Problem 1.

Given

0: λf.λx.x
succ: λn.λf.λx.(f ((n f) x))
n: if n is a natural number then its semantics is the result of n applications of succ on 0.
true: λx.λy.x
false: λx.λy.y
second: λx.λy.λz.y
g: λn.((n second) false)

What is the result of

(a) (g n) when n is 0.

(b) (g n) when n results from some application succ on 0.

(c) What mathematical/logical operation is computed by g.

Solution

Part (a)

(g 0)
(g λf.λx.x)
(λn.((n second) false) λf.λx.x)
((λf.λx.x second) false)
(λx.x false)
false

Part (b)

First consider what is happening when n applications of succ are taken on 0.

Consider n = 1 application of succ on 0.
(succ 0)
(λn.λf.λx.(f ((n f) x)) λf.λx.x)
λf.λx.(f ((λf.λx.x f) x)
λf.λx.(f (λx.x x))
λf.λx.(f x)

Now consider n = 2 applications of succ on 0.
(succ (succ 0))
(succ λf.λx.(f x))
(λn.λf.λx.(f ((n f) x)) λf.λx.(f x))
λf.λx.(f ((λf.λx.(f x) f) x))
λf.λx.(f (λx.(f x) x))
λf.λx.(f (f x))

As discussed in class, this pattern will continue.
That is, n applications of succ on 0 can be written as the following.
λf.λx.(f^1 (... (f^n x)))
Note the superscript denotes the occurrence index for the lambda function f.

Now consider (g n)
(g n)
(g ($succ_1$ (...($succ_n$ 0))))
(λn.((n second) false) ($succ_1$ (...($succ_n$ 0))))
((($succ_1$ (...($succ_n$ 0))) second) false)
((λf.λx.(f^1 (... (f^n x))) second) false)
((λf.λx.(f^1 (... (f^n x))) λx.λy.λz.y) false)
As these are lambda expressions, subscripts denote both the number of occurrences as well as the uniqueness of each lambda variable.
(λx.(λx_1.λy_1.λz_1.y_1 (... (λx_n.λy_n.λz_n.y_n x))) false)
(λx_1.λy_1.λz_1.y_1 (... (λx_n.λy_n.λz_n.y_n false)))
(λx_1.λy_1.λz_1.y_1 (... (λx_{n-1}.λy_{n-1}.λz_n.y_{n-1} (λx_n.λy_n.λz_n.y_n false)))
(λx_1.λy_1.λz_1.y_1 (... (λx_{n-1}.λy_{n-1}.λz_n.y_{n-1} λy_n.λz_n.y_n)))
Note that λy.λz.y is defined as 'true'.
(λx_1.λy_1.λz_1.y_1 (... (λx_{n-1}.λy_{n-1}.λz_n.y_{n-1} true)))
If n = 1 we would be done here and have a result of 'true'
(as the 1...n-1 terms would not exist).
If n > 1, we apply another beta reduction
(λx_1.λy_1.λz_1.y_1 (... (λx_{n-2}.λy_{n-2}.λz_n.y_{n-2} λy_{n-1}.λz_n.y_{n-1}))
(λx_1.λy_1.λz_1.y_1 (... (λx_{n-2}.λy_{n-2}.λz_n.y_{n-2} true)))
(λx_1.λy_1.λz_1.y_1 (... λy_{n-2}.λz_n.y_{n-2}))
(λx_1.λy_1.λz_1.y_1 (... (λx_{n-3}.λy_{n-3}.λz_n.y_{n-3} true)))
Again note that if n = 2, we would be done here and have a result of 'true'.
If n > 3 this trend would continue recursively until resolving to true.

Part (c)

As g resolves to false when n = 0, and true otherwise
it then computes n > 0 or n $\neq$ 0 on the domain of natural numbers.

Problem 2.

Consider the following λ-expression:

$$Y : \lambda\text{t}.(\lambda\text{x}.(\text{t (x x)})\ \lambda\text{x}.(\text{t (x x)}))$$

Prove/disprove that (Y t) after application of several β-reductions results in (t (Y t)).

Solution

Setup

We are given (Y t), apply the definition of Y to this given lambda expression.
(λt.(λx.(t (x x)) λx.(t (x x))) t)
Denote Q: λx.(t (x x))
Note that (Q Q) after a single β-reduction results in (t (Q Q))
Thus we are starting with (t (Q Q)).

Claim

let n $\in N$
If n β-reductions are taken of (t (Q Q)) denoted R
then R $\neq$ (t (Y t))

Proof by Induction on n

Base Case (n = 0)

This is true as Q $\neq$ Y $\wedge$ Q $\neq$ t
Thus (Q Q) $\neq$ (Y t)
Therefore (t (Q Q)) $\neq$ (t (Y t))

Induction Hypothesis

Assume (t_0 (t_1 (… (t_n (Q Q))))) $\neq$ (t (Y t))

Induction Step

We have (t_0 (t_1 (… (t_n (Q Q))))) $\neq$ (t (Y t))
Apply a single β-reduction to the left hand side producing
(t_0 (t_1 (… (t_n (t_{n+1} (Q Q))))))
Note that (t_0 (t_1 (… (t_n (t_{n+1} (X)))))) $\neq$ (t (X)) for any X
as (t_n (…)) is not a lambda expression that can be β-reduced.
Therefore (t_0 (t_1 (… (t_n (t_{n+1} (Q Q)))))) $\neq$ (t (Y t)) $\square$