

**Optimization Methods in
Management Science /
Operations Research
15.053/058**

**Introduction to
LP formulations**

Linear Programming

Optimization is an important and fascinating area of management science and operations research. It helps to do less work, but gain more.

Linear programming (LP) is a central topic in optimization. It provides a powerful tool in modeling many applications. LP has attracted most of its attention in optimization during the last six decades for two main reasons:

- Applicability: There are many real-world applications that can be modeled as linear programming;
- Solvability: There are theoretically and practically efficient techniques for solving large-scale problems.



Hi! My name is Cathy. I will guide you in tutorials during the semester.

In this tutorial, we introduce the basic elements of an LP and present some examples that can be modeled as an LP. In the next tutorials, we will discuss solution techniques.

Basic Components of an LP:

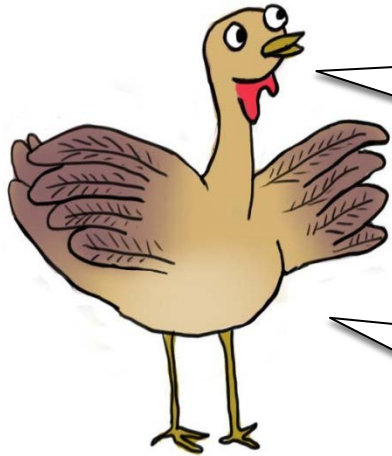
Each optimization problem consists of three elements:

- **decision variables:** describe our choices that are under our control;
- **objective function:** describes a criterion that we wish to minimize (e.g., cost) or maximize (e.g., profit);
- **constraints:** describe the limitations that restrict our choices for decision variables.

Formally, we use the term “**linear programming (LP)**” to refer to an optimization problem in which the objective function is linear and each constraint is a linear inequality or equality. I’ll discuss these features soon.



An Introductory Example



I am a bit confused about the LP elements. Can you give me more details.

Oh! I forgot to introduce myself. I am Tom; a new member of the 15.053 class. I am interested in learning linear programming. I will be with you during the tutorials.

Let's start with an example. I'll describe it first in words, and then we'll translate it into a linear program.



An Introductory Example

Problem Statement: A company makes two products (say, P and Q) using two machines (say, A and B). Each unit of P that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Q that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B.

Machine A is going to be available for 40 hours and machine B is available for 35 hours. The profit per unit of P is \$25 and the profit per unit of Q is \$30. Company policy is to determine the production quantity of each product in such a way as to maximize the total profit given that the available resources should not be exceeded



Task: The aim is to formulate the problem of deciding how much of each product to make in the current week as an LP.

Step 1: Defining the Decision Variables

We often start with identifying *decision variables* (i.e., what we want to determine among those things which are under our control).

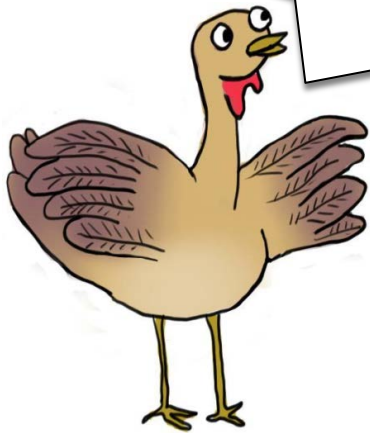
Tom! Can you identify the decision variables for our example?



The company wants to determine the optimal product to make in the current week. So there are two decision variables:

x : the number of units of P

y : the number of units of Q



Good job! Let's move on to the second step.

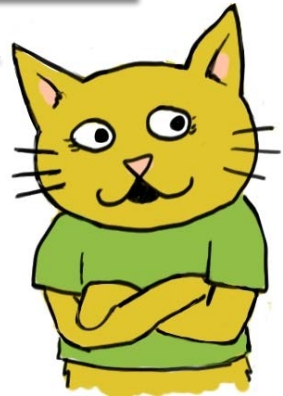
Step 2: Choosing an Objective Function

We usually seek a criterion (or a measure) to compare alternative solutions. This yields the objective function.

Tom! It is now your turn to identify the objective function.

