

HOMEWORK 2

If you are using technology (Matlab, Mathematica, etc.), include a printout of the relevant code and output or plot.

What I don't want to see is page after page of columns of numbers. That is just a waste of paper. If you are computing a numerical solution at 500 points, either print only the first few and last few numbers, or (even better) plot the curve.

1. Find the characteristic equation and reduced characteristic equation of the 3-step method

$$y_{n+1} = -\frac{27}{11}y_n + \frac{27}{11}y_{n-1} + y_{n-2} + h \left[\frac{3}{11}f_{n+1} + \frac{27}{11}f_n + \frac{27}{11}f_{n-1} + \frac{2}{11}f_{n-2} \right].$$

Find the (exact) roots of the reduced characteristic equation, and determine if this method is stable for small $h\lambda$ or not.

Notes: Iserles says this method has order 6. Let's not check that, and believe him. The reduced characteristic equation is a cubic polynomial equation. You can factor out the known root $r = 1$ and reduce it to a quadratic equation, which can be solved in exact form.

2. The IVP

$$\begin{aligned} y' &= -y + 2e^{-t} \cos(2t) \\ y(0) &= 0 \end{aligned}$$

has the true solution $y(t) = e^{-t} \sin(2t)$.

Solve it numerically from $t = 0$ to $t = 5$, using

(a) the classical 4-stage Runge-Kutta method with $h = 0.2$

(b) a PECE predictor-corrector method based on 4-step AB, 3-step AM, $h = 0.2$.

Use the RK values from part (a) for the startup.

(c) the Matlab built-in `ODE45`. This routine will pick its own steps.

Plot all 3 curves in one picture.

Print out the values at $t = 3$ for all methods, and the error at that point. You don't need to print out the rest of the values.

Run the RK and ABM programs again with $h = 0.1$, and compare the error at $t = 3$ for $h = 0.2$ and $h = 0.1$.

3. This is a very simple problem, just a preparation for the following problem. Use Matlab routine `fzero` or something comparable to find the two solutions of

$$e^x - x - 2 = 0.$$

There is one positive and one negative solution.

4. Solve the boundary value problem

$$\begin{aligned} y''(t) &= \frac{3}{2}y^2(t) \\ y(0) &= 4 \\ y(1) &= 1 \end{aligned}$$

by a shooting method. The required steps are as follows:

(a) Write a function `phi(s)` which calculates the amount by which the solution with initial conditions

$$\begin{aligned} y(0) &= 4 \\ y'(0) &= s \end{aligned}$$

misses the value $y(1) = 1$. In other words: Use `ode45` (with default accuracy) to solve the problem with initial condition $y'(0) = s$ numerically, and return $y(1) - 1$. (Here $y(1)$ means the value of y at the point 1, not the first entry in the vector).

(b) Plot `phi(s)` for $s \in [-40, -5]$ and read off initial guesses for the correct s . Use `fzero` to find the exact values of s , to default accuracy. There are two solutions.

(c) Plot the two solution curves corresponding to these two values of s in one picture. Obviously, both of them should satisfy the boundary conditions.

5. A *multivibrator* is a circuit that consists of a pair of transistor switches cross-coupled to provide an oscillator. Such circuits have a variety of uses depending on their type.

- The *monostable* multivibrator always returns to the same stable state.
- The *bistable* multivibrator (flip-flop) has two stable steady states. Once it is set to either, one it remains there until an external signal switches it to the opposite state. Such devices can be used in memory registers, for example.
- The *astable* multivibrator has two steady states as well, but they are both unstable, so it switches periodically between them. Such a device can be used as a clock.

Your task in this project is to simulate the operation of an astable multivibrator by solving the governing differential equations. During the integration interval, you will observe a short transient from the device's initial state, followed by a complete cycle including two switching events.

The two transistors central to the device are connected in the “common collector” mode. Their emitter and drain voltages make up the state variables V_1 , V_2 , V_3 , and V_4 of the device. The governing differential equations are derived from Kirchhoff's Voltage Law; the exponential terms come from the nonlinear device characteristics.

$$\begin{aligned} \frac{dV_1}{dt} &= p_1(V_{cc} - V_1) + p_2(V_{cc} - V_2) - p_3(e^{V_3/V_T} - 1) + \\ &\quad p_4(e^{(V_3-V_1)/V_T} - 1) - p_5(e^{V_2/V_T} - 1) - p_6(e^{(V_2-V_4)/V_T} - 1) \\ \frac{dV_2}{dt} &= p_7(V_{cc} - V_1) + p_2(V_{cc} - V_2) - p_8(e^{V_3/V_T} - 1) + \\ &\quad p_9(e^{(V_3-V_1)/V_T} - 1) - p_5(e^{V_2/V_T} - 1) - p_6(e^{(V_2-V_4)/V_T} - 1) \\ \frac{dV_3}{dt} &= p_2(V_{cc} - V_3) + p_7(V_{cc} - V_4) - p_5(e^{V_3/V_T} - 1) - \\ &\quad p_6(e^{(V_3-V_1)/V_T} - 1) - p_8(e^{V_2/V_T} - 1) + p_9(e^{(V_2-V_4)/V_T} - 1) \\ \frac{dV_4}{dt} &= p_2(V_{cc} - V_3) + p_1(V_{cc} - V_4) - p_5(e^{V_3/V_T} - 1) - \\ &\quad p_6(e^{(V_3-V_1)/V_T} - 1) - p_3(e^{V_2/V_T} - 1) + p_4(e^{(V_2-V_4)/V_T} - 1) \end{aligned}$$

In these equations, $V_T = 0.026$ is the *thermal voltage*, $V_{cc} = 5.0$ is a fixed voltage; the parameters p_1, \dots, p_9 are characteristic of the device and have values of

$$\begin{aligned} p_1 &= 1100000, & p_2 &= 20000, & p_3 &= 0.000011, \\ p_4 &= 0.000022, & p_5 &= 0.0000001, & p_6 &= 0.00001, \\ p_7 &= 1000000, & p_8 &= 0.00001, & p_9 &= 0.00002. \end{aligned}$$

The device commences operation at time $t = 0$ with the following voltage values:

$$V_1(0) = 0.3, \quad V_2(0) = -5.0, \quad V_3(0) = 0.6, \quad V_4(0) = 5.0.$$

To make life easier and avoid typos, I have typed the values of the constants into a file `params.m` that you can copy from the course web site.

(a) This is a stiff ODE. Solve the equations on the interval $[0, 0.001]$ (1 millisecond), using a stiff solver such as `ode15s`. Plot the result (all 4 components in the same picture).

(b) Answer the following questions.

- How many steps did the stiff solver take? How much computer time? (Hint: check out the `tic` and `toc` commands in Matlab).
- How many steps does a non-stiff solver, such as `ode113`, take? How much computer time does it take? (Note: on my machine, the RK-based solvers `ode23` and `ode45` can't handle this problem at all). You don't have to plot it again, it will look the same.
- What is the clock period?