Topics in Macro, Part 2: Incomplete Financial Markets

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Outline

Models with incomplete markets and aggregate uncertainty

- Krusell and Smith (1998)
- Alternative solution methods
- Alternative models

Krusell and Smith (1998)

Introduction

- This week, we introduce aggregate uncertainty in models with incomplete financial markets.
- This may seem like a small step, but it turns out there are some tricky issues.
- Krusell and Smith (1998) showed how to solve such models.

Aggregate uncertainty

- Suppose we would introduce aggregate uncertainty in the model of Aiyagari (1994).
- For example, we could introduce a stochastic aggregate productivity variable z_t. The production function becomes:

$$y_t = z_t k_t^{\alpha} l_t^{1-\alpha}$$

where z_t follows some stochastic process.

Aggregate uncertainty

The decision problem of the individual agent becomes:

$$\max_{c_t, k_{t+1}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

where

$$\begin{aligned} c_t + k_{t+1} &= w_t l_t + (1 - \delta + r_t) k_t \\ c_t &\geq 0 \\ k_t &\geq b \end{aligned}$$

Q. What is the key dierence with the model without aggregate uncertainty?

Aggregate uncertainty

- In order to make optimal decisions, the agent needs to make forecasts of r_t and w_t. That is, the agent needs to form beliefs on the law of motions for these two variables.
- Under rational expectations, the laws of motion for r_t and w_t as perceived by the agents coincide with the actual equilibrium laws of motion.
- Recall that in the Aiyagari model, aggregate labor supply equals one. Profit maximization by the firms implies

$$\begin{aligned} r_t &= \alpha z_t k_t^{\alpha-1} \\ w_t &= (1-\alpha) z_t k_t^{\alpha} \end{aligned}$$

- Hence, it is sufficient for agents to form beliefs about z_t and the law of motion for the aggregate capital stock, K_t.

What helps to predict the aggregate capital stock?

- The aggregate capital stock, K_t, is an equilibrium object defined by:

$$K_t = \int_i k'(k_{i,t-1}, l_{i,t-1}, \Theta_{i,t-1}) d_i$$

- where $k'(.)$ is the individual policy rule for capital, which depends on individual capital, $k_{i,t}$, labor endowment status, $l_{i,t-1}$ and Θ_t, which contains the aggregate state (which includes for example z_t).
- When $k'(.)$ is nonlinear, K_t depends on the entire cross-sectional distribution of capital holdings and labor endowments. Given that this distribution is time-varying, this entire distribution becomes part of the aggregate state Θ_t.

Krusell Smith (1998)

- Thus, in order to make optimal decisions, any individual agent needs to keep track of infinitely many state variables. Needless to say, the computational problem to be solved becomes much, much harder.
- The idea of Krusell and Smith (1998) is to characterize the cross-sectional wealth distribution using only a finite number of moments.
- They show that by discarding all information about the distribution except for its mean, agents can actually do a pretty good job in predicting the future capital stock (and thus prices and wages).
- Below, we discuss their model, which is essentially an extension of Aiyagari (1994).

Krusell Smith (1998)

The model

- Aggregate productivity follows a two-state Markov process, switching between a good state ($z_t = z_g$) and a bad state ($z_t = z_b$) with $z_g > z_b$.
- Agents can be either employed ($e_t = 1$) or unemployed ($e_t = 0$). The employment status follows a Markov process, with transition probabilities that depend on the productivity state z_t. Transition probabilities are such that in a boom ($z_t = z_g$), fewer agents are unemployed than in a downturn ($z_t = z_b$).
- Agents rent out their capital to the firms, at a rate r_t. Employed agents supply one unit of labor to the firms and receive a wage w_t.
- No borrowing is allowed, that is, $k_t \geq 0$ during each period.

Transition probabilities

- Note that when employment status transition probabilities are time-varying, the number of people in each of the two states becomes part of the state-space as well.
- Krusell and Smith (1998) chose the transition probabilities in a smart way, avoiding extra state variables.
- In particular, they chose transition probabilities such that the unemployment rate is always u_g in the good state and u_b in the bad state.

