Problem 1.

(10 points) Find the least squares polynomial of degree 2 for the data in the following table. Compute the error E. Graph the data and the polynomial.

x_i	1.0	1.1	1.3	1.5	1.9	2.1
y_i	1.84	1.96	2.21	2.45	2.94	3.18

Solution

$$\text{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.1 & 1.1^2 \\ 1 & 1.3 & 1.3^2 \\ 1 & 1.5 & 1.5^2 \\ 1 & 1.9 & 1.9^2 \\ 1 & 2.1 & 2.1^2 \end{bmatrix} b = \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}$$

$$A^T Aa = A^T b \Rightarrow (A^T A)^{-1}(A^T A)a = (A^T A)^{-1}A^T b \Rightarrow a = (A^T A)^{-1}A^T b$$

$$a = \begin{bmatrix} 66.841 & -90.719 & 28.748 \\ -90.719 & 124.544 & -39.811 \\ 28.748 & -39.811 & 12.832 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1.1 & 1.3 \\ 1.5 & 1.9 & 2.1 \\ 1 & 1.21 & 1.69 \\ 2.25 & 3.61 & 4.41 \end{bmatrix} \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}$$

$$= \begin{bmatrix} 4.86939 & 1.83451 & -2.51039 & -4.55547 & -1.74620 & 3.10816 \\ -5.98639 & -1.89239 & 3.90693 & 6.52134 & 2.19542 & -4.74490 \\ 1.76871 & 0.48237 & -1.32035 & -2.09647 & -0.56895 & 1.773469 \end{bmatrix} \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}$$

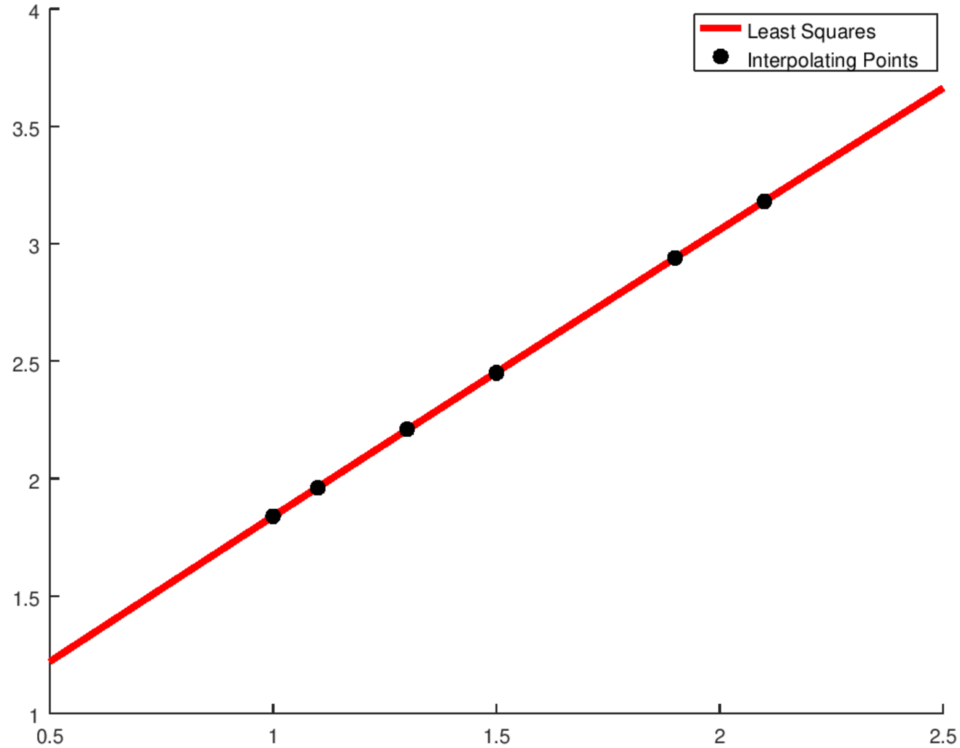
$$= \begin{bmatrix} 0.596581 \\ 1.253293 \\ -0.010853 \end{bmatrix}$$

This gives us the interpolating polynomial as $P_2(x) = -0.010853x^2 + 1.253293x + 0.596581$. Evaluating our initial points using our polynomial produces the results...

x_i	1.0	1.1	1.3	1.5	1.9	2.1
y_i	1.84	1.96	2.21	2.45	2.94	3.18
$P(x_i)$	1.839	1.9621	2.2075	2.4521	2.9387	3.1806
$\lvert P(x_i) - y_i\rvert$	-9.79e-4	2.0712e-3	2.4797e-3	2.1012e-3	1.3416e-3	6.3457e-4

Taking the sum of the differences finds the total error across all interpolating points as $E = 6.69 \times 10^{-6}$.

