Floyd’s Algorithm

All pairs shortest path

All pairs shortest path

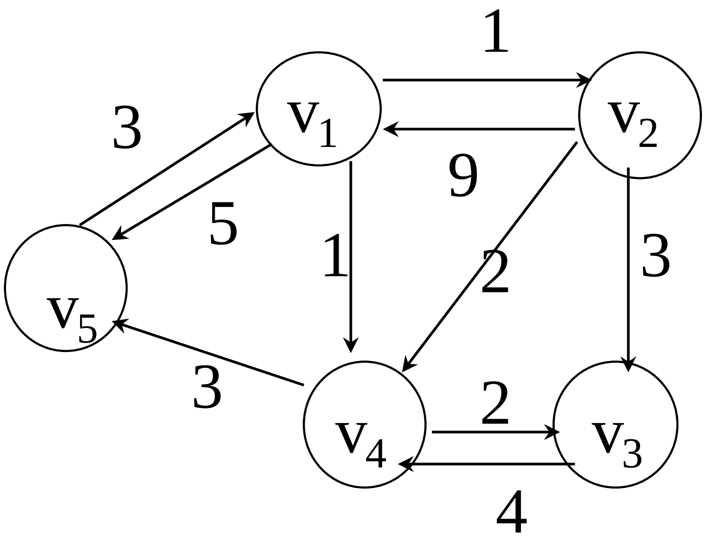
- ***The problem:*** **find the shortest path between every pair of vertices of a graph**

- ***The graph*****: may contain negative edges but no negative cycles**

- ***A representation*****: a weight matrix where**
  W(i,j)=0 if i=j.
  W(i,j)=∞ if there is no edge between i and j.
  W(i,j)="weight of edge"
- **Note: we have shown principle of optimality applies to shortest path problems**

The weight matrix and the graph

	1	2	3	4	5
1	0	1	∞	1	5
2	9	0	3	2	∞
3	∞	∞	0	4	∞
4	∞	∞	2	0	3
5	3	∞	∞	∞	0



The subproblems

- **How can we define the shortest distance $d_{i,j}$ in terms of "smaller" problems?**
- **One way is to restrict the paths to only include vertices from a restricted subset.**
- **Initially, the subset is empty.**
- **Then, it is incrementally increased until it includes all the vertices.**

The subproblems

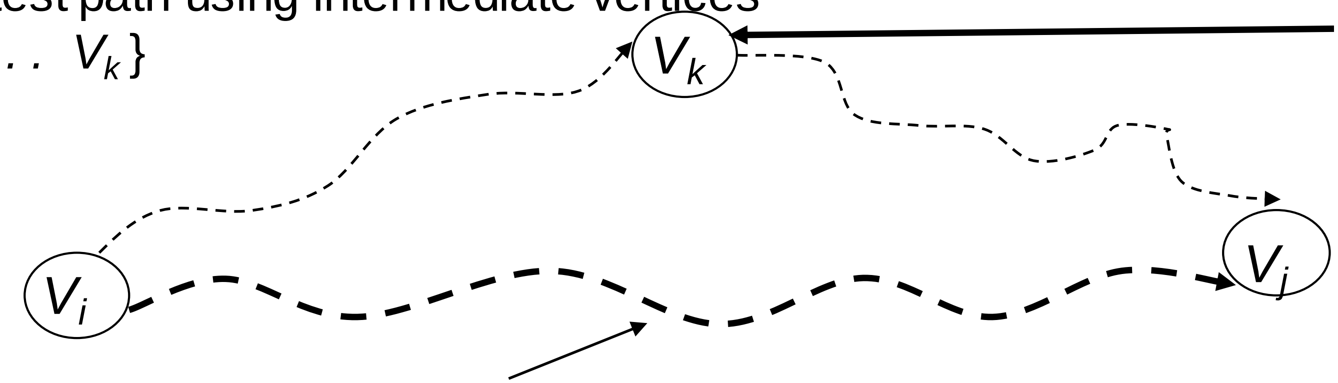
- **Let $D^{(k)}[i,j]$=weight of a shortest path from v_i to v_j using only vertices from $\{v_1,v_2,\ldots,v_k\}$ as intermediate vertices in the path**
  - $D^{(0)}=W$
  - $D^{(n)}=D$ **which is the goal matrix**
- **How do we compute $D^{(k)}$ from $D^{(k-1)}$?**

The Recursive Definition:

Case 1: A shortest path from v_i to v_j restricted to using only vertices from $\{v_1, v_2, \ldots, v_k\}$ as intermediate vertices does not use v_k. Then $D^{(k)}[i,j] = D^{(k-1)}[i,j]$.

