)]}, \quad \text{where } \lambda_1(\bar{w}) = 0, \tag{18}$$

[5] Although, strictly speaking, this cost function violates Assumption 3, it lies at the boundary of the set of functions defined by the assumption.

is the needed boundary condition. Readily available numerical methods exist that yield approximate solutions to such an equation.

After solving (18) for the optimal offer arrival rate when employed, the associated offer arrival rate when not employed and the reservation wage can be obtained by using the following facts:

$$c_0'(\lambda_0(R(b))) = c_1'(\lambda_1(R(b))), \tag{19}$$

and

$$\begin{aligned} b + \lambda_0(R(b))c_0'(\lambda_0(R(b))) - c_0(\lambda_0(R(b))) \\ = R(b) + \lambda_1(R(b))c_1'(\lambda_1(R(b))) - c_1(\lambda_1(R(b))), \end{aligned} \tag{20}$$

where equation (19) is an implication of equations (7) and equation (20) is a consequence of (6)–(8).

The marginal response of the reservation wage to an increase in the unemployment benefit reflects the relative magnitude of the job spell and unemployment spell hazard rates in the general model. Specifically, differentiating (20) with respect to b yields

$$1 + \lambda_0(R)c_0''(\lambda_0(R))\lambda_0'(R)R'(b) = R'(b) + \lambda_1(R)c_1''(\lambda_1(R))\lambda_1'(R)R'(b).$$

Consequently, equations (18) and (19) and equations (12) and (13), respectively, imply that $R(b)$ is the solution to the following differential equation:

$$R'(b) = \frac{r + \lambda_1(R)[1 - F(R)]}{r + \lambda_0(R)[1 - F(R)]} = \frac{r + \eta_1(R)}{r + \eta_0(R)}, \quad \text{where } R(\bar{w}) = \bar{w}. \tag{21}$$

Empirical evidence on the magnitudes of $R'(b)$ and the unemployment spell hazard rate $\eta_0(R)$ provides a means of obtaining an estimated lower bound on the size of β, the modulus of the contraction T. A range of 8–16 weeks easily incorporates most estimates of the average completed unemployment spell found in the empirical literature. Since expected unemployment duration is $1/\eta_0(R)$, the implied unemployment hazard estimate is 0.33 per month at the midpoint of this range. Feldstein and Poterba (1984) report estimates of the responsiveness of reservation wage rate to unemployment insurance (UI) benefits received, $R'(b)$, that range around 0.33. These numbers, equation (21), and an interest rate of 1 percent per month imply the following estimate of the job duration hazard per month when employed at the reservation wage:

$$\eta_1(R) = [r + \eta_0(R)]R'(b) - r = (0.01 + 0.33)(0.33 - 0.01) = 0.102.$$

Hence, the argument used in the proof to Lemma 3 (see the appendix) and the fact that

$$\lambda_1(R) \geq \lambda_1(R)[1 - F(R)] = \eta_1(R)$$

imply

$$\beta \geq \lambda_1(R)/[r+\lambda_1(R)] \geq \eta_1(R)/[r+\eta_1(R)] = 0.102/0.112 = 0.91.$$

In short, the contraction T can be expected to converge quite slowly. Hence, the alternative computational procedure implicit in equations (18)–(20) may well be much faster than that associated with iterating the contraction map.

The evidence that the job duration hazard rate is large relative to the interest rate suggests another approach to estimation. Namely, the offer arrival rate when employed can be approximated by its zero interest rate limit when the latter is sufficiently large relative to the former. Formally, as $r \to 0$, the limiting optimal offer arrival rate function converges to the solution

$$c_1''(\lambda_1)\lambda_1'(w) = -1/\lambda_1, \quad \text{where } \lambda_1(\bar{w}) = \bar{w}.$$

Since $\lambda c''(\lambda)$ is the derivative of $\lambda c'(\lambda) - c(\lambda)$ with respect to λ, the limiting solution for the offer arrival rate solves

$$\lambda_1(w)c_1'(\lambda_1(w)) - c_1(\lambda_1(w)) = \bar{w} - w \quad \text{when } r = 0. \tag{22}$$

Since the left side is strictly increasing in λ_1 given $c_1''(s) > 0$, the optimal offer arrival rate converges to an increasing function of the difference between the largest wage offer and the current wage earned as the interest rate becomes small.

