Sample Exercises for Microeconomics II*

May 27, 2020

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*This set of exercises are collected by Vahid Rostamkhani for Microeconomics II course presented by Dr Farshad Hagh-Panah at TeIAS. Please feel free to inform me if there are any flaws in exercises or solutions.

1 Normal and Extensive Form Games

1.1 Normal Form Games

Exercise 1. Person 1 cares both about her income and about person 2's income. Precisely, the value she attaches to each unit of her own income is the same as the value she attaches to any two units of person 2's income. How do her preferences order the outcomes $(1,4)$, $(2,1)$, and $(3,0)$, where the first component in each case is person 1's income and the second component is person 2's income? Give a payoff function consistent with these preferences.

1.2 Iterated Deletion of Dominated Strategies

Exercise 2. Consider the Cournot duopoly example.

1. Characterize the set of strategies that survive iterated deletion of strictly dominated strategies.

2. Formulate the game when there are $n \geq 3$ firms and identify the set of strategies surviving iterated deletion of strictly dominated strategies.

1.3 Extensive Form Games

Exercise 3. Suppose $T = \{a, b, c, d, e, f, g, h\}$ and $\prec$ is given by (i) $a < b\ c, d, e, f, g, h$,(i) $b \prec c, e$, and (iii) $d \prec f, g$

1. Draw the implied tree. Be sure to label all nodes.

2. Suppose this is a two player game, with player 2 owning b and d. Specify an action labeling for the game, and a pair of strategies for each of players 1 and 2 with the property that the four resulting strategy profiles have precisely c, f, g as outcomes.

3. Suppose now that player 2 cannot distinguish between the two nodes b and d. Describe player 2's information set(s). Is it possible to specify an action labeling for the game, and a pair of strategies for each of players 1 and 2 with the property that the four resulting strategy profiles have precisely c, f, g as outcomes? Why or why not?

2 A First Look at Equilibrium

2.1 Nash Equilibrium

Exercise 4. Suppose that there are infinitely many firms, all of which have the same cost function C Assume that $C(0) = 0$, and for $q > 0$ the function $C(q)/q$ has a unique minimizer $\underline{q}$; denote the minimum of $C(q)/q$ by $\underline{p}$. Assume that the inverse demand function P is decreasing. Show that in any Nash equilibrium the firms' total output Q^* satisfies

$$P\left(Q^* + \underline{q}\right) \leq \underline{p} \leq P\left(Q^*\right)$$

(That is, the price is at least the minimal value $\underline{p}$ of the average cost, but is close enough to this minimum that increasing the total output of the firms by $\underline{q}$ would reduce the price to at most $\underline{p}$.) To establish these inequalities, show that if $P\left(Q^*\right) < \underline{p}$ or $P\left(Q^* + \underline{q}\right) > \underline{p}$ then Q^* is not the total output of the firms in a Nash equilibrium, because in each case at least one firm can deviate and increase its profit.

Exercise 5. In the canonical Stackelberg model, there are two firms, I and II producing the same good. Their inverse demand function is $P = 6 - Q$, where Q is market supply. Each firm has a constant marginal cost of \$4 per unit and a capacity constraint of 3 units (the latter restriction will not affect optimal behavior, but assuming it eliminates the possibility of negative prices). Firm I chooses its quantity first. Firm II, knowing firm I's quantity choice, then chooses its quantity. Thus, firm I's strategy space is $S_1 = [0, 3]$ and firm II 's strategy space is $S_2 = \{\tau_2 | \tau_2 : S_1 \to \text{[0,3]}\}$. A strategy profile is $(q_1, \tau_2) \in S_1 \times S_2$, i.e., an action (quantity choice) for I and a specification for every quantity choice of I of an action (quantity choice) for II.

1. What are the outcome and payoffs of the two firms implied by the strategy profile (q_1, τ_2)?

2. Show that the following strategy profile does not constitute a Nash equilibrium: $\left(\frac{1}{2}, \tau_2\right)$, where $\tau_2\left(q_1\right) = \left(2 - q_1\right)/2$ Which firm(s) is (are) not playing a best response?

3. Prove that the following strategy profile constitutes a Nash equilibrium: $\left(\frac{1}{2}, \hat{\tau}_2\right)$, where $\hat{\tau}_2\left(q_1\right) = \frac{3}{4}$ if $q_1 = \frac{1}{2}$ and $\hat{\tau}_2\left(q_1\right) = 3$ if $q_1 \neq \frac{1}{2}$, i.e., II threatens to flood the market unless produces exactly $\frac{1}{2}$. Is there any other Nash equilibrium which gives the outcome path $\left(\frac{1}{2}, \frac{3}{4}\right)$? What are the firms payoffs in this equilibrium?

