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Microeconomics 2 Homework 2

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Exercise 1.

Proof. (a) In order to prove strictly partial order property for H_i we must prove following properties:

- Asymmetry: Suppose $h' \prec^\star h$ and $h \prec^\star h'$, then $\exists t' \in h'$, $\exists t \in h$ such that $t' \prec t$ and $t \prec t'$. This is impossible, so $h = h'$.
- Transitivity: Suppose $h \prec^\star h'$ and $h' \prec^\star h''$, then $\exists t \in h$, $\exists t' \in h'$ and $\exists t'' \in h''$ such that $t \prec t'$, $t' \prec t''$. By transitivity of nodes, we have $t \prec t''$. Consequently $h \prec^\star h''$.
- Irreflexivity: Suppose $h \prec^\star h'$, thus $\exists t \in h$ and $\exists t' \in h'$ such that $t \prec t'$ and also $t' \not\prec t$, then $h' \not\prec^\star h$.

(b) Consider the tree below in which transitivity does not hold:

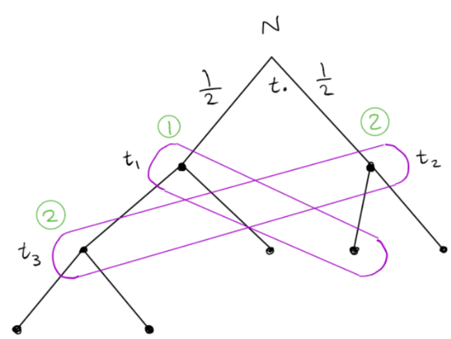

Figure 1: Exercise 1 extensive form game tree

(c) If $h' \prec^\star h$ then $\exists t'_m \in h'$ and $\exists t_m \in h$ such that $t'_m \prec t_m$. By perfect recall property, if $t'_l \in h(t_m)$, because $t'_m \prec t_m$ we conclude $\exists t'_l \in h(t'_m)$ such that $t'_l \prec t_l$. □

Exercise 2.

Proof. (a) The tree is as follows:

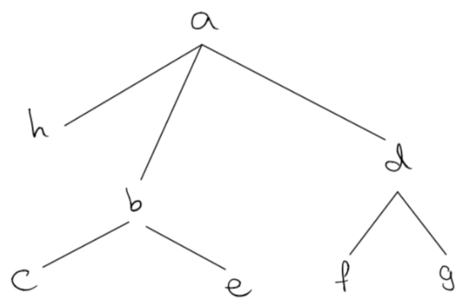

Figure 2: Section (a) extensive form game tree

(b) As it is shown in the tree, $S_1 = \{l, c, r\}$ and $S_2 = \{mx, my, nx, ny\}$. If players play the pair of strategies $S_1' = \{c, r\}$ and $S_2' = \{mx, my\}$, the resulting strategy profiles $\hat{S} = \{(c, mx), (c, my), (r, mx), (r, my)\}$ will have c, c, f and g as outcomes respectively.

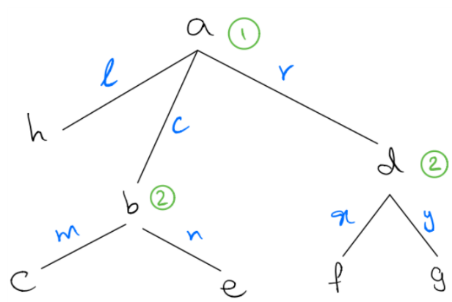

Figure 3: Section (b) extensive form game tree

(c) The tree is as follows: It is not possible, because player 2 in the information

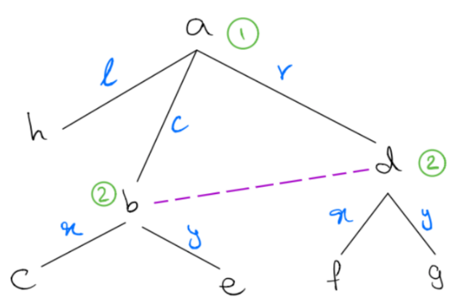

Figure 4: Section (c) extensive form game tree

set cannot distinguish between node b and d, So whenever outcome g is realized, outcome e must also be realized.