Case 2: A shortest path from v_i to v_j restricted to using only vertices from $\{v_1, v_2, \ldots, v_k\}$ as intermediate vertices does use v_k. Then $D^{(k)}[i,j] = D^{(k-1)}[i,k] + D^{(k-1)}[k,j]$.

Shortest path using intermediate vertices $\{V_1, \ldots V_k\}$



Shortest Path using intermediate vertices $\{V_1, \ldots V_{k-1}\}$

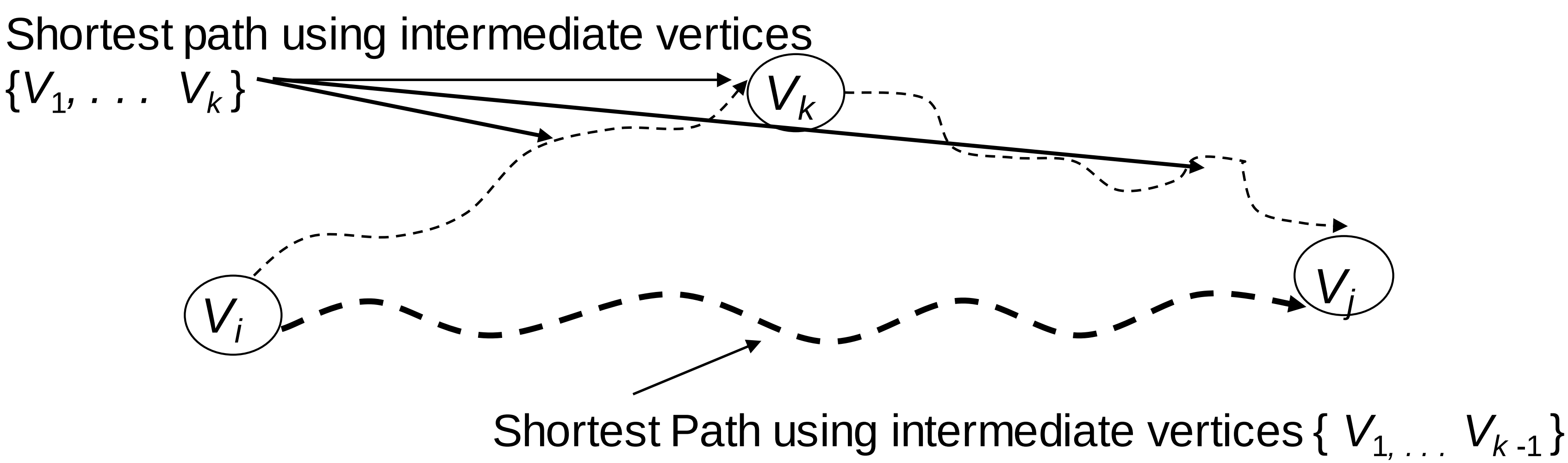
The recursive definition

- **Since**

$$D^{(k)}[i,j]= D^{(k-1)}[i,j] \text{ or}$$
$$D^{(k)}[i,j]= D^{(k-1)}[i,k]+ D^{(k-1)}[k,j].$$

We conclude:

$$D^{(k)}[i,j]= \min\{ D^{(k-1)}[i,j], D^{(k-1)}[i,k]+ D^{(k-1)}[k,j] \}.$$



The pointer array P

- **Used to enable finding a shortest path**
- **Initially the array contains 0**

- **Each time that a shorter path from *i* to *j* is found the *k* that provided the minimum is saved (highest index node on the path from *i* to *j*)**

