Complete asymptotic expansions of the Fermi–Dirac integrals F p ()=1/(p+1) 0 [p /(1+e)]d

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Citation: Journal of Mathematical Physics **42**, 1860 (2001); doi: 10.1063/1.1350634
View online: http://dx.doi.org/10.1063/1.1350634
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/42/4?ver=pdfcov
Published by the AIP Publishing

Complete asymptotic expansions of the Fermi–Dirac integrals $\mathcal{F}_p(\eta)=1/\Gamma(p+1)\int_0^\infty[\epsilon^p/(1+e^{\epsilon-\eta})]d\epsilon$

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(Received 23 October 2000; accepted for publication 2 January 2001)

The complete asymptotic expansions, that is to say expansions which include any exponentially small terms lying beyond all orders of the asymptotic power series, are calculated for the Fermi–Dirac integrals. We present two methods to accomplish this, the first in the complex plane utilizing Mellin transforms and Hankel's representation of the gamma function, and the second on the real line using the known asymptotic expansions of the confluent hypergeometric functions. The complete expansions of $\mathcal{F}_p(\eta)$ are then used to investigate the effect that these traditionally neglected exponentially small terms have on physical systems. It is shown that for a 2 dimensional nonrelativistic ideal Fermi gas, the subdominant exponentially small series becomes dominant.
[DOI: 10.1063/1.1350634]

I. INTRODUCTION

If one adheres to the conventional definition of the asymptotic series for a function given by Poincaré, all terms in the series are algebraic in the asymptotically small variable ϵ. This implies that transcendental exponentially small terms are not captured by the series in the limit $\epsilon\to 0$, and hence these transcendentally small terms have traditionally been neglected in asymptotics. These small terms are said to lie beyond all orders of the asymptotic expansion. By neglecting such terms in the asymptotic expansion of a given function, it is clear that the resultant series cannot give an exact representation of this function. It has been demonstrated however, most notably by Dingle[1] via his derivations of integral representations of the remainder terms of a wide class asymptotic series, that asymptotic series are capable of being precisely interpreted. Thus it became apparent in the late 1950s that the definition of Poincaré needed to be replaced. This led Dingle[1] to define the *complete asymptotic expansion* of a function $f(x)$ as an expansion constructed from asymptotic series, which formally exactly obeys—throughout a certain phase sector—all those relations satisfied by $f(x)$ which do not involve any finite numerical value of x. Dingle found that in practice a suitably rigorous analysis would yield, in a certain phase sector, the complete asymptotic expansion of a function, including any transcendentally small terms which may be present. Dingle's definition suggested that if methods were developed whereby the divergent sequence of late terms in an asymptotic expansion could be interpreted, then asymptotic expansions could become an exact representation of a function. The construction of such methods was pioneered by Dingle[1] with his theory of terminants, and has been further developed by Berry *et al.* in their work on *hyperasymptotics*.[2,3]

There has been an increasing amount of research undertaken in recent years which demonstrates the practical importance of obtaining complete asymptotic expansions as opposed to Poincaré expansions. For example, problems in crystal growth, viscous flow, quantum tunnelling, and optics[2,3] have demonstrated the physical manifestations of the terms lying beyond all orders in a complete asymptotic expansion; and work on ordinary differential equations and such problems as

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the Generalized Euler–Jacobi inversion formula[2] has reinforced their application to fundamental problems in classical analysis.

Our aim in this paper is to demonstrate that the transcendentally small terms appearing in a complete asymptotic expansion are capable of producing important physical effects, that can be manifest even when the expansion is truncated at an early order. In particular, we develop the complete asymptotic expansion of $\mathcal{F}_p(\eta)$, a function which is of fundamental importance in the statistical mechanics of Fermi systems. We then use this to demonstrate that exponentially small terms dominate the low temperature expansions of the thermodynamic functions of an ideal Fermi gas, when the number of spatial dimensions is even. We investigate the particular case of a two dimensional ideal Fermi gas in detail, and show that using our complete expansions we recover the exact result obtained by May[4] in which closed form solutions were obtained for the integrals occurring in the number equation, and internal energy.