(d) The tree is drawn like below:

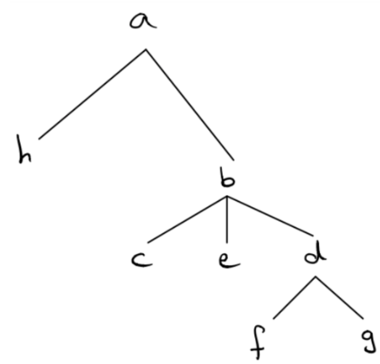

Figure 5: Section (d) extensive form game tree

□

Exercise 3.

Proof. $\hat{s_1}$ is a weakly dominated strategy for player 1, meaning:

$$\forall s_1' \in S_1 \quad \text{and} \quad \forall s_{-1} \in S_{-1}: \quad u_1(s_1', s_{-1}) \geq u_1(\hat{s_1}, s_{-1}) \tag{1}$$

$$\exists s_1'' \in S_1 \quad \text{and} \quad \forall s_{-1} \in S_{-1}: \quad u_1(s_1'', s_{-1}) \geq u_1(\hat{s_1}, s_{-1})$$

And if s is a Nash equilibrium of $\{(S_i', U_i)_{i=1}^n\}$ then we have:

$$\begin{cases} u_1(s_1, s_{-1}) \geq u_1(s_1', s_{-1}) & \forall s_1' \in S_1' \\ u_2(s_2, s_{-2}) \geq u_2(s_1', s_{-2}) & \forall s_2' \in S_2' = S_2 \\ \vdots \qquad\qquad \vdots & \vdots \\ u_n(s_n, s_{-n}) \geq u_n(s_n', s_{-n}) & \forall s_n' \in S_n' = S_n \end{cases} \tag{2}$$

Using inequalities above and (1), we conclude:

$$u_1(s_1, s_{-1}) \geq u_1(\hat{s_1}, s_{-1}) \tag{3}$$

If we notice to combination of first inequality in (2) and (3), and also to (n-1) inequalities in (2) we can say non of the players have motivation to deviate from s, thus it is a Nash equilibrium of $\{(S_i, U_i)_{i=1}^n\}$. □

Exercise 4.

Proof. Game setup is:

$$P(Q) = max\{a - Q, 0\}, \quad Q = q_1 + q_2 + \ldots + q_n, \quad C(q_i) = cq_i,$$

$$0 < c < a, \quad S_i = \mathbb{R}_+, \quad U_i(q_1, \ldots, q_n) = (P(q_1+q_2+\ldots+q_n)-c)q_i = (a-(q_1+q_2+\ldots+q_n)-c)q_i$$

Firm i maximizes its utility given the strategy of other firms. Therefore:

$$\frac{\partial U_i}{\partial q_i} = (a-c) - \sum_{j \neq i}^{n} q_j - 2q_i = 0 \tag{4}$$

$$\Rightarrow q_i = \frac{(a-c) - \sum_{j \neq i}^{n} q_j}{2}$$

Because the problem is symmetric, we must have $q_j = q_i$ .From (4):

$$2q_i = a - c - (n-1)q_i$$

$$\Rightarrow (n+1)q_i = a - c \quad \Rightarrow q_i = \frac{a-c}{n+1}$$

So Nash equilibrium strategy profile will be:

$$s = (\frac{a-c}{n+1}, \frac{a-c}{n+1}, , \ldots \frac{a-c}{n+1})$$

If n approaches infinity the quantity produced by every firm converges to zero and market clearing price will be:

$$P = \lim_{n \to \infty} a - (q_1 + q_2 + \ldots + q_n) = \lim_{n \to \infty} a - \frac{n}{n+1}(a-c) = c$$

That means market price will converge to marginal cost of the firms. These are the main properties of a competitive equilibrium market. Note that in this case the utility of the firms will also be zero. (Because utility functions are ordinal, that doesn't necessarily mean firms do not gain any utility at all.) □

Exercise 5.