4. Prove that the following strategy profile constitutes a Nash equilibrium: $(0, \tilde{\tau}_2)$, where $\tilde{\tau}_2(q_1) = 1$ if $q_1 = 0$ and $\tilde{\tau}_2(q_1) = 3$ if $q_1 \neq 0$, i.e., II threatens to flood the market unless I produces exactly 0. What are the firms' payoffs in this equilibrium?

5. Given $q_1 \in [0, 2]$, specify a Nash equilibrium strategy profile in which I chooses q_1. Why is it not possible to do this for $q_1 \in (2, 3]$?

6. What is the unique backward induction solution of this game?

2.2 Subgame Perfection

Exercise 6. A group of n people have to share k objects that the value differently. Each person assigns values to the objects; no one assigns the same value to two different objects. Each person evaluates a set of objects according to the sum of the values she assigns to the objects in the set. The following procedure is used to share the objects. The players are ordered 1 through n. Person 1 chooses an object, then person 2 does so, and so on; if $k > n$ then after person n chooses an object, person 1 chooses a second object, then person 2 chooses a second object, and so on. Objects are chosen until none remain. (In Canada and the USA professional sports teams use a similar procedure to choose new players.) Denote by $G(n, k)$ the extensive game that models this procedure. If $k \leq n$ then obviously $G(n, k)$ has a subgame perfect equilibrium in which each player's strategy is to choose her favorite object among those remaining when her turn comes. Show that if $k > n$ then $G(n, k)$ may have no subgame perfect equilibrium in which person 1 chooses her favorite object on the first round. (You can give an example in which $n = 2$ and $k = 3$.) Now fix $n = 2$. Define x_k to be the object least preferred by the person who does not choose at stage k (i.e. who does not choose the last object); define x_{k-1} to be the object, among all those except x_k, least preferred by the person who does not choose at stage $k - 1$. Similarly, for any j with $2 \leq j \leq k$, given $x_j, \ldots, x_k$, define x_{j-1} to be the object, among all those excluding $\{x_j, \ldots, x_k\}$, least preferred by the person who does not choose at stage $j - 1$. Show that the game $G(2, 3)$ has a subgame perfect equilibrium in which for every $j = 1, \ldots, k$ the object x_j is chosen at stage j. (This result is true for $G(2, k)$ for all values of k.) If $n \geq 3$ then interestingly a person may be better off in all subgame perfect equilibria of $G(n, k)$ when she comes later in the ordering of players. (An example, however, is difficult to construct; one is given in Brams and Straffin (1979).)

Exercise 7. Two firms, A and B, are in a market that is declining in size. The game starts in period 0, and the firms can compete in periods $0, 1, 2, 3, \ldots$ (i.e., indefinitely) if they so choose.

Duopoly profits in period t for firm A are equal to $105 - 10t$, and they are $10.5 - t$ for firm B. Monopoly profits (those if a firm is the only one left in the market) are $510 - 25t$ for firm A and $51 - 2t$ for firm B.

Suppose that at the start of each period, each firm must decide either to ”stay in ” or ”exit” if it is still active (they do so simultaneously if both are still active). Once a firm exits, it is out of the market forever and earns zero in each period thereafter. Firms maximize their (undiscounted) sum of profits.

What is this game's subgame perfect Nash equilibrium outcome (and what are the firms' strategies in the equilibrium)?

2.3 Mixing

Exercise 8. Consider the following variant of a sealed bid auction: There are two bidders who each value the object at v, and simultaneously submit bids. As usual, the highest bid wins and in the event of a tie, the object is awarded on the basis of a fair coin toss. But now all bidders pay their bid. (This is an all-pay auction.)

1. Formulate this auction as a normal form game.

2. Show that there is no equilibrium in pure strategies.

3. This game has an equilibrium in mixed strategies. What is it? (You should verify that the strategies do indeed constitute an equilibrium).

