

**Problem 1.**

Given

$0: \lambda f. \lambda x. x$   
 $\text{succ}: \lambda n. \lambda f. \lambda x. (f ((n f) x))$   
 $n$ : if  $n$  is a natural number then its semantics is the result of  $n$  applications of  $\text{succ}$  on  $0$ .  
 $\text{true}: \lambda x. \lambda y. x$   
 $\text{false}: \lambda x. \lambda y. y$   
 $\text{second}: \lambda x. \lambda y. \lambda z. y$   
 $g: \lambda n. ((n \text{ second}) \text{ false})$

What is the result of

- (a)  $(g \ 0)$  when  $n$  is  $0$ .
- (b)  $(g \ n)$  when  $n$  results from some application  $\text{succ}$  on  $0$ .
- (c) What mathematical/logical operation is computed by  $g$ .

**Solution****Part (a)**

$(g \ 0)$   
 $(g \ \lambda f. \lambda x. x)$   
 $(\lambda n. ((n \text{ second}) \text{ false}) \ \lambda f. \lambda x. x)$   
 $((\lambda f. \lambda x. x \text{ second}) \text{ false})$   
 $(\lambda x. x \text{ false})$   
 $\text{false}$

**Part (b)**

First consider what is happening when  $n$  applications of  $\text{succ}$  are taken on  $0$ .

Consider  $n = 1$  application of  $\text{succ}$  on  $0$ .

$(\text{succ } 0)$   
 $(\lambda n. \lambda f. \lambda x. (f ((n f) x))) \ \lambda f. \lambda x. x$   
 $\lambda f. \lambda x. (f ((\lambda f. \lambda x. x f) x))$   
 $\lambda f. \lambda x. (f (\lambda x. x x))$   
 $\lambda f. \lambda x. (f x)$

Now consider  $n = 2$  applications of  $\text{succ}$  on  $0$ .

$(\text{succ } (\text{succ } 0))$   
 $(\text{succ } \lambda f. \lambda x. (f x))$   
 $(\lambda n. \lambda f. \lambda x. (f ((n f) x))) \ \lambda f. \lambda x. (f x)$   
 $\lambda f. \lambda x. (f ((\lambda f. \lambda x. (f x f) x)))$   
 $\lambda f. \lambda x. (f (\lambda x. (f x x)))$   
 $\lambda f. \lambda x. (f (f x))$

As discussed in class, this pattern will continue.

That is,  $n$  applications of `succ` on 0 can be written as the following.

$\lambda f. \lambda x. (f^1 (\dots (f^n x)))$

Note the superscript denotes the occurrence index for the lambda function  $f$ .

Now consider  $(g\ n)$

$(g\ n)$

$(g\ (\text{succ}_1 (\dots (\text{succ}_n\ 0))))$

$(\lambda n. ((n\ \text{second})\ \text{false})\ (\text{succ}_1 (\dots (\text{succ}_n\ 0))))$

$((\text{succ}_1 (\dots (\text{succ}_n\ 0)))\ \text{second})\ \text{false})$

$((\lambda f. \lambda x. (f^1 (\dots (f^n\ x))))\ \text{second})\ \text{false})$

$((\lambda f. \lambda x. (f^1 (\dots (f^n\ x))))\ \lambda x. \lambda y. \lambda z. y)\ \text{false})$

As these are lambda expressions, subscripts denote both the number of occurrences as well as the uniqueness of each lambda variable.

$(\lambda x. (\lambda x_1. \lambda y_1. \lambda z_1. y_1 (\dots (\lambda x_n. \lambda y_n. \lambda z_n. y_n\ x))))\ \text{false})$

$(\lambda x_1. \lambda y_1. \lambda z_1. y_1 (\dots (\lambda x_n. \lambda y_n. \lambda z_n. y_n\ \text{false}))))$

$(\lambda x_1. \lambda y_1. \lambda z_1. y_1 (\dots (\lambda x_{n-1}. \lambda y_{n-1}. \lambda z_{n-1}. y_{n-1}\ (\lambda x_n. \lambda y_n. \lambda z_n. y_n\ \text{false}))))$

$(\lambda x_1. \lambda y_1. \lambda z_1. y_1 (\dots (\lambda x_{n-1}. \lambda y_{n-1}. \lambda z_{n-1}. y_{n-1}\ \lambda y_n. \lambda z_n. y_n))))$

Note that  $\lambda y. \lambda z. y$  is defined as 'true'.

$(\lambda x_1. \lambda y_1. \lambda z_1. y_1 (\dots (\lambda x_{n-1}. \lambda y_{n-1}. \lambda z_{n-1}. y_{n-1}\ \text{true}))))$

If  $n = 1$  we would be done here and have a result of 'true'

(as the  $1 \dots n-1$  terms would not exist).

If  $n > 1$ , we apply another beta reduction

$(\lambda x_1. \lambda y_1. \lambda z_1. y_1 (\dots (\lambda x_{n-2}. \lambda y_{n-2}. \lambda z_{n-2}. y_{n-2}\ \lambda y_{n-1}. \lambda z_{n-1}. y_{n-1})))$

$(\lambda x_1. \lambda y_1. \lambda z_1. y_1 (\dots (\lambda x_{n-2}. \lambda y_{n-2}. \lambda z_{n-2}. y_{n-2}\ \text{true}))))$

$(\lambda x_1. \lambda y_1. \lambda z_1. y_1 (\dots \lambda y_{n-2}. \lambda z_{n-2}. y_{n-2})))$

$(\lambda x_1. \lambda y_1. \lambda z_1. y_1 (\dots (\lambda x_{n-3}. \lambda y_{n-3}. \lambda z_{n-3}. y_{n-3}\ \text{true}))))$

Again note that if  $n = 2$ , we would be done here and have a result of 'true'.

If  $n > 3$  this trend would continue recursively until resolving to true.

### Part (c)

As  $g$  resolves to false when  $n = 0$ , and true otherwise

it then computes  $n > 0$  or  $n \neq 0$  on the domain of natural numbers.

**Problem 2.**

Consider the following  $\lambda$ -expression:

$$Y : \lambda t. (\lambda x. (t (x x)) \lambda x. (t (x x)))$$

Prove/disprove that  $(Y t)$  after application of several  $\beta$ -reductions results in  $(t (Y t))$ .

**Solution**

First expand  $(Y t)$

$$(\lambda t. (\lambda x. (t (x x)) \lambda x. (t (x x))) t)$$

Apply our first  $\beta$ -reduction:

$$(\lambda x. (t (x x)) \lambda x. (t (x x)))$$

Apply a second  $\beta$ -reduction:

$$(t (\lambda x. (t (x x)) \lambda x. (t (x x))))$$

Now consider  $(t (Y t))$

We can  $\beta$ -reduce outside then inside or inside then outside, in this case, do the inside first.

We already know what  $(Y t)$   $\beta$ -reduces to from the above, thus we produce:

$$(t (\lambda x. (t (x x)) \lambda x. (t (x x))))$$

Note that this is equivalent to the  $\beta$ -reduction we found for  $(Y t)$ , thus after the given  $\beta$ -reductions,  $(Y t)$  will result in  $(t (Y t))$ .  $\square$