Sample Linear Programming Problem

A furniture manufacturer makes two types of furniture – chairs and sofas. The production of the sofas and chairs requires three operations – carpentry, finishing, and upholstery. Manufacturing a chair requires 3 hours of carpentry, 9 hours of finishing, and 2 hours of upholstery. Manufacturing a sofa requires 2 hours of carpentry, 4 hours of finishing, and 10 hours of upholstery. The factory has allocated at most 66 labor hours for carpentry, 180 labor hours for finishing, and 200 labor hours for upholstery. The profit per chair is $90 and the profit per sofa is $75. How many chairs and how many sofas should be produced each day to maximize the profit?

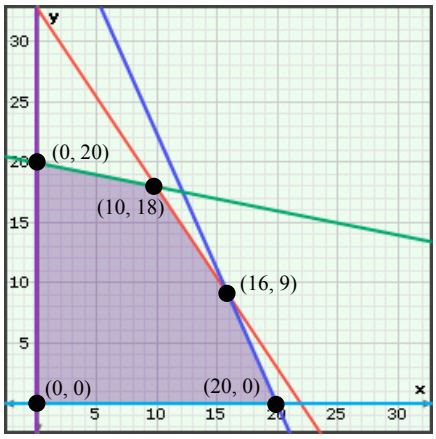
The Simplex Method: Standard Maximization Problems

A standard maximization problem is one in which the objective function is to be maximized, all the variables involved in the problem are nonnegative, and each linear constraint may be written so that the expression involving the variables is less than or equal to a nonnegative constant.

Example 1: A furniture manufacturer makes two types of furniture – chairs and sofas. The production of the sofas and chairs requires three operations – carpentry, finishing, and upholstery. Manufacturing a chair requires 3 hours of carpentry, 9 hours of finishing, and 2 hours of upholstery. Manufacturing a sofa requires 2 hours of carpentry, 4 hours of finishing, and 10 hours of upholstery. The factory has allocated at most 66 labor hours for carpentry, 180 labor hours for finishing, and 200 labor hours for upholstery. The profit per chair is \$90 and the profit per sofa is \$75. How many chairs and how many sofas should be produced each day to maximize the profit?

Linear Programming Method Of Corners Solution: If x is the number of chairs produced and y is the number of sofas produced we will want to maximize $P = 90x + 75y$ subject to the following constraints.

$$\begin{cases} 3x + 2y \le 66 \\ 9x + 4y \le 180 \\ 2x + 10y \le 200 \\ x \ge 0,\ y \ge 0 \end{cases} = \begin{cases} y \le -1.5x + 33 \\ y \le -2.25x + 45 \\ y \le -0.2x + 20 \\ x \ge 0,\ y \ge 0 \end{cases}$$



(Note: Solution set is the shaded region)

corner point	$P = 90x + 75y$
(0, 0)	0
(0, 20)	1500
(10, 18)	2250
(16, 9)	2115
(20, 0)	1800

To maximize the profit the manufacturer must produce 10 chairs and 18 sofas.

The Furniture Problem Using Simplex Method

Write the constraint inequalities above as equations by introducing "slack variables". Slack variables are the variables that "pick up" the extra value needed to convert the inequalities into equalities. (Note: Slack variables must always be positive.) Use the equations that contain the slack variables to set up a simplex tableau. Make a simplex tableau for the furniture problem.

In this initial simplex tableau, determine the basic and nonbasic variables. Find the solution that corresponds to this initial tableau.

Pivot the simplex tableau about the nine and then find the values of each variable.

x	y	s_1	s_2	s_3	P	
3	2	1	0	0	0	66
(9)	4	0	1	0	0	180
2	10	0	0	1	0	200
–90	–75	0	0	0	1	0

→

x	y	s_1	s_2	s_3	P	
0	2/3	1	–1/3	0	0	6
1	4/9	0	1/9	0	0	20
0	82/9	0	–2/9	1	0	160
0	–35	0	10	0	1	1800

Pivot the simplex tableau about the two-thirds and then find the values of each variable.

x	y	s_1	s_2	s_3	P	
0	(2/3)	1	–1/3	0	0	6
1	4/9	0	1/9	0	0	20
0	82/9	0	–2/9	1	0	160
0	–35	0	10	0	1	1800

→

x	y	s_1	s_2	s_3	P	
0	1	3/2	–1/2	0	0	9
1	0	–2/3	1/3	0	0	16
0	0	–41/3	13/3	1	0	78
0	0	105/2	–15/2	0	1	2115

Pivot the simplex tableau about the thirteen-thirds and then find the values of each variable.

