Problem 1.

(30 points)

(a) Use the forward-difference and backward-difference formulas to determine each missing entry in the following table with $f(x) = \sin x$, compute the errors and find error bounds using error formulas.

(b) Choose your favorite function f, nonzero number x. Generate approximations of $f'_n(x)$ to $f'(x)$ by

$$f'_n(x) = \frac{f(x + 10^{-n}) - f(x)}{10^{-n}}$$

for $n = 1, 2, \ldots, 10$ and describe what happens.

Solution

Part (a)

x	$f(x)$	$f'(x)$	$\cos x$	$\varepsilon = \mid \cos x - f'(x) \mid$
0.5	0.4794	0.85200	0.87758	0.025580
0.6	0.5646	0.79600	0.82534	0.029340
0.7	0.6442	0.79600	0.76484	0.031160

We find the error term $\mid \frac{h}{2} f''(\xi) \mid$ from the following formula

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f''(\xi)$$

This error bound will be the same for forward and backward difference. We have $f''(x) = -\sin x$ and $h = 0.1$ in all cases, thus our error is given by $\mid \frac{0.1}{2} \sin \xi \mid$ where ξ is a value within the bounds of our difference. For each, we therefore want to maximize the value $\sin \xi$.

$f'(0.5)$: $max_{0.5,0.6}\ \sin \xi = 0.564642 \Rightarrow = \frac{0.1}{2} 0.564642 = 0.028232$
Thus we have an error bound of 0.02832 which is just a bit over our actual error of 0.025580.

$f'(0.6)$: $max_{0.6,0.7}\ \sin \xi = 0.644218 \Rightarrow = \frac{0.1}{2} 0.644218 = 0.03221$
Thus we have an error bound of 0.03221 a little bit worse than before. Our actual error of 0.029340 fits nicely within this bound.

$f'(0.7)$: $max_{0.6,0.7}\ \sin \xi = 0.644218 \Rightarrow = \frac{0.1}{2} 0.644218 = 0.03221$
Since we are taking the backward difference, we come to the same error bound as the forward difference for $f'(0.6)$, that is 0.03221. Our actual error for this value is the worst at 0.031160 but still within the error bounds like we would expect.

Part (b)

Let $f(x) = \sin(x)$, and $x = 3$.
We have $f'(x) = \cos(x)$, thus $f'(3) = \cos(3) = -0.9899924966$.

n	$f'_n(x)$	$\mid f'_n(x) - f'(x) \mid$
1	-0.995393456265767	0.005400959665322
2	-0.990681590968298	0.000689094367853
3	-0.990062891599724	0.000070394999279
4	-0.989999550952969	0.000007054352524
5	-0.989993202188399	0.000000705587954
6	-0.989992567285158	0.000000070684713
7	-0.989992501865267	0.000000005264821
8	-0.989992490763036	0.000000005837409
9	-0.989992560151975	0.000000063551530
10	-0.989992532396400	0.000000035795954

In general, for larger values of n we would expect the error between our approximation and the actual derivative to go down. However, if we in the above example we can see the error actually increase from $n = 7$ to $n = 8$ and from $n = 8$ to $n = 9$. We do see a decrease again from $n = 9$ to $n = 10$ however we still don't have the precision found at $n = 7$ right before our estimates started to get worse.

Problem 2.

(60 points)

(a) Approximate the following integral using the Trapezoidal rule, find a bound for the error using error formula and compare this to the actual error:

$$\int_{0.5}^{1} x^4 dx.$$

(b) Repeat part (a) using Simpson's rule.

(c) Repeat part (a) using Composite Trapezoidal rule with $n = 4$.

(d) Repeat part (a) using Composite Simpson rule with $n = 4$.

(e) Write a code to implement part (c) and (d) in MATLAB.

(f) Write a function $v = \text{CompositeTrapezoidalRule}(f, a, b, n)$ to implement Composite trapezoidal rule with a given n for

$$\int_{a}^{b} f(x)dx,$$

verify your code with part (c).

Solution

Problem 3.

(20 points)

(a) Show that the following quadrature formula has a degree of precision equal to 3,

$$\int_{-1}^{1} f(x)dx = f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}).$$

(b) Approximate the following integral using Gaussian quadrature with $n = 2, 3, 5$,

$$\int_{1}^{1.5} x^2 \ln x dx$$

Solution

Problem 4.

(20 points)

(a) Use Composite Simpson rule with $n = 4$ to approximate the following integral

$$\int_0^1 x^{-1/4} \sin x dx$$

(b) Use Composite Simpson rule with $n = 6$ to approximate the following integral

$$\int_1^\infty \frac{\cos x}{x^3} dx$$

Solution

Problem 5.

(30 points)

(a) Use the Gaussian Elimination with Backward substitution to solve the following linear system (must show intermediate steps).

$$\begin{aligned} 2x_1 - 1.5x_2 &+ 3x_3 = 1 \\ -x_1 &+ 2x_3 = 3 \\ 4x_1 - 4.5x_2 &+ 5x_3 = 1 \end{aligned}$$

(b) Repeat (a) using the Gaussian Elimination with partial pivoting and Backward substitution (must show intermediate steps).

(c) Repeat (a) using Gaussian Elimination with scaled partial pivoting and Backward substitution (must show intermediate steps).

Solution

Problem 6.

(20 points)

(a) Solve the following linear system with forward and backward substitution,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

(b) Use the LU factorization (no permutation) with forward and backward substitution to solve the following linear system (Use the sample MATLAB codes posted on classpage as discussed in class),

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 2 \\ 4x_1 + 4x_2 - x_3 &= -1 \\ -2x_1 - 3x_2 + 4x_3 &= 1 \end{aligned}$$

Solution

Problem 7.

(20 points)

(a) Use the $A = LDL^t$ factorization to solve the following linear system,

$$\begin{aligned} 2x_1 - x_2 \quad\quad &= 2 \\ -x_1 + 2x_2 - x_3 &= -1 \\ - x_2 + 2x_3 &= 1 \end{aligned}$$

(b) Repeat (a) with Cholesky factorization $A = LL^t$.

Solution