Problem 1.

(a) Show that $x - (\ln x)^x = 0$ has at least one solution on $[4, 5]$.

(b) Use the Intermediate Value Theorem and Rolle's Theorem to show that the graph of $f(x) = x^3 + 2x + k$ crosses the x-axis exactly once, regardless of the value of the constant k.

Solution

Part (a)

Note that the function is continous, in particular, it is continous over $[4, 5]$. Further as we have that $f(4) = 4 - ln(4)^4 = 0.306638422$ and $f(5) = 5 - ln(5)^5 = -5.798691578$, and $-5.79869 < 0 < 0.306638422$, then by the Intermediate Value Theorem,
$\exists x \in [4, 5]$ such that $f(x) = 0$.

Part (b)

Proof goes here.

Problem 2.

Let $f(x) = 2x\cos(2x) - (x-2)^2$ and $x_0 = 0$.

(a) Find the third Taylor polynomial $P_3(x)$, and use it to approximate $f(0.4)$

(b) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_3(0.4)|$. Compute the actual error.

Solution

Part (a)

$P_3(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{6}(x-x_0)^3$

$$\begin{aligned} f(0) &= 2\times 0\cos(0) - (-2)^2 = -4 \\ f'(0) &= -2x - 4x\sin(2x) + 2\cos(2x) + 4 = 2\cos 0 + 4 = 2 + 4 = 6 \\ f''(0) &= -2(4\sin(2x) + 4x\cos(2x) + 1) = -2(0+0+1) = -2 \\ f^{(3)}(0) &= 8(2x\sin(2x) - 3\cos(2x)) = 8(0-3) = -24 \end{aligned}$$

Thus, the third Taylor polynomial about $x_0 = 0$ is given by

$$P_3(x) = -4 + 6x + -x^2 + -4x^3$$

Then

$$f(0.4) \approx P_3(0.4) = -2.016$$

Part (b)

We have that $f^{(4)}(x) = 32(2\sin(2x) + x\cos(2x))$.
The error is then given by the following (for some $c_x \in (0, 0.4)$):

$$\mathcal{E}(x) = \frac{f^{(4)}(c_x)}{24}x^4$$

Thus, to find an upper bound of the error, we must find an upper bound for $f^{(4)}(c_x)$. This is found at $f^{(4)}(0.4) = 54.8286$. Giving us an error of $\frac{54.8286}{24}(0.4)^4 = 0.05848384$.
Actual error is given by $|f(0.4) - P_3(0.4)| = |-2.00263 - -2.016| = 0.0133654$.

Problem 3.

Let $f \in C[a, b]$ be a function whose derivative exists on (a, b). Suppose f is to be evaluated at x_0 in (a, b), but instead of computing the actual value $f(x_0)$, the approximation value, $\widetilde{f}(x_0)$, is the actual value of f at $x_0 + \epsilon$, that is $\widetilde{f}(x_0) = f(x_0 + \epsilon)$.

(a) Use the Mean Value Theorem to estimate the absolute error $|f(x_0) - \widetilde{f}(x_0)|$ and the relative error $|f(x_0) - \widetilde{f}(x_0)|/|f(x_0)|$, assuming $f(x_0) \neq 0$.

(b) If $\epsilon = 5 \times 10^{-6}$ and $x_0 = 1$, find bounds for the absolute and relative errors for

(i) $f(x) = e^x$

(ii) $f(x) = \sin x$

Solution

Part (a)

For absolute error, we have $|f(x_0) - \widetilde{f}(x_0)| = |f(x_0) - f(x_0 + \epsilon)|$ by the mean value theorem it follows that $|f(x_0) - f(x_0 + \epsilon)| = |f'(c)\epsilon|, c \in (x_0, x_0 + \epsilon)$.
Therefore, our bound for absolute error is given by $\epsilon \times \sup|\{f'(c) : c \in (x_0, x_0 + \epsilon)\}|$.
Similarly, the bound for relative error is thus given by
$\epsilon \times \sup|\{f'(c) : c \in (x_0, x_0 + \epsilon)\}|/|f(x_0)|$.

Part (b)

(i) $f(x) = e^x$
To find absolute error by the formula given in part (a), we simply need to find the maximum derivative of $f(x)$ over $(1, 1 + 5 \times 10^{-6})$. As e^x is non-decreasing, and $e^{x\prime} = e^x$ the maximum is given by $e^{(1+5\times10^{-6})} \approx 2.7182954199$ we then multiply this by ϵ to get the absolute error $= e^{1+5\times10^{-6}} 5 \times 10^{-6} \approx 0.000013591477$.

Relative error is then $(e^{1+5\times10^{-6}} 5 \times 10^{-6})/(e^1) \approx 5.000025 \times 10^{-6}$.