3 Games with Nature

3.1 Games with Nature

Exercise 9. There are two firms, 1 and 2, producing the same good. The in verse demand curve is given by $P = \theta - q_1 - q_2$, where $q_i \in \mathbb{R}_+$ is firm i 's output. (Note that we are allowing negative prices.) There is demand uncertainty with nature determining the value of θ, assigning probability $\alpha \in (0, 1)$ to $\theta = 3$, and complementary probability $1 - \alpha$ to $\theta = 4$. Firm 2 knows (is informed of) the value of θ while firm 1 is not. Finally, each firm has zero costs of production. As usual, assume this description is common knowledge. Suppose the two firms choose quantities simultaneously. Define a strategy profile for this game. Describe the Nash equilibrium behavior and interim payoffs (which may be unique).

3.2 Auctions and Related Games

Exercise 10. Consider the following variant of a sealed-bid auction in a setting of independent private values. The highest bidder wins, and pays a price determined as the weighted average of the highest bid and second highest bid, with weight $\alpha \in (0,1)$ on the highest bid (ties are resolved by a fair coin). Suppose there are two bidders, with bidder i's value v_i randomly drawn from the interval $[\underline{v}_i, \bar{v}_i]$ according to the distribution function F_i with density f_i.

1. What are the interim payoffs of player i ?

2. Suppose (σ_1, σ_2) is a Nash equilibrium of the auction, and assume σ_i is a strictly increasing and differentiable func tion, for $i = 1, 2$. Describe the pair of differential equations the strategies must satisfy.

3.3 Games of Incomplete Information

Exercise 11. Consider the following strategic situation. Two opposed armies are poised to seize an island. Each army's general can choose either "attack" or "not attack." In addition, each army is either "strong" or "weak" with equal probability (the draws for each army are independent) and an army's type is known only to its general. Payoffs are as follows: The island is worth M if captured. An army can capture the island either by attacking when its opponent does not or by attacking when its rival does if it is strong and its rival is weak. If two armies of equal strength both attack, neither captures the island. An army also has a "cost" of fighting, which is s if it is strong and w if it is weak, where $s < w$. There is no cost of attacking if its rival does not. Identify all pure strategy Bayesian Nash equilibria of this game

Exercise 12. Consider a model in which each of two players does not know whether she is the only witness, or whether there is another witness. Denote by π the probability each player assigns to being the sole witness. Model this situation as a Bayesian game with three states: one in which player 1 is the only witness, one in which player 2 is the only witness, and one in which both players are witnesses. Find a condition on π under which the game has a pure Nash equilibrium in which each player chooses Call (given the signal that she is a witness) When the condition is violated, find the symmetric mixed strategy Nash equilibrium of the game.

Exercise 13. Firm 1's costs are private information, while firm 2 's are public. Nature determines the costs of firm 1 at the beginning of the game, with $\Pr(c_1 = c_L) = \theta \in (0,1)$. The terminal nodes of the extensive form are given by (c_1, q_1, q_2). As in Example 1.1.3, firm i 's ex post profit is

$$\pi_i (c_1, q_1, q_2) = [(a - q_1 - q_2) - c_i] q_i$$

where c_i is firm i's cost.

Assume $c_L, c_H, c_2 < a/2$. A strategy for player 2 is a quantity q_2. A strategy for player 1 is a function $q_1 : \{c_L, c_H\} \to \mathbb{R}_+$. For simplicity, write q_L for $q_1 (c_L)$ and q_H for $q_1 (c_H)$

Note that for any strategy profile $((q_L, q_H), q_2)$, the associated outcome distribution is

$$\theta \circ (c_L, q_L, q_2) + (1 - \theta) \circ (c_H, q_H, q_2)$$

1. Find the Nash equilibrium of the game described.

2. Now we modify the game to capture the possibility that firm 2 may know that firm 1 has low costs, c_L. This can be done as follows: Firm 1's space of uncertainty (types) is, as before, $\{c_L, c_H\}$, while firm 2 's is $\{t_I, t_U\}$. Nature determines the types according to the distribution

$$\Pr (t_1, t_2) = \begin{cases} 1 - p - q, & \text{if } (t_1, t_2) = (c_L, t_I) \\ p, & \text{if } (t_1, t_2) = (c_L, t_U) \\ q, & \text{if } (t_1, t_2) = (c_H, t_U) \end{cases}$$

   where $0 < p, q$ and $p + q < 1$. Firm 2 's type, t_I or t_U, does not affect his payoffs. Firm l's type is just his cost, c_1.