Transition probabilities

- Let $\pi_{ss'}$ denote the transition probability of going from aggregate state z_s to $z_{s'}$ with $z_s, z_{s'} \in \{z_b, z_g\}$
- Moreover, let $\pi_{ss'ee'}$ denote the transition probability of going from state $z_{s,e}$ to $z_{s',e'}$.
- The transition probabilities are required to satisfy the following restrictions:

$$\pi_{ss'00} + \pi_{ss'01} = \pi_{ss'10} + \pi_{ss'11} = \pi_{ss'}$$

and

$$u_s \frac{\pi_{ss'00}}{\pi_{ss'}} + (1 - u_{ss}) \frac{\pi_{ss'10}}{\pi_{ss'}} = u_{s'}$$

- for all possible values of s and s_0.

Decision problem individual

Formulated recursively, the individual agent becomes:

$$V(k, e, ; \Theta) = \max_{c, k'} \frac{c^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}[V(k', e'; \Theta') | k, e; \Theta]$$

where

$$\begin{aligned} c + k &= we + (1 - \delta + r)k \\ c_t &\geq 0 \\ k_t &\geq b \\ \Theta' &= G(\Theta) \end{aligned}$$

where $\sigma > 0$, Θ denotes the aggregate state, primes denote next periodis variables, and $G(\Theta)$ denotes the transition law of the aggregate state.

Firms

- Profit maximization by a representative competitive firm implies that

$$
\begin{aligned}
r_t &= \alpha z_t k_t^{\alpha-1} L_t^{1-\alpha} \\
w_t &= (1-\alpha) z_t k_t^{\alpha} L_t^{-\alpha}
\end{aligned}
$$

- where K_t denotes aggregate capital, as defined as above, and L_t is aggregate labour supply, which follows directly from the transition probabilities.

Aggregate state

- What should be included in the aggregate state Θ_t?
- From our discussion above, it is clear that Θ_t should at least include z_t and the cross-sectional distribution of capital holdings and employment status, which we will denote by Γ_t.
- Following the equilibrium definition of Krusell and Smith (1998) we let $\Theta_t = (z_t, \Gamma_t)$.

Wealth recursive equilibrium

Definition (Krusell and Smith(1998)): A wealth recursive equilibrium is a joint transition law $G(z,\Gamma)$, policy rules $c(k,e;z,\Gamma)$ and $k'(k,e;z,\Gamma)$ and price functions $r(k,e;z,\Gamma)$ and $w(k,e;z,\Gamma)$ such that:

- $c(k,e;z,\Gamma)$ and $k'(k,e;z,\Gamma)$ solve the individuals' problems
- $r(z,\Gamma)$ and $w(z,\Gamma)$ are competitive (i.e. they satisfy the firms' optimality conditions)
- $G(z,\Gamma)$ is consistent with the individuals' policy rules, as well as the individual and aggregate transition probabilities.

Existence and uniqueness?

Some words of caution:

- Existence and uniqueness of a wealth recursive equilibrium, as defined by Krusell (1998), has not been proven.
- In fact, Kubler and Schmedders (2006) provide examples in which including the wealth distribution in the aggregate state is not enough.
- Miao (2006) shows existence of an equilibrium for a general class of heterogeneous agent economies, provided that expected payoffs are included in the aggregate state.

Calibration

Krusell and Smith use the following parameter values:

- A period is one quarter
- $\beta = 0.99$ and $\delta = 0.025$
- $\sigma = 1$ (log utility), $\alpha = 0.36$ (capital share in income of 36%)
- $z_g = 1.01$ and $z_u = 0.99$
- $u_g = 0.04$, $u_b = 0.1$.
- Transition probabilities imply that the average unemployment duration is 2.5 quarters in a recession and 1.5 quarters in a boom.

Bounded rationality

- How do we deal with the infinite dimensionality of the wealth distribution Γ?
- Krusell and Smith (1998) assume that agents are boundedly rational and use only the first I moments of Γ, labeled $m = \{m_1, m_2, ..., m_I\}$.
- Agents then form beliefs on the joint transition law for z and m, which we call $G_I(m, z)$.
- In the simplest case, only the mean of the distribution is used. Let's assume agents perceive a law of motion of the following type:

$$\begin{aligned} \ln K' &= a_{0,g} + a_{1,g} \ln K \quad \text{in good times} \\ \ln K' &= a_{0,b} + a_{1,b} \ln K \quad \text{in bad times} \end{aligned}$$

