

**Problem 1.**

(30 points)

- (a) Use the forward-difference and backward-difference formulas to determine each missing entry in the following table with  $f(x) = \sin x$ , compute the errors and find error bounds using error formulas.
- (b) Choose your favorite function  $f$ , nonzero number  $x$ . Generate approximations of  $f'_n(x)$  to  $f'(x)$  by

$$f'_n(x) = \frac{f(x + 10^{-n}) - f(x)}{10^{-n}}$$

for  $n = 1, 2, \dots, 10$  and describe what happens.

**Solution****Part (a)**

$x$	$f(x)$	$f'(x)$	$\cos x$	$\varepsilon =  \cos x - f'(x) $
0.5	0.4794	0.85200	0.87758	0.025580
0.6	0.5646	0.79600	0.82534	0.029340
0.7	0.6442	0.79600	0.76484	0.031160

We find the error term  $|\frac{h}{2}f''(\xi)|$  from the following formula

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2}f''(\xi)$$

This error bound will be the same for forward and backward difference. We have  $f''(x) = -\sin x$  and  $h = 0.1$  in all cases, thus our error is given by  $|\frac{0.1}{2}\sin \xi|$  where  $\xi$  is a value within the bounds of our difference. For each, we therefore want to maximize the value  $\sin \xi$ .

$$f'(0.5): \max_{0.5, 0.6} \sin \xi = 0.564642 \Rightarrow \frac{0.1}{2} 0.564642 = 0.028232$$

Thus we have an error bound of 0.02832 which is just a bit over our actual error of 0.025580.

$$f'(0.6): \max_{0.6, 0.7} \sin \xi = 0.644218 \Rightarrow \frac{0.1}{2} 0.644218 = 0.03221$$

Thus we have an error bound of 0.03221 a little bit worse than before. Our actual error of 0.029340 fits nicely within this bound.

$$f'(0.7): \max_{0.6, 0.7} \sin \xi = 0.644218 \Rightarrow \frac{0.1}{2} 0.644218 = 0.03221$$

Since we are taking the backward difference, we come to the same error bound as the forward difference for  $f'(0.6)$ , that is 0.03221. Our actual error for this value is the worst at 0.031160 but still within the error bounds like we would expect.

**Part (b)**

Let  $f(x) = \sin(x)$ , and  $x = 3$ .

We have  $f'(x) = \cos(x)$ , thus  $f'(3) = \cos(3) = -0.9899924966$ .

n	$f'_n(x)$	$ f'_n(x) - f'(x) $
1	-0.995393456265767	0.005400959665322
2	-0.990681590968298	0.000689094367853
3	-0.990062891599724	0.000070394999279
4	-0.98999550952969	0.000007054352524
5	-0.989993202188399	0.000000705587954
6	-0.989992567285158	0.000000070684713
7	-0.989992501865267	0.000000005264821
8	-0.989992490763036	0.000000005837409
9	-0.989992560151975	0.000000063551530
10	-0.989992532396400	0.000000035795954

In general, for larger values of  $n$  we would expect the error between our approximation and the actual derivative to go down. However, if we in the above example we can see the error actually increase from  $n = 7$  to  $n = 8$  and from  $n = 8$  to  $n = 9$ . We do see a decrease again from  $n = 9$  to  $n = 10$  however we still don't have the precision found at  $n = 7$  right before our estimates started to get worse.

**Problem 2.**

(60 points)

- (a) Approximate the following integral using the Trapezoidal rule, find a bound for the error using error formula and compare this to the actual error:

$$\int_{0.5}^1 x^4 dx.$$

- (b) Repeat part (a) using Simpson's rule.
- (c) Repeat part (a) using Composite Trapezoidal rule with  $n = 4$ .
- (d) Repeat part (a) using Composite Simpson rule with  $n = 4$ .
- (e) Write a code to implement part (c) and (d) in MATLAB.
- (f) Write a function  $v = \text{CompositeTrapezoidalRule}(f, a, b, n)$  to implement Composite trapezoidal rule with a given  $n$  for

$$\int_a^b f(x) dx,$$

verify your code with part (c).

