Microeconomics 2

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Not For Distribution Beyond the Class

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Incomplete information version of Cournot example

Firm 1's costs are private information, while firm 2's are public. Nature determines the costs of firm 1 at the beginning of the game, with $\Pr(c_1 = c_H) = \theta \in (0, 1)$. Firm i's profit is

$$\pi_i(q_1, q_2; c_i) = [(a - q_1 - q_2) - c_i]q_i,$$

where c_i is firm i's cost. Assume $c_L, c_H, c_2 < a/2$. A strategy for player 2 is a quantity q_2. A strategy for player 1 is a function $q_1 : \{c_L, c_H\} \to \mathbb{R}_+$. For simplicity, write q_L for $q_1(c_L)$ and q_H for $q_1(c_H)$.

Note that for any strategy profile $((q_H, q_L), q_2)$, the associated outcome is

$$\theta \circ (q_H, q_2) + (1-\theta) \circ (q_L, q_2),$$

that is, with probability θ, the terminal node (q_H, q_2) is realized, and with probability $1-\theta$, the terminal node (q_L, q_2) is realized.
To find a Nash equilibrium, we must solve for three numbers q_L^*, q_H^*, and q_2^*.

Assume interior solution. We must have:

$$(q_H^*, q_L^*) = \arg\max_{q_H, q_L} \theta[(a - q_H - q_2^*) - c_H]q_H + (1 - \theta)[(a - q_L - q_2^*) - c_L]q_L.$$

This implies pointwise maximization, i.e.,

$$\begin{aligned} q_H^* &= \arg\max_{q_1} [(a - q_1 - q_2^*) - c_H]q_1 \\ &= \frac{1}{2}(a - q_2^* - c_H). \end{aligned}$$

and

$$q_L^* = \frac{1}{2}(a - q_2^* - c_L).$$

We must also have

$$
\begin{aligned}
q_2^* = &\arg\max_{q_2} \; \theta[(a - q_H^* - q_2 - c_2]q_2 \\
&\qquad\qquad + (1-\theta)[(a - q_L^* - q_2 - c_2]q_2 \\
= &\arg\max_{q_2} \; [(a - \theta q_H^* - (1-\theta)q_L^* - q_2) - c_2]q_2 \\
= &\frac{1}{2}\left(a - c_2 - \theta q_H^* - (1-\theta)q_L^*\right).
\end{aligned}
$$

Solving,

$$
\begin{aligned}
q_H^* &= \frac{a - 2c_H + c_2}{3} + \frac{1-\theta}{6}\left(c_H - c_L\right) \\
q_L^* &= \frac{a - 2c_L + c_2}{3} - \frac{\theta}{6}\left(c_H - c_L\right) \\
q_2^* &= \frac{a - 2c_2 + \theta c_H + \left(1-\theta\right) c_L}{3}
\end{aligned}
$$

Suppose payoffs of a two player two action game are given by one of these two bimatrices :

	H	T
H	$1,1$	$0,0$
T	$0,1$	$1,0$

	H	T
H	$1,x$	$0,1$
T	$0,0$	$1,1$

Example

	H	T
H	$1,1$	$0,0$
T	$0,1$	$1,0$

	H	T
H	$1,x$	$0,1$
T	$0,0$	$1,1$

Suppose first that $x = 0$. Either player II has dominant strategy to play H or a dominant strategy to play T. Suppose that II knows his own payoffs but player I thinks there is probability α that payoffs are given by the left matrix, probability $1-\alpha$ that they are given by the right matrix. Say that player II is of type 1 if payoffs are given by the left matrix, type 2 if payoffs are given by the right matrix.

Example

	H	T
H	$1,1$	$0,0$
T	$0,1$	$1,0$

	H	T
H	$1,x$	$0,1$
T	$0,0$	$1,1$

Clearly equilibrium must have: *II* plays H if *II* is of type 1, T if *II* is of type 2; *I* plays H if $\alpha > \frac{1}{2}$, T if $\alpha < \frac{1}{2}$.

Example

	H	T
H	$1, 1$	$0, 0$
T	$0, 1$	$1, 0$

	H	T
H	$1, x$	$0, 1$
T	$0, 0$	$1, 1$

But now suppose $x = 2$. Should player *II* still feel comfortable playing T if his type is t_2? The optimality of *II*'s action choice of T depends on his beliefs over *I*'s beliefs α. In particular, T is optimal only if *II* assigns probability of no more than $\frac{1}{2}$ to α being at least $\frac{1}{2}$.
This seems to lead us into an infinite regress, since now *I*'s beliefs about *II*'s beliefs about *I*'s beliefs become relevant! So, how to analyze such problems in general?

Definition [Harsanyi]

A *game of incomplete information* or *Bayesian game* is the collection $\{(A_i, T_i, p_i, u_i)_{i=1}^n\}$, where

- A_i is i's action space,
- T_i is i's type space,
- $p_i : T_i \to \Delta\left(\prod_{j\neq i} T_j\right)$ is i's subjective beliefs about the other players' types, given i's type and $\longrightarrow$ $p_i(t_i)$
- $u_i : \prod_j A_j \times \prod_j T_j \to \mathbb{R}$ is i's payoff function.

Definition

A strategy for i is

$$s_i : T_i \to A_i.$$

Let $s(t) \equiv (s_1(t_1), \ldots, s_n(t_n))$, etc.

Definition

The profile $(\hat{s}_1, \ldots, \hat{s}_n)$ is a *Bayesian* (or *Bayes-Nash*) *equilibrium* if, for all i and all $t_i \in T_i$,

$$E_{t_{-i}}[u_i(\hat{s}(t), t)] \geq E_{t_{-i}}[u_i(a_i, \hat{s}_{-i}(t_{-i}), t)], \quad \forall a_i \in A_i, \tag{1}$$

where the expectation over t_{-i} is taken with respect to the probability distribution $p_i(t_i)$.

If the type spaces are finite, then the probability i assigns to the vector $t_{-i} \in \prod_{j \neq i} \equiv T_{-i}$ when his type is t_i can be denoted $p_i(t_{-i}; t_i)$, and (1) can be written as

$$\sum_{t_{-i}} u_i(\hat{s}(t), t) p_i(t_{-i}; t_i) \geq \sum_{t_{-i}} u_i(a_i, \hat{s}_{-i}(t_{-i}) p_i(t_{-i}; t_i), \quad \forall a_i \in A_i.$$

Definition

The subjective beliefs are *consistent* or are said to satisfy the *Common Prior Assumption* (*CPA*) if there exists a probability distribution $p \in \Delta\left(\prod_i T_i\right)$ such that $p_i(t_i)$ is the probability distribution on T_{-i} conditional on t_i implied by p.

If the type spaces are finite, this is equivalent to

$$p_i(t_{-i}; t_i) = p(t_{-i}|t_i) = \frac{p\left(t\right)}{\sum_{t'_{-i}} p\left(t'_{-i}, t_i\right)}.$$

If beliefs are consistent, Bayesian game can be interpreted as having an initial move by nature, which selects $t \in T$ according to p.

Myerson, R. B. (1991)

- Is it possible to construct a situation for which there are no sets of types large enough to contain all the private information that players are supposed to have, so that no Bayesian game could represent this situation?
- Mertens and Zamir (1985) showed, under some technical assumptions, that no such counterexample to the generality of the Bayesian game model can be constructed, because a universal belief space can be constructed that is always big enough to serve as the set of types for each player.
- Although constructing an accurate model for any given situation may be extremely difficult, we can at least be confident that no one will ever be able to prove that some specific conflict situation cannot be described by any sufficiently complicated Bayesian game.

Myerson, R. B. (1991)

These results about the existence of universal belief spaces and universal Bayesian games constitute a theoretical proof that there should exist a Bayesian-game model that describes all the relevant structure of information and incentives in any given situation with incomplete information, so we can be confident that there is nothing intrinsically restrictive about the structure of the Bayesian game. We can be sure that no one will ever be able to take a real life conflict situation and prove that it would be impossible to describe by any Bayesian-game model.

Myerson, R. B. (1991)

- The Bayesian games are equivalent iff, for every possible type of every player, the two games impute probability and utility functions that are decision-theoretically equivalent.
- Using this equivalence criterion, we find that any Bayesian game with finite type sets is equivalent to a Bayesian game with consistent beliefs.
- Thus, consistency of beliefs and independence of types cannot be a problematic assumption in our analysis as long as we consider finite Bayesian games with general utility functions.

Myerson, R. B. (1991)

- There are several basic issues in a conflict situation about which players might have different information:
  - How many players are actually in the game?
  - What moves or actions are feasible for each player?
  - How will the outcome depend on the actions chosen by the players?
  - What are the players' preferences over the set of possible outcomes?