As has been discussed by Dingle,[1,5] the commonly accepted Sommerfeld method of asymptotically expanding the Fermi–Dirac integrals (see, e.g., Refs. 6, 7), does not yield the complete asymptotic expansion and thus leads to erroneous conclusions regarding the exponentially small terms. For general p, the complete asymptotic expansion of $\mathcal{F}_p(\eta)$ consists of an asymptotic power series, as well as a subdominant exponentially small series which is traditionally neglected. In the particular case when p is an odd half integer, this exponentially small series vanishes identically, and so the usual assertion accompanying the Sommerfeld treatment that exponentially small terms are being neglected is, in this case, incorrect. When p is an integer, however, the asymptotic power series truncates into a finite sum, and the exponentially small terms then become dominant. This will be shown to be the cause of the aforementioned behavior of the even dimensional Fermi gas.

Although the complete asymptotic expansion of $\mathcal{F}_p(\eta)$ is known,[1,5] albeit not well known, we feel that our proofs being simultaneously both perspicuous, and rigorous and complete, elucidate the construction of this seemingly esoteric yet fundamentally important expansion. We present the derivations from two alternative angles. The first method is performed using contour integration, and highlights the mechanisms producing the various idiosyncrasies displayed by the expansion of $\mathcal{F}_p(\eta)$. We begin with the standard Mellin/inverse Mellin transform method, and use this to obtain a representation of $\mathcal{F}_p(\eta)$ along Hankel's contour. By paying careful attention to the various limits involved, we then show how the Hankel contour representation of $\mathcal{F}_p(\eta)$ is related to Hankel's representation of the gamma function. The second method makes use of standard hypergeometric theory, and the known asymptotic expansions of the confluent hypergeometric functions, and provides an expedient alternative construction.

II. THE COMPLETE ASYMPTOTIC EXPANSION OF $\mathcal{F}_p(\eta)$ USING MELLIN TRANSFORMS

We define $\mathcal{F}_p(\eta)$, following Dingle,[5] via

$$\mathcal{F}_p(\eta)=\frac{1}{\Gamma(p+1)}\int_0^\infty \frac{\epsilon^p}{1+e^{\epsilon-\eta}}d\epsilon. \tag{2.1}$$

We seek the complete asymptotic expansion for $\mathcal{F}_p(\eta)$ for the case of large positive real η, and $p>-1$.

To obtain a contour integral representation of $\mathcal{F}_p(\eta)$, we express it as the inverse Mellin transform of it's Mellin transform,[5,1] which results in

$$\mathcal{F}_p(\eta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi e^{\eta s}\,ds}{s^{p+1}\sin(\pi s)},\quad 0<c<1. \tag{2.2}$$

By changing the sign of s in (2.2) and closing the contour to the right, we arrive at

$$\mathcal{F}_p(\eta)=\frac{1}{2i}\int_{\infty}^{(0+)}\frac{e^{-\eta s}\,ds}{(-s)^{p+1}\sin(\pi s)}, \tag{2.3}$$

where the integration is performed over Hankel's contour, viz. the contour begins at infinity in the first quadrant, encircles the origin in the positive direction and proceeds to infinity in the fourth quadrant in such a manner as to avoid enclosing any of the poles on the negative real axis.[8]

To proceed further, we shrink the curve onto the positive real axis from both above and below, indenting the curve around the simple poles at $s=1, 2, 3, \dots$, and also about the origin which will either be a pole if p is an integer or both a pole and a branch point if p is a noninteger. In either case we can choose $-s=se^{-i\pi}$ on the top of the curve to ensure that the generally many valued function $(-s)^{-p-1}$ becomes definite. We choose the indentations of the curve around the simple poles, both above and below the real axis, to be semi-circles of radius ρ, centered at the poles. We denote the contour of the indentation at the simple pole $s=n$ above the real axis by $above_{\rho}(n)$, and that below the real axis by $below_{\rho}(n)$. The indentation around the origin is taken to be along the curve Ω_{δ}, where we define Ω_{δ} to be the circle of radius $\delta<1$ centered at the origin and traversed in the positive direction.[9]

