

**Problem 1.**

Reproduce the text and figure exactly as it appears, including the rectangular border.

**Solution**

This is an inline equation:  $x + y = 3$ .

This is a displayed equation:

$$x + \frac{y}{z - \sqrt{3}} = 2.$$

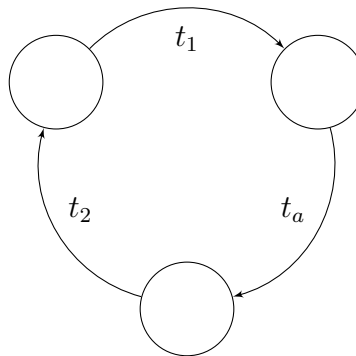
This is how you can define a piece-wise linear function:

$$f(x) = \begin{cases} 3x + 2 & \text{if } x < 0 \\ 7x + 2 & \text{if } x \geq 0 \text{ and } x < 10 \\ 5x + 22 & \text{otherwise.} \end{cases}$$

This is a matrix:

9	9	9	9
6	6	6	
3		3	3

This is a figure incorporated in a LaTeX file



**Problem 2.**

Show that  $\mathbb{N}$  and  $\mathbb{Z}$  are equinumerous.

**Solution**

*Proof.* The sets  $\mathbb{N}$  and  $\mathbb{Z}$  are equinumerous iff  $|\mathbb{N}| = |\mathbb{Z}|$ . By definition of cardinality, this is true iff there exists some function  $f$  which forms a bijection between the given sets.

Consider the function  $f : \mathbb{N} \rightarrow \mathbb{Z}$  defined as follows

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ x/2 & \text{if } x \text{ is even} \\ -(x-1)/2 & \text{otherwise} \end{cases}$$

**Injective.**

Assume  $f(x) = f(y)$

case  $f(x) = f(y) = 0$ :

Note that  $x$  is clearly not even, as  $f(x) = x/2 > 0$ .

Further,  $x$  is not odd and greater than 1 as  $(x-1) > 0 \Rightarrow f(x) = -(x-1)/2 < 0$ .

Therefore, we conclude  $x = 1$  as  $f(1) = 0$ . By the same logic,  $y = 1$ .

Thus we have that  $x = y = 1$ .

case  $f(x) = f(y) > 0$ :

Note that neither  $x, y$  cannot be 1, else  $f(x) = f(y) = 0$ . Further  $x, y$  must be even, else  $f(x) < 0$ .

Thus,  $f(x) = f(y)$

$$\Rightarrow x/2 = y/2$$

$$\Rightarrow 2x/2 = 2y/2$$

$$\Rightarrow x = y.$$

case  $f(x) = f(y) < 0$ :

Following the same argument as above, we note that  $x, y$  must both be odd and  $\neq 1$ .

Thus,  $f(x) = f(y)$

$$\Rightarrow -(x-1)/2 = -(y-1)/2$$

$$\Rightarrow -(-2)(x-1)/2 = -(-2)(y-1)/2$$

$$\Rightarrow x-1 = y-1$$

$$\Rightarrow x-1+1 = y-1+1$$

$$\Rightarrow x = y$$

**Surjective.**

let  $y \in \mathbb{Z}$

case  $y = 0$ :

choose  $x \in \mathbb{N} = 1$ , then  $f(x) = f(1) = 0$ .

case  $y > 0$ :

choose  $x = 2y$ , where  $y > 0 \wedge y \in \mathbb{Z} \Rightarrow y \in \mathbb{N} \Rightarrow x = 2y \in \mathbb{N}$ .

Further note that  $x = 2y$  is even.

Thus  $f(x) = f(2y) = 2y/2 = y$ .

case  $y < 0$ :

choose  $x = -2y + 1$ , where  $y < 0 \wedge y \in \mathbb{Z} \Rightarrow -y \in \mathbb{N} \Rightarrow x = -2y + 1 \in \mathbb{N}$

Further note that  $x = -2y + 1$  is odd.

Thus  $f(x) = f(-2y + 1) = -(-2y + 1 - 1)/2 = -(-2y)/2 = 2y/2 = y$ .

Thus, as  $f$  forms a bijection on  $\mathbb{N}$  and  $\mathbb{Z}$  we conclude that  $\mathbb{N}$  and  $\mathbb{Z}$  are equinumerous.  $\square$

**Problem 3.**

Let  $f : S \rightarrow S$  be a total function. Prove that if  $S$  is infinite,  $f$  can be (a) one-one without being onto, and (b) onto without being one-one.

**Solution**

*Proof.* For both examples, assume  $f : \mathbb{N} \rightarrow \mathbb{N}$ , where  $\mathbb{N}$  serves as an example of an infinite set  $S$ .

