Simplex of [set] B $= \Delta(B) \triangleq P = (p(b_1), \dots, (b_J)]$

where $\forall i,\ b_i \in B$ and $p(b_i) \geq 0$ and $\sum_{j=1}^{J} p(b_j) = 1$

Strictly dominated $\overset{\checkmark}{\Longrightarrow}$ NWBR (for pure actions)

$\not\Leftarrow$ (for pure actions converse is not true but for mix action convers is true)

NWBR = never a weak best response

- تقاطع BR های همه بازیکنان بازی NE است.

- برای بدست آوردن NE تمام حالت های ممکن BR را می‌نویسیم و در تمامی حالت ها بررسی می‌کنیم و اشتراک می‌گیریم تا نهایتا تمام NE ها بدست می‌آید.

- از بین استراتژی های strictly dominated هیچ NE خالص یا mixed نداریم!

↳ اما ممکن است در نتایج BR بیاید اما نهایتا NE نخواهد بود.

- برای اینکه چک کنیم یک استراتژی می‌تواند توسط ترکیبی از استراتژی های دیگر strictly dominate شود

باید ببینیم به ازای همه استراتژی های بازیکن دیگر آیا max payoff جزو این استراتژی هست یا نه؟ زیر

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توسط ترکیب خطی استراتژی های دیگر فضای بین آن ها ساخته می‌شود و می‌تواند از مقدار MAX بیشتر باشد پس اگر max payoff در ستون آن استراتژی نباشد احتمالاً می‌توان با ترکیب استراتژی های دیگر آن را strictly dominate کرد اما اگر max باشد نه! اگر max نباشد لزوماً وجود ندارد ولی احتمالاً ممکن است ترکیبی از استراتژی های دیگر وجود دارد که این استراتژی را strictly dominate کند.

← برای بدست آوردن BR حالت کلی را بنویس و با هم تقاطع بده برای بدست آوردن تمام NE ها.

← برای شمردن تعداد استراتژی های یک بازیکن برای هر information set آن بازیکن n انتخاب داریم. باید برای مشخص کردن تعادل های SPNE باید off the path ها را مشخص کنیم.

• در جاهایی که ~~BR~~ بازیکن بی‌تفاوت یا indifferent است BR چند تابع می‌شود که در آن موقعیت در هر کدام از تابع ها یک استراتژی متفاوت انتخاب می‌کند.

• تعادل نش شرط کافی نیست شرط لازم است در واقع نمی‌گوید بازی حتماً این شکلی می‌شود یا چطور این شکلی می‌شود بلکه می‌گوید اگر نش نباشد حتماً نمی‌شود.

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sequentially rational : به یک استراتژی sequentiall rational است اگر در تمام desicion node ها

مثلاً x ، اگر $\sigma(x|\sigma)$ بهترین پاسخ (BR) به $\sigma(x|\sigma)$ باشد اگر برای تمام نودها این برقرار باشد آنگاه استراتژی پروفایل σ ، sequential rational است.

SPNE $\equiv$ no one-shot deviation $\equiv$ Sequential rational

weak ~~beysian~~ sequentiall eq. (σ, μ) $\equiv$ PBE

(1) μ is beysian given σ

(2) σ is sequentially rational given μ

Sequential equilibrium: (σ, μ)

(i) μ is KW-consistent

(ii) exists sequence of completely mixed strategies:

(1) $\lim_{k \to \infty} \sigma^k = \sigma$

(2) μ^k is unique beysian belief for σ^k and $\lim_{k \to \infty} \mu^k = \mu$

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SE => PBE => SPNE => NE

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GT Note summary & practical:

Strictly dominated: A rational player will never choose a strictly dominated action.

. Expected utility maximization: A rational player will always choose an action that maximizes expected utility given beliefs P.

. If set of actions A is finite, there is always at least one best action.

. NWBR: A rational player will not choose a NWBR:

NWBR $\triangleq$ never a weak best response: that is, it is not a best response to any belief.

. prop: A strictly dominated action is $\Rightarrow$ ✓ NWBR

unless it is mixed $\not\Leftarrow$ M could be NWBR but undominated by pure action.