**Solution****Part (a)**

By the trapezoidal rule, our approximation is given by  $\frac{h}{2}[f(x_0) + f(x_1)]$ . Where  $x_0 = 0.5, x_1 = 1$  gives us  $f(x_0) = 0.0625, f(x_1) = 1.0$ . The estimate is given by  $1.0625 \frac{0.5}{2} = 0.26562$ . The actual value of the integral is 0.19375, thus our actual error is  $|0.26562 - 0.19375| = 0.071875$ .

The trapezoidal rule has an error term of  $\frac{h^3}{12}f''(\xi)$ . Where  $f''(\xi) = 12\xi^2$ , maximizing this on the bounds  $(0.5, 1)$  gives us 12, thus our error bound =  $\frac{0.5^3}{12}12 = 0.5^3 = 0.125$ . Note that our actual error 0.071875 is within this error bound.

**Part (b)**

First we compute the values needed by Simpson's rule. We have  $a = 0.5, b = 1.0$  thus  $h = (1.0 - 0.5)/2 = 0.25, x_0 = a = 0.5, x_1 = a + h = 0.75, x_2 = 1.0$ . Simpson's rule is given by  $\frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$ , filling in our values gives us  $\frac{0.25}{3}[0.5^4 + 4 \times 0.75^4 + 1^4] = 0.08333[0.0625 + 1.2656 + 1] = 0.08333[2.3281] = 0.194$ . Given the actual value of the integral as 0.19375 our actual error is then  $|0.194 - 0.19375| = 0.00025057$ .

Simpson's rule has an error term of  $\frac{h^5}{90}f^{(4)}(\xi)$ . Where  $f^{(4)}(\xi) = 24$ . Our error bound is then  $24 \frac{0.25^5}{90} = 0.00026042$ . Our actual error sneaks just under this error bound.

**Part (c)**

For Composite Trapezoidal Rule with  $n = 4$ , we have

$h = (1.0 - 0.5)/4 = 0.125$ ,  $x_i = [0.5, 0.625, 0.75, 0.875, 1.0]$ ,  $i \in [0, 1, \dots, 4]$ . The Composite Trapezoidal Approximation is given by  $\frac{h}{2}[f(a) + 2\sum_{j=1}^{n-1}f(x_j) + f(b)]$ . Plugging in our values gives us

$$\begin{aligned}\int_{0.5}^1 x^4 dx &\approx \frac{0.125}{2}[f(0.5) + 2(f(0.625) + f(0.75) + f(0.875)) + f(1.0)] \\ &= \frac{0.125}{2}[0.0625 + 2(1.0557) + 1] \\ &= \frac{0.125}{2}[3.1739] \\ &= 0.19837\end{aligned}$$

With an actual value of 0.19375 our actual error is  $0.19837 - 0.19375 = 0.0046188$ .

The error term is given by  $\frac{b-a}{12}h^2f''(\mu)$ . From part (a) we know we can maximize  $f''(\mu)$  as 12. Thus the error bound is  $12\frac{1-0.5}{12}0.125^2 = 0.0078125$ . Again, our actual is nicely within this bound.

**Part (d)**

For Composite Simpson's Rule with  $n = 4$ , we have

$h = (1.0 - 0.5)/4 = 0.125$ ,  $x_i = [0.5, 0.625, 0.75, 0.875, 1.0]$ ,  $i \in [0, 1, \dots, 4]$ , these are the same values obtained from Composite Trapezoidal Rule. The desired approximation is given by  $\frac{h}{3}[f(a) + 2\sum_{j=1}^{(n/2)-1}f(x_{2j}) + 4\sum_{j=1}^{n/2}f(x_{2j-1}) + f(b)]$ . Plugging in our values gives us

$$\begin{aligned}\int_{0.5}^1 x^4 dx &\approx \frac{0.125}{3}[f(0.5) + 2(f(0.75)) + 4(f(0.625) + f(0.875)) + f(1.0)] \\ &= \frac{0.125}{3}[0.0625 + 2(0.31641) + 4(0.73877) + 1.0] \\ &= \frac{0.125}{3}[4.6504] \\ &= 0.193766276\end{aligned}$$

With an actual value of 0.19375 our actual error is  $0.193766276 - 0.19375 = 0.0000162760416666519$ .

