Problem 1.

Use the pumping lemma to prove that $L = \{ww : w \in \{0,1\}*\}$ is not regular.

Solution

Proof. via contradiction
Assume that L is regular, that is, the pumping lemma holds on L. First, note that L is clearly infinite, thus the pumping lemma will not vacuously hold.

Let $p \in \mathbb{Z}^+$ such that the pumping lemma holds forr p.
Now define w such that $|w| = 4p$ and
$w_i = 0$ for $0 \leq i < p$,
$w_i = 1$ for $p \leq i < 2p$,
$w_i = 0$ for $2p \leq i < 3p$,
$w_i = 1$ for $3p \leq i < 4p$
Note that $w \in L$.
Then, by the pumping lemma we must have that $w = xyz$ where
(1) $|y| \geq 1 \wedge$ (2) $|xy| \leq p \wedge$ (3) $xy^n z \in L\ \forall n \geq 0$.
Note that by construction of w, xy will be a sequence of repeated 0's, as $|xy| \leq p$ and $\forall\ 0 < p, w_i = 0$.

Now consider $xy^n z$ where $n \geq 0$. By assumption, we have that $xy^0 z \in L$, that is $xz \in L$, where $xy = 0^k 1^p 0^p 1^p$, for some $0 \leq k < p$ (as we have removed at least 1 zero by removing y). We know $w \in L$ by the pumping lemma, thus let $w = vv, v \in \{0,1\}^*$ then $|v| = \frac{3p+k}{2}$.
$|v| = \frac{3p+k}{2} = 1.5p + 0.5k$, then $p < k \Rightarrow 0.5p < 0.5k \Rightarrow p + k < 1.5p + 0.5k = |v|$
further $|v| = 1.5p + 0.5k < 2p$

Thus we have, $p + k < |v| < 2p$. That is to say, v splits our string somewhere in the first group of 1's. Then $v = 0^k 1^p 0^m = 0^n 1^p$, where $m, n \in \mathbb{Z}^+, m + n = p$.
However, $0^k 1^p 0^m$ clearly ends in 0 where $0^n 1^p$ clearly ends in $1 \Rightarrow v \neq v$. This is a contradiction, the pumping lemma fails, and this language must be nonregular. $\square$

Problem 2.

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Prove that $L = \{w \in \{a,b\}^* : |w|_a \neq |w|_b\}$ is nonregular.

Solution

Proof. via contradiction
Assume that L is regular. Then, by the closure properties of regular languages, we have that $\overline{L}$ is regular. Where $\overline{L} = \{w \in \{a,b\}^* : |w|_a = |w|_b\}$ for $U = \{a,b\}^*$. As $\overline{L}$ is regular, the pumping lemma must hold on $\overline{L}$. Let p justify this fact.
Now define $w = a^p b^p$. Note that $|w|_a = |w|_b = p$, thus $w \in \overline{L}$. Apply the pumping lemma on w, by selecting $w = xyz$. As $|xy| \leq p$ we have that $xy = a^{|xy|}$ then $y = a^{|y|}$. Define $k = |p| > 0$, then $y = a^k$. Now, pump y for $n = 0$. Thus we create a new string $w' = xz$, $w' \in \overline{L}$ by the pumping lemma. In w we had $|w|_a = p$, but we have now removed $k > 0$ of them, thus $|w'|_a = (p-k) \neq p = |w|_b = |w'|_b$ (note we haven't removed any b's). As we now know $|w'|_a \neq |w'|_b$ we conclude $w \notin \overline{L}$. This contradicts the pumping lemma, thus via contradiction, $\overline{L}$ is not regular, therefore L is not regular (again by the closure properties). □