Asymptotic Expansions: Their Derivation and Interpretation

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Prologue

Throughout this book, the designation "asymptotic series" will be reserved for those series in which for large values of the variable at all phases the terms first progressively decrease in magnitude, then reach a minimum and thereafter increase. The theory of such ultimately divergent series will be called the discipline of "asymptotics". (Frequently, in both mathematics and physics, these phrases have been allowed wider meanings; see footnote to Section 5 of Chapter I). Some examples of such series were discovered in the early eighteenth century by James Stirling, Léonard Euler and Colin Maclaurin, but for over a hundred years were rarely regarded seriously by pure mathematicians. The pure mathematicians of the early nineteenth century, and above all Augustin-Louis Cauchy and Neils Abel, placed too great a stress on convergence to trust such runaway expansions. However, notwithstanding the continued absence of a thorough general investigation, a number of important asymptotic expansions were soon being found. Outstanding discoveries of this period were Pierre Laplace's (1812) evaluation of integrals by expanding the integrand about its limits or stationary points, Joseph Liouville and George Green's solutions to second-order linear differential equations (both 1837), and the observation made by Sir George Stokes (1864) that "constant" multiplying factors in asymptotic expansions can jump discontinuously as the phase of the variable is changed.

Two important general investigations were published in 1886. Henri Poincaré proposed a definition, according to which $\Sigma\, a_r/x^r$ is taken to be the asymptotic power series for a function $f(x)$ if

$$\lim_{|x|\to\infty} x^n\left\{f(x) - \sum_0^n a_r/x^r\right\} \to 0 \quad \text{for all zero and positive } n.$$

In his paper, Thomas Stieltjes established two points which had been observed empirically much earlier: namely that the error resulting from truncating an asymptotic series is of the same order as the last retained term; and that for an alternating series—one in which successive terms alternate in sign—a more exact value results from terminating the series with only half the least term. In the same paper, Stieltjes estimated the remainder term $f(x) - \sum_0^n a_r/x^r$ for several functions defined by integral representations. Adopting now logical rather than historical order, we add here that in 1952 J. C. P. Miller demonstrated how the remainder term can be estimated for a function defined by a differential equation, supplemented where necessary by a difference equation. Thus in their estimation of a remainder term, Stieltjes and Miller both had resort to information supplementary to that contained within the asymptotic expansion itself.

J. R. Airey showed in 1937 how remainder terms for a simple alternating asymptotic power series could be estimated if the general term in the series were known, without resort to supplementary information. In a group of papers published in 1958–9 the present author derived and evaluated integral representations for the remainder terms appropriate to a large number of important asymptotic power series, both alternating and single-sign, likewise just from the general term in each series. These demonstrations that asymptotic power series can be precisely interpreted, despite their ultimate divergence, implied that Poincaré's prescription had now either to be supplemented or replaced. For since

$$\lim_{x\to\infty} (x^n \times \text{any exponentially decreasing function of } x) \to 0 \quad \text{for all } n,$$

literal adherence to the prescription permits abandonment in an asymptotic expansion of any portions which are multiplied by an exponentially decreasing function of the variable. If such terms were actually discarded, interpretation of the abridged asymptotic expansion could not of course reproduce the original function exactly. This theoretic failure in uniqueness was solely doctrinarian, the result of an over-permissive prescription; in actual practice, a sufficiently detailed analysis for a particular function in a phase sector will yield its "complete asymptotic expansion" (Dingle 1962, Olver 1964) including any sets of exponentially small terms which may be present. But, because of past history, extant derivations of asymptotic expansions must always be scrutinized to make sure such terms have not been neglected or rejected in literal adherence to Poincaré's prescription, and to check the phase range over which each expansion is entire.

Thus by 1959 there was more than a glimpse of an exciting new prospect for the future rôle of asymptotic expansions in analysis and computation. If ways of deriving *all* such expansions—not just, as hitherto, simple asymptotic power series in which the general term is explicitly given—could be extended so as to yield not only the usual exact values for the first few terms but also an accurate expression for the general late term† enabling the divergent part of the expansion to be interpreted, asymptotics would break

† A few isolated examples were already known, outstanding amongst these being Meissel's estimate in 1891 of the general term in the expansion of a Bessel function with equally large order and argument.

free from its earlier drawbacks of vagueness and concomitant severe limitation in accuracy and range of applicability, and be elevated to a discipline eliciting precise answers. As a theoretical physicist I was strongly attracted by this prospect, for two distinct reasons: first, and the more obviously, on account of the anticipated greater ease of handling accurately the large variety of mathematical functions needed throughout my subject; and secondly, because it was becoming increasingly evident to me that there is a definite link between the asymptotic approach in mathematics and theoreticians' criteria for suitable methods to tackle physical problems—in other words, it is no mere accident that the majority of expansions encountered in physics turn out on close examination to be asymptotic rather than convergent. To the prospect of increased power in direct mathematical calculation and analysis there was thereby added the likelihood of being able to enrich in range and accuracy physicists' staple procedures such as phase-integral and perturbation theories. Plainly, for example, only a more highly developed theory of asymptotics could decisively clear up the two crucial challenges to those current perturbation techniques which in effect rearrange terms such that the first few orders form a decreasing sequence, without actually guaranteeing like improvement of later orders: (i) Has the sequence been successfully arranged as a legitimate expansion, or is the initial semblance of appropinquation fortuitous? (ii) Bearing in mind that a change in ordering in an infinite series—even a convergent one—can alter its sum†, what evidence is there for the perturbation sequence pointing to the *right* sum? Questions on the convergence and summability of perturbation expansions are indeed continually being raised in the literature (e.g. Kato, 1949, 1966; Simon, 1970; Gunson and Ng, 1972); and it is especially interesting to note that the third reference quoted is based on Borel summation, the starting point likewise of my interpretative theory of asymptotic expansions (Dingle, 1958; Chapters XXI–XXVI of this book).

The present book, the first of two volumes summarizing my researches according to this ambitious programme, is in large measure restricted to the classic problems of asymptotics, namely the investigation of asymptotic expansions of various types—power, large-order, transitional and uniform—which can be derived from convergent series, integral representations and second-order linear differential equations. In the second volume the theory will be extended to homogeneous and inhomogeneous differential equations with non-linear variation of the "wave-number" near a turning point, then to the broad field of eigenvalue problems—to the zeros of functions, eigenvalues dictated by boundary conditions, periodic functions such as solutions to the Mathieu equation, and on to perturbation

† e.g. Whittaker and Watson, 1927, Section 2.4. There is a misprint in their example. The correct equation for S is $S_{2n} = \sigma_{2n} - \sigma_n$, as is most easily seen by subtracting S_{2n} from σ_{2n}.

expansions and newly-opened opportunities for physical prediction. Extension will also be effected to difference equations.

The four main species of task confronting us in this first volume may be outlined as follows:

(1) To elucidate the origin and nature of asymptotic expansions; and through such understanding to construct definitions which avoid the ambiguities and resulting inexactitudes of Poincaré's specification.

(2) To formulate methods of deriving asymptotic expansions from convergent series, integral representations and second-order linear differential equations—homogeneous and inhomogeneous—in such a way that exponentially small parts do not get lost, and that a substantial number of terms in each component series can be found with ease. For until comparatively recently *any* way of obtaining the dominant term or two appeared adequate, there being little sense in continuing to higher terms expansions believed to be incorrigibly inaccurate.

Most effort will have to be expended on the Whittaker functions for moderate and large orders. These are perhaps the most important and versatile of all the classic higher transcendental functions, but their asymptotic behaviour has hitherto been difficult to elicit accurately, and—despite valiant efforts by earlier researchers—what published results there are tend to be unreliable, mainly through imprecise identifications.

(3) To extend these methods to provide an expression for the general late term in an asymptotic series. In the usual outcome, the general late term is itself first found as a complete asymptotic series.

(4) To develop a systematic theory for interpreting asymptotic expansions beyond their least term, resting on the assurance from (1) that these series of increasing terms are nonetheless symbolically exact representations.

Arising out of these investigations, I have thought it desirable to introduce three new descriptive names. The general late term in an asymptotic series is found to depend critically on one quantity, a certain measure of the distance to the next nearest stationary point in the integral representation for the function, or to the nearest turning point in the differential equation it satisfies: this measure will be called "the singulant"; once it is known, the general late term can usually be written down from inspection of the first few terms in one of the main asymptotic expansions. As is well known, phase-integral ("W.K.B.") solutions to a differential equation $y'' - Xy = 0$ depend strongly on how $X(x)$ behaves close to "a turning point" x_0 at which $X(x_0) = 0$; somewhat analogously, asymptotic solutions to $y'' + fy' + gy = 0$ where y'' is usually insignificant depend strongly on how

$f(x)$ behaves close to a point x_0 at which $f(x_0) = 0$; this will be called "an extinction point". In his pioneer work published in 1937, Airey introduced the name "converging factor" for the function which, multiplying a term in an asymptotic series, would correctly terminate that series. At the time his choice was a reasonably good one; unfortunately this and still more the slight variant "convergence factor" have since that time far more commonly been applied to largely arbitrary multipliers introduced into integrals to enforce convergence at awkward limits. I propose that Airey's phrase be replaced by the shorter and distinct "terminant".

To restrict the volume to readable proportions, exposition will be heuristic and descriptive rather than rigorously doctrinaire. At the present stage of development, the paramount aims should surely be to explain lines of argument, to derive new results, and to attempt to remove prevalent misunderstandings, of which asymptotics has throughout its history attracted far more than its fair share. By comparison, usefully rigorous exposition demands strict uniformity throughout in degree of rigour, pertinence to development, and of course absolute freedom from fallacy through error or omission. Fulfilment of these taxing demands can seldom keep pace with practical development. Indeed the student of asymptotics must in fairness be warned that close examination of attempted rigour in this difficult field not infrequently discloses lapses of varying severity. To mention one from each category above: a rigorous account of steepest descents for a single saddle-point, followed without serious discussion by a tacit assumption that contributions from saddle-points are strictly additive (untrue in general); meticulous derivation of an upper bound which is not pertinent to theoretical or practical development, at worst one which is not even of the same structure as the actual error; and failure to recognise that an unequivocal definition of an asymptotic expansion must be capable of *fixing* a given set of exponentially small terms, not just *permitting* those found by some procedure. The construction of rigorous proofs in asymptotics is a challenging field for the future, one which will need to start from the view-point of derivation and interpretation as two partly symbolic, but intrinsically exact, complementary processes. In this setting, most extant proofs are essentially calculations of error bounds relating to truncation, the roughest attempt at interpretation; the current position is therefore largely one of rigorously proven results for approximate asymptotics, and formal heuristically-derived results for precision asymptotics.

As a further curb to length, exposition will be single-mindedly directed towards those derivations and expansions found most suited to subsequent interpretation. Related investigations will just be indicated in the notes, exercises or references concluding each chapter. The book is a research-orientated monograph on the inter-relation between the derivation and interpretation of asymptotic expansions, not an attempt at a treatise or equitable review of the whole subject of asymptotics.

At the outset I wish to express my deep gratitude to those who have encouraged and helped me in these investigations: above all Professor F. W. J. Olver and Dr. J. C. P. Miller for their readiness to discuss issues in asymptotics old and new, and the former for sending me advance copies of some chapters of his forthcoming book "Asymptotics and Special Functions"; and those of my research students who have been associated at one period or another with my work in this field, especially Dr. Harald Müller (expansions of Mathieu and other periodic wave functions and their eigenvalues), Dr. Siebe Jorna (expansions from differential equations) and Dr. S. Malaviya (expansions from integral representations). I need hardly stress my manifest debt to earlier texts on asymptotics, particularly those of van der Corput (1954–5), Erdélyi (1956), Jeffreys (1962), Copson (1965) and Wasow (1965), since in this research-orientated book I have felt it unnecessary to repeat basic material and theorems already cogently expounded in these texts. Incidentally, the book does nevertheless include comprehensive discussions on a number of fundamental issues, all too often lightly skimmed over, whose study deepens our understanding of the asymptotic form, such as the origin and nature of the divergence of asymptotic series, how, why and where Stokes discontinuities occur, the relation between their onset and the behaviour of dominant and recessive series included in a complete asymptotic expansion, and continuation rules through consecutive phase sectors.

A substantial part of this work was written up in 1968–9 while I was on study leave at the Departments of Physics of the Universities of Alberta, California (La Jolla) and Western Australia. The hospitality of their staffs did much to transform this effort from a chore into an enjoyment. Finally I should like to thank my wife for sympathetic understanding of the ups and downs of this protracted quest, Dr. Neil C. McGill for numerous clarifications in exposition and generous help in proof reading, Linda MacLean and Cynthia Williams for expert typing of the intricate manuscript, and the publishers for their long-standing patience in awaiting the text.

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Chapter I

Asymptotics—A Behavioural Survey

1. Origin and Nature of Asymptotic Expansions

To help fix ideas, let us start by examining types of expansion which may be developed for the well-known error function

$$\phi(x) = 2\pi^{-\frac{1}{2}} \int_0^x e^{-u^2}\, du. \tag{1}$$

This function is important in its own right, and of especial interest in asymptotics through having provided one of the earliest examples historically of a Stokes discontinuity (Stokes, 1864).

Expansion of the exponential as a power series, followed by term by term integration, leads to the absolutely convergent series

$$\phi(x) = \frac{2x}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{(-x^2)^s}{s!(2s+1)}. \tag{2}$$

Though theoretically exact for all magnitudes and phases of the variable, such a convergent series can prove dismally inconvenient except for small values. For instance, in the series (2), individual terms do not begin to decrease until $s \sim |x^2|$, and their sum does not even begin to approximate the function well until about three times as many terms have been assembled. More seriously, for large $|x|$ the final sum is far smaller than the largest individual terms, which therefore have to be calculated to many extra significant figures. In more advanced examples than (2), the presence within the summation of a factor which is not so simple—e.g. a zeta-function or worse—can render this a daunting task.

Fortunately, the alternative "asymptotic" approach produces a series in which, by contrast, ease of calculation to a prescribed accuracy increases with $|x|$. We shall first deal with the phase sector $|\text{ph}\, x| < \frac{1}{2}\pi$.

The observation $2\pi^{-\frac{1}{2}}\int_0^\infty e^{-u^2}\,du = 1$ enables us to write

$$\phi(x) = 1 - 2\pi^{-\frac{1}{2}}\int_x^\infty e^{-u^2}\,du\,. \tag{3}$$

In the new integral, e^{-u^2} is of significant magnitude only near the lower limit $u = x$, and can therefore be expanded about this point. It is convenient to choose as expansion parameter the variable $f = u^2 - x^2$, whereupon

$$\phi(x) = 1 - \frac{e^{-x^2}}{x\sqrt{\pi}}\int_0^\infty e^{-f}\left(1 + \frac{f}{x^2}\right)^{-\frac{1}{2}} df. \tag{4}$$

Without pausing at this stage to examine the validity of ensuing steps, the expansion about the point $f = 0$ required for insertion is

$$\left(1 + \frac{f}{x^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(r-\frac{1}{2})!}{r!}\left(-\frac{f}{x^2}\right)^r, \tag{5}$$

so

$$\phi(x) = 1 - \frac{e^{-x^2}}{x\pi}\sum_0^\infty\frac{(r-\frac{1}{2})!}{r!\,(-x^2)^r}\int_0^\infty e^{-f}f^r\,df$$

$$= 1 - \frac{e^{-x^2}}{x\pi}\sum_0^\infty\frac{(r-\frac{1}{2})!}{(-x^2)^r}, \qquad |\mathrm{ph}\,x| < \tfrac{1}{2}\pi\,. \tag{6}$$

As will become clear when we examine the general question of definition in Section 5, it is advisable at the outset to make a careful distinction in terminology between:

(i) the right-hand side of the equation (6), "the asymptotic expansion of $\phi(x)$ in this phase range",

(ii) the second contribution, "a component asymptotic series", and

(iii) the summation itself, "an asymptotic power series".

This is also semantically strict, since in such a context "expansion" refers to an algebraic dilatation however composed, whereas the word "series" means more narrowly "an ordered sequence of systematically constructed terms".

The terms in the asymptotic power series $\Sigma(r-\frac{1}{2})!/(-x^2)^r$ behave in a radically different way from those in the convergent series $\Sigma(-x^2)^s/s!(2s+1)$. For moderate or large $|x|$, the terms in the former first progressively decrease in magnitude, then reach a minimum around $r \sim |x^2|$ and thereafter increase; while those in the latter first increase, reach a maximum around $s \sim |x^2|$ and thereafter decrease. Because of the ultimate progressive increase in magnitude of its terms, an asymptotic power series is divergent. Nevertheless, even if only crudely broken off at its least term (thereby retaining only the first few terms), it produces remarkably accurate results, especially for large values of the variable.

Why does such a series derived from the asymptotic approach end by diverging? To answer this, let us examine the steps in the foregoing derivation more closely. The binomial theorem for expanding $(1 + f/x^2)^\alpha$ is valid only if *either* the expansion terminates (α a positive integer) *or* $|f/x^2| < 1$. The former is not relevant to $(1 + f/x^2)^{-\frac{1}{2}}$ in (5), and yet we see that in the next step (6) we did suppose the binomial expansion to hold not only for f from 0 to x^2, but further from x^2 to ∞. The first terms in (6) yield highly accurate values for $\phi(x)$ when x is large because the exponential factor e^{-f} makes the integral negligible long before f reaches x^2 and the questionable region beyond. But however large x may be, sufficiently late terms diverge because of the extension of the binomial expansion beyond its circle of convergence.

More generally, the asymptotic expansion for a function $\int e^{-F(u)}G(u)\,du$, where G is slowly varying, is ascertained by first separating out the ranges of integration, U say, through which e^{-F} decreases monotonically right down to zero (equation (3) in our example), then within these ranges changing the variable of integration to F, thus

$$\int_U e^{-F}\,G\,du = \int_{F_0}^{\infty} e^{-F}\,[G/(dF/du)]\,dF$$

(cf. (4)), expanding in each $G/(dF/du)$ as a Taylor series (cf. (5)), and integrating term by term (cf. (6)). Each resultant asymptotic series is ultimately divergent because in its progress from F_0 to ∞ the variable F reaches and then exceeds the radius of convergence in the F-plane of this Taylor series for $G/(dF/du)$.

Next, we examine the consequences of outstepping the circle of convergence. No error of magnitude or phase has been incurred; when $|f/x^2| \geqslant 1$ the binomial expansion (5) retains perfect precision of meaning, namely that the series is to be summed [to $(1 + f/x^2)^{-\frac{1}{2}}$] in exactly the same way as if it had lain within the circle of convergence. The ultimate convergence failure in an asymptotic power series thus has its origin in a solely-symbolic mechanism of continuation, not involving any numerical inexactitude. Moreover the technical misdemeanour in continuation can be exactly atoned by applying a reverse process of symbolic continuation when interpreting late terms. This conclusion is in stark contrast with the long accepted *non-sequitur* according to which such an expansion must contain an inherent vagueness and inaccuracy because its late terms are not comprehensible as they stand†.

By a theorem of Darboux (1878), late terms in a Taylor series originate from the singularity in the function expanded which lies closest to the origin of expansion (Chapter VII, Section 2). For example, late terms in (5) originate from the branch point of $(1+f/x^2)^{-\frac{1}{2}}$ located at $f=-x^2$. More generally, if α is a positive integer $(1+f/x^2)^\alpha$ possesses no singularities and its expansion correspondingly terminates and so has no late terms; whereas if α is a negative integer so the function has a pole at $f=-x^2$, or if α is fractional so it has a branch point there, the expansion does not terminate and its late terms are dictated by the singularity. According to the binomial theorem,

$$\left(1+\frac{f}{x^2}\right)^\alpha = \frac{1}{(-\alpha-1)!}\sum_0^\infty \frac{(r-\alpha-1)!}{r!}\left(-\frac{f}{x^2}\right)^r, \tag{7}$$

showing the late terms to be alike whether the singularity consists of a pole or a branch point. By the Darboux theorem, late terms in a Taylor series for some more complicated function of f will also be of similar form, since they depend only on the behaviour of that function in the immediate neighbourhood of its singularity closest to the origin of expansion at $f=0$. Reference to the derivation of (6) then leads to the expectation that, barring pathological cases, sufficiently late terms $r \gg 1$ of *any* asymptotic power series will transpire to be expressible in a standard limiting form $(r+\text{constant})!/(\text{variable})^r$, the accuracy of this limiting representation increasing with r. (The full representation for finite r will consist of a decreasing sequence of like contributions). This conclusion, which will be verified in varying contexts throughout our investigation, is critically important in two ways: first, because it provides a valuable lead on how asymptotic power series and expansions containing them might best be defined; and second, because it shows that substantially a single theory of interpretation will apply equally to late terms of all such asymptotic series (Chapters XXI onwards).

† There are indications that some mathematicians, active in the field before the alleged vagueness got written into the theory by Poincaré's prescription, were unsure of this inference. Especially interesting is the reservation in parenthesis in the following extract from Stokes' own account of the Stokes discontinuity: "A semiconvergent series (considered numerically, and apart from its analytic form) defines a function only subject to a certain amount of vagueness which is so much the smaller as the modulus of the variable is larger". Stokes may well have realised that while the sum of an asymptotic series can be determined only approximately from the numerical values of its terms, this does not exclude the possibility that precise information might be extractable from their analytic form.

2. Stokes Discontinuities, and Continuation Rules for Asymptotic Expansions

The foregoing derivation of the asymptotic expansion for $\phi(x)$ was based on the understanding that $|\mathrm{ph}\, x| < \frac{1}{2}\pi$. Next let us suppose x to be a positive imaginary, $x = iy$ where y is real and positive. Correspondingly we set $u = iv$, so

$$\phi(iy) = 2i\,\pi^{-\frac{1}{2}} \int_0^y e^{v^2}\,dv\,. \tag{8}$$

The integrand is large only close to the upper limit $v = y$, and can therefore be expanded about this point. It is convenient to choose as expansion parameter the variable $f = y^2 - v^2$. In this notation

$$\phi(iy) = \frac{i\,e^{y^2}}{y\,\sqrt{\pi}} \int_0^{y^2} e^{-f}\left(1 - \frac{f}{y^2}\right)^{-\frac{1}{2}} df\,. \tag{9}$$

Since the integrand is real throughout the specified range of integration 0 to y^2, but would be a pure imaginary over the range y^2 to ∞, this can be expressed as

$$\begin{aligned}\phi(iy) &= \frac{i\,e^{y^2}}{y\sqrt{\pi}}\,\Re \int_0^{\infty} e^{-f}\left(1 - \frac{f}{y^2}\right)^{-\frac{1}{2}} df = \frac{i\,e^{y^2}}{y\pi}\,\Re \sum_0^{\infty} \frac{(r-\frac{1}{2})!}{r!\,y^{2r}} \int_0^{\infty} e^{-f} f^r df \\ &= \frac{i\,e^{y^2}}{y\pi} \sum_0^{\infty} \frac{(r-\frac{1}{2})!}{y^{2r}}\,.\end{aligned} \tag{10}$$

Note that the injunction to retain only the real part was in the event no longer required after substituting the binomial expansion; as one of its limitations, this expansion does not produce the expected imaginary terms when $f > y^2$, i.e. outwith its circle of convergence.

A virtually identical argument covers the case where x is a negative imaginary. In both cases the result is

$$\phi(x) = -\frac{e^{-x^2}}{x\pi} \sum_0^{\infty} \frac{(r-\frac{1}{2})!}{(-x^2)^r}\,, \qquad |\mathrm{ph}\, x| = \tfrac{1}{2}\pi\,, \tag{11}$$

which differs from (6) by absence of the unit term. Indeed it is obvious both from the original definition (1) and its absolutely convergent series (2) that $\phi(x)$ is purely imaginary when x is imaginary, so it would make nonsense at these phases for the real unit term to be included in the asymptotic expansion. The incipient absurdity is obviated by the abrupt disappearance of the unit contribution as $|\mathrm{ph}\, x|$ reaches $\frac{1}{2}\pi$, a type of discontinuity discovered by Stokes (1864)†. More generally, at a certain phase drawn in the complex plane as a "Stokes ray", an "associated function" appears, disappears or changes its numerical multiplier. This associated function is usually most conveniently expressed in series form; in the great majority of cases this "associated series" is asymptotic, but there is no governing rule and a few terminate as in the present example, or are conditionally convergent. It cannot be too strongly emphasized that such Stokes discontinuities are only in the *form* of the asymptotic expansion, not in its summed value; indeed, as we shall see in Section 3, the discontinuities in form are forced by the necessity of preserving continuity in the function represented.

Returning to our example, the discontinuity is completed as $|\mathrm{ph}\, x|$ leaves $\frac{1}{2}\pi$ in proceeding towards numerically larger phases. To see this we now deal with the remaining phase sectors $\frac{1}{2}\pi < |\mathrm{ph}\, x| \leqslant \pi$. Reversing the sign of u in the defining integral (1),

$$\phi(x) = -2\pi^{-\frac{1}{2}} \int_0^{(-x)} e^{-u^2}\, du = -\phi(-x), \tag{12}$$

a "continuation formula". In particular, (6) transforms to

$$\phi(x) = -1 - \frac{e^{-x^2}}{x\pi} \sum_0^\infty \frac{(r-\frac{1}{2})!}{(-x^2)^r}, \qquad \tfrac{1}{2}\pi < |\mathrm{ph}\, x| < \tfrac{3}{2}\pi. \tag{13}$$

Rules

The foregoing derivations suggest a set of rules for locating Stokes rays and continuing asymptotic expansions across them. These rules will later (Chapter XXI, Section 6) be verified from our general interpretative theory of asymptotic expansions (Dingle 1958) to be developed in chapters XXI onwards. [Credit for emphasizing that Stokes discontinuities follow a recognisable pattern is largely due to Heading (1957) in his investigation of the Stokes phenomenon in solutions to certain nth order differential equations. Later (1962) he set out rules for continuing the Liouville–Green approximations (cf. Chapter XIII)].

