

# MRes Macroeconomics

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Solving general equilibrium models with incomplete financial markets:

- endowment economy: Huggett (1993)
- production economy: Aiyagari (1994)

*Huggett (1993)*

# Introduction

- Many models in macroeconomics assume the existence of a representative agent. In these models, the wealth distribution collapses to the wealth level of the representative agent.
- The assumption that underlies the representative agent assumption is that financial markets are complete, that is, all state-contingent securities can be traded.
- The complete markets assumption is obviously unrealistic. In particular, we see a lot of wealth heterogeneity in the data.

## Huggett (1993)

- Standard representative agent models have great difficulties explaining why the real risk-free rate is low relative to the returns on equity (Mehra and Prescott (1985)) and given observed consumption growth (Mankiw and Zeldes (1989)).
- One reason could be that when agents are allowed to perfectly insure against idiosyncratic shocks, their precautionary demand for bonds can be expected to be reduced substantially.
- Thus, one would expect that in an incomplete markets world, the demand for bonds would be higher. When the net supply of bonds is zero, this would imply a lower equilibrium interest rate than under complete markets.
- Huggett (1993) investigates the quantitative effect of incomplete markets on the long-run equilibrium interest rate.

## Key features of the model

- Endowment economy
- Idiosyncratic income shocks
- One-period bonds are the only tradable financial assets. On top of that, agents' individual borrowing is subject to an exogenous limit.
- General equilibrium
- No aggregate uncertainty

# Idiosyncratic income risk

- There is a continuum of agents with identical preferences. Their total measure is one.
- In every period, each agent receives an exogenous endowment that can be either high ( $e_h$ ) or low ( $e_l$ ), with  $e_h > e_l$ .
- The individual endowment follows a Markov process with strictly positive transition probabilities:

$$\pi(e'|e) = P[e_{t+1} = e'|e_t = e] > 0$$

$$e, e' \in \{e_l, e_h\}$$

# Preferences and budget constraint

- Preferences of the agents are given by

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where  $\beta \in (0, 1)$  is the discount factor and  $c_t$  denotes consumption of non-durables.  $u(c_t)$  is the isoelastic utility function, i.e.

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

- The budget constraint is given by:

$$c_t + q_t a_{t+1} = e_t + a_t$$

where  $a_t$  is the amount of bonds, and  $q_t$  is price of a bond paying off 1 in period  $t + 1$ . Note that  $a_t$  is chosen in period  $t - 1$ .



# Borrowing limit

- Agents face an exogenous borrowing limit:

$$a_{t+1} \geq \underline{a}$$

- Consumption must be positive in each period, i.e.

$$c_t \geq 0$$

# State variables

What are the individual state variables?

- income:  $e_t$
- bond holdings:  $a_t$

# Agent's problem

- We are interested in the steady-state properties of the model. Hence, we can treat the bond price  $q$  as a constant (but of course it is endogenous so we need to solve for it).
- The optimization problem of an individual agent can now be characterized by the following Bellman equation:

$$V(e, a, q) = \max_{c, a'} u(c) + \beta \sum_{e'} V(e', a', q) \pi(e'|e)$$

where

$$c + qa' = e + a, \quad a' \geq \underline{a}, \quad c \geq 0$$

where subscripts are removed and primes denote variables one period ahead.

- The solution of the individual's problem involves finding the policy functions  $c = c(e, a, q)$  and  $a' = a(e, a, q)$ .

# Stationary equilibrium

- We want to study the long-run properties of the model. Given that agents face idiosyncratic shocks that cannot be fully insured, individual income and wealth levels will not be constant.
- However, there does exist a unique stationary equilibrium in which aggregate variables and the wealth distribution are constant (for a proof, see Hopenhayn and Prescott (1992)).
- Let  $\phi$  be a probability measure on  $S$  where,  $S = [\underline{a}, \bar{a}] \times \{e_l, e_h\}$ . Hence,  $\phi$  represents the wealth distribution.

# Stationary equilibrium

**Definition:** a stationary equilibrium is  $c(e, a)$ ,  $a(e, a)$ ,  $q$  and  $\phi$  such that:

- 1  $c(e, a)$  and  $a(e, a)$  solve the individual agents problem given  $q$ ,
- 2 goods and asset markets clear, i.e.

$$\int_S c(x) d\phi = \int_S e d\phi$$

$$\int_S a(x) d\phi = 0$$

- 3  $\phi$  is stationary, i.e.  $\phi(B) = \int_S P((e, a), B) d\phi$  for all  $B \in B_S$ . where  $P((e, a), B)$  is a transition probability from state  $(e, a)$  to somewhere in  $B$ .

