MRes Macroeconomics

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Week 3

Outline

Solving general equilibrium models with incomplete financial markets:

- endowment economy: Huggett (1993)
- production economy: Aiyagari (1994)

Huggett (1993)

Introduction

- Many models in macroeconomics assume the existence of a representative agent. In these models, the wealth distribution collapses to the wealth level of the representative agent.

There is no distribution at all we just have a representative agent.

- The assumption that underlies the representative agent assumption is that financial markets are complete, that is, all state-contingent securities can be traded.
- The complete markets assumption is obviously unrealistic. In particular, we see a lot of wealth heterogeneity in the data.

Huggett (1993)

The real interest rate were far lower than what the models predict.

- Standard representative agent models have great difficulties explaining why the real risk-free rate is low relative to the returns on equity (Mehra and Prescott (1985)) and given observed consumption growth (Mankiw and Zeldes (1989)).
- One reason could be that when agents are allowed to perfectly insure against idiosyncratic shocks, their precautionary demand for bonds can be expected to be reduced substantially.
- Thus, one would expect that in an incomplete markets world, the demand for bonds would be higher. When the net supply of bonds is zero, this would imply a lower equilibrium interest rate than under complete markets.
- Huggett (1993) investigates the quantitative effect of incomplete markets on the long-run equilibrium interest rate.

Huggett (1993)

Key features of the model

- Endowment economy
- Idiosyncratic income shocks
- One-period bonds are the only tradable financial assets. On top of that, agents' individual borrowing is subject to an exogenous limit.
- General equilibrium
- No aggregate uncertainty

Idiosyncratic income risk

- There is a continuum of agents with identical preferences. Their total measure is one.
- In every period, each agent receives an exogenous endowment that can be either high (e_h) or low (e_l), with $e_h > e_l$.
- The individual endowment follows a Markov process with strictly positive transition probabilities:

$$\pi(e'|e) = P[e_{t+1} = e'|e_t = e] > 0$$

$$e, e' \in \{e_l, e_h\}$$

Preferences and budget constraint

- Preferences of the agents are given by

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t)$$

where $\beta \in (0,1)$ is the discount factor and c_t denotes consumption of non-durables. $u(c_t)$ is the isoelastic utility function, i.e.

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

\sigma is the level of risk aversion higher sigma associates with higher risk aversion.

- The budget constraint is given by:

$$c_t + q_t a_{t+1} = e_t + a_t$$

where a_t is the amount of bonds, and q_t is price of a bond paying off 1 in period $t+1$. Note that a_t is chosen in period $t-1$.

Borrowing limit

- Agents face an exogenous borrowing limit:

$$a_{t+1} \geq \underline{a}$$

- Consumption must be positive in each period, i.e.

$$c_t \geq 0$$

State variables

What are the individual state variables?

- income: e_t
- bond holdings: a_t

Agent's problem

- We are interested in the steady-state properties of the model. Hence, we can treat the bond price q as a constant (but of course it is endogenous so we need to solve for it).
- The optimization problem of an individual agent can now be characterized by the following Bellman equation:

$$V(e, a, q) = \max_{c, a'} u(c) + \beta \sum_{e'} V(e', a', q)\pi(e'|e)$$

  where

$$c + qa' = e + a, \quad a' \geq \underline{a}, \quad c \geq 0$$

  where subscripts are removed and primes denote variables one period ahead.
- The solution of the individualis problem involves finding the policy functions $c = c(e, a, q)$ and $a' = a(e, a, q)$.

Stationary equilibrium

- We want to study the long-run properties of the model. Given that agents face idiosyncratic shocks that cannot be fully insured, individual income and wealth levels will not be constant.
- However, there does exist a unique stationary equilibrium in which aggregate variables and the wealth distribution are constant (for a proof, see Hopenhayn and Prescott (1992)).
- Let ϕ be a probability measure on S where, $S = [\underline{a}, \bar{a}] \times \{e_l, e_h\}$. Hence, ϕ represents the wealth distribution.

