## Problem 1.

Use Matlab (or some other program) to calculate the polynomial which interpolates the function

$$f(x) = \frac{1}{1 + 25x^2} \tag{1}$$

at equally spaced points between -1 and 1, in steps of 0.25. You can use Matlab built-in to find the polynomial.

Plot the original function (in default color blue) and the interpolating polynomial (in red), and put circles around the interpolation points. Make sure you use enough points so that you get a smooth-looking curve.

**Solution**
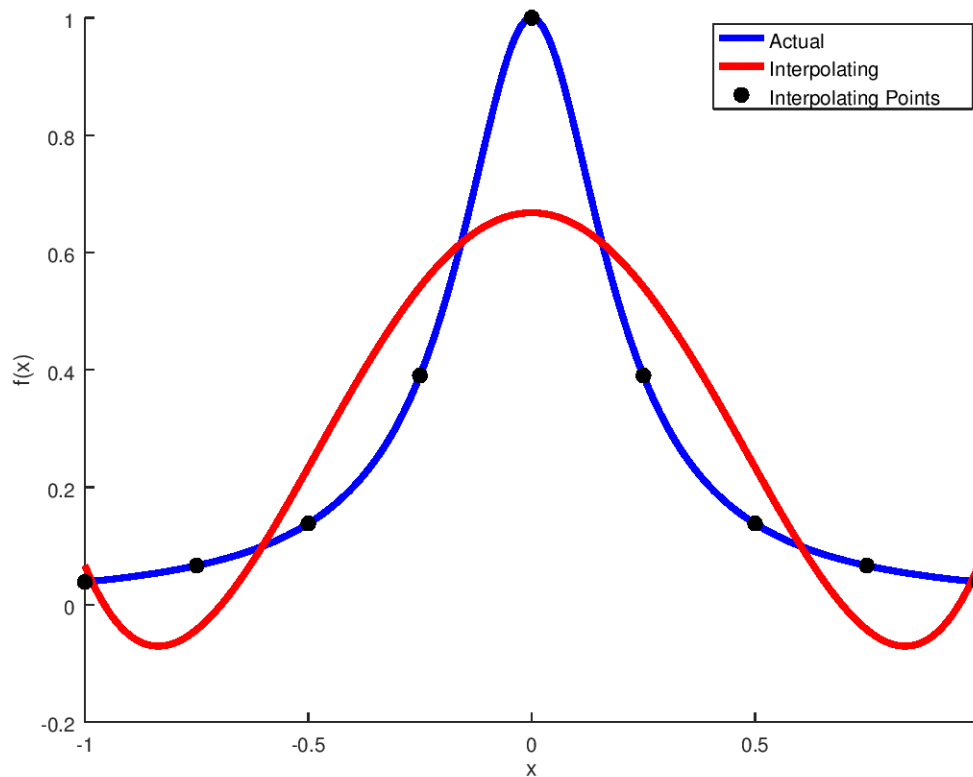


```
1    function p1
2
3    clear all;
4    close all;
5    hold on;
6
```

```matlab
7    f = @(x) 1 ./ (1 + 25 * x .^ 2);

8

9    % Actual curve.
10   _x = linspace(-1, 1, 200);
11   _y = f(_x);

12

13   % Interpolating points.
14   x = -1:0.25:1;
15   y = f(x);

16

17   % Fit a polynomial to the interpolating points.
18   p = polyval(polyfit(x, y, 5), _x);

19

20   plot(_x, _y, 'b', 'DisplayName', 'Actual', 'LineWidth', 3);
21   plot(_x,  p, 'r', 'DisplayName', 'Interpolating', 'LineWidth', 3);
22   scatter(x, y, 50, 'k', 'filled', 'DisplayName', 'Interpolating Points');
23   legend('show');
24   xlabel('x');
25   ylabel('f(x)');

26

27   end
```

## Problem 2.

(a) Given a point $x_0$ and a step size h, use Lagrange polynomials to find the quadratic polynomial which interpolates a function f(x) at the points $x_0$-2h, $x_0$, and $x_0$+h.

(b) Integrate the polynomial on $[x_0$-2h, $x_0$+h] to produce an integration formula. Use the formula to estimate $\int_0^3 e^{-x} dx$.