Individual problem

The optimization problem of the boundedly rational agent becomes:

$$V(k, e; K, Z) = \max_{c,k'} \frac{c^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}[V(k', e'; K', Z')|k, e; K, Z]$$

where

$$\begin{aligned}
c + k &= w(K, Z)e + (1 - \delta + r(K, Z))k \\
\ln K' &= a_{0,g} + a_{1,g} \ln K \\
\ln K' &= a_{0,b} + a_{1,b} \ln K \\
c_t &\geq 0 \\
k_t &\geq b
\end{aligned}$$

The idea now is to find coefficent $a_{0,g}$, $a_{1,g}$, $a_{0,b}$ and $a_{1,b}$ that give minimal prediction errors.

Krusell-Smith algorithm

1. Guess initial values for $a_{0,g}$, $a_{1,g}$, $a_{0,b}$ and $a_{1,b}$.
2. Solve the problem of the individual agent.
3. Use the individualis decision rules to simulate a panel of N agents over T time periods (with N and T both being large numbers).
4. Discard the first n iterations and run a regression to update the values of $a_{0,g}$, $a_{1,g}$, $a_{0,b}$ and $a_{1,b}$.
5. Go back to step (2) unless the coefficients have converged (according to a certain criterion).
6. If coefficients have converged, evaluate the accuracy of the transition law. If accuracy is not satisfactory, include more moments in the transition law and start again.

Krusell-Smith: results

- The laws of motion for aggregate capital found by Krusell and Smith are:

$$\begin{aligned} \ln K' &= 0.095 + 0.962 \ln K \\ \ln K' &= 0.085 + 0.965 \ln K \end{aligned}$$

- But what is the accuracy of these laws of motion?

Accuracy

Krusell and Smith report the following accuracy measures:

a) R^2 of the regression: 0.999998 for both (1) and (2).

b) $\hat{\sigma}$: standard deviation of the regression error: 0.0028% for (1) and 0.0036% for (2)

c) maximum forecast error 25 years ahead: 0.1%.

Accuracy

d) Plot of simulated capital (using the policy rules):

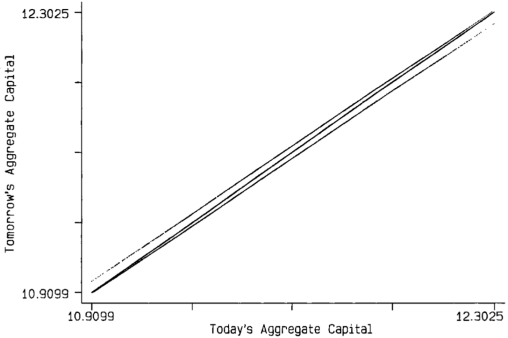


FIG. 1.—Tomorrow's vs. today's aggregate capital (benchmark model)

Figure: Top line: capital in the good state. Bottom line: capital in the bad state. Middle line: 45 degree line.

Accuracy

- In many papers after Krusell and Smith (1998), only the R^2 and $\hat{\sigma}$ are reported. There have been concerns about this practice (see Den Haan (2009)).
- The main problems with these measure are:
  - They are based on the forecast error one period ahead. In the model, agents use the laws of motion to forecast infinitely many periods ahead and small forecast errors may build up to become large ones.
  - They are based on average forecast errors, possibly hiding that for certain parts of the state-space forecast errors are large.

Krusell and Smiths (1998)

Intuition

- Back to Krusell and Smith (1998), whose results turn out to pass all accuracy tests pretty well.

Q. Why can the aggregate capital stock be forecasted so well using only the mean of the distribution?

Krusell and Smiths (1998)

Individual decision rule

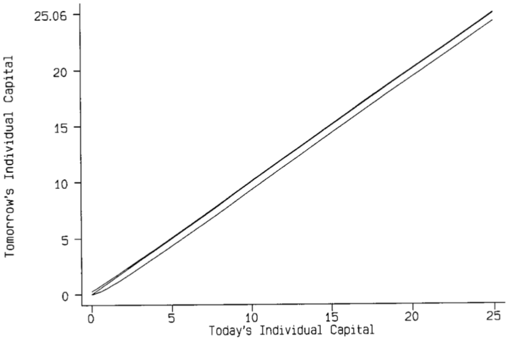


FIG. 2.—An individual agent's decision rules (benchmark model, aggregate capital = 11.7, good aggregate state).