(ii) $f(x) = \sin x$
$f'(x) = \cos(x)$, thus to determine bound for absolute error, I need the maximum value of $\cos(x)$ over $(1, 1 + 5 \times 10^{-6})$. Between 0 and $\pi/2$ $\cos(x)$ is positive and decreasing. As we have $0 < 1 < 1 + 5 \times 10^{-6} < \pi/2$ we will find our maximum value at $cos(1) \approx 0.5403023$. We then multiply this by ϵ to get the absolute error $= \cos(1) \times \epsilon \approx 2.701511529 \times 10^{-6}$.

Relative error is then $\cos(1) \times \epsilon/\sin(1) \approx 3.210463 \times 10^{-6}$.

Problem 4.

The equation $4x^2 - e^x - e^{-x} = 0$ has four solutions $\pm x_1$ and $\pm x_2$. Use Newton's method to approximate the solution to within 10^-5 with the following values of y_0 : (a)$p_0 = -10$, (b) $p_0 = -5$, (c) $p_0 = -3$, (d) $p_0 = -1$, (c) $p_0 = 1$, (g) $p_0 = 3$, (h) $p_0 = 5$, (i) $p_0 = 10$. Comment on the results.

Solution

Here is the code I am using to run newton's method

```
function p4

clear all;
close all;
hold on;

% This is the function we want to find 0's for.
f = @(t) (4 * t.^2) - e.^t - e.^(-t);
df = @(t) (8 * t) - e.^t + e.^(-t);
% Iterate via Newton's method until we are within 'err' of 0.
ERR = 10^-(5);
% List of starting points to use with Newton's method.
% Change the order so they show up better when I plot the results.
_y0 = [-10, 10, -5, 5, -3, 3, -1, 1];
% Here's some colors for plotting to help make things clear
COL = [
        [243 54 51] ./ 255,
        [234 78 92] ./ 255,
        [155 234 51] ./ 255,
        [51 234 118] ./ 255,
        [51 234 222] ./ 255,
        [51 88 234] ./ 255,
        [130 51 234] ./ 255,
        [234 51 152] ./ 255
]

% Loop through each starting point to run Newton's method.
for y0 = _y0

        % Steps will contain a list of each point in the iteration.
        % We can plot this, as well as examine it's size to see
        % how well each initial point is performing.
        steps = [y0];

        % loop while we still have too much error
```

```
        % infinite loop if we don't converge... but we do
        % for these y, so it's cool
        while abs(f(steps(end))) > ERR
                tmp = steps(end);
                nxt = tmp - f(tmp)/df(tmp);
                steps = [steps nxt];
        end

        y0
        steps
        plot(steps, f(steps), 'LineWidth', 3, 'Color', COL(end, :), 'DisplayName', int2s
        COL = COL(1:end-1, :);

end

legend('show');

end
```

Below I have formatted the text output from that code, we can immediately see some symmetry about the axis $x = 0$. Using the initial y_0 values, I found a total of 4 zeros at $x = \{\pm 4.3062, \pm 0.82450\}$. The closer I start to the guess, the faster (less steps) Newton's method will converge to the result. Interestingly, $x = \pm 3$ overshoot the zero on their side of the axis $x = 0$, that is $+3$ finds -0.82450 and -3 finds $+0.82450$. All others initial values find the zero nearest to them.

```
y0 = -10 => [-10.0000, -9.0146, -8.0456, -7.1093, -6.2339, -5.4634,
         -4.8572, -4.4752, -4.3269, -4.3066,  -4.3062]

y0 = -5 => [-5.0000  -4.5533  -4.3478  -4.3076  -4.3062  -4.3062]

y0 = -3 => [-3.00000   1.00194   0.83852   0.82461   0.82450]

y0 = -1 => [-1.00000  -0.83825  -0.82460  -0.82450]

y0 = 1 => [1.00000   0.83825   0.82460   0.82450]

y0 = 3 => [3.00000  -1.00194  -0.83852  -0.82461  -0.82450]

y0 = 5 => [5.0000   4.5533   4.3478   4.3076   4.3062   4.3062]

y0 = 10 => [10.0000, 9.0146, 8.0456, 7.1093, 6.2339, 5.4634,
         4.8572, 4.4752, 4.3269, 4.3066, 4.3062]
```

Problem 5.

Use Newton's method and Modified Newton's method to find the solution accurate to within 10^-5 for

$$1 - 4x\cos x + 2x^2 + \cos 2x = 0, \text{ for } 0 \leq x \leq 1.$$

Comment on the performance of the methods.

Solution

TODO

Problem 6.

Use each of the following methods to find a solution in $[0.1, 1]$ accurate to within 10^-4 for

$$600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0.$$

(a) Bisection method

(b) Newton's method

(c) Secant method

(d) Müller's method

Comment on the performance of these methods.

Solution

TODO

Problem 7.

Solution

An object falling vertically through the air is subjected to viscous resistance as well as the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17 ft/s^2$ and k represents the coefficient of air resistance in lb-s/ft. Suppose s_0=300ft, $m = 0.25$lb, and $k = 0.1$ lb-s/ft. Find to within 0.01s the time it takes this quarter-pounder to hit the ground. Use Fixed-point iteration, Steffensen's method and Newton's method to find the solution.