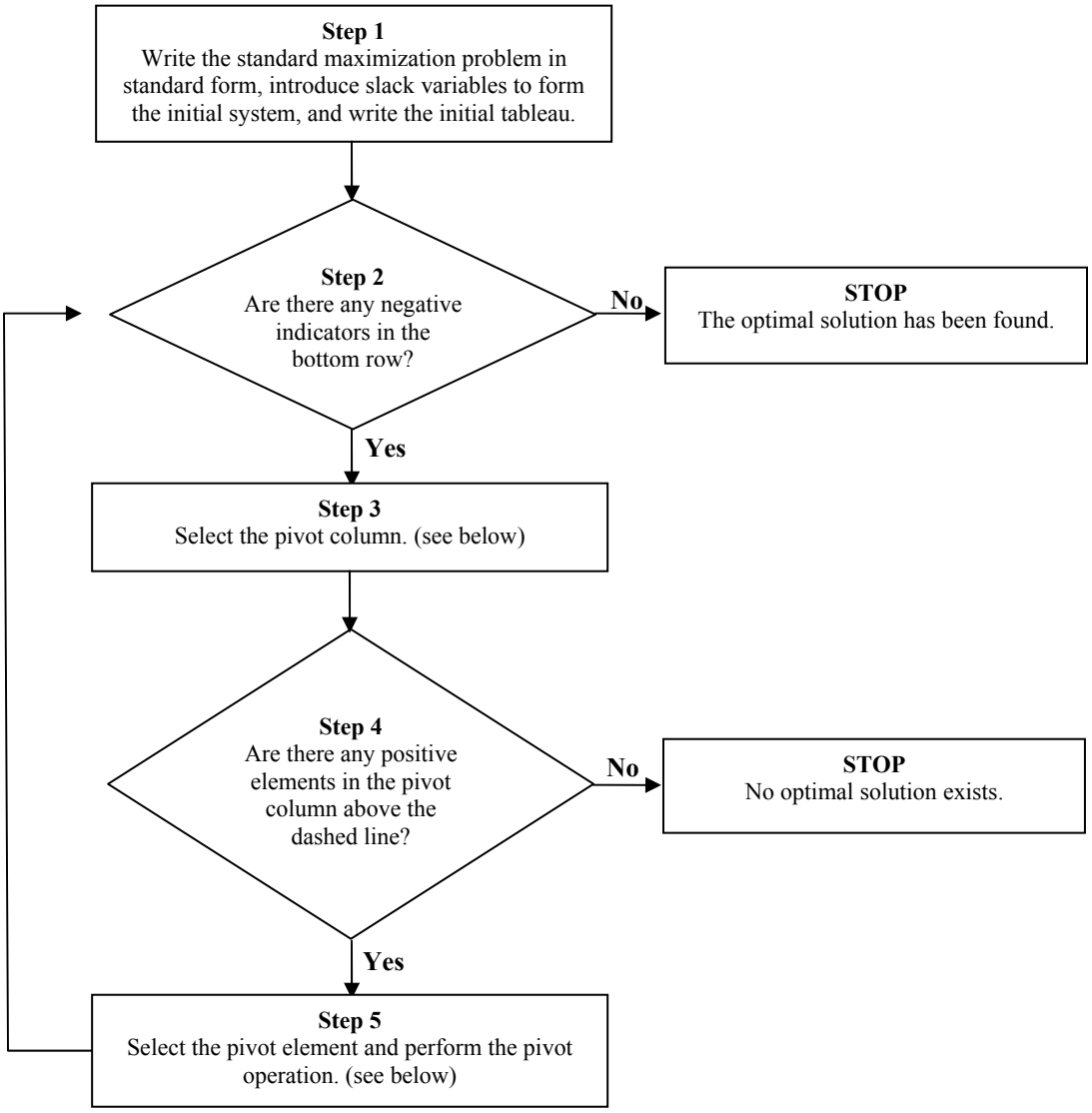
x	y	s_1	s_2	s_3	P	
0	1	3/2	–1/2	0	0	9
1	0	–2/3	1/3	0	0	16
0	0	–41/3	(13/3)	1	0	78
0	0	105/2	–15/2	0	1	2115