The essence of the derivation that follows is to split up the $\int_{\infty}^{(0+)}$ integral so that we can deal with the simple poles, and the problem of the origin separately. Having dealt with the simple poles we will be left with integrals whose integrand is only nonanalytic at the origin, and we will then be in a position to identify these remaining pieces with Hankel's representation of the gamma function. To begin then, we split up (2.3) as follows:

$$\begin{aligned}\int_{\infty}^{(0+)}\frac{e^{-\eta s}\,ds}{2i(-s)^{p+1}\sin(\pi s)}=&\sum_{n=1}^{\infty}\int_{n+1-\rho}^{n+\rho}\frac{e^{-\eta s}\,ds}{2ie^{-i\pi(p+1)}s^{p+1}\sin(\pi s)}+\int_{1-\rho}^{\delta}\frac{e^{-\eta s}\,ds}{2ie^{-i\pi(p+1)}s^{p+1}\sin(\pi s)}\\&+\sum_{n=1}^{\infty}\int_{above_{\rho}(n)}\frac{e^{-\eta s}\,ds}{2ie^{-i\pi(p+1)}s^{p+1}\sin(\pi s)}\\&+\sum_{n=1}^{\infty}\int_{n+\rho}^{n+1-\rho}\frac{e^{-\eta s}\,ds}{2ie^{i\pi(p+1)}s^{p+1}\sin(\pi s)}+\int_{\delta}^{1-\rho}\frac{e^{-\eta s}\,ds}{2ie^{i\pi(p+1)}s^{p+1}\sin(\pi s)}\\&+\sum_{n=1}^{\infty}\int_{below_{\rho}(n)}\frac{e^{-\eta s}\,ds}{2ie^{i\pi(p+1)}s^{p+1}\sin(\pi s)}\\&+\int_{\Omega_{\delta}}\frac{e^{-\eta s}\,ds}{2i(-s)^{p+1}\sin(\pi s)}.\end{aligned} \tag{2.4}$$

The first and third lines of (2.4) are the contributions from along the real line from above and below, respectively.

We now take the limit $\rho\to 0$. The contribution of the indentations around the simple poles becomes

$$\frac{(e^{i\pi(p+1)}+e^{-i\pi(p+1)})}{2i}\sum_{l=1}^{\infty} i\pi\,\mathrm{Res}(e^{-\eta s}s^{-p-1}\mathrm{cosec}(\pi s)|s=l)=\cos(\pi p)\mathcal{F}_p(-\eta), \tag{2.5}$$

where $\mathrm{Res}(f(s)|s=l)$ signifies the residue of $f(s)$ at $s=l$, and where we have used

$$\mathcal{F}_p(-\eta)=\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu^{p+1}}e^{-\nu\eta}, \quad \eta>0. \tag{2.6}$$

For the contributions along the real axis we obtain

$$-\frac{[e^{i\pi(p+1)}-e^{-i\pi(p+1)}]}{2i}\lim_{\rho\to 0}\left[\sum_{l=1}^{\infty}\int_{l+\rho}^{l+1-\rho}\frac{e^{-\eta s}}{s^{p+1}\sin(\pi s)}ds+\int_{\delta}^{l-\rho}\frac{e^{-\eta s}}{s^{p+1}\sin(\pi s)}ds\right]$$
$$=\sin(\pi p)\mathcal{P}\int_{\delta}^{\infty}\frac{e^{-\eta s}ds}{s^{p+1}\sin(\pi s)}. \tag{2.7}$$

$\mathcal{P}$ signifies the Cauchy principal value of the integral.

Since $\delta<1$, we can substitute into the Ω_δ integral, the result[10]

$$\frac{\pi s}{\sin(\pi s)}=\sum_{\nu=0}^{\infty}2\tau_{2\nu}s^{2\nu},\quad |s|<1, \tag{2.8}$$

where

$$\tau_n\equiv\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu^n}=(1-2^{1-n})\zeta(n), \tag{2.9}$$

and $\zeta(n)$ is the Riemann zeta function.