We can see that the least squares fit finds a polynomial which is almost linear, but not quite. From both the graph below, and the error above, the polynomial is clearly doing a good job (though not perfect) of fitting to the points.



Problem 2.

(20 points)

(a) Find the least squares polynomial of degree 2 for the following function $f(x)$ on the indicated interval

$$f(x) = e^x,\ [-1, 1]$$

Compute the error E. Graph the function and the polynomial.

(b) Repeat (a) with Legendre Polynomials.

Solution

Part (a)

I will be using the $x = (-1, 0, 1)$ as interpolating points to find my least squares polynomial, thus we want to use the following matrices in our system.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} b = \begin{bmatrix} \frac{1}{e} \\ 1 \\ e \end{bmatrix}$$

$A^T A a = A^T b \Rightarrow (A^T A)^{-1}(A^T A)a = (A^T A)^{-1} A^T b \Rightarrow a = (A^T A)^{-1} A^T b$

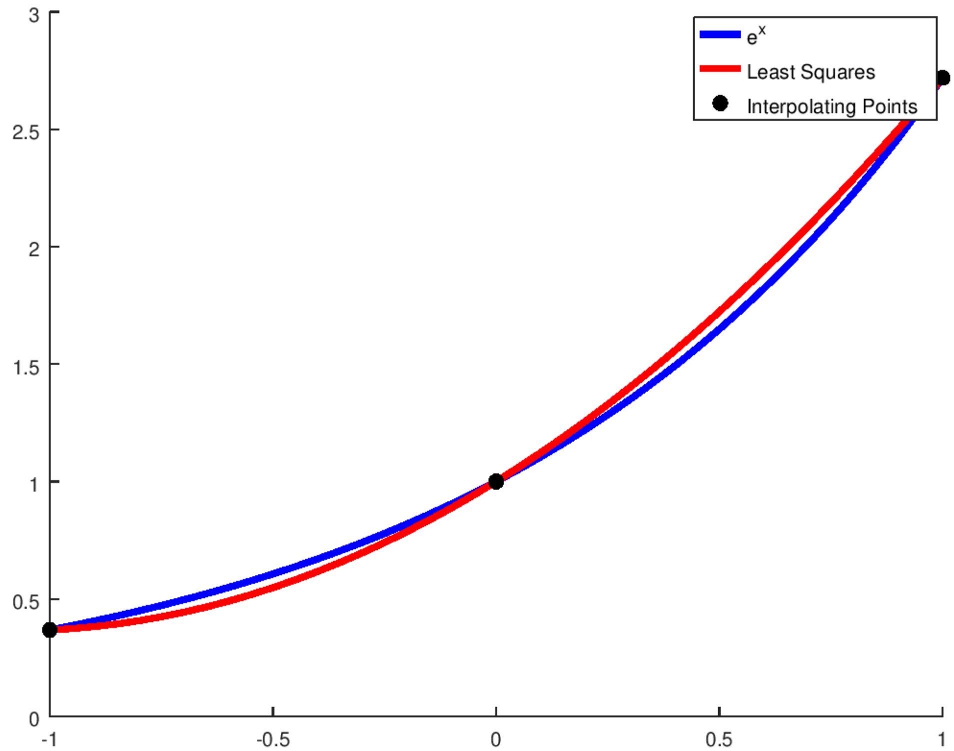
$$\begin{aligned} a &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0.5 & 0 \\ -1 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{e} \\ 1 \\ e \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{1}{e} \\ 1 \\ e \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1.7520 \\ 0.54308 \end{bmatrix} \end{aligned}$$

This gives us the interpolating polynomial $P_2(x) = 0.54308x^2 + 1.17520x + 1$. Evaluating our initial points using our polynomial produces the results...

x_i	-1.0	0	1.0
y_i	$\frac{1}{e}$	1	e
$P(x_i)$	0.36788	1.0	2.71828
$\lvert P(x_i) - y_i\rvert$	5.5883e-7	0	1.8285e-6

Taking the sum of the differences finds the total error across all interpolating points as $E = 2.3873 \times 10^{-6}$. If we look over the range $[-1, 1]$ and want to compute the total error (not just the interpolating points), we simply compute $\int_{-1}^{1} |P(x) - e^x| dx = 0.0890387$.

From the graph below, we can see that the interpolating polynomial closely matches, but diverges slightly from e^x. Notably, the polynomial switches from underestimating to overestimating after crossing $x = 0$.