$\Rightarrow A$) policy function for capital very linear, except for agents close to borrowing constraint.

Krusell and Smiths (1998)

Lorenz curve

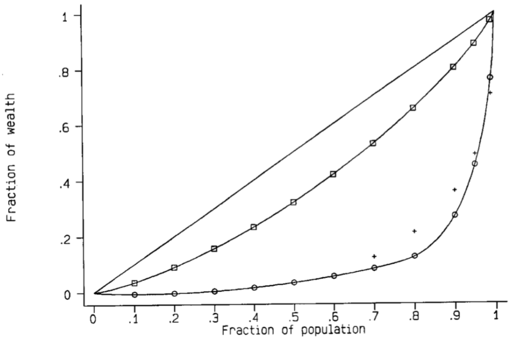


FIG. 3.—Lorenz curves for wealth holdings (+ refers to the data, □ to the benchmark model, and ○ to the stochastic-β model).

$\Rightarrow B)$ agents close to the borrowing constraint own a very small fraction of the aggregate capital stock, so they are not very important for the aggregate.

margina saving propensities

- The utility costs from accepting fluctuations in consumption are very small
  - even when these fluctuations are several times larger than for aggregate consumption
- Access to one aggregate asset is sufficient for providing the agent with very good insurance in utility terms
- In the stationary state, most agents have enough capital that their savings behavior is guided mainly by intertemporal concerns rather than by insurance motives

Statement

Consider the following statement:

The results of Krusell and Smith show that wealth heterogeneity is not of first-order importance for the dynamics of aggregate variables. Therefore, one can safely use a representative agent model to study the business cycle.

Q. What do you think?

Alternative solution methods

Going forward

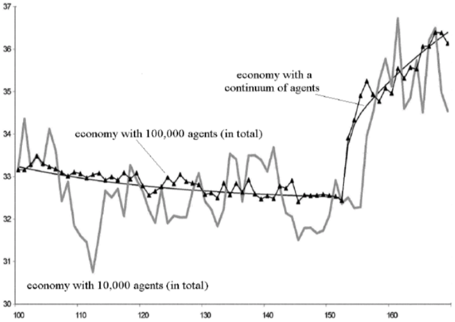
- Computational constraints are still limiting the exploration of heterogeneous agent economies.
- However, computers are becoming faster and faster.
- Moreover, some techniques that can significantly speed up the algorithm have been developed.
- Krusell and Smith use value function iteration (with interpolation) to solve the individualis problem.

Updating the aggregate law of motion

- Krusell and Smith use a Monte Carlo simulation to update the coefficients of the aggregate law of motion. But we know Monte Carlo methods are typically not very efficient.
- We discuss two possible improvements:
  a) Simulating a continuum of agents
  b) Explicit aggregation

Drawbacks of Monte Carlo

Example



Notes: This graph plots the simulated aggregate capital stock of the unemployed using either a finite number (10,000) or a continuum of agents. It displays a subset of the observations shown in Figure 1.

Figure: Source: Algan et al. (2008)

Simulating a continuum of agents

- Rios-Rull (1999) simulates a continuum of agents rather than a finite number of agents, avoiding sampling errors due to idiosyncratic shocks:

    (i) construct a grid for capital: $K_0, K_1, ..., K_N$, where $K_0 = 0$ in our case (the borrowing constraint).

    (ii) Suppose the distribution has only mass at the grid points. Let $p_{e,i,t}$ denote the mass of agents with employment status e and capital stock k_i in the beginning of period t. Guess an initial distribution at $t = 0$, compute the mean and draw a series of aggregate shocks $z_0, z_1, ...z_T$.

Simulating a continuum of agents

- Before I write down the next step formally, let me explain what happens.
- We want to know the distribution of capital at the end of the period using the individual policy rules. Note that these choices are not necessarily on the grid
- Suppose agents who are located at a certain gridpoint chose k_{t+1} in between the grid points, i.e. $k_j < k_{t+1} < k_{j+1}$, then they are "split up" over the two grid points, with weights determined according to the distance of their choice to the nodes.