# Calibration

Huggett (1993) considers the following calibration:

- The model period is two months.
- $e_h = 1$ ,  $e_l = 0.1$ ,  $\pi(e_h|e_h) = 0.925$  and  $\pi(e_h|e_l) = 0.5$ . These numbers roughly match the standard deviation of income measured in the cross-section (20% on an annual basis) and the average unemployment duration (12.3 weeks).
- $\beta$  is set to 0.99332 (0.96 on an annual basis).
- $\sigma$  is set to 1.5 following Mehra and Prescott (1985)
- Regarding the borrowing limit a range of values for  $a$  is considered by Huggett: -2, -4, -6, and -8. To interpret these numbers, it is useful to know that a borrowing limit of  $a = -5.3$  correspond to one year of average annual earnings.

# Solving the model - Outline

Huggettis solution procedure involves the following steps:

- 1 Guess an initial asset price  $q$ .
- 2 Guess an initial value functions  $V_0$ .
- 3 Solve the individual agents problem, given  $q$ .
- 4 Iterate on value functions till convergence.
- 5 Iterate on  $\phi$  until it converges, starting from some initial  $\phi_0$ .
- 6 Compute the aggregate demand for bonds. Update  $q$  as follows: if it is negative, lower  $q$ . If it is positive, increase  $q$ .
- 7 Go back to step 2 until the absolute change in  $q$  is smaller than some convergence criterion.

## Solving the model - initial guess for $q$

- How can we come up with a good initial guess for the bond price  $q$ ?
- Note that in a complete-markets version of the model, the representative agents problem would imply the following Euler equation:

$$q_t c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}$$

- Hence,  $q = \beta$  seems like a reasonable initial guess.



## Solving the model - individual's problem

- Following Huggett (1993), I use value function iteration to solve the individual's problem.
- I discretize  $S$  and use 1000 equidistant grid points on the interval  $[a, -6]$  (and verify ex post that the interval is large enough).
- As a stopping criterion I use that the root mean squared change in the value function should not exceed  $10^{-6}$ .

## Solving the model - distribution

- To find the stationary distribution, given the individual policy rules, I iterate on the following equation:

$$\phi_{t+1}(a_i, e_j) = \sum_{k=1}^n \sum_{m \in \{h, l\}} I[a' = a_i | a_k, e_m] \pi(e' = e_j | e_m) \phi_t(a_k, e_m)$$

where  $\phi_t(a_i, e_j)$  denotes the mass of agents at the grid point  $(a_i, e_j)$ , with  $i = 1, 2, \dots, n$  and  $j \in \{h, l\}$  and  $I$  is the indicator function.

- As a convergence rule, I use that  $\|\phi_{t+1} - \phi_t\| < 10^{-6}$

## Solving the model - q

- Given a stationary distribution, I compute aggregate demand for bonds as:

$$D = \sum_{k=1}^n \sum_{m \in \{h,l\}} a(a_k, e_m) \phi_t(a_k, e_m)$$

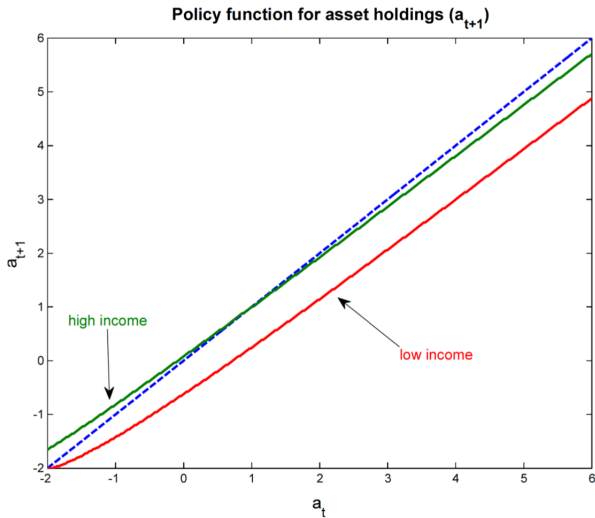
where  $a(a_k, e_m)$  is the policy rule for the asset choice.