Part (b)

$$b = \begin{bmatrix} \int_{-1}^{1} e^x P_0(x)dx \\ \int_{-1}^{1} e^x P_1(x)dx \\ \int_{-1}^{1} e^x P_2(x)dx \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} e^x dx \\ \int_{-1}^{1} e^x x dx \\ \int_{-1}^{1} e^x \frac{1}{2}(3x^2-1)dx \end{bmatrix} = \begin{bmatrix} e - \frac{1}{e} \\ \frac{2}{e} \\ 0.143126 \end{bmatrix} = \begin{bmatrix} 2.3504 \\ 0.73576 \\ 0.143126 \end{bmatrix}$$

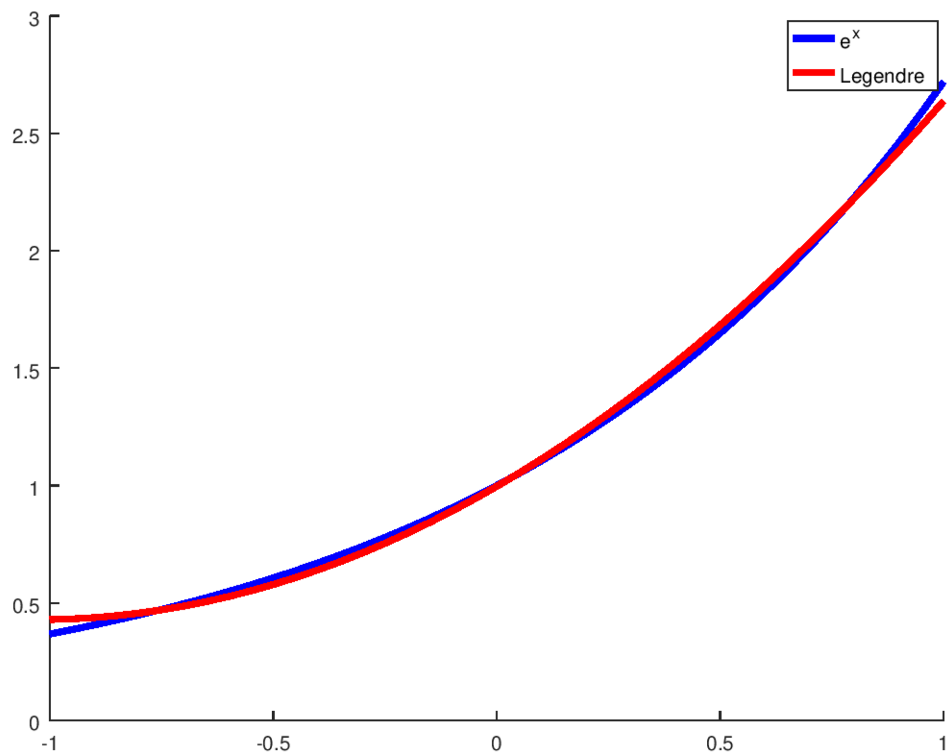
$$d = \begin{bmatrix} \frac{2(1)-1}{2} b_1 \\ \frac{2(2)-1}{2} b_2 \\ \frac{2(3)-1}{2} b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \times 2.3504 \\ \frac{3}{2} \times 0.73576 \\ \frac{5}{2} \times 0.143126 \end{bmatrix} = \begin{bmatrix} 1.1752 \\ 1.1036 \\ 0.35781 \end{bmatrix}$$

The Legendre Polynomial Approximation is then given by...

$$\begin{aligned} 1.1752P_0 + 1.1036P_1 + 0.35781P_2 &= 1.1752 + 1.1036x + 0.35781\frac{1}{2}(3x^2 - 1) \\ &= 0.536715x^2 + 1.1036x + 0.996295 \end{aligned}$$

We can compute error over the bounds by taking $^{1}_{-1}|L_2(x) - e^x|dx = 0.0458856$. Recall the error for least squares was 0.0890387, so we have approximately halved our error despite both polynomials having degree 2. Note that we may have found a better least squares polynomial if we had used different interpolating points, these were not necessary for the Legendre approximation.

Below we can visually confirm that this polynomial is doing a decent job of approximating e^x on our given bounds. It appears to be more consistent and accurate than the least squares polynomial but with greater error towards the end points.



Problem 3.

(20 points)

(a) Use the zeros of $\overline{T}_4$ to construct an interpolating polynomial of degree 3 for the following function on the interval $[-1, 1]$:

$$f(x) = e^x$$

Find a bound for the maximum error of the approximation.

(b) Repeat (a) on interval $[0, 2]$.