We want to maximize the total profit. The profit per each unit of product P is \$25 and profit per each unit of product Q is \$30. Therefore, the total profit is $25x+30y$ if we produce x units of P and y units of Q. This leads to the following objective function:

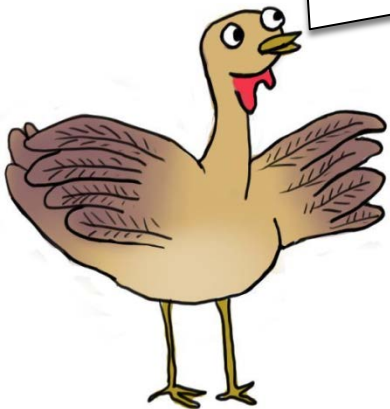
~~$\max 40x+35y$~~



Note that:

- 1: The objective function is linear in terms of decision variables x and y (i.e., it is of the form $ax + by$, where a and b are constant).
- 2: We typically use the variable z to denote the value of the objective. So the objective function can be stated as:

$$\max z = 25x + 30y$$



Step 3: Identifying the Constraints

In many practical problems, there are limitations (such as resource / physical / strategic / economical) that restrict our decisions. We describe these limitations using mathematical constraints.

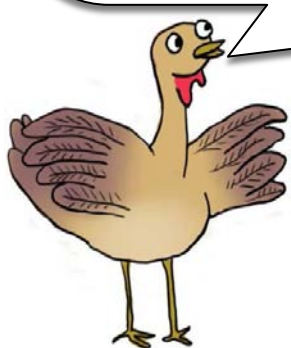
Tom! What are the constraints in our example?

The amount of time that machine A is available restricts the quantities to be manufactured. If we produce x units of P and y units of Q, machine A should be used for $50x + 24y$ minutes since each unit of P requires 50 minutes processing time on machine A and each unit of Q requires 24 minutes processing time on machine A. On the other hand, machine A is available for 40 hours or equivalently for 2400 minutes. This imposes the following constraint:

$$50x + 24y \leq 2400.$$

Similarly, the amount of time that machine B is available imposes the following constraint:

$$30x + 33y \leq 2100.$$

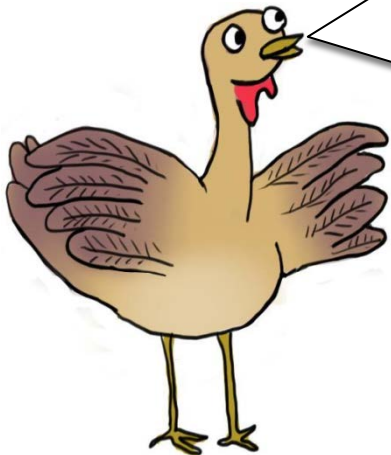


These constraints are linear inequalities since in each constraint the left-hand side of the inequality sign is a linear function in terms of the decision variables x and y and the right hand side is constant.

Step 3: Identifying the Constraints

Note: In most problems, the decision variables are required to be *nonnegative*, and this should be explicitly included in the formulation. This is the case here. So you need to include the following two non-negativity constraints as well:

$$x \geq 0 \text{ and } y \geq 0$$



I see your point. So the constraints we are subject to (s.t.) are :

$$50x + 24y \leq 2400, \quad (\text{machine A time})$$

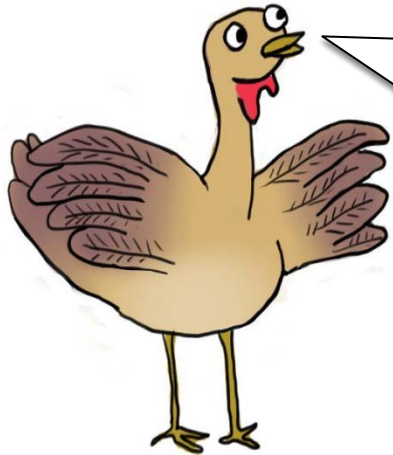
$$30x + 33y \leq 2100, \quad (\text{machine B time})$$

$$x \geq 0,$$

$$y \geq 0.$$

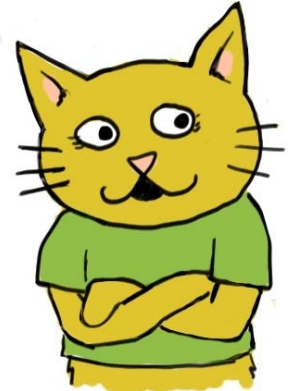
LP for the Example

Well done Tom! You did a good job. Now, write the LP by putting all the LP elements together.