- I update q using

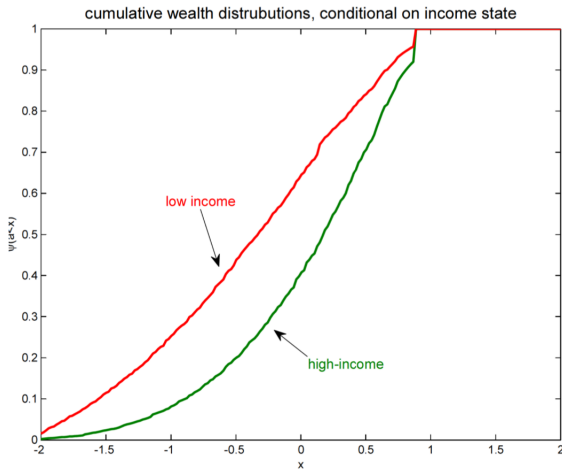
$$q_{s+1} = (1 + \lambda D_s) q_s$$

where  $s$  is the index of the iteration and  $\lambda$  is a scalar to be satisfied. I set  $\lambda = 0.005$ .

# Policy Function: Capital



# Upper limit on bond holdings



# Summery

- Very few agents at the borrowing constraint.
- Still, the introduction of incomplete financial markets can have a large negative effect on the equilibrium interest rate.
- But what if aggregate savings are possible?

*Aiyagari (1994)*

# Aiyagari (1994)

- Suppose there is an aggregate savings technology (like capital) but financial markets are still incomplete.
- One might expect that the precautionary savings motive will lead to a larger steady-state level of capital, relative to a model with complete financial markets.
- Aiyagari (1994) constructed a model to investigate this issue.



# Aiyagari (1994)

Key features of the model:

- capital investment (no bonds) idiosyncratic income shocks
- general equilibrium
- no aggregate uncertainty

## Problem of the individual

The optimization problem of the individual is given by

$$\max_{c_t, k_{t+1}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

where

$$\begin{aligned} c_t + k_{t+1} &= w l_t + (1 - \delta + r) k_t \\ c_t &\geq 0 \\ k_t &\geq b \end{aligned}$$

where  $\beta \in (0, 1)$  is the discount factor and  $c_t$  denotes consumption of non-durables,  $k_t$  is the amount of capital,  $l_t$  is labor endowment,  $w_t$  is the wage per efficiency unit of labor,  $r_t$  is the rental rate of capital,  $\delta$  is the rate of depreciation and  $b < 0$  is a borrowing limit.

# Labor endowment process

- The labor endowment is subject to idiosyncratic shocks.
- It will be assumed that  $l_t$  has bounded support  $[l_{min}, l_{max}]$  with  $l_{min} > 0$ .
- In particular, it will be assumed to follow a Markov process that is such that  $\mathbb{E}[l_t] = 1$ , that is, aggregate supply of labor equals one in every period.

# Borrowing limit

- The borrowing limit  $b$  is set to the natural borrowing limit.
- The idea behind the natural borrowing limit is that the agent should always be able to pay back her debt, no matter the future realizations of her labor endowment shocks
- The worst-case scenario is that agent will have  $l_{t+j} = l_{min}$  for any  $j = 1, 2, \dots$
- How do we determine  $b$ ?

## Borrowing limit

- Suppose the agent is exactly at the borrowing limit in period  $t$ , i.e.  $k_t = b$  and the worst-case scenario occurs, i.e.  $l_{t+j} = l_{min}$  for any  $j = 1, 2, \dots$
- Then being exactly able to pay back her debt means that the agent must have zero consumption in all future periods, i.e.  $c_{t+j} = 0$  for  $j = 1, 2, \dots$
- From the budget constraint it follows that

$$k_{t+j+1} = wl_{min} + (1 + r)k_{t+j}$$

- With  $k_{t+j+1} = k_{t+j} = b$ . It follows that

$$b = -\frac{wl_{min}}{r}$$

- Because the utility function satisfies the Inada conditions, the borrowing limit will never actually bind, given that initial assets are above the borrowing limit.

# Production

- A representative firm operate on a Cobb-Douglas production function:

$$y = k_{d,t}^{\alpha} l_{d,t}^{1-\alpha}$$

where "d" stands for "demand".