The limiting solution for $\lambda_1(w)$ together with the associated limiting solutions for $\lambda_0(R)$ and $R(b)$ can be computed directly using equations (19), (20), and (22). Indeed, closed-form limiting solutions can be obtained in interesting cases. For example, in the case of a quadratic cost of search of the form

$$c_i(s) = \gamma_i s^2, \quad i = 0, 1, \tag{23}$$

one obtains square-root representations of the optimal offer arrival rates,

$$\lambda_1(w) = [(\bar{w} - w)/\gamma_1]^{1/2}, \tag{24.a}$$

$$\lambda_0(R(b)) = (\gamma_1/\gamma_0)\lambda_1(R(b)) = [(\bar{w} - b)/\gamma_0]^{1/2}, \tag{24.b}$$

and a linear reservation wage function of the form

$$R(b) = (\gamma_0/\gamma_1)b + (1 - \gamma_0/\gamma_1)\bar{w}. \tag{24.c}$$

3 Monte Carlo evidence on estimating structural duration models

In this section, a series of estimation experiments using Monte-Carlo-generated data are conducted. The experiments are designed to check the performance of the estimation procedures outlined previously in the constant search intensity case. The series include experiments that study the accuracy of the estimates obtained when the limiting zero interest rate solution is used as an approximation to the true reservation wage. For this purpose, we assume that the wage offer distribution is known. The other experiments deal with the problem of estimating the other structural parameters of the job search model jointly with the parameters of the offer distribution using the true optimal job search strategy.

The question of the quality of the estimates obtained when the zero interest rate approximation is used is of interest for three related reasons. First, the limiting solution can be expressed as closed-form differentiable functions of the structural parameters. Consequently, maximum-likelihood estimates of the parameters can be obtained using standard methods, for example, Newton's. Second, the available empirical evidence suggests that the limiting solution may be an adequate approximation. Third, estimation using the true model requires the computation of the search strategy function for each trial structural parameter vector. The number of iterations required in the search for the maximum-likelihood estimates may be significantly reduced by using the estimates initially obtained for the limiting solution as starting values.

How well does the approximate model recover the true parameter values? We attempt to answer the question for sample sizes of the sort that are likely to be encountered in practice. For simplicity, only the case of a constant search intensity is considered, which corresponds to the special case of zero search costs up to some bound on search intensity. Offer arrival rates when employed or not are λ_1 and λ_0, respectively, and the offer distribution is the cumulative density function (c.d.f.) $F(x)$. The optimal job acceptance strategy for an unemployed worker receiving unemployment benefit b is to accept all offers equal to or greater than the unique reservation wage $R(b)$ that solve equation (16), restated here as

$$R(b)=b+(\lambda_0-\lambda_1)\int_{R(b)}^{\bar{w}}\frac{1-F(x)}{r+\lambda_1[1-F(x)]}\,dx. \tag{25}$$

Obviously,

$$R(b)\to(\lambda_1/\lambda_0)b+(1-\lambda_1/\lambda_0)\bar{w}\quad\text{as }\ r/\lambda_1\to 0. \tag{26}$$

In other words, when the interest rate is small relative to the offer arrival rate, the approximate reservation wage is a weighted average of the unemployment benefit and the highest wage offer, where the weights reflect the relative offer arrival rates when employed and when not employed.

Each Monte Carlo data sample is generated using the true reservation wage, the solution to (25). For each individual, denoted by $i=1,\dots,N$, the realized completed length of an unemployment spell, T_0, and the unemployment benefit received is observed together with the completed length of the subsequent employment spell, T_1, and the wage earned during the spell. By assumption, the individuals in the sample are drawn independently from an identical population. Since T_0 and T_1 are exponential variables with hazards $\lambda_0[1-F(R(b))]$ and $\lambda_1[1-F(w)]$, respectively, the log likelihood of the sample of observed spell length pairs (T_{0i}, T_{1i}) conditional on the observed benefit-wage pairs (b_i, w_i), $i=1,\dots,N$, is

$$\begin{aligned}\ln L = &\sum_i [\ln\{\lambda_0[1-F(R(b_i))]-\lambda_0[1-F(R(b_i))]T_{0i}] \\ &+\sum_i [\ln\{\lambda_1[1-F(w_i)]-\lambda_1[1-F(w_i)]T_{1i}].\end{aligned} \tag{27}$$

The estimates reported for each experiment are maximum likelihood in the sense that they maximize this function.