First, what determines phases of Stokes rays? The argument culminating in (10) predicts breaks as the line of integration starting at the origin $f = 0$ sweeps through the branch point $f = y^2$ of the function $(1 - f/y^2)^{-\frac{1}{2}}$. Now according to the binomial expansion (7), late terms $r \gg \alpha + 1$ in the expansion of a form $(1 - f/y^2)^\alpha$ are all of the same sign

† First noticed in 1857 while investigating asymptotic expansions of Bessel functions, but not published until 1864.

and phase when f/y^2 is real, i.e. along just the same phase ray drawn from $f = 0$ through $f = y^2$. Conversely, we have the following prescription:

A. *The Stokes rays for an asymptotic series are determined by those phases for which successive late terms are homogeneous in phase and all of the same sign.*

For example, the terms in $\Sigma(r - \frac{1}{2})!/(-x^2)^r$ are all of the same phase and sign when $\operatorname{ph} x = \pm\frac{1}{2}\pi$, which therefore specify the Stokes rays.

An alternative criterion is suggested by the observation from (6), (11) and (13) that at these same phases $\operatorname{ph} x = \pm\frac{1}{2}\pi$ the contribution of the component asymptotic series to $\phi(x)$, varying essentially as $e^{(-x^2)}$, reaches peak exponential dominance (for fixed $|x|$) over the associated function, unity. The generalized rule may be expressed as follows:

B. *The Stokes rays for an asymptotic series are determined by those phases for which the series (including its multiplier) attains peak exponential dominance over its associated function.*

For an asymptotic series known to satisfy a second-order homogeneous differential equation, the associated function is simply that second solution to the equation which displays opposite exponential behaviour to the first. In such cases B enables Stokes rays to be located just from the leading terms of the two series solutions, whereas A calls for knowledge of the signs of late terms in the one series.

Combining B and A,

BA. *Relative to its associate, an asymptotic series is dominant where late terms are of uniform sign, and recessed where late terms alternate in sign.*

Comparison between (6), (11) and (13) leads to a collection of inferences concerning the continuation of asymptotic expansions across Stokes rays. Thus:

C. *The factor multiplying an asymptotic series is analytically continued across its own Stokes ray.*

This factor will remain correct up to the first-encountered Stokes ray of the associated series if asymptotic, or singularity if conditionally convergent. Likewise, the factor multiplying this associated series will continue back to the Stokes ray of the first series. Hence C and B may be combined as under:

CB. *In the phase sector lying between a Stokes ray of one asymptotic series and a neighbouring Stokes ray of its associate, if asymptotic, their combined contribution to a complete asymptotic expansion is the sum of the series dominant near one ray and the series dominant near the other.*

This is a handy rule in practice, since it is often far easier to reckon multipliers of dominant series than of recessive. Furthermore, once Stokes rays have been located by criteria A and B, it furnishes the key to converting into complete asymptotic expansions the Poincaré expansions in earlier literature, provided of course these have been correctly quoted over several phase regions. For, vague and imprecise as it undoubtedly is in some respects, Poincaré's prescription does at least guarantee correctness of dominant component series.

D. *On crossing its Stokes ray, an asymptotic series generates a discontinuity in form which is, on the ray, $\frac{1}{2}\pi$ out of phase with the series and proportional to its associated function.*

Rephrasing part of this through B, an asymptotic series at peak dominance generates a discontinuity in form proportional to its (relatively recessed) associated function. We may aptly view the rule as a principle of minimal fractional break in form.

E. *Half the discontinuity in form occurs on reaching the Stokes ray, and half on leaving it the other side.*

Combining this with the latter part of D,

ED. *On the Stokes ray the factor multiplying the associated function is the mean of the factors to either side of the ray.*

It will be appreciated from the three separate arguments leading to (6), (11) and (13), which relate to the simple error function $\phi(x)$, that for more complicated functions separate direct derivations of the asymptotic expansion appropriate to each phase sector and Stokes ray would mount up to a formidable programme. Fortunately, our general interpretative theory will obviate the need for this. Henceforth we shall therefore concentrate on carrying through a derivation for only one phase sector, or frequently for specific phases—as on Stokes rays—whichever can be treated most easily.

With ingenuity, complete asymptotic expansions for each new phase range can often be determined by recourse just to the rules set out above. This approach is particularly simple where only two component asymptotic series can be involved, as for example in the extremely important case of solutions to second-order homogeneous differential equations. The following advanced example illustrates how much can be accomplished this way, starting from the barest minimum of information. (Further examples are given in the exercises. An extremely important application—to phase-integral connection formulae—is examined in detail in Chapter XIII, Section 3).

Let $D_p(x)$ and $\mathscr{D}_p(x)$ denote those solutions to the parabolic cylinder equation $d^2y/dx^2 + (p + \frac{1}{2} - \frac{1}{4}x^2)y = 0$ which are respectively exponentially decreasing and increasing when x is real and positive. By trial substitution these are easily found to be

$$D_p(x) = x^p D_-, \qquad \mathscr{D}_p(x) = x^{-p-1}D_+, \qquad \operatorname{ph} x = 0,$$

where

$$D_- = \frac{e^{-\frac{1}{4}x^2}}{(-p-1)!}\sum_0^\infty \frac{(2r-p-1)!}{r!(-2x^2)^r}, \qquad D_+ = \frac{e^{\frac{1}{4}x^2}}{p!}\sum_0^\infty \frac{(2r+p)!}{r!(2x^2)^r}. \tag{14}$$

The asymptotic expansions at higher phases can be determined as under:

(i) Near $\operatorname{ph} x = 0$, D_- is recessed compared with the associated series D_+ and so has no Stokes rays in the neighbourhood (B). This is also clear from the alternation in signs in its series (A). The nearest Stokes ray is at $\frac{1}{2}\pi$ ((ii) below), so the representation can be extended in phase to read

$$D_p(x) = x^p D_-, \qquad 0 \leqslant \operatorname{ph} x < \tfrac{1}{2}\pi.$$

(ii) At $\operatorname{ph} x = \frac{1}{2}\pi$, D_- is at peak dominance relative to D_+, so has a Stokes ray there (B). This also follows from the like signs and phases of all late terms (A). In the asymptotic expansion for D_p, the factor multiplying D_- is analytically continued across this ray (C); but there will be generated a formal discontinuity proportional to $x^{-p-1}D_+ = e^{-\frac{1}{2}i\pi(p+1)}|x|^{-p-1}D_+$ and $\frac{1}{2}\pi$ out of phase with $x^p D_- = e^{\frac{1}{2}i\pi p}|x|^p D_-$ (D). Half this discontinuity occurs on reaching the ray, half on leaving it (E). Hence

$$\begin{aligned} D_p(x) &= e^{\frac{1}{2}i\pi p}\{|x|^p D_- + \tfrac{1}{2}\alpha\, e^{\frac{1}{2}i\pi}\,|x|^{-p-1}\,D_+\} \\ &= x^p D_- + \tfrac{1}{2}\alpha\, e^{i\pi(p+1)}x^{-p-1}D_+, \qquad \operatorname{ph} x = \tfrac{1}{2}\pi. \end{aligned}$$

$$D_p(x) = x^p D_- + \alpha\, e^{i\pi(p+1)}\,x^{-p-1}D_+, \qquad \tfrac{1}{2}\pi < \operatorname{ph} x < \pi.$$

The magnitude of the real constant α is unknown at this stage of the argument.

(iii) At $\operatorname{ph} x = \pi$, D_+ is at peak dominance relative to D_-, so has a Stokes ray there (B). This is also clear from the like signs and phases of late terms (A). In the expansion for D_p, the factor multiplying D_+ is analytically continued on crossing this ray (C), and on the ray itself can be written $\alpha|x|^{-p-1} = \alpha(-x)^{-p-1}$; but there will be a formal discontinuity generated proportional to $x^p D_- = e^{i\pi p}(-x)^p D_-$ and $\frac{1}{2}\pi$ out of phase with $x^{-p-1}D_+ = e^{-i\pi(p+1)}(-x)^{-p-1}D_+$ (D).

Half this discontinuity occurs on reaching the ray, half on leaving it (E). Hence

$$D_p(x) = x^p D_- + \alpha\{(-x)^{-p-1} D_+ + \tfrac{1}{2} i\beta(-x)^p D_-\}$$
$$= (e^{i\pi p} + \tfrac{1}{2} i\alpha\beta)(-x)^p D_- + \alpha(-x)^{-p-1} D_+, \qquad \operatorname{ph} x = \pi.$$
$$D_p(x) = (e^{i\pi p} + i\alpha\beta)(-x)^p D_- + \alpha(-x)^{-p-1} D_+,$$
$$\pi < \operatorname{ph} x < \tfrac{3}{2}\pi,$$

where β is real.

(iv) Keeping to real parameters and variable, an initially real function cannot acquire an imaginary part if its derivatives are finite and continuous, as is certainly the case here where finite second and later derivatives follow from the differential equation. Credit for emphasizing how this can help evaluate Stokes discontinuities is largely due to Zwaan (1929) in his pioneer work on phase-integral (W.K.B.) connection formulae.

By its genesis $D_p(x)$ is real when x is real, and therefore in particular at $\operatorname{ph} x = \pi$. Thus the multiplier of $(-x)^p D_-$ is $\cos \pi p$, and $\alpha\beta = -2 \sin \pi p$. Reference to the series forms shows that, apart from involving a different phase of x, D_- and D_+ differ only through $-p-1$ appearing in the first where p appears in the second. The discontinuities they engender at Stokes rays, α and β respectively, must be similarly related, i.e. $\beta(p) = \alpha(-p-1)$. The determining equation is thereby reduced to

$$\alpha(p)\ \alpha(-p-1) = -2 \sin \pi p. \tag{15}$$

Recognizing this as the reflection formula for factorials, the solution is

$$\alpha(p) = (2\pi)^{\frac{1}{2}}/(-p-1)!, \qquad \beta(p) = (2\pi)^{\frac{1}{2}}/p!,$$

or vice-versa. To distinguish between them, we argue that Stokes multipliers such as α and β are dictated by late terms in asymptotic series. From the definitions, late terms in D_- necessarily carry the outer factor $1/(-p-1)!$, which must therefore appear as the major factor in $\alpha(p)$; and late terms in D_+ carry the outer factor $1/p!$, which must then be the major factor in $\beta(p)$.

The expansion at $\operatorname{ph} x = \pi$ is especially interesting. Explicitly,

$$D_p(x) = \cos \pi p\ (-x)^p D_-$$
$$+ (2\pi)^{\frac{1}{2}}(-x)^{-p-1} D_+/(-p-1)!, \qquad \operatorname{ph} x = \pi, \tag{16}$$

showing that a solution can be exponentially decreasing both for

$x \to \infty$ and $x \to -\infty$ only if $p = 0, 1, 2, \ldots$—e.g. quantization of a simple harmonic oscillator—and then $D_p(x) = (-1)^p D_p(-x)$.

(v) Pursuing the argument, the asymptotic expansions for $D_p(x)$ can easily be found at $\text{ph}\, x = \frac{3}{2}\pi$ and for the next phase sector $\frac{3}{2}\pi < \text{ph}\, x < 2\pi$.

(vi) At $\text{ph}\, x = 0$, D_+ is at peak dominance relative to D_-, so has a Stokes ray there (B). The formal discontinuity on moving off this Stokes ray, towards a higher phase, is the same as that on reaching the other ray at $\text{ph}\, x = \pi$, since the series are identical at these two rays. Thus, by (iii),

$$\mathscr{D}_p(x) = x^{-p-1}D_+ + \tfrac{1}{2}i\beta x^p D_-, \qquad 0 < \text{ph}\, x < \tfrac{1}{2}\pi.$$

(vii) At $\text{ph}\, x = \frac{1}{2}\pi$, D_- has a Stokes ray. By (ii),

$$\begin{aligned}\mathscr{D}_p(x) &= x^{-p-1}D_+ + \tfrac{1}{2}i\beta\,\{x^p D_- + \tfrac{1}{2}\alpha\, e^{i\pi(p+1)}\, x^{-p-1}D_+\}\\ &= \{1 + \tfrac{1}{4}i\alpha\beta\, e^{i\pi(p+1)}\}x^{-p-1}D_+ + \tfrac{1}{2}i\beta x^p D_-, \qquad \text{ph}\, x = \tfrac{1}{2}\pi.\end{aligned}$$

$$\mathscr{D}_p(x) = \{1 + \tfrac{1}{2}i\alpha\beta e^{i\pi(p+1)}\}\, x^{-p-1}D_+ + \tfrac{1}{2}i\beta x^p D_-, \qquad \tfrac{1}{2}\pi < \text{ph}\, x < \pi.$$

(viii) At $\text{ph}\, x = \pi$, D_+ has a Stokes ray. By (iii),

$$\begin{aligned}\mathscr{D}_p(x) &= \{e^{-i\pi(p+1)} + \tfrac{1}{2}i\alpha\beta\}\,\{(-x)^{-p-1}D_+ + \tfrac{1}{2}i\beta(-x)^p D_-\}\\ &\quad + \tfrac{1}{2}i\beta x^p D_-\\ &= \{-e^{-i\pi p} + \tfrac{1}{2}i\alpha\beta\}\,(-x)^{-p-1}D_+\\ &\qquad - \beta\{\sin \pi p + \tfrac{1}{4}\alpha\beta\}\,(-x)^p D_-, \qquad \text{ph}\, x = \pi.\end{aligned}$$

(ix) By its genesis $\mathscr{D}_p(x)$ is real when x is real, and therefore in particular at $\text{ph}\ x = \pi$. Thus the multiplier of $(-x)^{-p-1}D_+$ is $-\cos \pi p$, and $\alpha\beta = -2 \sin \pi p$. This relation between α and β has already been found in (iv), and α and β determined therefrom.

The expansion at $\text{ph}\, x = \pi$ is noteworthy. Explicitly,

$$\mathscr{D}_p(x) = -\cos \pi p\, (-x)^{-p-1}D_+ - (\tfrac{1}{2}\pi)^{\frac{1}{2}} \sin \pi p\, (-x)^p D_-/p!, \qquad \text{ph}\, x = \pi, \tag{17}$$

showing that a solution can be purely exponentially increasing both for $x \to \infty$ and $x \to -\infty$ only if $p =$ integer, and then $\mathscr{D}_p(x) = (-1)^{p+1}\mathscr{D}_p(-x)$.

(x) Pursuing the argument, the asymptotic expansions for $\mathscr{D}_p(x)$ can easily be developed for $\pi < \text{ph}\, x < \frac{3}{2}\pi$, $\text{ph}\, x = \frac{3}{2}\pi$, and $\frac{3}{2}\pi < \text{ph}\, x < 2\pi$.

3. Cause of Stokes Discontinuities

Why do Stokes discontinuities occur? The short answer is that they simply have to if the asymptotic expansion is to comply with the variation of the continuous function represented. For such a function is already expansible as a convergent series of unchanging form, with an outer factor describing the behaviour at branch point or pole. A second expansion, in different powers of the variable and with a different outer factor, cannot indefinitely keep pace in magnitude and phase with the first save by periodic breaks in form. In fact an asymptotic expansion does keep pace in the most efficient and least disruptive manner imaginable: by undergoing increments in form which are $\frac{1}{2}\pi$ out of phase with the continuing portion, located where this continuing portion is at a peak relative to the increment.

Features in a defining function which enforce Stokes discontinuities in its asymptotic expansion include the following, arranged in roughly ascending order of intricacy:

Limits. To illustrate by our familiar example, the asymptotic expansion of $\phi(x)$ must be of the form

$$\phi(x) = A + Bx^{-1}e^{-x^2}\sum_0^{\infty}(r-\tfrac{1}{2})!/(-x^2)^r$$

because only these two component series, the first here a bare single term, satisfy the second-order differential equation for $\phi(x)$, $d^2\phi/dx^2 + 2x\,d\phi/dx = 0$. The definition $\phi(x) = 2\pi^{-\frac{1}{2}}\int_0^x e^{-u^2}du$ dictates two obvious limits: when $x \to +\infty$, $\phi \to 1$, so $A = 1$ in this phase neighbourhood; when $x \to -\infty$, $\phi \to -1$, so $A = -1$ in this second phase neighbourhood.

Phases. As immediate consequences of its definition $\phi(x)$ is real when $\operatorname{ph} x = 0, \pm\pi$, so A and B are then real; and $\phi(x)$ is imaginary when $\operatorname{ph} x = \pm\frac{1}{2}\pi$, so then B is still real but A must be imaginary or zero.

Continuation formulae. According to (12), $\phi(x) = -\phi(-x)$. Thus when the phase of x is changed by $\pm\pi$, B is unaffected but A is reversed in sign.

Convention on fractional powers. If the defining function introduces some parameter μ in such a way that fractional powers appear in results, a convention has to be imposed from the start making the generally multivalued function $(-x)^{\mu}$ definite. Throughout, we shall adhere to the customary convention

$$-1 \equiv e^{\mp i\pi}, \qquad \operatorname{ph} x \gtrless 0,$$

i.e., the complex plane cut from 0 to ∞, and hence write

$$(-x)^{\mu} \equiv \begin{cases} x^{\mu} e^{-i\pi\mu}, & \operatorname{ph} x > 0 \\ x^{\mu} e^{i\pi\mu}, & \operatorname{ph} x < 0 \\ x^{\mu} \cos \pi\mu, & \operatorname{ph} x = 0. \end{cases}$$

In asymptotic expansions these will naturally be classed amongst the Stokes discontinuities.

Changing number of contributing stationary points. To illustrate, the Schläfli integral representations for Bessel functions contain the integrand $\exp(x \sinh w - pw)$, which has stationary points

$$w = \pm \cosh^{-1} p/x \qquad (0 < x < p), \qquad w = \pm i \cos^{-1} p/x \qquad (x > p).$$

But the defining contour for $J_p(x)$ is such that in the first range only the positive point contributes, resulting in a single asymptotic series; whereas in the second range both points contribute, resulting in two additive asymptotic series multiplied by different exponentials.

Likewise, an integral representation involving a finite limit, $\int_0^\infty e^{-F(x,w)}\, dw$ say, will produce a single asymptotic series over those ranges of x for which $F(x, w)$ increases monotonically with w, but two when F displays a minimum between $w = 0$ and $w = \infty$.

4. Types of Series in Mathematics and Physics

To turn for a moment from mathematics to physics, it is characteristic of the theoreticians' approach to the solution of physical problems to favour a method of successive approximation in which the very first term will provide at least a rough value for the answer; and to accept from the start that this method of approach will probably be a good one only over a limited range. These properties are typical of a mathematically asymptotic expansion: compare the initial decrease in magnitude of terms, the small number of terms required to supply a close approximation, and the limitations in range resulting from (a) the size of the least term, and (b) the Stokes discontinuity. By contrast these physical desiderata are not parallelled by a mathematically absolutely convergent expansion: compare (for all but very small values of the variable) the latter's initial increase in magnitude of terms, the large number of terms required to give a close approximation, but compensated by the universal validity. Such comparative analysis suggests that the most commonly applied methods of successive approximation in theoretical physics will turn out on close scrutiny to be asymptotic rather than convergent. The evidence available to date shows this to be largely true. Phase-integral (W.K.B.) methods, with their extensive applicability in almost all branches of physics—from tides and earthquakes through optics to ionospheric theory and quantum mechanics† —are intrinsically asymptotic, as we shall see in chapters XIII and XIV. So also are methods hinging on expansion about a stationary point (VI, VII), as in theories of the de Haas-van Alphen effect and the rainbow (Berry 1966, 1967). Perturbation expansions, vitally important in varied fields through often affording the only feasible mode of solution, are in the majority asymptotic (second volume). One elementary example of this tendency towards asymptotic form is worth singling out on account of its familiarity: the omnipresence, especially in statistical mechanics, of the Stirling–Laplace asymptotic formula $x! \sim (2\pi)^{\frac{1}{2}} e^{-x} x^{x+\frac{1}{2}}$ for a factorial, compared with the virtual non-appearance in scientific application of the corresponding convergent expansion $x! = 1 - {\cdot}5772x + \dots$.

By far the most frequent exception to the predilection towards asymptotics in physics is the hypergeometric form, in which sufficiently late terms are

$$\frac{(r+a)!}{(r+b)!(\text{variable})^r} \cdots\cdots\cdots \begin{array}{l}\text{hypergeometric type}\\ \text{(conditionally convergent)}\end{array} \tag{19}$$

as compared with

$$\frac{(r+a)!}{(\text{variable})^r} \cdots\cdots\cdots \begin{array}{l}\text{asymptotic}\\ \text{(divergent)}\end{array} \tag{20}$$

and

$$\frac{1}{(r+b)!(\text{variable})^r} \cdots\cdots\cdots \begin{array}{l}\text{exponential type}\\ \text{(absolutely convergent).}\end{array} \tag{21}$$

As expected from their form, hypergeometric series disport properties midway between those of asymptotic and exponential type. Their appearance in physics, second in prevalence to the asymptotic, is correlated with theoreticians' resigned acceptance of a limitation in range; for series of hypergeometric type have a well-defined radius of convergence, frequently directly associated in physics with some transition point. Their commonest appearance is correspondingly in the theory of co-operative phenomena (especially phase transitions), e.g. the Ising model, ferromagnetism, and electron screening such as in Fermi–Thomas calculations. They are endemic in modern hydrodynamics, with its emphasis on boundary layers and abrupt changes in flow patterns. As remarked above, the majority of

† The most recent review of applications to quantum mechanics is by Berry and Mount (1972).

perturbation expansions in theoretical physics turn out to be asymptotic; the hypergeometric behaviour displayed by others arises wherever the effect investigated concerns a change merely in effective force parameters on a macroscopic scale, such as a change in effective dielectric constant because of screening.

Series of hypergeometric type do not share with asymptotic forms the property that, over a wide range of values of the variable, relatively few terms provide a close approximation to the function represented. Accordingly, physical theories involving them tend to be notorious for the exorbitant number of terms which need to be calculated in series expansions before confident predictions can be made from them—e.g. Ising series. Nonetheless, quite unlike asymptotic expansions which are divergent whatever the magnitude and sign of the variable, granted enough terms of a power series of hypergeometric type, convergence can systematically and reliably be accelerated by all manner of algebraic rearrangements; to mention a few, Euler and similar transformations, Padé approximants, taking an inverse power so as to give the original series a chance to display its radius of convergence, and successive subtraction of recognized summable sequences. This fundamental dissimilarity with asymptotic theory makes it unprofitable to include parallel discussions of the hypergeometric form in our present investigation.

The distinction in type drawn above related primarily to simple power series. Extension to transcendental functions of the variable can give rise to instances of striking, albeit superficial, likeness in isolated features. Consider for example the expansion

$$\frac{(x-a-1)!\,(x-b-1)!}{x!}$$
$$= \frac{1}{a!\,b!}\sum_{s=0}(-1)^s\,(x-s-a-b-2)!\,\frac{(s+a)!\,(s+b)!}{s!}.$$

Though hypergeometric in type as far as coefficients are concerned, its behaviour for large positive x shares with asymptotic forms the drop in successive terms by a factor $0(x^{-1})$ near the start, changing to divergence after $s \sim x$. But this instance of accidental similarity is limited in scope. Whereas an asymptotic form would diverge whatever the magnitude and phase of x, the continuation of this hypergeometric-type expansion to negative x is conditionally convergent, since with $x = -y$ the expansion is replaced by

$$\frac{(y-1)!}{(y+a)!\,(y+b)!} = \frac{1}{a!\,b!}\sum_{s=0}\frac{1}{(y+s+a+b+1)!}\,\frac{(s+a)!\,(s+b)!}{s!}.$$

5. Problems of Defining Asymptotic Series and Expansions

Poincaré's specification of an asymptotic power series

As already noted in the Prologue, a formal definition of asymptotic power series—a sub-group of the more general asymptotic series and expansions—was proposed in 1886 by Henri Poincaré. According to this, an ultimately-divergent series† $\Sigma a_r/x^r$ is taken to be the asymptotic power series for a function $f(x)$ if

$$\lim_{|x|\to\infty} x^n \left\{ f(x) - \sum_0^n a_r/x^r \right\} \to 0 \qquad \text{for all zero and positive } n. \qquad (22)$$

Without implying any belittlement of Poincaré's contribution—his definition having indeed furnished the point of departure for most subsequent theoretical researches—it is imperative to ferret out its shortcomings, and find how to surmount or circumvent them, before we can be satisfied that ascription of a latently exact meaning to complete asymptotic expansions rests on a firm theoretical footing as well as on verification in manipulation.

The observation

$$\lim_{x\to\infty} (x^n \times \text{any exponentially decreasing function of } x) \to 0 \text{ for all } n$$

illustrates the central deficiency of (22). According to Poincaré's specification the same series could serve equally for $f(x)$ and any function differing from $f(x)$ by some exponentially smaller contribution; this is the source of the frequent misunderstanding that asymptotic expansions cannot represent unique functions. In the form (22) the specification thereby permits, though it does not enforce, abandonment in an asymptotic expansion of any portions which are multiplied by an exponentially decreasing function of the variable in the phase region considered.

From before the time Poincaré proposed his definition, and for long afterwards partly as a deduction from it, it was widely believed that asymptotic power series were limited in meaning and therefore accuracy to about the magnitude of their least term. Consequently, in dealing with

† This restriction to ultimately divergent series is explicitly included in some authoritative texts, such as Whittaker and Watson's "Modern Analysis". The restriction has often been omitted from Poincaré's prescription, with the result that convergent series which obey (22) have then formally also been included amongst the class of series labelled "asymptotic". As remarked at the start of the prologue, we shall throughout this book restrict the meaning of "asymptotic" so as to exclude convergent expansions. This narrower meaning actually corresponds the more closely with normal descriptive practice; one would, for example, never knowingly speak of "deriving an asymptotic expansion" where "deriving the convergent expansion" was meant.

asymptotic expansions it became more or less standard practice for exponentially small portions to be discarded on the grounds that they were exceeded in magnitude by this supposed inherent inexactitude of component asymptotic power series. Looking back, one can see how this apparent simplification actually confused and complicated issues in all directions! Quite apart from introducing a descriptive uncertainty into the meaning of later more successful interpretations of component asymptotic series, it complicated the tracing of discontinuities in form across consecutive Stokes rays, and it imparted a perplexing non-additive property to the specification. For if the complete expansions of two functions X and Y were originally determined as

$$X = \Sigma\, a_r/x^r + e^{-x}\Sigma\, b_s/x^s, \qquad Y = \Sigma\, a_r/x^r + e^{-x}\Sigma\, c_s/x^s,$$

then, with the exponentially small portions discarded, X and Y *when treated separately* were each said to possess the asymptotic expansion $\Sigma a_r/x^r$. But if the starting point had been the combined function $e^x(X - Y)$, this would be said to possess the asymptotic expansion $\Sigma(b_s - c_s)/x^s$, not zero as anticipated from the separate treatments. An important special case of this inconsistency—in the eigenvalues of the Mathieu equation—was pointed out by Goldstein in 1929.