. A mixed action cannot yield a payoff higher than the best pure actions in it's support. Since mixed action payoff is a convex combination of pure payoffs

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prop: A mixed action σ is a best response iff ($\Rightarrow$, $\Leftarrow$) every pure action in it's support is a best response.

↳ IF it is not we can put more weights on the better and having a better payoff

prop: NWBR. complement to the previous prop:

a is dominated (by mixed or pure action) $\overset{\text{iff}}{\Longleftrightarrow}$ a is NWBR.

pay, out come, risk: payoffs are not outcomes, they give the utility from outcomes. payoffs already include risk aversion and other concerns.

So a rational player will maximize his or her own expected utility.

1 solution method/concept

→ Itrated deletion of strictly dominated actions:

- requires:
  - rational players
  - Complete information (knowing each other payoffs)
  - knowing that ther other one knows his payoff .... as many needed.
  - They know that other player is rational.
  - they know that they know that the other is rational and ...— as many rounds as deletion rounds.

- Order: In deletion of strictly dominated strategies order does not matter.

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- Common knowledge $\triangleq$ unlimited knowing that sth is true and they know they know and...

- order in weak dominated: In deletion of weakly dominated strategies

  order does matter. $\Rightarrow$ so it is not a good way for solving.

NWBR deletion:

- In a 2player game NWBR deletion result $\overset{\text{exactly}}{=}$ strictly dominated strategies

- Nash equilibrium:

  a pure strategy profile $a^* = (a_1^*, \dots, a_N^*)$ is NE if for each player i

Expected U

$$NE \triangleq \forall i, \quad u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \text{ for all } a_i \in A_i$$

$$\Updownarrow$$

$$\forall i, \quad a_i^* \in BR(a_{-i}^*)$$

A mixed strategy profile $\sigma^* = (\sigma_1^*, \dots, \sigma_N^*)$ is NE if

$$\forall i, \quad \sigma_i^* \in BR(\sigma_{-i}^*) \equiv a_i \in BR(\sigma_{-i}) \ \forall a_i \in supp(\sigma_i^*)$$

player

prop: σ^* is NE $\Rightarrow$ σ^* survives iterated deletion of strictly dominated actions

$\parallel$ $\quad$ $\parallel$

[every pure action]: $\forall a \in supp(\sigma)$ survives "

pure action

- For finding mixed strateies that are NE we have to make the other player indifferent.

prop.: Every finite game $\begin{cases} \text{finite players} \\ \text{finite actions} \end{cases}$ has at ~~last~~ one (maybe mixed) NE.

- NE may not be unique.
- NE has nothing to do with pareto.
- NE could involve weakly <u>dominated</u> strategies.
- NE could involve pareto <u>dominated</u> by another N equilibrium.
- برای بدست آوردن BR حالت کلی را بنویسیم همچنین برای بدست آوردن NE.

Common knowledge : knowing the other person is rational and so on. (آگاهی عمومی)

NWBR deletion must be done for any belief ! → rationalizable strategy

. For all forms of game we have two assumptions : 1- rationality
2- common knowledge

. Beliefs (or special form of it : common knowledge) is present in any game form we have.

~ In nash equilibrium we don't argue about how the equilibrium is achieved and mechanism of game but only about equilibrium itself.

. Compact = closed and bounded set

*A rational player will not play a strictly dominated strategy.

• Def. strictly dominated: In normal form game. $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$

s_i' and s_i'' are feasible:

s_i' is strictly dominated by s_i'' iff $u_i(s_1, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_n) \overset{\text{strict}}{<} u_i(s_i'')$ $\forall s_j$

→ Iterated elimination of strictly dominated strategies method for solving:

→ 1- rational player don't play strictly dominated strategies.

2- Each step requires a further assumption of other player rationality. → common knowledge → common knowledge definition

common knowledge ≜ infinity (∞) number of knowing other player is rational.

• Nash eq is a stronger concept than iterated elimination

√√★ Nash eq always survive iterated elimination but converse is not true chapter 1: static games of complete information

static ≡ Simultaneuos _ move

Complete information ≡ each player's pay of function is common-knoweldge among all the players.