**Solution**

**Part (a)**

The quadratic Lagrange polynomial P(t) is given by

$$P(t) = f(x_0 - 2h)\frac{(t - t_1)(t - t_2)}{(t_0 - t_1)(t_0 - t_2)} + f(x_0)\frac{(t - t_0)(t - t_2)}{(t_1 - t_0)(t_1 - t_2)}$$
$$+ f(x_0 + h)\frac{(t - t_0)(t - t_1)}{(t_2 - t_0)(t_2 - t_1)}$$

In this case we have $t_0 = x_0 - 2h$, $t_1 = x_0$, and $t_2 = x_0 + h$.
Next we make the necessary substitutions and simplify the different parts of P(t).

$$
\begin{aligned}
(t - t_1)(t - t_) &= (t - x_0)(t - (x_0 + h)) \\
(t - t_0)(t - t_2) &= (t - x_0 + 2h)(t - x_0 - h) \\
(t - t_0)(t - t_1) &= (t - x_0 + 2h)(t - x0) \\
(t_0 - t_1)(t_0 - t_2) &= ((x_0 - 2h) - x_0)((x_0 - 2h) - (x_0 + h)) \\
&= (-2h)(x_0 - 2h - x_0 - h) \\
&= (-2h)(-3h) \\
&= 6h^2 \\
&= (t - x_0)(t - x_0 - h) \\
(t_1 - t_0)(t_1 - t_2) &= (x_0 - (x_0 - 2h))(x_0 - (x_0 + h)) \\
&= (2h)(-h) \\
&= -2h^2 \\
(t_2 - t_0)(t_2 - t_1) &= ((x_0 + h) - (x_0 - 2h))((x_0 - 2h) - x0) \\
&= (3h)(h) \\
&= 3h^2
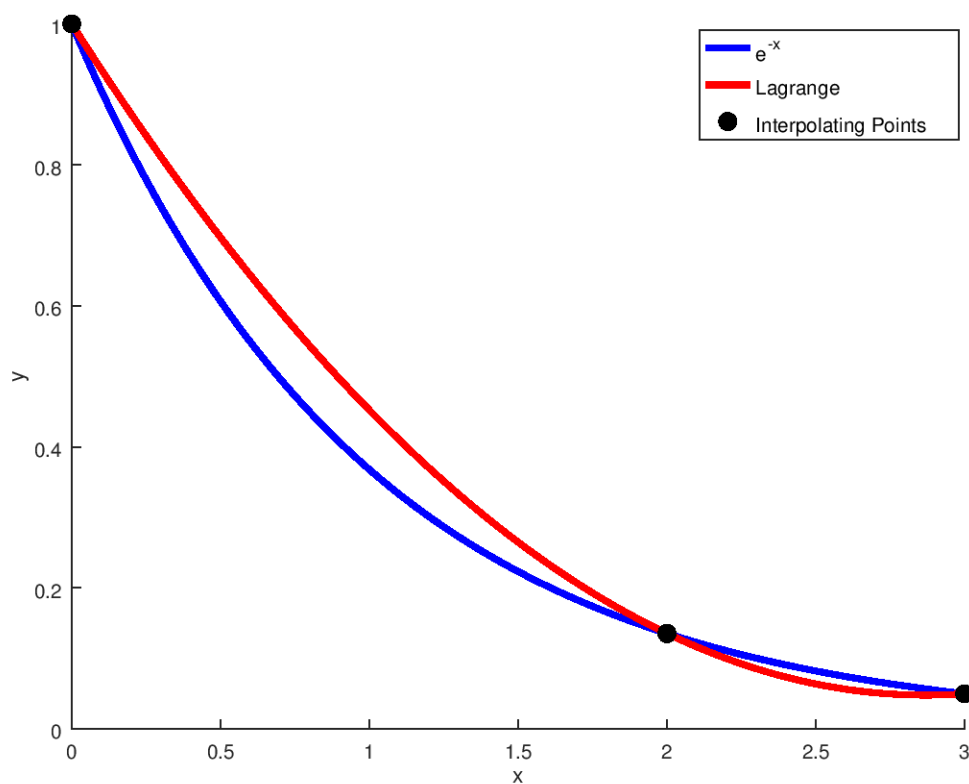\end{aligned}
$$

Thus we arrive at the following equation

$$P(t) = \frac{f(x_0 - 2h)}{6h^2}(t - x_0)(t - x_0 - h) + \frac{f(x_0)}{-2h^2}(t - x_0 + 2h)(t - x_0 - h)$$

$$+ \frac{f(x_0 + h)}{3h^2}(t - x_0 + 2h)(t - x_0)$$

$$= \frac{f(x_0 - 2h)}{6h^2}(t^2 - (2x_0 + h)t + (hx_0 + x_0^2)) + \frac{f(x_0)}{-2h^2}(t^2 + (h - 2x_0)t + (x_0^2 - hx_0 - 2h^2))$$

$$+ \frac{f(x_0 + h)}{3h^2}(t^2 + (2h - x_0)t + (x_0^2 - 2hx_0))$$

Here we can see the that P(t) is quadratic with respect to t (as expected), solving for the coefficients of $P(t) = At^2 + Bt + C$ produces

$$A = f(x_0 - 2h)\frac{1}{6h^2} + f(x_0)\frac{1}{-2h^2} + f(x_0 + h)\frac{1}{3h^2}$$

$$B = f(x_0 - 2h)\frac{-2x_0 - h}{6h^2} + f(x_0)\frac{h - 2x_0}{-2h^2} + f(x_0 + h)\frac{2h - 2x_0}{3h^2}$$

$$C = f(x_0 - 2h)\frac{hx_0 + x_0^2}{6h^2} + f(x_0)\frac{x_0^2 - hx_0 - 2h^2}{-2h^2} + f(x_0 + h)\frac{x_0^2 - 2hx_0}{3h^2}$$