The error term is given by  $\frac{b-a}{180}h^4f^{(4)}(\mu)$ . From part (b) we know we  $f^{(4)}(\mu) = 24$ . Putting it all together yields  $24\frac{1.0-0.5}{180}0.125^4 = 0.0000162760416666667$ . Here we have a pretty accurate error bound and our actual value is falling just under it, though at this number of significant digits we might expect computer precision to be affecting our results.

**Part (e)**

For CompositeTrapezoidalRule see my answer to part (f).

```
function v = CompositeSimpsonsRule(f, a, b, n)

    h = (b-a)/n;
    x = (a+h):h:b;

    i = 1:(n/2)-1;
    j = 1:(n/2);

    part1 = 2 * sum(f(x(2.*i)));
    part2 = 4 * sum(f(x(2.*j-1)));
    inner = f(a) + part1 + part2 + f(b);

    v = (h/3) * inner;
end
```

Running

```
CompositeSimpsonsRule(@(x)x.^4, 0.5, 1, 4)
```

produces a result of 0.19377 which is the same I found in part (d).

**Part (f)**

```
function v = CompositeTrapezoidalRule(f, a, b, n)

    h = (b-a)/n;
    x = (a+h):h:b;

    j = 1:(n-1);

    part = 2 * sum(f(x(j)));
    inner = f(a) + part + f(b);

    v = (h/2) * inner;
end
```

Running

```
CompositeTrapezoidalRule(@(x)x.^4, 0.5, 1, 4)
```

produces a result of 0.19830 which is roughly the same I found in part (c). There is some variation at later significant digits, but I would attribute to truncation when computing by hand.

**Problem 3.**

(20 points)

- (a) Show that the following quadrature formula has a degree of precision equal to 3,

$$\int_{-1}^1 f(x)dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

- (b) Approximate the following integral using Gaussian quadrature with
- $n = 2, 3, 5$
- ,

$$\int_1^{1.5} x^2 \ln x dx$$

**Solution****Part (a)**

*Proof.* If the quadrature formula has a degree of precision equal to 3, then  $x^k$  must be exact for each  $k = 0, 1, 2, 3$ . Further 3 must be the largest integer for which this is true. Thus if it is in fact true for  $k = 0, 1, 2, 3$  (as we will show), it then must not be true for  $k = 4$  else the quadrature formula would have a degree of precision equal to 4.

**k=0:**  $\int_{-1}^1 x^0 dx = \int_{-1}^1 dx = 2$ , further  $(-\frac{\sqrt{3}}{3})^0 + \frac{\sqrt{3}}{3}) = 1 + 1 = 2$ . Therefore we have at least a quadrature with degree of precision 0.

**k=1:**  $\int_{-1}^1 x dx = 0$ , further  $(-\frac{\sqrt{3}}{3}) + \frac{\sqrt{3}}{3}) = 0 \Rightarrow$  our degree of precision is at least 1.

**k=2:**  $\int_{-1}^1 x^2 dx = \frac{2}{3}$ , further  $(-\frac{\sqrt{3}}{3})^2 + \frac{\sqrt{3}}{3})^2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \Rightarrow$  our degree of precision is at least 2.

**k=3:**  $\int_{-1}^1 x^3 dx = 0$ , further  $(-\frac{\sqrt{3}}{3})^3 + \frac{\sqrt{3}}{3})^3 = -(\frac{\sqrt{3}}{3})^3 + \frac{\sqrt{3}}{3})^3 = 0 \Rightarrow$  our degree of precision is at least 3.

**k=4:**  $\int_{-1}^1 x^4 dx = \frac{4}{10}$ , further  $(-\frac{\sqrt{3}}{3})^4 + \frac{\sqrt{3}}{3})^4 = 2(\frac{\sqrt{3}}{3})^4 = 2\frac{9}{81} = 2\frac{1}{9} = \frac{2}{9}$ . But as  $\frac{4}{10} \neq \frac{2}{9} \Rightarrow$  our degree of precision is less than 4.