Myerson, R. B. (1991)

- Harsanyi (1967-68) argued that all of these issues can be modeled in a unified way.
  - Uncertainty about whether a player is "in the game" can be converted into uncertainty about his set of feasible actions, by allowing him only one feasible action ("nonparticipation") when he is supposed to be "out of the game."
  - Uncertainty about whether a particular action is feasible for player i can in turn be converted into uncertainty about how outcomes depend on actions, by saying that player i will get some very bad outcomes if he uses an action that is supposed to be infeasible.
    - Alternatively, whenever an action c_i, is supposed to be infeasible for player i, we can simply identify some feasible other action d_i, and suppose that the outcome from using c_i, in our game model is the same as for d_i. (so c_i, in our game model is reinterpreted as "Do c_i, if you can, otherwise do d_i").

Myerson, R. B. (1991)

- Harsanyi (1967-68) argued that all of these issues can be modeled in a unified way.
  - Uncertainty about how outcomes depend on actions and uncertainty about preferences over outcomes can be unified by modeling each player's utility as a function directly from the set of profiles of players' actions to the set of possible utility payoffs (representing the composition of the function from actions to outcomes with the function from outcomes to utility payoffs).
- Thus, we can model all the basic uncertainty in the game as uncertainty about how utility payoffs depend on profiles of actions. This uncertainty can be represented formally by introducing an unknown parameter θ into the utility functions.

$A_i = R_+$

$T_1 = \{t_1, t_2\}$ $T_2 = \{t_3\}$ P_1 trivial

Incomplete information version of Cournot example

The Cournot game is represented as a game of incomplete information, as follows: The action spaces are $A_i = \mathbb{R}_+$. Firm 1's type space is $T_1 = \{t_1', t_1''\}$ while firm 2's type space is a singleton $T_2 = \{t_2\}$. The belief mapping p_1 for firm 1 is trivial: both types assign probability one to the type t_2 (since T_2 is a singleton, there is no alternative), while the belief mapping for firm 2 is

$$p_2(t_2) = \theta \circ t_1' + (1 - \theta) \circ t_1'' \in \Delta(T_1).$$

Finally, payoffs are

$$u_1(q_1, q_2, t_1, t_2) = \begin{cases} [(a - q_1 - q_2) - c_H]q_1, & \text{if } t_1 = t_1', \\ [(a - q_1 - q_2) - c_L]q_1, & \text{if } t_1 = t_1'', \end{cases}$$

and

$$u_2(q_1, q_2, t_1, t_2) = [(a - q_1 - q_2) - c_2]q_2.$$

In this example, it is of course more natural to denote the type t_1' by c_H and t_1'' by c_L.

Example: Selten's horse

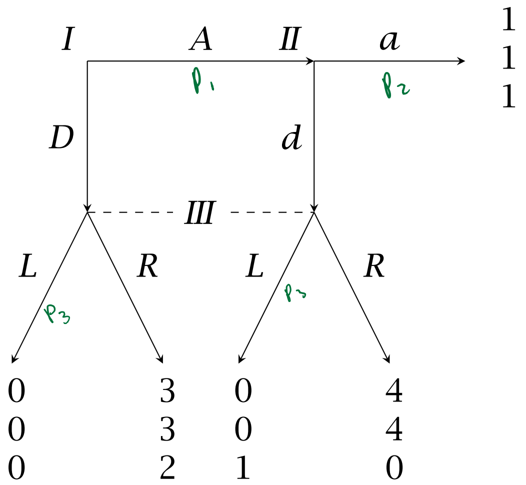


Figure: Selten's horse

Let (p_1, p_2, p_3) denote the mixed strategy profile where

$$\begin{aligned}
\Pr(I \text{ plays } A) &= p_1, \\
\Pr(II \text{ plays } a) &= p_2, \text{ and} \\
\Pr(III \text{ plays } L) &= p_3.
\end{aligned}$$

Consider the Nash equilibrium profile $(0, 1, 0)$ (i.e., DaR). This profile is subgame perfect, and yet player *II* is not playing *sequentially rationally*. It is also *not* trembling hand perfect: playing a is not optimal against any mixture close to DR.

The only trembling hand perfect equilibrium outcome is Aa. The set of Nash equilibria with this outcome is $\{(1, 1, p_3) : \frac{3}{4} \leq p_3 \leq 1\}$. In these equilibria, player III's information set is not reached, and so the profile cannot be used to obtain beliefs for III. However, each Nash equilibrium in the set is trembling hand perfect: Fix an equilibrium $(1, 1, p_3)$. Suppose first that $p_3 \in [\frac{3}{4}, 1)$ (so that $p_3 \neq 1!$) and consider the completely mixed profile

$$\begin{aligned} p_1^n &= 1 - \frac{1}{n}, \\ p_2^n &= 1 - \frac{2}{(n-1)}, \\ \text{and} \qquad p_3^n &= p_3. \end{aligned}$$

Note that $p_1^n, p_2^n \to 1$ as $n \to \infty$. Suppose $n \geq 4$. Easy to verify that both I and II are playing optimally against the mixed profile in $(1, 1, p_3)$. What about III? The probability that III is reached is

$$\frac{1}{n} + \frac{(n-1)}{n} \times \frac{2}{(n-1)} = 3/n,$$

and so the induced beliefs for III at his information set assign probability $\frac{1}{3}$ to the left node and $\frac{2}{3}$ to the right. Player III is therefore indifferent and so willing to randomize.

The same argument shows that $(1, 1, 1)$ is trembling hand perfect, using the trembles

$$\begin{aligned} p_1^n =& 1 - \frac{1}{n}, \\ p_2^n =& 1 - \frac{2}{(n-1)}, \\ \text{and} \qquad p_3^n =& 1 - \frac{1}{n}. \end{aligned}$$

Indeed, any sequence of trembles satisfying $p_1^n \to 1$, $p_2^n \to 1$, and $p_3^n \to 1$ will work, providing

$$\limsup_{n\to\infty} \frac{(1-p_1^n)}{(1-p_1^n p_2^n)} \leq \frac{1}{3}.$$

(It is not even necessary for $(1-p_1^n)/(1-p_1^n p_2^n)$ to have a well-defined limit.)

Definition

A *system of beliefs* μ in a finite extensive form is a specification of a probability distribution over the decision nodes in every information set, i.e., $\mu : X \to [0, 1]$ such that

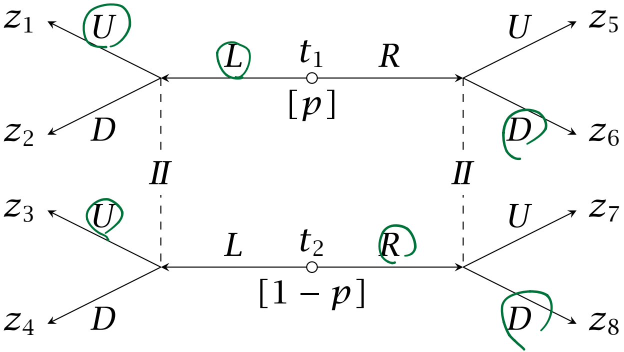
$$\sum_{x \in h} \mu(x) = 1, \qquad \forall h.$$

Note that $\mu \in \prod_{h \in \cup_i H_i} \Delta(h)$, a compact set. We can write μ as a vector $(\mu_h)_h$ specifying $\mu_h \in \Delta(h)$ for each $h \in \cup_i H_i$.

We interpret μ as describing player beliefs. In particular, if h is player i's information set, then $\mu_h \in \Delta(h)$ describes i's beliefs over the nodes in h.

Let $\mathbf{P}^b$ denote the probability distribution on the set of terminal nodes Z implied by the behavior profile b (and nature ρ).

Consider the profile (LR, UD) in the game displayed bellow:



The label $[p]$ indicates that nature chooses the node t_1 with probability p (so that $\rho(t_1) = p$ and $\rho(t_2) = 1 - p$). The induced distribution $\mathbf{P}^b$ on Z is $p \circ z_1 + (1 - p) \circ z_8$.

The expected payoff to player i is

$$E[u_i|b] \equiv \sum_{z\in Z} u_i(z)\mathbf{P}^b(z).$$

Let $Z(h) = \{z \in Z : \exists x \in \overset{h}{\cancel{H}}, x \prec z\}$. Let $\mathbf{P}^{\mu,b}(\cdot|h)$ denote the probability distribution on $Z(h)$ implied by $\mu_h \in \Delta(h)$ (the beliefs specified by μ over the nodes in h), the behavior profile b (interpreted as describing behavior at information set h and any that could be reached from h), and nature ρ (if there are any moves of nature following h). By setting $\mathbf{P}^{\mu,b}(z|h) = 0$ for all $z \notin Z(h)$, $\mathbf{P}^{\mu,b}(\cdot|h)$ can be interpreted as the distribution on Z, "conditional" on h being reached. Note that $\mathbf{P}^{\mu,b}(\cdot|h)$ only depends on μ through μ_h; it does not depend on $\mu_{h'}$ for any $h' \neq h$. Player i's expected payoff conditional on h is

$$\begin{aligned} E^{\mu,b}[u_i|h] &\equiv \sum_{z\in Z} u_i(z)\mathbf{P}^{\mu,b}(z|h). \\ &= \sum_{x\in h}\sum_{\{z:x\prec z\}} \mu(x)\mathbf{P}^b(z \mid x)u_i(z) \end{aligned}$$

Definition

A behavior strategy profile $\hat{b}$ in a finite extensive form is *sequentially rational at* $h \in H_i$, *given a system of beliefs* μ, if

$$E^{\mu,\hat{b}}[u_i \mid h] \geq E^{\mu,(b_i,\hat{b}_{-i})}[u_i \mid h],$$

for all b_i.