Adding these pieces together we see that (2.3) can be written in the following more illuminating form:

$$\mathcal{F}_p(\eta)=\cos(\pi p)\mathcal{F}_p(-\eta)+\sin(\pi p)\mathcal{P}\int_{\delta}^{\infty}\frac{e^{-\eta s}ds}{s^{p+1}\sin(\pi s)}$$
$$+\frac{i}{2\pi}\sum_{\nu=0}^{\infty}\frac{2\tau_{2\nu}}{\eta^{2\nu-p-1}}\int_{\Omega_\delta}(-s)^{2\nu-p-2}e^{-s}\,ds. \tag{2.10}$$

It is to be noted that no asymptotic analysis has yet been performed. In order to obtain the complete asymptotic expansion of $\mathcal{F}_p(\eta)$ for arbitrary $p>-1$, we need to asymptotically expand the contribution from (2.7).

In the special case when p is an integer, it will be noticed that the contribution from (2.7) vanishes, and it should be obvious that this is merely a result of the fact that $(-s)^{-p-1}$ is in fact single valued in this case. Also, the Ω_δ integral then simply yields the residues at the origin for the finite number of terms in which the integrand is meromorphic, and vanishes for the remainder of the terms since their integrands are analytic. Hence for integer p the asymptotic expansions to be derived will agree precisely with the exact results. This observation is in agreement with Dingle's definition of a complete asymptotic expansion, which requires that the correct asymptotic expansion should agree with exact result for those special special values of p in which direct integration of $\mathcal{F}_p(\eta)$ is possible.

If p is a noninteger, then the contribution from (2.7) must be asymptotically expanded. However, nothing in the following argument relies on p being a noninteger, and it is equally true for integer p; $p\in Z$ just happens to be a particularly simple special case of the general result to be now derived.

Consider then

$$\mathcal{P}\int_{\delta}^{\infty}\frac{e^{-\eta s}ds}{s^{p+1}\sin(\pi s)}, \tag{2.11}$$

and note that $s^{-p-1}e^{-\eta s}\to 0$ as $s\to\infty$. The dominant contribution obviously comes from the neighborhood of $s=\delta<1$. We develop the desired asymptotic power series then, by substituting

(2.8) into (2.11), and integrating term by term. By making this substitution we are essentially ignoring the poles of cosec(πs) at $s=1, 2, 3, \ldots$, and thus the resulting integrals are no longer of the Cauchy principle value type. We thus obtain

$$\mathcal{F}_p(\eta)\sim\sum_{\nu=0}^{\infty}\frac{2\tau_{2\nu}}{\eta^{2\nu-p-1}}\frac{i}{2\pi}\left[\int_{\Omega_\delta}(-t)^{2\nu-p-2}e^{-t}dt-2i\sin(\pi p)\int_{\delta}^{\infty}(-t)^{2\nu-p-2}e^{-t}dt\right]$$
$$+\cos(\pi p)\mathcal{F}_p(-\eta). \tag{2.12}$$

The term in parentheses in the above power series will be recognized as the function $-2i\sin[(2\nu-p-1)\pi]\Gamma(2\nu-p-1)$,[11] viz. it is simply Hankel's expression for the gamma function. Inserting this result into (2.12), we arrive at

$$\mathcal{F}_p(\eta)\sim\sum_{\nu=0}^{\infty}2\tau_{2\nu}\frac{\sin[(2\nu-p-1)\pi]}{\pi}\Gamma(2\nu-p-1)\eta^{p+1-2\nu}+\cos(\pi p)\mathcal{F}_p(-\eta), \tag{2.13}$$

or upon utilizing the reflection formula for the gamma function,

$$\mathcal{F}_p(\eta)\sim\sum_{\nu=0}^{[(p+1)/2]}\frac{2\tau_{2\nu}}{\Gamma(p+2-2\nu)}\eta^{p+1-2\nu}+\frac{\sin(\pi p)}{\pi}\sum_{[(p+3)/2]}^{\infty}2\tau_{2\nu}\Gamma(2\nu-p-1)$$
$$\times\eta^{p+1-2\nu}+\cos(\pi p)\mathcal{F}_p(-\eta), \tag{2.14}$$

$[x]$ represents the integer part of the real number x.