Note that by a generalization of the argument provided for  $k = 3$  we know that the quadrature formula will be true for all odd  $k$ , but for a degree of precision of  $k$  we need it to be true for all integers less than or equal to  $k$ . Thus as the formula is not exact at  $k = 4$  and is exact for  $k = 0, 1, 2, 3$  we know that we must have a degree of precision equal to 3.  $\square$

**Part (b)**

Before we begin, we must first transform the given integral on the bounds 1, 1.5 to the bounds  $-1, 1$  through the following transformation

$$\begin{aligned}\int_1^{1.5} f(x)dx &= \int_{-1}^1 f\left(\frac{(1.5-1)t + 1 + 1.5}{2}\right) \frac{(1.5-1)}{2} dt \\ &= \int_{-1}^1 f\left(\frac{0.5t + 2.5}{2}\right) 0.25 dt \\ &= \int_{-1}^1 0.25 f(0.25t + 1.25) dt\end{aligned}$$

Thus define  $g(x) = 0.25f(0.25x + 1.25)$ . Therefore the approximation via Gaussian quadrature is given by  $\sum_{i=1}^n c_i g(x_i)$ . Note that the actual value of the integral we are approximating is 0.192259.

**n=2:**

From the table in the lecture notes  $c = [1.0, 1.0]$ . The points we would like to consider are  $x_i = [-1, 1]$ , plugging these into the formula above gives us

$$\begin{aligned}\int_1^{1.5} x^2 \ln x dx &\approx c_1 g(x_1) + c_2 g(x_2) \\ &= g(-1) + g(1) \\ &= 0.22807\end{aligned}$$

This is an error of 0.035811.

**n=3:**

From the table in the lecture notes  $c = [\frac{5}{9}, \frac{8}{9}, \frac{5}{9}]$ . The points we would like to consider are  $x_i = [-1, 0, 1]$ , plugging these into the formula above gives us

$$\begin{aligned}\int_1^{1.5} x^2 \ln x dx &\approx c_1 g(x_1) + c_2 g(x_2) + c_3 g(x_3) \\ &= \frac{5}{9}g(-1) + \frac{8}{9}g(0) + \frac{5}{9}g(1) \\ &= 0.0 + 0.07748 + 1.2671 \\ &= 0.20419\end{aligned}$$

This is an error of 0.011931.

**n=5:**

From the table in the lecture notes

$c = [0.2369268850, 0.4786286705, 0.5688888889, 0.4786286705, 0.2369268850]$ . The points we would like to consider are  $x_i = [-1, -0.5, 0, 0.5, 1]$ , plugging these into the formula above gives us

$$\begin{aligned}\int_1^{1.5} x^2 \ln x dx &\approx c_1 g(x_1) + c_2 g(x_2) + c_3 g(x_3) + c_4 g(x_4) + c_5 g(x_5) \\ &= 0.2369268850g(-1.0) + 0.4786286705g(-0.5) + \\ &\quad 0.5688888889g(0.0) + 0.4786286705g(0.5) + 0.2369268850g(1.0) \\ &= 0.000000 + 0.017837 + 0.049587 + 0.072043 + 0.054037 \\ &= 0.19350\end{aligned}$$

This is an error of 0.0012410.

We can see that as we increase  $n$  we get a more accurate approximation of the actual integral.



**Problem 4.**

(20 points)

- (a) Use Composite Simpson rule with
- $n = 4$
- to approximate the following integral

$$\int_0^1 x^{-1/4} \sin x dx$$

- (b) Use Composite Simpson rule with
- $n = 6$
- to approximate the following integral

$$\int_1^\infty \frac{\cos x}{x^3} dx$$

**Solution****Part (a)**

We have values  $a = 0, b = 1, n = 4$  from this we can derive  $h = (b - a)/n = 0.25$ . With points  $x_i = [0, 0.25, 0.5, 0.75, 1.0], i \in [0, 1, \dots, 4]$ .

Note that  $x^{-1/4} \sin x$  has a singularity at it's left endpoint, even though the integral exists.