The objection of non-additivity can easily be pushed aside by distinguishing carefully between an "asymptotic expansion" and its component "asymptotic series". The asymptotic expansion of a function $f(x)$ within a phase sector might then be written

$$f(x) = A(x)\,\Sigma\, a_r/x^r + B(x)\,\Sigma\, b_s/x^s + \dots$$

where the summations constitute component asymptotic power series and the multipliers $A(x)$, $B(x)$, etc. are analytic functions of x. But the central deficiency of the specification remains: if in attempting to specify the component series we rely on Poincaré's prescription, there would be an exponentially small arbitrariness introduced by each; in particular, if $B(x)$ were exponentially smaller than $A(x)$, the "definition" would fail to specify or confirm, rather than just passively *permit*, the series $\Sigma b_s/x^s$ on account of the arbitrariness attributed to the meaning of $\Sigma a_r/x^r$.

Poincaré- Watson definition

At the cost of considerable complication the central deficiency of Poincaré's specification can be removed, except near Stokes rays, on supplementing (22) by the requirement that it must apply unchanged over a sufficiently broad range of phase in x. To see this, let us return to our familiar example written in the form

$$\phi(x^{\frac{1}{2}}) = 1 - \frac{e^{-x}}{\pi x^{\frac{1}{2}}} \sum_{0}^{\infty} \frac{(r-\frac{1}{2})!}{(-x)^r}, \qquad -\pi < \operatorname{ph} x < \pi. \tag{23}$$

Over the phase range $-\frac{1}{2}\pi < \operatorname{ph} x < \frac{1}{2}\pi$ the second contribution in (23) is exponentially smaller than the first; whereas over the ranges $-\pi < \operatorname{ph} x < -\frac{1}{2}\pi$ and $\frac{1}{2}\pi < \operatorname{ph} x < \pi$ the unit term is exponentially smaller than the second contribution. Hence, for this example, if the Poincaré specification is supplemented by the requirement of applying unchanged over a total range of phase of x exceeding π (as indeed assured in (23)), no error or ambiguity can arise from

$$\lim_{x\to\infty} \ (x^n \times \text{any exponentially decreasing function of } x) \to 0 \text{ for all } n,$$

since neither contribution in (23) remains exponentially small, relatively to the other, right throughout the broad range. An important theorem based on this simple idea was proved by Watson in 1911. Let there be two functions $f_1(x)$ and $f_2(x)$ which are analytic for $|x| \geqslant \gamma$ and $\alpha \leqslant \operatorname{ph} x \leqslant \beta$, and let them be such that in this region they possess the asymptotic power series

$$f_1(x) = \sum_{0}^{n} a_r/x^r + R_n, \qquad f_2(x) = \sum_{0}^{n} a_r/x^r + S_n, \tag{24}$$

where for all n

$$|a_n| < A\rho^n(kn)!, \qquad |x^{n+1}R_n| < B\sigma^n(ln)!, \qquad |x^{n+1}S_n| < B\sigma^n(ln)!.$$

Then, if $\beta - \alpha > \pi l$, $f_1(x) \equiv f_2(x)$ and a unique function is therefore defined by the asymptotic power series.

Watson's demonstration that an asymptotic power series of the form (20) uniquely defines an analytic function is assuredly of crucial importance. But the conditions attached to his theorem as it stands, and the existence of Stokes discontinuities, combine to make it an unduly intricate *definition* except for simple asymptotic expansions derived by elementary means. In the first place the formulation (24) presupposes independent estimation of the remainder term, i.e. information supplementary to that contained within the unterminated asymptotic expansion itself; a logically consistent but circuitous procedure can in fact be devised in which the definition is first adopted tacitly assuming the remainder condition to be satisfied, the coefficients a next found and the R_n then predicted from these, enabling the required inequality and minimum phase range for validity to be established *a posteriori*. More awkward are complications arising from the liability of Stokes discontinuities. For Watson's theorem to be directly applicable the phase range has to comply with two stipulations: first, it has to be sufficiently narrow for the complete asymptotic expansion to remain unaffected throughout that range by any Stokes discontinuity in form; second, it has to be amply wide for every exponential multiplier to veer from being exponentially small to exponentially large. In a relatively complicated expansion comprising several component series, these conditions are likely to conflict. In principle, this objection could also be overcome by augmentation in detail and sophistication. A complete asymptotic expansion could be dissected into separate parts, the only summations included in any individual part being one asymptotic power series $\Sigma a_r/x^r$ and its associated series $\Sigma A_r/x^r$; Watson's theorem would then be applicable to each part separately over the phase range for which that part remains free from a Stokes discontinuity. In this way the uniqueness of every part could be established while allowing the phase region over which the theorem is invoked to vary somewhat from part to part; though the combined result would represent the complete asymptotic expansion only over that phase range shared by all its parts.

Enough has been said to exemplify the involved nature of this definition, the three main problems being those of the remainder, liability of conflicting conditions on the phase range, and non-applicability near Stokes rays. Confirmation along these lines of a complete asymptotic expansion demands too much advance and advanced knowledge of the form of the coefficients, the remainder, and indeed of the whole theory of the properties of asymptotic series over an extended phase range, to make the idea, deceptively straightforward as it appears at root, a workable basis of definition except for simple asymptotic expansions derived by elementary means such as those of chapters II and IV.

The "non-numerical compliance" definition of a complete asymptotic expansion

A radically different approach to formal definition is based on the observation that asymptotic expansions are normal, immediately comprehensible, functions of their variables *in so far as functional form is concerned*. The dissimilarity with convergent expansions lies in the impossibility of assigning a precise *numerical* value to the sum (except in the infinite limit) before interpretation of the divergent sequence of late terms. A complete asymptotic expansion of a function $f(x)$ may therefore be defined (Dingle 1962) as an expansion containing asymptotic series, which formally exactly obeys—throughout a certain phase sector—all those relations satisfied by $f(x)$ which do not involve any numerical value of x other than on the infinite circle $|x| \to \infty$: for instance,

(i) Functional form as $|x| \to \infty$, i.e. boundary conditions on f and its derivatives at infinity.

(ii) Differential, difference and integral equations.

(iii) Relations involving other parameters incorporated, such as recurrence relations between orders.

For example, expansions (6), (11) and (13) for the error function are each easily confirmed as complete and unique by (i) and (ii) taken in conjunction; namely by compliance with the correct behaviour as $x \to \infty$, $x \to \pm i\infty$ and $x \to -\infty$ in the three phase sectors, and with the differential coefficient $d\phi/dx = 2\pi^{-\frac{1}{2}}e^{-x^2}$ for all x.

The supremacy of this definition in actual practice shows up when tested on expansions over which there has been overt doubt or dispute in the past. For instance the complete expansion for the Fermi-Dirac integral is (Dingle 1957)

$$\mathscr{F}_p(x) = \frac{1}{p!}\int_0^\infty \frac{\varepsilon^p d\varepsilon}{e^{\varepsilon - x} + 1} = 2x^{p+1}\sum_0^\infty \frac{t_{2r}}{(p+1-2r)!x^{2r}}$$

$$+ (\cos \pi p)\sum_1^\infty \frac{(-1)^{r-1}e^{-rx}}{r^{p+1}}, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi, \tag{25}$$

where $t_0 = \frac{1}{2}$, $t_r = \sum_1^\infty (-1)^{\mu-1}/\mu^r$. The second contribution was omitted by Sommerfeld (1928) and by later authors. Both versions display the correct behaviour as $x \to \infty$ and satisfy the known relation $\mathscr{F}_p'(x) = \mathscr{F}_{p-1}(x)$; but (iii) of the definition immediately confirms (25) as the correct version, since on taking p to be an integer it agrees with the direct integrations which are then possible (Rhodes 1950); for instance $\mathscr{F}_0(x) = x + \ln(1 + e^{-x})$.

Again, the complete expansion for large integer n of

$$I(n) = \int_0^{m\pi} \frac{\cos nu}{u^2 + 1}\, du \tag{26}$$

where $m = 1, 2, 3, \ldots,$ is† (Chapter VIII, question 1.)

$$I(n) = \tfrac{1}{2}\pi e^{-|n|} - (-1)^{mn}\sum_{r=1,3,5,}^\infty \frac{(-1)^{\frac{1}{2}(r-1)}r!(2m\pi)^r}{n^{r+1}(m^2\pi^2+1)^{r+1}}$$

$$\times \sum_{s=0}^{\frac{1}{2}(r-1)} \frac{(r-s)!}{s!(r-2s)!}\left(-\frac{m^2\pi^2+1}{4m^2\pi^2}\right)^s. \tag{27}$$

† The case $m = 1$ was considered by Olver (1964).

Disconcertingly, a naive straight integration by parts of (26) as it stands appears to give an expansion omitting the exponential contribution to (27)†. Part (iii) of the above definition immediately confirms the true presence of the exponential term, since on passing to infinite limit in the parameter, $m \to \infty$, the defining integral becomes $\int_0^\infty (u^2+1)^{-1} \cos nu\, du = \frac{1}{2}\pi\, e^{-|n|}$.

Incidentally, these two examples underline the advisability of retaining generality in deriving asymptotic expansions. A contributory reason historically for the acceptance of the inaccurate versions of (25) and (27) was that attention had been fastened on special cases where their inexactitudes were not evident, namely on half-integral values of p for the Fermi-Dirac integral, when the second contribution happens to vanish identically; and on the special case $m = 1$ in (26), preventing passage to the simply examinable case of infinite upper limit of integration.

So far we have defined a complete asymptotic expansion on the supposition that we can already recognise a component asymptotic series, distinguishing it from a convergent series. There is no difficulty here. Almost invariably such a component series of an asymptotic expansion can be expressed as a power series in some variable for which sufficiently late terms are of the standard form $(r+a)!/(\text{variable})^r$. In the remaining cases where such a form cannot yet be established, or more likely where it is too troublesome to try and elucidate the precise form of late terms, it suffices to characterize an asymptotic series as an ordered series of systematically constructed terms (cf. section 1) which—for all large values of the variable, at all phases—first progressively decrease in magnitude, then reach a minimum and thereafter progressively increase.

† The mistake is one easily made in integrations by parts, and has really little or nothing to do with asymptotics. Successive integration by parts of (26), by integrating the trigonometrical factor and differentiating the algebraic, is more rigorously equivalent to the introduction of a Taylor series for the algebraic factor about each of these limits. Now a Taylor series of the form

$$\frac{1}{u^2+1} = \frac{1}{u_0{}^2+1} - (u-u_0)\frac{2u_0}{(u_0{}^2+1)^2}\cdots$$

can be adopted *either* when $|u_0| > 1$ (e.g. $u_0 = m\pi$ at upper limit), *or* when $|u_0| < 1$ (*cf.* $u_0 = 0$ at lower limit of (26) as it stands); provided of course that $(u-u_0)$ is sufficiently small in each case. But although the forms of Taylor series in the two regions are similar, this is no guarantee that on integration they produce the same integration constants. In fact they do not, because of the poles of $(u^2+1)^{-1}$ on the circle $|u| = 1$.

Thus if the asymptotic expansion is to be derived through repeated integration by parts, the integral must first be split into $\int_0^\infty - \int_{m\pi}^\infty$, and integrations by parts performed in the second integral where both limits lie on the same side of the circle of convergence $|u| = 1$.

As an amusing and instructive verification of this explanation, it is observed that when $m = 0$ the exponential contribution to (27) is indeed absent, because in this trivial case both limits 0 and $m\pi$ of the original integral already lie on the same side of $|u| = 1$, namely within this circle.

Exercises

1. If p is an integer, $(1-f^2)^{p-\frac{1}{2}}$ becomes a pure imaginary for $f > 1$. Hence prove that for real and positive x,

$$F_p(x) = \int_0^1 e^{-xf}(1-f^2)^{p-\frac{1}{2}}\,df = \Re \int_0^\infty e^{-xf}(1-f^2)^{p-\frac{1}{2}}\,df.$$

Expand $(1-f^2)^{p-\frac{1}{2}}$ by the binomial theorem and integrate term by term to arrive at the asymptotic power series

$$F_p(x) = \frac{1}{x\sqrt{\pi}\,(-p-\frac{1}{2})!}\sum_0^\infty \frac{(r-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{(\frac{1}{4}x^2)^r}, \qquad \operatorname{ph} x = 0.$$

2. Two independent solutions to the differential equation $d^2y/dz^2 = zy$ are the important Airy functions. For real positive z their asymptotic power series are

$$Ai(z) = z^{-\frac{1}{4}}E_-, \qquad Bi(z) = 2z^{-\frac{1}{4}}E_+, \qquad \operatorname{ph} z = 0,$$

where

$$E_\pm(z) = \frac{e^{\pm(2/3)z^{3/2}}}{2\sqrt{\pi}\,(-\frac{1}{6})!\,(-\frac{5}{6})!}\sum_0^\infty \frac{(r-\frac{1}{6})!\,(r-\frac{5}{6})!}{r!(\pm\frac{4}{3}z^{3/2})^r}.$$

Find the asymptotic expansion of $Ai(z)$ in the phase sector $\frac{2}{3}\pi < \operatorname{ph} z < \frac{4}{3}\pi$ by developing the following argument:

(i) Near $\operatorname{ph} z = 0$, E_- is recessed compared with the associated series E_+ and so has no Stokes rays in the neighbourhood. Hence extension in phase range is permitted up to the nearest Stokes ray:

$$Ai(z) = z^{-\frac{1}{4}}E_-, \qquad 0 \leqslant \operatorname{ph} z < \tfrac{2}{3}\pi.$$

(ii) At $\operatorname{ph} z = \frac{2}{3}\pi$, $z^{3/2} = -|z|^{3/2}$, so E_- is at peak dominance relative to E_+ and has a Stokes ray there. The same conclusion follows on noting the equality in sign and phase of all late terms in E_-.

In the asymptotic expansion for Ai, the factor multiplying E_- is continued on crossing this ray, but there will be a formal discontinuity generated proportional to E_+ and $\frac{1}{2}\pi$ out of phase with E_-. Hence

$$Ai(z) = z^{-\frac{1}{4}}(E_- + \alpha e^{\frac{1}{2}i\pi}E_+), \qquad \tfrac{2}{3}\pi < \operatorname{ph} z < \tfrac{4}{3}\pi,$$

where the magnitude of the real quantity α is as yet unknown.

(iii) By its genesis $Ai(z)$ is real when z is real, and thus in particular at $\operatorname{ph} z = \pi$. At this phase

$$E_\pm = \frac{e^{\mp(2/3)i(-z)^{3/2}}}{2\sqrt{\pi}(-\frac{1}{6})!\,(-\frac{5}{6})!}\sum_0^\infty \frac{(r-\frac{1}{6})!\,r-\frac{5}{6})!}{r!\{\mp\frac{4}{3}i(-z)^{3/2}\}^r},$$

i.e. the two E's are complex conjugates. The reality condition for

$$Ai(z) = (-z)^{-\frac{1}{4}}(e^{-\frac{1}{4}i\pi}E_- + \alpha\, e^{\frac{1}{4}i\pi}E_+), \qquad \operatorname{ph} z = \pi,$$

therefore requires $\alpha = 1$.

3. Starting from $Bi(z) = 2z^{-\frac{1}{4}}E_+$ at $\operatorname{ph} z = 0$, find its asymptotic expansion in the phase sector $\frac{2}{3}\pi < \operatorname{ph} z < \frac{4}{3}\pi$ by developing the following argument:

(i) At $\operatorname{ph} z = 0$, E_+ is at peak dominance relative to its associated series E_-, so has a Stokes ray there. The same conclusion follows on noting the equality in sign and phase of all late terms in E_+.

In the asymptotic expansion for Bi, the factor multiplying E_+ is continued as we move off this ray, but there will be a formal discontinuity generated proportional to E_- and $\frac{1}{2}\pi$ out of phase with E_+, so

$$Bi(z) = 2z^{-\frac{1}{4}}(E_+ + \tfrac{1}{2}\beta e^{\frac{1}{2}i\pi}E_-), \qquad 0 < \operatorname{ph} z < \tfrac{2}{3}\pi.$$

(Because of the symmetry between E_+ and E_-, the full Stokes discontinuity on passing right across the Stokes ray of E_+ at $\operatorname{ph} z = 0$ should be the same as the discontinuity on crossing the Stokes ray of E_- at $\operatorname{ph} z = \frac{2}{3}\pi$, so $\beta = \alpha$. But to display the power and generality of the method, this identity will not be invoked here).

(ii) E_+ does not again reach peak dominance, or its late terms assume same signs and phases, until $\operatorname{ph} z = \frac{4}{3}\pi$, so this series has no Stokes rays in the phase sector $0 < \operatorname{ph} z < \frac{4}{3}\pi$.

(iii) The discontinuity caused by crossing the Stokes ray of E_- at $\operatorname{ph} z = \frac{2}{3}\pi$ has already been evaluated as $e^{\frac{1}{2}i\pi}E_+$. Hence

$$\begin{aligned} Bi(z) &= 2z^{-\frac{1}{4}}\{E_+ + \tfrac{1}{2}\beta\, e^{\frac{1}{2}i\pi}(E_+ + e^{\frac{1}{2}i\pi}E_+)\} \\ &= z^{-\frac{1}{4}}\{(2-\beta)E_+ + \beta\, e^{\frac{1}{2}i\pi}E_-\}, \qquad \tfrac{2}{3}\pi < \operatorname{ph} z < \tfrac{4}{3}\pi. \end{aligned}$$

(iv) By its genesis $Bi(z)$ is real when z is real. Now for real negative z, i.e. for $\operatorname{ph} z = \pi$,

$$Bi(z) = (-z)^{-\frac{1}{4}}\{(2-\beta)\,e^{-\frac{1}{4}i\pi}E_+ + \beta\, e^{\frac{1}{4}i\pi}E_-\}.$$

Since at this phase the two E's are complex conjugates, the reality condition demands $2 - \beta = \beta$, i.e. $\beta = 1$.

4. The asymptotic expansion of the confluent hypergeometric function is

$$F(a, c, x) = x^{-a} e^{i\pi a} F_- + x^{a-c} F_+, \qquad 0 < \operatorname{ph} x < \pi,$$

where

$$F_- = \frac{(c-a)!}{(c-a-1)!\,(a-1)!\,(a-c)!} \sum_0^\infty \frac{(r+a-1)!\,(r+a-c)!}{r!(-x)^r},$$

$$F_+ = \frac{(c-1)!\,e^x}{(a-1)!\,(-a)!\,(c-a-1)!} \sum_0^\infty \frac{(r-a)!\,(r+c-a-1)!}{r!\,x^r}.$$

Develop the following argument for determining $F(a, c, -x)$ when x is real and positive. At $\operatorname{ph} x = \pi$ the first series is at peak dominance relative to the second (which is $\propto e^x$, exponentially small), so has a Stokes ray there. In the expansion for F, the factor multiplying F_- is analytically continued across this ray—and can be written $(-x)^{-a}$ on the ray itself—but there will be a formal discontinuity generated proportional to $x^{a-c}F_+ = e^{i\pi(a-c)}(-x)^{a-c}F_+$ and $\frac{1}{2}\pi$ out of phase with $(-x)^{-a}F_-$, i.e. an imaginary discontinuity. Hence

$$F(a, c, x) = (-x)^{-a}F_- + \{e^{i\pi(a-c)} + \tfrac{1}{2}i\alpha\}(-x)^{a-c}F_+,$$

where α is real but as yet undetermined in magnitude,

By its genesis, provided a and c are real $F(a, c, x)$ is real when x is real, and thus in particular at $\operatorname{ph} x = \pi$. Hence

$$F(a, c, x) = (-x)^{-a}F_- + \cos \pi(a-c) \quad (-x)^{a-c}F_+, \qquad \operatorname{ph} x = \pi,$$

and $\alpha = -2 \sin \pi(a-c)$.

5. Continuing the argument of question 4, find the asymptotic expansion of $F(a, c, x)$ in the phase sector $\pi < \operatorname{ph} x < 2\pi$ by adding in the second half of the discontinuity at $\operatorname{ph} x = \pi$, that on leaving the Stokes ray, producing

$$F(a, c, x) = (-x)^{-a}F_- + \{e^{i\pi(a-c)} + i\alpha\}(-x)^{a-c}F_+, \qquad \pi < \operatorname{ph} x < 2\pi.$$

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Chapter II

Derivation of Asymptotic Power Series from Convergent Series

1. Defining Functions by Convergent series

The designation "convergent series" was first introduced by James Gregory in 1667, but it was not until a century later that serious work was started on evolving rigorous general tests for convergence. The best known of these were discovered by Jean D'Alembert (1768), Augustin-Louis Cauchy (1821), Neils Abel (1826), J. L. Raabe (1832), Augustus De Morgan (1842) and P. G. Lejeune Dirichlet (1862). Nonetheless, from the latter part of the seventeenth century onwards it became the practice to define functions by series forms. This mode of definition remains common; for instance, hypergeometric functions are invariably so defined, and the simple exponential frequently so. Moreover, in examining the solutions of a differential equation, one of the simplest procedures is to solve the equation in series by the method of Ferdinand Frobenius (1873) and then invoke these series—two for a linear second-order equation—to define and then investigate the independent solutions.

Such a defining series

$$S(x) = \sum_0^\infty a_s x^s \tag{1}$$

may become inconvenient over some range of ph x, though remaining convergent. Consider for example the absolutely convergent series

$$S(x) = \sum_0^\infty (-x)^s/s!(2s+1) \tag{2}$$

characterising the error function

$$\phi(x) = 2\pi^{-\frac{1}{2}} x\, S(x^2)\,. \tag{3}$$

If $|x| \gg 1$ the contribution in (2) from large late terms $s \sim |x|$ is of type $\Sigma(-x)^s/s! = e^{-x}$. When $\Re(x)$ is positive this contribution is exponentially small, so $S(x)$ is largely controlled in magnitude by its early terms and therefore does not as a whole vary exponentially. On the other hand, when $\Re(x)$ is negative the bulk contribution from late terms is exponentially large, far outweighing the contribution from early terms; this is not only inconvenient computationally, but also analytically because it corresponds to an essential singularity in the sum as $\Re(x) \to -\infty$.

When examining such phase regions it is therefore expedient to modify the defining series by removing the exponential dependence. Writing

$$S(x) = e^{-\mu x}\bar{S}(x), \qquad \bar{S}(x) = \sum_0^\infty \bar{a}_t x^t, \tag{4}$$

we have

$$\bar{S}(x) = \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{(\mu x)^r}{r!} a_s x^s.$$

On putting $r = t - s$, the required coefficients are seen to be

$$\bar{a}_t = \sum_{s=0}^t \mu^{t-s} a_s/(t-s)!. \tag{5}$$

The technical problem of evaluating such finite summations is deferred to Section 9. Here we note only that for our example of the error function, $a_s = (-1)^s/s!(2s+1)$, $\mu = 1$ and the new coefficients are

$$\bar{a}_t = \sum_0^t \frac{(-1)^s}{(t-s)!\,s!(2s+1)} = \frac{\sqrt{\pi}}{2(t+\frac{1}{2})!}\,. \tag{6}$$

2. Method of Mellin Transforms

Given a function defined by a convergent power series free from an essential singularity at infinity within the phase range under consideration, the most direct method for deriving an equivalent asymptotic power series is by way of its Mellin transform. If more recondite asymptotic forms are required, such as those remaining directly usable when some additional parameter is very large, it is better to convert the given convergent series into a Laplace-type representation (Chapter III) and proceed from there (Chapters IV–XI).

Let us consider the Mellin integral representation

$$S(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} M(m)\, x^{-m}\, dm, \tag{7}$$

with a view to determining the conditions under which it is equivalent to the convergent power series $\Sigma a_s x^s$. The primary requirement is clearly that $M(m)$ must possess single poles at $m = 0, -1, -2, -3, \ldots\ldots$, with residues

a_{-m}. The factorial function $(m-1)!$ has single poles at just these points, but with residues $(-1)^m/(-m)!$ Hence a suitable choice for $M(m)$ is

$$M(m) = a_{-m}(-1)^m (m-1)!\,(-m)! \tag{8}$$

provided the poles of $(-m)!$ at $m = 1, 2, 3, \ldots$, are excluded by imposing the condition $0 < \gamma < 1$ and imagining the contour to envelop the left (negative) half plane anticlockwise. The same result, differently expressed, can alternatively be derived by noting that the poles of $1/\sin \pi m$ at integer values of m leave residues $1/\pi \cos \pi m = (-1)^m/\pi$, leading to the choice $M(m) = a_{-m}\pi(-1)^m/\sin \pi m$, precisely equivalent to (8). In our example of the error function, the two Mellin representations are thus

$$S(x) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} \dfrac{(m-1)!}{1-2m}\, x^{-m}\, dm\,, & |\mathrm{ph}\, x| < \tfrac{1}{2}\pi, \quad (9) \\[2ex] \dfrac{e^{-x}}{4\sqrt{\pi}\, i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} \dfrac{(m-1)!(-m)!}{(-m+\tfrac{1}{2})!}\,(-x)^{-m}\, dm\,, & |\mathrm{ph}\,(-x)| < \tfrac{1}{2}\pi. \quad (10) \end{cases}$$

Here the phase sectors have been ascertained by writing $x = |x|\, e^{i\mathrm{ph} x}$ and noting that towards the limits of integration $m = \gamma \pm i\mathscr{m}$, $\mathscr{m} \to \infty$,

$$(\gamma \pm i\mathscr{m} - 1)! \sim (2\pi)^{\frac{1}{2}} \mathscr{m}^{\gamma-\frac{1}{2}}\, e^{-\frac{1}{2}\pi\mathscr{m}},$$

so the convergence of (9) is controlled at the limits by the factors $e^{-\mathscr{m}(\frac{1}{2}\pi \mp \mathrm{ph} x)}$, $\mathscr{m} \to \infty$. As might have been anticipated, for each form the phase sector is that within which the sum represented by the Mellin integral exhibits no essential singularity at infinity.

