Problem 1.

Use the pumping lemma to prove that $L = \{ww : w \in \{0,1\}*\}$ is not regular.

Solution

Proof. via contradiction
Assume that L is regular, that is, the pumping lemma holds on L. First, note that L is clearly infinite, thus the pumping lemma will not vacuously hold.

Let $p \in \mathbb{Z}^+$ such that the pumping lemma holds by our assumption on p.
Now define w such that $|w| = 4p$ and
$w_i = 0$ for $0 \leq i < p$,
$w_i = 1$ for $p \leq i < 2p$,
$w_i = 0$ for $2p \leq i < 3p$,
$w_i = 1$ for $3p \leq i < 4p$
Note that $w \in L$.
Then, by the pumping lemma we must have that $w = xyz$ where
(1) $|y| \geq 1 \wedge$ (2) $|xy| \leq p \wedge$ (3) $xy^n z \in L \ \forall n \geq 0$.
Note that by construction of w, xy will be a sequence of repeated 0's, as $|xy| \leq p$ and $\forall\ 0 < p, w_i = 0$.

Now consider $xy^n z$ where $n \geq 0$. By assumption, we have that $xy^0 z \in L$, that is $xz \in L$, where $xy = 0^k 1^p 0^p 1^p$, for some $0 \leq k < p$ (as we have removed at least 1 zero by removing y). Assume $w = vv, v \in \{0,1\}^*$ by definition of L, then $|v| = \frac{3p+k}{2}$.
$|v| = \frac{3p+k}{2} = 1.5p + 0.5k$, then $p < k \Rightarrow 0.5p < 0.5k \Rightarrow p + k < 1.5p + 0.5k = |v|$
further $|v| = 1.5p + 0.5k < 2p$

Thus we have, $p + k < |v| < 2p$. That is to say, v splits our string somewhere in the first group of 1's. Then $v = 0^k 1^p 0^m = 0^n 1^p$, where $m, n \in \mathbb{Z}^+, m + n = p$.
However, $0^k 1^p 0^m$ clearly ends in 0 where $0^n 1^p$ clearly ends in $1 \Rightarrow v \neq v$. This is a contradiction, the pumping lemma fails, and this language must be nonregular. □

Problem 2.

Prove that $L = \{w \in \{a, b\}^* : |w|_a \neq |w|_b\}$ is nonregular.

Solution

Proof. TODO $\square$