Solution

Part (a)

First, note that we will have 4 zeros of T_4 given by $\cos(\frac{2k-1}{2n}\pi)$. These are the values $x_0 = \{0.92388, 0.38268, -0.38268, -0.92366\}$. Evaluating our function at these points produces $f(x_0) = \{2.51904, 1.46621, 0.68203, 0.39698\}$. Next we set up our linear equation to find the coefficients of our polynomial.

$$\begin{bmatrix} 0.92388^3 & 0.92388^2 & 0.92388 & 1 \\ 0.38268^3 & 0.38268^2 & 0.38268 & 1 \\ -0.38268^3 & -0.38268^2 & -0.38268 & 1 \\ -0.92388^3 & -0.92388^2 & -0.92388 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2.51904 \\ 1.46621 \\ 0.68203 \\ 0.39696 \end{bmatrix}$$

Solving this yields

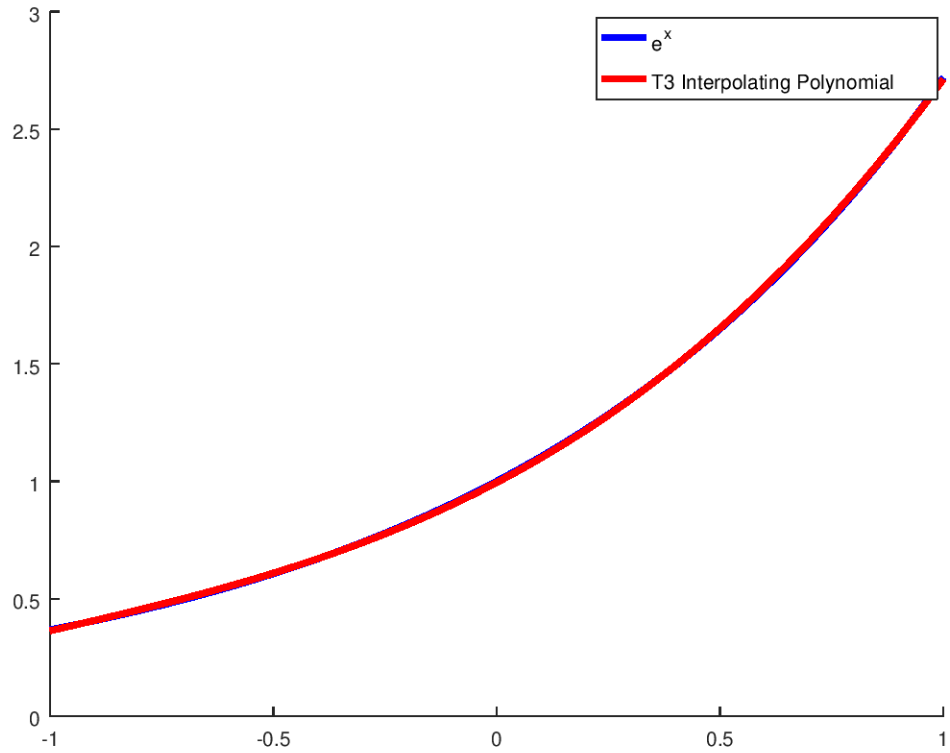
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0.17518 \\ 0.54290 \\ 0.99893 \\ 0.99462 \end{bmatrix}$$

Which produces the 4th degree polynomial $0.17518x^3 + 0.54290x^2 + 0.99893x + 0.99462$. To find an upper bound for the error on the bounds I simply need to solve

$$\max_{-1,1}(|0.17518x^3 + 0.54290x^2 + 0.99893x + 0.99462 - e^x|)$$

This occurs at $x = 1$ which evaluates to an error of $e - \frac{271163}{100000} \approx 0.00665183$.

Once printed out you may not be able to see e^x below the interpolating polynomial, clearly this method is doing a pretty decent job of approximating on the given bounds.



Part (b)

We can follow the same process as part a, but instead use the 4 zeros given by $1 + \cos(\frac{2k-1}{2n}\pi)$. These are the zeros $x_0 = \{1.923880, 1.382683, 0.617317, 0.076120\}$. Evaluating our function at these points produces $f(x_0) = \{6.8475, 3.9856, 1.8539, 1.0791\}$. Again, we set up and solve our linear equation to find the coefficients of our polynomial.

$$\begin{bmatrix} 1.92388^3 & 1.92388^2 & 1.92388 & 1 \\ 1.38268^3 & 1.38268^2 & 1.38268 & 1 \\ 0.617317^3 & 0.617317^2 & 0.617317 & 1 \\ 0.076120^3 & 0.076120^2 & 0.076120 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 6.8475 \\ 3.9856 \\ 1.8539 \\ 1.0791 \end{bmatrix}$$