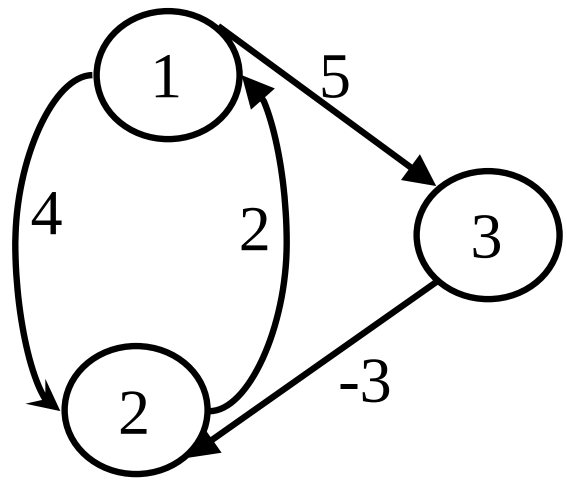
- **To print the intermediate nodes on the shortest path a recursive procedure that print the shortest paths from *i* and *k*, and from *k* to *j* can be used**

Floyd's Algorithm Using n+1 *D* matrices

Floyd//Computes shortest distance between all pairs of
//nodes, and saves P to enable finding shortest paths

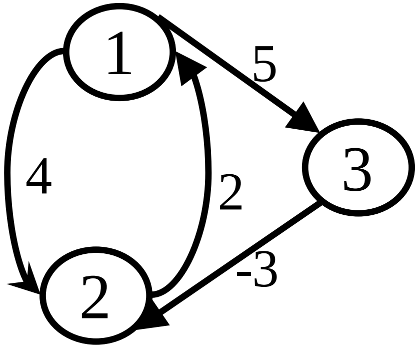
1. $D^0 \leftarrow W$ // initialize *D* array to *W* [ ]
2. $P \leftarrow 0$ // initialize P array to [0]
3. **for** $k \leftarrow 1$ **to** n
4. **do for** $i \leftarrow 1$ **to** n
5. **do for** $j \leftarrow 1$ **to** n
6. **if** $(D^{k-1}[i, j] > D^{k-1}[i, k] + D^{k-1}[k, j])$
7. **then** $D^k[i, j] \leftarrow D^{k-1}[i, k] + D^{k-1}[k, j]$
8. $P[i, j] \leftarrow k$;
9. **else** $D^k[i, j] \leftarrow D^{k-1}[i, j]$

Example



$$W = D^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & \infty \\ 3 & \infty & -3 & 0 \end{array}$$