→

x	y	s_1	s_2	s_3	P	
0	1	–1/13	0	3/26	0	18
1	0	5/13	0	–1/13	0	10
0	0	–41/3	1	3/13	0	18
0	0	375/13	0	45/16	1	2250

Write the solution to the furniture problem.

The Simplex Procedure (taken from Barnett 10e)



Selecting the Pivot Element (*Note*: The pivot element is always positive and is never in the bottom row.)

Step 1: Locate the most negative number in the bottom row of the tableau. The column containing this element is the *pivot column*. If there is a tie for the most negative number, choose either.

Step 2: Divide each positive element in the pivot column above the dashed line into the corresponding element in the last column. The *pivot row* is the row corresponding to the smallest quotient obtained. If there is a tie for the smallest quotient, choose either. If the pivot column above the dashed line has not positive elements, there is no solution, and we stop.

Step 3: The *pivot element* is the element in the intersection of the pivot column and pivot row.

Remember… If it is not possible to select a pivot column, the simplex method stops and we conclude that the optimal solution has been found. If the pivot column has been selected and it is not possible to select a pivot row, the simplex method stops and we conclude that there is no optimal solution.

Selecting the Basic and Nonbasic Variables in the Tableau

Step 1: Determine the basic variables.

a) The number of basic variables is equal to the number of equations.

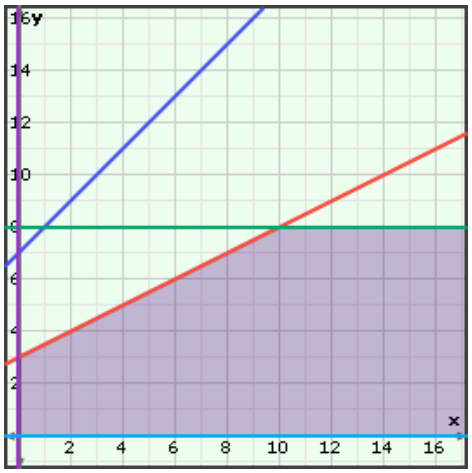
b) A variable can be selected as a basic variable only if it corresponds to a column in the tableau that has exactly one nonzero element (usually 1) and the nonzero element in the column is not in the same row as the nonzero element in the column of another basic variable. (Note: The objective function variable is always selected as a basic variable.)

Step 2: Determine the nonbasic variables.

a) After the basic variables are selected, the remaining variables are selected as nonbasic variables. (The tableau columns under the nonbasic variables usually contain more than one nonzero element.)

Example 2: Solve using the simplex method.

Maximize $P = 6x + 8y$ subject to $\begin{cases} -x + 2y \le 6 \\ -x + y \le 7 \\ y \le 8 \\ x, y \ge 0 \end{cases}$



x	y	s_1	s_2	s_3	P	
–1	2	1	0	0	0	6
–1	1	0	1	0	0	7
0	1	0	0	1	0	8
–6	–8	0	0	0	1	0

x	y	s_1	s_2	s_3	P	
–1/2	1	1/2	0	0	0	3
–1/2	0	–1/2	1	0	0	4
1/2	0	–1/2	0	1	0	5
–10	0	4	0	0	1	24

x	y	s_1	s_2	s_3	P	
0	1	0	0	1	0	8
0	0	–1	1	1	0	9
1	0	–1	0	2	0	10
0	0	–6	0	20	1	124