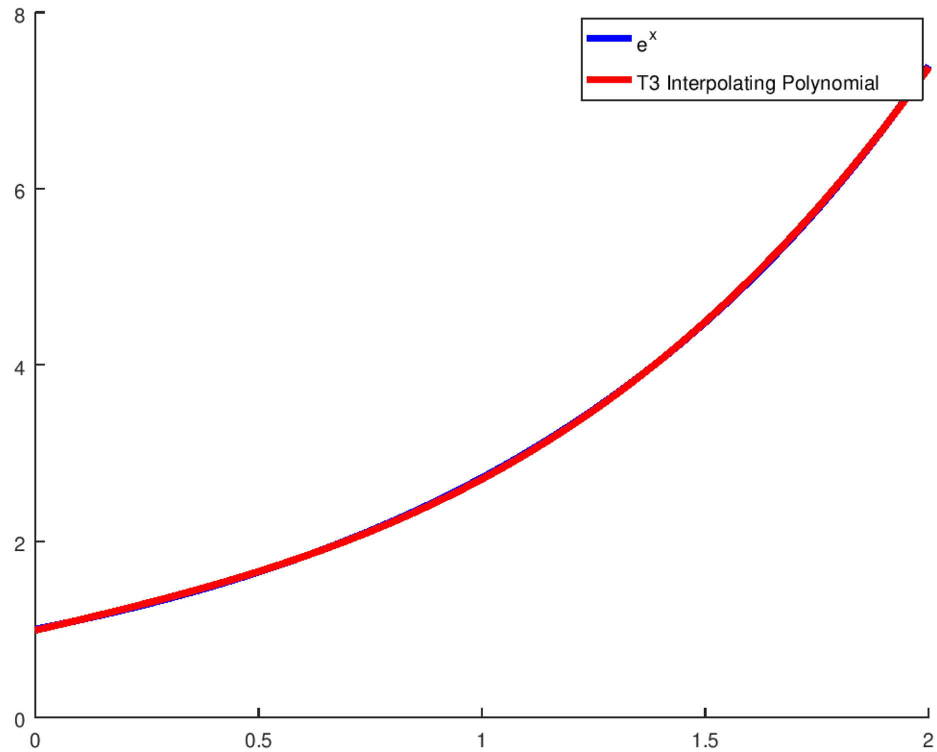
Solving this yields

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0.476177 \\ 0.047226 \\ 1.192398 \\ 0.987843 \end{bmatrix}$$

Which gives us the polynomial $0.476177x^3 + 0.047226x + 1.192398x + 0.987843$. To find an upper bound for the error on the bounds I simply need to solve

$$\max_{0,2}(|0.476177x^3 + 0.047226x + 1.192398x + 0.987843 - e^x|)$$

This occurs at $x = 2$ which evaluates to an error of $e^2 - \frac{1474191}{200000} \approx 0.018101$. We can see that as e^x starts to blow up, our polynomial will obviously start to perform worse relative to the bounds of part (a).



Problem 4.

(20 points) Find all the Chebyshev rational approximations of degree 2 for $f(x) = e^{-x}$. Graph the function and the polynomial. (Use MATLAB routines on classpage)

