Problem 1.

(10 points) Find the least squares polynomial of degree 2 for the data in the following table. Compute the error E. Graph the data and the polynomial.

x_i	1.0	1.1	1.3	1.5	1.9	2.1
y_i	1.84	1.96	2.21	2.45	2.94	3.18

Solution

$$\text{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.1 & 1.1^2 \\ 1 & 1.3 & 1.3^2 \\ 1 & 1.5 & 1.5^2 \\ 1 & 1.9 & 1.9^2 \\ 1 & 2.1 & 2.1^2 \end{bmatrix} b = \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}$$

$$A^T A a = A^T b \Rightarrow (A^T A)^{-1}(A^T A)a = (A^T A)^{-1} A^T b \Rightarrow a = (A^T A)^{-1} A^T b$$

$$a = \begin{bmatrix} 66.841 & -90.719 & 28.748 \\ -90.719 & 124.544 & -39.811 \\ 28.748 & -39.811 & 12.832 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1.1 & 1.3 \\ 1.5 & 1.9 & 2.1 \\ 1 & 1.21 & 1.69 \\ 2.25 & 3.61 & 4.41 \end{bmatrix} \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}$$

$$= \begin{bmatrix} 4.86939 & 1.83451 & -2.51039 & -4.55547 & -1.74620 & 3.10816 \\ -5.98639 & -1.89239 & 3.90693 & 6.52134 & 2.19542 & -4.74490 \\ 1.76871 & 0.48237 & -1.32035 & -2.09647 & -0.56895 & 1.773469 \end{bmatrix} \begin{bmatrix} 1.84 \\ 1.96 \\ 2.21 \\ 2.45 \\ 2.94 \\ 3.18 \end{bmatrix}$$

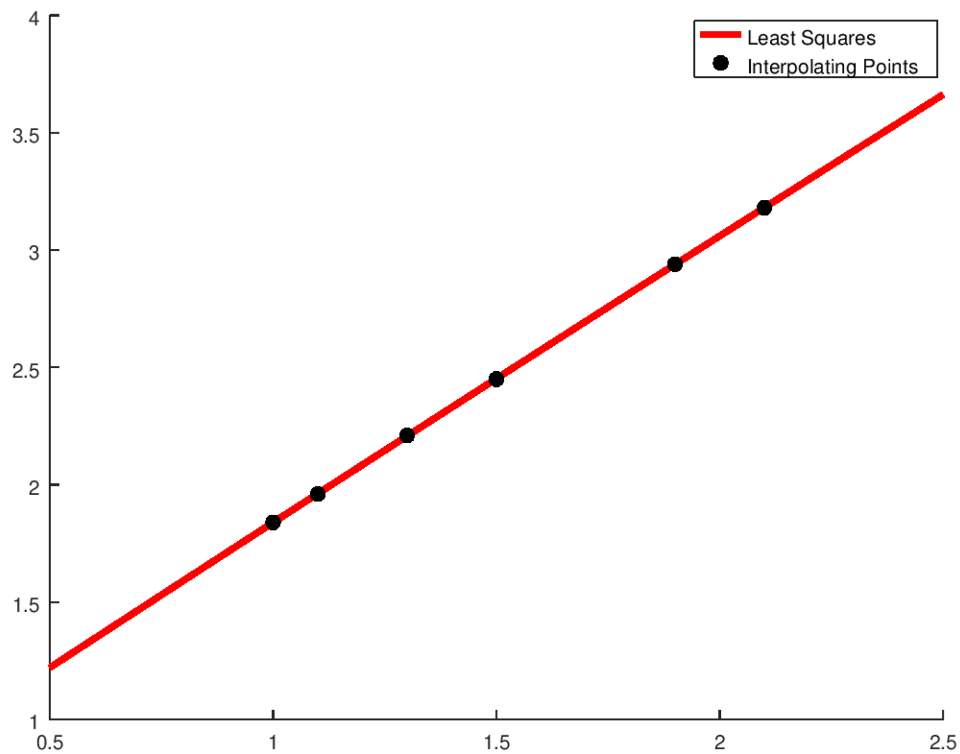
$$= \begin{bmatrix} 0.596581 \\ 1.253293 \\ -0.010853 \end{bmatrix}$$

This gives us the interpolating polynomial as $P_2(x) = -0.010853x^2 + 1.253293x + 0.596581$. Evaluating our initial points using our polynomial produces the results...

x_i	1.0	1.1	1.3	1.5	1.9	2.1
y_i	1.84	1.96	2.21	2.45	2.94	3.18
$P(x_i)$	1.839	1.9621	2.2075	2.4521	2.9387	3.1806
$\lvert P(x_i) - y_i \rvert$	-9.79e-4	2.0712e-3	2.4797e-3	2.1012e-3	1.3416e-3	6.3457e-4

Taking the sum of the differences finds the total error across all interpolating points as $E = 6.69 \times 10^{-6}$.