Note that given h, $x_0$, and the interpolating points $f(x_0 - 2h)$, $f(x_0)$, and $f(x_0 + h)$ each of the terms A, B, C are constant with respect to t.
Included below is a graph comparing $f(x) = e^{-x}$ with P(t) given $x_0 = 2$, h = 1.

## Part (b)

$$\int_{x_0-2h}^{x_0+h} P(t) = \int_{x_0-2h}^{x_0+h} At^2 + Bt + C\,dt$$

$$= (\frac{1}{3}At^3 + \frac{1}{2}Bt^2 + Ct)\Big|_{x_0-2h}^{x_0+h}$$

$$= (\frac{1}{3}A((x_0+h)^3 - (x_0-2h)^3) + \frac{1}{2}B((x_0+h)^2 - (x_0-2h)^2) + C((x_0+h) - (x_0-2h))$$

Expanding and simplifying our terms

$$(x_0 + h) - (x_0 - 2h) = x_0 - x_0 + h + 2h$$
$$= 3h$$
$$(x_0 + h)^2 - (x_0 - 2h)^2 = x_0^2 + 2x_0h + h^2 - x_0^2 + 4x_0h - 4h^2$$
$$= 6x_0h - 3h^2$$
$$(x_0 + h)^3 - (x_0 - 2h)^3 = x_0^3 + 3x_0^2h + 3x_0h^2 + h^3$$
$$- x_0^3 + 6x_0^2h - 12x_0h^2 + 8h^3$$
$$= 9x_0^2 - 9x_0h^2 + 9h^3$$
$$= 9(x_0^2h - x_0h^2 + h^3)$$

This gives us the following equation

$$\int_{x_0-2h}^{x_0+h} P(t) = 3hA(x_0^2 - x_0h + h^2) + \frac{1}{2}hB(6x_0 - 3h) + C3h$$

Using the same constant coefficients A, B, C that were derived in part (A).
We can use this formula to estimate $\int_0^3 e^{-x}dx$ by setting $x_0 = 2$, $h = 1$ and plugging those values into our equation for $\int_{x_0-2h}^{x_0+h} P(t)$.
By doing so we produce the following approximation:

$$\int_0^3 P(t)dt = 1.0545$$

$$\approx$$

$$\int_0^3 e^{-x}dx = 0.95021$$

The estimate of the integral via the Lagrange polynomial has an error of 0.10429.

```
1   function p2
2
3   clear all;
4   close all;
5   hold on;
6
7   % Constants used for testing.
8   h = 1.0;
9   x0 = 2.0;
10
```

```matlab
11   % Example function to test our interpolation method.
12   f = @(x) e .^ (-x);
13   _x = linspace(0, 3, 200);
14
15   % List of our interpolation points.
16   x = [x0 - 2*h, x0, x0 + h];
17
18   % Plot the target curve.
19   plot(_x, f(_x), 'b', 'DisplayName', 'e^{-x}', 'LineWidth', 3);
20
21   % Interpolating points.
22   F0 = f(x0 - 2*h);
23   F1 = f(x0);
24   F2 = f(x0 + h);
25
26   % These are the constant scalars of the quadratic formula.
27   A = (F0/(6*h^2)) + (F1/(-2*h^2)) + (F2/(3*h^2));
28   B = (F0*(-2*x0-h))/(6*h^2) + (F1*(h-2*x0))/(-2*h^2) +
     ↪   (F2*(2*h-2*x0))./(3*h^2);
29   C = (F0*(h*x0 + x0^2))/(6*h^2) + (F1*(x0^2-h*x0-2*h^2))/(-2*h^2) +
     ↪   (F2*(x0^2-2*h*x0))/(3*h^2);
30
31   % And this is the actual interpolating polynomial.
32   P = @(x) A*(x.^2) + B*x + C;
33   plot(_x, P(_x), 'r', 'DisplayName', 'Lagrange', 'LineWidth', 3);
34   scatter(x, f(x), 75, 'k', 'filled', 'DisplayName', 'Interpolating Points');
35
36   % Evaluate the integral of P(t) from (x0-2h) to (x0+h).
37   INT = 3*h*A*(x0^2-x0*h+h^2) + 0.5*h*B*(6*x0-3*h) + C*3*h % = 1.0545
38   % Built in numerical solution for the integral of f we are estimating.
39   quad(f, 0, 3) % = 0.95021
40
41   legend('show');
42   xlabel('x');
43   ylabel('y');
44
45   end
```