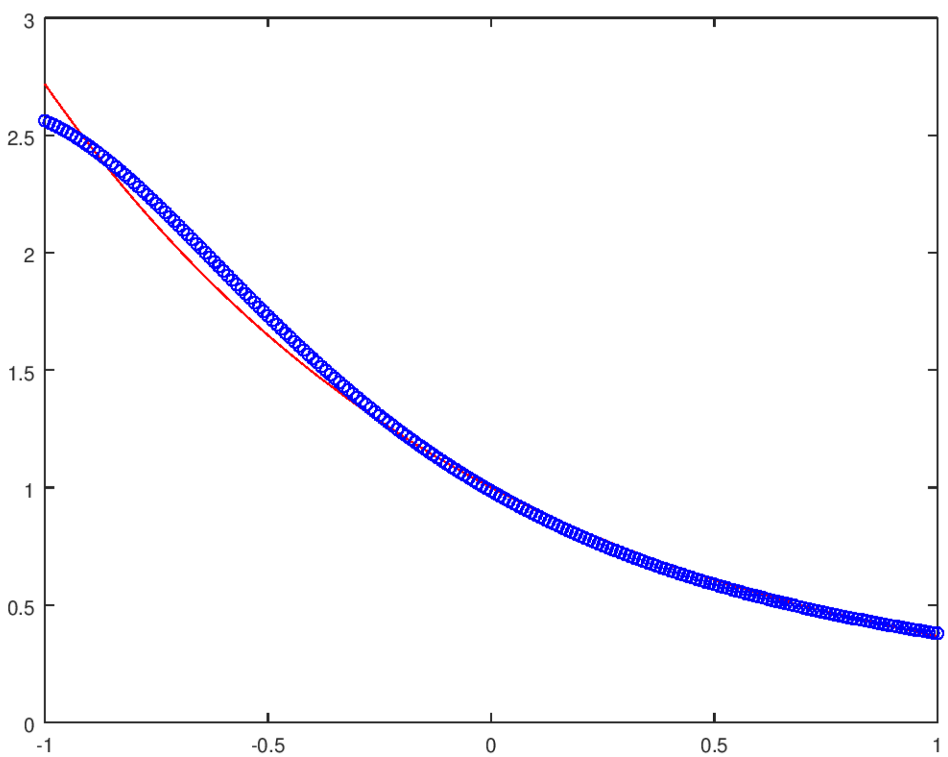
Solution

From the sample code posted on the class page we can derive these pretty easily. As noted in the code there are three possible cases.

(n=0, m=2)
Here we have the vectors $p = [0.79040]$ and $q = [1, 0.88911, 0.19759]$.
Our polynomial then takes the form...

$$\frac{T_{0,n}(x)p}{T_{0,m}(x)q} = \frac{0.79040}{1 + 0.88911x + (2x^2 - 1)0.19759}$$
$$= \frac{0.79040}{0.39518x^2 + 0.8891x + 0.80241}$$

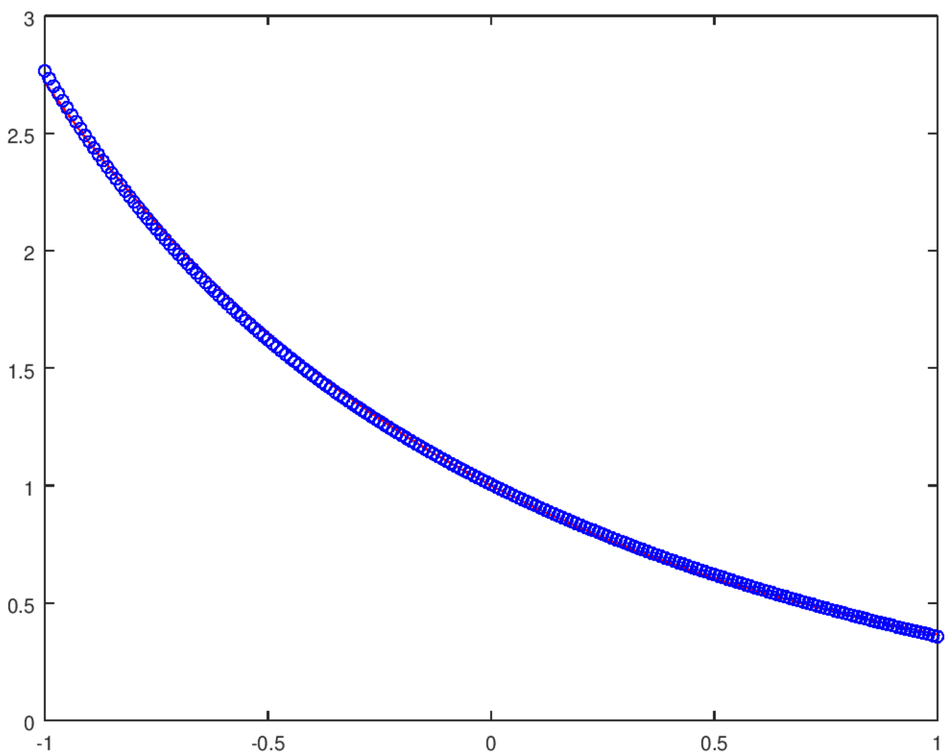
We can graph this to see how well of an approximation this is. We know from the MATLAB code provided on the course page our error is about 0.15604.