A behavior strategy profile $\hat{b}$ in an extensive form is *sequentially rational, given a system of beliefs* μ, if for all players i and all information sets $h \in H_i$, $\hat{b}$ is sequentially rational at h.

A behavior strategy profile $\hat{b}$ in an extensive form is *sequentially rational* if it is sequentially rational given some system of beliefs.

Definition

A *one-shot deviation* by player i from $\hat{b}$ is a strategy b_i' with the property that there exists a (necessarily unique) information set $h' \in H_i$ such that $\hat{b}_i(h) = b_i'(h)$ for all $h \neq h'$, $h \in H_i$, and $\hat{b}_i(h') \neq b_i'(h')$.

A one-shot deviation b_i' (from b, given a system of beliefs μ) is *profitable* if

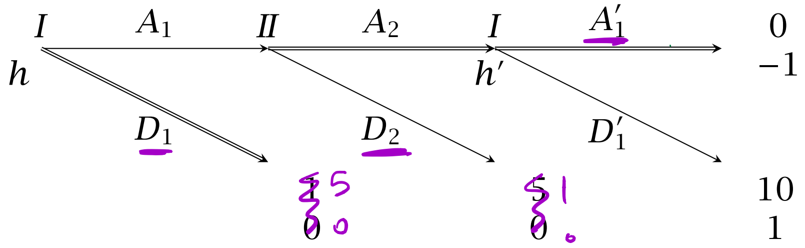
$$E^{\mu,(b_i',b_{-i})}[u_i \mid h'] > E^{\mu,b}[u_i \mid h'],$$

$\neq$

where $h' \in H_i$ is the information set for which $b_i'(h') \neq b_i(h')$.

Example

Consider the profile $((D_1, A'_1), A_2)$ in the game in figure bellow. Player *I* is not playing sequentially rationally at his first information set *h*, but does *not* have a profitable one-shot deviation there. Player *I* does have a profitable one-shot deviation at his second information set *h'*. Player *II* also has a profitable one-shot deviation.



Consider, now the profile $((D_1, A'_1), D_2)$. Now, player *I* is playing sequentially rationally at *h*, even though he still has a profitable one-shot deviation from the specified play at *h'*.

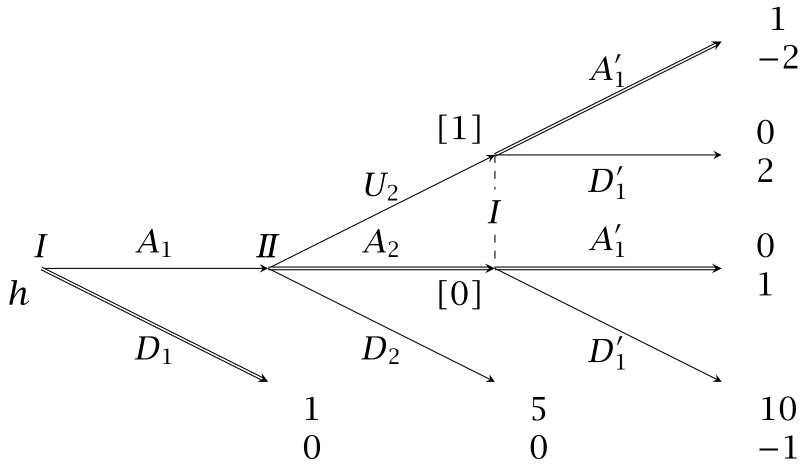
Lemma

If $\hat{b}$ is sequentially rational given μ, then there are no profitable one-shot deviations.

Without further restrictions on μ, a profile may fail to be sequentially rational and yet have no profitable one-shot deviations.

Example

Consider the profile$((D_1, A_1'), A_2)$ in the following game.



Player I is not playing sequentially rationally at his first information set h, but does *not* have a profitable one-shot deviation at *any* information set, given the system of beliefs indicated.

Remark

A game of perfect information has singleton information sets. In such a case, the system of beliefs is trivial, and sequential rationality is equivalent to subgame perfection.

In finite game of perfect information:
subgame perfect
=
sequentially rational
=
no one-shot deviation

The following is the first instance of what is often called the *one-shot deviation principle*

Theorem

A strategy profile b in a finite game of perfect information is subgame perfect if and only if it is sequentially rational. A strategy profile b in a finite game of perfect information is sequentially rational if and only if there are no profitable one-shot deviations.

Proof.

The equivalence of subgame perfection and sequential rationality for finite games of perfect information is immediate.
It is also immediate that a sequentially rational strategy profile has no profitable one-shot deviations.
The proof of the other direction is left as an exercise. □

Without some restrictions connecting beliefs to behavior, even Nash equilibria need *not* be sequentially rational. For any distribution $\mathbf{P} \in \Delta(Z)$, for any $x \in X$, define

$$\mathbf{P}(x) = \sum_{\{z \in Z : x \prec z\}} \mathbf{P}(z).$$

Definition

The information set h in a finite extensive form game is *reached with positive probability under* b, or is *on the path-of-play*, if

$$\mathbf{P}^b(h) = \sum_{x \in h} \mathbf{P}^b(x) > 0.$$

Theorem

?

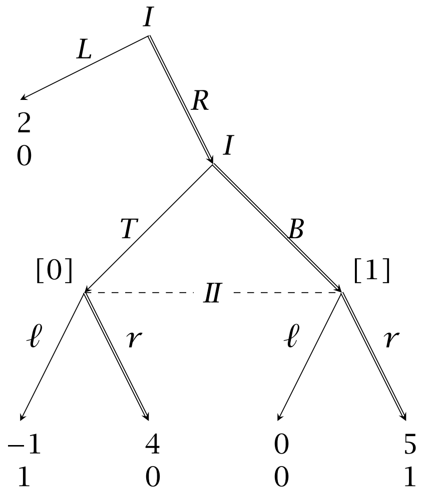
The behavior strategy profile b of a finite extensive form game is Nash if and only if it is sequentially rational at every information set on the path of play, given a system of beliefs μ obtained using Bayes' rule at those information sets, i.e., for all h on the path of play,

?

$$\mu(x) = \frac{\mathbf{P}^b(x)}{\mathbf{P}^b(h)} \quad \forall x \in h.$$

b is NE if it is SR on the path given μ from Bayes on the path.

b is weak PBE if it is so everywhere given μ from Bayes on the path.


I
L
R
2
0
I
T
B
[0]
[1]
II
ℓ
r
ℓ
r
−1
1
4
0
0
0
5
1

The label [p] indicates that the player owning that information set assigns probability p to the labeled node. The profile RBr (illustrated) is Nash and satisfies the conditions of the theorem.

In Theorem 4, sequential rationality is only imposed at information sets on the path of play. Strengthening this to all information sets yields:

Definition

A strategy profile b of a finite extensive form game is a *weak perfect Bayesian equilibrium* if there exists a system of beliefs μ such that

1. b is sequentially rational given μ, and
2. for all h on the path of play,

$$\mu(x) = \frac{\mathbf{P}^b(x)}{\mathbf{P}^b(h)} \quad \forall x \in h.$$

Note that a strategy profile b is a weak perfect Bayesian equilibrium, if and only if, it is a Nash equilibrium that is sequentially rational.
Using Bayes' rule "where possible" yields something even stronger. The phrase "where possible" is meant to suggest that we apply Bayes' rule in a conditional manner.

We first need (recall that information sets are not partially ordered by precedence):

Definition

The information set h *follows* h' if for all $x \in h$, there exists $x' \in h'$ such that $x' \prec x$.
An information set h (following h') is *reached with positive probability from* h' *under* (μ, b) if

$$\mathbf{P}^{\mu,b}(h \mid h') = \sum_{x \in h} \mathbf{P}^{\mu,b}(x \mid h') > 0.$$

Note that for any two information sets owned by the same player, $h.h' \in H_i$, h follows h' in the sense of Definition 6 if, and only if, $h' \prec^* h$.

h'

h

Definition

A strategy profile b of a finite extensive form game is an *almost perfect Bayesian equilibrium* if there exists a system of beliefs μ such that

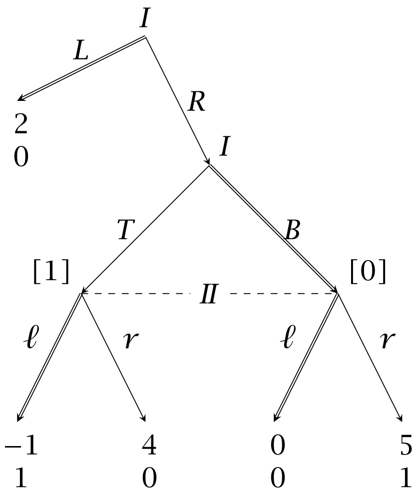
1. b is sequentially rational given μ, and
2. for any information set h' and following information set h reached with positive probability from h' under (μ, b),

$$\mu(x) = \frac{\mathbf{P}^{\mu,b}(x \mid h')}{\mathbf{P}^{\mu,b}(h \mid h')} \quad \forall x \in h.$$

Theorem

Every almost perfect Bayesian equilibrium is subgame perfect.