Here is the LP:

$$\begin{array}{ll}\max & z = 25x + 30y \\ \text{s.t.} & 50x + 24y \leq 2400, \\ & 30x + 33y \leq 2100, \\ & x \geq 0, \\ & y \geq 0.\end{array}$$



Note: To be realistic, we would require *integrality* for the decision variables x and y . It will lead to an integer program if we include integrality. Integer programs are harder to solve and we will consider them in later lectures. For the moment, we leave out integrality and consider this LP in this tutorial.

Now let's consider a generalization of this example in order to test whether you have understood the concepts.

A Manufacturing Example

Problem Statement: An operations manager is trying to determine a production plan for the next week. There are three products (say, P, Q, and R) to produce using four machines (say, A and B, C, and D). Each of the four machines performs a unique process. There is one machine of each type, and each machine is available for 2400 minutes per week. The unit processing times for each machine is given in Table 1.



Table 1: Machine Data

Unit Processing Time (min)				
Machine	Product P	Product Q	Product R	Availability (min)
A	20	10	10	2400
B	12	28	16	2400
C	15	6	16	2400
D	10	15	0	2400
Total processing time	57	59	42	9600

A Manufacturing Example

Problem Statement (cont.): The unit revenues and maximum sales for the week are indicated in Table 2. Storage from one week to the next is not permitted. The operating expenses associated with the plant are \$6000 per week, regardless of how many components and products are made. The \$6000 includes all expenses except for material costs.

Table 2: Product Data

Item	Product P	Product Q	Product R
Revenue per unit	\$90	\$100	\$70
Material cost per unit	\$45	\$40	\$20
Profit per unit	\$45	\$60	\$50
Maximum sales	100	40	60

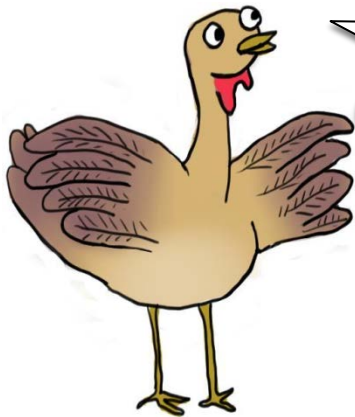


Task: Here we seek the “optimal” product mix-- that is, the amount of each product that should be manufactured during the present week in order to maximize profits. Formulate this as an LP.

Step 1: Defining the Decision Variables

Tom! You are supposed to do this example by yourself. Remember that the first step is to define the decision variables. Try to come up with the correct definition.

If you need help or want to check your solution, click here to see the answer. Otherwise, you can continue by identifying the objective function.



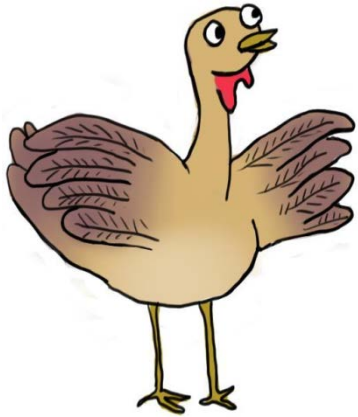
Good idea! Let me try. It should not be a difficult task.

We are trying to select the optimal product mix, so we define three decision variables as follows:

p: number of units of product P to produce,
q: number of units of product Q to produce,
r: number of units of product R to produce.

Step 2: Choosing an Objective Function

Click [here](#) to if you want to see the objective function. Otherwise, continue to describe the constraints.