**Part (a)**

Define

$$f(x) = x + 1$$

Note that  $f$  is defined  $\forall x \in \mathbb{N}$  thus  $f$  is total.

If we let  $f(x) = f(y)$  then  $x + 1 = y + 1 \Rightarrow x = y$ . Clearly  $f$  is one-one.

However, consider  $f(x) = 1$ , then  $x + 1 = 1 \Rightarrow x = 0$  but  $0 \notin \mathbb{N}$ .

Therefore  $f$  is not onto as  $1 \in \mathbb{N}$  is not reachable from its (infinite) domain.

**Part (b)**

Define

$$f(x) = \begin{cases} 1 & x \text{ is even} \\ x/2 & x \text{ is odd} \end{cases}$$

Note that  $f$  is defined  $\forall x \in \mathbb{N}$  thus  $f$  is total.

Let  $y \in \mathbb{N}$ , then  $f(2y) = (2y)/2 = y$ , noting  $2y$  even, therefore  $f$  is onto.

However,  $f(1) = f(3) = f(5) = \dots = 1$ , clearly  $f$  is not one-one. □

**Problem 4.**

Show that the relation  $R$  defined

$$\forall m, n \in \mathbb{N}, (m, n) \in R \iff (m - n) \bmod 3 = 0$$

is an equivalence relation, and describe its equivalence classes.

**Solution**

*Proof.* An equivalence relation is one that is reflexive, symmetric, and transitive...

**Reflexive.**

let  $a \in \mathbb{N}$

$$(a - a) \bmod 3 = 0 \bmod 3 = 0 \Rightarrow (a, a) \in R$$

**Symmetric.**

let  $a, b \in \mathbb{N}$

assume  $(a, b) \in R$  then  $(a - b) \bmod 3 = 0$

note that  $(a - b) = -(b - a)$

$$\text{then } (b - a) \bmod 3 = -(a - b) \bmod 3 = -((a - b) \bmod 3) = -0 = 0$$

therefore  $(b, a) \in R$ .

**Transitive.**

let  $a, b, c \in \mathbb{N}$

assume  $(a, b) \in R \wedge (b, c) \in R$ .

$$\text{then } (a - b) \bmod 3 = 0 \wedge (b - c) \bmod 3 = 0.$$

$$\text{consider } (a - c) \bmod 3 = (a - c + b - b) \bmod 3 = ((a - b) + (b - c)) \bmod 3$$

we can distribute the modulus operator...

$$\begin{aligned} (a - c) \bmod 3 &= ((a - b) \bmod 3) + ((b - c) \bmod 3) \bmod 3 \\ &= (0 + 0) \bmod 3 = 0 \bmod 3 = 0 \end{aligned}$$

therefore  $(a, c) \in R$ .

□

Each equivalence class contains any numbers whose differences form multiples of 3. Thus there are 3 equivalence classes. Natural numbers who are offset by  $3k$  from 0, those who are offset by  $3k$  from 1, and those who are offset by  $3k$  from 2 (for some  $k \in \mathbb{N}$ ).

$$\{0, 3, 6, 9, 12, \dots\}$$

$$\{1, 4, 7, 10, 13, \dots\}$$

$$\{2, 5, 8, 11, 14, \dots\}$$

**Problem 5.**

Show that  $\sum_{i=1}^n i^2 = (2n+1)(n+1)n/6$ .

**Solution**

*Proof. via induction on  $n$*

**Induction Basis.**

let  $n = 1$

then  $\sum_{i=1}^n i^2 = n^2 = 1^2 = 1$

and  $(2n+1)(n+1)n/6 = (2 \cdot 1 + 1)(1 + 1) \cdot 1/6$   
 $= 3(2)/6 = 6/6 = 1$

**Induction Basis.**

$\exists k \in \mathbb{N} : \sum_{i=1}^k i^2 = (2k+1)(k+1)k/6$ .

**Induction Step.**

assume  $\exists n \in \mathbb{N} : \sum_{i=1}^n i^2 = (2n+1)(n+1)n/6$  via Induction Hypothesis.

then  $\sum_{i=1}^n i^2 + (n+1)^2 = (2n+1)(n+1)n/6 + (n+1)^2$

$= \sum_{i=1}^{(n+1)} i^2 = (2n+1)(n+1)n/6 + n(n+2) + 1$

$= (2n+1)(n+1)n/6 + 6(n(n+2) + 1)/6$

$= ((2n^3 + 3n^2 + n) + (6n^2 + 12n + 6))/6$

$= (2n^3 + 9n^2 + 13n + 6)/6$

$= (2n+3)(n+2)(n+1)/6$

$= \sum_{i=1}^{n+1} i^2 = (2(n+1)+1)((n+1)+1)(n+1)/6$

□