Example



Theorem

The behavior strategy profile b of a finite extensive form game is Nash if and only if it is sequentially rational at every information set on the path of play, given a system of beliefs μ obtained using Bayes' rule at those information sets, i.e., for all h on the path of play,

$$\mu(x) = \frac{\mathbf{P}^b(x)}{\mathbf{P}^b(h)} \quad \forall x \in h.$$

The profile $LB\ell$ (illustrated) is weak perfect Bayesian, but not almost perfect Bayesian. Note that $LT\ell$ is *not* weak perfect Bayesian. The only subgame perfect eq is RBr. ? 150 ?

Definition

A strategy profile b of a finite extensive form game is a *weak perfect Bayesian equilibrium* if there exists a system of beliefs μ such that

1. b is sequentially rational given μ, and
2. for all h on the path of play,

$$\mu(x) = \frac{\mathbf{P}^b(x)}{\mathbf{P}^b(h)} \quad \forall x \in h.$$

Definition

A strategy profile b of a finite extensive form game is an *almost perfect Bayesian equilibrium* if there exists a system of beliefs μ such that

1. b is sequentially rational given μ, and
2. for any information set h' and following information set h reached with positive probability from h' under (μ, b),

$$\mu(x) = \frac{\mathbf{P}^{\mu,b}(x \mid h')}{\mathbf{P}^{\mu,b}(h \mid h')} \quad \forall x \in h.$$

We are not yet at perfect Bayesian equilibrium, because we still need to address the phenomenon illustrated by MWG Example 9.C.4. While it is straightforward to directly deal with the example, the conditions that deal with the general phenomenon are complicated and hard to interpret. It is rare for the complicated conditions to be used in practice.

The term *perfect Bayesian equilibrium* (or *PBE*) is often used in applications to describe the collections of restrictions on the system of beliefs that “do the right/obvious thing,” and as such is one of the more abused notions in the literature.

Sender (informed player) types $t \in T \subset \mathbb{R}$. T may be finite. Probability distribution $\rho \in \Delta(T)$.
Sender chooses $m \in M \subset \mathbb{R}$. M may be finite.
Responder chooses $r \in R \subset \mathbb{R}$. R may be finite.
Payoffs: $u(m, r, t)$ for sender and $v(m, r, t)$ for responder.
Strategy for sender, $\tau : T \to M$.
Strategy for responder, $\sigma : M \to R$.

Definition

The pure strategy profile $(\hat{\tau}, \hat{\sigma})$ is a *perfect Bayesian equilibrium* of the signaling game if

1. for all $t \in T$,
$$\hat{\tau}(t) \in \arg\max_{m \in M} \; u(m, \hat{\sigma}(m), t),$$
2. for all m, there exists some $\mu \in \Delta(T)$ such that
$$\hat{\sigma}(m) \in \arg\max_{r \in R} \; E^{\mu}[v(m, r, t)],$$
where E^{μ} denotes expectation with respect to μ, and
3. for $m \in \hat{\tau}(T)$, μ in part 2 is given by
$$\mu(t) = \rho\{t \mid m = \hat{\tau}(t)\}.$$

Perfect Bayesian Equilibrium*

Since the different information sets for player II are not ordered by $\prec^*$, consistency places no restrictions on beliefs at different information sets of player II. This implies the following result:

Theorem

Suppose T, M, and R are finite. A profile is a perfect Bayesian equilibrium if, and only if, it is a sequential equilibrium.

Example

(bq, fr) is a separating eq.

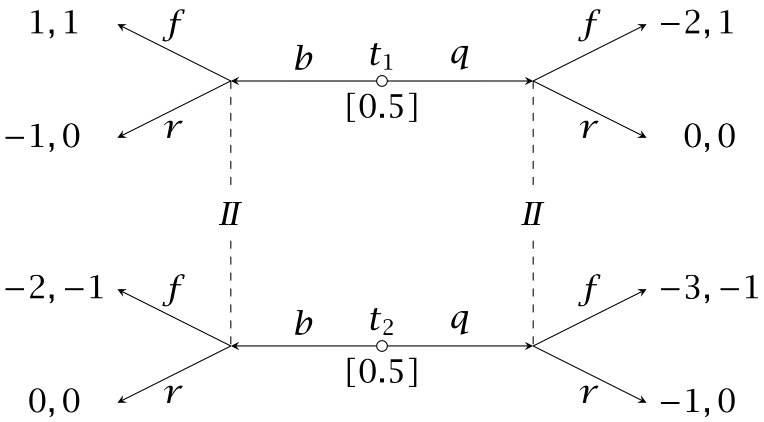


Figure: A signaling game

Example: Beer-Quiche

(bb, rf) and (qq, fr) are both pooling eq.

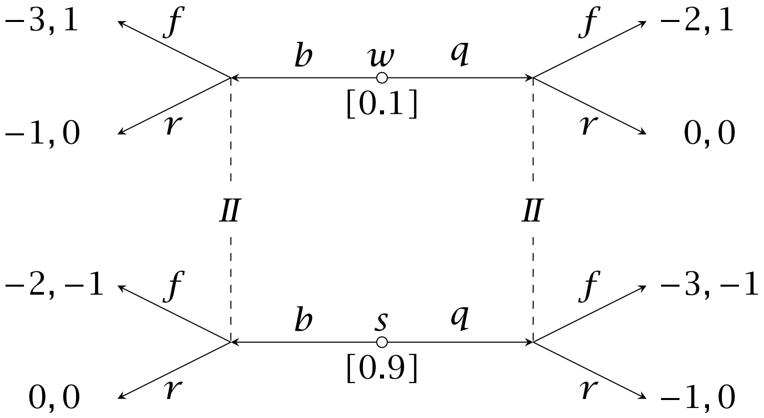


Figure: The Beer-Quiche game.

The eq in which the types pool on q is often argued to be unintuitive: Would the w type ever "rationally" deviate to b. In this pooling eq, w receives 0, and this is strictly larger than his payoff from b *no matter* how II responds. On the other hand, if by deviating to B, s can "signal" that he is indeed s, he is strictly better off, since II's best response is r, yielding payoff of 0. This is an example of the *intuitive criterion*, or of the *test of equilibrium domination*.

Intuitive Criterion*

Let $BR(T', m)$ be the set of best replies to m of the responder, when the beliefs have support in T', i.e.,

$$\begin{aligned} BR(T', m) &= \{r \in R : \exists \mu \in \Delta(T'), r \in \arg\max_{r' \in R} E^{\mu}[v(m, r', t)]\} \\ &= \bigcup_{\mu \in \Delta(T')} \arg\max_{r' \in R} E^{\mu}[v(m, r', t)]. \end{aligned}$$

Intuitive Criterion*

Definition

Fix a perfect Bayesian equilibrium $(\hat{\tau}, \hat{\sigma})$, and let $\hat{u}(t) = u(\hat{\tau}(t), \hat{\sigma}(\hat{\tau}(t)), t)$. Define $D(m) \subset T$ as the set of types satisfying

$$\hat{u}(t) > \max_{r \in BR(T,m)} u(m, r, t).$$

The equilibrium $(\hat{\tau}, \hat{\sigma})$ *fails the intuitive criterion* if there exists m' (necessarily not in $\hat{\tau}(T)$, i.e., an unsent message) and a type t' (necessarily not in $D(m)$) such that

$$\hat{u}(t') < \min_{r \in BR(T \setminus D(m'), m')} u(m', r, t').$$

Job Market Signaling

Worker with private ability $\theta \in \Theta \subset \mathbb{R}$.
Worker can signal ability through choice of level of education, $e \in \mathbb{R}_+$.
Worker utility

$$w - c(e, \theta),$$

w is wage, and c is disutility of education. Assume c is $\mathcal{C}^2$ and satisfies *single-crossing*:

$$\frac{\partial^2 c(e, \theta)}{\partial e \partial \theta} < 0.$$

Also assume $c(e, \theta) \geq 0$, $c_e(e, \theta) \equiv \partial c(e, \theta)/\partial e \geq 0$, $c_e(0, \theta) = 0$, $c_{ee}(e, \theta) > 0$, and $\lim_{e \to \infty} c_e(e, \theta) = \infty$.

Two identical firms competing for worker. Each firm values worker of type θ with education e at $f(e,\theta)$. In any discretization of the game, in any almost perfect Bayesian equilibrium, after any e, firms have identical beliefs $\mu \in \Delta(\Theta)$ about worker ability. Consequently, the two firms are effectively playing a sealed bid common value first price auction, and so both firms bid their value $E_\mu f(e,\theta)$. To model as a game, replace the two firms with a single uninformed receiver (the "market") with payoff

$$-(f(e,\theta) - w)^2.$$

Strategy for worker, $e : \Theta \to \mathbb{R}_+$.
Strategy for "market", $w : \mathbb{R}_+ \to \mathbb{R}_+$.
Assume f is $\mathcal{C}^2$. Assume $f(e,\theta) \geq 0$, $f_e(e,\theta) \equiv \partial f(e,\theta)/\partial e \geq 0$, $f_\theta(e,\theta) > 0$, $f_{ee}(e,\theta) \leq 0$, and $f_{e\theta}(e,\theta) \geq 0$.