Expressed in this form, the behavior of $\mathcal{F}_p(\eta)$ in the special cases p an integer and p an odd half integer become obvious. The beauty of this derivation lies in the fact that it was not necessary to treat the integer and noninteger cases separately, and that it makes manifest the mechanism accounting for the structural differences of the expansions in these two cases, namely the single valued or many valuedness of the integrand, respectively.

III. THE COMPLETE ASYMPTOTIC EXPANSION OF $\mathcal{F}_p(\eta)$ USING CONFLUENT HYPERGEOMETRIC FUNCTIONS

The complete asymptotic expansion of $\mathcal{F}_p(\eta)$ can be obtained in a more expedient and less complicated, although less enlightening fashion by simply using the known asymptotic expansions for the confluent hypergeometric functions. By splitting the range of integration in (2.1) into $[0,\eta]$ and $[\eta,\infty]$, and inserting the binomial expansions of $(1+e^{x-\eta})^{-1}$ we obtain

$$\Gamma(p+1)\mathcal{F}_p(\eta)=\sum_{k=0}^{\infty}(-1)^k\int_0^{\eta}e^{-k(\eta-x)}x^p dx+\sum_{k=1}^{\infty}(-1)^{k+1}\int_{\eta}^{\infty}e^{-k(x-\eta)}x^p dx. \tag{3.1}$$

We then change the variable of integration to $t\equiv\pm(\eta-x)/\eta$ in the first and second summands, respectively, thus obtaining

$$\Gamma(p+1)\mathcal{F}_p(\eta)=\sum_{k=0}^{\infty}(-1)^k\eta^{p+1}\int_0^1 e^{(-\eta k)t}t^{1-1}(1-t)^{p+2-1-1}dt$$
$$+\sum_{k=1}^{\infty}(-1)^{k+1}\eta^{p+1}\int_0^{\infty}e^{-(\eta k)t}t^{1-1}(1+t)^{p+2-1-1}dt. \tag{3.2}$$

The integrals in the first and second summands are then recognized as $[1/(p+1)]M(1,p+2,-\eta k)$ and $U(1,p+2,\eta k)$, respectively, where $M(a,b,z)\equiv{}_1F_1(a;b;z)\equiv\Phi(a,b,z)$ and

$U(a,b,z)\equiv z^{-a}\,{}_2F_0(a;1+a-b;-1/z)\equiv\Psi(a,b,z)$ are the two independent solutions of Kummer's equation discussed in Ref. 10. Hence $\mathcal{F}_p(\eta)$ can be written in terms of these confluent hypergeometric functions as

$$\mathcal{F}_p(\eta)=\frac{\eta^{p+1}}{\Gamma(p+2)}+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\eta^{p+1}}{\Gamma(p+1)}\left[U(1,p+2,\eta k)-\frac{1}{p+1}M(1,p+2,-\eta k)\right]. \tag{3.3}$$

To derive the complete asymptotic expansion for $\mathcal{F}_p(\eta)$ we then substitute into (3.3) the known asymptotic expansions for $M(1,p+2,-\eta k)$ and $U(1,p+2,\eta k)$,[10]

$$\frac{1}{p+1}M(1,p+2,-\eta k)\sim\sum_{n=0}^{\infty}\frac{(-p)_n}{k^{n+1}}\eta^{-n-1}-\frac{\Gamma(p+1)}{\eta^{p+1}}(-1)^{-p}\frac{e^{-\eta k}}{k^{p+1}}, \tag{3.4}$$

$$U(1,p+2,\eta k)\sim\sum_{n=0}^{\infty}(-1)^n\frac{(-p)_n}{k^{n+1}}\eta^{-n-1}, \tag{3.5}$$

where $(a)_n$ is Pochammer's symbol, defined by[10]

$$\begin{gathered}(a)_0=1,\\ (a)_n=a(a+1)(a+2)\cdots\ (a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)}.\end{gathered} \tag{3.6}$$