- Profit maximization implies:

$$\begin{aligned} r_t &= \alpha k_t^{\alpha-1} l_t^{1-\alpha} \\ w_t &= (1 - \alpha) k_t^{\alpha} l_t^{-\alpha} \end{aligned}$$

- Since  $l_t$  is exogenously given,  $r_t$  and  $w_t$  are directly given as functions of the capital stock.
- It is easy to show that firms make no profits.

# Market clearing

- Let individuals be indexed by  $i$  and let their total measure be equal to one. The clearing of the capital and labor market implies that:

$$k_{d,t} = \int_i k_{i,t} di$$

$$l_{d,t} = \int_i l_{i,t} di$$

# Stationary equilibrium

- We are interested in the steady-state properties of this model. So from now on I will drop time subscripts.
- By checking the conditions provided in Hopenhayn and Prescott (1992) for the Aiyagari model, it can be shown that a unique stationary equilibrium exists.
- Here, I will focus on the computation and the quantitative results.



# Solving the model

- 1 Guess an initial aggregate capital stock  $K_j$  with  $j = 0$  and specify a converge level  $\epsilon$ .
- 2 Compute the associated rental rate  $r$  and wage  $w$ :

$$\begin{aligned}r &= \alpha K_j^{\alpha-1} \\w &= (1 - \alpha) K_j^{\alpha}\end{aligned}$$

- 3 Solve the individual's problem, given  $r$  and  $w$  using your preferred method. That is, find the policy rule for the capital choice  $k_{+1}(k, l; r, w) = k_{t+1}$ .

## Solving the model

- 4 Using the policy rule, compute the stationary distribution of capital.
- 5 Compute a new value for aggregate capital as

$$\hat{K} = \int_i k_i di$$

- 6 Update the capital stock as  $K_{j+1} = \lambda K_b + (1 - \lambda)K_j$ , where  $\lambda \in (0, 1]$  is a dampening parameter and  $j$  is the index of the iteration. Go back to step 2) until  $\|K_{j+1} - K_j\| < \epsilon$ .

## Solving the model - an alternative approach

- An alternative is to iterate on  $r$  instead of on  $K$ . In that case, it is important to make sure that  $\beta(1 + r - \delta) < 1$  as otherwise the capital stock will "explode".
- One could start with some initial guess  $r_0 = \frac{1}{\beta} - 1 + \delta - \epsilon$ , where  $\epsilon$  is a very small number, and compute the associated values for capital demand  $k_d$  and for  $w$ .
- Then, solve the individual's problem and find the implied stationary distribution.
- Next, compute the aggregate capital supply  $\hat{K}$ , and compute  $\hat{r} = \alpha \hat{K}^{\alpha-1}$
- Update the rental rate of capital as  $r_1 = \frac{r_0 + \hat{r}}{2}$

## Solving the model - an alternative approach

- Given  $r_1$ , solve the individual's problem, compute (i) the stationary capital distribution (ii) aggregate capital supply  $\hat{K}$ , (iii) the associated rental rate  $\hat{r}$ .
- Finally compute excess supply of capital as  $K_{\text{excess}} = \hat{K} - k_d$ , where  $k_d$  is the aggregate demand for capital consistent with  $r_1$ .
- Update the  $r$  as follows:
  - If  $K_{\text{excess}} > 0$ , set  $r_2 = \frac{r_1 + \hat{r}}{2}$
  - If  $K_{\text{excess}} < 0$ , set  $r_2 = \frac{r_0 + \hat{r}}{2}$
- Continue until  $K_{\text{excess}} < \epsilon$ , where  $\epsilon$  is a pre-specified convergence criterion.

## Results - aggregate saving rate

- Aiyagari compares the equilibrium savings rate in (i) the incomplete markets model and (ii) a version of the model with full insurance.
- He finds that for plausible parameter values, savings rate is very similar in the two models.
- The reason is that self-insurance is quite an effective substitute for insurance contracts between agents.

## Results - inequality

- Aiyagari's model also qualitatively matches some facts present in the data:
  - consumption inequality is smaller than income inequality,
  - the income and wealth distributions are positively skewed (median < mean).
- However, the model generates too little dispersion in income and wealth. The table below displays Gini coefficients, for the model and the data:

	Gini income	Gini Wealth
model	.12	.32
data (U.S)	.4	.8