In all experiments conducted, the true offer arrival rates are $\lambda_0=3$ and $\lambda_1=1$ and the interest rate is either 0.01 or 0.04. The unemployment benefit received by an individual in the sample is either 2 or 4. To generate the data for each individual in a sample, equation (25) was first solved for the true reservation wage $R(b_i)$, where $b_i=2$ and $b_i=4$ depending on the individual, given the true parameter values stated above. With this reservation wage, the unemployment hazard rate $\lambda_0[1-F(R(b_i))]$ was computed, and an exponentially distributed completed unemployment spell length was generated for the individual. (All computations were performed using the program GAUSS running on an IBM personal computer.) An acceptable wage $w_i \geq R(b_i)$ was then generated for the individual by drawings from a truncated binomial distribution, with the truncation point equal to the reservation wage. Given this acceptable wage, a new exponentially distributed random variable (length of the worker's subsequent job spell) was generated using $\lambda_1[1-F(w_i)]$ as the hazard. This procedure was then repeated for each individual in the sample.

The sample size was fixed at 500 observations for each replication of each estimation experiment, on the belief that this was close to the best that could be expected in practical situations. One thousand samples were generated and estimates were computed for each sample in each experiment. Tables 1 and 2 report the mean and standard deviation of the simulated distribution of maximum-likelihood estimates. To avoid problems of discontinuity in the likelihood function, which complicate the computation of estimates as well as the proof of consistency of the estimators, we treat the discrete binomial distribution as an approximation to a true continuous distribution. Finally, in all experiments the interest rate and the largest wage offer are treated as known. Hence, the structural parameters to be estimated include the two offer arrival rates λ_0 and λ_1 and the offer distribution parameter p.

Table 1. *Monte Carlo results: approximate reservation wage model means of MLE estimates*

	λ_0 (true value 3.0)	λ_1 (true value 1.0)	Percentage outside (95% normal confidence interval) λ_0	λ_1
$r=0.01$	2.9714 (0.1389)	1.0237 (0.0924)	0.059	0.055
$r=0.04$	2.9434 (0.1887)	1.1107 (0.1256)	0.069	0.063

Note: Standard errors (in parentheses) are computed from frequency distribution of computed estimates.

Table 2. *Monte Carlo results: true reservation wage model means of MLE estimates*

	λ_0 (true value 3.0)	λ_1 (true value 1.0)	p (true value 0.5)
Case 1: wage distribution known	3.0152 (0.1056)	1.0100 (0.1009)	–
Case 2: wage distribution estimated	2.9537 (0.2108)	1.1201 (0.1737)	0.5351 (0.0456)

Notes: $r=0.01$ in both cases. Standard errors (in parentheses) are computed from frequency distribution of computed estimates.

In Table 1, the estimation procedure utilizes the approximate reservation wage given in equation (26). In this case, the approximate reservation wage is a twice-differentiable function of the arrival rates and the largest wage offer. Consequently, the likelihood function is also twice differentiable given a twice-differentiable form for F that fits the binomial distribution c.d.f. To focus on the essential aspect of estimating the structure of the job search model using the approximation, we treat the wage distribution, form, and parameters as known. Two cases are considered: an interest rate of 0.01 and of 0.04.

The estimates of the offer arrival parameters obtained seem satisfactory in both cases. The estimators are centered near their true values; the mean of the 1,000 estimates is less than one standard deviation away from the true value. There is a noticeable increase in variability and some change in the distribution of the estimates in the higher interest rate case, but the effects are not large. Column 3 of Table 1 shows the fraction of times that the estimated coefficients would lie outside a 95 percent confidence interval around the true value. For 1,000 replications a 95 percent confidence interval around the 5 percent critical level would cover 3.6–6.4 percent. Thus, there is some evidence that the empirical distribution of the estimates becomes more fat-tailed as the interest rate gets larger.

The results in Table 1 suggest that the use of the zero interest rate approximation need not cause a significant loss in accuracy, at least in the estimation of the arrival rates of wage offers. There is some evidence that the method overestimates the reservation wage and hence underestimates the arrival rate of offers while unemployed (cf. column 1, Table 1). An interesting question is whether the alternative method using the true reservation wage for each parameter configuration yields significantly better results. It is also of interest to see how well these methods work when the parameters of the wage distribution are jointly estimated.

The first panel in Table 2 provides results comparable to the first panel in Table 1. The only difference is that the Table 1 procedure uses the approximate reservation wage at each parameter configuration while the procedure used to generate Table 2 uses the true reservation wage, the solution to equation (25). Computationally, this change in procedure was not difficult because there are only two values of b to consider. However, analytical derivatives of the reservation wage function are no longer available. Without them, we employed numerical first and second derivatives in maximizing the likelihood function. As case 1 in Table 2 indicates, there is only slight gain in precision attributable to using the true rather than the approximate reservation wage.