Interchanging the order of summation in the resultant double sum we thus obtain

$$\begin{aligned}\mathcal{F}_p(\eta)\sim{}&\frac{\eta^{p+1}}{\Gamma(p+2)}+\sum_{n=0}^{\infty}(-1)^n\frac{(-p)_n}{\Gamma(p+1)}\eta^{p-n}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{n+1}}\\ &-\sum_{n=0}^{\infty}\frac{(-p)_n}{\Gamma(p+1)}\eta^{p-n}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{n+1}}+(-1)^{-p}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{p+1}}e^{-\eta k}.\end{aligned} \tag{3.7}$$

Making use of (2.6) and (2.9) this becomes

$$\mathcal{F}_p(\eta)\sim(-1)^{-p}\mathcal{F}_p(-\eta)+\frac{\eta^{p+1}}{\Gamma(p+2)}+\sum_{n=1}^{\infty}2\tau_{2n}\left[\frac{-(-p)_{2n-1}}{\Gamma(p+1)}\right]\eta^{p+1-2n}. \tag{3.8}$$

Since $\mathcal{F}_p(\eta)$ and the above power series are real for real p and η, taking the real part of the above equation results in the correct exponential series in (2.14), $\cos(\pi p)\mathcal{F}_p(-\eta)$. We also note, using (3.6), that

$$\left[\frac{-(-p)_{2n-1}}{\Gamma(p+1)}\right]=\frac{1}{\Gamma(p+2-2n)}. \tag{3.9}$$

Hence we arrive at the following expression which is entirely equivalent to (2.13) and (2.14):

$$\mathcal{F}_p(\eta)\sim\cos(\pi p)\mathcal{F}_p(-\eta)+\sum_{\nu=0}^{\infty}\frac{2\tau_{2\nu}}{\Gamma(p+2-2\nu)}\eta^{p+1-2\nu}. \tag{3.10}$$

This method is certainly far more expedient than the previous method since all the genuine asymptotic analysis has already been performed for us in the tabulation of the complete asymptotic expansions of the functions U and M.

IV. LOW TEMPERATURE BEHAVIOR OF AN ARBITRARY DIMENSIONAL FERMI GAS

We proceed now to use our complete asymptotic expansion of $\mathcal{F}_p(\eta)$ to investigate the statistical mechanics of an ideal Fermi gas in d spatial dimensions. We will see that in even dimensions, the subdominant series in (3.10) produces very important physical effects in the expansions of the thermodynamic functions.

For an ideal nonrelativistic spin 1/2 Fermi gas at temperature T in d dimensions, the internal energy, U, and average number density, n, are expressed in terms of $\mathcal{F}_p$ via

$$n=\frac{2}{\lambda_T^d}\frac{1}{\Gamma\left(\frac{d}{2}+1\right)}\left(\frac{\epsilon_F}{kT}\right)^{d/2}=\frac{2}{\lambda_T^d}\mathcal{F}_{d/2-1}(\beta\mu), \tag{4.1}$$

$$\frac{U}{N\epsilon_{\mathrm{F}}}=\frac{d}{2}\Gamma(d/2+1)\left(\frac{kT}{\epsilon_{\mathrm{F}}}\right)^{d/2+1}\mathcal{F}_{d/2}(\beta\mu), \tag{4.2}$$

where ϵ_{F} is the Fermi energy, $\mu(T)$ is the chemical potential, k is Boltzmann's constant, $\lambda_T \equiv \sqrt{2\pi\hbar/mkT}$, and $\beta\equiv 1/kT$. The dependence of ϵ_{F} on n and d is stated implicitly in (4.1).