1. **Unproductive education**. When f is independent of e, we can interpret θ as the productivity of the worker, and so assume $f(e, \theta) = \theta$.
2. **Productive education.** $f_e(e, \theta) > 0$.

If market believes worker has ability $\hat{\theta}$, firm pays wage $f(e, \hat{\theta})$. The result is a signaling game as described in Section **??**, and so we can apply equilibrium notion of perfect Bayesian as defined there.

Full Information

If firm *knows* worker has ability θ, worker chooses e to maximize

$$f(e,\theta) - c(e,\theta). \tag{2}$$

For each θ there is a unique e^* maximizing (2). That is,

$$e^*(\theta) = \arg\max_{e \geq 0} f(e,\theta) - c(e,\theta).$$

Assuming $f_e(0,\theta) > 0$ (together with the assumption on c above) is sufficient to imply that $e^*(\theta)$ is interior for all θ and so

$$\frac{de^*}{d\theta} = -\frac{f_{e\theta}(e,\theta) - c_{e\theta}(e,\theta)}{f_{ee}(e,\theta) - c_{ee}(e,\theta)} > 0.$$

Incomplete Information

Define

$$U(\theta, \hat{\theta}, e) \equiv f(e, \hat{\theta}) - c(e, \theta).$$

Note that

$$e^*(\theta) = \arg\max_{e \geq 0} U(\theta, \theta, e). \tag{3}$$

We are first interested in *separating* perfect Bayesian equilibria.
The outcome associated with a profile Suppose $(\hat{e}, \hat{w})$ is

$$(\hat{e}(\theta), \hat{w}(\hat{e}(\theta)))_{\theta \in \Theta}.$$

If $e' = \hat{e}(\theta')$ for some $\theta' \in \Theta$, then $\hat{w}(e') = f(e', (\hat{e})^{-1}(e')) = f(e', \theta')$, and so the payoff to the worker of type θ is

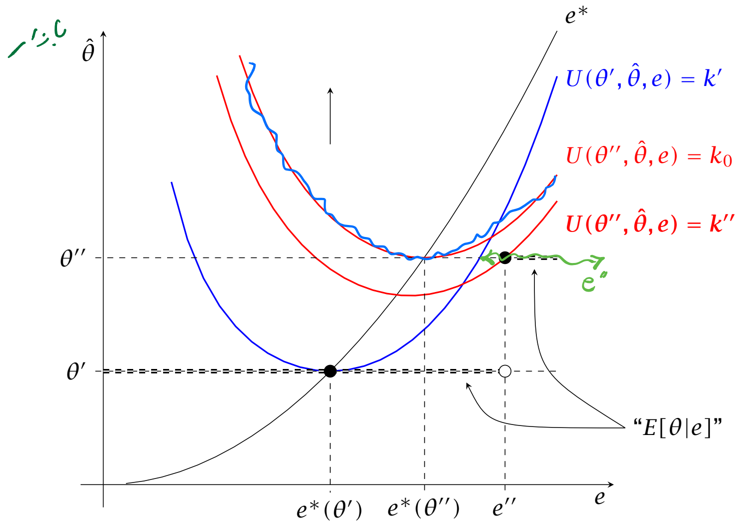
$$\hat{w}(e') - c(e', \theta) = f(e', \theta') - c(e', \theta) = U(\theta, \theta', e').$$

A separating (one-to-one) strategy $\hat{e}$ is *incentive compatible* if no type strictly benefits from mimicking another type, i.e.,

$$U(\theta', \theta', \hat{e}(\theta')) \geq U(\theta', \theta'', \hat{e}(\theta'')), \qquad \forall \theta', \theta'' \in \Theta. \tag{4}$$

بازار   بازار

Figure **??** illustrates the case $\Theta = \{\theta', \theta''\}$.



e^*
$\hat{\theta}$
$U(\theta', \hat{\theta}, e) = k'$
$U(\theta'', \hat{\theta}, e) = k_0$
$U(\theta'', \hat{\theta}, e) = k''$
θ''
θ'
e''
"$E[\theta|e]$"
$e^*(\theta')$
$e^*(\theta'')$
e''
e

Indifference curves in $\hat{\theta} - e$ space. Space of types is $\Theta = \{\theta', \theta''\}$. Note $k' = U(\theta', \theta', e^*(\theta'))$, $k'' = U(\theta'', \theta'', e'')$, and $k_0 = U(\theta'', \theta'', e^*(\theta''))$, and that incentive compatibility is satisfied at the indicated points: $U(\theta'', \theta'', e'') \geq U(\theta'', \theta', e^*(\theta'))$ and $U(\theta', \theta', e^*(\theta')) \geq U(\theta', \theta'', e'')$. For any $e < e''$, firms believe $\theta = \theta'$, and for any $e \geq e''$, firms believe $\theta = \theta''$.
The figures in this subsection are drawn using the production function $f(e, \theta) = e\theta$ and cost function $c(e, \theta) = e^5/(5\theta)$, so that $U(\theta, \hat{\theta}, e) = e\hat{\theta} - (e^5)/(5\theta)$. The set of possible θ's is $\{1, 2\}$, with full information optimal educations of 1 and $\sqrt{2}$.

$$\left.\begin{aligned} f(e,\theta) &= e\theta \\ c(e,\theta) &= \frac{e^5}{5\theta} \end{aligned}\right\} \Rightarrow U(\theta, \hat{\theta}, e) = e\hat{\theta} - \frac{e^5}{5\theta}$$

$\Theta = \{1, 2\}$

Definition

The separating strategy profile $(\hat{e}, \hat{w})$ is a *perfect Bayesian equilibrium* if

1. $\hat{e}$ satisfies (4), i.e., is incentive compatible,
2. $\hat{w}(e) = f(e, \hat{e}^{-1}(e))$ for all $e \in \hat{e}(\Theta)$,
3. for all $e \in \mathbb{R}_+$ and all $\theta \in \Theta$,

$$\hat{w}(e) - c(e, \theta) \leq U(\theta, \theta, \hat{e}(\theta)),$$

   and
4. $\hat{w}$ is sequentially rational, i.e., for all $e \in \mathbb{R}_+$, there is some $\mu \in \Delta(\Theta)$ such that

$$\hat{w}(e) = E_\mu f(e, \theta).$$

For $e \in \hat{e}(\Theta)$, μ is of course given by the belief that assigns probability one to the type for which $e = \hat{e}(\theta)$, i.e. $\hat{\theta} = \hat{e}^{-1}(e)$). Sequential rationality restricts wages for $w \notin \hat{e}(\Theta)$.
Note that the Intermediate Value Theorem implies that for all $e \in \mathbb{R}_+$ and all $\mu \in \Delta(\Theta)$, there exists a unique $\hat{\theta} \in conv(\Theta)$ such that

$$f(e, \hat{\theta}) = E_\mu f(e, \theta).$$

Thus, condition 4 in Definition of eq. can be replaced by: for all $e \in \mathbb{R}_+$, there is some $\hat{\theta} \in conv(\Theta)$ such that

$$\hat{w}(e) = f(e, \hat{\theta}).$$

In particular, if $\Theta = \{\theta', \theta''\}$, then $\hat{\theta} \in [\theta', \theta'']$. (This is useful in interpreting the figures in this section.)

Let $\underline{\theta} = \min \Theta$. Sequential rationality implies

$$\hat{e}(\underline{\theta}) = e^*(\underline{\theta}). \tag{5}$$

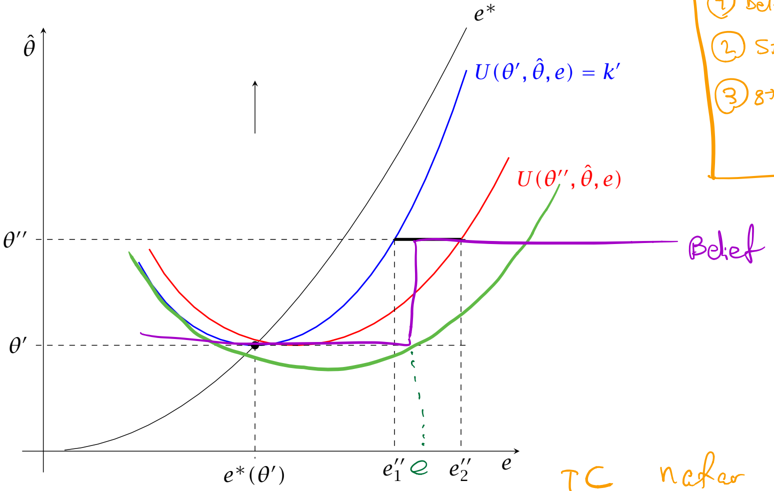
Separating Perfect Bayesian Equilibrium Outcomes

