Problem 1.

Find the characteristic equation and reduced characteristic equation of the 3-step method

$$y_{n+1} = -\frac{27}{11}y_n + \frac{27}{11}y_{n-1} + y_{n-2} + h\left[\frac{3}{11}f_{n+1} + \frac{27}{11}f_n + \frac{27}{11}f_{n-1} + \frac{2}{11}f_{n-2}\right]$$

Find the (exact) roots of the reduced characteristic equation, and determine if this method is stable for small $h\lambda$ or not.

Solution

First let us rewrite the method as

$$y_{n+1} + \frac{27}{11}y_n - \frac{27}{11}y_{n-1} - y_{n-2} = h\left[\frac{3}{11}f_{n+1} + \frac{27}{11}f_n + \frac{27}{11}f_{n-1} + \frac{2}{11}f_{n-2}\right]$$

Note that for this method we have

$$a_0 = -1, a_1 = -\frac{27}{11}, a_2 = \frac{27}{11}$$

Which produces the characteristic polynomial

$$\begin{aligned} p(z) &= z^3 + \frac{27}{11}z^2 - \frac{27}{11} - 1 \\ &= \frac{1}{11}(z-1)(11z^2 + 38z + 11) \\ &= \frac{1}{11}(z-1)(z + \frac{1}{11}(19 + 4\sqrt{15}))(z - \frac{1}{11}(4\sqrt{15} - 19)) \end{aligned}$$

From this we find the zeros for the characteristic polynomial

$$z = 1, z = -\frac{1}{11}(19 + 4\sqrt{15}), z = \frac{1}{11}(4\sqrt{15} - 19)$$

Note that $|-\frac{1}{11}(19 + 4\sqrt{15})| \approx 3.1356303 > 1$.
This implies that this method is not stable for small values of $h\lambda$.

Problem 2.

The IVP

$$y' = -y + 2e^{-t}\cos(2t)$$
$$y(0) = 0$$

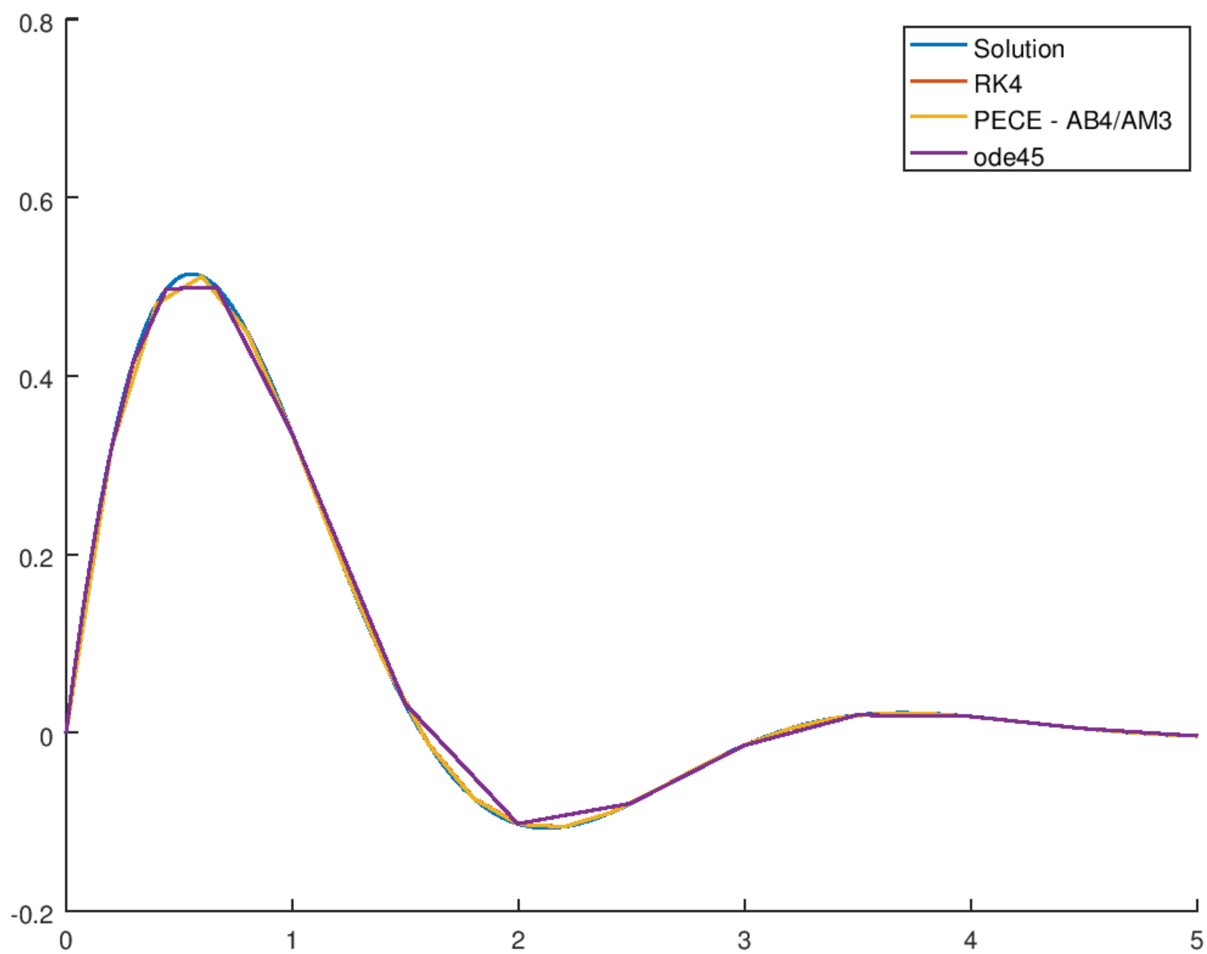
has the true solution $y(t) = e^{-t}\sin(2t)$.
Solve it numerically from t=0 to t=5, using

