# Problem 1.

(a) Show that $x - (\ln x)^x = 0$ has at least one solution on $[4, 5]$.

(b) Use the Intermediate Value Theorem and Rolle's Theorem to show that the graph of $f(x) = x^3 + 2x + k$ crosses the x-axis exactly once, regardless of the value of the constant k.

### Solution

### Part (a)

Note that the function is continous, in particular, it is continous over $[4, 5]$. Further as we have that $f(4) = 4 - ln(4)^4 = 0.306638422$ and $f(5) = 5 - ln(5)^5 = -5.798691578$, and $-5.79869 < 0 < 0.306638422$, then by the Intermediate Value Theorem,
$\exists x \in [4, 5]$ such that $f(x) = 0$.

### Part (b)

Proof goes here.

## Problem 2.

Let $f(x) = 2x\cos(2x) - (x-2)^2$ and $x_0 = 0$.

(a) Find the third Taylor polynomial $P_3(x)$, and use it to approximate $f(0.4)$

(b) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_3(0.4)|$. Compute the actual error.

### Solution

### Part (a)

$P_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{6}(x-x_0)^3$

$$\begin{aligned} f(0) &= 2 \times 0\cos(0) - (-2)^2 = -4 \\ f'(0) &= -2x - 4x\sin(2x) + 2\cos(2x) + 4 = 2\cos 0 + 4 = 2 + 4 = 6 \\ f''(0) &= -2(4\sin(2x) + 4x\cos(2x) + 1) = -2(0+0+1) = -2 \\ f^{(3)}(0) &= 8(2x\sin(2x) - 3\cos(2x)) = 8(0-3) = -24 \end{aligned}$$

Thus, the third Taylor polynomial about $x_0 = 0$ is given by

$$P_3(x) = -4 + 6x + -x^2 + -4x^3$$

Then

$$f(0.4) \approx P_3(0.4) = -2.016$$

### Part (b)

We have that $f^{(4)}(x) = 32(2\sin(2x) + x\cos(2x))$.
The error is then given by the following (for some $c_x \in (0, 0.4)$):

$$\mathcal{E}(x) = \frac{f^{(4)}(c_x)}{24}x^4$$

Thus, to find an upper bound of the error, we must find an upper bound for $f^{(4)}(c_x)$. This is found at $f^{(4)}(0.4) = 54.8286$. Giving us an error of $\frac{54.8286}{24}(0.4)^4 = 0.05848384$.
Actual error is given by $|f(0.4) - P_3(0.4)| = |-2.00263 - -2.016| = 0.0133654$.

## Problem 3.

Let $f \in C[a, b]$ be a function whose derivative exists on $(a, b)$. Suppose $f$ is to be evaluated at $x_0$ in $(a, b)$, but instead of computing the actual value $f(x_0)$, the approximation value, $\widetilde{f}(x_0)$, is the actual value of $f$ at $x_0 + \epsilon$, that is $\widetilde{f}(x_0) = f(x_0 + \epsilon)$.

(a) Use the Mean Value Theorem to estimate the absolute error $|f(x_0) - \widetilde{f}(x_0)|$ and the relative error $|f(x_0) - \widetilde{f}(x_0)|/|f(x_0)|$, assuming $f(x_0) \neq 0$.

(b) If $\epsilon = 5 \times 10^{-6}$ and $x_0 = 1$, find bounds for the absolute and relative errors for

  (i) $f(x) = e^x$

  (ii) $f(x) = \sin x$

### Solution

### Part (a)

For absolute error, we have $|f(x_0) - \widetilde{f}(x_0)| = |f(x_0) - f(x_0 + \epsilon)|$ by the mean value theorem it follows that $|f(x_0) - f(x_0 + \epsilon)| = |f'(c)\epsilon|, c \in (x_0, x_0 + \epsilon)$.
Therefore, our bound for absolute error is given by $\epsilon \times \sup|\{f'(c) : c \in (x_0, x_0 + \epsilon)\}|$.
Similarly, the bound for relative error is thus given by
$\epsilon \times \sup|\{f'(c) : c \in (x_0, x_0 + \epsilon)\}|/|f(x_0)|$.

