

Problem 1.

- (a) Show that $x - (\ln x)^x = 0$ has at least one solution on $[4, 5]$.
- (b) Use the Intermediate Value Theorem and Rolle's Theorem to show that the graph of $f(x) = x^3 + 2x + k$ crosses the x-axis exactly once, regardless of the value of the constant k .

Solution**Part (a)**

Note that the function is continuous, in particular, it is continuous over $[4, 5]$. Further as we have that $f(4) = 4 - \ln(4)^4 = 0.306638422$ and $f(5) = 5 - \ln(5)^5 = -5.798691578$, and $-5.79869 < 0 < 0.306638422$, then by the Intermediate Value Theorem, $\exists x \in [4, 5]$ such that $f(x) = 0$.

Part (b)

Proof goes here.

Problem 2.

Let $f(x) = 2x \cos(2x) - (x - 2)^2$ and $x_0 = 0$.

- (a) Find the third Taylor polynomial $P_3(x)$, and use it to approximate $f(0.4)$
- (b) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_3(0.4)|$. Compute the actual error.

Solution**Part (a)**

$$P_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{6}(x - x_0)^3$$

$$f(0) = 2 \times 0 \cos(0) - (-2)^2 = -4$$

$$f'(0) = -2x - 4x \sin(2x) + 2 \cos(2x) + 4 = 2 \cos 0 + 4 = 2 + 4 = 6$$

$$f''(0) = -2(4 \sin(2x) + 4x \cos(2x) + 1) = -2(0 + 0 + 1) = -2$$

$$f^{(3)}(0) = 8(2x \sin(2x) - 3 \cos(2x)) = 8(0 - 3) = -24$$

Thus, the third Taylor polynomial about $x_0 = 0$ is given by

$$P_3(x) = -4 + 6x - x^2 - 4x^3$$

Then

$$f(0.4) \approx P_3(0.4) = -2.016$$

Part (b)

We have that $f^{(4)}(x) = 32(2 \sin(2x) + x \cos(2x))$.

The error is then given by the following (for some $c_x \in (0, 0.4)$):

$$\mathcal{E}(x) = \frac{f^{(4)}(c_x)}{24} x^4$$

Thus, to find an upper bound of the error, we must find an upper bound for $f^{(4)}(c_x)$. This is found at $f^{(4)}(0.4) = 54.8286$. Giving us an error of $\frac{54.8286}{24}(0.4)^4 = 0.05848384$. Actual error is given by $|f(0.4) - P_3(0.4)| = |-2.00263 - -2.016| = 0.0133654$.

Problem 3.

Let $f \in C[a, b]$ be a function whose derivative exists on (a, b) . Suppose f is to be evaluated at x_0 in (a, b) , but instead of computing the actual value $f(x_0)$, the approximation value, $\tilde{f}(x_0)$, is the actual value of f at $x_0 + \epsilon$, that is $\tilde{f}(x_0) = f(x_0 + \epsilon)$.

- (a) Use the Mean Value Theorem to estimate the absolute error $|f(x_0) - \tilde{f}(x_0)|$ and the relative error $|f(x_0) - \tilde{f}(x_0)|/|f(x_0)|$, assuming $f(x_0) \neq 0$.
- (b) If $\epsilon = 5 \times 10^{-6}$ and $x_0 = 1$, find bounds for the absolute and relative errors for
 - (i) $f(x) = e^x$
 - (ii) $f(x) = \sin x$

Solution**Part (a)**

For absolute error, we have $|f(x_0) - \tilde{f}(x_0)| = |f(x_0) - f(x_0 + \epsilon)|$ by the mean value theorem it follows that $|f(x_0) - f(x_0 + \epsilon)| = |f'(c)\epsilon|$, $c \in (x_0, x_0 + \epsilon)$.

Therefore, our bound for absolute error is given by $\epsilon \times \sup\{|f'(c)| : c \in (x_0, x_0 + \epsilon)\}$.