$\text{payoff function}_i = \text{payoff fuction}_i\underbrace{(\text{combination of actions})}_{\text{action profile}}$

1.1 Normal form game:
1- players
2- strategies for each player → 2- $S_1, \dots, S_n$
3- payoff function of each player → 3- $u_1, \dots, u_n$
*4- simultaneous act (without knoweledge about other players strategy. → 4- without knowledge
1- n player

bi-matrix ≡ two number in each cell

In bi-matrix by convention first number is for so-called row player.

$S_i \triangleq$ set of strategies available to player i ≡ i's strage space.

$s_i \triangleq$ an arbitrary member of set S_i

- Simultaneity $\triangleq$ suffices that each choose his action without knowledge of the other's choices.

. Iterated elimination of strictly dominated strategy
→ 1- requires assumption of common Knowledge
→ 2- can be impercise

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* In Nash equilibrium each player strategy is best response (BP) to other players specified strategies.

- pure strategy Vs. mixed strategy / Nash is stronger concept than iterated elimination.

- For finding NE . brute-force approach → check every combination
  ↳ underlining every number

best response for each player <=> satisfies ✓ NE condition

because

. Nash equilibrium ⇒ ✓ iterated elimination solution
  ⇐ ✗ (Nash always survive iterated elimination of strictly dominated strategies but converse is not true.)

. finite game = { players finite
  strategy sets finite

Nash Theorem : In any finite game, there exist at least one Nash equilibrium. /

game solutions by far → iterated elimination
→ brute force for nash underlining BP

. NE is not unique necessarily.

. proposition : If Iterated elimination eliminate all but one action profile or set of strategies, then this profile is the unique (NE).

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- proposition: If $(s_1^*, \dots, s_n^*)$ are a Nash equilibrium then they survive iterated elimination of strictly dominated strategies.

- Nash eqs survive iterated but anything that survive is not nash eq necessarily unless it is unique!

- There may be games which game theory doesn't provide a unique solution and no convention will develop.

- Cournot game → normal form game, 2 firm, choose quantities

- In Nash equilibrium each agent strategy is the best response to other firms equilibrium strategy.

- A strategy is strictly dominated if there is no belief about the other player's choices for which the strategy is a best response.

- Bertrand → different game from Cournot., duopoly price choosing!

- pure strategy : elements of $S_i = s_i$

- mixed strategy is probability distribution over strategies in S_i

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Repeated games: Same interaction (game) repeatedly. Each period action
infinite

Stage game

choices is commonly observed at the end of the period.

payoff = sum of discounted of stages payoff with fixed δ
discount rate.

$G = \{ P, \{A_i\}_{i \in P}, \{u_i\}_{i \in P} \}$ = stage game

$a_i \in A_i$ pure action

$a_i \in \Delta A_i$ mixed action

$\alpha \in \prod_{i \in P} \Delta A_i$ a mixed action profile

$G^{\infty}(\delta) = \{ P, \{S_i\}_{i \in P}, \{\pi_i\}_{i \in P}, \delta \}$ = infinitely repeated game

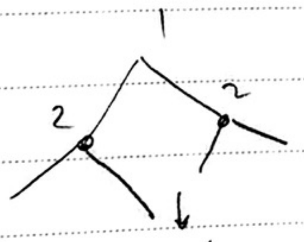
. Extensive form games : moves may ocurr sequentially.

chance events

without moves by nature

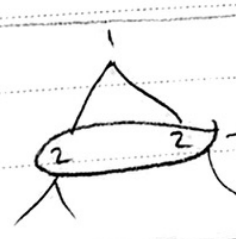
Etensive form games → perfect information → actions are always immediately observed by opponents.

→ imperfect information



nodes are identified with the sequence of actions that lead to them.

player 2 observe player 1 choice then make her choice.