To find the asymptotic expansion from such a Mellin pair we move the path of integration in each to the right until all poles have been passed. (Or, to find the remainder after a large but finite number of terms as needed in the "Poincaré–Watson" definition of a complete asymptotic expansion, the path of integration is moved past only the required number of poles). For each representation the result takes the form

$$S(x) = -\,2\pi i\,\Sigma(\text{residues at poles}) + R\,. \tag{11}$$

Appeal to the theory developed in Chapter VI to follow shows the remainder integral R to be exponentially small. In the first phase sector, for instance, retaining only fast-varying factors the dominant contribution from the stationary point (s.p.) is

$$R \sim \frac{1}{2\pi i}\int_{\mathrm{s.p.}} (m-1)!x^{-m}\, dm \sim (x-1)!x^{-x} \sim e^{-x}, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi\,.$$

Having now the leading series in two contiguous phase sectors, we can invoke rule CB of the preceding chapter: in the phase sector lying between a Stokes ray of one asymptotic series and a neighbouring Stokes ray of its associate, if asymptotic, their combined contribution to a complete asymptotic expansion is the sum of the series dominant near one ray and the series dominant near the other.

Returning to our example, the only pole passed in (9) is that at $m = \frac{1}{2}$, residue $-\frac{1}{2}$, whereby

$$S(x) = \tfrac{1}{2}\sqrt{\pi}\, x^{-\frac{1}{2}} + O(x^{-1} e^{-x}), \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{12}$$

The poles passed in (10) are those of $(-m)!$ at $m = 1 + r$ $(r = 0, 1, 2, 3, \ldots,)$, residues $-(-1)^r/r!$, and

$$S(x) = \tfrac{1}{2}\sqrt{\pi}\, e^{-x} \sum_0^\infty \frac{(-1)^r}{(-r-\frac{1}{2})!}(-x)^{-1-r} + O(x^{-\frac{1}{2}})$$

$$= \tfrac{1}{2}\pi^{-\frac{1}{2}} e^{-x} \sum_0^\infty (r-\tfrac{1}{2})!\,(-x)^{-1-r} + O(x^{-\frac{1}{2}}), \qquad |\mathrm{ph}\,(-x)| < \tfrac{1}{2}\pi. \tag{13}$$

The main contribution to (12) has of course no Stokes ray, while those of the series in (13) lie at $|\mathrm{ph}\, x| = \pi$. The function satisfies a second-order linear differential equation, so these two are the only contributing solutions. Hence by rule CB,

$$S(x) = \tfrac{1}{2}\sqrt{\pi}\, x^{-\frac{1}{2}} + \tfrac{1}{2}\pi^{-\frac{1}{2}} e^{-x} \sum_0^\infty (r-\tfrac{1}{2})!(-x)^{-1-r}, \qquad |\mathrm{ph}\, x| < \pi, \tag{14}$$

or through (3)

$$\phi(x) = 1 - \frac{e^{-x^2}}{\pi x} \sum_0^\infty \frac{(r-\frac{1}{2})!}{(-x^2)^r}, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{15}$$

More generally, by far the commonest poles encountered are those of the factorial function, its logarithmic derivative $\Psi(m) = d \ln m!/dm$, the cosecant, and the Riemann zeta function and its generalization $\zeta(m,\alpha) = \sum_0^\infty (s+\alpha)^{-m}$. The following limits enable residues to be calculated at single and double poles of an integrand containing these functions:

$$\lim_{\varepsilon \to 0} (-r-1+\varepsilon)! = \frac{(-1)^r}{r!}\left\{\frac{1}{\varepsilon} + \Psi(r)\right\}, \qquad r = 0, 1, 2, 3, \ldots \tag{16}$$

$$\lim_{\varepsilon \to 0} \Psi(-r-1+\varepsilon)! = -\frac{1}{\varepsilon} + \Psi(r), \qquad r = 0, 1, 2, 3, \ldots \tag{17}$$

$$\lim_{\varepsilon \to 0} \operatorname{cosec} \pi(r + \varepsilon) = \frac{(-1)^r}{\pi\varepsilon}, \qquad r = 0, \pm 1, \pm 2, \ldots \tag{18}$$

$$\lim_{\varepsilon \to 0} \zeta(1 + \varepsilon) = \frac{1}{\varepsilon} + C, \qquad C = 0{\cdot}5772 \ldots \tag{19}$$

$$\lim_{\varepsilon \to 0} \zeta(1 + \varepsilon, \alpha) = \frac{1}{\varepsilon} - \Psi(\alpha - 1). \tag{20}$$

The foregoing procedure is easily adapted to more complicated series than (1), in particular to series which include negative or fractional powers. The two sequences of poles contained in their Mellin representations—those equivalent to the given convergent series in ascending powers of x, and those determining the asymptotic series in descending powers—may then overlap in the m-plane, so that no straight vertical dividing line $\gamma - i\infty$ to $\gamma + i\infty$ can be drawn. In such cases the contour is to be indented to enforce complete separation of the two sequences of poles.

The following examples well illustrate the scope of the method and the special problems which may arise; at the same time they nearly all result in such straightforward expressions that the reader should experience little difficulty in verifying they indeed constitute complete asymptotic expansions according to both the "Poincaré–Watson" and "non-numerical compliance" definitions discussed in the preceding chapter. Only for the comparatively intricate confluent hypergeometric function (Section 5) will we reconfirm the expansions from definition in the text.

3. Incomplete Factorial Function

This function is normally defined by its integral representation

$$(p, x)! = \int_0^x t^p e^{-t}\, dt, \qquad p > -1, \tag{21}$$

but replacement of the exponential by its series and subsequent term by term integration at once gives an equivalent definition as a convergent series,

$$(p, x)!/x^{p+1} = S(x) = \sum_0^\infty (-x)^s / s!(s + p + 1). \tag{22}$$

Incidentally, the error function corresponds to the particular case $\phi(x) = \pi^{-\frac{1}{2}}(-\frac{1}{2}, x^2)!$

If $|x| \gg 1$ the bulk contribution from late terms $s \sim |x|$ is of type e^{-x}, so the sum possesses an essential singularity as $\Re(x) \to -\infty$. In the phase region $\Re(x) < 0$ it is therefore expedient to write

$$S(x) = e^{-x}\bar{S}(x), \qquad \bar{S}(x) = \sum_0^\infty \bar{a}_t x^t,$$

where (Section 9)

$$\bar{a}_t = \sum_0^t \frac{(-1)^s}{(t-s)!\,s!\,(s+p+1)} = \frac{p!}{(t+p+1)!}. \tag{23}$$

The equivalent Mellin integrals can be written down at once from (8) as

$$(p,x)! = \begin{cases} \dfrac{x^{p+1}}{2\pi i}\displaystyle\int \frac{(m-1)!}{p+1-m}\,x^{-m}dm, & |\mathrm{ph}\, x| < \tfrac{1}{2}\pi \quad (24) \\[2ex] \dfrac{p!\,x^{p+1}e^{-x}}{2\pi i}\displaystyle\int \frac{(m-1)!\,(-m)!}{(p+1-m)!}\,(-x)^{-m}dm, & |\mathrm{ph}(-x)| < \tfrac{1}{2}\pi. \quad (25) \end{cases}$$

Moving the path of integration in each to the right past the poles,

$$(p,x)! = \begin{cases} p! + O(x^p e^{-x}), & |\mathrm{ph}\, x| < \tfrac{1}{2}\pi \quad (26) \\[2ex] p!\,x^{p+1}\,e^{-x}\displaystyle\sum_0^\infty \frac{(-1)^r}{(p-r)!}(-x)^{-1-r} + O(1), & |\mathrm{ph}(-x)| < \tfrac{1}{2}\pi. \quad (27) \end{cases}$$

The main contribution to (26) has no Stokes ray since it is a closed function, while those of the series in (27) lie at $|\mathrm{ph}\, x| = \pi$. Hence

$$(p,x)! = p! - [p,x]! \tag{28}$$

where

$$[p,x]! = \int_x^\infty t^p\,e^{-t}dt = \frac{x^p\,e^{-x}}{(-p-1)!}\sum_0^\infty \frac{(r-p-1)!}{(-x)^r}, \qquad |\mathrm{ph}\, x| < \pi. \tag{29}$$

4. Exponential Integral

As a more advanced example we treat the series

$$S(x) = \sum_0^\infty \Psi(s) x^s/s!. \tag{30}$$

In fact

$$S(x) = e^x\left\{\ln x - Ei(-x)\right\}, \qquad -Ei(-x) = \int_x^\infty u^{-1}\,e^{-u}\,du, \tag{31}$$

this exponential integral being a re-definition of the incomplete factorial function for the special case $p = -1$ not covered directly by (21) or (22). $S(x)$ is not the simplest convergent series specifying $-Ei-(x)$, but it admirably illustrates how asymptotic power series can be developed from convergent series containing the Ψ-function.

Since $\Psi(s) = -C + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + s^{-1}$ is slowly varying for large s, if $|x| \gg 1$ the bulk contribution from late terms $s \sim |x|$ in (30) is of type e^x, so the sum possesses an essential singularity as $\Re(x) \to \infty$. Writing

$$S(x) = e^x \bar{S}(x), \qquad \bar{S}(x) = \sum_0^\infty \bar{a}_t x^t,$$

the new coefficients are

$$\bar{a}_t = (-1)^t \sum_0^t \frac{(-1)^s}{(t-s)!\,s!}\,\Psi(s). \tag{32}$$

A general method for evaluating summations of this class is formulated in Section 9. For the particular case (32),

$$\bar{a}_0 = -C, \qquad \bar{a}_{t \neq 0} = -(-1)^t/t!\,t. \tag{33}$$

The Mellin integral representing the original series (30) can be written down at once from (8) as

$$S(x) = (2\pi i)^{-1} \int (m-1)!\,\Psi(-m)\,(-x)^{-m}\,dm, \qquad |\mathrm{ph}(-x)| < \tfrac{1}{2}\pi. \tag{34}$$

In the second representation, the anomalous coefficient $\bar{a}_0$ compels us to make a special evaluation of the contribution to a convergent series from the pole at $m = 0$ in the corresponding result from (33), $\bar{M}(m) = (m-1)!/m$. Near $m = 0$,

$$\frac{(m-1)!}{m}\,x^{-m} = \frac{1}{m^2}\,\{1 + m\Psi(O)\ldots\}\,\{1 - m \ln x \ldots\},$$

giving a contribution to the Mellin integral of

$$\Psi(O) - \ln x = -C - \ln x$$

in place of the correct though anomalous value $\bar{a}_0 = -C$. Hence

$$S(x) = e^x \left[\ln x + \frac{1}{2\pi i} \int \frac{(m-1)!}{m}\,x^{-m}\,dm\right], \qquad |\mathrm{ph}\,x| < \tfrac{1}{2}\pi. \tag{35}$$

Moving the path of integration in (35) and (34) to the right past the poles,

$$S(x) = \begin{cases} e^x \ln x + O(x^{-1}), & |\mathrm{ph}\, x| < \frac{1}{2}\pi, \quad (36) \\ -\sum_0^\infty r!\,(-x)^{-1-r} + O(e^x \ln x), & |\mathrm{ph}(-x)| < \frac{1}{2}\pi. \quad (37) \end{cases}$$

The main contribution to (36) has no Stokes ray since it is a closed function, while those of the series in (37) lie at $|\mathrm{ph}\, x| = \pi$. Hence

$$S(x) = e^x \ln x - \sum_0^\infty r!\,(-x)^{-1-r}, \qquad |\mathrm{ph}\, x| < \pi, \tag{38}$$

or through (31)

$$-Ei(-x) = \frac{e^{-x}}{x} \sum_0^\infty \frac{r!}{(-x)^r}, \qquad |\mathrm{ph}\, x| < \pi. \tag{39}$$

5. Confluent Hypergeometric Function $F(a, c, x)$

This function is still normally defined by its convergent series

$$F(a, c, x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \ldots = \frac{(c-1)!}{(a-1)!} \sum_0^\infty \frac{(s+a-1)!}{s!(s+c-1)!} x^s. \tag{40}$$

If $|x| \gg 1$ the bulk contribution from late terms $s \sim |x|$ is of type e^x, so the sum possesses an essential singularity as $\Re(x) \to \infty$. Writing

$$S(x) = e^x \bar{S}(x), \qquad \bar{S}(x) = \sum_0^\infty \bar{a}_t x^t,$$

the new coefficients are (cf. Section 9)

$$\begin{aligned} \bar{a}_t &= (-1)^t \frac{(c-1)!}{(a-1)!} \sum_0^t \frac{(-1)^s (s+a-1)!}{(t-s)!\,s!\,(s+c-1)!} \\ &= \frac{(-1)^t (c-1)!\,(t+c-a-1)!}{(c-a-1)!\,t!\,(t+c-1)!}. \end{aligned} \tag{41}$$

Incidentally, comparison of (40) and (41) establishes the continuation formula

$$F(a, c, x) = e^x F(c-a, c, -x) \tag{42}$$

originally discovered by Ernst Kummer (1837).

From (8), the equivalent Mellin integrals are

$$F(x) = \begin{cases} \dfrac{(c-1)!}{(c-a-1)!}\dfrac{e^x}{2\pi i}\displaystyle\int \dfrac{(m-1)!\,(-m+c-a-1)!}{(-m+c-1)!}\, x^{-m}\,dm, \\ \hfill |\mathrm{ph}\, x| < \tfrac{1}{2}\pi \qquad (43) \\ \dfrac{(c-1)!}{(a-1)!}\dfrac{1}{2\pi i}\displaystyle\int \dfrac{(m-1)!\,(-m+a-1)!}{(-m+c-1)!}\,(-x)^{-m}dm, \\ \hfill |\mathrm{ph}(-x)| < \tfrac{1}{2}\pi. \qquad (44) \end{cases}$$

Moving the path of integration to the right past the poles, and introducing the reflection formula for factorials,

$$F(x) = \begin{cases} \dfrac{(c-1)!}{(a-1)!}\,\dfrac{x^{a-c}\,e^x}{(-a)!\,(c-a-1)!}\displaystyle\sum_0^\infty \dfrac{(r-a)!\,(r+c-a-1)!}{r!\,x^r} \\ \hfill +\,O(x^{-a}), \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi \qquad (45) \\ \dfrac{(c-1)!}{(c-a-1)!}\,\dfrac{(-x)^{-a}}{(a-1)!\,(a-c)!}\displaystyle\sum_0^\infty \dfrac{(r+a-1)!\,(r+a-c)!}{r!\,(-x)^r} \\ \hfill +\,O(x^{a-c}e^x), \qquad |\mathrm{ph}(-x)| < \tfrac{1}{2}\pi. \qquad (46) \end{cases}$$

The asymptotic series in (45) has a Stokes ray at $\mathrm{ph}\, x = 0$, while those of the series in (46) lie at $\mathrm{ph}\, x = \pm\pi$. Because of the ray at $\mathrm{ph}\, x = 0$, the complete asymptotic expansions are in principle different for the three phase regions $0 < \mathrm{ph}\, x < \pi$, $0 > \mathrm{ph}\, x > -\pi$ and $\mathrm{ph}\, x = 0$, though the differences here amount only to differing readings of $(-x)^{-a}$. With the customary convention $-1 \equiv e^{\mp i\pi}$, $\mathrm{ph}\, x \gtrless 0$, making multivalued fractional powers definite (Chapter I, Section 3),

$$F(a, c, x) = \frac{(c-1)!}{(a-1)!}\,\frac{x^{a-c}\,e^x}{(-a)!\,(c-a-1)!}\sum_0^\infty \frac{(r-a)!\,(r+c-a-1)!}{r!\,x^r}$$
$$+ \begin{pmatrix} e^{i\pi a} \\ e^{-i\pi a} \\ \cos \pi a \end{pmatrix} \frac{(c-1)!}{(c-a-1)!}\,\frac{x^{-a}}{(a-1)!\,(a-c)!}\sum_0^\infty \frac{(r+a-1)!\,(r+a-c)!}{r!\,(-x)^r},$$
$$\left.\begin{matrix} 0 < \mathrm{ph}\, x < \pi \\ 0 > \mathrm{ph}\, x > -\pi \\ \mathrm{ph}\, x = 0 \end{matrix}\right\}. \qquad (47)$$

In reconfirming these expansions directly from our definition of non-numerical compliance, we would first verify that both parts of (47)—independently or together—satisfy

(i) the second-order differential equation in x, $xF'' + (c - x)F' - aF = 0$,

(ii) the second-order difference equation in a, $(c - a)F(a - 1, c, x) + (2a - c + x)F(a, c, x) - aF(a + 1, c, x) = 0$,

(iii) the second-order difference equation in c, $c(c - 1)F(a, c - 1, x) - c(c - 1 + x)F(a, c, x) + (c - a)xF(a, c + 1, x) = 0$.

Since in this example we have to hand as yet no integral representation (Chapter III, Section 4), we cannot at this stage verify the multipliers the easy way, by examining limits $|x| \to \infty$ in the various phase sectors. Instead, we concentrate on special values of the parameters:

(iv) When $-a = 0, 1, 2, \ldots,$ the defining series (40) reduces to a polynomial in x of degree $(-a)$. In the asymptotic expansions (47) the first part is wiped out by $[(a - 1)!]^{-1}$, and the second is the same polynomial written backwards. As expected, this argument cannot distinguish between the phase factors $e^{\mp i\pi a}, \cos \pi a$, these following (Chapter I, Section 3) from *conventional* readings of the otherwise multivalued power $(-x)^{-a}$.

(v) When $a - c = 0, 1, 2, \ldots,$ the defining series (40) is easily summed to e^x times a polynomial in x of degree $a - c$. In (47) the second part is wiped out by $[(c - a - 1)!]^{-1}$, and the first is e^x times the same polynomial written backwards.

There could, however, still be some multiplier with unit period in $a - c$, such as $e^{\pm 2i\pi(a-c)}$ (or their mean $\cos 2\pi(a - c)$), corresponding to a switch in reading from x to $xe^{\pm 2i\pi}$. Such queries are inescapable in asymptotics (in particular), where fractional powers are liable to appear even though none had been involved in the defining function (Chapter I, Section 3). Fortunately, to dispose of such possibilities we have only to glance at the defining series in (say) the simple case of real and positive a, c and x, and note that F is then always real, positive and non-zero; whereas a factor $e^{\pm 2i\pi(a-c)}$ would send the asymptotic expansion complex for non-integral $a - c$, while $\cos 2\pi(a - c)$ would produce negative values and periodic zeros.

Extension to further phase sectors is most easily accomplished through the continuation formula (42).

6. Modified Bessel Function

One series solution to the modified Bessel equation

$$\frac{d^2I}{dx^2} + \frac{1}{x}\frac{dI}{dx} - \left(1 + \frac{p^2}{x^2}\right) I = 0 \tag{48}$$

is easily found to be

$$I_p(x) = (\tfrac{1}{2}x)^p \sum_0^\infty \frac{(\frac{1}{4}x^2)^s}{s!\,(s+p)!}. \tag{49}$$

Provided p is not an integer, a second independent solution is $I_{-p}(x)$.

It is clear from the duplication formula $s!(s-\frac{1}{2})! = \sqrt{\pi}\,4^{-s}(2s)!$ that if $|x| \gg 1$ the contribution from large terms $s \sim |x|$ is of type $\Sigma x^{2s}/(2s! = \cosh x = \frac{1}{2}(e^x + e^{-x})$, so the sum possesses essential singularities as $\Re(x) \to \pm\infty$. Writing

$$I_p(x) = (\tfrac{1}{2}x)^p\, e^{\pm x} S^\pm(x), \qquad S^\pm(x) = \sum_0^\infty a_t^\pm x^t,$$

the new coefficients are (Section 9)

$$a_t^\pm = (\mp 1)^t \sum_0^{\frac{1}{2}t, \frac{1}{2}(t-1)} \frac{(\frac{1}{4})^s}{(t-2s)!\,s!\,(s+p)!} = \frac{4^p}{\sqrt{\pi}} \frac{(\mp 2)^t\,(t+p-\frac{1}{2})!}{t!\,(t+2p)!}. \tag{50}$$

There is no need to pursue this example independently, because comparison with the series (40) defining $F(a, c, x)$ shows that

$$p!\,I_p(x) = (\tfrac{1}{2}x)^p\, e^{\pm x}\, F(p + \tfrac{1}{2}, 2p + 1, \mp 2x). \tag{51}$$

The asymptotic expansion can now simply be written down from (47):

$$I_p(x) = \frac{1}{(2\pi x)^{\frac{1}{2}}\,(p-\frac{1}{2})!\,(-p-\frac{1}{2})!} \left[e^x \sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(2x)^r} \right.$$
$$\left. + \begin{pmatrix} e^{i\pi(p+\frac{1}{2})} \\ e^{-i\pi(p+\frac{1}{2})} \\ -\sin \pi p \end{pmatrix} e^{-x} \sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(-2x)^r} \right] \left\{ \begin{matrix} 0 < \mathrm{ph}\, x < \pi \\ 0 > \mathrm{ph}\, x > -\pi \\ \mathrm{ph}\, x = 0 \end{matrix} \right\}. \tag{52}$$

Extension to further phase sectors is most easily accomplished through the continuation formula

$$I_p(x) = e^{\pm i\pi p} I_p(x\, e^{\mp i\pi}), \tag{53}$$

which immediately follows from the defining series.

7. Logarithm of Factorial

The factorial is commonly defined as the infinite product

$$x! = \prod_{\nu=1}^{\infty} \frac{(1 + 1/\nu)^x}{1 + x/\nu} = e^{-Cx} \prod_{1}^{\infty} \frac{e^{x/\nu}}{1 + x/\nu} \tag{54}$$

where C is the Euler constant 0·5772.... Taking the logarithm,

$$\ln(x!) = -Cx + \sum_{1}^{\infty} \{x/\nu - \ln(1 + x/\nu)\}$$

$$= -Cx + \sum_{2}^{\infty} s^{-1}\zeta(s)(-x)^s, \qquad |x| < 1. \tag{55}$$

This convergent series is free from exponential variation with x, so by (8) the appropriate Mellin transform is

$$M(m) = \zeta(-m)\,(m - 1)!\,(-m - 1)! = -\pi\zeta(-m)/m \sin \pi m\,. \tag{56}$$

The anomalous coefficients a_0 and a_1 in (55) compel us to make special evaluations of the contribution to a convergent series from the poles at $m = 0$ and $m = -1$. Near $m = 0$,

$$-\frac{\pi\,\zeta(-m)\,x^{-m}}{m \sin \pi m} = -\frac{1}{m^2}\{\zeta(0) - m\zeta'(0)\ldots\}\{1 - m \ln x \ldots\},$$

contributing to the Mellin integral a term

$$\zeta'(0) + \zeta(0) \ln x = -\tfrac{1}{2} \ln 2\pi x$$

instead of the correct anomalous coefficient $a_0 = 0$. With $m = -1 + \varepsilon$,

$$-\frac{\pi\,\zeta(-m)\,x^{-m}}{m \sin \pi m} = \frac{x(1 - \varepsilon C\ldots)(1 - \varepsilon \ln x\ldots)}{\varepsilon^2(1 - \varepsilon)},$$

contributing a term $x(1 - C - \ln x)$ instead of the correct anomalous value $a_1 x = -Cx$. Hence

$$\ln(x!) = \tfrac{1}{2} \ln 2\pi x + x(\ln x - 1)$$

$$+ (2\pi i)^{-1} \int \zeta(-m)\,(m - 1)!\,(-m - 1)!\,x^{-m} dm, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{57}$$

Moving the path of integration to the right past the poles at $m = 1, 2, 3, \ldots.,$ and simplifying the residues through the Riemann relation

$$\zeta(-m) = -\frac{m!}{\pi(2\pi)^m}\zeta(m+1)\sin\tfrac{1}{2}\pi m, \tag{58}$$

we obtain the Stirling asymptotic expansion

$$\ln(x!) = \tfrac{1}{2}\ln 2\pi x + x(\ln x - 1) + \frac{1}{\pi}\sum_{1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{1}{2}(m-1)}(m-1)!\,\zeta(m+1)}{(2\pi x)^m}, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{59}†$$

Late terms in the asymptotic power series contained in (59) are all of the same sign and phase when $|\mathrm{ph}\, x| = \frac{1}{2}\pi$, which therefore specify the Stokes rays. Extension to further phase sectors is most easily accomplished through the continuation formula

$$\ln(x!) = -\ln\{(x\,e^{\pm i\pi})!\} + \ln(\pi x/\sin \pi x). \tag{60}$$

This can be deduced directly from the definition (54), since this yields

$$x!(-x)! = \prod_1^{\infty}(1 - x^2/v^2)^{-1}$$

which displays exactly the same poles and residues as $\pi x/\sin \pi x$ and equals it by the Mittag–Leffler theorem (1880).

8. Fermi–Dirac Integral

This is defined by

$$\mathscr{F}_p(x) = \frac{1}{p!}\int_0^{\infty}\frac{t^p dt}{e^{t-x}+1}, \qquad p > -1. \tag{61}$$

Writing

$$(e^{t-x}+1)^{-1} = e^{x-t}(1+e^{x-t})^{-1} = \sum_1^{\infty}(-1)^{s-1}e^{s(x-t)}$$

and integrating term by term leads to a series convergent for negative x,

$$\mathscr{F}_p(x) = \sum_1^{\infty}(-1)^{s-1}e^{xs}/s^{p+1}, \qquad \Re(x) \leqslant 0. \tag{62}$$

† Warning: the commonly quoted range $|\mathrm{ph}\, x| < \pi$ refers to acceptance in the Poincaré approximation.

For the purposes of this chapter and the next, we shall regard (62) and its analytic continuation as specifying the function.

Regarded as a function of e^x, (62) does not behave exponentially. By (8),

$$\mathscr{F}_p(x) = -\frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{-xm}\,dm}{(-m)^{p+1}\sin \pi m}, \qquad |\Im(x)| \leqslant \pi, \tag{63}$$

where $-1 < \gamma < 0$ here because in (62) there is no term $s = 0$ corresponding to a pole at $m = 0$. The restriction on the imaginary part $\Im(x)$ in (63) is of no consequence since it could be avoided by changing the integration variable to (xm).