Let me review the problem statement to write the constraints



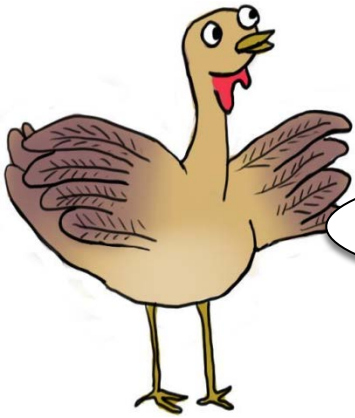
Our objective is to maximize profit:

$$\begin{aligned}\text{Profit} &= (90-45)p + (100-40)q + (70-20r - 6000) \\ &= 45p + 60q + 50r - 6000\end{aligned}$$

Note: The operating costs are not a function of the variables in the problem. If we were to drop the \$6000 term from the profit function, we would still obtain the same optimal mix of products. Thus, the objective function is

$$z = 45p + 60q + 50r$$

Step 3: Identifying the Constraints



Click **here** to if you want to see the constraints.



It is good to see the complete model now to see my answer.

The amount of time a machine is available and the maximum sales potential for each product restrict the quantities to be manufactured. Since we know the unit processing times for each machine, the constraints can be written as linear inequalities as follows:

$$20p + 10q + 10r \leq 2400 \quad (\text{Machine A})$$

$$12p + 28q + 16r \leq 2400 \quad (\text{Machine B})$$

$$15p + 6q + 16r \leq 2400 \quad (\text{Machine C})$$

$$10p + 15q + 0r \leq 2400 \quad (\text{Machine D})$$

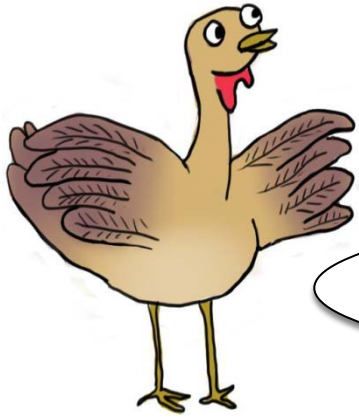
Observe that the unit for these constraints is minutes per week. Both sides of an inequality must be in the same unit. The market limitations are written as simple upper bounds.

$$\text{Market constraints: } P \leq 100, Q \leq 40, R \leq 60.$$

Logic indicates that we should also include nonnegativity restrictions on the variables .

$$\text{Nonnegativity constraints: } P \geq 0, Q \geq 0, R \geq 0.$$

A Manufacturing Example



Click [here](#) to if you want to see complete model.



Let me review the problem statement to write the constraints

By combining the objective function and the constraints, we obtain the LP model as follows:

$$\begin{array}{ll}\max & z=45p+60q+50r \\ \text{s.t.} & 20p+10q+10r \leq 2400 \\ & 12p+28q+16r \leq 2400 \\ & 15p+6q+16r \leq 2400 \\ & 10p+15q+0r \leq 2400 \\ & 0 \leq p \leq 100 \\ & 0 \leq q \leq 40 \\ & 0 \leq r \leq 60\end{array}$$

A Manufacturing Example: Optimal Solution

The optimal solution to this LP is

$$P=81.82, Q=16.36, R=60$$

with the corresponding objective value $z=\$7664$. To compute the profit for the week, we reduce this value by \$6000 for operating expenses and get \$1664. By setting the production quantities to the amount specified by the solution, we find machine usage as shown in the following table.



Machine	Available usage(min)	Actual usage (min)
A	2400	2400
B	2400	2400
C	2400	2285
D	2400	1064

Product	Maximum sales	Units produced
P	100	81.82
Q	40	16.63
R	60	60

Post Office Problem:

Let's consider one more example.

Problem Statement: A post office requires different numbers of full-time employees on different days of the week. Each full-time employee must work five consecutive days and then receive two days off.

In the following table, the number of employees required on each day of the week is specified.



Day	Number of full-time Employees Required
1=Monday	17
2=Tuesday	13
3=Wednesday	15
4=Thursday	19
5=Friday	14
6=Saturday	16
7=Sunday	11

Task: Try to formulate an LP that the post office can use to minimize the number of full-time employees who are needed to satisfy these constraints.

Step 1: Defining the Decision Variables

Remember that we often start with identifying decision variables. There might be several ways for defining decision variables, and you should be careful to pick the right one.

In our example, we want to decide how many employees we must hire. One may think of defining decision variables as follows:

y_i : number of employees who work on day i .



You can check this is not the right choice. To see why, notice that we cannot enforce the constraint that each employee works five consecutive days. In this case, there is a choice of decision variables that ensures we satisfy the constraint that each employee works five consecutive days.