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}$$



$D^0 =$

	1	2	3
1	0	4	5
2	2	0	∞
3	∞	-3	0

k = 1
Vertex 1 can be intermediate node

$D^1 =$

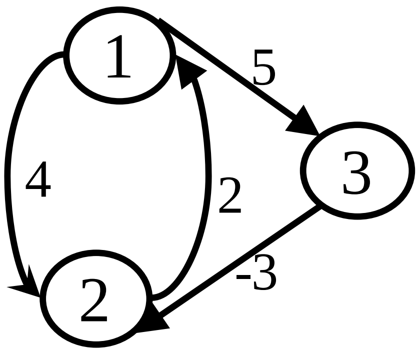
	1	2	3
1	0	4	5
2	2	0	7
3	∞	-3	0

$D^1[2,3] = \min(D^0[2,3], D^0[2,1]+D^0[1,3])$
$= \min (\propto, 7)$
$= 7$

P =

	1	2	3
1	0	0	0
2	0	0	1
3	0	0	0

$D^1[3,2] = \min(D^0[3,2], D^0[3,1]+D^0[1,2])$
$= \min (-3,\propto)$
$= -3$



$D^1 =$

	1	2	3
1	0	4	5
2	2	0	7
3	∞	-3	0

k = 2
Vertices 1, 2 can be intermediate

$D^2 =$

	1	2	3
1	0	4	5
2	2	0	7
3	-1	-3	0

$D^2[1,3] = \min(\, D^1[1,3],\, D^1[1,2]+D^1[2,3] \,)$

$= \min(5,\, 4+7)$

$= 5$

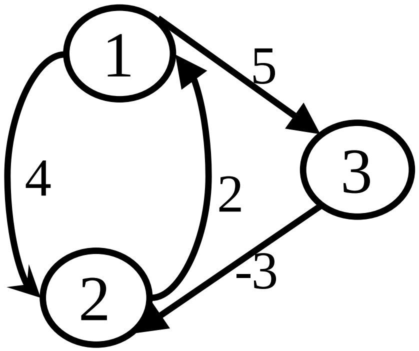
$P =$

	1	2	3
1	0	0	0
2	0	0	1
3	2	0	0

$D^2[3,1] = \min(\, D^1[3,1],\, D^1[3,2]+D^1[2,1] \,)$

$= \min(\propto,\, -3+2)$

$= -1$



$D^2 =$

	1	2	3
1	0	4	5
2	2	0	7
3	-1	-3	0

k = 3
Vertices 1, 2, 3 can be intermediate

$D^3 =$

	1	2	3
1	0	2	5
2	2	0	7
3	-1	-3	0

$$D^3[1,2] = \min(D^2[1,2], D^2[1,3]+D^2[3,2])$$
$$= \min(4, 5+(-3))$$
$$= 2$$

$P =$

	1	2	3
1	0	3	0
2	0	0	1
3	2	0	0

$$D^3[2,1] = \min(D^2[2,1], D^2[2,3]+D^2[3,1])$$
$$= \min(2, 7+(-1))$$
$$= 2$$

Floyd's Algorithm: Using 2 D matrices

Floyd

1. ***D*** $\leftarrow$ ***W*** // initialize *D* array to *W* [ ]

2. ***P*** $\leftarrow$ 0 // initialize P array to [0]

3. **for *k*** $\leftarrow$ **1 to *n***

// Computing D' from D

4. do for *i* $\leftarrow$ **1 to *n***

5. do for *j* $\leftarrow$ **1 to *n***

6. if (*D*[*i*, *j*] > *D*[*i*, *k*] + *D*[*k*, *j*])

7. then *D'*[*i*, *j*] $\leftarrow$ ***D*[*i*, *k*] + *D*[*k*, *j*]**

8. *P*[*i, j*] $\leftarrow$ ***k*;**

9. else *D'*[*i*, *j*] $\leftarrow$ ***D*[*i*, *j*]**

10. *Move D' to D.*

Can we use only one D matrix?

- ***D*[*i,j*] depends only on elements in the *k*th column and row of the distance matrix.**
- **We will show that the *k*th row and the *k*th column of the distance matrix are unchanged when D^k is computed**
- **This means *D* can be calculated *in-place***

The main diagonal values

- **Before we show that *k*th row and column of *D* remain unchanged we show that the main diagonal remains 0**

- $$\begin{aligned} D^{(k)}[\, j,j \,] &= \min\{\, D^{(k-1)}[\, j,j \,] \,,\ \ D^{(k-1)}[\, j,k \,] + D^{(k-1)}[\, k,j \,] \,\} \\ &= \min\{\, 0,\ \ D^{(k-1)}[\, j,k \,] + D^{(k-1)}[\, k,j \,] \,\} \\ &= 0 \end{aligned}$$

- **Based on which assumption?**

The *k*th column

- *k*th column of D^k is equal to the *k*th column of $D^{k\text{-}1}$
- *Intuitively true -* a path from i to k will not become shorter by adding k to the allowed subset of intermediate vertices
- For all i, $D^{(k)}[i,k] =$
  $= \min\{ D^{(k-1)}[i,k],\ D^{(k-1)}[i,k] + D^{(k-1)}[k,k] \}$
  $= \min\{ D^{(k-1)}[i,k], D^{(k-1)}[i,k]+0 \}$
  $= D^{(k-1)}[i,k]$

The *k*th row

- *k*th row of D^k is equal to the *k*th row of D^{k-1}

For all j, $D^{(k)}[k,j] =$

$$= \min\{ D^{(k-1)}[k,j], D^{(k-1)}[k,k] + D^{(k-1)}[k,j] \}$$
$$= \min\{ D^{(k-1)}[k,j], 0 + D^{(k-1)}[k,j] \}$$
$$= D^{(k-1)}[k,j]$$

Floyd's Algorithm using a single *D*

Floyd

1. ***D*** $\leftarrow$ ***W*** // initialize *D* array to *W* [ ]

2. ***P*** $\leftarrow$ 0 // initialize P array to [0]

3. **for** ***k*** $\leftarrow$ **1 to** ***n***

4. do for ***i*** $\leftarrow$ **1 to** ***n***

5. do for ***j*** $\leftarrow$ **1 to** ***n***

6. if (*D*[*i*, *j*] > *D*[*i*, *k*] + *D*[*k*, *j*])