Moving the path of integration to the right past the poles, those of $1/\sin \pi m$ at $m = 1, 2, 3, \ldots$, give contributions

$$\sum_1^\infty (-1)^m e^{-xm}/(-m)^{p+1} = (-1)^{-p}\mathscr{F}_p(-x), \qquad \Re(x) \geqslant 0. \tag{64}$$

That from the multiple pole at $m = 0$ is most easily evaluated through the expansion

$$\frac{1}{\sin \pi m} = \frac{2}{\pi}\sum_0^\infty t_{2r}\, m^{2r-1}, \qquad |m| < 1, \tag{65}$$

where

$$t_0 = \tfrac{1}{2}, \qquad t_r = \sum_1^\infty (-1)^{s-1}/s^r = (1 - 2^{1-r})\zeta(r). \tag{66}$$

Taking a contour $\int_\infty^{(0-)}$ starting from a point at infinity on the positive real axis, encircling the origin once clockwise, then returning to its starting point, the required contribution is

$$\frac{1}{i\pi}\sum_0^\infty t_{2r}\int_\infty^{(0-)} \frac{e^{-xm}\,dm}{(-m)^{-2r+p+2}} = 2\sum_0^\infty \frac{t_{2r}}{x^{2r-p-1}(-2r+p+1)!} \tag{67}$$

by Hankel's integral representation of the factorial function. This series is asymptotic, rather than convergent, because the range of integration extended beyond the circle of convergence of the power series (65).

Late terms in (67) are all of the same sign and phase when $\operatorname{ph} x = 0, \pm\pi$, so these are Stokes rays. The contribution (64), being expressed in closed analytic form, can exhibit no Stokes discontinuity. Because of the ray at $\operatorname{ph} x = 0$, the complete asymptotic expansions are in principle different for the three phase regions $0 < \operatorname{ph} x < \pi$, $0 > \operatorname{ph} x > -\pi$ and $\operatorname{ph} x = 0$, though the differences amount only to different readings of $(-1)^{-p}$. With the customary convention $-1 \equiv e^{\mp i\pi}$, $\operatorname{ph} x \gtrless 0$, making multivalued fractional powers definite (Chapter I, Section 3),

$$\mathscr{F}_p(x) = \frac{2\sin \pi p}{\pi} \sum_0^\infty \frac{(2r-p-2)!\, t_{2r}}{x^{2r-p-1}}$$
$$+ \begin{pmatrix} e^{i\pi p} \\ e^{-i\pi p} \\ \cos \pi p \end{pmatrix} \mathscr{F}_p(-x), \qquad \left.\begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 \end{matrix}\right\}. \tag{68}$$

9. Evaluation of Summations $\sum_{s=0}^{t} f(s)/(t-s)!$

These summations arise wherever an exponential factor has to be removed from a power series (Section 1). The simplest example of the type is the binomial expansion

$$\sum_0^t \frac{(\pm u)^s}{(t-s)!\, s!} = \frac{(1 \pm u)^t}{t!}, \tag{69}$$

and usually the most expeditious way of evaluating more complicated examples is to reduce the summation to this fundamental form by representing $f(s)$ as some integral with respect to u. For instance, in the summations (6) and (23), namely

$$\bar{a}_t = \sum_0^t \frac{(-1)^s}{(t-s)!\, s!\, (s+\sigma)},$$

we write

$$(s+\sigma)^{-1} = \int_0^1 u^{s+\sigma-1}\, du,$$

whence

$$\bar{a}_t = \int_0^1 u^{\sigma-1} du \sum_0^t \frac{(-u)^s}{(t-s)!\, s!} = \frac{1}{t!}\int_0^1 u^{\sigma-1}(1-u)^t du = \frac{(\sigma-1)!}{(t+\sigma)!}. \tag{70}$$

Similarly, summations of the type (41), i.e.

$$A = \sum_0^t \frac{(-1)^s}{(t-s)!\, s!} \frac{(s+\alpha)!}{(s+\beta)!},$$

can be reduced to the form (69) by writing

$$\frac{(s+\alpha)!}{(s+\beta)!} = \frac{1}{(\beta-\alpha-1)!}\int_0^1 u^{s+\alpha}(1-u)^{\beta-\alpha-1}du\,.$$

Then

$$\begin{aligned}(\beta-\alpha-1)!\,A &= \int_0^1 u^{\alpha}(1-u)^{\beta-\alpha-1}\,du \sum_0^t \frac{(-u)^s}{(t-s)!\,s!}\\ &= \frac{1}{t!}\int_0^1 u^{\alpha}(1-u)^{t+\beta-\alpha-1}\,du\\ &= \frac{\alpha!\,(t+\beta-\alpha-1)!}{t!\,(t+\beta)!}\,.\end{aligned} \tag{71}$$

Note especially the case $\alpha = 0$, where

$$\sum_0^t \frac{(-1)^s}{(t-s)!\,(s+\beta)!} = \frac{1}{(\beta-1)!\,t!\,(t+\beta)}\,. \tag{72}$$

Summations involving the function $\Psi(s) = d\ln s!/ds$, such as (32), can be deduced from (71) by differentiation. For example,

$$\begin{aligned}B = \frac{\partial}{\partial\alpha}A &= \sum_0^t \frac{(-1)^s}{(t-s)!\,s!}\frac{(s+\alpha)!}{(s+\beta)!}\Psi(s+\alpha)\\ &= \frac{\alpha!\,(t+\beta-\alpha-1)!}{(\beta-\alpha-1)!\,t!\,(t+\beta)!}\{\Psi(\beta-\alpha-1)+\Psi(\alpha)-\Psi(t+\beta-\alpha-1)\}\,.\end{aligned} \tag{73}$$

The special case of (73) in which $\alpha = \beta = 0$ requires some care. When $t = 0$, it is clear from both the original summation and the summed form that $B(t=0) = \Psi(0) = -C$. When $t \neq 0$, the factor $(\beta-\alpha-1)!$ in the denominator of (71), which becomes infinite when $\beta \to \alpha$, wipes out any contributions from the last two Ψ's since these remain finite, leaving

$$B(t\neq 0) = \frac{1}{t!\,t}\lim_{\varepsilon\to 0}\frac{\Psi(\varepsilon-1)}{(\varepsilon-1)!} = \frac{1}{t!\,t}\lim_{\varepsilon\to 0}\frac{\varepsilon\,\Psi(\varepsilon-1)}{\varepsilon!} = -\frac{1}{t!\,t}$$

since $\Psi(\varepsilon-1)$ has residue -1 at its poles, in particular at $\varepsilon = 0$.

Similarly,

$$C = -\frac{\partial}{\partial\beta}A = \sum_0^t \frac{(-1)^s}{(t-s)!\,s!}\frac{(s+\alpha)!}{(s+\beta)!}\Psi(s+\beta) = \frac{\alpha!\,(t+\beta-\alpha-1)!}{(\beta-\alpha-1)!\,t!\,(t+\beta)!}$$
$$\times\{\Psi(\beta-\alpha-1)+\Psi(t+\beta)-\Psi(t+\beta-\alpha-1)\}. \tag{74}$$

The summation (50), essentially

$$\bar{a}_t = \sum_0^{\frac{1}{2}t,\frac{1}{2}(t-1)} \frac{(\frac{1}{4})^s}{(t-2s)!\,s!\,(s+p)!}, \tag{75}$$

can be reduced to the form A by means of the duplication and reflection formulae for the factorial function. For since

$$(2\mu)! = \pi^{-\frac{1}{2}}4^\mu\mu!(\mu-\tfrac{1}{2})!, \qquad (-\mu-\tfrac{1}{2})! = \pi/(\mu-\tfrac{1}{2})!\cos\pi\mu,$$
$$\frac{(\frac{1}{4})^s}{(t-2s)!} = \frac{\cos\frac{1}{2}\pi t}{2^t\sqrt{\pi}}\,\frac{(-1)^s(s-\frac{1}{2}t-\frac{1}{2})!}{(\frac{1}{2}t-s)!}.$$

Hence by (71)

$$\bar{a}_t = \frac{\cos\frac{1}{2}\pi t}{2^t\sqrt{\pi}}\,\frac{(-\frac{1}{2}t-\frac{1}{2})!\,(t+p-\frac{1}{2})!}{(\frac{1}{2}t)!\,(\frac{1}{2}t+p)!\,(\frac{1}{2}t+p-\frac{1}{2})!},$$

or, on reintroducing the reflection and duplication formulae,

$$\bar{a}_t = \frac{4^p\,2^t(t+p-\frac{1}{2})!}{\sqrt{\pi}\;t!\,(t+2p)!}. \tag{76}$$

Alternatively, and more directly, we can start by rewriting (75) as

$$\bar{a}_t = \frac{1}{\sqrt{\pi}}\sum_0^{\frac{1}{2}t,\frac{1}{2}(t-1)} \frac{1}{(t-2s)!\,(2s)!}\,\frac{(s-\frac{1}{2})!}{(s+p)!}. \tag{77}$$

Combining the representation

$$\frac{(s-\frac{1}{2})!}{(s+p)!} = \frac{2}{(p-\frac{1}{2})!}\int_0^1 u^{2s}(1-u^2)^{p-\frac{1}{2}}\,du \tag{78}$$

with the summation

$$\sum_0^{\frac{1}{2}t,\frac{1}{2}(t-1)} \frac{u^{2s}}{(t-2s)!\,(2s)!} = \frac{(1+u)^t+(1-u)^t}{2(t!)}, \tag{79}$$

we easily find

$$\bar{a}_t = \frac{1}{\sqrt{(\pi)}\,t!\,(p-\frac{1}{2})!}\int_{-1}^1 (1+u)^{t+p-\frac{1}{2}}(1-u)^{p-\frac{1}{2}}\,du. \tag{80}$$

The integration reproduces (76).

Corresponding summations with an additional quotient of factorials can be evaluated in closed form only if the factorials are inter-related. The following results can be deduced from theorems due to Saalschütz (1891), Dixon (1903), Watson (1924) and Whipple (1925):

$$\sum_0^t \frac{(-1)^s (s+\alpha)!\,(s+\beta)!}{(t-s)!\,s!\,(s+\gamma)!\,(s-t+\alpha+\beta-\gamma+1)!}$$

$$= \frac{(-1)^t \alpha!\,\beta!\,(t+\gamma-\alpha-1)!\,(t+\gamma-\beta-1)!}{(\gamma-\alpha-1)!\,(\gamma-\beta-1)!\,(\alpha+\beta-\gamma+1)!\,t!\,(t+\gamma)!}. \tag{81}$$

$$\sum_0^t \frac{(-1)^s (s+\alpha)!\,(s+\beta)!}{(t-s)!\,s!\,(s+t+\alpha+1)!\,(s+\alpha-\beta)!}$$

$$= \frac{(\frac{1}{2}\alpha-\frac{1}{2})!\,\beta!\,(t+\frac{1}{2}\alpha-\frac{1}{2}-\beta)!}{2(\frac{1}{2}\alpha-\frac{1}{2}-\beta)!\,t!\,(t+\frac{1}{2}\alpha+\frac{1}{2})!\,(t+\alpha-\beta)!}. \tag{82}$$

$$\sum_0^t \frac{(-1)^s (s+\alpha)!\,(s+\beta)!}{(t-s)!\,s!\,(s-t-\alpha-1)!\,(s-t-\beta-1)!}$$

$$= \begin{cases} \dfrac{(\frac{1}{2}t+\alpha)!\,(\frac{1}{2}t+\beta)!\,(t+\alpha+\beta+1)!}{(-\alpha-1)!\,(-\beta-1)!\,(\frac{1}{2}t)!\,(\frac{1}{2}t+\alpha+\beta+1)!}, & t \text{ even} \\ 0, & t \text{ odd.} \end{cases} \tag{83}$$

$$\sum_0^t \frac{(s+\alpha)!\,(s+\beta)!\,(2t-s)!}{(t-s)!\,s!\,(s+\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2})!}$$

$$= \frac{4^t \alpha!\,\beta!\,(t+\frac{1}{2}\alpha)!\,(t+\frac{1}{2}\beta)!}{(\frac{1}{2}\alpha)!\,(\frac{1}{2}\beta)!\,(t+\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2})!} \tag{84}$$

$$\sum_0^t \frac{(-1)^s (s+\alpha)!\,(s+\beta)!}{(t-s)!\,s!\,(s-\frac{1}{2}t+\frac{1}{2}\alpha)!\,(s+2\beta+1)!}$$

$$= \begin{cases} \dfrac{(-1)^{\frac{1}{2}t} \alpha!\,\beta!\,(\beta+\frac{1}{2})!\,(\frac{1}{2}t-\frac{1}{2}\alpha+\beta)!}{2^t (\frac{1}{2}\alpha)!\,(2\beta+1)!\,(-\frac{1}{2}\alpha+\beta)!\,(\frac{1}{2}t)!\,(\frac{1}{2}t+\beta+\frac{1}{2})!}, & t \text{ even} \\ 0, & t \text{ odd}. \end{cases} \tag{85}$$

$$\sum_0^t \frac{(-1)^s (s+\alpha)!\,(s-\alpha-1)!}{(t-s)!\,s!\,(s+\beta)!\,(s-2t-\beta-1)!}$$

$$= \frac{\alpha!\,(-\alpha-1)!\,4^t\,(t+\frac{1}{2}\alpha+\frac{1}{2}\beta)!\,(t-\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{2})!}{\beta!\,(-\beta-1)!\,(\frac{1}{2}\alpha+\frac{1}{2}\beta)!\,(-\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{2})!\,t!} \tag{86}$$

$$\sum_0^t \frac{(-1)^s (s+\alpha)!\,(s+t)!}{(t-s)!\,s!\,(s+\beta)!\,(s+2\alpha-\beta+1)!}$$

$$= \frac{\alpha!\,(\frac{1}{2}\beta)!\,(\frac{1}{2}\beta-\frac{1}{2})!\,(\alpha-\frac{1}{2}\beta+\frac{1}{2})!\,(\alpha-\frac{1}{2}\beta)!}{\beta!\,(2\alpha-\beta+1)!\,(\frac{1}{2}t+\frac{1}{2}\beta)!\,(-\frac{1}{2}t+\frac{1}{2}\beta-\frac{1}{2})!\,(\frac{1}{2}t+\alpha-\frac{1}{2}\beta+\frac{1}{2})!\,(-\frac{1}{2}t+\alpha-\frac{1}{2}\beta)!} \tag{87}$$

There are very few summations of the type

$$\sum_0^t u^s F(s)/(t-s)!$$

which lead to simple functions for general values of u. The following list covers those which can be reduced to functions which have been extensively investigated in the literature. To shorten the list, t-dependent factorials other than $(t-s)!$ have been put in the *numerator* only. Further results can be derived therefrom by such transformations as setting $r = t - s$ and summing over r instead of over s, introducing the reflection formula for factorials, or differentiating or integrating over u with a weighting factor.

Summations from $s = 0$ to $s = t$

Incomplete factorial

$$\sum u^s/s! = e^u[t, u]!/t!$$

Binomial expansion

$$\sum u^s/(t-s)!\,s! = (1+u)^t/t!$$

$$\sum \frac{u^s}{(t-s)!\,(s+m)!} = \frac{1}{u^m}\left\{\frac{(1+u)^{t+m}}{(t+m)!} - \sum_{r=0}^{m-1} \frac{u^r}{(t+m-r)!\,r!}\right\}$$

Tschebychef polynomials

$$\sum \frac{(-u)^s\,(s+t-1)!}{(t-s)!\,s!\,(s-\frac{1}{2})!} = \frac{\cos(2t\sin^{-1}\sqrt{u})}{t\sqrt{\pi}} = \frac{1}{t\sqrt{\pi}}\,T_t(1-2u)$$

$$\sum \frac{(-u)^s\,(s+t)!}{(t-s)!\,s!\,(s+\frac{1}{2})!} = \frac{2}{\sqrt{\pi}}\,\frac{\sin\{(2t+1)\sin^{-1}\sqrt{u}\}}{(2t+1)\sqrt{u}} = \frac{2}{(t+1)\sqrt{\pi}}\,U_t(1-2u)$$

Legendre polynomials

$$\sum \frac{(-u)^s (s+t)!}{(t-s)!\,(s!)^2} = P_t(1-2u)$$

$$\sum \frac{(-u^2)^s (s+t-\frac{1}{2})!}{(t-s)!\; s!\; (s-\frac{1}{2})!} = (-1)^t P_{2t}(u)$$

$$\sum \frac{(-u^2)^s (s+t+\frac{1}{2})!}{(t-s)!\; s!\; (s+\frac{1}{2})!} = \frac{(-1)^t}{u} P_{2t+1}(u)$$

$$\sum \frac{(-u)^s (s+t)!}{(t-s)!\, s!\, (s-m)!} = \left\{\frac{u}{1-u}\right\}^{\frac{1}{2}m} P_t^m(1-2u)$$

$$\sum \frac{(-u^2)^s (s-t+m-\frac{1}{2})!}{(t-s)!\, s!\, (s-\frac{1}{2})!} = \frac{(-4)^t}{(-2)^m (1-u^2)^{\frac{1}{2}m-t}} P_m^{\,m-2t}(u)$$

$$\sum \frac{(-u^2)^s (s-t+m+\frac{1}{2})!}{(t-s)!\, s!\, (s+\frac{1}{2})!} = \frac{(-4)^t}{(-2)^m u(1-u^2)^{\frac{1}{2}m-t}} P_{m+1}^{m-2t}(u)$$

Gegenbauer polynomials

$$\sum \frac{(-u^2)^s (s+t+v-1)!}{(t-s)!\, s!\, (s-\frac{1}{2})!} = \frac{(-1)^t (v-1)!}{\sqrt{\pi}} C_{2t}^{\,v}(u)$$

$$\sum \frac{(-u^2)^s (s+t+v)!}{(t-s)!\; s!\; (s+\frac{1}{2})!} = \frac{(-1)^t (v-1)!}{u\sqrt{\pi}} C_{2t+1}^{v}(u)$$

$$\sum \frac{(-u)^s (s+t+2v-1)!}{(t-s)!\, s!\, (s+v-\frac{1}{2})!} = \frac{(-1)^t (2v-1)!}{(v-\frac{1}{2})!} C_t^{\,v}(2u-1)$$

Jacobi polynomials

$$\sum \frac{(-u)^s (s+t+\alpha-1)!}{(t-s)!\, s!\, (s+\gamma-1)!} = \frac{(t+\alpha-1)!}{t!\,(\gamma-1)!} \mathscr{F}_t(\alpha, \gamma, u)$$

Hypergeometric function

$$\sum \frac{(-u)^s (s+\alpha)!}{(t-s)!\, s!\, (s+\beta)!} = \frac{\alpha!}{t!\,\beta!} F(-t, \alpha+1; \beta+1; u)$$

$$\sum \frac{(s+\alpha)!}{(-u)^s (t-s)!\, s!\, (s+\beta)!} = \frac{(t+\alpha)!}{(-u)^t\, t!\, (t+\beta)!} F(-t, -t-\beta; -t-\alpha; u)$$

$$\sum \frac{1}{(-u)^s (T-s)!\, s!} = \frac{1}{(-u)^t (T-t)!\, t!} F(-t, 1; T-t+1; u)$$

Hermite polynomials

$$\sum \frac{(-\frac{1}{2}u^2)^s}{(t-s)!\,s!\,(s-\frac{1}{2})!} = \frac{1}{(-2)^t\,t!\,(t-\frac{1}{2})!} H_{2t}(u)$$

$$\sum \frac{(-\frac{1}{2}u^2)^s}{(t-s)!\,s!\,(s+\frac{1}{2})!} = \frac{1}{u(-2)^t\,t!\,(t+\frac{1}{2})!} H_{2t+1}(u)$$

Laguerre polynomials; Confluent hypergeometric function

$$\sum \frac{(-u)^s}{(t-s)!\,s!\,(s+\alpha)!} = \frac{1}{(t+\alpha)!} L_t^{(\alpha)}(u) = \frac{1}{t!\,\alpha!} F(-t, \alpha+1, u$$

Spherical Bessel function

$$\sum \frac{(s+t)!}{(2u)^s\,(t-s)!\,s!} = \left(\frac{2u}{\pi}\right)^{\frac{1}{2}} e^u\, K_{t+\frac{1}{2}}(u)$$

Neumann–Lommel polynomials

$$\sum \frac{(s+t-1)!}{(\frac{1}{4}u^2)^s\,(t-s)!} = \frac{u}{t} O_{2t}(u)$$

$$\sum \frac{(s+t)!}{(\frac{1}{4}u^2)^s\,(t-s)!} = \frac{u^2}{2t+1} O_{2t+1}(u)$$

$$\sum \frac{(s+t-1)!\,s!}{u^{2s}\,(t-s)!\,(s-\frac{1}{2})!} = \frac{u^2}{t\sqrt{\pi}} \Omega_t(u)$$

Stirling numbers of the first kind

$$\sum u^s S_t^s = u!/(u-t)!$$

Bernoulli polynomials

$$\sum \frac{1}{u^s\,(t-s)!\,s!} B_s = \frac{1}{u^t\;t!} B_t(u)$$

Euler polynomials

$$\sum \frac{1}{(2u)^s\,(t-s)!\,s!} E_s = \frac{1}{(u+\frac{1}{2})^t\,t!} E_t(u+\frac{1}{2})$$

Summations from $s = 0$ to $s = \frac{1}{2}t$ or $\frac{1}{2}(t-1)$

Binomial expansion

$$\sum \frac{u^{2s}}{(t-2s)!\,(2s)!} = \frac{(1+u)^t + (1-u)^t}{2\,t!}$$

$$\sum \frac{u^{2s}}{(t-2s)!\,(2s+1)!} = \frac{(1+u)^{t+1} - (1-u)^{t+1}}{2u\;(t+1)!}$$

Hermite polynomials

$$\sum \frac{1}{(-2u^2)^s\,(t-2s)!\,s!} = \frac{1}{u^t\;t!} H_t(u)$$

Bernoulli polynomials

$$\sum \frac{\zeta(2s)}{(-4\pi^2u^2)^s\,(t-2s)!} = -\frac{1}{4u\,(t-1)!} - \frac{1}{2u^t\;t!} B_t(u)$$

Tschebychef polynomials

$$\sum \frac{(t-s)!}{(-4u^2)^s\,(t-2s)!\,s!} = \frac{1}{(2u)^t} U_t(u)$$

$$\sum \frac{(t-s-1)!}{(-4u^2)^s\,(t-2s)!\,s!} = \frac{2}{(2u)^t\,t} T_t(u)$$

Gegenbauer polynomials

$$\sum \frac{(t-s+\nu)!}{(-4u^2)^s\,(t-2s)!\,s!} = \frac{\nu!}{(2u)^t} C_t^{\nu+1}(u)$$

Legendre polynomials

$$\sum \frac{(t-s-\frac{1}{2})!}{(-4u^2)^s\,(t-2s)!\,s!} = \frac{\sqrt{\pi}}{(2u)^t} P_t(u)$$

$$\sum \frac{(t-s+\nu-\frac{1}{2})!}{(-4u^2)^s\,(t-2s)!\,s!} = \frac{\sqrt{\pi}}{(2u)^t\,\{-2\,(1-u^2)^{\frac{1}{2}}\}^\nu} P_{t+\nu}^{\nu}(u)$$

Lommel polynomials

$$\sum \frac{(-\frac{1}{4}u^2)^s\,(t-s)!\,(t-s+\nu-1)!}{(t-2s)!\,s!\,(s+\nu-1)!} = (\tfrac{1}{2}u)^t\,R_{t,\nu}(u)$$

Exercises

1. According to the fundamental theorem in Mellin transforms,

$$F(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} M(m)x^{-m}dm, \qquad M(m) = \int_0^\infty F(x)x^{m-1}\,dx.$$

Evaluating $M(m)$ by integrating over x first and u second, show that the Mellin representation of the integral

$$I(x) = \int_0^\infty e^{-xu(1+u)^{1/3}}\,du$$

is

$$I(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(\frac{4}{3}m-2)!\,(m-1)!}{(\frac{1}{3}m-1)!}(-m)!\,x^{-m}dm, \qquad |\mathrm{ph}\,x| < \pi,$$

where $\frac{3}{4} < \gamma < 1$.

Determining the residues at the poles of $(-m)!$, derive the asymptotic series

$$I(x) = -\sum_0^\infty \left(\frac{4r-2}{3}\right)!\Big/\left(\frac{r-2}{3}\right)!\,(-x)^{r+1}, \qquad |\mathrm{ph}\,x| < \pi.$$

Determining the residues at the poles of $(\frac{4}{3}m-2)!$ and $(m-1)!$, derive the absolutely convergent expansion

$$I(x) = \tfrac{3}{4}\sum_{s\neq 1,5,9,\ldots}\left(\frac{s-1}{4}\right)!\,x^{\frac{3}{4}(s-1)}/s!$$

$$-\sum_{s\neq 0,3,6,\ldots}(\tfrac{1}{3}s)!\,x^s/(\tfrac{4}{3}s+1)! - \tfrac{1}{2}\sum_0^\infty s!\,x^{3s}/(4s+1)!.$$

2. Show that the Mellin representation of the integral

$$I(x) = \int_0^\infty \frac{(x^2+u)^{\frac{1}{2}}-x}{e^u-1}\,du = \int_0^\infty \frac{e^{-\sinh^{-1}(xu^{-1/2})}}{e^u-1}\,u^{\frac{1}{2}}\,du$$

is

$$I(x) = -\frac{1}{4\pi^{3/2}i}\int_{\gamma-i\infty}^{\gamma+i\infty}(m-1)!\,(m+\tfrac{1}{2})!\,\zeta(m+\tfrac{3}{2})\,(-m-\tfrac{3}{2})!\,x^{-2m}dm,$$

$$|\mathrm{ph}\,x| < \tfrac{1}{2}\pi,$$

where $0 < \gamma < \frac{1}{2}$.