Example 3: A contractor is planning to build a new housing development consisting of colonial, split-level, and ranch-style houses. A colonial house requires one-half acre of land, $60,000 capital and 4,000 labor-hours to construct, and returns a profit of $20,000. A split-level house requires one-half acre of land $60,000 capital, and 3000 labor-hours to construct and returns a profit of $18,000. A ranch house requires 1 acre of land, $80,000 capital, and 4,000 labor-hours to construct, and returns a profit of $24,000. The contractor has available 30 acres of land, $3,200,000 capital, and 180,000 labor-hours. (Barnett 10e)*

a) How many houses of each type should be constructed to maximize the contactor's profit? What is the maximum profit.

Solution: Set up the objective function, constraints, and simplex tableau.

$$\left[\begin{array}{ccccccc|c} 1/2 & 1/2 & 1 & 1 & 0 & 0 & 0 & 30 \\ 2 & 2 & 0 & -8 & 1 & 0 & 0 & 80 \\ 2 & 1 & 0 & -4 & 0 & 1 & 0 & 60 \\ \hline -8000 & -6000 & 0 & 24000 & 0 & 0 & 1 & 720000 \end{array}\right]$$

$$\left[\begin{array}{ccccccc|c} 0 & 1/4 & 1 & 2 & 0 & -1/4 & 0 & 15 \\ 0 & 1 & 0 & -4 & 1 & -1 & 0 & 20 \\ 1 & 1/2 & 0 & -2 & 0 & 1/2 & 0 & 30 \\ \hline 0 & -2000 & 0 & 8000 & 0 & 4000 & 1 & 960000 \end{array}\right]$$

$$\left[\begin{array}{ccccccc|c} 0 & 0 & 1 & 3 & -1/4 & 0 & 0 & 10 \\ 0 & 1 & 0 & -4 & 1 & -1 & 0 & 20 \\ 1 & 0 & 0 & 0 & 1/2 & 1 & 0 & 20 \\ \hline 0 & 0 & 0 & 0 & 2000 & 2000 & 1 & 1000000 \end{array}\right]$$

b) A decrease in demand for colonial houses causes the profit on a colonial house to drop from $20,000 to $17,000. Discuss the effect of this change on the number of houses built and on the maximum profit.

Solution:

$$\left[\begin{array}{ccccccc|c} 1/2 & 1/2 & 1 & 1 & 0 & 0 & 0 & 30 \\ 2 & 2 & 0 & -8 & 1 & 0 & 0 & 80 \\ 2 & 1 & 0 & -4 & 0 & 1 & 0 & 60 \\ \hline -5000 & -6000 & 0 & 24000 & 0 & 0 & 1 & 720000 \end{array}\right]$$

$$\left[\begin{array}{ccccccc|c} 0 & 0 & 1 & 3 & -1/4 & 0 & 0 & 10 \\ 1 & 1 & 0 & -4 & 1/2 & 0 & 0 & 40 \\ 1 & 0 & 0 & 0 & -1/2 & 1 & 0 & 20 \\ \hline 1000 & 0 & 0 & 0 & 3000 & 0 & 1 & 960000 \end{array}\right]$$

A company manufactures three-speed, five-speed, and ten-speed bicycles. Each bicycle passes through three departments; fabrication, painting & plating, and final assembly. The relevant manufacturing data are given in the table below.

	Labor Hours per Bicycle			Maximum Labor-Hours Available per Day
	Three-Speed	Five-Speed	Ten-Speed	
Fabrication	3	4	5	120
Painting & Plating	5	3	5	130
Final Assembly	4	3	5	120
Profit per bicycle	$80	70	100	

(A) How many bicycles of each type should the company manufacture per day in order to maximize its profit? What is the maximum profit?

(B) Discuss the effect on the solution to part (A) if the profit on a ten-speed bicycle increases to $110 and all other data in part (A) remains the same.

(C) Discuss the effect on the solution to part (A) if the profit on a five-speed bicycle increases to $110 and all other data in part (A) remains the same.