One important aspect of the use of an approximate solution for the reservation wage is the computation time saved. In the cases considered in panel 1 of Table 2 and in Table 1, a typical estimation problem with 500 observations required 3 minutes of CPU time on an IBM personal computer using the approximate solution, whereas the corresponding time required is 14.5 minutes using the true reservation wage.

The second panel in Table 2 reports the results of estimating the parameter p of the wage distribution jointly with the arrival rates. As in the first panel, the true reservation wage function is computed for each parameter configuration. There is a noticeable decrease in precision of the estimates of the arrival rates, and the parameter p is, on average, too high. Since the hazard rates out of unemployment and employment, respectively, are $\lambda_0[1-F(R)]$ and $\lambda_1[1-F(w)]$, overestimation of p should result in an underestimate of λ_0 and an underestimate of λ_1. Both patterns are seen in the second panel of Table 2.

4 Summary

Although both job search and job turnover models based on intertemporal optimization theory and appropriate data for estimating their structure have been available for some time, there are few attempts to do so in the literature. The principal reason for the dearth of activity on the topic is the fact that the behavioral strategies implied by any realistic parameterization of the structure of the underlying dynamic programming problem cannot be represented as closed-form functions of the regressors and parameters. In this chapter, the nature of the optimal search strategy implied by a general job search model is studied in more detail. The purpose of the study is to find solutions to the representation problem that may prove useful as part of an estimation procedure.

Three approaches are suggested. The first is based on the fact that the optimal search and reservation wage functions can be computed given the value function and the fact that the value function can be computed as a fixed point of a known contraction map. The second approach is derivative of the fact that the optimal search intensity is the solution to a known first-order differential equation that can be solved numerically. Finally, the model considered has a limiting solution for the optimal strategy that can be characterized in closed form for realistic parameterizations of the structure. This solution is an approximation to the optimal strategy when the interest rate is small relative to the frequency with which the worker receives wage offers, a condition which is consistent with available empirical evidence.

For a special case of the model, that of fixed search intensities, the properties of the approaches are studied using Monte Carlo techniques. In this case, the results imply that a procedure that uses the limiting solution as an approximation of the optimal strategy recovers estimates of the structural parameters that are quite close to their true values for a wide range of the critical parameter, the interest rate. Although using the true optimal strategy for each parameter candidate in an iterative computation procedure is shown to be computationally feasible on a standard IBM personal computer, in this special case there is little gain in accuracy relative to the alternative computationally cheaper procedure in the experiment considered.

In many ways, the paper is just a first step. Realistic search and matching models are much richer than that considered in Monte Carlo studies reported in this paper. Further Monte Carlo testing of the approaches using a variable search intensity version of this model is in progress. In the future, studies that incorporate matching phenomena along the lines of that developed in Jovanovic (1984) and Mortensen (1984) are planned. Nevertheless, we feel that the results obtained in this initial effort are encouraging and we look forward to the successful application of the techniques using real data.

Appendix

Proof of Lemma 1: Either equation (3.a) or (3.b) can be written as

$$v(y) = \max_{s \in \mathbb{R}_+} \Big[[1-\beta(s)][y-c(s)] + \beta(s) \int \max[v_1(x), v(y)]\, dF(x) \Big], \tag{A.1}$$

where $y = b$ and $c(s) = c_0(s)$ in the case of (3.a) and $y = w$ and $c(s) = c_1(s)$ in the case of (3.b). Given the definition of $\beta(s)$ in (4), equation (A.1) is equivalent to

$$rv(y) = \max_{s \in \mathbb{R}_+} \Big[r[y-c(s)] + s \int \max[v_1(x) - v(y), 0]\, dF(x) \Big]. \tag{A.2}$$

The continuity of $v(y)$ follows by virtue of the continuity of the maximum function and the implicit function theorem. Finally, were $v(y)$ nonincreasing in y, the right side of (A.2) would be strictly increasing in y, which is a contradiction. Q.E.D.