7. then ***D*[*i*, *j*]** $\leftarrow$ ***D*[*i*, *k*] + *D*[*k*, *j*]**

8. *P*[*i*, *j*] $\leftarrow$ ***k*;**

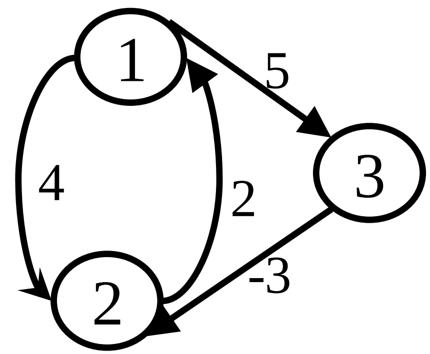
Printing intermediate nodes on shortest path from q to r

```
path(index q, r)
    if (P[ q, r ]!=0)
        path(q, P[q, r])
        println( "v"+ P[q, r])
        path(P[q, r], r)
        return;
    //no intermediate nodes
    else return
```

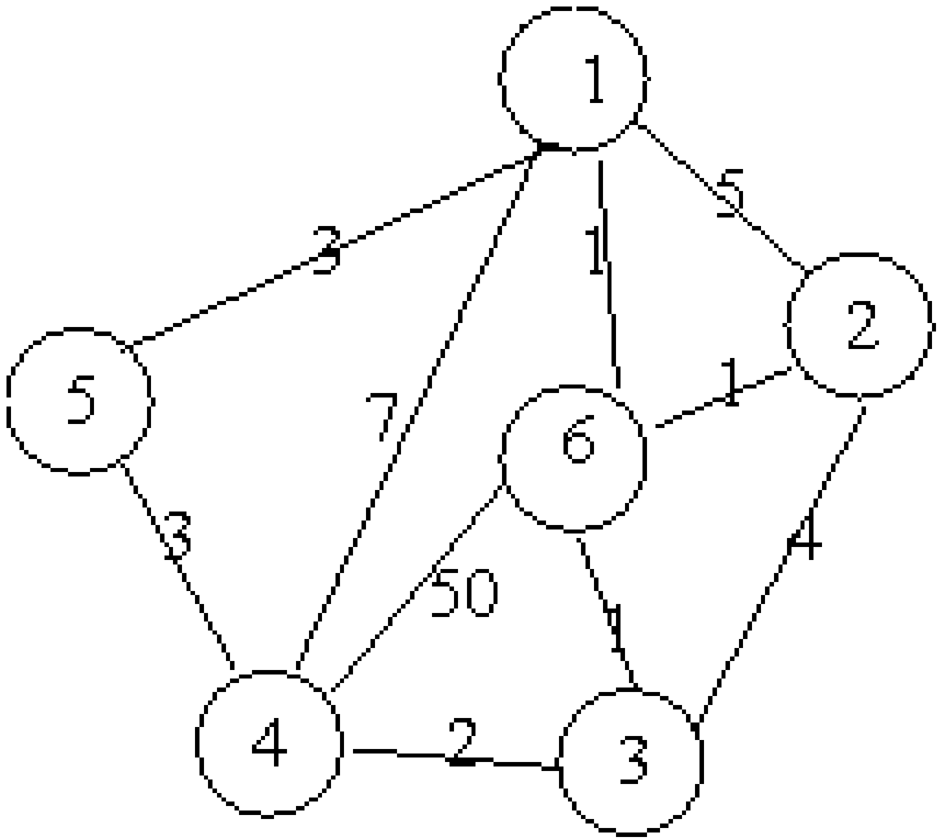
P =

	1	2	3
1	0	3	0
2	0	0	1
3	2	0	0

Before calling path check D[q, r] < ∞, and print node q, after the call to path print node r