(a) the classical 4-stage Runge-Kutta method with h=0.2

(b) a PECE predictor-corrector method based on 4-step AB, 3-step AM, h=0.2
Use RK values from part (a) for the startup.

(c) the Matlab built-in ODE45. This routine will pick its own steps.

Solution



	Actual	RK4	PECE - AB4/AM3	ODE45 (h unknown)
h = 0.2				
Value	-0.013911	-0.013911	-0.013776	-0.014068
Error		$2.1822 * 10^{-7}$	$1.3560 * 10^{-4}$	$1.5663 * 10^{-4}$
h = 0.1				
Value	-0.013911	-0.013911	-0.013908	
Error		$1.2907 * 10^{-8}$	$3.3871 * 10^{-6}$	

```
function p2

clear all;
close all;
hold on;

% Actual solution.
_f = @(t) e.^(-t) .* sin(2 .* t);
_t = linspace(0, 5, 200);
_y = _f(_t);

plot(_t, _y, 'DisplayName', 'Solution', 'LineWidth', 1);
val = _f(3)

% Numerical Solutions
f = @(t, y) -y + 2 .* e^(-t) * cos(2*t);
h = 0.2; % Step Size
t = 0:h:5; % Interval
y = [0]; % Initial point
steps = length(t);

% RK4
for n = 1:(steps-1)

    % 4-Step
    k1 = f(t(n),        y(n));
    k2 = f(t(n) + h/2,        y(n) + (h/2) * k1);
    k3 = f(t(n) + h/2,        y(n) + (h/2) * k2);
    k4 = f(t(n) + h,        y(n) + h * k3);

    % Compute y_{n+1}
    y(n+1) = y(n) + (h/6) * (k1 + 2*k2 + 2*k3 + k4);

end

val = y((3/h) + 1)
err = norm(_f(3) - val, inf)
```

```
plot(t, y, 'DisplayName', 'RK4', 'LineWidth', 1);

% PECE
y = y(1:4); % Pull startup values from RK4

for n = 5:steps

    % P: 4-Step AB
    y(n) = y(n-1) + h * (
        (55/24) * f(t(n-1), y(n-1)) -
        (59/24) * f(t(n-2), y(n-2)) +
        (37/24) * f(t(n-3), y(n-3)) -
        (3/8)        * f(t(n-4), y(n-4))
    );

    % C: 3-Step AM
    y(n) = y(n-1) + h * (
        (3/8)        * f(t(n), y(n)) +
        (19/24)        * f(t(n-1), y(n-1)) -
        (5/24)        * f(t(n-2), y(n-2)) +
        (1/24)        * f(t(n-3), y(n-3))
    );

end

val = y((3/h) + 1)
err = norm(_f(3) - val, inf)
plot(t, y, 'DisplayName', 'PECE - AB4/AM3', 'LineWidth', 1);

% ODE45
% Note I am using octave, so the implementation may differ slightly
% from the one provided by Matlab.

[solx, soly] = ode45(f, [0 5], [0 0]);
val = interp1(solx, soly(:,1), 3, method='linear')
err = norm(_f(3) - val, inf)
plot(solx, soly(:,1), 'DisplayName', 'ode45', 'LineWidth', 1);

legend('show');

end
```

Problem 3.