$U' = \frac{4}{5}$

$\begin{cases} U(1,1,e_1^*) \geq U(1,2,e); \\ 2e_1'' - \frac{e_1''^2}{10} = \frac{4}{5} \Rightarrow e_1'' = \end{cases}$

$\begin{cases} U(2,2,e_2'') = 2e_2'' - \frac{e_2''^2}{10} \\ U(2,1,e^*) = e^*\theta - \frac{e^{*2}}{10} \end{cases}$

$U(2,2,e_2'') \geq U(2,1,e^*)$

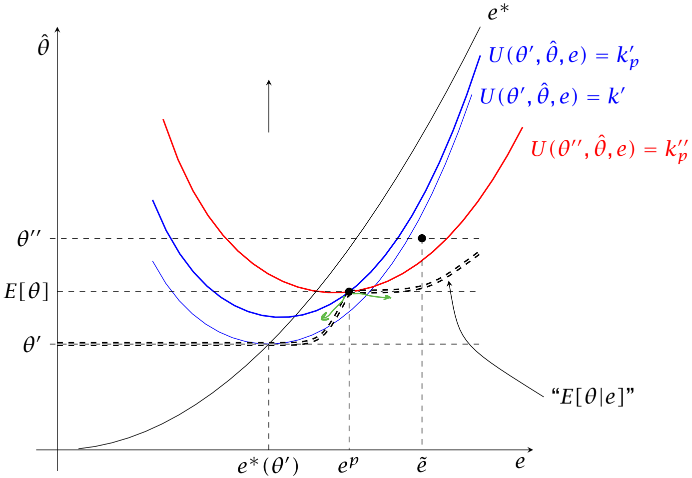


① Belief
② Strategy of firms
③ strategy of worker

IC nafar dovom

1. low type $\rightarrow$ first best
2. wasati
3. akhari

Separating equilibria when space of types is $\Theta = \{\theta', \theta''\}$. The set of separating perfect Bayesian equilibrium outcomes is given by $\{((e^*(\theta'), f(e^*(\theta'), \theta')),\ (e'', f(e'', \theta''))) : e'' \in [e_1'', e_2'']\}$. Note that θ'' cannot receive a lower payoff than $\max_e U(\theta'', \theta', e)$.


$\hat{\theta}$
e^*
$U(\theta', \hat{\theta}, e) = k'_p$
$U(\theta', \hat{\theta}, e) = k'$
$U(\theta'', \hat{\theta}, e) = k''_p$
θ''
$E[\theta]$
θ'
"$E[\theta|e]$"
$e^*(\theta')$
e^p
$\tilde{e}$
e

$q_2 = \frac{1}{2}$: $\hat{\theta} = 3$

q_2 : o.w. $\hat{\theta} = 4$

اونی که first best بازی می‌کنه O.w.

اونی که نه :

A pooling outcome at $e = e^p$. $k_p' = U(\theta', E\theta, e^p)$, $k_p'' = U(\theta'', E\theta, e^p)$. Note that $E[\theta|e]$, firms' beliefs after potential deviating e's must lie below the θ' and θ'' indifference curves indexed by k_p' and k_p'', respectively.

solution (3)

Moral Hazard

- Information asymmetry occurs after contract has been signed
- Natural interpretation: hidden action
- Applications:
  - owner — manager
  - manager — worker
  - insuree — insured
  - buyer — seller
  - advisee — advisor (patient — doctor, client — lawyer, ...)
  - teacher — students
  - tax agency — individuals
  - voters — politicians
  - landlord — tenant

Optimal Risk Sharing

- 2 individuals with utility functions $u(c_1)$ and $v(c_2)$
- N states of nature $\longrightarrow j$
- Wealth in each state: $x_{i,j}$
- Aggregate wealth: $X_j = x_{1,j} + x_{2,j}$
- Resource constraint: $X_j = c_{1,j} + c_{2,j}$

Optimal Risk Sharing

- The Pareto Optimal allocations solve:

$$\max_{\{c_{1j},c_{2j}\}_{j=1}^{N}} \sum_{j=1}^{N} \pi_j u(c_{1j})$$

s.t.

$$\sum_{j=1}^{N} \pi_j v(c_{2j}) \geq \bar{u},$$

$$X_j = c_{1j} + c_{2j} \forall j.$$

Optimal Risk Sharing

- Lagrangian:

$$\mathcal{L} = \sum_{j=1}^{N} \pi_j u(c_{1j}) + \lambda(\sum_{j=1}^{N} \pi_j v(X_j - c_{1j}) - \bar{u}).$$

- FOCs:

$$\pi_j u'(c_{1j}) = \lambda \pi_j v'(X_j - c_{1j}) \quad \forall j,$$

$$\Rightarrow \frac{v'(X_j - c_{1j})}{u'(c_{1j})} = \frac{1}{\lambda} \quad \forall j.$$

- Since λ is not a function of j, we obtain:

$$\frac{v'(X_j - c_{1j})}{u'(c_{1j})} = \frac{v'(X_{\tilde{j}} - c_{1\tilde{j}})}{u'(c_{1\tilde{j}})} \quad \forall j, \tilde{j}. \qquad (\textit{Borch Rule})$$

Optimal Risk Sharing

- Ratio of marginal utilities is constant across states (perfect risk sharing)
  - If one party is risk neutral and the other is risk averse, then the risk neutral party fully insures the risk averse
  - If there's no aggregate uncertainty ($X_1 = X_2 = ... = X_N$) and at least one party is risk averse, then the consumption of both parties is constant
- Main testable implication:
  - Idiosyncratic shocks should not affect consumption, only aggregate shocks should matter
- General conclusion: there is some consumption smoothing, but it's far from complete

Basic Model

- 2 outcomes: x_L and x_H
- Continuum of efforts $e \in \mathcal{R}_+$
- Effort determines the probability of each outcome:

$$Pr(x = x_H | e) = p(e),$$

$$Pr(x = x_L | e) = 1 - p(e),$$

$$\text{where} \quad p' > 0 > p'', p(0) = 0, p(\infty) = 1.$$

- Preferences:

$$\text{Principal:} \quad v(x - w), v' > 0 \geq v'',$$

$$\text{Agent:} \quad u(w) - e, u' > 0 > u''.$$

First Best

- Suppose the principal can observe e.
- The principal's program is

$$\max_{e,w_L,w_H} p(e)v(x_H - w_H) + [1 - p(e)]v(x_L - w_L)$$

s.t.

$$p(e)u(w_H) + [1 - p(e)]u(w_L) - e \geq 0. \quad \text{(IR)} \longrightarrow \text{individual rationality}$$

- FOCs with respect to w_L and w_H:

$$\frac{v'(x_H - w_H)}{u'(w_H)} = \lambda,$$

$$\frac{v'(x_L - w_L)}{u'(w_L)} = \lambda,$$

First Best

- Combining both conditions, we obtain:

$$\frac{v'(x_H - w_H)}{u'(w_H)} = \frac{v'(x_L - w_L)}{u'(w_L)}. \quad \text{(Borch \quad Rule)}$$

- Example: risk-neutral principal $\longrightarrow v' = 1$
  - $w_H = w_L$: principal fully insures the agent.

Second Best

- Program when e is not observed by the Principal:

$$\max_{e,w_L,w_H} p(e)v(x_H - w_H) + [1 - p(e)]v(x_L - w_L)$$

s.t.

$$e \in \arg\max_{\tilde{e}} p(\tilde{e})u(w_H) + [1 - p(\tilde{e})]u(w_L) - \tilde{e}, \quad \text{(IC)}$$

$$p(e)u(w_H) + [1 - p(e)]u(w_L) - e \geq 0. \quad \text{(IR)}$$

- (IC) implicitly assumes that the agent chooses the effort preferred by the principal when indifferent

Second Best

- First- and second-order conditions from (IC):

$$p'(e)[u(w_H) - u(w_L)] = 1 \quad \text{(IC FOC)},$$

$$p''(e)[u(w_H) - u(w_L)] < 0 \quad \text{(IC SOC)}.$$

- It must be the case that $w_H > w_L$ (otherwise, the solution is $e = 0$). Therefore, (IC SOC) is always satisfied.