   - What is the probability firm 2 assigns to $c_1 = c_L$ when his type is t_I? When his type is t_U?
   - What is the probability firm 1 assigns to firm 2 knowing firm 1's cost? [This may depend on 1's type.]
   - Solve for the values of p and q that imply that t_U has the beliefs of player 2 in part (1) and c_L assigns probability $1 - \alpha$ to t_I.
   - For these values of p and q, solve for the Nash equilibrium of this game. Compare your analysis to that of part (1).

4 Nash Equilibrium Refinements in Dynamic Games

4.1 Sequential Rationality and Perfect Bayesian Equilibrium

Exercise 14. A plaintiff, Ms. P, files a suit against Ms. D (the defendant). If Ms. P wins, she will collect π dollars in damages from Ms. D. Ms. D knows the likelihood that Ms. P will win,

$\lambda \in [0,1]$, but Ms. P does not (Ms. D might know if she was actually at fault). They both have strictly positive costs of going to trial of c_p and c_d. The prior distribution of λ has density $f(\lambda)$ (which is common knowledge). Suppose pretrial settlement negotiations work as follows: Ms. P makes a take-it-or-leave-it settlement offer (a dollar amount) to Ms. D. If Ms. D accepts, she pays Ms. P and the game is over. If she does not accept, they go to trial.

1. What are the (pure strategy) weak perfect Bayesian equilibria of this game?

2. What effects do changes in c_p, c_d, and π have?

3. Now allow Ms. D, after having her offer rejected, to decide not to go to court after all. What are the weak perfect Bayesian equilibria? What about the effects of the changes in (2)?

4. What are the sequential equilibria?

Exercise 15. A buyer and a seller are bargaining. The seller owns an object for which the buyer has value $v > 0$ (the seller's value is zero). This value is known to the buyer but not to the seller The value's prior distribution is common knowledge. There are two periods of bargaining. The seller makes a take-it-or-leave-it offer (i.e., names a price) at the start of each period that the buyer may accept or reject. The game ends when an offer is accepted or after two periods whichever comes first. Both players discount period 2 payoffs with a discount factor of $\delta \in (0,1)$ Assume throughout that the buyer always accepts the seller's offer whenever she is indifferent.

1. Characterize the (pure strategy) weak perfect Bayesian equilibria for a case in which v can take two values v_L and v_H, with $v_H > v_L > 0$, and where $\lambda = \text{Prob}\,(v_H)$.

2. Do the same for the case in which v is uniformly distributed on $[\underline{v}, \bar{v}]$.

3. What are the sequential equilibria?

5 Signaling

Exercise 16.

1. Verify that the signaling game illustrated in Figure ?? has no Nash equilibrium in pure strategies.

2. For the game in Figure ?? suppose $p = \frac{1}{2}$. Describe a perfect Bayesian equilibrium.

3. How does your answer to part (2) change if $p = 0.1$?

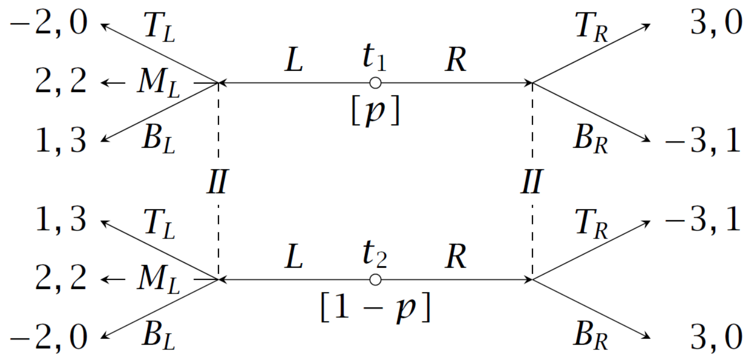

Figure 1

Exercise 17. Assume a single firm and a single consumer. The firm's product may be either high or low quality and is of high quality with probability λ. The consumer cannot observe quality before purchase and is risk neutral. The consumer's valuation of a high-quality product is $v_{\boldsymbol{H}}$ her valuation of a low-Quality product is v_L. The costs of production for high (H) and low (L) quality are c_H and c_L, respectively. The consumer desires at most one unit of the product. Finally, the firm's price is regulated and is set at p. Assume that $v_H > p > v_L > c_H > c_L$.

1. Given the level of p, under what conditions will the consumer buy the product?

2. (b) Suppose that before the consumer decides whether to buy, the firm (which knows its type can advertise. Advertising conveys no information directly, but consumers can observe the total amount of money that the firm is spending on advertising, denoted by A. Can there be a separating perfect Bayesian equilibrium, that is, an equilibrium in which the consumer rationally expects firms with different quality levels to pick different levels of advertising?