The convergent power series expansion of $\mathcal{F}_p(\eta)$, (2.6), can be used to examine the behavior of U and n for negative $\beta\mu$, i.e., in the classical region. We focus on the degenerate region in which $\beta\mu$ is large and positive. Using (2.14), we find in this case that n and U have the following expansions.

$$\begin{aligned}\frac{n\lambda_T^d}{2}\sim&\sum_{\nu=0}^{[d/4]}\frac{2\tau_{2\nu}}{\Gamma(d/2+1-2\nu)}(\beta\mu)^{d/2-2\nu}-\sin(d\pi/2)\sum_{\nu=[(d+4)/4]}^{\infty}2\tau_{2\nu}\frac{\Gamma(2\nu-d/2)}{\pi}\\&\times(\beta\mu)^{d/2-2\nu}-\cos(d\pi/2)\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu^{d/2}}e^{-\nu\beta\mu},\quad(\beta\mu)\to\infty\end{aligned} \tag{4.3}$$

$$\begin{aligned}\frac{U}{N\epsilon_{\mathrm{F}}}\left(\frac{\epsilon_{\mathrm{F}}}{kT}\right)^{d/2+1}\frac{1}{\frac{d}{2}\Gamma(d/2+1)}\sim&\sum_{\nu=0}^{[(d+2)/4]}\frac{2\tau_{2\nu}}{\Gamma(d/2+2-2\nu)}(\beta\mu)^{d/2+1-2\nu}+\sin(d\pi/2)\\&\times\sum_{\nu=[(d+6)/4]}^{\infty}2\tau_{2\nu}\frac{\Gamma(2\nu-d/2-1)}{\pi}(\beta\mu)^{d/2+1-2\nu}\\&+\cos(d\pi/2)\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu^{d/2+1}}e^{-\nu\beta\mu},\quad(\beta\mu)\to\infty.\end{aligned} \tag{4.4}$$

From these expansions we see clearly that n and U behave quite differently depending on whether the number of spatial dimensions is odd or even. For an odd number of dimensions the exponentially small series vanishes and we simply obtain a conventional asymptotic power series. For an even number of dimensions, however, the asymptotic power series truncates, and the exponentially small series thus dominates.

It is instructive to compare the expansions of n and U for the physically important cases $d=2$ and 3. The number density n goes like

$$d=2,\qquad \frac{n\lambda_T^2}{2}\sim\beta\mu+\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu}e^{-\nu\beta\mu}, \tag{4.5}$$

$$d=3,\qquad \frac{n\lambda_T^3}{2}\sim\sum_{\nu=0}^{\infty}\frac{2\tau_{2\nu}}{\Gamma(5/2-2\nu)}(\beta\mu)^{3/2-2\nu}. \tag{4.6}$$

For the energy we obtain

$$d=2, \qquad \frac{U}{N\epsilon_{\mathrm{F}}}\left(\frac{\epsilon_{\mathrm{F}}}{kT}\right)^{2} \sim \frac{(\beta\mu)^{2}}{2}+\frac{\pi^{2}}{6}-\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu^{2}}e^{-\nu\beta\mu}, \tag{4.7}$$

$$d=3, \qquad \frac{U}{N\epsilon_{\mathrm{F}}}\left(\frac{\epsilon_{\mathrm{F}}}{kT}\right)^{5/2} \sim \frac{8}{9\sqrt{\pi}}\sum_{\nu=0}^{\infty}\frac{2\tau_{2\nu}}{\Gamma(7/2-2\nu)}(\beta\mu)^{5/2-2\nu}. \tag{4.8}$$

The $d=3$ result is simply the standard result first obtained by Sommerfeld,[7] but it is interesting to note that in this case the subdominant exponential series vanishes identically. The usual statement that this result neglects terms $O(e^{-\beta\mu})$, e.g., Ref. 6, is therefore incorrect. When $d=2$ it is the power series in (4.3) and (4.4) which vanish identically, leaving only a finite sum and the exponentially small series. Thus the subdominant exponentially small series which should have been neglected in strict adherence to Poincaré's definition has become dominant. We shall now investigate the physical effects of this dominance of the exponentially small series in the two dimensional Fermi gas in greater detail.

V. THE TWO DIMENSIONAL FERMI GAS

In 2 spatial dimensions we have seen that only the first term in the power series of the large $\beta\mu$ expansion of n is nonzero, and thus that the exponentially small series is dominant.

To obtain suitable asymptotic expansions in the ultra degenerate limit, $kT/\epsilon_F \ll 1$, for U and the heat capacity at constant volume, C_v, we invert (4.5) for $\beta\mu$ and substitute this into (4.7).