We can get around this by first computing the fourth Taylor Polynomial  $P_4$  for  $g(x) = \sin x$ .

$$\begin{aligned} P_4(x) &= g(0) + g'(0)(x) + \frac{g''(0)}{2}(x)^2 + \frac{g'''(0)}{6}(x)^3 + \frac{g^{(4)}(0)}{24}(x)^4 \\ &= 0 + x + 0 + \frac{-x^4}{24} = x + \frac{-x^4}{24} \end{aligned}$$

We now write

$$\int_0^1 \frac{\sin x}{x^{1/4}} dx = \int_0^1 \frac{\sin(x) - (x + \frac{-x^4}{24})}{x^{1/4}} dx + \int_0^1 \frac{x + \frac{-x^4}{24}}{x^{1/4}} dx$$

Note that the second integral above is an integral over a polynomial, we can compute this exactly as 0.56266. Thus we now have

$$\int_0^1 \frac{\sin x}{x^{1/4}} dx = \int_0^1 \frac{\sin(x) - (x + \frac{-x^4}{24})}{x^{1/4}} dx + 0.56266$$

We now define  $G(x)$  and use Composite Simpson's rule on  $[0, 1]$  to evaluate the remaining integral.

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{\sin(x) - (x + \frac{-x^4}{24})}{x^{1/4}} & \text{otherwise} \end{cases}$$

The approximation by Composite Simpson's Rule on  $G(x)$  is given by

$$\frac{h}{3}[G(a) + 2\sum_{j=1}^{(n/2)-1} G(x_{2j}) + 4\sum_{j=1}^{n/2} G(x_{2j-1}) + G(b)].$$

Plugging in our values gives us

$$\begin{aligned} \int_0^1 x^{-1/4} \sin x dx &\approx \frac{0.25}{3}[G(a) + 2(G(x_2)) + 4(G(x_1) + G(x_3)) + G(b)] \\ &= \frac{1}{12}[G(0) + 2(G(0.5)) + 4(G(0.25) + G(0.75)) + G(1)] \\ &= \frac{1}{12}[0 - 0.042741 - 0.25093 - 11686] \\ &= \frac{1}{12}[-0.41054] \\ &= -0.034211 \end{aligned}$$

Putting it all together and we arrive at

$$\int_0^1 \frac{\sin x}{x^{1/4}} dx \approx -0.034211 + 0.56266 = 0.52845$$

This is not too far off from the actual value of 0.56266 with an actual error of 0.034211.

### Part (b)

We can convert this right endpoint of infinity into a limit with a left endpoint singularity at 0 via the following substitution

$$\int_a^\infty f(x) dx = \int_0^{1/a} t^{-2} f\left(\frac{1}{t}\right) dt$$

In this particular instance we have  $f(x) = \frac{\cos x}{x^3}$  giving us the following integral to solve  $\int_0^1 t^{-2} \frac{\cos(1/t)}{(1/t)^3} dt = \int_0^1 t \cos(1/t) dt$ .

**Problem 5.**

(30 points)

- (a) Use the Gaussian Elimination with Backward substitution to solve the following linear system (must show intermediate steps).

$$2x_1 - 1.5x_2 + 3x_3 = 1$$

$$-x_1 \qquad \qquad + 2x_3 = 3$$

$$4x_1 - 4.5x_2 + 5x_3 = 1$$

- (b) Repeat (a) using the Gaussian Elimination with partial pivoting and Backward substitution (must show intermediate steps).
- (c) Repeat (a) using Gaussian Elimination with scaled partial pivoting and Backward substitution (must show intermediate steps).

**Solution**

**Problem 6.**

(20 points)

- (a) Solve the following linear system with forward and backward substitution,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

- (b) Use the LU factorization (no permutation) with forward and backward substitution to solve the following linear system (Use the sample MATLAB codes posted on classpage as discussed in class),

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 2 \\ 4x_1 + 4x_2 - x_3 &= -1 \\ -2x_1 - 3x_2 + 4x_3 &= 1 \end{aligned}$$

**Solution**

**Problem 7.**

(20 points)

- (a) Use the  $A = LDL^t$  factorization to solve the following linear system,

$$\begin{aligned} 2x_1 - x_2 &= 2 \\ -x_1 + 2x_2 - x_3 &= -1 \\ -x_2 + 2x_3 &= 1 \end{aligned}$$

- (b) Repeat (a) with Cholesky factorization  $A = LL^t$ .

**Solution**