Stationary equilibrium

Definition: a stationary equilibrium is $c(e,a), a(e,a), q$ and ϕ such that:

Which maximizes their utility

1. $c(e,a)$ and $a(e,a)$ solve the individual agents problem given q,
2. goods and asset markets clear, i.e.

$$\int_S c(x) d\phi = \int_S e d\phi$$

$$\int_S a(x) d\phi = 0$$

3. ϕ is stationary, i.e. $\phi(B) = \int_S P((e,a),B) d\phi$ for all $B \in B_S$. where $P((e,a),B)$ is a transition probability from state (e,a) to somewhere in B.

Calibration

Huggett (1993) considers the following calibration:

- The model period is two months.
- $e_h = 1, e_l = 0.1, \pi(e_h|e_h) = 0.925$ and $\pi(e_h|e_l) = 0.5$. These numbers roughly match the standard deviation of income measured in the cross-section (20% on an annual basis) and the average unemployment duration (12.3 weeks).
- β is set to 0.99332 (0.96 on an annual basis).
- σ is set to 1.5 following Mehra and Prescott (1985)
- Regarding the borrowing limit a range of values for a is considered by Huggett: -2, - 4, - 6, and -8. To interpret these numbers, it is useful to know that a borrowing limit of a= -5.3 correspond to one year of average annual earnings.

Solving the model - Outline

Huggettis solution procedure involves the following steps:

1. Guess an initial asset price q.
2. Guess an initial value functions V_0.
3. Solve the individual agents problem, given q.
4. Iterate on value functions till convergence.
5. Iterate on ϕ until it converges, starting from some initial ϕ_0.
6. Compute the aggregate demand for bonds. Update q as follows: if it is negative, lower q. If it is positive, increase q.
7. Go back to step 2 until the absolute change in q is smaller than some convergence criterion.

Solving the model - initial guess for q

- How can we come up with a good initial guess for the bond price q?
- Note that in a complete-markets version of the model, the representative agents problem would imply the following Euler equation:

$$q_t c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}$$

- Hence, $q = \beta$ seems like a reasonable initial guess.

Solving the model - individual's problem

- Following Huggett (1993), I use value function iteration to solve the individualis problem.

Bins have equal length

- I discretize S and use 1000 equidistant grid points on the interval [a, -6] (and verify ex post that the interval is large enough).
- As a stopping criterion I use that the root mean squared change in the value function should not exceed 10^{-6}.

Solving the model - distribution

- To find the stationary distribution, given the individualsipolicy rules, I iterate on the following equation:

$$\phi_{t+1}(a_i, e_j) = \sum_{k=1}^{n} \sum_{m \in \{h,l\}} I[a' = a_i | a_k, e_m] \pi(e' = e_j | e_m) \phi_t(a_k, e_m)$$

  where $\phi_t(a_i, e_j)$ denotes the mass of agents at the grid point (a_i, e_j), with $i = 1, 2, ..., n$ and $j \in \{h, l\}$ and I is the indicator function.
- As a convergence rule, I use that $||\phi_{t+1} - \phi_t|| < 10^{-6}$

Solving the model - q

- Given a stationary distribution, I compute aggregate demand for bonds as:

$$D = \sum_{k=1}^{n} \sum_{m \in \{h,l\}} a(a_k, e_m)\phi_t(a_k, e_m)$$

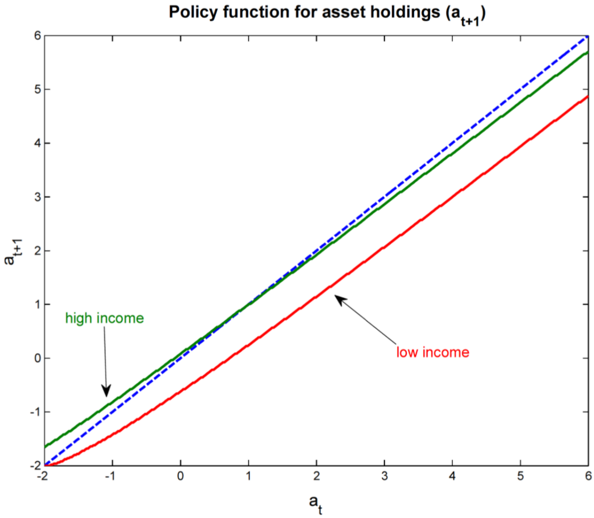
  where $a(a_k, e_m)$ is the policy rule for the asset choice.