Example

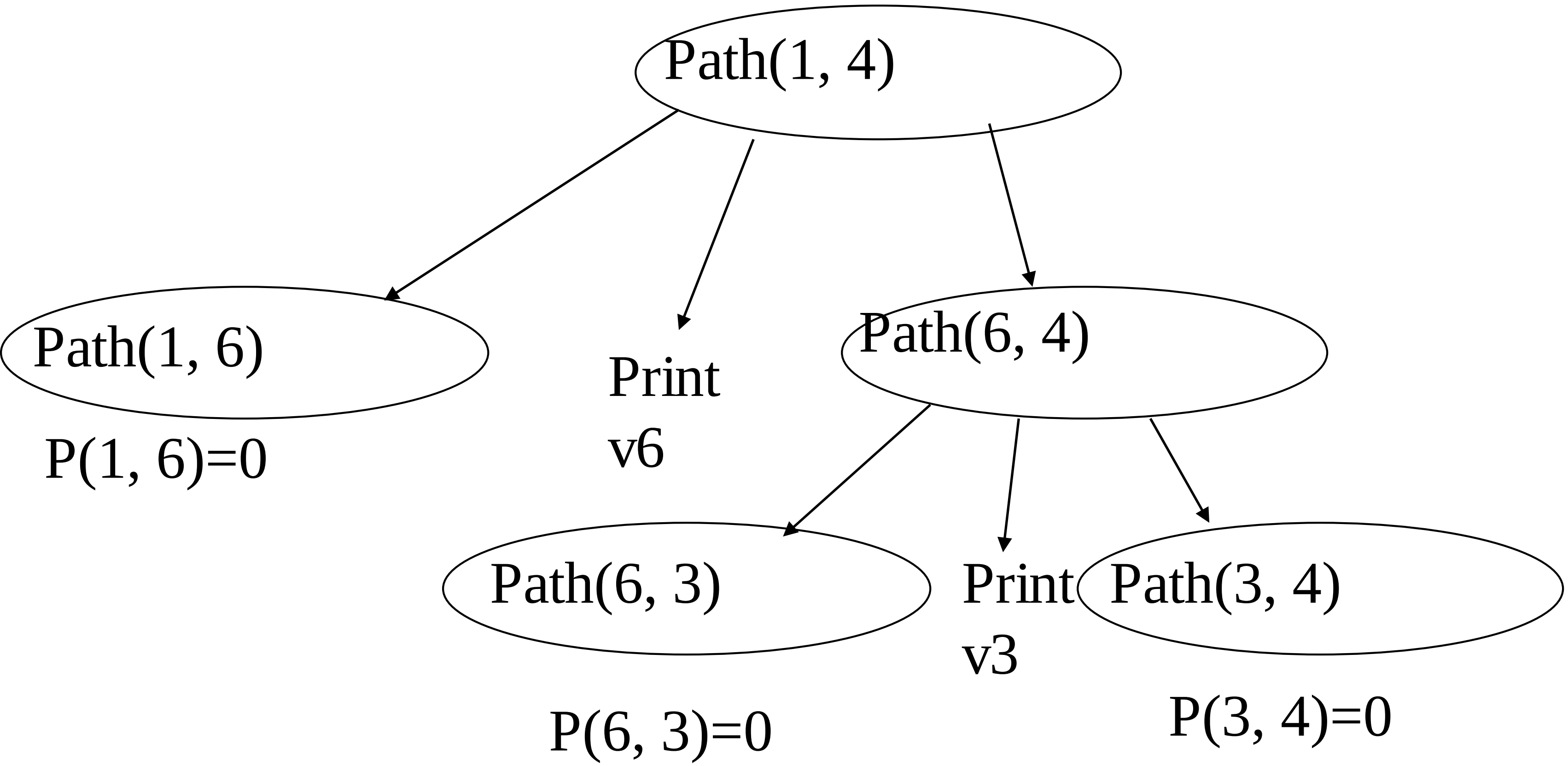


The final distance matrix and P

$$D^6 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 2(6) & 2(6) & 4(6) & 3 & 1 \\ 2 & 2(6) & 0 & 2(6) & 4(6) & 5(6) & 1 \\ 3 & 2(6) & 2(6) & 0 & 2 & 5(4) & 1 \\ 4 & 4(6) & 4(6) & 2 & 0 & 3 & 3(3) \\ 5 & 3 & 5(6) & 5(4) & 3 & 0 & 4(1) \\ 6 & 1 & 1 & 1 & 3(3) & 4(1) & 0 \end{array}$$

The values in parenthesis are the non zero P values.

The call tree for Path(1, 4)



The intermediate nodes on the shortest path from 1 to 4 are v6, v3.
The shortest path is v1, v6, v3, v4.