## Problem 3.

For k=2, Iserles calls these the Nystrom and Milne methods. That is, start with

$$y(t_{n+2}) = y(t_n) + \int_{t_n}^{t_{n+2}} f(\tau, y(\tau))d\tau, \tag{2}$$

and interpolating at $t_n$, $t_{n+1}$ (explicit) or $t_n$, $t_{n+1}$, $t_{n+2}$ (implicit).
Find the leading error term, and the order of the method, for both cases.

### Solution

### Part (a)

Given $y(t_{n+2}) = y(t_n) + \int_{t_n}^{t_{n+2}} f(\tau, y(\tau))d\tau$, to solve for the explicit case (and thus find the error), we must approximate $\int_{t_n}^{t_{n+2}} f(\tau, y(\tau))d\tau$ using a Lagrange polynomial interpolating the points $f(t_n, y_n)$ and $f(t_{n+1}, y(t_{n+1}))$.
That is $y(t_{n+2}) = y(t_n) + \int_{t_n}^{t_{n+2}} f(\tau, y(\tau))d\tau \approx y(t_n) + \int_{t_n}^{t_{n+2}} P(\tau)d\tau$. Where P(t) is the Lagrange polynomial.

$$
\begin{aligned}
P(t) &= f(t_n, y_n)\frac{t - t_{n+1}}{t_n - t_{n+1}} + f(t_{n+1}, y(t_{n+1}))\frac{t - t_n}{t_{n+1} - t_n} \\
&= \frac{f(t_n, y_n)}{-h}(t - t_{n+1}) + \frac{f(t_{n+1}, y(t_{n+1}))}{h}(t - t_n)
\end{aligned}
$$

Next we integrate each side from $t_n$ to $t_{n+2}$ with respect to t

$$
\begin{aligned}
\int_{t_n}^{t_{n+2}} P(\tau)d\tau &= \frac{f(t_n, y_n)}{-h}\int_{t_n}^{t_{n+2}}(\tau - t_{n+1})d\tau + \frac{f(t_{n+1}, y(t_{n+1}))}{h}\int_{t_n}^{t_{n+2}}(\tau - t_n)d\tau \\
&= \frac{f(t_n, y_n)}{-h}(\frac{1}{2}\tau^2 - (t_{n+1})\tau)\Big|_{t_n}^{t_{n+2}} + \frac{f(t_{n+1}, y(t_{n+1}))}{h}(\frac{1}{2}\tau^2 - (t_n)\tau)\Big|_{t_n}^{t_{n+2}}
\end{aligned}
$$

Evaluating the integrals...

$$\left. (\frac{1}{2}\tau^2 - (t_{n+1})\tau) \right|_{t_n}^{t_{n+2}} = \frac{1}{2}(t_{n+2}^2) - (t_{n+1})(t_{n+2}) - \frac{1}{2}t_n^2 + (t_{n+1})t_n$$

$$= \frac{1}{2}(t_{n+2}^2 - t_n^2) + (t_{n+1})(t_n - t_{n+2})$$

$$= \frac{1}{2}(t_{n+2} + t_n)(t_{n+2} - t_n) + -2h(t_{n+1})$$

$$= h(t_{n+2} + t_n) + -2h(t_{n+1})$$

$$= h(t_{n+2} + t_n - t_{n+1} - t_{n+1})$$

$$= h(t_{n+2} - t_{n+1} + t_n - t_{n+1})$$

$$= h(h - h)$$

$$= 0$$

$$\left. (\frac{1}{2}\tau^2 - (t_n)\tau) \right|_{t_n}^{t_{n+2}} = \frac{1}{2}(t_{n+2})^2 - (t_n)(t_{n+2}) - \frac{1}{2}t_n^2 - (t_{n+1})(t_{n+2})$$

$$= \frac{1}{2}(t_{n+2}^2 - t_n^2) + (t_n)(t_n - t_{n+2})$$

$$= \frac{1}{2}(t_{n+2} + t_n)(t_{n+2} - t_n) + -2h(t_n)$$

$$= h(t_{n+2} + t_n) + -2h(t_n)$$

$$= h(t_{n+2} + t_n + -t_n - t_n)$$

$$= h(t_{n+2} - t_n)$$

$$= h(2h)$$

Substituting these results into our original equation

$$\int_{t_n}^{t_{n+2}} P(\tau)d\tau = \frac{f(t_n, y_n)}{-h}0 + \frac{f(t_{n+1}, y(t_{n+1}))}{h}h(2h)$$

$$= 2hf(t_{n+1}, y(t_{n+1}))$$

From

$$y(t_{n+2}) = y(t_n) + \int_{t_n}^{t_{n+2}} f(\tau, y(\tau))d\tau \approx y(t_n) + \int_{t_n}^{t_{n+2}} P(\tau)d\tau = y(t_n) + 2hf(t_{n+1}, y(t_{n+1}))$$