→ simultaneous move : 2 don't observe player 1 move

→ Desicion nodes have said to be in the same informatio set

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Normal form game notation : $G = \{P, \{S_i\}_{i\in P}, \{u_i\}_{i\in P}\}$

capital

$P = \{1, \dots, n\}$: finite n player set

$S_1 \leftarrow \{\{s_{11}, s_{12}\}, \{s_{21}, s_{22}, s_{23}\}\} \rightarrow S_2$

$\{S_i\}_{i\in P}$, finite set of pure strategies for each player i

↳ set of n sets one for each player i

$\{u_i\}$: for each playe utility function $u_i : S \to \mathbb{R}$ where

$S = \underbrace{\Pi_{i\in P} S_i}$ is the set of pure strategy profile

$S_1 \times S_2 \times \dots \times S_n$

every combination of S_i = every combination of strategy = all pure strategy profiles ! | a combination of strategies != profile

. Proposition: Every extensive form game τ has perfect information

if every information set of τ is a sigleton, and if τ

contains no moves by nature.

. The subgame of τ, starting from x denoted by τ_x is the game

defined by the portion of τ that strats from decision node

$x \in D$ and inclued all subsequent nodes.

↳ * subgame should be able to be analyzed without reference

to the rest of the game, so in imperfect information game

not every desicion node is the begining of subgame !

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If σ is a strategy profile $\sigma|_x$ is the strategy profile that σ induces in subgame Γ_x.

Strategy profile σ is a subgame perfect equilibrium of Γ if for every subgame of Γ_x, $\sigma|_x$ is a Nash equilibrium.

- Sequentially rational

- No profitable one-shot deviation → for every node x cannot improve his payoff in Γ_x by changing action in x.

- let Z a finite set, $\Delta Z :=$ set of probability distributions over Z : $\Delta Z := \{P : Z \to \mathbb{R}_+ \mid \sum_{z \in Z} P(z) = 1\}$

If p,q is relations on ΔZ, and $p \succeq q$, $\succeq$ on Z admits an expected utility representation if there is $u : Z \to \mathbb{R}$ such that:

$$p \succeq q \iff \sum_{z \in Z} u(z)p(z) \geq \underbrace{\sum u(z)q(z)}$$

→ expected of u with respect to q

:= expected utility function

u is called Bernoulli utility function

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. Normal form game : Simultaneous move

. Extensive form : moves over time

. Repeated game : a normal form game played repeatedly, with all previous moves being observed before each round by all agents.

. Beysian games : players recieve private information before play begins.

. pure strategy profiles : $S = S_1 \times S_2$

. A = finite set , $\Delta A = \{ P : A \to \mathbb{R}^+ \mid \sum_{a \in A} P(a) = 1 \} \rightarrow$

set of all probability distribution over A

$\sigma_i \in \Delta A \equiv$ mixed strategy

$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \prod_{i \in P} \Delta S_i$ = mixed strategy profile

we assume players randomize independently. $\rightarrow f(\sigma) = f(\sigma_1) \times f(\sigma_2) \times \dots \times f(\sigma_n)$

. Only a pure strategy can be strictly dominant.

. strictly dominated by another strategy is for any belief BUT it suffices to check for all $\underbrace{\text{strategy}}_{\text{pure}}$ profiles of others = S_{-i}

easier to check

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. Iterated strict dominance:

i) first remove all dominated pure strategies

ii) check all remaining mixed strategies.

• Removing a pure strategy implicitly removes the mixed strategies that use it.

. A mixed strategy can be dominated even if none of the pure strategies it uses are.

• Strategies remained doesn't depend on the order in which strategies removed.

. Iterated strict dominance doesn't completely solve the game necessarily!

. weakly dominated $u_i(\sigma_i, \mu_i) \geq u_i(\sigma_i', \mu_i)$ for all $\mu_i \in \Delta S_{-i}$

and

$u_i(\sigma_i, \mu_i') > u_i(\sigma_i', \mu_i')$ for some $\mu_i' \in \Delta S_{-i}$

$\Downarrow$

it is enough to consider only pure strategy of oppenents.

. Order of weakly dominanted strategies can matter.