### Part (b)

**(i)** $f(x) = e^x$
To find absolute error by the formula given in part (a), we simply need to find the maximum derivative of $f(x)$ over $(1, 1 + 5 \times 10^{-6})$. As $e^x$ is non-decreasing, and $e^{x\prime} = e^x$ the maximum is given by $e^{(1+5\times10^{-6})} \approx 2.7182954199$ we then multiply this by $\epsilon$ to get the absolute error $= e^{1+5\times10^{-6}} 5 \times 10^{-6} \approx 0.000013591477$.

Relative error is then $(e^{1+5\times10^{-6}} 5 \times 10^{-6})/(e^1) \approx 5.000025 \times 10^{-6}$.

**(ii)** $f(x) = \sin x$
$f'(x) = \cos(x)$, thus to determine bound for absolute error, I need the maximum value of $\cos(x)$ over $(1, 1 + 5 \times 10^{-6})$. Between 0 and $\pi/2$ $\cos(x)$ is positive and decreasing. As we have $0 < 1 < 1 + 5 \times 10^{-6} < \pi/2$ we will find our maximum value at $cos(1) \approx 0.5403023$. We then multiply this by $\epsilon$ to get the absolute error $= \cos(1) \times \epsilon \approx 2.701511529 \times 10^{-6}$.

Relative error is then $\cos(1) \times \epsilon/\sin(1) \approx 3.210463 \times 10^{-6}$.

## Problem 4.

The equation $4x^2 - e^x - e^{-x} = 0$ has four solutions $\pm x_1$ and $\pm x_2$. Use Newton's method to approximate the solution to within $10^-5$ with the following values of $y_0$ : (a)$p_0 = -10$, (b) $p_0 = -5$, (c) $p_0 = -3$, (d) $p_0 = -1$, (c) $p_0 = 1$, (g) $p_0 = 3$, (h) $p_0 = 5$, (i) $p_0 = 10$. Comment on the results.

### Solution

Here is the code I am using to run newton's method

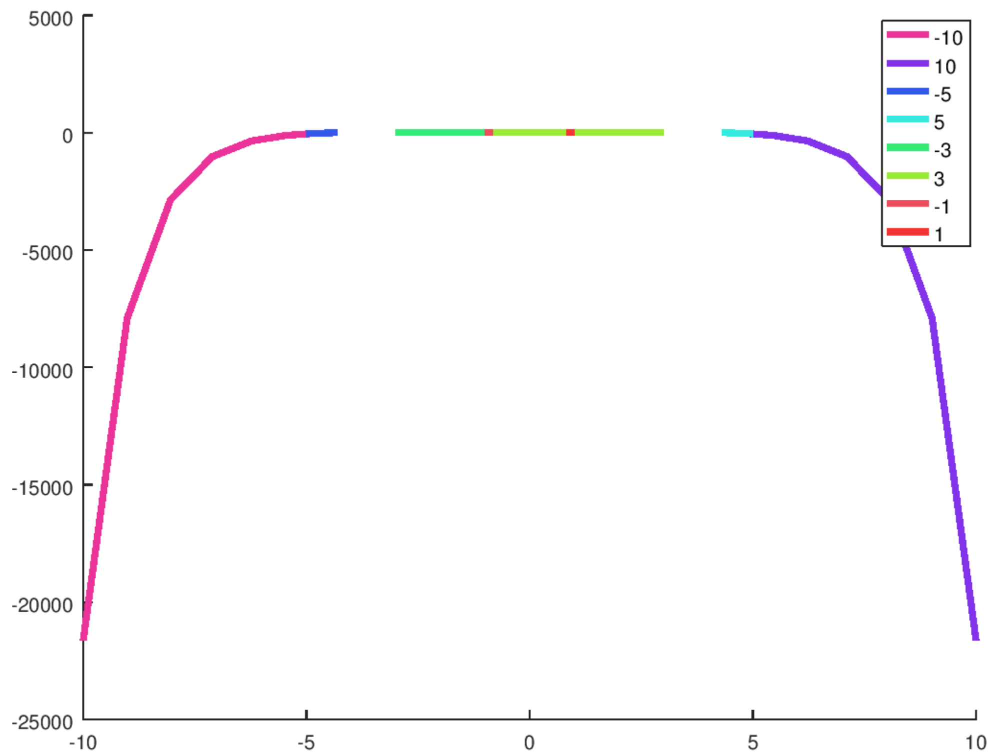
```
function p4

clear all;
close all;
hold on;

% This is the function we want to find 0's for.
f = @(t) (4 * t.^2) - e.^t - e.^(-t);
df = @(t) (8 * t) - e.^t + e.^(-t);
% Iterate via Newton's method until we are within 'err' of 0.
ERR = 10^-(5);
% List of starting points to use with Newton's method.
% Change the order so they show up better when I plot the results.
_y0 = [-10, 10, -5, 5, -3, 3, -1, 1];
% Use these colors for plotting Newton's method for each _y0.
COL = [
        [243 54 51] ./ 255,
        [234 78 92] ./ 255,
        [155 234 51] ./ 255,
        [51 234 118] ./ 255,
        [51 234 222] ./ 255,
        [51 88 234] ./ 255,
        [130 51 234] ./ 255,
        [234 51 152] ./ 255
]

% Loop through each starting point to run Newton's method.
for y0 = _y0

        % Steps will contain a list of each point in the iteration.
        % We can plot this, as well as examine it's size to see
        % how well each initial point is performing.
        steps = [y0];

        % loop while we still have too much error
```

```
        % infinite loop if we don't converge... but we do
        % for these y, so it's cool
        while abs(f(steps(end))) > ERR
                tmp = steps(end);
                nxt = tmp - f(tmp)/df(tmp);
                steps = [steps nxt];
        end

        y0
        steps
        plot(steps, f(steps), 'LineWidth', 3, 'Color', COL(end, :), 'DisplayName', int2s
        COL = COL(1:end-1, :);

end

legend('show');

end
```