We get our k-step method as follows

$$y_{n+2} = y_n + 2hf(t_{n+1}, y(t_{n+1}))$$

The coefficients for the general k-step then are $a_1 = -1, a_1 = 0, b_2 = 0, b_1 = 2, b_0 = 0$. Thus $p(w) = w^2 - 1$ and $\sigma(w) = 2w$.

$$p(e^h) - h\sigma(e^h) = e^{2h} - 1 - h2e^h$$

$$= \frac{1}{3}h^3 + \frac{1}{3}h^4 + \frac{11}{60}h^5 + \dots$$

$$= \frac{1}{3}h^3 + O(h^4)$$

This implies that the Nystrom method has order 2, with a leading error term $\frac{1}{3}h^3$.

**Part (b)**

For the implicit case, the Lagrange polynomial approximating $\int_{t_n}^{t_{n+2}} f(\tau, y(\tau))d\tau$ will use interpolating points $f(t_n, y_n)$, $f(t_{n+1}, y(t_{n+1}))$, and $f(t_{n+2}, y(t_{n+2}))$.
That is $y(t_{n+2}) = y(t_n) + \int_{t_n}^{t_{n+2}} f(\tau, y(\tau))d\tau \approx y(t_n) + \int_{t_n}^{t_{n+2}} P(\tau)d\tau$. Where P(t) is the Lagrange polynomial.

$$P(t) = f(t_n, y(t_n))\frac{(t - t_{n+1})(t - t_{n+2})}{(t_n - t_{n+1})(t_n - t_{n+2})} + f(t_{n+1}, y(t_{n+1}))\frac{(t - t_n)(t - t_{n+2})}{(t_{n+1} - t_n)(t_{n+1} - t_{n+2})}$$

$$+ f(t_{n+2}, y(t_{n+2}))\frac{(t - t_n)(t - t_{n+1})}{(t_{n+2} - t_n)(t_{n+2} - t_{n+1})}$$

$$= \frac{f(t_n, y(t_n))}{2h^2}(t - t_{n+1})(t - t_{n+2}) + \frac{f(t_{n+1}, y(t_{n+1}))}{-h^2}(t - t_n)(t - t_{n+2})$$

$$+ \frac{f(t_{n+2}, y(t_{n+2}))}{2h^2}(t - t_n)(t - t_{n+1})$$

Next we integrate P(t)

$$\int_{t^n}^{t_{n+2}} P(\tau)d\tau = \frac{f(t_n, y(t_n))}{2h^2}\int_{t^n}^{t_{n+2}}(\tau - t_{n+1})(\tau - t_{n+2})d\tau + \frac{f(t_{n+1}, y(t_{n+1}))}{-h^2}\int_{t^n}^{t_{n+2}}(\tau - t_n)(\tau - t_{n+2})d\tau$$

$$+ \frac{f(t_{n+2}, y(t_{n+2}))}{2h^2}\int_{t^n}^{t_{n+2}}(\tau - t_n)(\tau - t_{n+1})d\tau$$

Where

$$\int_{t^n}^{t_{n+2}}(\tau - t_{n+1})(\tau - t_{n+2})d\tau = \frac{2}{3}h^3$$

$$\int_{t^n}^{t_{n+2}}(\tau - t_n)(\tau - t_{n+2})d\tau = -\frac{4}{3}h^3$$

$$\int_{t^n}^{t_{n+2}}(\tau - t_n)(\tau - t_{n+1})d\tau = \frac{2}{3}h^3$$

Thus we have

$$\int_{t^n}^{t_{n+2}} P(\tau)d\tau = \frac{1}{3}hf(t_n, y_n) + \frac{4}{3}hf(t_{n+1}, y(t_{n+1})) + \frac{1}{3}hf(t_{n+2}, y(t_{n+2}))$$

Giving us our k-step method

$$y_{n+2} = y_n + h(\frac{1}{3}f(t_n, y_n) + \frac{4}{3}f(t_{n+1}, y_{n+1}) + \frac{1}{3}f(t_{n+2}, y_{n+2}))$$

The coefficients for the general k-step then are $a_1 = -1, a_1 = 0, b_2 = \frac{1}{3}, b_1 = \frac{4}{3}, b_0 = \frac{1}{3}$. Thus $p(w) = w^2 - 1$ and $\sigma(w) = \frac{1}{3} + \frac{4}{3}w + \frac{1}{3}w^2$.