6 Moral Hazard

Exercise 18. Consider the following hidden action model with three possible actions $E = \{e_1, e_2, e_3\}$. There are two possible profit outcomes: $\pi_H = 10$ and $\pi_L = 0$. The probabilities of π_H conditional on the three effort levels are $f(\pi_H|e_1) = \frac{2}{3}$, $f(\pi_H|e_2) = \frac{1}{2}$, and $f(\pi_H|e_3) = \frac{1}{3}$. The agent's effort cost function has $g(e_1) = \frac{5}{3}$, $g(e_2) = \frac{8}{5}$, $g(e_3) = \frac{4}{3}$. Finally, $v(w) = \sqrt{w}$, and the manager's reservation utility is $\bar{u} = 0$.

1. What is the optimal contract when effort is observable?

2. (b) Show that if effort is not observable, then e_2 is not implementable. For what levels of $g(e_2)$ would e_2 be implementable? [Hint: Focus on the utility levels the manager will get for the two outcomes, v_1 and v_2, rather than on the wage payments themselves.]

3. What is the optimal contract when effort is not observable?

4. Suppose, instead, that $g(e_1) = \sqrt{8}$, and let $f(\pi_H|e_1) = x \in (0,1)$. What is the optimal contract if effort is observable as x approaches 1? What is the optimal contract as x approaches if it is not observable? As x approaches 1 , is the level of effort implemented higher or lower when effort is not observable than when it is observable?

Exercise 19. Consider a relationship between a principal and an agent in which only two results, valued at 50,000 and 25,000 are possible. The agent must choose between three possible efforts. The probability of each of the results contingent on the efforts is given below

	$25,000$	$50,000$
e^1	0.25	0.75
e^2	0.50	0.50
e^3	0.75	0.25

Assume that the principal is risk-neutral and that the agent is risk-averse, with their respective preferences described by the following functions:

$$B(x,w) = x - w \quad U(w,e) = \sqrt{w} - v(e)$$

with $v(e^1) = 40, v(e^2) = 20$, and $v(e^3) = 5$. The reservation utility level of the agent is $\underline{U} = 120$

1. Write down the optimal contracts under symmetric information for each effort level and the profits obtained by the principal in each case. What effort level does the principal prefer?

2. Write down the optimal contracts when there exists a moral hazard problem. What is the optimal effort level and the contract chosen by the principal?

Exercise 20. Consider a relationship between a principal and an agent in which there are only two possible results, one high, x_2, and the other low, x_1. The frequency with which each result occurs depends on the agent's effort, $e \in [0,1]$, and a random state variable. Assume that the probability of the high result is the same as the effort, i.e., $\Pr(x = x_2|e) = e$, so that $\Pr(x = x_1|e) = 1 - e$ The agent's utility is of the form $U(w,e) = u(w) - v(e)$, where $u(\cdot)$ is increasing and concave, and $v(\cdot)$

is increasing and convex. The principal's objective function is $B(x-w)$, which is increasing and concave (that is, she could be risk averse)

1. Write down the constrained maximization problem of the principal, and find the conditions that determine the optimal contract.

2. Now we assume that the agent's effort is not publicly known. Write down the constrained maximization problem that defines the optimal contract in this case. Is the first-order approach valid in this example? Describe the relationship between the optimal contract's wages and the differences in this contract compared to part (1).

7 Adverse Selection

Exercise 21. Read and summarize Akerlof (1978) in two pages.

Exercise 22. Read and summarize Akerlof (2002) in two pages (This interesting paper is George A. Akerlof's Prize Lecture in 2001).

Exercise 23. The quality in the used cars market is variable. Suppose, that the quality of a given car, measured in monetary units, can be any number between 2.000 and 6.000, with uniform probability. The distribution of quality is known to everybody and owners know the quality of the car they are selling, but not the buyer. All the agents are risk neutral.

1. If a buyer thinks that all cars are in the market, what is the maximum price he is willing to pay?

2. At the previous price, what is the quality of the cars in the market?

3. Find the equilibrium in this market.

4. Suppose now buyers value each car at a price 20% higher than the owner. The sell prices for the owners are as before. Suppose that there are more buyers than sellers, so that at the competitive price each buyer pays his reservation price. What is the new equilibrium?

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