- I update q using

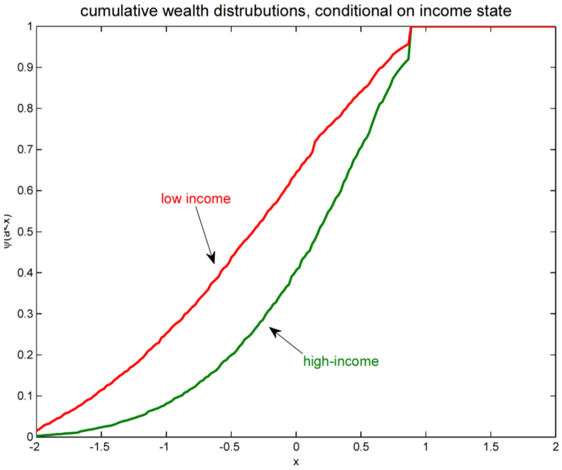
$$q_{s+1} = (1 + \lambda D_s) q_s$$

  where s is the index of the iteration and λ is a scalar to be satisfied. I set $\lambda = 0.005$.

Policy Function: Capital



Upper limit on bond holdings



Summery

- Very few agents at the borrowing constraint.
- Still, the introduction of incomplete financial markets can have a large negative effect on the equilibrium interest rate.
- But what if aggregate savings are possible?

Aiyagari (1994)

Aiyagari (1994)

- Suppose there is an aggregate savings technology (like capital) but financial markets are still incomplete.
- One might expect that the precautionary savings motive will lead to a larger steady-state level of capital, relative to a model with complete financial markets.
- Aiyagari (1994) constructed a model to investigate this issue.

Aiyagari (1994)

Key features of the model:

- capital investment (no bonds) idiosyncratic income shocks
- general equilibrium
- no aggregate uncertainty

Problem of the individual

The optimization problem of the individual is given by

$$\max_{c_t, k_{t+1}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

where

$$\begin{aligned} c_t + k_{t+1} &= wl_t + (1 - \delta + r)k_t \\ c_t &\geq 0 \\ k_t &\geq b \end{aligned}$$

where $\beta \in (0, 1)$ is the discount factor and c_t denotes consumption of non-durables, k_t is the amount of capital, l_t is labor endowment, w_t is the wage per efficiency unit of labor, r_t is the rental rate of capital, δ is the rate of depreciation and $b < 0$ is a borrowing limit.

Labor endowment process

- The labor endowment is subject to idiosyncratic shocks.
- It will be assumed that It has bounded support $[l_{min}, l_{max}]$ with $l_{min} > 0$.
- In particular, It will be assumed to follow a Markov process that is such that $\mathbb{E}[l_t] = 1$, that is, aggregate supply of labor equals one in every period.

Borrowing limit

- The borrowing limit b is set to the natural borrowing limit.
- The idea behind the natural borrowing limit is that the agent should always be able to pay back her debt, no matter the future realizations of her labor endowment shocks
- The worst-case scenario is that agent will have $l_{t+j} = l_{min}$ for any $j = 1, 2, ...$
- How do we determine b?

Borrowing limit

- Suppose the agent is exactly at the borrowing limit in period t, i.e. $k_t = b$ and the worst-case scenario occurs, i.e. $l_{t+j} = l_{min}$ for any $j = 1, 2, ...$
- Then being exactly able to pay back her debt means that the agent must have zero consumption in all future periods, i.e. $c_{t+j} = 0$ for $j = 1, 2, ...$
- From the budget constraint it follows that

$$k_{t+j+1} = wl_{min} + (1+r)k_{t+j}$$

- With $k_{t+j+1} = k_{t+j} = b$. It follows that

$$b = -\frac{wl_{min}}{r}$$

- Because the utility function satisfies the Inada conditions, the borrowing limit will never actually bind, given that initial assets are above the borrowing limit.