Below I have formatted the text output from that code, we can immediately see some symmetry about the axis $x = 0$. Using the initial $y_0$ values, I found a total of 4 zeros at $x = \{\pm 4.3062, \pm 0.82450\}$. The closer I start to the guess, the faster (less steps) Newton's method will converge to the result. Interestingly, $x = \pm 3$ overshoot the zero on their side of the axis $x = 0$, that is $+3$ finds $-0.82450$ and $-3$ finds $+0.82450$. All others initial values find the zero nearest to them.

```
y0 = -10 => [-10.0000, -9.0146, -8.0456, -7.1093, -6.2339, -5.4634,
         -4.8572, -4.4752, -4.3269, -4.3066,  -4.3062]

y0 = -5 => [-5.0000  -4.5533  -4.3478  -4.3076  -4.3062  -4.3062]

y0 = -3 => [-3.00000   1.00194   0.83852   0.82461   0.82450]

y0 = -1 => [-1.00000  -0.83825  -0.82460  -0.82450]

y0 = 1 => [1.00000   0.83825   0.82460   0.82450]

y0 = 3 => [3.00000  -1.00194  -0.83852  -0.82461  -0.82450]

y0 = 5 => [5.0000   4.5533   4.3478   4.3076   4.3062   4.3062]

y0 = 10 => [10.0000, 9.0146, 8.0456, 7.1093, 6.2339, 5.4634,
         4.8572, 4.4752, 4.3269, 4.3066, 4.3062]
```

And here is a graph to visualize what Newton's method is doing for each starting point.


5000
0
-5000
-10000
-15000
-20000
-25000
-10
-5
0
5
10
-10
10
-5
5
-3
3
-1
1

## Problem 5.

Use Newton's method and Modified Newton's method to find the solution accurate to within $10^-5$ for

$$1 - 4x\cos x + 2x^2 + \cos 2x = 0, \text{ for } 0 \leq x \leq 1.$$

Comment on the performance of the methods.

### Solution

TODO

## Problem 6.

Use each of the following methods to find a solution in $[0.1, 1]$ accurate to within $10^-4$ for

$$600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0.$$

(a) Bisection method

(b) Newton's method

(c) Secant method

(d) Müller's method

Comment on the performance of these methods.

### Solution

TODO

# Problem 7.

### Solution

An object falling vertically through the air is subjected to viscous resistance as well as the force of gravity. Assume that an object with mass $m$ is dropped from a height $s_0$ and that the height of the object after $t$ seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17 ft/s^2$ and $k$ represents the coefficient of air resistance in lb-s/ft. Suppose $s_0$=300ft, $m = 0.25$lb, and $k = 0.1$ lb-s/ft. Find to within 0.01s the time it takes this quarter-pounder to hit the ground. Use Fixed-point iteration, Steffensen's method and Newton's method to find the solution.