$$\begin{aligned}
p(e^h) - h\sigma(e^h) &= (e^{2h} - 1) - h(\frac{1}{3} + \frac{4}{3}e^h + \frac{1}{3}e^{2h}) \\
&= -\frac{1}{90}h^5 - \frac{1}{90}h^6 - \frac{23}{3780}h^7 \cdots \\
&= -\frac{1}{90}h^5 + O(h^6)
\end{aligned}$$

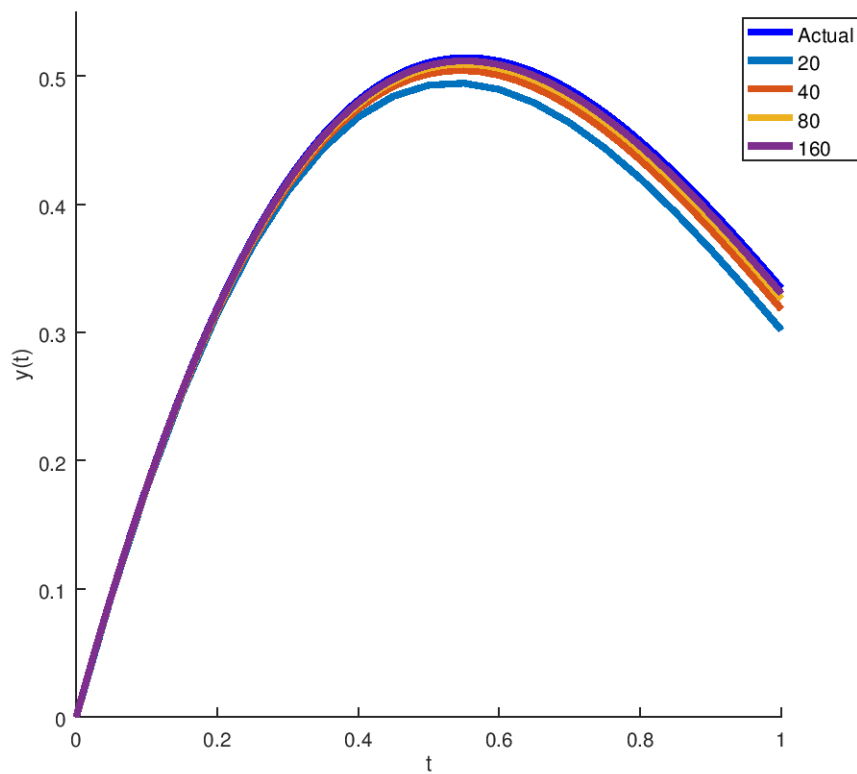This implies that the Milne method has order 4, with a leading error term $-\frac{1}{90}h^5$.
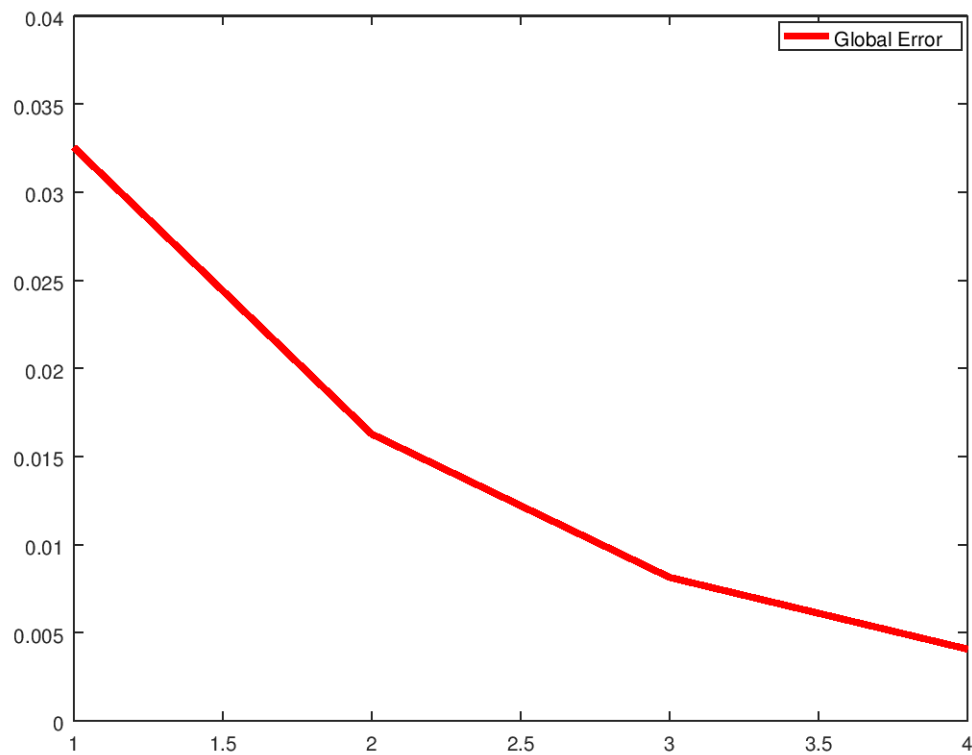
## Problem 4.