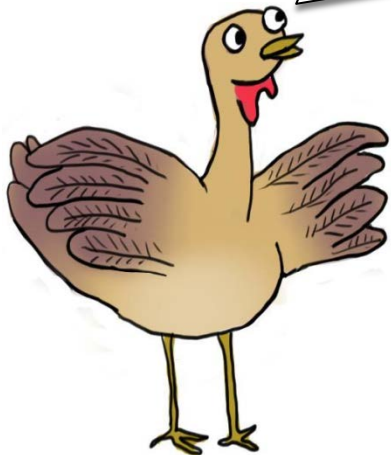
Tom! Can you come up with decision variables that can be used to formulate this problem?

Step 1: Defining the Decision Variables

Cathy, I think I see what you mean. I think we need to define seven decision variables as follows:

x_i : number of employees whose five consecutive days of work begin on day i . (They work on days $i, i+1, i+2, i+3$, and $i+4$). We define it for $i=1, 2, \dots, 7$.

For example, x_1 is the number of people beginning work on Monday.

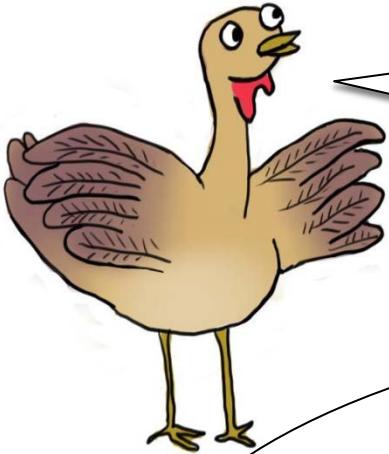


Great choice! That's exactly what I was looking for.

Now, try to formulate the objective function and the constraints.



Step 2 & 3: Choosing an Objective and Identifying the Constraints



Our aim is to minimize the number of hired employees. The objective function is

$$\mathbf{min} \quad z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

The post office must ensure that enough employees are working on each day of the week. For example, at least 17 employees must be working on Monday. To ensure that at least 17 employees are working on Monday, we require that the constraint

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 17$$

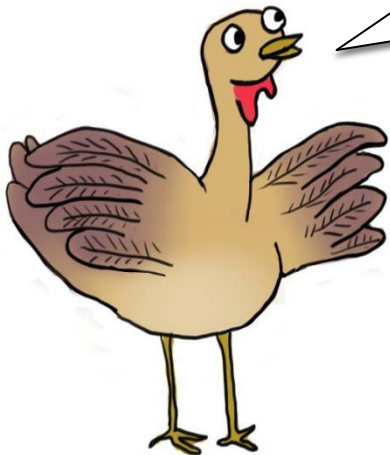
be satisfied. We have similar constraints for other days.

Well done, Tom. You can now put together all the constraints and the objective function to have an LP for the post office problem.



The LP Model for the Post Office Problem

$$\begin{aligned} \min \quad & z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\ \text{s.t.} \quad & x_1 + x_4 + x_5 + x_6 + x_7 \geq 17 && \text{(Monday constraint)} \\ & x_1 + x_2 + x_5 + x_6 + x_7 \geq 13 && \text{(Tuesday constraint)} \\ & x_1 + x_2 + x_3 + x_6 + x_7 \geq 15 && \text{(Wednesday constraint)} \\ & x_1 + x_2 + x_3 + x_4 + x_7 \geq 19 && \text{(Thursday constraint)} \\ & x_1 + x_2 + x_3 + x_4 + x_5 \geq 14 && \text{(Friday constraint)} \\ & x_2 + x_3 + x_4 + x_5 + x_6 \geq 16 && \text{(Saturday constraint)} \\ & x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 && \text{(Sunday constraint)} \\ & x_i \geq 0 \quad \text{for } i=1,2,\dots,7 && \text{(Nonnegativity constraints)} \end{aligned}$$



Having formulated the LP as above, I am curious what the optimal solution is.

Optimal Solution for the Post Office Problem

The optimal solution to this LP is

$$z=67/3, \quad x_1=4/3, \quad x_2=10/3, \quad x_3=2, \\ x_4=22/3, \quad x_5=0, \quad x_6=10/3, \quad x_7=5.$$

Because we are only allowing full-time employees, however, the variables must be integers (to be realistic). If we include integrality, we get an integer program.