(n=1, m=1)
Again, using the sample code provided on the class page, we have the vectors
$p = [1, -0.48232]$ and $q = [1, 0.46626]$.
Our polynomial then takes the form...

$$\begin{aligned}\frac{T_{0,n}(x)p}{T_{0,m}(x)q} &= \frac{1 - 0.48232x}{1 + 0.44626x} \\ &= \frac{4.66276}{x + 2.24085} - 1.0808\end{aligned}$$

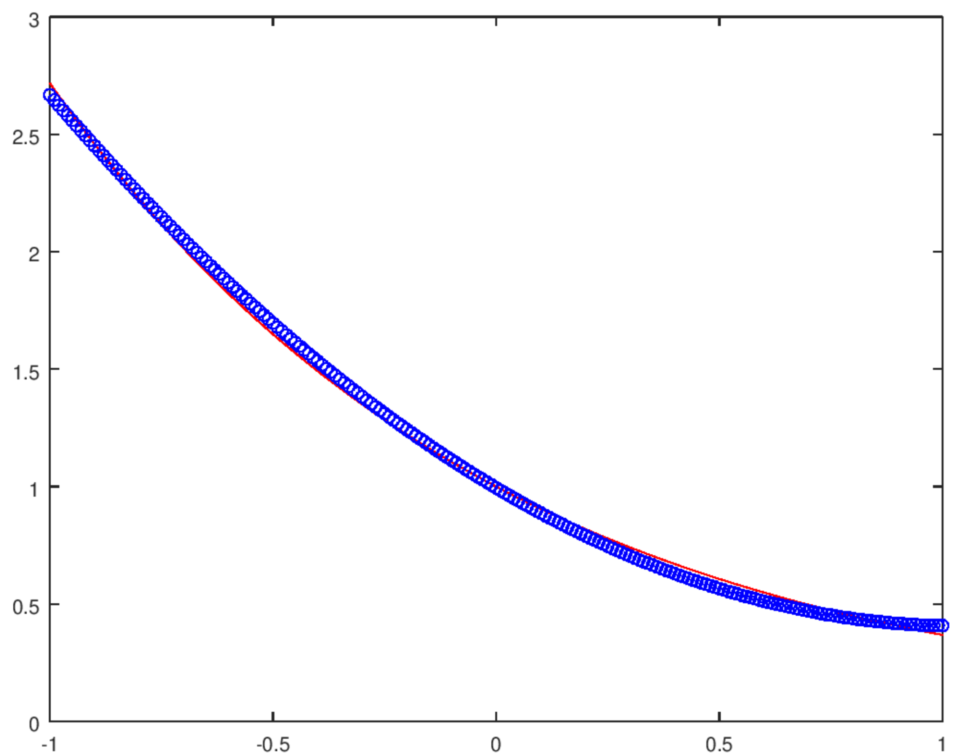
We can graph this to see how well of an approximation we have. From the MATLAB code we can see an error of 0.047232 a significant improvement over the first case.



(n=2, m=0)
In the final case, we have the vectors $p = [1.26607, -1.13032, 0.27150]$ and $q = [1]$. Giving us the polynomial...

$$\begin{aligned}\frac{T_{0,n}p}{T_{0,m}q} &= \frac{1.26607 - 1.13032x + 0.27150(2x^2 - 1)}{1} \\ &= 0.543x^2 - 1.13032x + 0.99457\end{aligned}$$

The graph of which is included below, from the MATLAB code we see it has an error of 0.050402, slightly worse than the error of the second case.



Problem 5.

(30 points) (Use MATLAB routines on classpage)

(a) Find the continous least squares trigonometric polynomial $S_3(x)$ for $f(x) = e^x$ on $[-\pi, \pi]$.

(b) Find the discrete least squares trigonometric polynomials $S_n(x)$ for $f(x) = e^x$ on $[-\pi, \pi]$ with $n = 3, m = 6$.

(c) Find the trigonometric interpolating polynomial $S_n(x)$ for $f(x) = e^x$ on $[-\pi, \pi]$ with $n = 8$.

Solution

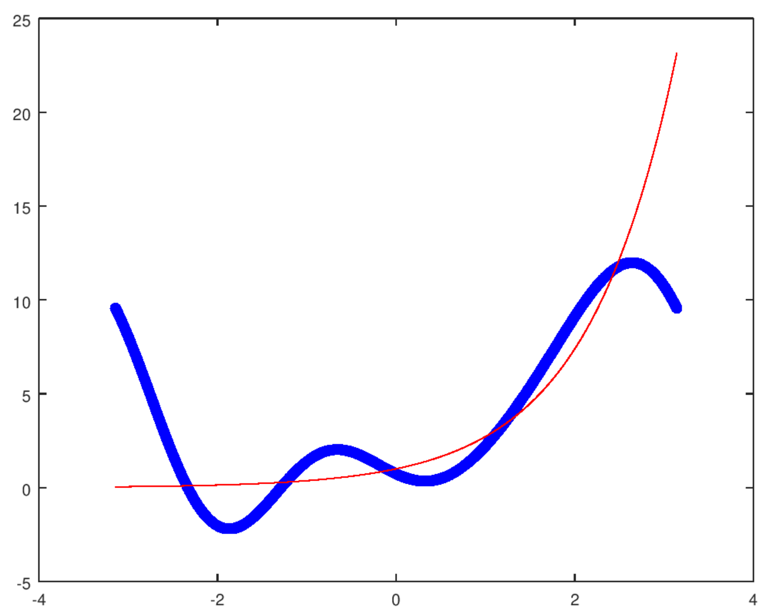
Part (a)