Proof of Lemma 2: It is sufficient to show that (A.2) has a unique solution for $v(y)$ for all $y \in B$. In other words, given the function

$$f(z) \equiv \max_{s \in \mathbb{R}_+} \Big[r[y-z-c(s)] + s \int \max[v_1(x) - z, 0]\, dF(x) \Big], \tag{A.3}$$

we need only show that $f(z) = 0$ has a unique solution. First, note that the optimization problem is well defined for all finite values of z given Assumption 3. Hence, $f: \mathbb{R} \to \mathbb{R}$ exists. Second, the fact that the maximum function is strictly increasing and continuous implies that $f(z)$ is strictly decreasing and continuous. Consequently, $f(z) = 0$ has no more than one root.

Because the support of the wage offer distribution W is bounded by Assumption 1 and $c(s)$ is nonnegative and $c(0) = 0$ by Assumption 3, $f(y) < 0$ for sufficiently large but finite values of z and $f(y) > 0$ for sufficiently small values of z. Indeed,

$$f(z) = r(y-z) < 0 \quad \forall z > \max_{w \in W}[v_1(w), y]$$

and

$$\begin{aligned} f(z) &= r(y-z) + \max_{s \in \mathbb{R}_+} \Big[s \int \max[v_1(x) - z, 0]\, dF(x) - c(s) \Big] \\ &\geq r(y-z) > 0 \quad \forall z < \min_{w \in W}[v_1(w), -y]. \end{aligned}$$

Finally, these facts and the continuity of $f(v)$ implies that $f(z)=0$ has a solution. Q.E.D.

Proof of Lemma 3: The transformation T is defined by

$$(Tv)(w)=\max_{s\geq 0}\left[[1-\beta(s)][w-c(s)] + \beta(s)\int \max[v(w),v(x)]\,dF(x)\right], \tag{A.4}$$

where

$$\beta(s)\equiv s/(r+s). \tag{A.5}$$

Under Assumption 3, the optimal search intensity function given v, $\lambda(w;v)$, is the unique solution to the first-order condition to the optimization problem on the right side of (A.4), that is,

$$(r+\lambda(w;v))c'(\lambda(w;v))-c(\lambda(w;v)) = \int \max[v(x),v(w)]\,dF(x)-w. \tag{A.6}$$

Claim (a) is an immediate implication of (A.4)–(A.6). Here, Tv is increasing and continuous given that v has those properties by virtue of the fact that the maximum function is continuous and increasing. If $v(\bar{w})=\bar{w}$ then $\lambda(\bar{w},v)=0$, which implies $(Tv)(\bar{w})=\bar{w}$ given the assumption that $c(0)=c'(0)=0$ and the fact that $\beta(0)=0$. Finally, $c(0)=\beta(0)=0$ implies $(Tv)(w)\geq w$.

Claim (b) holds if a number $0\leq\beta<1$ exists such that $\|Tv-Tu\|\leq\beta\|v-u\|$, where v and u are any two elements of $\mathfrak{F}$ and $\|\cdot\|$ represents the supremum norm. Note that (A.6) implies $\lambda(w;v)\leq\lambda(w;\bar{w})$ for all v such that $w\leq v(w)\leq\bar{w}$, where $\lambda(w;\bar{w})$ is the continuous increasing function defined by

$$(r+\lambda(w;\bar{w}))c_1'(\lambda(w;\bar{w}))-c_1(\lambda(w;\bar{w}))=\bar{w}-w. \tag{A.7}$$

In what follows we establish the required inequality for

$$\beta\equiv\sup_{w\in W}[\beta(\lambda(w;\bar{w}))]=\beta(\lambda(w;\bar{w}))\equiv\frac{\lambda(\underline{w};\bar{w})}{r+\lambda(\underline{w};\bar{w})}, \tag{A.8}$$

which is both nonnegative and strictly less than 1 given W bounded. The equality in (A.8) is an implication of the fact that $\lambda(w;v)$ is strictly decreasing in w by virtue of (A.7) and Assumption 3. Because (A.4) implies

$$(Tu)(w)\geq[1-\beta(s(w,v))][w-c(s(w,v))] + \beta(s(w,v))\int\max[u(w),u(x)]\,dF(x),$$

v and u both increasing respectively yield

$$(Tv)(w)-(Tu)(w)$$

$$\leq \beta(s(w,v))\left[\int \max[v(w),v(x)]\,dF(x)-\int \max[u(w),u(x)]\,dF(x)\right]$$

$$=\beta(s(w,v))\left[[v(w)-u(w)]F(w)+\int_{w}^{\bar{w}}[v(x)-u(x)]\,dF(x)\right].$$

Finally, because the first term in the final product satisfies $0\leq\beta(s(w,v))\leq\beta$ and because the absolute value of the second term is less than or equal to $\sup|v(w)-u(w)|$,

$$\|Tv-Tu\|\equiv\sup_{w\in W}|(Tv)(w)-(Tu)(w)|\leq|\beta|\sup_{x\in W}|v(x)-u(x)|$$

$$\equiv\beta\|v-u\|. \qquad \text{Q.E.D.}$$

REFERENCES

Benhabib, J. and C. Bull. 1983. "Job Search: The Choice of Intensity." *Journal of Political Economy* 91: 747–64.

