Problem 1.

Given

0: $\lambda f.\lambda x.x$
succ: $\lambda n.\lambda f.\lambda x.$(f ((n f) x))
n: if n is a natural number then its semantics is the result of n applications of succ on 0.
true: $\lambda x.\lambda y.x$
false: $\lambda x.\lambda y.y$
second: $\lambda x.\lambda y.\lambda z.y$
g: $\lambda n.$((n second) false)

What is the result of

(a) (g n) when n is 0.

(b) (g n) when n results from some application succ on 0.

(c) What mathematical/logical operation is computed by g.

Solution

Problem 2.

Consider the following λ-expression:

$$Y : \lambda\text{t.}(\lambda\text{x.}(\text{t (x x)) } \lambda\text{x.(t (x x)))}$$

Prove/disprove that (Y t) after application of several β-reductions results in (t (Y t)).

Solution

Setup

We are given (Y t), apply the definition of Y to this given lambda expression.
(λt.(λx.(t (x x)) λx.(t (x x))) t)
Denote Q: λx.(t (x x))
Note that (Q Q) after a single β-reduction results in (t (Q Q))
Thus we are starting with (t (Q Q)).

Claim

let n $\in N$
After n β-reductions of (t (Q Q)) denoted R
then R $\neq$ (t (Y t))

Proof by Induction on n

Base Case (n = 0)
This is true as Q $\neq$ Y $\wedge$ Q $\neq$ t
Thus (Q Q) $\neq$ (Y t)
Therefore (t (Q Q)) $\neq$ (t (Y t))

Induction Hypothesis
Assume (t_0 (t_1 (... (t_n (Q Q))))) $\neq$ (t (Y t))

Induction Step
We have (t_0 (t_1 (... (t_n (Q Q))))) $\neq$ (t (Y t))
Apply a single β-reduction to the left hand side producing
(t_0 (t_1 (... (t_n (t_{n+1} (Q Q))))))
(Q Q) $\neq$ (Y t) as proven in the base case.
(t_0 (t_1 (... (t_n (t_{n+1} (X)))))) $\neq$ (t (X)) for any X
as (t_n (...)) is not a lambda expression that can be β-reduced.
Therefore (t_0 (t_1 (... (t_n (t_{n+1} (Q Q)))))) $\neq$ (t (Y t)) $\square$