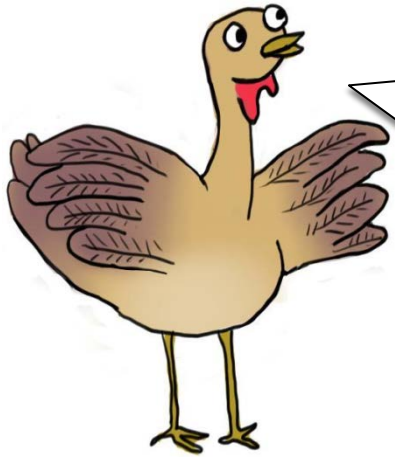


Integer programming techniques can be used to show that an optimal solution to the post office problem is

$$z=23, \quad x_1=4, \quad x_2=4, \quad x_3=2, \quad x_4=6, \quad x_5=0, \quad x_6=6, \quad x_7=3.$$

Note: Bartholdi, Orlin and Ratliff (1980) have developed an efficient technique to determine the minimum number of employees required when each worker receives two consecutive days off.

Creating a Fair Schedule for Employees



The optimal solution we found requires 4 workers to start on Monday, 4 on Tuesday, and so on. The workers who start on Saturday will be unhappy because they never receive a weekend day off. This is not *fair*.

Good point Tom! By rotating the schedules of the employees over a 23-week period, a fairer schedule can be obtained. To see how this is done, consider the following schedule:

- ☐ Weeks 1-4: start on Monday
- ☐ Weeks 5-8: start on Tuesday
- ☐ Weeks 9-10: start on Wednesday
- ☐ Weeks 11-16: start on Thursday
- ☐ Weeks 17- 20: start on Saturday
- ☐ Weeks 21-23: start on Sunday



Creating a Fair Schedule for Employees (cont.)

Employee 1 follows this schedule for a 23-week period. Employee 2 starts with week 2 of the schedule (starting on Monday for 3 weeks, then on Tuesday for 4 weeks, and closing with 3 weeks starting on Sunday and 1 week on Monday). We continue in this fashion to generate a 23-week schedule for each employee.

For example, employee 13 will have the following schedule :

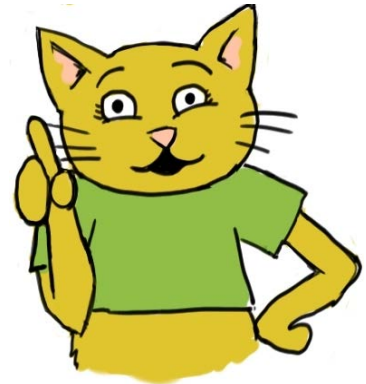
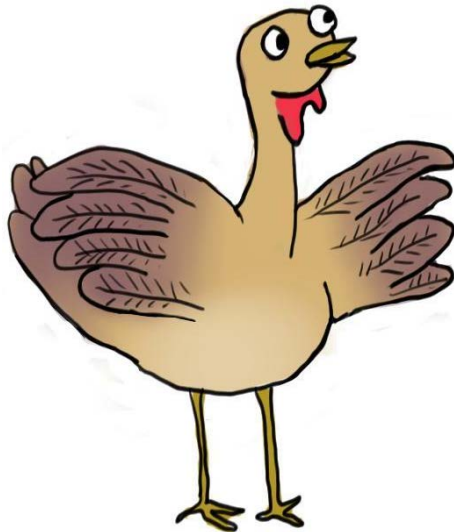
- ☐ Weeks 1-4 : start on Thursday
- ☐ Weeks 5-8: start on Saturday
- ☐ Weeks 9-11: start on Sunday
- ☐ Weeks 12-15: start on Monday
- ☐ Weeks 16-19 :start on Tuesday
- ☐ Weeks 20-21:start on Wednesday
- ☐ Weeks 22-23 :start on Thursday

This method of scheduling treats each employee equally.



So, is there anything more to say?

We will continue our discussion on algebraic formulation of linear programming in the second tutorial. We can say goodbye to the students and wish them well.



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