Proof. (a) Game setup is:

$$P(Q) = a - Q = a - q_1 - q_2, \quad U_1(q_1, q_2) = (a - q_1 - q_2 - c_1)q_1$$

First we assume $P > 0$.

$$\Rightarrow \frac{\partial U_1}{\partial q_1} = a - c_1 - q_2 - 2q_1 = 0$$

$$\Rightarrow q_1 = \frac{a - c_1 - q_2}{2}$$

Similarly for q_2 we have:

$$q_2 = \frac{a - c_2 - q_1}{2}$$

In Nash equilibrium we must have $q_2^\star = \phi_1(q_2^\star)$. Hence:

$$a - c_1 - \frac{a - c_2 - q_1}{2} - 2q_1 = 0$$

We can do the same procedure for q_2 and obtain Nash equilibrium strategy profile:

$$\begin{cases} q_1 = \frac{a-2c_1+c_2}{3} \\ q_2 = \frac{a-2c_2+c_1}{3} \end{cases}$$

Now we check if assumption $P > 0$ holds:

$$q_1 + q_2 = \frac{2a - (c_1 + c_2)}{3}$$

$$\Rightarrow P = a - \frac{2a - (c_1 + c_2)}{3} = \frac{a + c_1 + c_2}{3}$$

That is greater than zero. Because $0 < c_i < a/2$ it is easy to show that $q_1, q_2 > 0$. From strategy profile it is clear that $\frac{\partial q_2}{\partial c_1} > 0$. Intuitively if marginal cost of firm 1 increases, utility of the firm from production decreases for a the same amount of production. Therefore it reduces its production and gives the opportunity to firm 2 to increase its production and its share of the market.

(b) If the assumption of the exercise holds, q_2 obtained in previous part will be negative. Hence it will not produce any amount of product. Therefore the first firm will maximize its utility given $q_2 = 0$.

$$\Rightarrow U_1(q_1, q_2) = (a - q_1 - c_1)q_1$$

$$\Rightarrow \frac{\partial U_1}{\partial q_1} = a - c_1 - 2q_1 = 0$$

$$\Rightarrow \begin{cases} q_1 = \frac{a-c_1}{2} \\ q_2 = 0 \end{cases}$$

The price will be:

$$P(Q) = a - Q = a - q_1 = \frac{a + c_1}{2}$$

That is positive. In this case we can say firm 1 is a monopolist.

□

Exercise 6.

Proof. First we examine some nodes from the end of game. The player who can replace 2 with 1 wins. The player in whose turn, the number on the board is 3 loses, because he must write 2 on the board and the other player wins. If we continue this process until 20, we find out some numbers have the property that if the player can replace it in his turn, he wins for sure. there are some other numbers that have reverse property, meaning that the player who has them loses. We call the first set of numbers *good numbers* and the second set of numbers *bad numbers*. We argue that the player who moves first has a winning strategy. We show 1000000 with $2k$. Consider the following game tree:

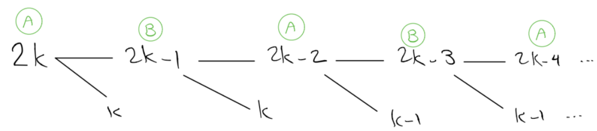

Figure 6: Exercise 6 extensive form game tree

There are two cases: 1. k is a *bad number*. In this case player A replaces $2k$ with it and player B loses and player A wins. 2. k is a *good number*. In this case player A doesn't write it on the board, because player B wins. So player A replaces $2k$ with $2k-1$. in the second round, player B doesn't replace $2k-1$ with k because we assumed k is a *good number* for player A. So he writes $2k-2$ on the board. Notice that player A again sees an even number on the board. so the same argument continues until player A sees number 2 on the board and replaces it with 1. Hence the player who moves first, has a winning strategy. $\square$