Simulating a continuum of agents

(iii) Let $f_{e,i,t}$ denote the mass of agents with employment status e and capital stock k_i at the end of period t. Then we have:

$$f_{e,j,t} = \sum_{i=1}^{N} p_{e,i,t} \alpha_{e,j,i,t}$$

with

$$\alpha_{e,j,i,t} = \begin{cases} 0 & \text{if} \quad k'(e, k_i; z, K_t) < k_{j-1} \\ \frac{k'(e,k_i;z,k_t) - k_{j-1}}{k_j - k_{j-1}} & \text{if} \quad k_{j-1} < k'(e, k_i; z, K_t) < k_j \\ 1 & \text{if} \quad k'(e, k_i; z, K_t) = k_j \\ \frac{k_{j+1} - k'(e,k_i;z,k_t)}{k_{j+1} - k_j} & \text{if} \quad k_j < k'(e, k_i; z, K_t) < k_{j+1} \\ 0 & \text{if} \quad k'(e, k_i; z, K_t) \geq k_{j+1} \end{cases}$$

Simulating a continuum of agents

(iv) Now use the realizations of the aggregate shock to compute the distribution at the beginning of the next period, $t+1$ using the transition probabilities. That is, compute $p_{e,i,t+1}$ at each of the grid points.

(v) Go back to step (iii) until the end of the simulation is reached $(t = T)$.

Perturbation-based approaches

- Perturbation often thought to be inappropriate when there is heterogeneity, because of accuracy issues.
- This is not necessarily true. For example, a perturbation may be very suitable if there is only ex-ante heterogneity (no idiosyncratic shocks).
- But even if there are idiosyncratic shocks it may be possible to perturb around a steady state that preserves the idiosyncratic risk. This is the approach in Campbell (1998), Reiter (2006) and Ravn and Sterk (2012).

Alternative solution methods

Market clearing

- A convenient aspect of the Krusell-Smith model is that markets clear trivially.
- Once the aggregate capital stock is known, r_t and w_t are simply computed as the marginal products of capital and labour, respectively.
- However, market clearing prices are not always obtained trivially and this can be a problem when simulating the economy.

Non-trivial market clearing

Example

- Consider a version of the Huggett (1993) model with aggregate uncertainty. Remember that in such a model, agents can only invest in one-period bonds. How to solve this model?
- Following the strategy of Krusell and Smith (1998), one could specify a perceived law of motion for the bond price q_t as a linear function of the aggregate state variables and then try to find the coefficients such that the perceived law of motion and the actual law of motion are very close (but not precisely equal).
- When simulating the model, aggregate demand for bonds will not be exactly equal. Small errors may build up over time so that after a number of periods there can be a serious violation of the asset market-clearing condition.

Imposing market clearing in the simulation

Strategy 1

One way to impose market clearing in a simulation is to do the following.

a) Make the bond price q_t a state variable in the individualis decision problem.

b) Use the perceived law of motion for q_t only for the next period.

c) At each point in the simulation, chose q_t such that the bond market clears.

Imposing market clearing in the simulation

Strategy 2

a) Define an auxiliary variable:

$$d_{i,t+1} \equiv a_{i,t+1} + q_t$$

where $a_{i,t+1}$ is the individualis choice of bond holdings.

b) Use the above definition to substitute out $a_{i,t+1}$ in the indivuals' problem, i.e. let the agent solve for $d_{i,t+1}$ instead of $a_{i,t+1}$.

c) Note that bond market clearing requires that

$$\int_i a_{i,t+1} d_i = \int_i d_{i,t+1} d_i - q_t = 0$$

d) In the simulation, solve for q_t using $q_t = \int_i d_{i,t+1} d_i$ and for individual bond holdings using $a_{i,t+1} = d_{i,t} - q_t$. Then the bond market is guaranteed to clear by construction.

Other models

Models with heterogeneous agents (and aggregate uncertainty) have been used in a variety of contexts. Just some examples are:

- Models of firm dynamics: Hopenhayn (1992), Hopenhayn and Rogerson (1993), Clementi and Palazzo (2010), Sedlacek (2011).
- Monetary policy: Imrohoroglu (1992), Cooley and Quadrini (2006).
- Welfare costs of business cycles: Storesletten, Telmer and Yaron (2001), Mukoyama and Sahin (2006), Krusell, Mukoyama, Sahin and Smith (2008).
- Job uncertainty and demand eects: Ravn and Sterk (2012).