Derive the asymptotic series

$$I(x) = \frac{1}{2\sqrt{\pi}\,x}\sum_0^\infty \frac{(r-\frac{1}{2})!\,\zeta(r+2)}{(-x^2)^r}, \qquad |\mathrm{ph}\,x| < \tfrac{1}{2}\pi,$$

and the absolutely convergent expansion

$$I(x) = 2x(\ln 2x - 1) + \tfrac{1}{2}\sqrt{(\pi)}\sum_0^\infty \{\zeta(\tfrac{3}{2}-s)\,x^{2s}/s! - \zeta(-s)\,x^{2s+3}/(s+\tfrac{3}{2})!\}.$$

3. The following class of convergent series has been extensively discussed by W. B. Ford (1916, 1928, 1936):

$$S(x) = \sum_0^\infty a_s x^s, \qquad a_s = \sum_0^\infty A_\mu/(s+\beta+\mu)!.$$

Show that if $|x| \gg 1$ the bulk contribution from late terms is of type e^x. Derive the alternative series

$$S(x) = e^x \sum_0^\infty \bar{a}_t x^t, \qquad \bar{a}_t = \frac{(-1)^t}{t!}\sum_0^\infty \frac{A_\mu}{(\beta+\mu-1)!\,(t+\beta+\mu)}.$$

Hence establish the Mellin representation

$$S(x) = \frac{e^x}{2\pi i}\sum_0^\infty \frac{A_\mu}{(\beta+\mu-1)!}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(m-1)!}{\beta+\mu-m}\,x^{-m}\,dm, \qquad |\mathrm{ph}\,x| < \tfrac{1}{2}\pi.$$

Moving the path of integration to the right past the poles, prove that the leading series when $|\mathrm{ph}\,x| < \frac{1}{2}\pi$ is

$$S(x) \sim e^x \sum_0^\infty A_\mu\, x^{-\beta-\mu}.$$

4. Prove that in Ford's class of series the leading series when $|\mathrm{ph}(-x)| < \frac{1}{2}\pi$ are

$$S(x) \sim \sum_1^\infty a_{-m}x^{-m} - \sum \{\text{residues at poles of } a_{-m} \text{ of } \pi a_{-m}/(-x)^m \sin \pi m\},$$

$$|\mathrm{ph}(-x)| < \tfrac{1}{2}\pi.$$

(The results of questions 3 and 4 cannot be combined to yield a complete asymptotic expansion without further knowledge of the form of the coefficients A_μ, and hence a_s. For a start, the Stokes rays of each of the component series would need to be determined).

5. Given the asymptotic expansion for $F(a, c, x)$ when $\operatorname{ph} x = 0$, show from the Kummer continuation formula that for real positive x

$$F(a, c, -x) = \frac{x^{-a}(c-1)!}{(a-1)!\,(a-c)!\,(c-a-1)!}\sum_0^\infty \frac{(r+a-1)!\,(r+a-c)!}{r!\,x^r}$$

$$+\frac{x^{a-c}e^{-x}(c-1)!\cos\pi(a-c)}{(-a)!\,(c-a-1)!\,(a-1)!}\sum_0^\infty \frac{(r-a)!\,(r+c-a-1)!}{r!\,(-x)^r}.$$

6. Find from the Kummer formula the asymptotic expansion of $F(a, c, x)$ in the phase sector $\pi < \operatorname{ph} x < 2\pi$.

7. Derive the following Mellin representations for modified Bessel functions:

$$I_p(x) = \begin{cases} \dfrac{(2x)^p e^{\pm x}}{\sqrt{\pi}}\dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} \frac{(m-1)!\,(-m+p-\frac{1}{2})!}{(-m+2p)!}(\pm 2x)^{-m}\,dm, \\ \qquad\qquad\qquad\qquad |\operatorname{ph}(\pm x)| < \frac{1}{2}\pi \\ \dfrac{e^x}{(2\pi x)^{\frac{1}{2}}}\dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} \frac{(m-1)!\,(-m+p-\frac{1}{2})!}{(m+p-\frac{1}{2})!}(2x)^m\,dm. \end{cases}$$

8. The Bessel function of the third kind is defined by

$$K_p(x) = \frac{\pi}{2\sin\pi p}\left(I_{-p}(x) - I_p(x)\right).$$

Show that its asymptotic expansion is

$$K_p(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \frac{e^{-x}}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!}\sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(-2x)^r},$$

$$|\operatorname{ph} x| < \pi.$$

9. Establish the Mellin representation

$$K_p(x) = (2\pi x)^{-\frac{1}{2}}e^{-x}\cos\pi p\ (2\pi i)^{-1}\int_{-i\infty}^{i\infty}(m-1)!\,(-m+p-\tfrac{1}{2})!$$

$$\times\,(-m-p-\tfrac{1}{2})!\,(2x)^m\,dm, \qquad |\operatorname{ph} x| < \tfrac{3}{2}\pi,$$

excepting when p is half an odd integer.

10. The function exponentially increasing when $x > 0$, analogous to the then exponentially decreasing $K_p(x)$, is

$$\mathscr{K}_p(x) = \tfrac{1}{2}\pi\left(I_{-p}(x) + I_p(x)\right).$$

Show that its asymptotic expansion is

$$\mathscr{K}_p(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \frac{1}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!} \left[e^x \sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(2x)^r} \right.$$

$$\left. + \begin{pmatrix} i \\ -i \\ 0 \end{pmatrix} \cos \pi p \; e^{-x} \sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(-2x)^r} \right] \left\{ \begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 . \end{matrix} \right\}.$$

11. The ordinary Bessel functions of the first and second kind, J and Y, are related to K through

$$J_p(x) - iY_p(x) = 2i\pi^{-1} e^{\frac{1}{2}i\pi p} K_p(xe^{\frac{1}{2}i\pi}).$$

Show that their asymptotic expansions are

$$J_p(x) = R\cos\phi - S\sin\phi, \qquad Y_p(x) = R\sin\phi + S\cos\phi, \qquad |\operatorname{ph} x| < \tfrac{1}{2}\pi,$$

where $\phi = x - \frac{1}{2}\pi p - \frac{1}{4}\pi$ and

$$R = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \frac{1}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!} \sum_{0,2,4,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}r}\,(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(2x)^r},$$

$$S = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \frac{1}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!} \sum_{1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(r-1)}\,(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(2x)^r}.$$

12. Given the asymptotic expansion of the modified Bessel function $I_p(x)$ in the sector $0 < \operatorname{ph} x < \pi$, show from the continuation formula that

$$I_p(x) = \frac{(2\pi x)^{-\frac{1}{2}}}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!} \left\{ e^{i\pi(p+\frac{1}{2})}\, e^{-x} \sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(-2x)^r} \right.$$

$$\left. + \, e^{2i\pi(p+\frac{1}{2})}\, e^x \sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(2x)^r} \right\}, \qquad \pi < \operatorname{ph} x < 2\pi.$$

13. Two independent solutions to the differential equation $d^2y/dz^2 = zy$ are the important Airy functions

$$Ai(z) = \tfrac{1}{3}z^{\frac{1}{2}}\{I_{-1/3}(\tfrac{2}{3}z^{3/2}) - I_{1/3}(\tfrac{2}{3}z^{3/2})\} = \frac{z^{\frac{1}{2}}}{\pi\sqrt{3}} K_{1/3}(\tfrac{2}{3}z^{3/2}),$$

$$Bi(z) = (\tfrac{1}{3}z)^{\frac{1}{2}}\{I_{-1/3}(\tfrac{2}{3}z^{3/2}) + I_{1/3}(\tfrac{2}{3}z^{3/2})\} = \frac{2z^{\frac{1}{2}}}{\pi\sqrt{3}} \mathscr{K}_{1/3}(\tfrac{2}{3}z^{3/2}).$$

Introducing known results for the modified Bessel functions, prove that the asymptotic expansions of the Airy functions are

$$Ai(z) = z^{-1/4}E_-\,, \qquad |\mathrm{ph}\,z| < \tfrac{2}{3}\pi$$

$$Bi(z) = z^{-1/4}\left[2E_+ + \begin{pmatrix} i \\ -i \\ 0 \end{pmatrix} E_-\right], \qquad \left.\begin{matrix} 0 < \mathrm{ph}\,z < \tfrac{2}{3}\pi \\ 0 > \mathrm{ph}\,z > -\tfrac{2}{3}\pi \\ \mathrm{ph}\,z = 0 \end{matrix}\right\},$$

where

$$E_\pm(z) = \frac{e^{\pm\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}\,(-\tfrac{1}{6})!\,(-\tfrac{5}{6})!}\sum_0^\infty \frac{(r-\tfrac{1}{6})!\,(r-\tfrac{5}{6})!}{r!\,(\pm\tfrac{4}{3}z^{3/2})^r}\,.$$

14. Combining the results of questions (12) and (13), prove that

$$Ai(z) = z^{-\frac{1}{4}}(E_- + iE_+)\,, \qquad Bi(z) = z^{-\frac{1}{4}}(E_+ + iE_-)\,, \qquad \tfrac{2}{3}\pi < \mathrm{ph}\,z < \tfrac{4}{3}\pi\,.$$

15. Show that for negative variable the asymptotic expansions of Ai and Bi can be written

$$Ai(-z) = P\sin\Psi - Q\cos\Psi\,, \qquad Bi(-z) = P\cos\Psi + Q\sin\Psi\,,$$

$$|\mathrm{ph}\,z| < \tfrac{1}{3}\pi\,,$$

where $\Psi = \frac{2}{3}z^{\frac{3}{2}} + \frac{1}{4}\pi$ and

$$P = \frac{z^{-\frac{1}{4}}}{\sqrt{\pi}}\frac{1}{(-\tfrac{1}{6})!\,(-\tfrac{5}{6})!}\sum_{0,2,4,\ldots}^\infty \frac{(-1)^{\frac{1}{2}r}(r-\tfrac{1}{6})!\,(r-\tfrac{5}{6})!}{r!\,(\tfrac{4}{3}z^{\frac{3}{2}})^r}\,,$$

$$Q = \frac{z^{-\frac{1}{4}}}{\sqrt{\pi}}\frac{1}{(-\tfrac{1}{6})!\,(-\tfrac{5}{6})!}\sum_{1,3,5,\ldots}^\infty \frac{(-1)^{\frac{1}{2}(r-1)}(r-\tfrac{1}{6})!\,(r-\tfrac{5}{6})!}{r!\,(\tfrac{4}{3}z^{\frac{3}{2}})^r}\,.$$

16. Show from the continuation formula that on crossing the Stokes rays $\mathrm{ph}\,x = \pm\frac{1}{2}\pi$ the formal discontinuities in the asymptotic expansion for $\ln(x!)$ are respectively $-\ln(1 - e^{\pm 2i\pi x})$. Verify that each occurs at that phase for which the fractional increment is least, and that this increment is then $\frac{1}{2}\pi$ out of phase with the component asymptotic power series. Prove that on the Stokes rays

$$\ln(\pm iy)! = \tfrac{1}{2}\ln(\pi y/\sinh \pi y) \pm i[y(\ln y - 1) + \tfrac{1}{4}\pi$$

$$- \pi^{-1}\sum_{1,3,5,\ldots}^\infty \zeta(m+1)\,(m-1)!/(2\pi y)^m]\,,$$

y being real and positive.

17. The Bose–Einstein integral is defined by

$$\mathscr{B}_p(x) = \frac{1}{p!}\int_0^\infty \frac{t^p dt}{e^{t-x} - 1},$$

taking the principal value when x is positive.

Prove that a convergent series is

$$\mathscr{B}_p(x) = \sum_1^\infty e^{xs}/s^{p+1}, \qquad \Re(x) < 0.$$

Starting from the definition, show by integrating first over x and then over t that the Mellin representation is

$$\mathscr{B}_p(x) = \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{-xm}\,dm}{(-m)^{p+1}\tan \pi m}, \qquad |\Im(x)| \leqslant \pi,$$

where $-1 < \gamma < 0$. Introducing the series

$$1/\tan \pi m = -2\pi^{-1}\sum_0^\infty \zeta(2r)\, m^{2r-1}, \qquad |m| < 1,$$

derive the asymptotic expansion

$$\mathscr{B}_p(x) = \frac{2\sin \pi p}{\pi}\sum_0^\infty \frac{(2r-p-2)!\,\zeta(2r)}{x^{2r-p-1}}$$

$$+ \begin{pmatrix} e^{i\pi p} \\ e^{-i\pi p} \\ \cos \pi p \end{pmatrix} \mathscr{B}_p(-x), \qquad \left.\begin{matrix} 0 < \mathrm{ph}\, x < \pi \\ 0 > \mathrm{ph}\, x > -\pi \\ \mathrm{ph}\, x = 0 \end{matrix}\right\}.$$

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Chapter III

Conversion of Power Series into Integral Representations

1. Conversion Procedure

The methods of the preceding chapter, based on Mellin representations, afford the most direct derivation of asymptotic power series for a function defined by its convergent series. If more involved forms of asymptotic expansion are required, it is better to convert the power series into an integral representation of the Laplace type, which does not involve any integration over factorial functions. Then the powerful methods to be developed in Chapters IV–XI can be applied to yield asymptotic expansions of all degrees of generality and complexity.

The principles to be followed in effecting this conversion are simple. Suppose the given power series is

$$S(x) = \sum_0^\infty a_s x^s, \tag{1}$$

and it is desired to find an integral representation for $S(x)$ which involves in its integrand some familiar series

$$S^0(x) = \sum_0^\infty a_s{}^0 x^s. \tag{2}$$

Then, following a procedure somewhat similar to that adopted in Section 9 of the preceding chapter, we have only to express the quotient $a_s/a_s{}^0$ as some integral over a variable u of $(\pm u)^s$, i.e. introduce the "sub-representation"

$$a_s/a_s{}^0 = U(\pm u)^s \tag{3}$$

where U is the appropriate integral operator, and (1) becomes

$$S(x) = U \sum_0^\infty a_s{}^0(\pm ux)^s = U\, S^0(\pm ux). \tag{4}$$

To facilitate this routine, two tables are given at the end of this chapter, one of convergent power series for familiar functions $S^0(x)$, the other of integral operators U for quotients a_s/a_s^0.

2. Incomplete Factorial Function

Starting from the convergent series II (22), namely

$$\frac{(p, x)!}{x^{p+1}} = S(x) = \sum_0^\infty \frac{(-x)^s}{s!\,(s+p+1)}, \tag{5}$$

we can choose as familiar function

$$S^0(x) = \sum_0^\infty (-x)^s/s! = e^{-x}.$$

The sub-representation needed is

$$(s+p+1)^{-1} = \int_0^1 u^{s+p}\,du.$$

With these,

$$S(x) = \int_0^1 u^p\,S^0(ux)\,du = \int_0^1 u^p\,e^{-ux}\,du, \qquad \Re(p) > -1,$$

so

$$(p, x)! = \int_0^x t^p\,e^{-t}\,dt, \qquad [p, x]! = \int_x^\infty t^p\,e^{-t}\,dt = p! - (p, x)!,$$
$$\Re(p) > -1. \tag{6}$$

Alternatively we could regard (5) as

$$S(x) = \sum_0^\infty \frac{(-x)^s\,(s+p)!}{s!\,(s+p+1)!}$$

and choose as familiar function

$$S^0(x) = \sum_0^\infty \frac{(-x)^s}{s!\,(s+p+1)!} = x^{-\frac{1}{2}p-\frac{1}{2}}\,J_{p+1}(2x^{\frac{1}{2}}).$$

We then need the sub-representation

$$(s+p)! = \int_0^\infty u^{s+p}\,e^{-u}\,du.$$

With these,

$$S(x) = \int_0^\infty u^p\, e^{-u}\, S^0(ux)\, du,$$

so

$$(p, x)! = x^{\frac{1}{2}p+\frac{1}{2}} \int_0^\infty u^{\frac{1}{2}p-\frac{1}{2}}\, e^{-u}\, J_{p+1}[2(ux)^{\frac{1}{2}}]\, du. \tag{7}$$

Starting from the other convergent series II (23),

$$\frac{e^x\,(p, x)!}{p!\,x^{p+1}} = S(x) = \sum_0^\infty \frac{x^s}{(s+p+1)!}, \tag{8}$$

we can choose

$$S^0(x) = \sum_0^\infty x^s/s! = e^x, \qquad \frac{s!}{(s+p+1)!} = \frac{1}{p!}\int_0^1 u^s(1-u)^p\, du.$$

Then

$$S(x) = \frac{1}{p!}\int_0^1 (1-u)^p\, e^{ux}\, du,$$

leading again to (6).

Alternatively we could regard (8) as

$$S(x) = \sum_0^\infty \frac{x^s\; s!}{s!\,(s+p+1)!}$$

and choose

$$S^0(x) = \sum_0^\infty \frac{x^s}{s!\,(s+p+1)!} = x^{-\frac{1}{2}p-\frac{1}{2}}\, I_{p+1}(2x^{\frac{1}{2}}), \qquad s! = \int_0^\infty u^s\, e^{-u}\, du.$$

With these,

$$(p, x)! = p!\, x^{\frac{1}{2}p+\frac{1}{2}}\, e^{-x} \int_0^\infty u^{-\frac{1}{2}p-\frac{1}{2}}\, e^{-u}\, I_{p+1}[2(ux)^{\frac{1}{2}}]\, du. \tag{9}$$

Starting from the asymptotic series II (29),

$$\frac{(-p-1)!\,e^x[p, x]!}{x^p} = S(x) = \sum_0^\infty \frac{(s-p-1)!}{(-x)^s}, \qquad |\mathrm{ph}\, x| < \pi, \tag{10}$$

we can choose

$$S^0(x) = \sum_0^\infty (-x)^{-s} = x/(1+x), \qquad (s-p-1)! = \int_0^\infty u^{s-p-1} e^{-u} \, du .$$

With these,

$$[p, x]! = \frac{x^{p+1} e^{-x}}{(-p-1)!} \int_0^\infty \frac{u^{-p-1} e^{-u} \, du}{u+x}, \qquad \Re(p) < 0, \qquad |\mathrm{ph}\, x| < \pi . \quad (11)$$

3. Exponential Integral

Starting from the convergent series II (30),

$$S(x) = \sum_0^\infty \Psi(s) \, x^s/s! = -C e^x + \sum_0^\infty [\Psi(s) + C] \, x^s/s! , \tag{12}$$

we choose

$$S^0(x) = \sum_0^\infty x^s/s! = e^x, \qquad \Psi(s) + C = \int_0^1 \frac{1-u^s}{1-u} \, du .$$

With these,

$$S(x) = -C e^x + \int_0^1 \frac{e^x - e^{ux}}{1-u} \, du ,$$

corresponding to

$$-Ei(-x) = -C - \ln x + \int_0^1 (1 - e^{-vx}) \, v^{-1} \, dv . \tag{13}$$

Combining this with a well-known integral representation for $C + \ln x$, namely

$$C + \ln x = \int_0^\infty \left(\frac{1}{1+v} - e^{-vx} \right) \frac{dv}{v}, \qquad \Re(x) > 0 ,$$

(13) reduces to

$$-Ei(-x) = \int_1^\infty \frac{e^{-vx}}{v} \, dv , \qquad \Re(x) > 0 , \tag{14}$$

which is in fact the customary definition of the exponential integral.

Starting from the other convergent series II (33),

$$S(x) = \sum_1^\infty (-x)^s/s!\,s\,, \tag{15}$$

we choose

$$S^0(x) = \sum_1^\infty (-x)^s/s! = e^{-x} - 1\,, \qquad s^{-1} = \int_0^1 u^{s-1}\,du\,.$$

Then

$$S(x) = \int_0^1 (e^{-ux} - 1)\,u^{-1}\,du\,,$$

leading again to (13).

Alternatively we could regard (15) as

$$S(x) = \sum_1^\infty (s-1)!\,(-x)^s/(s!)^2\,,$$

and choose

$$S^0(x) = \sum_1^\infty (-x)^s/(s!)^2 = J_0(2x^{\frac{1}{2}}) - 1\,, \qquad (s-1)! = \int_0^\infty u^{s-1}\,e^{-u}\,du\,.$$

Then

$$-Ei(-x) = -C - \ln x - \int_0^\infty \{J_0[2(ux)^{\frac{1}{2}}] - 1\}\,u^{-1}\,e^{-u}\,du\,. \tag{16}$$

The asymptotic series II (39) is simply (10) with $p = -1$, so (11) reduces to

$$-Ei(-x) = [-1, x]! = e^{-x}\int_0^\infty \frac{e^{-u}\,du}{u+x} = \int_x^\infty \frac{e^{-v}\,dv}{v}\,, \qquad |\mathrm{ph}\,x| < \pi\,, \tag{17}$$

which is (14) again.

4. Confluent Hypergeometric Function

The theory is similar to that for the incomplete factorial function, which indeed represents the special case

$$(p+1)\,(p, x)! = x^{p+1}\,e^{-x}\,F(1, p+2, x)\,. \tag{18}$$

Starting from the convergent series II (40) and choosing

$$S^{\circ}(x) = \sum_{0}^{\infty} \frac{x^s}{s!} = e^x,$$

$$\frac{(s+a-1)!}{(s+c-1)!} = \frac{1}{(c-a-1)!} \int_0^1 u^{s+a-1}(1-u)^{c-a-1}\, du,$$

we have

$$F(a, c, x) = \frac{(c-1)!}{(a-1)!\,(c-a-1)!} \int_0^1 u^{a-1}\,(1-u)^{c-a-1}\, e^{ux}\, du. \tag{19}$$

Incidentally, a change of variable in this integral representation to $v = 1 - u$ leads to the Kummer continuation formula

$$F(a, c, x) = e^x\, F(c-a, c, -x). \tag{20}$$

Alternatively we can choose

$$S^0(x) = \sum_{0}^{\infty} x^s/s!\,(s+c-1)! = x^{\frac{1}{2}-\frac{1}{2}c}\, I_{c-1}(2x^{\frac{1}{2}}),$$

$$(s+a-1)! = \int_0^{\infty} u^{s+a-1}\, e^{-u}\, du,$$

and find

$$F(a, c, x) = \frac{(c-1)!}{(a-1)!} x^{\frac{1}{2}-\frac{1}{2}c} \int_0^{\infty} u^{a-\frac{1}{2}c-\frac{1}{2}}\, e^{-u}\, I_{c-1}[2(ux)^{\frac{1}{2}}]\, du. \tag{21}$$

In this representation, let the phase of x be increased by π through the relation $I_{c-1}(iy) = i^{c-1} J_{c-1}(y)$. Transforming back to $F(a, c, x)$ via the Kummer relation (20) leads to the variant

$$F(a, c, x) = \frac{(c-1)!}{(c-a-1)!} x^{\frac{1}{2}-\frac{1}{2}c}\, e^x \int_0^{\infty} u^{\frac{1}{2}c-a-\frac{1}{2}}\, e^{-u}\, J_{c-1}[2(ux)^{\frac{1}{2}}]\, du. \tag{22}$$

Starting from the asymptotic expansion II (47), we can choose in the second series

$$S^0(x) = \sum_{0}^{\infty} (s+a-c)!/s!\,(-x)^s = (a-c)!\,(1+x^{-1})^{c-a-1}$$

and introduce the sub-representation for $(s + a - 1)!$. With these,

$$\psi(a, c, x) = \frac{1}{(a-1)!\,(a-c)!}\sum_0^\infty \frac{(s+a-1)!\,(s+a-c)!}{s!\,(-x)^s}$$

$$= \frac{1}{(a-1)!}\int_0^\infty u^{a-1}\left(1+\frac{u}{x}\right)^{c-a-1} e^{-u}\,du, \qquad |\mathrm{ph}\,x| < \pi. \tag{23}$$

Incidentally, this form establishes the important identity

$$\psi(a, c, x) = \psi(a - c + 1, 2 - c, x). \tag{24}$$

The complete asymptotic expansion II (47) can therefore be represented as follows:

$$F(a, c, x) = \frac{(c-1)!}{(a-1)!}x^{a-c}\,e^x\,\psi(1-a, 2-c, -x)$$
$$+\begin{pmatrix} e^{i\pi a} \\ e^{-i\pi a} \\ \cos \pi a \end{pmatrix}\frac{(c-1)!}{(c-a-1)!}\,x^{-a}\,\psi(a, c, x), \qquad \left.\begin{matrix} 0 < \mathrm{ph}\,x < \pi \\ 0 > \mathrm{ph}\,x > -\pi \\ \mathrm{ph}\,x = 0 \end{matrix}\right\}, \tag{25}$$

where at $\mathrm{ph}\,x = 0$ the principal value of the integral $\psi(1-a, 2-c, -x)$ is to be understood. We shall see in the next chapter that this powerful representation (25) also follows from (19).

Solution of (25) for $\psi(a, c, x)$, followed by insertion of the Kummer formula, leads to the important result

$$\psi(a, c, x) = \frac{(-c)!}{(a-c)!}x^a\,F(a, c, x) + \frac{(c-2)!}{(a-1)!}x^{a-c+1}\,F(a-c+1, 2-c, x). \tag{26}$$

This enables us to deduce further integrals for ψ from those already derived for F. In particular, (21) yields

$$\psi(a, c, x) = \frac{4\,x^{a-\frac{1}{2}c+\frac{1}{2}}}{(a-1)!\,(a-c)!}\int_0^\infty v^{2a-c}\,e^{-v^2}\,K_{c-1}(2v\sqrt{x})\,dv \tag{27}$$

where $K_p = (\frac{1}{2}\pi/\sin \pi p)\,(I_{-p} - I_p)$ is the Bessel function of the third kind, or modified Hankel function. Moreover, application to (22) leads to the following representation which will prove most valuable in Chapter VIII, Section 8:

$$\psi(a, c, x) = 4\,x^{a-\frac{1}{2}c+\frac{1}{2}}\,e^x\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} v^{c-2a}\,e^{v^2}\,K_{c-1}(2v\sqrt{x})\,dv. \tag{28}$$

5. Modified Bessel Function

The relation II (51), namely

$$p!\,I_p(x) = (\tfrac{1}{2}x)^p\,e^{\pm x}\,F(p+\tfrac{1}{2}, 2p+1, \mp 2x), \tag{29}$$

enables us to specialize all the results of the preceding section to the modified Bessel function. Thus (19) reduces to the "Poisson integrals":

$$\begin{aligned} I_p(x) &= \frac{(2x)^p\,e^{-x}}{\sqrt{\pi}\,(p-\frac{1}{2})!}\int_0^1 (u-u^2)^{p-\frac{1}{2}}\,e^{2ux}\,du \\ &= \frac{(\frac{1}{2}x)^p}{\sqrt{\pi}\,(p-\frac{1}{2})!}\int_{-1}^1 (1-v^2)^{p-\frac{1}{2}}\,e^{-vx}\,dv \\ &= \frac{(\frac{1}{2}x)^p}{\sqrt{\pi}\,(p-\frac{1}{2})!}\int_0^\pi (\sin\omega)^{2p}\,e^{-x\cos\omega}\,d\omega, \end{aligned} \tag{30}$$

on substituting successively $u = \frac{1}{2}(1-v)$, $v = \cos\omega$.