Second Best

- Because the second-order condition is (globally) satisfied in this case, we can replace (IC) by the first-order condition (IC FOC) — we'll return to this later:

$$\begin{aligned}\mathcal{L} = & p(e)v(x_H - w_H) + [1 - p(e)]v(x_L - w_L) \\ & + \lambda\big(p(e)u(w_H) + [1 - p(e)]u(w_L) - e\big) \\ & + \mu\big(p'(e)[u(w_H) - u(w_L)] - 1\big)\end{aligned}$$

- FOC with respect to w_H:

$$-p(e)v'(x_H - w_H) + \lambda p(e)u'(w_H) + \mu p'(e)u'(w_H) = 0$$

$$\Rightarrow \frac{v'(x_H - w_H)}{u'(w_H)} = \lambda + \mu \frac{p'}{p(e)}. \tag{6}$$

- FOC with respect to w_L:

$$\frac{v'(x_L - w_L)}{u'(w_L)} = \lambda - \mu \frac{p'}{1 - p(e)}. \tag{7}$$

- Claim: $\mu > 0$ as long as $e = 0$ is suboptimal.
- Whenever the optimal effort e is interior, it must satisfy the following FOC:

$$\begin{gathered} p'(e)[v(x_H - w_H) - v(x_L - w_L)] \\ + \lambda[p'(e)(u(w_H) - u(w_L)) - 1] \\ + \mu[p''(e)(u(w_H) - u(w_L)] = 0. \end{gathered}$$

- Recall (IC FOC): $p'(e)(u(w_H) - u(w_L)) = 1$.
- Suppose, in order to obtain a contradiction, that $\mu = 0$. Then, the previous equation becomes:

$$p'(e)[v(x_H - w_H) - v(x_L - w_L)] = 0$$

- which, because $p'(e) > 0$, yields: $x_H - w_H = x_L - w_L$.
- Substituting in equations (6) and (7), we obtain: $w_H = w_L$. Hence, $e = 0$.
- Thus, whenever the optimal effort is positive, we have $\mu > 0$.

Second Best

- Substituting $\mu > 0$ in the FOCs with respect to w_L and w_H, we obtain:

$$\frac{v'(x_H - w_H)}{u'(w_H)} > \frac{v'(x_L - w_L)}{u'(w_L)}.$$

  (incomplete risk-sharing)
- Example: risk-neutral principal
  - $w_L < w_H$: agent gets "rewarded" for high outcomes and "punished" for low outcomes

A More General Model: Grossman-Hart

- Main idea: split the problem in two steps:
  - Step 1: find the least costly way to implement each given action,
  - Step 2: pick the action that maximizes the difference between benefits and costs.

2×2 **model:**

- For simplicity, suppose there are 2 effort levels e_L and e_H and 2 states x_L and x_H
- $Pr(x = x_H | e_H) = p_H, Pr(x = x_H | e_L) = p_L$

- In the first step, for each $i \in \{L, H\}$, the principal solves:

$$\begin{aligned} C(e_i) &= \min_{w(.)} p_i w(x_H) + (1 - p_i) w(x_L) \\ \text{s.t.} \quad & i \in \arg\max_{j} p_j u(w(x_H)) + (1 - p_j) u(w(x_L)) - c(e_j), \quad \text{(IC)} \\ & p_i u(w(x_H)) + (1 - p_i) u(w(x_L)) - c(e_i) \geq 0. \quad \text{(IR)} \end{aligned}$$

- In the second step, the principal chooses the effort that maximizes his payoff

$$p_i x_H + (1 - p_i) x_L - C(e_i).$$

- Since u is strictly increasing, we can define $h = u^{-1}$ and write the first-step program in terms of the agent's indirect utility instead of wages:

$$\min_{v_L, v_H} p_i h(v_H) + (1 - p_i) h(v_L)$$

$$\text{s.t.} \quad p_i v_H + (1 - p_i) v_L - c(e_i) \geq p_j v_H + (1 - p_j) v_L - c(e_j), j \in \{L, H\}, \quad \text{(IC)}$$

$$p_i v_H + (1 - p_i) v_L - c(e_i) \geq 0. \quad \text{(IR)}$$

- This is a simple convex programming problem: minimize a convex function subject to linear constraints

- Solution of the second-step program exists and is unique.
- (IR) binds.
- FB = SB if either: (i) shirking is first-best optimal, or (ii) both principal and agent are risk-neutral (make the agent the residual claimant)

- We want to show that (IC) binds. Consider the relax problem:

$$\min_{v_L, v_H} p_H h(v_H) + (1 - p_H) h(v_L)$$
$$\text{s.t. } p_H v_H + (1 - p_H) v_L - c(e_H) \geq 0. \quad \text{(IR)}$$

- Suppose $(\tilde{v}_H, \tilde{v}_L)$ is a solution to the relax problem.
- Define $v_H^\dagger = \tilde{v}_H + \frac{\epsilon}{p_H}$ and $v_L^\dagger = \tilde{v}_L - \frac{\epsilon}{1 - p_H}$.
- Note that (IR) is satisfied at $(v_H^\dagger, v_L^\dagger)$.

- Since $(\tilde{v}_H, \tilde{v}_L)$ is a solution, the following program must be optimized at $\epsilon = 0$:

$$\min_{\epsilon} p_H h(\tilde{v}_H + \frac{\epsilon}{p_H}) + (1 - p_H) h(\tilde{v}_L - \frac{\epsilon}{1 - p_H})$$
$$\text{s.t. } p_H(\tilde{v}_H + \frac{\epsilon}{p_H}) + (1 - p_H)(\tilde{v}_L - \frac{\epsilon}{1 - p_H}) - c(e_H) \geq 0. \quad \text{(IR)}$$

- Taking first order condition with respect to ϵ, evaluated at $\epsilon = 0$:

$$h'(\tilde{v}_H) - h'(\tilde{v}_L) = 0.$$

- Which implies: $\tilde{v}_H = \tilde{v}_L$.
- Which is a contradiction with (IC). Hence (IC) binds.

The Market for Lemons

- Continuum of buyers (mass 1) and sellers (mass 1).
- Good's quality: $q \in \{L, H\}$.
- Each seller only knows quality of his/her product.
- Cost of production of good of quality q is c_q.
- Buyers value quality L at v_L and quality H at v_H.
- α percent of sellers are of high quality.

Assumptions

- $$v_H > c_H > c_L > v_L.$$
- $$c_H > \overset{E(v)}{\alpha v_H + (1-\alpha) v_L} > \overset{E(c)}{\alpha c_H + (1-\alpha) c_L}.$$
- If no one has private information: gains from trade exists.
- If quality is observable: for high quality gains from trade exists.

- Consider competitive market with price p:
  - Seller of quality q sells only if $p \geq c_q$.
    1. $p < v_L$: zero supply.
    2. $c_H > p \geq c_L$: only low value sellers will produce the good, so the buyers' expected value of v is:

       $$E[v|p] = v_L < c_L.$$

       Hence zero demand.
    3. $p \geq c_H$: the buyers' expected value of v is:

       $$E[v|p] = \alpha v_H + (1-\alpha) v_L < c_H \leq p.$$

       Hence zero demand.

Insurance

- Non-smokers on average live longer than smokers.
- If a life insurance company does not discriminate, its life insurance will be more valuable for smokers than for non-smokers.
- Smokers will purchase insurance more than non-smokers. Hence, the average mortality rate increases.
- Therefore, the insurer will pay more.
- In response, the company may increase premiums. However, higher prices cause more non-smoking customers to cancel their insurance.
- As more smokers compared with non-smokers take out life insurance, the prices will continue to rise, which mean fewer non-smokers will purchase insurance.
- As this process continues, the life insurance market will collapse.

A bargaining problem is a pair $< S, d >$, $S \subset \mathbb{R}^2$ compact and convex, $d \in S$, and $\exists s \in S$ such that $s_i > d_i$ for $i = 1, 2$. Let $\mathcal{B}$ denote the collection of bargaining problems. While d is often interpreted as a disagreement point, this is not the role it plays in the axiomatic treatment. It only plays a role in INV (where its role has the flavor of a normalization constraint) and in SYM. The appropriate interpretation is closely linked to noncooperative bargaining. It is *not* the value of an outside option!

Definition

A *bargaining solution* is a function $f : \mathcal{B} \to \mathbb{R}^2$ such that $f(S, d) \in S$.

The Axioms (1)

INV (Invariance to Equivalent Utility Representations)
Given $< S, d >$, and a pair of constants (α_i, β_i) with $\alpha_i > 0$ for each individual $i = 1, 2$, let $< S', d' >$ be the bargaining problem given by

$$S' = \{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) : (s_1, s_2) \in S\}$$

and

$$d_i' = \alpha_i d_i + \beta_i, \quad i = 1, 2.$$

Then

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i, \quad i = 1, 2.$$

The Axioms(2)

SYM (Symmetry)
If $d_1 = d_2$ and $(s_1, s_2) \in S \iff (s_2, s_1) \in S$, then

$$f_1(S, d) = f_2(S, d).$$

The Axioms(3)

IIA (Independence of Irrelevant Alternatives)
If $S \subset T$ and $f(T, d) \in S$, then

$$f(S, d) = f(T, d).$$

The Axioms(4)

PAR (Pareto Efficiency)
If $s \in S$, $t \in S$, $t_i > s_i$, $i = 1, 2$, then

$$f(S, d) \neq s.$$

Theorem (Nash)

If $f : \mathcal{B} \to \mathbb{R}^2$ satisfies INV, SYM, IIA, and PAR, then

$$f(S,d) = \underset{(d_1,d_2)\le(s_1,s_2)\in S}{\arg\max}\,(s_1 - d_1)(s_2 - d_2) \equiv f^N(S,d).$$

Rubinstein Bargaining

- Two agents bargain over $[0, 1]$.
- Time is indexed by t, $t = 1, 2, \ldots$.
- A proposal is a division of the pie $(x, 1 - x)$, $x \geq 0$.
- The agents take turns to make proposals. Player I makes proposals on odd t and II on even t. If the proposal $(x, 1 - x)$ is agreed to at time t, I's payoff is $\delta_1^{t-1} x$ and II's payoff is $\delta_2^{t-1}(1 - x)$.
- Perpetual disagreement yields a payoff of $(0, 0)$.
- Impatience implies $\delta_i < 1$.

