MWG_8.D.7.a: prove $\boxed{v_i \geq w_i}$

maximin: $$\underline{w_i} = \max_{\alpha_i}\Big[\overbrace{\min_{\alpha_{-i}} u_i(\alpha_i, \alpha_{-i})}^{f(\alpha_i)}\Big]$$

minimax: $$\underline{v_i} = \min_{\alpha_{-i}}\Big[\underbrace{\max_{\alpha_i} u_i(\alpha_i, \alpha_{-i})}_{f(\alpha_{-i})}\Big]$$

$$\underline{\arg\min_{\alpha_{-i}} u_i(\alpha_i, \alpha_{-i}) = \alpha_{-i}^*} \quad , \quad \underline{\arg\max_{\alpha_i} u_i(\alpha_i, \alpha_{-i}^*) = \alpha_i^*}$$

$$\rightarrow \max_{\alpha_i} u_i(\alpha_i, \alpha_{-i}) \geq \max_{\alpha_i} u_i(\alpha_i, \alpha_{-i}^*) = u_i(\alpha_i^*, \alpha_{-i}^*) = \underline{w_i}$$

$$\overset{\text{Min}}{\Longrightarrow} \min_{\alpha_{-i}}\Big[\max_{\alpha_i} u_i(\alpha_i, \alpha_{-i})\Big] = \underline{v_i} \geq u_i(\alpha_i^*, \alpha_{-i}^*) = \underline{w_i}$$

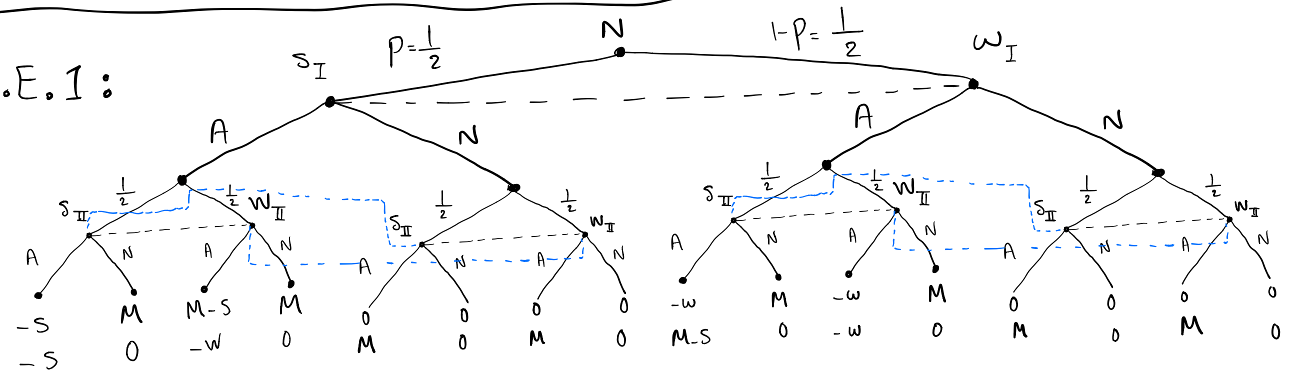
---

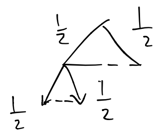
or: $$\arg\max_{\alpha_i} u_i(\alpha_i, \alpha_{-i}) = \underline{\alpha_i^\dagger} \quad , \quad \arg\min_{\alpha_{-i}} u_i(\alpha_i^\dagger, \alpha_{-i}) = \underline{\alpha_{-i}^\dagger}$$

$$\min_{\alpha_{-i}} u_i(\alpha_i, \alpha_{-i}) \leq \min_{\alpha_{-i}} u_i(\alpha_i^\dagger, \alpha_{-i}) = u_i(\alpha_i^\dagger, \alpha_{-i}^\dagger) = \underline{v_i}$$

$$\overset{\text{Max}}{\Longrightarrow} \max_{\alpha_i}\Big[\min_{\alpha_{-i}} u_i(\alpha_i, \alpha_{-i})\Big] = \underline{w_i} \leq u_i(\alpha_i^\dagger, \alpha_{-i}^\dagger) = \underline{v_i}$$

MWG_8.E.1:





also, another form can be written...