We can see that the least squares fit finds a polynomial which is almost linear, but not quite. From both the graph below, and the error above, the polynomial is clearly doing a good job (though not perfect) of fitting to the points.



Problem 2.

(20 points)

(a) Find the least squares polynomial of degree 2 for the following function $f(x)$ on the indicated interval

$$f(x) = e^x,\ [-1, 1]$$

Compute the error E. Graph the function and the polynomial.

(b) Repeat (a) with Legendre Polynomials.

Solution

Part (a)

I will be using the $x = (-1, 0, 1)$ as interpolating points to find my least squares polynomial, thus we want to use the following matrices in our system.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} b = \begin{bmatrix} \frac{1}{e} \\ 1 \\ e \end{bmatrix}$$

$$A^T Aa = A^T b \Rightarrow (A^T A)^{-1}(A^T A)a = (A^T A)^{-1} A^T b \Rightarrow a = (A^T A)^{-1} A^T b$$

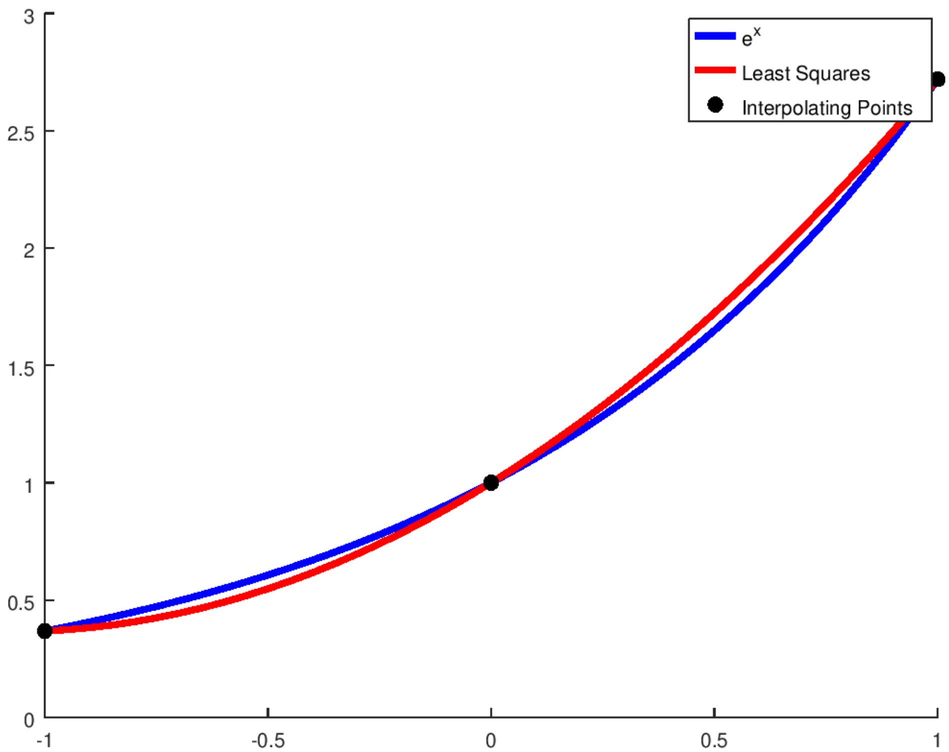
$$\begin{aligned} a &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0.5 & 0 \\ -1 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{e} \\ 1 \\ e \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{1}{e} \\ 1 \\ e \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1.7520 \\ 0.54308 \end{bmatrix} \end{aligned}$$

This gives us the interpolating polynomial $P_2(x) = 0.54308x^2 + 1.17520x + 1$. Evaluating our initial points using our polynomial produces the results...

x_i	-1.0	0	1.0
y_i	$\frac{1}{e}$	1	e
$P(x_i)$	0.36788	1.0	2.71828
$\lvert P(x_i) - y_i\rvert$	5.5883e-7	0	1.8285e-6

Taking the sum of the differences finds the total error across all interpolating points as $E = 2.3873 \times 10^{-6}$. If we look over the range $[-1, 1]$ and want to compute the total error (not just the interpolating points), we simply compute $\int_{-1}^{1} |P(x) - e^x| dx = 0.0890387$.

From the graph below, we can see that the interpolating polynomial closely matches, but diverges slightly from e^x. Notably, the polynomial switches from underestimating to overestimating after crossing $x = 0$.