. Best response $\triangleq$ σ_i is BP to reduced conjecture $\mu_i \in \Delta S_{-i}$ if

$u_i(\sigma_i, \mu_i) \geq u_i(\sigma_i', \mu_i)$ for all $\sigma_i' \in \Delta S_i$

setvalued map or correspondence not function ?

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$B_i : \Delta S_{-i} \Rightarrow \Delta S_i$ ≡ best response correspondance $\Longleftrightarrow$ $\sigma_i \in B_i(\mu_i)$

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labor - mechanism design - bargaining - moral hazard
signal

. σ_i is a best response to μ_i iff every pure strategy s_i in the support of σ_i is a best response to μ_i.

. The rationalizable strategies are those that remain after we iteratively remove all strategies that are not best response to any allowable conjecture.

→ iteratively removing strategies that cannot be best responses.

Computing player's rationalizable strategies:

i) Remove pure strategies that are never a best response to any allowable conjecture.

ii) Remove mixed strategies that are never a best response ...

. Computing all mixed strategy nash equilibria:

(i) Eliminate pure strategies that are not rationalizable.

(ii) for each profile of supports, find all equilibria!

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Extensive form game → perfect information → choices are observed w/o chance or moves by nature
→ imperfect information

→ i.e 2 doesn't observe player 1 move befor her decision.
→ = same information set

$I_i \subseteq D_i$: set of information set

When play reaches player i's information set I, i knows that some node in I has been reached, but he cannot tell which.

If $x, y \in I \iff A_x = A_y = A_I$

when information set is singleton { like $\{x\}$

. Γ is perfect information game if every information set is sg

. Nontrivial information set = we cannot tell which node player is

. perfect recall : each player remembers everthing !

. $e < y$: when the path from the root node to y includes edge e

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Behavior strategy: $\beta_i \in \prod_{I \in \mathcal{I}_i} \Delta A_I$, each time a player reaches an information set, he randomizes over the actions in A_I.

. principle of sequential behavior: predictions of play in extensive form games should require optimal behavior strating from every information set not just those on the equilibrium path.

. σ is a strategy profile in Γ $\Rightarrow$ $\sigma|_x$ is the strategy profile that σ induces in subgame Γ_x.

. SPE: subgame perfect equilibrium: strategy profile σ is subgame perfect equilibrium if in each subgam Γ_x of Γ, $\sigma|_x$ is a Nash equilibrium.

. Def $\triangleq$ strategy σ_i is sequentially rational given σ_{-i} if for each descison node x of player i, $(\sigma|_x)_i$ is a best response to $(\sigma|_x)_{-i}$ in Γ_x. If this is true for every player, we say that straty profile σ itself is sequentially rational

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Def: strategy σ_i admits no profitable one-shot deviations given σ_{-i} if for each decision node x of player i, player i cannot improve his payoff in T_x by changing his action at nodes x but otherwise following strategy $(\sigma|_x)_i$. If this is true for every player we say that strategy profile σ itself admits no profitable one-shot deviations.

1

. Theorem: In a finite perfect information game, subgame perfect equilibrium, sequentially rational and no-profitable one-shot deviatio are equivalent.

→ backward induction procedure:

σ admits no-profitable one-shot deviation $\Longleftrightarrow$ survive backward induction

. A pure subgame perfect equilibrium always exists, because of backward induction.

. Common certainty of rationality? "knowledge" refers to beliefs that are both certain and correct, certainty rather than knowledge allows for the possibility that a player discovers that his beliefs are incorrect during course of play.

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Belief in: The belief profile μ is beysian given profile σ if $\mu_i(x) = \frac{P_\sigma(x)}{P_\sigma(I)}$

where $P_\sigma(I) = \sum_{x \in I} P_\sigma(x)$

. At information set I on the path of play, Beysian beliefs describe conditional probabilities of nodes reached.

. σ_i is rational strating from information set $I \in \mathbb{I}_i$ given σ_{-i} and μ_i

if $\sum_{x \in I} \mu_i(x) u_i(\sigma_i, \sigma_{-i} | x) \geq \sum_{x \in I} \mu_i(x) u_i(\hat{\sigma}_i, \sigma_{-i} | x)$ for all $\hat{\sigma}_i$

if This holds for every information set $I \in \mathbb{I}_i$ we call σ_i:

Sequentially rational given σ_{-i} and μ_i

↳ if for all player hold we call σ sequentially rational given μ.