Working with the provided MATLAB code, we have following vectors to use to build our trigonometric polynomial (note we do not need $b(3)$):

$$a_0 = 7.3522, a = [-3.67608, 1.47043, -0.73522], b = [3.6761, -2.9409]$$

We can then follow the codes example to build the trigonometric polynomial.

$$\begin{aligned} S_3(x) &= \frac{a0}{2} + \cos(xk)a + \sin(xk(1:n-1))b \\ &= 3.6761 - 3.67608\cos(x) + 1.47043\cos(2x) - 0.73522\cos(3x) \\ &\;3.6761\sin(x) - 2.9440\sin(2x) \end{aligned}$$



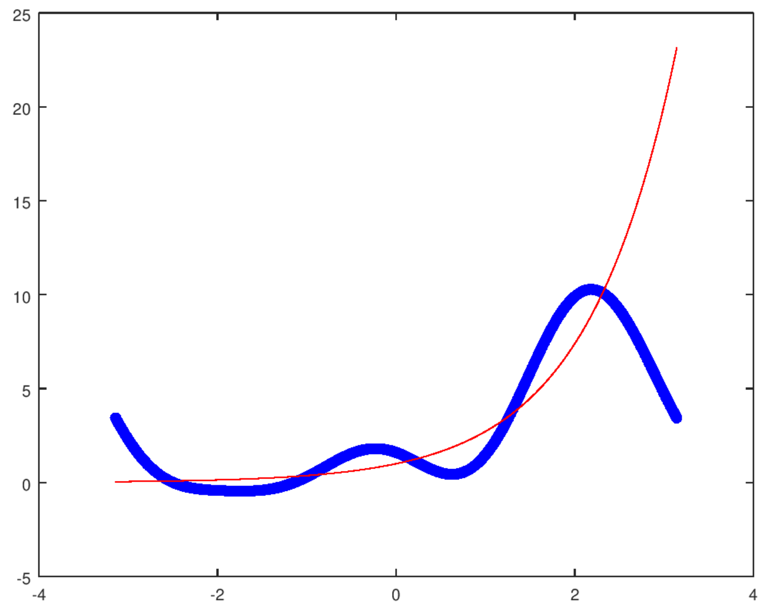
Part (b)

We follow a similar process here, first we have the following vectors derived from the code:

$$a_0 = 5.5946, a = [-1.92077, -0.27774, 0.99998], b = [3.5097, -2.6036]$$

We can then follow the codes example to build the trigonometric polynomial.

$$\begin{aligned} S_{3,6}(x) &= \frac{a0}{2} + \cos(xk)a + \sin(xk(1:n-1))b \\ &= 2.7973 - 1.92077\cos(x) - 0.27774\cos(2x) + 0.99998\cos(3x) \\ &\ 3.5097\sin(x) - 2.6036\sin(2x) \end{aligned}$$



Part (c)

This polynomial is obviously the most complex, but we can follow the same approach as before to derive it from the MATLAB code. Letting MATLAB run it's initial computations, we arrive at the following set of vectors which will be used to construct the trigonometric interpolating polynomial $S_8(x)$.

$$\begin{aligned} aa_0 &= 6.0028 \\ aa_n &= -1.16373 \\ aa &= [-2.32745, 0.12403, 0.60729, -0.90414, 1.04546, -1.11782, 1.15309] \\ bb &= [3.58208, -2.75145, 1.91784, -1.33901, 0.91301, -0.57180, 0.27595] \end{aligned}$$

Where $S_8(x)$ is given by

$$\frac{aa_0}{2} + \frac{aa_n}{2\cos(nx)} + \cos(xk(1:n-1))aa + \sin(xk(1:n-1))bb$$

Putting these together yields

$$S_8(x) = 3.0014 + \frac{-1.16373}{2\cos(8x)} + -2.32745\cos(x) + 0.12403\cos(2x) + 0.60729\cos(3x) + \\ -0.90414\cos(4x) + 1.04546\cos(5x) + -1.11782\cos(6x) + 1.15309\cos(7x) + \\ 3.58208\sin(x) + -2.75145\sin(2x) + 1.91784\sin(3x) + -1.33901\sin(4x) + \\ 0.91301\sin(5x) + -0.57180\sin(6x) + 0.27595\sin(7x) +$$