I strong \ II strong	A	N
A	$-S, -S$	$M, 0$
N	$0, M$	X $0, 0$

I strong \ II weak	A	N
A	$M-S, -W$	$M, 0$
N	$0, M$	$0, 0$ X

$\frac{1}{4}$

I weak \ II strong	A	N
A	$-W, M-S$	$M, 0$
N	$0, M$	$0, 0$ X

I weak \ II weak	A	N
A	$-W, -W$	$M, 0$
N	$0, M$	$0, 0$ X

II

I \ II	AA	AN	NA	NN
AA	X $\frac{M}{4} - \frac{S+W}{2}, \frac{M}{4} - \frac{S+W}{2}$ X	$\frac{M}{2} - \frac{S+W}{4}, \frac{M}{4} - \frac{S}{2}$	$\frac{3M}{4} - \frac{S+W}{4}, -\frac{W}{2}$ X	$M, 0$
AN	$\boxed{\frac{M}{4} - \frac{S}{2}, \frac{M}{2} - \frac{S+W}{4}}$	$\frac{M}{4} - \frac{S}{4}, \frac{M}{4} - \frac{S}{4}$	$\frac{M}{2} - \frac{S}{4}, \frac{M}{4} - \frac{W}{4}$ X	X $\frac{M}{2}, 0$
NA	X $-\frac{W}{2}, \frac{3M}{4} - \frac{S+W}{4}$	X $\frac{M}{4} - \frac{W}{4}, \frac{M}{2} - \frac{S}{4}$	$\boxed{\frac{M}{4} - \frac{W}{4}, \frac{M}{4} - \frac{W}{4}}$	X $\frac{M}{2}, 0$
NN	$0, M$	$0, \frac{M}{2}$ X	$0, \frac{M}{2}$ X	X $0, 0$ X

اول X ها رو حذف می‌کنیم.

AA : $\frac{M}{4} - \frac{S}{2} = 0 \Rightarrow M = 2S$

AN : $\frac{M}{2} - \frac{S+W}{4}, \frac{M}{4} - \frac{S}{4}, 0 \Rightarrow M = W, 2M = S + W$

$M = S$

$\left[\frac{2M - S - W}{4}\right]$

NA : $\frac{3M}{4} - \frac{S+W}{4}, \frac{M}{2} - \frac{S}{4}, \frac{M}{4} - \frac{W}{4}, 0 \Rightarrow M = W, 2M = S, 2M = W + S$

$M = S - W, 2M = S$

$M = W$

$\left[\frac{2M - S}{4}\right]$

(AN, AA) is Nash iff : $M \geq 2S$ & $M \geq w$, $2M \geq S + w$ redundant $\iff$ $M \geq \max\{2S, w\}$

(AA, AN)

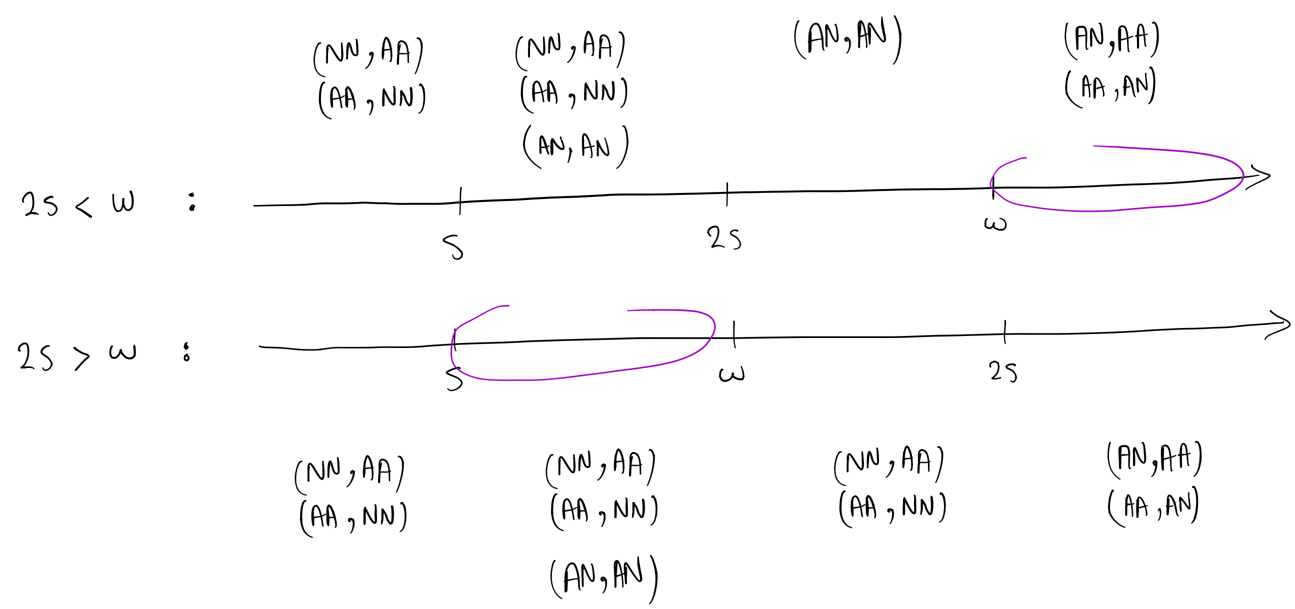
(NN, AA) is Nash iff : $M \leq 2S$

(AA, NN)

(AN, AN) is Nash iff : $M \leq w$, $M \geq S$ $\iff$ $S \leq M \leq w$

(NA, NA) is Nash iff : $2M \leq S \Rightarrow M \leq \frac{S}{2}$, $M \leq S - w \Rightarrow M \leq S - w < 0$, $M \geq w \Rightarrow M \geq w$ $\}$ $\times$ $\iff$ could not be Nash

Note that multiple cases are possible :