Likewise the representation (25) becomes

$$I_p(x) = (2\pi x)^{-\frac{1}{2}}\left[e^x\,\psi_p(-x) - \begin{pmatrix}\sin\pi p - i\cos\pi p\\ \sin\pi p + i\cos\pi p\\ \sin\pi p\end{pmatrix} e^{-x}\,\psi_p(x)\right],$$

$$\left.\begin{matrix} 0 < \mathrm{ph}\,x < \pi \\ 0 > \mathrm{ph}\,x > -\pi \\ \mathrm{ph}\,x = 0 \end{matrix}\right\}, \tag{31}$$

where by (24) and (23)

$$\begin{aligned} \psi_p(x) &= \psi_{-p}(x) = \psi(p+\tfrac{1}{2}, 2p+1, 2x) \\ &= \frac{1}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!}\sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(-2x)^r} \\ &= \frac{1}{(p-\frac{1}{2})!}\int_0^\infty \left(u+\frac{u^2}{2x}\right)^{p-\frac{1}{2}} e^{-u}\,du, \qquad |\mathrm{ph}\,x| < \pi. \end{aligned} \tag{32}$$

At $\mathrm{ph}\,x = 0$ the principal value of the integral $\psi_p(-x)$ is to be understood.

The function $\psi_p(x)$ is closely related to the Bessel function

$$K_p(x) = \frac{\pi}{2\sin \pi p}\{I_{-p}(x) - I_p(x)\} = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\psi_p(x)$$

$$= \frac{\sqrt{\pi}\,(\frac{1}{2}x)^p}{(p-\frac{1}{2})!}\int_1^\infty (v^2-1)^{p-\frac{1}{2}} e^{-vx}\,dv$$

$$= \frac{\sqrt{\pi}\,(\frac{1}{2}x)^p}{(p-\frac{1}{2})!}\int_0^\infty (\sinh\omega)^{2p} e^{-x\cosh\omega}\,d\omega, \qquad \Re(x) > 0. \tag{33}$$

This is Hobson's representation (1898).

A further set of representations for the modified Bessel functions can be found by going back to the defining series, II (49),

$$(\tfrac{1}{2}x)^{-p} I_p(x) = \sum_0^\infty (\tfrac{1}{4}x^2)^s/s!\,(s+p)!.$$

Choosing

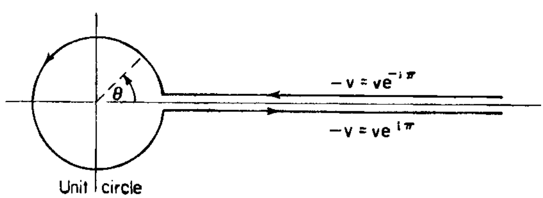
$$S^0(x) = \sum_0^\infty \frac{(\frac{1}{4}x^2)^s}{s!} = e^{\frac{1}{4}x^2}, \qquad \frac{1}{(s+p)!} = -\frac{1}{2\pi i}\int_\infty^{(0+)} \frac{e^{-u}\,du}{(-u)^{s+p+1}},$$

we have

$$I_p(x) = -(\tfrac{1}{2}x)^p \frac{1}{2\pi i}\int_\infty^{(0+)} \frac{e^{-u-\frac{1}{4}x^2/u}}{(-u)^{p+1}}\,du$$

$$= -\frac{1}{2\pi i}\int_\infty^{(0+)} \frac{e^{-\frac{1}{2}x(v+v^{-1})}}{(-v)^{p+1}}\,dv, \qquad \Re(x) > 0. \tag{34}$$

It is possible to express this in terms of simpler integrals by specializing the contour as in the diagram:



Writing $-v = e^{i\theta}$ on the unit circle, where θ runs from $-\pi$ to π, the contribution to (34) is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}\, e^{-ip\theta}\, d\theta.$$

With $-v = e^{\omega}\, e^{-i\pi}$ on the upper straight and $e^{\omega}\, e^{i\pi}$ on the lower, where ω runs from 0 to ∞ as v runs from 1 to ∞, the contribution is

$$\frac{1}{2\pi i}\left\{\frac{1}{e^{-i\pi(p+1)}} - \frac{1}{e^{i\pi(p+1)}}\right\}\int_0^{\infty} e^{-x\cosh\omega - p\omega}\, d\omega.$$

Hence

$$I_p(x) = \frac{1}{\pi}\int_0^{\pi} \cos p\theta\; e^{x\cos\theta}\, d\theta - \frac{\sin \pi p}{\pi}\int_0^{\infty} e^{-x\cosh\omega - p\omega}\, d\omega\,, \qquad \Re(x) > 0, \tag{35}$$

a representation due to Schläfli (1871, 1873) and Sonine (1870, 1880). On subtracting I_{-p} the first contribution is removed, leaving the following notably simple representation for the Bessel function of the third kind (Schläfli 1873):

$$K_p(x) = \int_0^{\infty} \cosh p\omega\; e^{-x\cosh\omega}\, d\omega, \qquad \Re(x) > 0. \tag{36}$$

6. Logarithm of Factorial

Starting from the conditionally convergent series II (55),

$$\ln(x!) + Cx = S(x) = \sum_2^{\infty} \zeta(s)\,(-x)^s/s, \qquad |x| < 1, \tag{37}$$

we choose

$$S^0(x) = \sum_2^{\infty}\frac{(-x)^s}{s!} = e^{-x} - 1 + x, \qquad (s-1)!\,\zeta(s) = \int_0^{\infty}\frac{u^{s-1}\,du}{e^u - 1}.$$

These lead to the representation

$$\ln(x!) = -Cx + \int_0^{\infty}\frac{e^{-ux} - 1 + ux}{e^u - 1}\,\frac{du}{u}, \qquad \Re(x) > 0. \tag{38}$$

Combination with the well-known integral

$$C = \int_0^\infty \frac{e^{-u} - 1 + u}{e^u - 1} \frac{du}{u}$$

leads to Malmstén's (c. 1840) representation

$$\ln (x!) = \int_0^\infty \left\{ \frac{e^{-ux} - 1}{e^u - 1} + xe^{-u} \right\} \frac{du}{u}, \qquad \Re(x) > 0. \tag{39}$$

The asymptotic expansion II (59) contains the asymptotic power series

$$S(x) = \sum_1^\infty (-1)^{s-1} (2s - 2)!\, \zeta(2s)/(2\pi x)^{2s-1}.$$

Choosing

$$S^0(x) = \sum_1^\infty \frac{(-1)^{s-1}}{(2s - 1)(2\pi x)^{2s-1}} = \tan^{-1}\left(\frac{1}{2\pi x}\right),$$

$$(2s - 1)!\, \zeta(2s) = \int_0^\infty \frac{u^{2s-1}\, du}{e^u - 1},$$

we obtain Binet's "second representation" (1839)

$$\ln (x!) = \tfrac{1}{2} \ln 2\pi x + x(\ln x - 1) + \frac{1}{\pi} \int_0^\infty \frac{\tan^{-1}(u/2\pi x)}{e^u - 1}\, du, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{40}$$

7. Fermi–Dirac Integral

Starting from the conditionally convergent series II (62),

$$\mathscr{F}_p(x) = S(x) = \sum_1^\infty (-1)^{s-1} e^{xs}/s^{p+1}, \qquad \Re(x) \leqslant 0, \tag{41}$$

we choose

$$S^0(x) = \sum_1^\infty (-1)^{s-1} e^{xs} = \frac{1}{1 + e^{-x}}, \qquad \frac{1}{s^{p+1}} = \frac{1}{p!} \int_0^\infty u^p e^{-us}\, du.$$

These lead to the representation

$$\mathscr{F}_p(x) = \frac{1}{p!}\int_0^\infty \frac{u^p\,du}{e^{u-x}+1}, \tag{42}$$

which is in fact the primary definition of the Fermi–Dirac integral.

The asymptotic expansion II (68) contains the asymptotic power series

$$S(x) = 2\sum_0^\infty t_{2r}/(-2r+p+1)!\,x^{2r-1}.$$

Choosing

$$S^0(x) = 2\sum_1^\infty \frac{1}{(-2r+p+1)!\,(2r-1)!\,x^{2r-1}}$$

$$= \frac{1}{p!}\left\{\left(1+\frac{1}{x}\right)^p - \left(1-\frac{1}{x}\right)^p\right\},$$

$$(2r-1)!\,t_{2r} = \int_0^\infty \frac{u^{2r-1}\,du}{e^u+1},$$

we have

$$S(x) = \frac{x}{(p+1)!} + \frac{1}{p!}\int_0^\infty \left\{\left(1+\frac{u}{x}\right)^p - \left(1-\frac{u}{x}\right)^p\right\}\frac{du}{e^u+1},$$

where for fractional p the power $(1-u/x)^p$ is to be made definite when $u > x$ by our phase convention (Chapter I, Section 3). This result simplifies on integrating by parts to give

$$\mathscr{F}_p(x) = \frac{x^{p+1}}{(p+1)!}\int_{-\infty}^\infty \left(1+\frac{u}{x}\right)^{p+1}\frac{e^u}{(e^u+1)^2}\,du + \begin{pmatrix} e^{i\pi p} \\ e^{-i\pi p} \\ \cos \pi p \end{pmatrix}\mathscr{F}_p(-x),$$

$$\left.\begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 \end{matrix}\right\}. \tag{43}$$

As will be seen in the next chapter, this important representation also follows from (42).

8. Table of Convergent Power Series†

Conditionally convergent series, $\lvert x \rvert < 1$	Function
$\sum_0^\infty x^s$	$1/(1-x)$
$\sum_0^\infty s x^s$	$x/(1-x)^2$
$\sum_0^\infty s^2 x^s$	$x(1+x)/(1-x)^3$
$\sum_1^\infty x^s/s$	$-\ln(1-x)$
$\sum_0^\infty x^{2s}/(2s+1)$	$x^{-1}\tanh^{-1} x$
$\sum_1^\infty \dfrac{x^s}{s+a}$	$x\int_0^1 \dfrac{u^a\,du}{1-ux}$
$\sum_1^\infty \dfrac{x^s}{s^2}$	$-\int_0^1 \ln(1-ux)\dfrac{du}{u}$

† Variants, such as those obtained by differentiation, or by changing the phase of x by π or $\frac{1}{2}\pi$, have been omitted unless of special interest.

$\sum_{1}^{\infty} \frac{x^s}{s^{p+1}}$	$\mathscr{B}_p(\ln x) = \frac{1}{p!}\int_0^\infty \frac{u^p\,du}{e^{u-\ln x} - 1}$, Bose–Einstein integral
$\sum_{1}^{\infty} \frac{(-1)^{s-1} x^s}{s^{p+1}}$	$\mathscr{F}_p(\ln x) = \frac{1}{p!}\int_0^\infty \frac{u^p\,du}{e^{u-\ln x} + 1}$, Fermi–Dirac integral
$\sum_{1}^{\infty} \frac{x^s}{s(s+1)}$	$1 + \frac{1-x}{x}\ln(1-x)$
$\sum_{0}^{\infty} \frac{x^{2s}}{(2s+1)(2s+3)}$	$\frac{x^2-1}{2x^3}\tanh^{-1} x + \frac{1}{2x^2}$
$\sum_{1}^{\infty} \frac{x^{2s}}{2s(2s+1)}$	$1 - x^{-1}\tanh^{-1} x - \frac{1}{2}\ln(1-x^2)$
$\sum_{0}^{\infty} \frac{x^{2s}}{(2s-1)(2s+1)(2s+3)}$	$\frac{(1-x^2)^2}{8x^3}\tanh^{-1} x - \frac{1}{8x^2} - \frac{1}{8}$
$\sum_{0}^{\infty} \frac{(s+\alpha)!}{s!} x^s$	$\alpha!\,(1-x)^{-\alpha-1}$
$\sum_{1}^{\infty} \frac{(s-\frac{1}{2})!}{s!\,s} x^{2s}$	$-2\pi^{\frac{1}{2}}(\operatorname{sech}^{-1} x + \ln \frac{1}{2}x)$

Conditionally convergent series, $\lvert x\rvert < 1$	Function
$\sum_{1}^{\infty} \frac{(s-\frac{1}{2})!}{s!\,s}(-x^2)^s$	$-2\pi^{\frac{1}{2}}(\operatorname{cosech}^{-1} x + \ln \frac{1}{2}x)$
$\sum_{0}^{\infty} \frac{(s-\frac{1}{2})!}{s!(2s+1)} x^{2s}$	$\pi^{\frac{1}{2}} x^{-1} \sin^{-1} x$
$\sum_{0}^{\infty} \frac{s!}{(s+\frac{1}{2})!} x^{2s}$	$\frac{2}{\sqrt{\pi}} \frac{\sin^{-1} x}{x(1-x^2)^{\frac{1}{2}}}$
$\sum_{0}^{\infty} \frac{(s-1)!}{(s+\frac{1}{2})!} x^{2s}$	$\frac{4}{\sqrt{\pi}}\left\{1-(1-x^2)^{\frac{1}{2}}\frac{\sin^{-1} x}{x}\right\}$
$\sum_{0}^{\infty} \frac{s!}{(s+\frac{1}{2})!\,(s+1)} x^{2s}$	$\frac{2}{\sqrt{\pi}}\left(\frac{\sin^{-1} x}{x}\right)^2$
$\sum_{0}^{\infty} \frac{(s+p-1)!\,(s+p-\frac{1}{2})!}{s!\,(s-\frac{1}{2})!} x^{2s}$	$\frac{(p-1)!\,(p-\frac{1}{2})!}{2\sqrt{\pi}}\{(1+x)^{-2p}+(1-x)^{-2p}\}$
$\sum_{0}^{\infty} \frac{(s+p-1)!\,(s-p-1)!}{s!\,(s-\frac{1}{2})!} x^{2s}$	$\frac{(p-1)!\,(-p-1)!}{\sqrt{\pi}}\cos(2p\sin^{-1} x)$

$\sum_0^\infty \frac{(s+p-\frac{1}{2})!\,(s-p-\frac{1}{2})!}{s!\,(s+\frac{1}{2})!}x^{2s}$	$\frac{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!}{\sqrt{\pi}\,px}\sin(2p\sin^{-1}x)$
$\sum_0^\infty \frac{(s+p-\frac{1}{2})!\,(s-p-\frac{1}{2})!}{s!\,(s-\frac{1}{2})!}x^{2s}$	$\frac{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!}{\sqrt{\pi}}\frac{\cos(2p\sin^{-1}x)}{\cos(\sin^{-1}x)}$
$\sum_0^\infty \frac{(s+p)!\,(s-p)!}{s!\,(s+\frac{1}{2})!}x^{2s}$	$\frac{p!\,(-p)!}{\sqrt{\pi}\,px}\frac{\sin(2p\sin^{-1}x)}{\cos(\sin^{-1}x)}$
$\sum_0^\infty \frac{(s+p-1)!\,(s+p-\frac{3}{2})!}{s!\,(s+2p-1)!}x^{s}$	$\frac{(p-1)!\,(p-\frac{3}{2})!}{(2p-1)!}\left\{\frac{1+(1-x)^{\frac{1}{2}}}{2}\right\}^{1-2p}$
$\sum_0^\infty \frac{(s+p-1)!\,(s+p-\frac{1}{2})!}{s!\,(s+2p-1)!}x^{s}$	$\frac{(p-1)!\,(p-\frac{1}{2})!}{(2p-1)!\,(1-x)^{\frac{1}{2}}}\left\{\frac{1+(1-x)^{\frac{1}{2}}}{2}\right\}^{1-2p}$
$\sum_0^\infty \left\{\frac{(s-\frac{1}{2})!}{s!}\right\}^2 x^{2s}$	$2F(x)=2\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-x^2\sin^2\theta)^{\frac{1}{2}}}$, a complete elliptic integral
$\sum_0^\infty \frac{(s-\frac{1}{2})!\,(s-\frac{3}{2})!}{(s!)^2}x^{2s}$	$-4E(x)=-4\int_0^{\frac{1}{2}\pi}(1-x^2\sin^2\theta)^{\frac{1}{2}}\,d\theta$, a complete elliptic integral
$\sum_0^\infty \frac{(s+m+n)!\,(s+m-n-1)!}{s!\,(s+m)!}x^{s}$	$\frac{(-1)^m(n-m)!\,(m-n-1)!}{[x(1-x)]^{\frac{1}{2}m}}P_n^{\,m}(1-2x)$, associated Legendre polynomial

Conditionally convergent series, $\|x\| < 1$	Function
$\sum_0^\infty \frac{(s+n)!\,(s-n-1)!}{s!\,(s+m)!} x^s$	$\frac{(-1)^m n!\,(-n-1)!\,(n-m)!}{(n+m)!} \left(\frac{1-x}{x}\right)^{\frac{1}{2}m} P_n^m(1-2x)$
$\sum_0^\infty \frac{(s+\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2})!\,(s+\frac{1}{2}m-\frac{1}{2}n-1)!}{s!\,(s+m)!} x^s$	$\frac{2^m(\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2})!\,(\frac{1}{2}m-\frac{1}{2}n-1)!\,(n-m)!}{(n+m)!\,x^{\frac{1}{2}m}} P_n^m[(1-x)^{\frac{1}{2}}]$
$\sum_0^\infty \frac{(s+a-1)!\,(s+b-1)!}{s!\,(s+c-1)!} x^s$	$\frac{(a-1)!\,(b-1)!}{(c-1)!} F(a, b; c; x)$, hypergeometric function
$\sum_1^\infty (1+\frac{1}{2}+\frac{1}{3}+\ldots 1/s)\,x^s = \sum_1^\infty (\Psi(s)+C)\,x^s$	$-\frac{\ln(1-x)}{1-x}$
$\sum_2^\infty \left(1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{s-1}\right)\frac{x^s}{s}$	$\frac{1}{2}[\ln(1-x)]^2$
$\sum_0^\infty \left(1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{2s+2}\right)\frac{x^{2s}}{2s+3}$	$-\frac{\tanh^{-1} x \ln(1-x^2)}{2x^3}$
$\sum_1^\infty \left(1-\frac{1}{2}+\frac{1}{3}-\ldots\frac{1}{2s-1}\right)\frac{x^{2s}}{s}$	$-\ln(1+x)\ln(1-x)$

$\sum_{1}^{\infty}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{2s-1}\right)\frac{x^{2s}}{s}$	$(\tanh^{-1}x)^2$
$\sum_{0}^{\infty}\left(1-\frac{1}{3}+\frac{1}{5}-\ldots\frac{1}{4s+1}\right)\frac{x^{4s}}{2s+1}$	$\frac{\tan^{-1}x\ \tanh^{-1}x}{x^2}$
$\sum_{2}^{\infty}\zeta(s)\frac{(-x)^s}{s}$	$\ln x! + Cx$
$\sum_{0}^{\infty}\zeta(p+1-s)\frac{(-x)^s}{s!}$	$\mathscr{B}_p(-x)+\frac{\pi x^p}{p!\sin\pi p}$, Bose–Einstein integral
$\sum_{0}^{\infty}\zeta(2s)\,x^{2s}$	$-\frac{1}{2}\pi x\cot\pi x$
$\sum_{0}^{\infty}\zeta(2s)\frac{x^{2s}}{s}$	$-\ln\left(\frac{\sin\pi x}{\pi x}\right)$
$\sum_{1}^{\infty}\zeta(2s)\frac{(-x^2)^s}{s(2s+1)}$	$1-\ln 2\pi x+\int_0^1\ln(2\sin\pi u x)\,du$
$\sum_{2}^{\infty}t_s(-x)^s/s;\quad t_0=\frac{1}{2},\quad t_s=(1-2^{1-s})\,\zeta(s)= =1-2^{-s}+3^{-s}-\ldots$	$\ln\{x!/[(\frac{1}{2}x)!]^2\}$

Conditionally convergent series, $\lvert x\rvert < 1$	Function
$\sum_{0}^{\infty} t_{p+1-s}\, x^s/s!$	$\mathscr{F}_p(x)$, Fermi–Dirac integral
$\sum_{0}^{\infty} t_{2s}\, x^{2s}$	$\pi x \operatorname{cosec} \pi x$
$\sum_{1}^{\infty} t_{2s}\, x^{2s}/s$	$\ln\left(\dfrac{\tan \frac{1}{2}\pi x}{\frac{1}{2}\pi x}\right)$
$\sum_{1}^{\infty}\left(1 + \dfrac{1}{3^{2s}} + \dfrac{1}{5^{2s}} + \dots\right) x^{2s} = \sum_{1}^{\infty} (1 - 2^{-2s})\, \zeta(2s)\, x^{2s}$	$\frac{1}{4}\pi x \tan \frac{1}{2}\pi x$
$\sum_{1}^{\infty}\left(1 + \dfrac{1}{3^{2s}} + \dfrac{1}{5^{2s}} + \dots\right) \dfrac{x^{2s}}{s}$	$-\ln(\cos \frac{1}{2}\pi x)$
$\sum_{0}^{\infty}\left(1 - \dfrac{1}{3^{2s+1}} + \dfrac{1}{5^{2s+1}} - \dots\right) x^{2s}$	$\frac{1}{4}\pi \sec \frac{1}{2}\pi x$
$\sum_{0}^{\infty}\left(1 - \dfrac{1}{3^{2s+1}} + \dfrac{1}{5^{2s+1}} - \dots\right) \dfrac{x^{2s}}{2s+1}$	$\dfrac{1}{2x} \ln(\sec \frac{1}{2}\pi x + \tan \frac{1}{2}\pi x)$
$\sum_{m}^{\infty} S_s^{m}\, x^s/s!$; Stirling numbers of the first kind	$[\ln(1 + x)]^m/m!$

Conditionally convergent series, various radii			Function
$\sum_0^\infty B_s x^s/s!$,	$\lvert x\rvert < 2\pi$,	Bernoulli numbers, $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \ldots$	$x/(e^x - 1)$. See also under $\zeta(2s) = -\frac{1}{2}B_{2s}(-4\pi^2)^s/(2s)!$
$\sum_0^\infty E_s x^s/s!$,	$\lvert x\rvert < \frac{1}{2}\pi$,	Euler numbers, $E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ \ldots$	sech x. See also under $1 - 1/3^{2s+1} + \ldots = \frac{1}{4}\pi E_{2s}(-\frac{1}{4}\pi^2)^s/(2s)!$
$\sum_m^\infty C_s^m x^{-s}$,	$\lvert x\rvert > m$,	Stirling numbers of the second kind	$(x - 1 - m)!/(x - 1)!$

Absolutely convergent series		Function
$\sum_0^\infty x^s/s!$		e^x
$\sum_m^\infty C_s^m x^s/s!$,	Stirling numbers of the second kind	$(e^x - 1)^m/m!$
$\sum_0^\infty x^{2s}/(2s)!$		$\cosh x$
$\sum_0^\infty x^{2s}/(2s + 1)!$		$x^{-1} \sinh x$

Absolutely convergent series	Function
$\sum_{0}^{\infty} \dfrac{x^{2s}}{(s+\frac{1}{2})!}$	$\dfrac{e^{x^2}\phi(x)}{x} = \dfrac{2}{\sqrt{\pi}} \dfrac{e^{x^2}\operatorname{Erf} x}{x}$, error function
$\sum_{0}^{\infty} \dfrac{(-x^2)^s}{(s+\frac{1}{2})!}$	$\dfrac{2}{\sqrt{\pi}} \dfrac{e^{-x^2}\operatorname{Erfi} x}{x}$
$\sum_{0}^{\infty} \dfrac{(-x^2)^s}{s!\,(2s+1)}$	$\dfrac{\sqrt{\pi}}{2x}\phi(x) = \dfrac{\operatorname{Erf} x}{x}$
$\sum_{0}^{\infty} \dfrac{x^{2s}}{s!\,(2s+1)}$	$\dfrac{\operatorname{Erfi} x}{x}$
$\sum_{0}^{\infty} \dfrac{(-x)^s}{(\frac{1}{2}s)!}$	$e^{x^2}\{1-\phi(x)\} = (2/\sqrt{\pi})e^{x^2}\operatorname{Erfc} x$
$\sum_{0}^{\infty} \dfrac{(\frac{1}{2}s+\frac{1}{2}r-\frac{1}{2})!\,(-x)^s}{s!}$	$\dfrac{r!}{2^{\frac{1}{2}r-\frac{1}{2}}} e^{\frac{1}{4}x^2} Hh_r(x/\sqrt{2}) = \sqrt{\pi}\; r!\, e^{\frac{1}{4}x^2} i^r \operatorname{erfc} x = \dfrac{r!}{2^{\frac{1}{2}r-\frac{1}{2}}}$ $\times e^{\frac{1}{8}x^2} D_{-r-1}(x/\sqrt{2})$
$\sum_{1}^{\infty} (-x)^s/s!\,s$	$Ei(-x) - \ln x - C$, exponential integral

$\sum_{0}^{\infty} \Psi(s)\, x^s/s!$	$e^x\{\ln x - Ei(-x)\}$
$\sum_{0}^{\infty} (-x)^s/s!(s+p+1)$	$x^{-p-1}(p, x)!$, incomplete factorial function
$\sum_{0}^{\infty} x^s/(s+p+1)!$	$x^{-p-1}\, e^x(p, x)!/p!$
$\sum_{0}^{\infty} (-x^2)^s/(2s+1)!\,(2s+1)$	$x^{-1}\, Si(x)$, sine integral
$\sum_{1}^{\infty} (-x^2)^s/(2s)!\,(2s)$	$Ci(x) - \ln x - C$, cosine integral
$\sum_{0}^{\infty} (-x^2)^s/(2s)!\,(2s+\frac{1}{2})$	$(2\pi/x)^{\frac{1}{2}}\, C(x) = \int_0^1 u^{-\frac{1}{2}} \cos ux\, du$, Fresnel integral
$\sum_{0}^{\infty} (-x^2)^s/(2s+1)!\,(2s+\frac{3}{2})$	$(2\pi/x^3)^{\frac{1}{2}}\, S(x) = x^{-1}\int_0^1 u^{-\frac{1}{2}} \sin ux\, du$, Fresnel integral
$\sum_{0}^{\infty} (-x^2)^s/(2s)!\,(2s+\alpha)$	$x^{-\alpha}\left\{(\alpha-1)!\cos\frac{1}{2}\pi\alpha - \int_x^{\infty} u^{\alpha-1}\cos u\, du\right\}$

