Problem 1.

Reproduce the text and figure exactly as it appears, including the rectangular border.

Solution

$$f(x) = \begin{cases} 2x + y \\ 3x + 2z \\ 4x + 2y + 3z \end{cases}$$

9	9	9	9
6	6	6	
3		3	3

Problem 2.

Show that $\mathbb{N}$ and $\mathbb{Z}$ are equinumerous.

Solution

Proof. The sets $\mathbb{N}$ and $\mathbb{Z}$ are equinumerous iff $|\mathbb{N}| = |\mathbb{Z}|$. By definition of cardinality, this is true iff there exists some function f which forms a bijection between the given sets.

Consider the function $f : \mathbb{N} \to \mathbb{Z}$ defined as follows

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ x/2 & \text{if } x \text{ is even} \\ -(x-1)/2 & \text{otherwise} \end{cases}$$

Injective.
Assume $f(x) = f(y)$

case $f(x) = f(y) = 0$:
Note that x is clearly not even, as $f(x) = x/2 > 0$.
Further, x is not odd and greater than 1 as $(x - 1) > 0 \Rightarrow f(x) = -(x - 1)/2 < 0$.
Therefore, we conclude $x = 1$. By the same logic, $y = 1$.
Thus we have that $x = y = 1$.
case $f(x) = f(y) > 0$:
Note that neither x, y cannot be 1, else $f(x) = f(y) = 0$. Further x, y must be even, else $f(x) < 0$.
Thus, $f(x) = f(y)$
$\Rightarrow x/2 = y/2$
$\Rightarrow 2x/2 = 2y/2$
$\Rightarrow x = y$.
case $f(x) = f(y) < 0$:
Following the same argument as above, we note that x, y must both be odd and $\neq 1$.
Thus, $f(x) = f(y)$
$\Rightarrow -(x - 1)/2 = -(y - 1)/2$
$\Rightarrow -(-2)(x - 1)/2 = -(-2)(y - 1)/2$
$\Rightarrow x - 1 = y - 1$
$\Rightarrow x - 1 + 1 = y - 1 + 1$

$\Rightarrow x = y$

Surjective.
let $y \in \mathbb{Z}$

case $y = 0$:
choose $x \in \mathbb{N} = 1$, then $f(x) = f(1) = 0$.
case $y > 0$:
choose $x = 2y$, where $y > 0 \wedge y \in \mathbb{Z} \Rightarrow y \in \mathbb{N} \Rightarrow x = 2y \in \mathbb{N}$.
Further note that $x = 2y$ is even.
Thus $f(x) = f(2y) = 2y/2 = y$.
case $y < 0$:
choose $x = -2y + 1$, where $y < 0 \wedge y \in \mathbb{Z} \Rightarrow -y \in \mathbb{N} \Rightarrow x = -2y + 1 \in \mathbb{N}$
Further note that $x = -2y + 1$ is odd.
Thus $f(x) = f(-2y + 1) = -(-2y + 1 - 1)/2 = -(-2y)/2 = 2y/2 = y$.

Thus, as f forms a bijection on $\mathbb{N}$ and $\mathbb{Z}$ we conclude that $\mathbb{N}$ and $\mathbb{Z}$ are equinumerous. $\square$

Problem 3.

Let $f : S \to S$ be a total function. Prove that if S is infinite, f can be (a) one-one without being onto, and (b) onto without being one-one.

Solution

Proof. For both examples, assume $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ serves as an example of an infinite set S.

Part (a)

Define

$$f(x) = x + 1$$

If we let $f(x) = f(y)$ then $x + 1 = y + 1 \Rightarrow x = y$. Clearly f is one-one.
However, consider $f(x) = 1$, then $x + 1 = 1 \Rightarrow x = 0$ but $0 \notin \mathbb{N}$.
Therefore f is not onto as $1 \in \mathbb{N}$ is not reachable from its (infinite) domain.

Part (b)

Define

$$f(x) = \begin{cases} 1 & x \text{ is even} \\ x/2 & x \text{ is odd} \end{cases}$$

Let $y \in \mathbb{N}$, then $f(2y) = (2y)/2 = y$, noting 2y even, therefore f is onto.
However, $f(1) = f(3) = f(5) = \cdots = 1$, clearly f is not one-one. $\square$

Problem 4.

Show that the relation R defined

$$\forall m, n \in \mathbb{N}, (m, n) \in R \iff (m - n) \bmod 3 = 0$$

is an equivalence relation, and describe its equivalence classes.

Solution

Proof. An equivalence relation is one that is reflexive, symmetric, and transitive...

Reflexive.
let $a \in \mathbb{N}$
$(a - a) \bmod 3 = 0 \bmod 3 = 0 \Rightarrow (a, a) \in R$

Symmetric.
let $a, b \in \mathbb{N}$
assume $(a, b) \in R$ then $(a - b) \bmod 3 = 0$
note that $(a - b) = -(b - a)$
then $(b - a) \bmod 3 = -(a - b) \bmod 3 = -((a - b) \bmod 3) = -0 = 0$
therefore $(b, a) \in R$.

Transitive.
let $a, b, c \in \mathbb{N}$
assume $(a, b) \in R \ \wedge \ (b, c) \in R$.
then $(a - b) \bmod 3 = 0 \ \wedge \ (b - c) \bmod 3 = 0$.
consider $(a - c) \bmod 3 = (a - c + b - b) \bmod 3 = ((a - b) + (b - c)) \bmod 3$
we can distribute the modulus operator...
$(a - c) \bmod 3 = ((a - b) \bmod 3) + ((b - c) \bmod 3) \bmod 3$
$= (0 + 0) \bmod 3 = 0 \bmod 3 = 0$
therefore $(a, c) \in R$.

$\square$

Each equivalence class contains any numbers who's differences form multiples of 3. Thus there are 3 equivalence class. Natural numbers who are offset by $3k$ from 0, those who are offset by $3k$ from 1, and those who are offset by $3k$ from 2 (for some $k \in \mathbb{N}$).
$\{0, 3, 6, 9, 12 \ldots\}$
$\{1, 4, 7, 10, 13 \ldots\}$
$\{2, 5, 8, 11, 14 \ldots\}$

Problem 5.

Show that $\sum_{i=1}^{n} i^2 = (2n+1)(n+1)n/6$.

Solution

Proof. via induction on n

Base Case.
let $n = 1$
then $\sum_{i=1}^{n} i^2 = n^2 = 1^2 = 1$
and $(2n+1)(n+1)n/6 = (2 \cdot 1 + 1)(1+1) \cdot 1/6$
$= 3(2)/6 = 6/6 = 1$

Induction Step.

assume $\exists\ n \in \mathbb{N} : \sum_{i=1}^{n} i^2 = (2n+1)(n+1)n/6.$

then $\sum_{i=1}^{n} i^2 + (n+1)^2 = (2n+1)(n+1)n/6 + (n+1)^2$

$= \sum_{i=1}^{(n+1)} i^2 = (2n+1)(n+1)n/6 + n(n+2) + 1$

$= (2n+1)(n+1)n/6 + 6(n(n+2)+1)/6$

$= ((2n^3 + 3n^2 + n) + (6n^2 + 12n + 6))/6$

$= (2n^3 + 9n^2 + 13n + 6)/6$

$= (2n+3)(n+2)(n+1)/6$

$= \sum_{i=1}^{n+1} i^2 = (2(n+1)+1)((n+1)+1)(n+1)/6$

$\square$