A pair (σ, μ): strategy profile σ and belief profile μ is an assessment

(σ, μ) is weak sequential equilibrium if:

(1) μ is beysian given σ

(ii) σ is sequentially rational given μ

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path of play $\equiv$ $P_\sigma(I) > 0$ $\equiv$ I is in the path of play

if (I) μ is beysian and (II) on the path of play σ_i is rational $\equiv$ Nash eq

. Weak sequential equilibrium is a refinement of NE.
||
perfect Beysian equilibrium

. perfect information ⟶ each I (information set) is singleton ≡ each player knows the history of play so far.

. Complete information ⟶ Structure of the game is common knowledge ≡ each player could draw the game tree with all the payoffs!

in complete : structure is unknown to at least some players.

. In a finite perfect information game

sequentially rational = SPE = admits no profitable one-shot deviation

weak sequentially rational = perfect Beysian

. The Belief profil μ is beysian given profile σ if $\mu_i(x) = \frac{P_\sigma(x)}{P_\sigma(x)}$

whenever $P_\sigma(I) > 0$, ≡ beliefs are determined by conditional probabilities whenever possible.

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Palyer i's beliefs $\triangleq$ $\mu_i : D_i \rightarrow [0,1]$, $\sum_{x \in I} \mu_i(x) = 1$, $\forall I \in \mathbb{I}_i$

= $x \in I$, $\mu_i(x)$ represents the probability that player i assigns to being at node x GIVEN that his information set I has been reached.

$P_\sigma(x) \triangleq$ Given strategy profile σ, $P_\sigma(x)$ is the probability that $x \in X$ is reached $\Rightarrow P_\sigma(I) = \sum_{x \in I} P_\sigma(x)$

μ is beysian if $\mu_i(x) = \dfrac{P_\sigma(x)}{P_\sigma(I)}$ wherever $P_\sigma(I) > 0$

Give a pair (σ, μ) consisting of strategy profile σ and belief profile μ is weak sequential equilibrium or perfect baysian equilibrium if:

(I) μ is Beysian given σ

(II) σ is sequentially rational given μ $\rightarrow$ at each I for each i u is maximized!

- σ is NE iff (i) μ is beysian given σ and (ii) for each information set I on the path of play for each player ($\equiv P_\sigma(I) > 0$) then σ_i is rational (= best response) given σ_{-i} and μ_i $\rightarrow$ so if I is not on the path of play, then player need not to play a best response given his beliefs.

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$P_{\sigma}(I) = \frac{1}{2} \times \sigma(I_n) \times 2 = \sigma(I_n)$

proposition : PBE => SPE

weak PBE need not $\not\Rightarrow$ SPE
↳ no restriction on beliefs off the eq path.

- Sequential equilibrium : (σ, μ) is a sequential equilibrium if

(i) μ is kw-consistent given σ : there exists a sequence of completly mixed strategy profiles $\{\sigma^k\}_{k=1}^{\infty}$ such that

(1) $\lim_{k\to\infty} \sigma^k = \sigma$, and

(2) if μ^k are the unique Baysian beliefs for σ^k, then

$$\lim_{k\to\infty} \mu^k = \mu$$

(ii) σ is sequentially rational given μ

Sequentail Eq => (σ,μ) is $\overset{\text{weak sequential equilibrium}}{\overset{\equiv}{\text{PBE}}}$ => σ is SPE => σ is NE
(σ, μ)

- Sequential rationality : starting from an information set I given σ_{-i} and μ_i

if $\sum_{x\in I} \mu_i(x) u_i(\sigma_i, \sigma_{-i} | x) \geq \sum_{x\in I} \mu_i(x) u_i(\hat{\sigma}_i, \sigma_{-i} | x)$ for all $\hat{\sigma}_i$

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In pooling we offer only one (D, P) but in seperating if I am Q_1 I off (D_1, P_1) and If am Q_2 (P_{2}, P_2)