- Histories are $h^t \in [0,1]^{\tau-1}$.
- Strategies for player I, $\tau_I^1 : \cup_{t \text{ odd}}[0,1]^{t-1} \to [0,1]$, $\tau_I^2 : \cup_{t \text{ even}}[0,1]^t \to \{A, R\}$, and for player II, $\tau_{II}^1 : \cup_{t \geq 2 \text{ even}}[0,1]^{t-1} \to [0,1]$ and $\tau_{II}^2 : \cup_{t \text{ odd}}[0,1]^t \to \{A, R\}$.
- Need to distinguish between histories in which all proposals have been rejected, and those in which all but the last have been rejected and the current one is being considered.

The Stationary Equilibrium

- All the subgames after different even length histories of rejected proposals are strategically identical. A similar comment applies to different odd length histories of rejected proposals. Finally, all the subgames that follow different even length histories of rejected proposals followed by the *same* proposal on the table are strategically identical. Similarly, all the subgames that follow different odd length histories of rejected proposals followed by the *same* proposal on the table are strategically identical.
- Consider first equilibria in history independent (or stationary) strategies. Recall that a strategy for player I is a pair of mappings, (τ_I^1, τ_I^2). The strategy τ_I^1 is stationary if, for all $h^t \in [0,1]^{t-1}$ and $\hat{h}^{\hat{t}} \in [0,1]^{\hat{t}-1}$, $\tau_I^1(h^t) = \tau_I^1(\hat{h}^{\hat{t}})$ (and similarly for the other strategies).

- If a strategy profile is a stationary equilibrium (with agreement), there is a pair (x^*, z^*), such that *I* expects x^* in any subgame in which *I* moves first and expects z^* in any subgame in which *II* moves first.
- In order for this to be an equilibrium, *I*'s claim should make *II* indifferent between accepting and rejecting: $1 - x^* = \delta_2(1 - z^*)$, and similarly *I* is indifferent, so $z^* = \delta_1 x^*$.
- Consider the first indifference. Player *I* won't make a claim that *II* strictly prefers to $1 - z^*$ next period, so $1 - x^* \leq \delta_2(1 - z^*)$. If *II* strictly prefers $(1 - z^*)$ next period, she rejects and gets $1 - z^*$ next period, leaving *I* with z^*. But *I* can offer *II* a share $1 - z^*$ this period, avoiding the one period delay.

- Solving yields

$$x^* = (1 - \delta_2)/(1 - \delta_1\delta_2),$$

  and

$$z^* = \delta_1(1 - \delta_2)/(1 - \delta_1\delta_2).$$

- The stationary subgame perfect equilibrium (note that backward induction is not well defined for the infinite horizon game) is for I to always claim x^* and accept any offer $\geq z^*$, and for II to always offer z^* and always accept any claim $\leq x^*$.

- While in principal, there could be nonstationary equilibria, it turns out that there is only one subgame perfect equilibrium.
- Denote by i/j the game in which i makes the initial proposal to j.
- Define:

$$M_i = \sup\left\{i\text{'s discounted expected payoff in any subgame perfect eq of } i/j\right\}$$

$$m_i = \inf\left\{i\text{'s discounted expected payoff in any subgame perfect eq of } i/j\right\}.$$

Claim: $m_j \geq 1 - \delta_i M_i$.

Proof.

Note first that i must, in equilibrium, accept any offer $> \delta_i M_i$.
The claim is proved by proving that in every equilibrium, player j's payoff in j/i is at least $1 - \delta_i M_i$. Suppose not, that is, suppose there exists an equilibrium yielding a payoff $u_j < 1 - \delta_i M_i$ to j. But this is impossible, since j has a profitable deviation in such an equilibrium: offer $\delta_i M_i + \varepsilon$, ε small. Player i must accept, giving j a payoff of $1 - \delta_i M_i - \varepsilon > u_j$, for ε sufficiently small. □

Claim: $M_j \leq 1 - \delta_i m_i$.

Proof.

If j makes an equilibrium offer that i accepts, then i must be getting at least $\delta_i m_i$ (since otherwise, i should reject the offer to receive a guaranteed discounted value of at least $\delta_i m_i$).
This implies that j gets no more than $1 - \delta_i m_i$.
The other possibility is that j makes an equilibrium offer that i rejects (in equilibrium). Then, in equilibrium, in the game i/j, i cannot offer j more than $\delta_j M_j$. In this case, j's payoff is no more than $\delta_j^2 M_j$.
So,

$$\begin{aligned} M_j &\leq \max\{\underbrace{1 - \delta_i m_i}_{\text{if } i \text{ accepts}}, \underbrace{\delta_j^2 M_j}_{\text{if } i \text{ rejects}}\} \\ &\implies M_j \leq 1 - \delta_i m_i. \end{aligned}$$

□

Combining the above two claims:

$$\begin{aligned} M_j &\leq 1-\delta_i(1-\delta_j M_j) \\ &\implies M_j \leq \frac{(1-\delta_i)}{(1-\delta_i\delta_j)},\ M_i \leq \frac{(1-\delta_j)}{(1-\delta_i\delta_j)}. \end{aligned}$$

This implies

$$m_i \geq 1-\delta_j \frac{(1-\delta_i)}{(1-\delta_i\delta_j)} = \frac{(1-\delta_j)}{(1-\delta_i\delta_j)}$$

and so

$$m_i = M_i = \frac{(1-\delta_j)}{(1-\delta_i\delta_j)}.$$

A formal model of an exchange economy

- L commodities $l \in \mathcal{L} = \{1, ..., L\}$.
- H agents $h \in \mathcal{H} = \{1, ..., H\}$.
- A bundle of commodities is a vector $c \in \mathcal{R}_+^{\mathcal{L}}$.
- Each agent h is characterized by endowments $e^h \in \mathcal{R}_+^{\mathcal{L}}$ and a utility function $u^h \in \mathcal{R}_+^{\mathcal{L}} \to \mathcal{R}$.

- An exchange economy is characterized by agentsindividual endowments and their utility functions.
- $E = ((u^h, e^h)_{h \in \mathcal{H}})$.
- Agents are assumed to take a market price $p \in \mathcal{R}_+^{\mathcal{L}}$ as given and then to choose a consumption bundle out of their budget set to maximize their utility.
- We will always assume that $u(.)$ is a continuous functions. Therefore, for $p >> 0$ there will be a solution to the agents maximization problem.

Equilibrium

A Walrasian equilibrium for the economy E is a vector $(p, (c_h)_{h \in \mathcal{H}}) \in \mathcal{R}^{\mathcal{L}} \times \mathcal{R}_{+}^{\mathcal{HL}}$ such that

1. Markets clear: $\sum_{h \in \mathcal{H}} (c^h - e^h) = 0$;
2. Each agent h maximizes her utility, subject to constraints.

- We want to explore under which conditions such equilibria exist and whether such equilibria are desirable.

Pareto-optimality

- A *feasible allocation* is a vector $(c_h)_{h\in\mathcal{H}} \in \mathcal{R}_+^{\mathcal{HL}}$ such that $\sum_{h\in\mathcal{H}}(c^h - e^h) \leq 0$.
- Given an economy E, an allocation $(c_h)_{h\in\mathcal{H}}$ is said to be Pareto-optimal (or Pareto efficient) if:
  1. there is no other feasible allocation $(\tilde{c}_h)_{h\in\mathcal{H}}$ with $u^h(\tilde{c}_h) \geq u^h(c_h)$ for all $h \in \mathcal{H}$ and
  2. $u^{h'}(\tilde{c}_{h'}) > u^{h'}(c_{h'})$ for at least one $h' \in \mathcal{H}$.

Assumptions

1. (A1) for all agents $h \in \mathcal{H}$: $e^h >> 0$.
2. (A2) for all agents $h \in \mathcal{H}$: utility is increasing.
3. (A3) for all agents $h \in \mathcal{H}$: utility is continuous on its domain.
4. (A4) for all agents $h \in \mathcal{H}$: utility is concave on its domain.

First Welfare Theorem

Theorem ((Arrow (1951), Debreu (1951)))

Let $(p, (c_h)_{h \in \mathcal{H}})$ be a Walrasian equilibrium for the economy E. If (A2) holds, then the allocation $(c_h)_{h \in \mathcal{H}}$ is Pareto-efficient.

Second Welfare Theorem

Theorem ((Arrow (1951), Debreu (1951)))

Given an economy E which satisfies (A1)-(A4). If $(e_h)_{h\in\mathcal{H}}$, (for all $h \in \mathcal{H}$) is Pareto-efficient then there exists a $p \in \mathcal{R}^{\mathcal{L}}_{+}$ such that $(p, (e_h)_{h\in\mathcal{H}})$ is a Walrasian equilibrium for the economy E.