Problem 1.

Assume we have a two-dimensional grid with equal spacing h in the x and y directions. The standard approximation to $\nabla^2 u = u_{xx} + u_{yy}$ with accuracy $O(h^2)$ is

$$\frac{1}{h^2}\begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix}$$

Consider a general 3 x 3 stencil. Because of symmetry considerations, we really only have 3 coefficients to play with:

$$\frac{1}{h^2}\begin{Bmatrix} a & b & a \\ b & c & b \\ a & b & a \end{Bmatrix}$$

(a) Write out the Taylor series expansion of this formula up to the 4th derivative terms. Derive the two equations that make this formula an approximation to $\nabla^2 u$ with error $O(h^2)$.

(b) Explain why there is no way to get a formula with accuracy higher than $O(h^2)$ out of this approach.

(c) However, there is a way to produce a formula where the error term is of the form

$$c \cdot (\nabla^4 u)h^2.$$

Here $\nabla^4 u = \nabla_2(\nabla_2 u)$. Find it.

Solution

Part (a)

Part (b)

Part (c)

Problem 2.

The standard second derivative approximation is

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = u_i'' + \frac{1}{12}u^{(4)}h^2 + \ldots.$$

(a) Solve the DE

$$\begin{aligned} u''(x) &= -sinx + e^x \\ u(0) &= 1 \\ u(1) &= sin1 + e \end{aligned}$$

using standard finite differences, with step size h = 1/4. The true solution $u(x) = sinx + e^x$. Find the maximum error. Repeat with step size h = 1/8, and verify that the error goes down by a factor of 4, as expected.

(b) Here is a way to get higher accuracy

$$\begin{aligned} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} &= \frac{1}{12}u_i^{(4*)}h^2 + O(h^4) \\ &= f_i + \frac{1}{12}f_i''h^2 + O(h^4) \\ &= f_i + \frac{1}{12}\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}h^2 + O(h^4) \end{aligned}$$

Repeat part (a) with this right-hand side, and verify that this time the error behaves like h^4.

Solution

Part (a)

Part (b)

Problem 3.

Instead of the Poisson equation $\nabla^2 u = f$ in two dimensions, we could consider an equation of the form $a(x, y)u_{xx} + b(x, y)u_{yy} = f$, with some coefficient functions a and b. That would be easy to program: just multiply your stencils with the appropriate values of a_{ij}, b_{ij}. What comes up much more frequently in applications is the equation

$$\nabla \cdot [a(x, y)\nabla u(x, y) = f.$$

For the one-dimensional version $[a(x)u'(x)]'$, a divided difference formula is given by

$$[a(x)u'(x)]'_i \approx \frac{a_{i+1/2}(u_{i+1} - u_i) - a_{i-1/2}(u_i - u_{i-1})}{h^2}.$$

This is derived by doing two centered differences in a row, with step size h/2.

(a) Determine the leading error term of this formula.

(b) Write out the system of linear equations that you get when you apply this formula in the BVP

$$\begin{aligned} [a(x)u'(x)]' &= f(x) on [0, 1] \\ u(0) &= 2 \\ u(1) &= 3 \end{aligned}$$

with step size h = 0.25. The functions a and f are not given, so your solution should contain general terms like $a_{3/2}$ and f_2. This will be a 3×3 system of equations.

Solution

Part (a)

Part (b)

Problem 4.

(a) Set up the finite difference equations for stepsize h = k = 0.25 for the Poisson equation

$$
\begin{aligned}
-\nabla^2 u &= x^2 + y^2 in[0,1] \times [0,1] \\
u(x,0) &= 0 \\
u(x,1) &= \frac{1}{2}x^2 \\
u(0,y) &= sin(\pi y) \\
u(1,y) &= e^{\pi} sin(\pi y) + \frac{1}{2}x^2 y^2
\end{aligned}
$$

The correct solution is

$$u(x,y) = e$$

(b) Solve the equation and plot the resulting surface. Solve it again with h = 0.05, and plot.

Solution

Part (a)

Part (b)