Inversion of (4.5) leads to an asymptotic expansion for the chemical potential,

$$\mu(T)=\epsilon_{\mathrm{F}}\left[1-\frac{e^{-\beta\epsilon_{\mathrm{F}}}}{\beta\epsilon_{\mathrm{F}}}-\frac{e^{-2\beta\epsilon_{\mathrm{F}}}}{2\beta\epsilon_{\mathrm{F}}}+O\left(\frac{e^{-3\beta\epsilon_{\mathrm{F}}}}{\beta\epsilon_{\mathrm{F}}}\right)\right]. \tag{5.1}$$

The first correction term to the chemical potential away from the Fermi energy is seen to be exponentially small in β, so that even as the system moves away from absolute zero the chemical potential stays essentially fixed at the Fermi energy. This demonstrates that this system is loath to move away from a perfect Fermi sphere configuration as T increases.

Substituting (5.1) into (4.7) we obtain

$$\begin{aligned} U=\frac{N\epsilon_{\mathrm{F}}}{2}\Bigg[&1+\frac{\pi^{2}}{3}\left(\frac{kT}{\epsilon_{\mathrm{F}}}\right)^{2}-2\left(\frac{kT}{\epsilon_{\mathrm{F}}}+\left[\frac{kT}{\epsilon_{\mathrm{F}}}\right]^{2}\right)e^{-\epsilon_{\mathrm{F}}/kT} \\ &-\left(\frac{kT}{\epsilon_{\mathrm{F}}}+\frac{1}{2}\left[\frac{kT}{\epsilon_{\mathrm{F}}}\right]^{2}\right)e^{-2(\epsilon_{\mathrm{F}}/kT)}\Bigg]+O\left(\frac{kT}{\epsilon_{F}}e^{-3\epsilon_{F}/kT}\right). \end{aligned} \tag{5.2}$$

Note that all the algebraic factors multiplying the exponential terms are in fact polynomials; they have not been truncated. Differentiating (5.2) we obtain the heat capacity at constant volume C_v,

$$\frac{C_v}{Nk}=\frac{\pi^{2}}{3}\frac{kT}{\epsilon_{\mathrm{F}}}-\left(\frac{\epsilon_{F}}{kT}+2+2\frac{kT}{\epsilon_{\mathrm{F}}}\right)e^{-\epsilon_{\mathrm{F}}/kT}-\left(\frac{\epsilon_{\mathrm{F}}}{kT}+1+\frac{1}{2}\frac{kT}{\epsilon_{\mathrm{F}}}\right)e^{-2(\epsilon_{\mathrm{F}}/kT)}+O\left(\frac{\epsilon_{\mathrm{F}}}{kT}e^{-3(\epsilon_{\mathrm{F}}/kT)}\right). \tag{5.3}$$

We see that to first order C_v vanishes linearly as $T\to 0$, as one would expect for a Fermi gas, just as in the 3 dimensional case. The interesting thing to note however is that unlike the 3 dimensional case, the small T behavior of C_v in 2 dimensions contains transcendental rather than algebraic correction terms. This lack of algebraic corrections to C_v is testament to the physical effects that the transcendentally small terms in a complete asymptotic expansion can have.

The results obtained here using the complete asymptotic expansion of $\mathcal{F}_p(\eta)$ are also seen to be in agreement with the closed form solutions obtained by May.[4] The reason for this is rather obvious from the derivation in Sec. II, since it was pointed out there that in the case of integer p, our result for the asymptotic expansion of $\mathcal{F}_p(\eta)$ is actually the exact solution of the integral. In effect the complete asymptotic expansion thus simultaneously performs the Sommerfeld treatment, as well as an exact treatment for the case of integer p. By using the complete asymptotic expansions we are thus able to recover Sommerfeld's asymptotic result for $d=3$, and May's exact result for $d=2$. This provides a concrete physical example of the usefulness of obtaining the complete asymptotic expansions for a given function, as opposed to the traditional algebraic series.

ACKNOWLEDGMENT

The authors would like to thank the Australian Research Council for funding this work.

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