Homework 3

1. Assume we have a two-dimensional grid with equal spacing h in the x and y directions. The standard approximation to $\nabla^2 u = u_{xx} + u_{yy}$ with accuracy $O(h^2)$ is

$$\frac{1}{h^2}\begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix}$$

Consider a general 3×3 stencil. Because of symmetry considerations, we really only have 3 coefficients to play with:

$$\frac{1}{h^2}\begin{Bmatrix} a & b & a \\ b & c & b \\ a & b & a \end{Bmatrix}$$

(a) Write out the Taylor series expansion of this formula up to the 4th derivative terms. Derive the two equations that make this formula an approximation to $\nabla^2 u$ with error $O(h^2)$.

Hint: Observe that for every term with $x + h$, there is a corresponding term with $x - h$, and likewise for $y \pm h$. The effect is that any term with an odd number of derivatives in x or y (or both) cancels. You only have to worry about u, u_{xx}, u_{yy}, u_{xxxx}, u_{xxyy} and u_{yyyy}.

(b) Explain why there is no way to get a formula with accuracy higher than $O(h^2)$ out of this approach.

(c) However, there is a way to produce a formula where the error term is of the form

$$c \cdot (\nabla^4 u)h^2.$$

Here $\nabla^4 u = \nabla^2(\nabla^2 u)$. Find it.

Remark: This formula can be used in an approach similar to the one used in problem 2. See Iserles' book for details.

2. This problem is a one-dimensional version of an approach described for two dimensions in the Iserles book.

The standard second derivative approximation is

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = u_i'' + \frac{1}{12}u_i^{(4)}h^2 + \cdots .$$

(a) Solve the DE

$$\begin{aligned} u''(x) &= -\sin x + e^x \\ u(0) &= 1 \\ u(1) &= \sin 1 + e \end{aligned}$$

using standard finite differences, with step size $h = 1/4$. The true solution is $u(x) = \sin x + e^x$. Find the maximum error.

Repeat with step size $h = 1/8$, and verify that the error goes down by a factor of 4, as expected.

(b) Here is a way to get higher accuracy:

$$\begin{aligned}
\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} &= u_i'' + \frac{1}{12} u_i^{(4)} h^2 + O(h^4) \\
&= f_i + \frac{1}{12} f_i'' \, h^2 + O(h^4) \\
&= f_i + \frac{1}{12} \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} \, h^2 + O(h^4)
\end{aligned}$$

Repeat part (a) with this right-hand side, and verify that this time the error behaves like h^4.

3. Instead of the Poisson equation $\nabla^2 u = f$ in two dimensions, we could consider an equation of the form $a(x,y)u_{xx} + b(x,y)u_{yy} = f$, with some coefficient functions a and b. That would be easy to program: just multiply your stencils with the appropriate values of a_{ij}, b_{ij}.

What comes up much more frequently in applications is the equation

$$\nabla \cdot [a(x,y)\nabla u(x,y)] = f.$$

For the one-dimensional version $[a(x)u'(x)]'$, a divided difference formula is given by

$$\big[a(x)u'(x)\big]_i' \approx \frac{a_{i+1/2}(u_{i+1} - u_i) - a_{i-1/2}(u_i - u_{i-1})}{h^2}.$$

This is derived by doing two centered differences in a row, with step size $h/2$.

(a) Determine the leading error term of this formula.

(b) Write out the system of linear equations that you get when you apply this formula to the BVP

$$\begin{aligned}
[a(x)u'(x)]' &= f(x) \quad \text{on } [0,1] \\
u(0) &= 2 \\
u(1) &= 3
\end{aligned}$$

with step size $h = 0.25$. The functions a and f are not given, so your solution should contain general terms like $a_{3/2}$ and f_2. This will be a 3×3 system of equations.

4. **(a)** Set up the finite difference equations for stepsize $h = k = 0.25$ for the Poisson equation

$$\begin{aligned}
-\nabla^2 u &= x^2 + y^2 \qquad \text{in } [0,1] \times [0,1] \\
u(x,0) &= 0 \\
u(x,1) &= \frac{1}{2}x^2 \\
u(0,y) &= \sin(\pi y) \\
u(1,y) &= e^{\pi} \sin(\pi y) + \frac{1}{2}y^2.
\end{aligned}$$

The correct solution is

$$u(x,y) = e^{\pi x} \sin(\pi y) + \frac{1}{2}x^2 y^2.$$

Print out the matrix and right-hand side. This should be a 9×9 matrix. You can use routine `poisson` to build the matrix. Basically all the work is in getting the right-hand side correct.

(b) Solve the equation and plot the resulting surface. Solve it again with $h = 0.05$, and plot.

Do not print out any values in this part, just do the surface plots.