Production

- A representative firm operate on a Cobb-Douglas production function:
$$y = k_{d,t}^{\alpha} l_{d,t}^{1-\alpha}$$
where "d" stands for "demand".
- Profit maximization implies:
$$\begin{aligned} r_t &= \alpha k_t^{\alpha-1} l_t^{1-\alpha} \\ w_t &= (1-\alpha) k_t^{\alpha} l_t^{-\alpha} \end{aligned}$$
- Since l_t is exogenously given, r_t and w_t are directly given as functions of the capital stock.
- It is easy to show that firms make no profits.

Market clearing

- Let individuals be indexed by i and let their total measure be equal to one. The clearing of the capital and labor market implies that:

$$k_{d,t} = \int_i k_{i,t} di$$

$$l_{d,t} = \int_i l_{i,t} di$$

Stationary equilibrium

- We are interested in the steady-state properties of this model. So from now on I will drop time subscripts.
- By checking the conditions provided in Hopenhayn and Prescott (1992) for the Aiyagari model, it can be shown that a unique stationary equilibrium exists.
- Here, I will focus on the computation and the quantitative results.

Solving the model

1. Guess an initial aggregate capital stock K_j with$j = 0$ and specify a converge level ϵ.
2. Compute the associated rental rate r and wage w:

$$
\begin{aligned}
r &= \alpha K_j^{\alpha-1} \\
w &= (1-\alpha) K_j^{\alpha}
\end{aligned}
$$

3. Solve the individualis problem, given r and w using your preferred method. That is, find the policy rule for the capital choice $k_{+1}(k, l; r, w) = k_{t+1}$.

Solving the model

4. Using the policy rule, compute the stationary distribution of capital.
5. Compute a new value for aggregate capital as

$$\hat{K} = \int_i k_i di$$

6. Update the capital stock as $K_{j+1} = \lambda K_b + (1-\lambda)K_j$, where $\lambda \in (0,1]$ is a dampening parameter and j is the index of the iteration. Go back to step 2) until $||K_{j+1} - K_j|| < \epsilon$.

Solving the model - an alternative approach

- An alternative is to iterate on r instead of on K. In that case, it is important to make sure that $\beta(1 + r - \delta) < 1$ as otherwise the capital stock will "explode".
- One could start with some initial guess $r_0 = \frac{1}{\beta} - 1 + \delta - \epsilon$, where ϵ is a very small number, and compute the associated values for capital demand k_d and for w.
- Then, solve the individualis problem and find the implied stationary distribution.
- Next, compute the aggregate capital supply $\hat{K}$, and compute $\hat{r} = \alpha \hat{K}^{\alpha-1}$
- Update the rental rate of capital as $r_1 = \frac{r_0 + \hat{r}}{2}$

Solving the model - an alternative approach

- Given r_1, solve the individual's problem, compute (i) the stationary capital distribution (ii) aggre capital supply $\hat{K}$, (iii) the associated rental rate $\hat{r}$.
- Finally compute excess supply of capital as $K_{excess} = \hat{K} - k_d$, where k_d is the aggregate demand for capital consistent with r_1.
- Update the r as follows:
  - If $K_{excess} > 0$, set $r_2 = \frac{r_1 + \hat{r}}{2}$
  - If $K_{excess} < 0$, set $r_2 = \frac{r_0 + \hat{r}}{2}$
- Continue until $K_{excess} < \epsilon$, where ϵ is a pre-specified convergence criterion.

Results - aggregate saving rate

- Aiyagari compares the equilibrium savings rate in (i) the incomplete markets model and (ii) a version of the model with full insurance.
- He finds that for plausible parameter values, savings rate is very similar in the two models.
- The reason is that self-insurance is quite an effective substitute for insurance contracts between agents.

Results - inequality

- Aiyagari's model also qualitatively matches some facts present in the data:
  - consumption inequality is smaller than income inequality,
  - the income and wealth distributions are positively skewed (median<mean).
- However, the model generates too little dispersion in income and wealth. The table below displays Gini coeffcients, for the model and the data:

	Gini income	Gini Wealth
model	.12	.32
data (U.S)	.4	.8