A2 : $S_i^1 = a_i$, $\forall t > 1 :$ $S_i^t(h^t) = \begin{cases} b_i & \text{if } (S_1^{t-1}, S_2^{t-1}) = (b_1, b_2) \text{ or } (c_1, c_2) \\ c_i & \text{if o.w.} \end{cases}$

if no one deviates : $u_1(\cdot) = 4 - \delta + \frac{2\delta^2}{1-\delta} = u_2(\cdot)$

now suppose player 1 deviates in first period :

b_1 : $u_1(b_1, \cdot) = 2 - \delta + \frac{2\delta^2}{1-\delta} \leq u_1(a_1, \cdot)$

c_1 : $u_1(c_1, \cdot) = 1 - \delta + \frac{2\delta^2}{1-\delta} \leq u_1(a_1, \cdot)$

now suppose player 1 deviates in second period :

a_1 : $u_1(\cdot, a_1, \cdot) = 1 - \delta + \frac{2\delta^2}{1-\delta} \leq -1 + \frac{2\delta}{1-\delta}$

$\Rightarrow \quad 2 - \delta \leq \frac{2\delta(1-\delta)}{1-\delta} \quad \Rightarrow \quad 2 - \delta \leq 2\delta \quad \Rightarrow \quad \underline{\delta \geq \frac{2}{3}}$

b_1 : $u_1(\cdot, b_1, \cdot) =$ similar

now suppose player 1 deviates in the third period :

a_1 : $3 - \delta + \frac{2\delta^2}{1-\delta} \leq \frac{2}{1-\delta} \quad \Rightarrow \quad 3 - \delta \leq \frac{2(1-\delta^2)}{1-\delta} = 2(1+\delta)$

$\Rightarrow \quad 3 - \delta \leq 2 + 2\delta \quad \Rightarrow \quad \underline{\delta \geq \frac{1}{3}}$

c_1 : $1 - \delta + \frac{2\delta^2}{1-\delta} \leq \frac{2}{1-\delta} \quad \Rightarrow \quad 1 - \delta \leq 2 + 2\delta \quad \Rightarrow \quad \underline{\delta \geq 0}$

$\Rightarrow$ if $\frac{2}{3} \leq \delta < 1$, no one will deviate and the strategy profile will be SPE.

A1 : define the minimax strategy as follows :

minmax payoff of player i : $\underline{\underline{v}_i = \min_{\alpha_{-i}} \Big[\overbrace{\max_{\alpha_i} u_i(\alpha_i, \alpha_{-i})}^{BR(\alpha_{-i})} \Big]}$

this is the least a player can receive. (worst of best responses)

$\underline{\text{the minmax strategy profile against } i : m^i_{-i} = \arg\min_{\alpha_{-i}} \Big[\max_{\alpha_i} u_i(\alpha_i, \alpha_{-i}) \Big]}$

and m^i_i is the strategy of player i that yields $u(m^i_i, m^i_{-i}) = \underline{v}_i$

suppose $\exists\, a' = (a'_1, \dots, a'_n)$ such that $u_i(a') = v_i > \underline{v}_i$ for every i.

$v_i > u_i(N) \geq \underline{v}_i$

Consider the following strategy:

* everyone plays a' as long as no one has deviated in previous periods.
if player j deviates, others play m^j_{-j} thereafter. *

Now let's check if player i has a profitable deviation.

obviously he will deviate to the strategy that maximizes his payoff: (in this period only)

$$\bar{v}_i = \max_{a_i} u_i(a_i, a'_{-i})$$

$\Rightarrow$ deviation payoff: $v_i + \delta v_i + \dots + \delta^{t-1} v_i + \delta^{t} \bar{v}_i + \delta^{t+1} \underline{v}_i + \delta^{t+2} \underline{v}_i + \dots$

deviation is not profitable only if: $\cancel{\dfrac{v_i}{1-\delta}} \geq \cancel{\dfrac{v_i}{1-\delta}} - \dfrac{\delta^t v_i}{1-\delta} + \delta^t \bar{v}_i + \dfrac{\delta^{t+1} \underline{v}_i}{1-\delta}$

$\Rightarrow \delta^t v_i \geq \delta^t (1-\delta) \bar{v}_i + \delta^{t+1} \underline{v}_i \Rightarrow v_i \geq (1-\delta)\bar{v}_i + \delta \underline{v}_i$

$\Rightarrow \delta \bar{v}_i - \delta \underline{v}_i \geq \bar{v}_i - v_i \Rightarrow \delta \geq \dfrac{\bar{v}_i - v_i}{\bar{v}_i - \underline{v}_i} = \underline{\delta}$

For $\delta > \underline{\delta}$, the defined strategy is SPE. Note that $\underline{\delta} \in (0,1)$ by definition.

A3:

(a) prisoner's dilemma $\longrightarrow$ (D,D) is the only Nash in every period,
since the game is finitely repeated $\longrightarrow$ SPE is (D,D) in every period

(b) straight result of Nash reversion pure strategy folk theorem.