Use the Matlab routine $fzero$ or something comparable to find the two solutions of

$$e^x - x - 2 = 0.$$

There is one positive and one negative solution.

Solution

Zeros exist at the points x = -1.8414 and x = 1.1462.

```
function p3

y = @(x) (e.^x) - x - 2;
% -1.8414
fzero(y, [-10, 0])
%  1.1462
fzero(y, [0, 10])

end
```

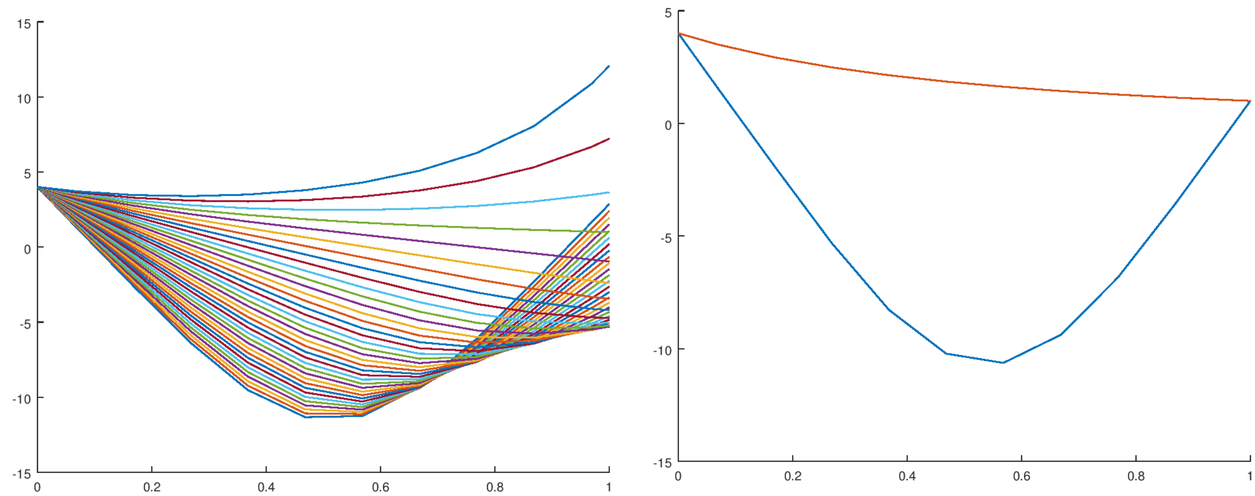
Problem 4.

Solve the boundary value problem

$$y''(t) = \frac{3}{2}y^2(t)$$
$$y(0) = 4$$
$$y(1) = 1$$

by a shooting method.

Solution



(a) s $\in [-40, -5]$ (b) Solutions

Using fzero to find zeros of the function phi(s) produces values of s = -35.859 and s = -8. Figure (a) shows the solutions for integer s values in the interval $[-40, -5]$, where figure (b) displays only the solutions for s = -35.859 and s = -8 that satisfy the IVP with y(0) = 4 and y(1) = 1. The code to produce these solutions may be found below.

```
function p4

clear all;
close all;
hold on;

for s = -40:-5
    [err, solx, soly] = phi(s);
    plot(solx, soly(:,1), 'DisplayName', 'ode45', 'LineWidth', 1);
end

% -35.859
zero0 = fzero(@phi, -40)
% -8
zero1 = fzero(@phi, -5)

[err, solx, soly] = phi(zero0);
plot(solx, soly(:,1), 'DisplayName', 'ode45', 'LineWidth', 1);
[err, solx, soly] = phi(zero1);
plot(solx, soly(:,1), 'DisplayName', 'ode45', 'LineWidth', 1);

end
```

```
function [err, solx, soly] = phi(s)

dy = @(t, y) [y(2), (3/2) * (y(1) .^ 2)];

[solx, soly] = ode45(dy, [0, 1], [4, s]);
est = interp1(solx, soly(:,1), 1, method='linear');
err = est - 1;

end
```

Problem 5.

(a) This is a stiff ODE. Solev the equations on the interval [0, 0.001], using a stiff solver such as ode15s. Plot the result.

(b) Answer the following questions.

- How many steps did the stiff solver take? How much computer time?
- How many steps does a non-stiff solver, such as ode113 take? How much computer time does it take?
- What is the clock period?

Solution

TODO