The IVP

$$y' = -y + e^{-t}\cos(2\text{t}) \tag{3}$$
$$y(0) = 0 \tag{4}$$

has the true solution $y(\text{t}) = e^{-t}\sin(2\text{t})$. Solve it numerically, using Euler's method with 20, 40, 80, 160 steps from 0 to 1, and compute the global error at the endpoint for each stepsize. Verify that the error goes down approximately linearly.

**Solution**

Note that the error term (graphed above) descends approximately linearly with respect to the step size.

```matlab
function p4

clear all;
close all;
hold on;
axis([0, 1, 0, 0.55], 'square');

y  = @(t) (e .^ -t) .* (sin(2 .* t));
dy = @(t, y) -y + (2 .* e .^ -t) .* cos(2 .* t);

_x = linspace(0, 1, 200);
_y = y(_x);
plot(_x, y(_x), 'DisplayName', 'Actual', 'b', 'LineWidth', 3);

% Where to start and stop the approximation.
start = 0;
stop = 1;

ERR = zeros(4, 1);
```

```matlab
steps = 20 .* 2 .^ (0:3);

for idx = 1:4

    h = (stop - start)/steps(idx);
    yk = zeros(steps + 1, 1);
    x = start:h:stop;

    yk(1) = 0; % Given initial point.
    for k = 1:steps(idx)
        tk = start + h * k;
        yk(k+1) = yk(k) + h * dy(tk, yk(k));
    end

    plot(x, yk, 'DisplayName', num2str(steps(idx)), 'LineWidth', 3);
    ERR(idx) = norm(yk - y(start:h:stop)', inf);
end

xlabel('t');
ylabel('y(t)');
legend('show');

%%%%%%%%%%%%%%%%%%%%%%
%% Graph the error. %%
%%%%%%%%%%%%%%%%%%%%%%
hold;
plot(1:4, ERR, 'r', 'LineWidth', 3, 'DisplayName', 'Global Error');
legend('show');

end
```

## Problem 5.

Solve

$$y''(t) = \begin{bmatrix} 0 \\ -10 \end{bmatrix} - 0.1y'(t),$$

$$y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$y'(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

by using Euler's method with stepsize h = 0.1, until $y_2$ becomes 0 or negative.
Plot $y_1$ and $y_)$ as functions of t (horizontal and vertical distance as functions of time), and
also plot $y_2$ versus $y_1$. That will show the path of the object. It should look a bit like a
parabola, but compressed on the right because of friction.