Burdett, K. 1978. "Employee Search and Quits." *American Economic Review* 68: 212–20.

Burdett, K. and D. T. Mortensen. 1978. "Labor Supply under Uncertainty." In *Research in Labor Economics,* Vol. 2, R. G. Ehrenberg, Ed. Greenwich, CT: JAI, pp. 109–58.

Burdett, K., N. Kiefer, D. T. Mortensen, and G. Neumann. 1984. "Earnings, Unemployment, and the Allocation of Time Over Time." *Review of Economic Studies* LI: 559–78.

Ehrenberg, R. G. and R. L. Oaxaca. 1976. "Unemployment Insurance, Duration of Unemployment, and Subsequent Wage Gain." *American Economic Review* 66: 754–66.

Feldstein, M. and J. Poterba. 1984. "Unemployment Insurance and Reservation Wages." *Journal of Public Economics* 23: 141–67.

Flinn, C. J. 1986. "Wages and Job Mobility of Young Workers." *Journal of Political Economy* 94: S88–S110.

Flinn, C. and J. J. Heckman. 1982. "New Methods for Analyzing Structural Models of Labor Force Dynamics." *Journal of Econometrics* 18: 115–68.

1983. "Are Unemployment and Out of the Labor Force Behaviorally Distinct Labor Force States?" *Journal of Labor Economics* 1: 28–42.

Heckman, J. J. and B. Singer. 1982. "The Identification Problem in Econometric Models for Duration Data." In *Advances in Econometrics,* W. Hildenbrand, Ed. Cambridge: Cambridge University Press, pp. 39–77.

Jovanovic, B. 1979a. "Job Matching and the Theory of Turnover." *Journal of Political Economy* 87: 972–90.

1979b. "Firm-Specific Capital and Turnover." *Journal of Political Economy* 87: 1246–60.

1984. "Matching, Turnover and Unemployment." *Journal of Political Economy* 92: 108–22.

Kiefer, N. and G. Neumann. 1979a. "An Empirical Job Search Model with a Test of the Contest Reservation Wage Hypothesis." *Journal of Political Economy* 87: 69–82.

1979b. "Estimation of Wage Offer Distributions and Reservation Wages." In *Studies in the Economics of Search,* S. A. Lippman and J. J. McCall, Eds. New York: North-Holland, pp. 171–89.

1981. "Individual Effects in a Nonlinear Model: Explicit Treatment of Heterogeneity in the Empirical Job Search Model." *Econometrica* 49: 965–80.

Lancaster, T. and S. Nickell. 1980. "The Analysis of Reemployment Probabilities for the Unemployed." *Journal of the Royal Statistical Society (Series A)* 135: 257–71.

Miller, R. A. 1984. "Job Matching and Occupational Choice." *Journal of Political Economy* 92: 1086–1120.

Mincer, J. and B. Jovanovic. 1981. "Labor Mobility and Wages." In *Studies in Labor Markets,* S. Rosen, Ed. NBER Conference Report, Chicago: University of Chicago Press, pp. 21–64.

Mortensen, D. T. 1984. "Quit Probabilities and Job Tenure: On the Job Training or Matching?" Center for Mathematical Studies Discussion Paper No. 630, Northwestern University.

Nickell, S. 1979. "Estimating the Probability of Leaving Unemployment." *Econometrica* 47: 39–57.

Pakes, A. 1986. "Patents as Options: Some Estimates of the Value of Holding European Patent Stocks." *Econometrica* 54: 755–84.

Ross, S. M. 1970. *Applied Probability Models with Optimization Applications.* San Francisco: Holden-Day.

Rust, J. 1984. "Maximum Likelihood Estimation of Controlled Discrete Choice Processes." Social Systems Research Institute, University of Wisconsin.

Topel, R. and M. P. Ward. 1985. "Job Mobility and the Careers of Young Men." Mimeo, University of Chicago and University of California at Los Angeles.