Part (b)

$$b = \begin{bmatrix} \int_{-1}^{1} e^x P_0(x)dx \\ \int_{-1}^{1} e^x P_1(x)dx \\ \int_{-1}^{1} e^x P_2(x)dx \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} e^x dx \\ \int_{-1}^{1} e^x x dx \\ \int_{-1}^{1} e^x \frac{1}{2}(3x^2-1)dx \end{bmatrix} = \begin{bmatrix} e - \frac{1}{e} \\ \frac{2}{e} \\ 0.143126 \end{bmatrix} = \begin{bmatrix} 2.3504 \\ 0.73576 \\ 0.143126 \end{bmatrix}$$

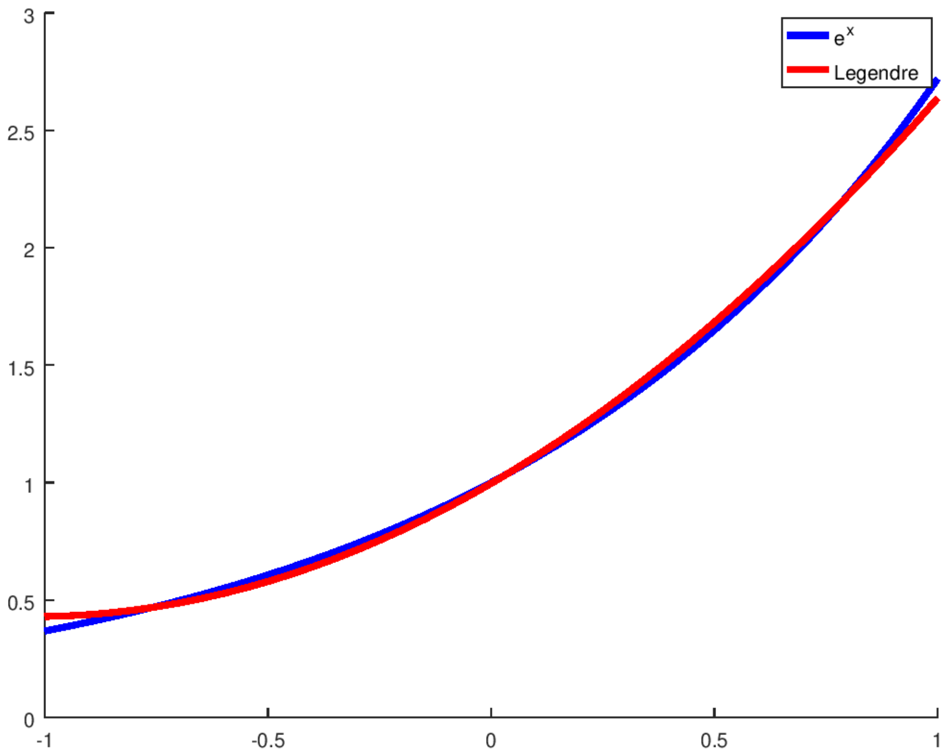
$$d = \begin{bmatrix} \frac{2(1)-1}{2} b_1 \\ \frac{2(2)-1}{2} b_2 \\ \frac{2(3)-1}{2} b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \times 2.3504 \\ \frac{3}{2} \times 0.73576 \\ \frac{5}{2} \times 0.143126 \end{bmatrix} = \begin{bmatrix} 1.1752 \\ 1.1036 \\ 0.35781 \end{bmatrix}$$

The Legendre Polynomial Approximation is then given by…

$$\begin{aligned} 1.1752P_0 + 1.1036P_1 + 0.35781P_2 &= 1.1752 + 1.1036x + 0.35781\frac{1}{2}(3x^2-1) \\ &= 0.536715x^2 + 1.1036x + 0.996295 \end{aligned}$$

We can compute error over the bounds by taking ${}_{-1}^{1}|L_2(x) - e^x|dx = 0.0458856$. Recall the error for least squares was 0.0890387, so we have approximately halved our error despite both polynomials having degree 2. Note that we may have found a better least squares polynomial if we had used different interpolating points, these were not necessary for the Legendre approximation.

Below we can visually confirm that this polynomial is doing a decent job of approximating e^x on our given bounds. It appears to be more consistent and accurate than the least squares polynomial but with greater error towards the end points.



Problem 3.

(20 points)

(a) Use the zeros of $\overline{T}_4$ to construct an interpolating polynomial of degree 3 for the following function on the interval $[-1, 1]$:

$$f(x) = e^x$$

Find a bound for the maximum error of the approximation.

(b) Repeat (a) on interval $[0, 2]$.

Solution

TODO

Problem 4.

(20 points) Find all the Chebyshev rational approximations of degree 2 for $f(x) = e^{-x}$. Graph the function and the polynomial. (Use MATLAB routines on classpage)

Solution

TODO

Problem 5.

(30 points) (Use MATLAB routines on classpage)

(a) Find the continous least squares trigonometric polynomial $S_3(x)$ for $f(x) = e^x$ on $[-\pi, \pi]$.

(b) Find the discrete least squares trigonometric polynomials $S_n(x)$ for $f(x) = e^x$ on $[-\pi, \pi]$ with $n = 3, m = 6$.

(c) Find the trigonometric interpolating polynomial $S_n(x)$ for $f(x) = e^x$ on $[-\pi, \pi]$ with $n = 8$.

Solution

TODO