**Solution**
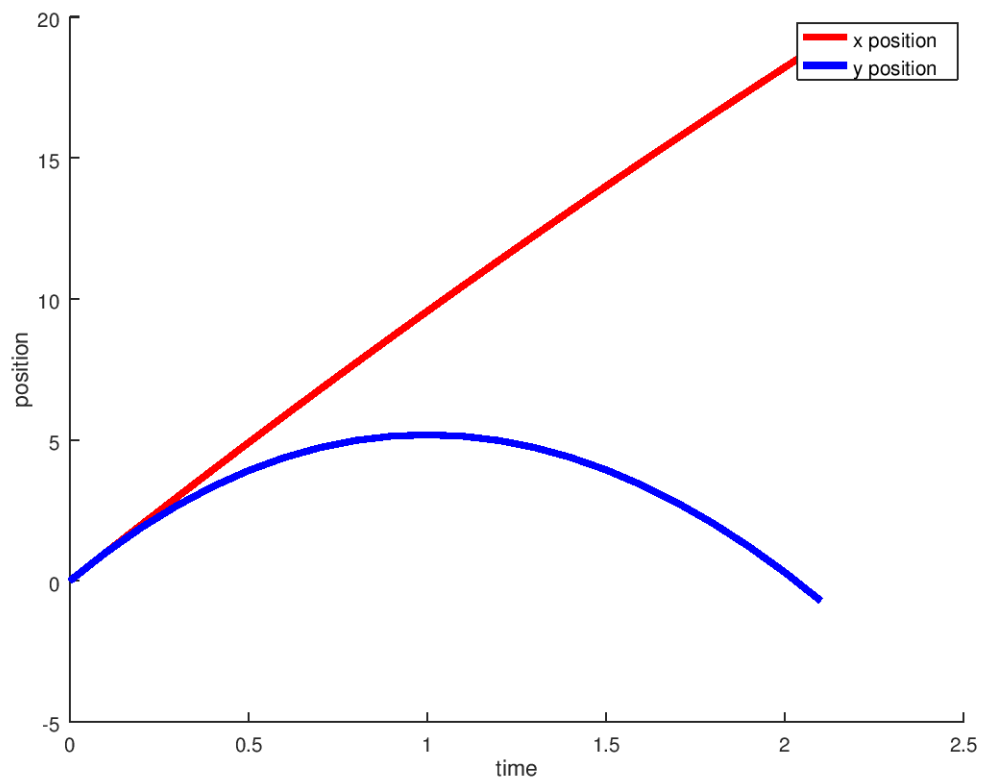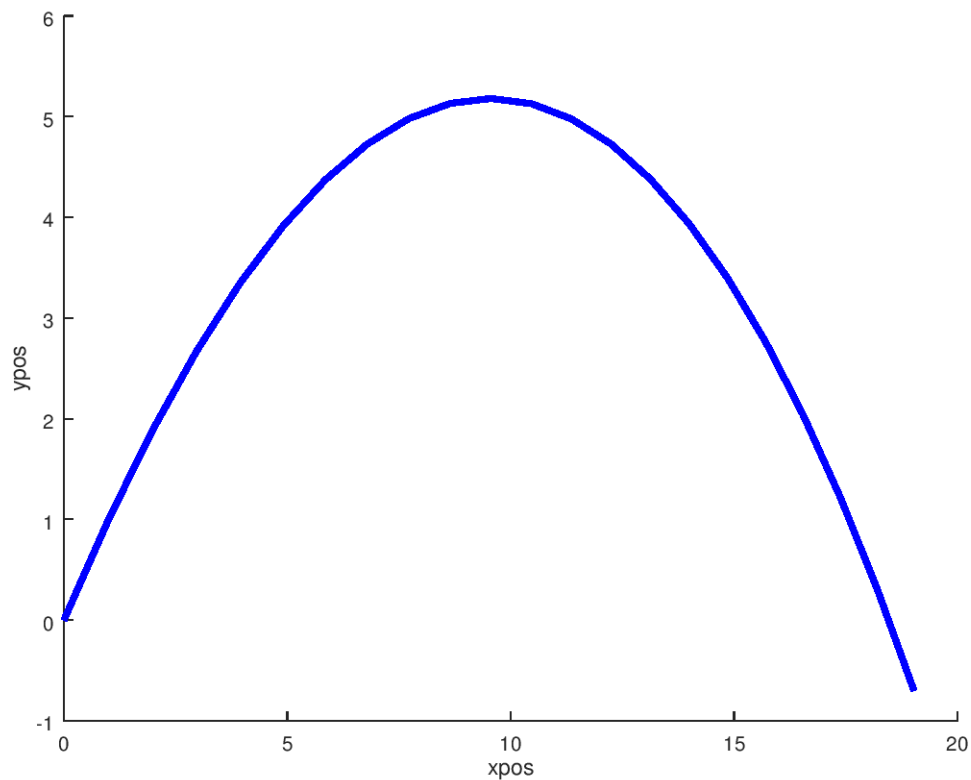
```
1   function p5
2
3   clear all;
4   close all;
5   hold on;
6
7   % Numerical solution initialized by the given initial values.
8   y1 = [[0; 0]];                      % y1 = y
9   y2 = [[10; 10]];         % y2 = y'
10
11  dy2 = @(t, y2) [0; -10] - (0.1 .* y2);
12  h = 0.1;          % Constant step size.
13  _t = [0];          % List of time steps for plotting.
14
15  % Need at least one pass to get off the ground...
16  do
17      % Get the current time step.
18      t = _t(length(_t));
19
20      % Previous elements we are stepping off from.
21      prev_y2 = y2(:, length( y2(2,:) ));
```

```matlab
22        prev_y1 = y1(:, length( y1(2,:) ));
23
24        % Step the simulation.
25        y1 = [y1, prev_y1 + h .* prev_y2];
26        y2 = [y2, prev_y2 + h .* dy2(t, prev_y2)];
27
28        % Step our method forward.
29        _t = [_t, t+h];
30
31    % Until we reach the ground again.
32    until y1(2, length(y1(1,:))) <= 0 % Octave not Matlab :)
33
34    % Plot horizontal and vertical position with respect to to time.
35    xlabel('time');
36    ylabel('position');
37    xplot = plot(_t, y1(1,:), 'LineWidth', 3, 'r');
38    yplot = plot(_t, y1(2,:), 'LineWidth', 3, 'b');
39    legend('x position', 'y position');
40
41    % Plot horizontal against vertical position.
42    hold;
43    xlabel('xpos');
44    ylabel('ypos');
45    plot(y1(1,:), y1(2,:), 'LineWidth', 3, 'b');
46
47    end
```