Similarly, the bound for relative error is thus given by

$$\epsilon \times \sup\{|f'(c)| : c \in (x_0, x_0 + \epsilon)\} / |f(x_0)|.$$

Part (b)

(i) $f(x) = e^x$

To find absolute error by the formula given in part (a), we simply need to find the maximum derivative of $f(x)$ over $(1, 1 + 5 \times 10^{-6})$. As e^x is non-decreasing, and $e^{x'} = e^x$ the maximum is given by $e^{(1+5 \times 10^{-6})} \approx 2.7182954199$ we then multiply this by ϵ to get the absolute error $= e^{1+5 \times 10^{-6}} 5 \times 10^{-6} \approx 0.000013591477$.

Relative error is then $(e^{1+5 \times 10^{-6}} 5 \times 10^{-6}) / (e^1) \approx 5.000025 \times 10^{-6}$.

(ii) $f(x) = \sin x$

$f'(x) = \cos(x)$, thus to determine bound for absolute error, I need the maximum value of $\cos(x)$ over $(1, 1 + 5 \times 10^{-6})$. Between 0 and $\pi/2$ $\cos(x)$ is positive and decreasing. As we have $0 < 1 < 1 + 5 \times 10^{-6} < \pi/2$ we will find our maximum value at $\cos(1) \approx 0.5403023$. We then multiply this by ϵ to get the absolute error $= \cos(1) \times \epsilon \approx 2.701511529 \times 10^{-6}$.

Relative error is then $\cos(1) \times \epsilon / \sin(1) \approx 3.210463 \times 10^{-6}$.

Problem 4.

The equation $4x^2 - e^x - e^{-x} = 0$ has four solutions $\pm x_1$ and $\pm x_2$. Use Newton's method to approximate the solution to within 10^{-5} with the following values of y_0 : (a) $p_0 = -10$, (b) $p_0 = -5$, (c) $p_0 = -3$, (d) $p_0 = -1$, (e) $p_0 = 1$, (f) $p_0 = 3$, (g) $p_0 = 5$, (h) $p_0 = 10$. Comment on the results.

Solution

```
function p4

clear all;
close all;
hold on;

% This is the function we want to find 0's for.
f = @(t) (4 * t.^2) - e.^t - e.^(-t);
df = @(t) (8 * t) - e.^t + e.^(-t);
% Iterate via Newton's method until we are within 'err' of 0.
ERR = 10^-(5);
% List of starting points to use with Newton's method.
_y0 = [-10, -5, -3, -1, 1, 3, 5, 10];

% Loop through each starting point to run Newton's method.
for y0 = _y0

    % Steps will contain a list of each point in the iteration.
    % We can plot this, as well as examine it's size to see
    % how well each initial point is performing.
    steps = [y0];

    % loop while we still have too much error
    while abs(f(steps(end))) > ERR
        tmp = steps(end);
        nxt = tmp - f(tmp)/df(tmp);
        steps = [steps nxt];
    end

    y0
    size(steps)
    plot(steps, f(steps), 'LineWidth', 3);

end

end
```

Problem 5.

Use Newton's method and Modified Newton's method to find the solution accurate to within 10^{-5} for

$$1 - 4x \cos x + 2x^2 + \cos 2x = 0, \text{ for } 0 \leq x \leq 1.$$

Comment on the performance of the methods.

Solution

TODO

Problem 6.

Use each of the following methods to find a solution in $[0.1, 1]$ accurate to within 10^{-4} for

$$600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0.$$

- (a) Bisection method
- (b) Newton's method
- (c) Secant method
- (d) Müller's method

Comment on the performance of these methods.

Solution

TODO

Problem 7.**Solution**

An object falling vertically through the air is subjected to viscous resistance as well as the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17 ft/s^2$ and k represents the coefficient of air resistance in lb-s/ft. Suppose $s_0=300ft$, $m = 0.25lb$, and $k = 0.1$ lb-s/ft. Find to within 0.01s the time it takes this quarter-pounder to hit the ground. Use Fixed-point iteration, Steffensen's method and Newton's method to find the solution.