Absolutely convergent series	Function
$\sum_0^\infty (-x^2)^s/2s+1)!\,(2s+1+\alpha)$	$x^{-1-\alpha}\left\{(\alpha-1)!\sin\frac{1}{2}\pi\alpha - \int_x^\infty u^{\alpha-1}\sin u\,du\right\}$
$\sum_0^\infty (\frac{1}{4}x^2)^s/s!\,(s+p)!$	$(\frac{1}{2}x)^{-p} I_p(x)$, modified Bessel function
$\sum_0^\infty (-\frac{1}{4}x^2)^s/s!\,(s+p)!$	$(\frac{1}{2}x)^{-p} J_p(x)$, Bessel function
$\sum_0^\infty \Psi(s)(\frac{1}{4}x^2)^s/(s!)^2$	$K_0(x) + I_0(x)\ln\frac{1}{2}x$
$\sum_0^\infty \Psi(s)(-\frac{1}{4}x^2)^s/(s!)^2$	$-\frac{1}{2}\pi Y_0(x) + J_0(x)\ln\frac{1}{2}x$
$\sum_0^\infty (-x^4)^s/[(2s)!]^2$	$ber(2x)$, Kelvin function
$\sum_0^\infty (-x^4)^s/[(2s+1)!]^2$	$x^{-2}\,bei(2x)$, Kelvin function
$\sum_0^\infty \Psi(2s)(-x^4)^s/[(2s)!]^2$	$ker(2x) - \frac{1}{4}\pi\,bei(2x) + (\ln x)\,ber(2x)$

$\sum_0^\infty \Psi(2s+1)(-x^4)^s/[(2s+1)!]^2$	$x^{-2}\{kei(2x) + \frac{1}{4}\pi\, ber(2x) + (\ln x)\, bei(2x)\}$
$\sum_0^\infty \frac{(s+p-\frac{1}{2})!}{s!(s+2p)!}(\pm 2x)^s$	$\sqrt{\pi}\; e^{\pm x}(2x)^{-p} I_p(x)$
$\sum_0^\infty \frac{(s+p-\frac{1}{2})!}{s!\,(s+p)!\,(s+2p)!}\, x^{2s}$	$\sqrt{\pi}\; x^{-2p} I_p{}^2(x)$
$\sum_0^\infty \frac{(s+p-\frac{1}{2})!}{s!\,(s+p)!\,(s+2p)!}\,(-x^2)^s$	$\sqrt{\pi}\; x^{-2p} J_p{}^2(x)$
$\sum_0^\infty \frac{(2s+p+q)!}{s!\,(s+p)!\,(s+q)!\,(s+p+q)!}\,(\frac{1}{4}x^2)^s$	$(\frac{1}{2}x)^{-p-q} I_p(x)\, I_q(x)$
$\sum_0^\infty \frac{(2s+p+q)!}{s!\,(s+p)!\,(s+q)!\,(s+p+q)!}\,(-\frac{1}{4}x^2)^s$	$(\frac{1}{2}x)^{-p-q} J_p(x)\, J_q(x)$
$\sum_0^\infty (\frac{1}{4}x^2)^s/(s+\frac{1}{2})!\,(s+p+\frac{1}{2})!$	$(\frac{1}{2}x)^{-p-1}\,\mathbf{L}_p(x)$, modified Struve function
$\sum_0^\infty (-\frac{1}{4}x^2)^s/(s+\frac{1}{2})!\,(s+p+\frac{1}{2})!$	$(\frac{1}{2}x)^{-p-1}\,\mathbf{H}_p(x)$, Struve function

Absolutely convergent series	Function
$\sum_0^\infty (-\frac{1}{4}x^2)^s/(s+\frac{1}{2}p)!\,(s-\frac{1}{2}p)!$	$\mathbf{J}_p(x)\cos\frac{1}{2}\pi p + \mathbf{E}_p(x)\sin\frac{1}{2}\pi p$, Anger and Weber functions
$\sum_0^\infty (-\frac{1}{4}x^2)^s/(s+\frac{1}{2}p+\frac{1}{2})!\,(s-\frac{1}{2}p+\frac{1}{2})!$	$(2/x)\{\mathbf{J}_p(x)\sin\frac{1}{2}\pi p - \mathbf{E}_p(x)\cos\frac{1}{2}\pi p\}$
$\sum_0^\infty (-\frac{1}{4}x^2)^s/(s+\frac{1}{2}p+\frac{1}{2}q+\frac{1}{2})!\,(s-\frac{1}{2}p+\frac{1}{2}q+\frac{1}{2})!$	$[4/\{x^{q+1}(\frac{1}{2}p+\frac{1}{2}q-\frac{1}{2})!\,(-\frac{1}{2}p+\frac{1}{2}q-\frac{1}{2})!\}]\, s_{q,p}(x)$, Lommel function
$\sum_0^\infty (\pm\frac{1}{2}x)^s/(\frac{1}{2}s)!\,(\frac{1}{2}s+p)!$	$(\frac{1}{2}x)^{-p}\{I_p(x) \pm \mathbf{L}_p(x)\}$

9. Table of Representation as a Power

Function	Representation
s^{-1}	$\int_0^1 u^{s-1}\,du$
s^{-2}	$-\int_0^1 u^{s-1}\ln u\,du$

$1/s^{\alpha}$	$\dfrac{1}{(\alpha-1)!}\displaystyle\int_0^{\infty} e^{-us} u^{\alpha-1}\, du$
s^{α}	$\alpha!\,(i/2\pi)\displaystyle\int_{\infty}^{(0+)} e^{-us}(-u)^{-\alpha-1}\, du$
s^{n}	$(u\,\partial/\partial u)^{n}\, u^{s}\vert_{u=1}$
$\dfrac{1}{(s+\alpha)(s+\beta)}$	$\dfrac{1}{\beta-\alpha}\displaystyle\int_0^{1} u^{s-1}(u^{\alpha}-u^{\beta})\, du$
$\dfrac{\lambda s+\mu}{(s+\alpha)(s+\beta)}$	$\dfrac{1}{\alpha-\beta}\displaystyle\int_0^{\infty} e^{-us}\{(\alpha\lambda-\mu)\,e^{-\alpha u}-(\beta\lambda-\mu)\,e^{-\beta u}\}\, du$
$\dfrac{\lambda s+\mu}{(s+\alpha)^{2}}$	$\displaystyle\int_0^{\infty} e^{-u(s+\alpha)}\{\lambda+(\mu-\alpha\lambda)\,u\}\, du$
$\dfrac{1}{s^{2}+\alpha^{2}}$	$\dfrac{1}{\alpha}\displaystyle\int_0^{\infty} e^{-us}\sin\alpha u\, du$
$\dfrac{s}{s^{2}+\alpha^{2}}$	$\displaystyle\int_0^{\infty} e^{-us}\cos\alpha u\, du$
$\dfrac{s}{s^{2}-1}$	$\displaystyle\int_0^{\infty} \frac{du}{\{(u^{2}+1)^{\frac{1}{2}}+u\}^{s}} = \int_0^{\infty} \{(u^{2}+1)^{\frac{1}{2}}-u\}^{s}\, du$

Function	Representation
$\dfrac{1}{(s^2+\alpha^2)^{\nu+\frac{1}{2}}}$	$\dfrac{\sqrt{\pi}}{(2\alpha)^{\nu}(\nu-\frac{1}{2})!}\displaystyle\int_0^{\infty} e^{-us}\,u^{\nu}\,J_{\nu}(\alpha u)\,du$
$\dfrac{s}{(s^2+\alpha^2)^{\nu+3/2}}$	$\dfrac{\sqrt{\pi}}{2(2\alpha)^{\nu}(\nu+\frac{1}{2})!}\displaystyle\int_0^{\infty} e^{-us}\,u^{\nu+1}\,J_{\nu}(\alpha u)\,du$
$\{(s^2+\alpha^2)^{\frac{1}{2}}-s\}^{\nu}$	$\nu\alpha^{\nu}\displaystyle\int_0^{\infty} e^{-us}\,u^{-1}\,J_{\nu}(\alpha u)\,du$
$e^{\alpha s^2}$	$\frac{1}{2}(\pi\alpha)^{-\frac{1}{2}}\displaystyle\int_{-\infty}^{\infty} e^{us}\,e^{-u^2/4\alpha}\,du$
$s^{-\frac{1}{2}}\,e^{-\alpha/s}$	$\pi^{-\frac{1}{2}}\displaystyle\int_0^{\infty} e^{-us}\,u^{-\frac{1}{2}}\cos(2\alpha^{\frac{1}{2}}u^{\frac{1}{2}})\,du$
$s^{-\nu-1}\,e^{-\alpha/s}$	$\alpha^{-\frac{1}{2}\nu}\displaystyle\int_0^{\infty} e^{-us}\,u^{\frac{1}{2}\nu}\,J_{\nu}(2\alpha^{\frac{1}{2}}u^{\frac{1}{2}})\,du$
$e^{-\alpha\sqrt{s}}$	$\frac{1}{2}\pi^{-\frac{1}{2}}\alpha\displaystyle\int_0^{\infty} e^{-us}\,u^{-3/2}\,e^{-\alpha^2/4u}\,du$
$s^{-\frac{1}{2}}\,e^{-\alpha\sqrt{s}}$	$\pi^{-\frac{1}{2}}\displaystyle\int_0^{\infty} e^{-us}\,u^{-\frac{1}{2}}\,e^{-\alpha^2/4u}\,du$

$s^{\frac{1}{2}} e^{-\alpha\sqrt{s}}$	$\pi^{-\frac{1}{2}} \int_0^\infty e^{-us}(\tfrac{1}{4}\alpha^2 u^{-5/2} - \tfrac{1}{2}u^{-3/2})\, e^{-\alpha^2/4u}\, du$
e^{-e^s}	$(2\pi i)^{-1} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{-us}(u-1)!\, du, \qquad \gamma > 0$
$\dfrac{1}{e^{s/\alpha}+1}$	$\dfrac{\alpha}{2i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{-us} \dfrac{du}{\sin \pi\alpha u}, \qquad 0 < \gamma < 1$
$\dfrac{1}{e^{s/\alpha}-1}$	$-\dfrac{\alpha}{2i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{-us} \dfrac{du}{\tan \pi\alpha u}, \qquad 0 < \gamma < 1$
$\ln s$	$\int_0^\infty (e^{-u} - e^{-us})\, u^{-1}\, du$
$\ln s + C$†	$\int_0^\infty \{(1+u)^{-1} - e^{-us}\}\, u^{-1}\, du$
$\ln\left(\dfrac{s+\alpha}{s+\beta}\right)$	$\int_0^\infty e^{-us}\left(\dfrac{e^{-\beta u} - e^{-\alpha u}}{u}\right) du$

† $C = -\Psi(0) = 0{\cdot}5772156665$

$$= -\int_0^\infty \ln u \; e^{-u}\, du = \int_0^\infty \{(1+u)^{-1} - e^{-u}\}\, u^{-1}\, du = \int_0^\infty \left(\frac{1}{e^u - 1} - \frac{e^{-u}}{u}\right) du = \tfrac{1}{2} + 2\int_0^\infty \frac{u\, du}{(u^2+1)(e^{2\pi u} - 1)}$$

Function	Representation
$\dfrac{\ln s}{s^{\alpha+1}}$	$\dfrac{1}{\alpha!}\displaystyle\int_0^\infty e^{-us}\{\Psi(\alpha) - \ln u\}\, u^\alpha\, du$
$(\ln s)^2/s$	$\displaystyle\int_0^\infty e^{-us}\{(\ln u + C)^2 - \tfrac{1}{6}\pi^2\}\, du$
$\ln(1 + s^2/\alpha^2)$	$2\displaystyle\int_0^\infty (1 - e^{-us})\, u^{-1} \cos \alpha u\, du$
$\ln(1 + e^{-s/\alpha})$	$\dfrac{1}{2i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-us} \frac{du}{u \sin \pi\alpha u}, \qquad 0 < \gamma < 1$
$\ln(1 - e^{-s/\alpha})$	$\dfrac{1}{2i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-us} \frac{du}{u \tan \pi\alpha u}, \qquad 0 < \gamma < 1$
$\ln(1 - e^{-e^s})$	$-\dfrac{1}{2\pi i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-us}(u-1)!\,\zeta(u+1)\, du, \quad \gamma > 0$
$\ln(1 + e^{-e^s})$	$\dfrac{1}{2\pi i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-us}(u-1)!\,(1 - 2^{-u})\,\zeta(u+1)\, du, \quad \gamma > 0$

$\tan^{-1}(s/\alpha)$	$\int_0^\infty (1 - e^{-us})\, u^{-1} \sin \alpha u\, du$
$\tan^{-1}(\alpha/s)$	$\int_0^\infty e^{-us} u^{-1} \sin \alpha u\, du$
$\sum_{\nu=0}^{\infty} (-1)^\nu/(s + 2\nu + 1)$	$\int_0^{\frac{1}{2}\pi} \tan^s u\, du$
$s!$	$\int_0^\infty u^s e^{-u}\, du = \dfrac{1}{2i \sin \pi s} \int_\infty^{(0+)} (-u)^s e^{-u}\, du$
$s!/\alpha^{s+1}$	$\int_0^\infty u^s e^{-\alpha u}\, du$
$1/s!$	$\dfrac{i}{2\pi} \int_\infty^{(0+)} \dfrac{e^{-u}\, du}{(-u)^{s+1}}$
$(\frac{1}{2}s)!$	$2 \int_0^\infty u^{s+1} e^{-u^2}\, du$
$s!/(\frac{1}{2}s)!$	$2\pi^{-\frac{1}{2}} \int_0^\infty (2u)^s e^{-u^2}\, du$

Function	Representation
$(s-\frac{1}{2})!/s!$	$2\pi^{-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\sin^{2s}u\,du = 2\pi^{-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\cos^{2s}u\,du$
$s!/(s-n)!$	$(\partial/\partial u)^n\,u^s\vert_{u=1}$
$(s-\frac{1}{2})!/s!\,s$	$4\pi^{-\frac{1}{2}}\int_0^1(1-u^2)^{s-1}\,u\sin^{-1}u\,du$
$(s+\alpha)!\,(s+\beta)!$	$4\int_0^\infty u^{2s+\alpha+\beta+1}\,K_{\alpha-\beta}(2u)\,du$
$\dfrac{(s+\alpha)!}{(s+\beta)!}$	$\dfrac{1}{(\beta-\alpha-1)!}\int_0^1 u^{s+\alpha}(1-u)^{\beta-\alpha-1}du$
	$= \dfrac{2^\alpha(\alpha-\beta)!}{\pi}\int_{-\infty}^\infty\left(\dfrac{2}{1+iu}\right)^s\dfrac{du}{(1+iu)^{\beta+1}(1-iu)^{\alpha-\beta+1}}$
$\dfrac{1}{(s+\alpha)!(s+\beta)!}$	$-\dfrac{1}{2\pi i\cos\pi\beta}\int_\infty^{(0^+)}\dfrac{J_{\alpha-\beta}(-2u)\,du}{(-u^2)^s(-u)^{\alpha+\beta+1}}$

$\dfrac{\{(\frac{1}{2}s-\frac{1}{2})!\}^2}{s!}$	$\displaystyle\int_0^1 \{u(1-u)\}^{\frac{1}{2}s-\frac{1}{2}}\,du = 2\int_0^{\frac{1}{2}\pi} (\tfrac{1}{2}\sin u)^s\,du$ $\displaystyle = 2\int_{-\infty}^{\infty} \frac{du}{(2\cosh u)^{s+1}}$
$\dfrac{(s!)^2}{(2s+1)!}$	$\displaystyle\int_0^1 \{u(1-u)\}^s\,du$
$\dfrac{(s-\frac{1}{2}+\alpha)!\,(s-\frac{1}{2}-\alpha)!}{(2s)!}$	$\displaystyle 2\int_0^{\frac{1}{2}\pi} \frac{(\frac{1}{2}\sin 2u)^{2s}\,du}{\tan^{2\alpha} u} = 4\int_0^{\infty} \frac{\cosh 2\alpha u}{(4\cosh^2 u)^{s+\frac{1}{2}}}\,du$
$(s+\alpha)!\,(s+\beta)!/(s+\gamma)!$	$\displaystyle \pi^{-1}\sin\pi\gamma\;(\alpha-\gamma)!\,(\beta-\gamma)!\int_0^{\infty} (-u)^{s-1}\,u^{\gamma}$ $\times\,\psi(\alpha-\gamma+1,\,\alpha-\beta+1,\,u)\,du$
$s!/\{(\frac{1}{2}s)!\}^2$	$\displaystyle 2\pi^{-1}\int_0^{\frac{1}{2}\pi} (2\cos u)^s\,du$
$(2s)!/(s!)^2$	$\displaystyle \frac{2}{\pi}\int_1^{\infty} \left(\frac{4}{u^2}\right)^s \frac{du}{u(u^2-1)^{\frac{1}{2}}}$
$(2s)!/(s+\gamma)!\,(s-\gamma)!$	$\displaystyle 2\pi^{-1}\int_0^{\frac{1}{2}\pi} (2\cos u)^{2s}\cos 2\gamma u\,du$

Function	Representation
$\dfrac{(2s)!}{(s+\gamma)!\,(s-\gamma)!}$	$\pi^{-1}\displaystyle\int_0^\pi (2\sin u)^{2s}\, e^{i\gamma(2u-\pi)}\, du$
$\dfrac{(s+\alpha)!}{(s+\beta)!\,(s+\gamma)!}$	$-\dfrac{(\alpha-\gamma)!}{2\pi i(\beta-\gamma)!}\displaystyle\int_\infty^{(0+)} \dfrac{F(\alpha-\gamma+1, \beta-\gamma+1, -u)\, du}{(-u)^{s+\gamma+1}}$
$\dfrac{(s+\alpha)!(s+\beta)!}{(s+\gamma)!\,(s+\delta)!}$	$-\dfrac{(\alpha-\gamma)!\,(\beta-\gamma)!}{2\pi i(\delta-\gamma)!}$ $\times\displaystyle\int_\infty^{(0+)} \dfrac{F(\alpha-\gamma+1, \beta-\gamma+1; \delta-\gamma+1; -u)\, du}{(-u)^{s+\gamma+1}}$
$\dfrac{1}{(s^2+\alpha^2)(s^2+3^2\alpha^2)\ldots\{s^2+(2n+1)^2\alpha^2\}}$	$\dfrac{1}{(2n+1)!\,\alpha^{2n+1}}\displaystyle\int_0^\infty e^{-us}\sin^{2n+1}\alpha u\, du$
$\dfrac{1}{s(s^2+2^2\alpha^2)(s^2+4^2\alpha^2)\ldots\{s^2+(2n)^2\alpha^2\}}$	$\dfrac{1}{(2n)!\,\alpha^{2n}}\displaystyle\int_0^\infty e^{-us}\sin^{2n}\alpha u\, du$
$(s-1)!\sin\frac{1}{2}\pi s$	$\displaystyle\int_0^\infty u^{s-1}\sin u\, du$
$(s-1)!\cos\frac{1}{2}\pi s$	$\displaystyle\int_0^\infty u^{s-1}\cos u\, du$

$(s-1)!\sin\alpha s$	$\int_0^\infty u^{s-1}e^{-u\cos\alpha}\sin(u\sin\alpha)\,du$
$(s-1)!\cos\alpha s$	$\int_0^\infty u^{s-1}e^{-u\cos\alpha}\cos(u\sin\alpha)\,du$
$(-s)!\sin(\frac{1}{2}\pi s-\alpha)$	$\int_0^\infty u^{-s}\cos(u+\alpha)\,du$
$(-s)!\cos(\frac{1}{2}\pi s-\alpha)$	$\int_0^\infty u^{-s}\sin(u+\alpha)\,du$
$1/\cos\frac{1}{2}\pi s$	$2\pi^{-1}\int_0^{\frac{1}{2}\pi}\tan^s u\,du$
$1/(s-1)!\cos\frac{1}{2}\pi s$	$-4\pi^{-1}\int_0^\infty u^{-s}\sin^2\frac{1}{2}u\,du$
$1/\sin(\pi s/n)$	$\frac{n}{\pi}\int_0^\infty\frac{u^{s-1}\,du}{u^n+1}$
$\Psi(s)=(d/ds)\ln s!$	$\int_0^\infty\left(e^{-u}-\frac{1}{(1+u)^{s+1}}\right)\frac{du}{u}=\int_0^\infty\left(\frac{e^{-u}}{u}-\frac{e^{-us}}{e^u-1}\right)du$

Function	Representation
$\Psi(s) + C$	$\int_0^1 \left(\frac{1-u^s}{1-u}\right) du = \int_0^\infty \left(\frac{1}{1+u} - \frac{1}{(1+u)^{s+1}}\right) \frac{du}{u}$
$\{\Psi(s) + C\}/s$	$-\int_0^1 u^{s-1} \ln(1-u)\, du$
$(s+1)!\, \Psi(s)$	$\int_0^\infty u^{s+1} e^{-u} \ln u\, du$
$\Psi(s) - \ln s$	$\int_0^\infty e^{-us}\left(\frac{1}{u} - \frac{1}{e^u - 1}\right) du$
$\Psi(s-1) - \ln s$	$\int_0^\infty e^{-us}\left(\frac{1}{u} - \frac{1}{1 - e^{-u}}\right) du$
$\Psi(s+\alpha) - \Psi(s+\beta)$	$\int_0^1 u^s\left(\frac{u^\beta - u^\alpha}{1-u}\right) du$
$\{\Psi(s+\alpha) - \Psi(\alpha)\}\, s!/(s+\alpha)!$	$\{(\alpha-1)!\}^{-1}\int_0^1 e^{-u(s+1)}\, u(1-e^{-u})^{\alpha-1}\, du$
$\Psi(\frac{1}{2}s + \frac{1}{2}) - \Psi(\frac{1}{2}s)$	$2\int_0^1 u^{s+1}(1+u)^{-1}\, du$

$\sum_{\nu=1}^{s}(-1)^\nu/\nu$	$2\int_0^{\frac{1}{4}\pi}(-\tan^2 u)^s \tan u\, du - \ln 2$
$\sum_{\nu=0}^{s}(-1)^\nu/(2\nu+1)$	$-\int_0^{\frac{1}{4}\pi}(-\tan^2 u)^{s+1}\, du + \frac{1}{4}\pi$
$(s-1)!\,\zeta(s)$	$\int_0^\infty u^{s-1}(e^u-1)^{-1}\, du$
$s!\,\zeta(s)$	$\int_0^\infty u^s e^u(e^u-1)^{-2}\, du = \frac{1}{4}\int_0^\infty u^s \sinh^{-2}\frac{1}{2}u\, du$
$\dfrac{\zeta(s)}{(-s)!}$	$\dfrac{i}{2\pi}\int_\infty^{(0^+)}\dfrac{(-u)^{s-1}\, du}{e^u-1}$
$\dfrac{\zeta'(s)}{s\zeta(s)}$	$-\int_0^\infty \dfrac{1}{(1+u)^{s+1}}\Psi(u)\, du$
$(-1)^{s-1}B_{2s}/s;\quad B_0=1,\quad B_2=1/6,\quad \ldots$	$4\int_0^\infty \dfrac{u^{2s-1}}{e^{2\pi u}-1}\, du$
$(s-1)!\,t_s;\quad t_0=\frac{1}{2},\quad t_s=(1-2^{1-s})\zeta(s) = 1-2^{-s}+3^{-s}-\ldots$	$\int_0^\infty \dfrac{u^{s-1}}{e^u+1}\, du$

Function	Representation
$s!\,t_s$	$\int_0^\infty u^s e^u(e^u+1)^{-2}\,du = \frac{1}{4}\int_0^\infty u^s \cosh^{-2}\frac{1}{2}u\,du$
$t_s/(-s)!$	$\frac{i}{2\pi}\int_\infty^{(0^+)} \frac{(-u)^{s-1}\,du}{e^u+1}$
$S_s^m/s!$, Stirling numbers of the first kind	$\frac{1}{m!}\frac{1}{2\pi i}\oint \frac{\{\ln(1+u)\}^m}{u^{s+1}}\,du$
$C_s^m/s!$, Stirling numbers of the second kind	$\frac{1}{m!}\frac{1}{2\pi i}\oint \frac{(e^u-1)^m}{u^{s+1}}\,du$
$B_s/s!$, Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = 1/6$, ...	$\frac{1}{2\pi i}\oint \frac{du}{u^s(e^u-1)}$
$E_s/s!$, Euler numbers $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, ...	$\frac{1}{2\pi i}\oint \frac{\operatorname{sech} u\,du}{u^{s+1}}$

EXERCISES

1. In $S(x) = \sum_0^\infty x^s/(s+p+1)!$ choose as familiar function

$$S^0(x) = \sum_0^\infty x^s = (1-x)^{-1}, \qquad |x| < 1,$$

and introduce the sub-representation

$$\frac{1}{(s+p+1)!} = \frac{i}{2\pi}\int_\infty^{(0+)} \frac{e^{-u}\,du}{(-u)^{s+p+2}}.$$

Hence derive the integrals

$$(p, x)! = p!\,x^{p+1}\,e^{-x}\frac{1}{2\pi i}\int_\infty^{(0+)} \frac{e^{-u}\,du}{(-u)^{p+1}\,(u+x)}, \qquad x \text{ enclosed},$$

$$[p, x]! = -p!\,x^{p+1}\,e^{-x}\frac{1}{2\pi i}\int_\infty^{(0+)} \frac{e^{-u}\,du}{(-u)^{p+1}\,(u+x)}, \qquad x \text{ not enclosed}.$$

2. In $S(x) = \sum_0^\infty x^s/(s+p+1)!$ choose as familiar function

$$S^0(x) = \sum_0^\infty \frac{x^s}{(s+p+1)!}\frac{(s+\frac{1}{2}p)!}{s!} = \frac{\sqrt{\pi}\,e^{\frac{1}{2}x}}{x^{\frac{1}{2}p+\frac{1}{2}}}\,I_{\frac{1}{2}p+\frac{1}{2}}(\tfrac{1}{2}x),$$

and introduce the sub-representation

$$\frac{s!}{(s+\frac{1}{2}p)!} = \frac{1}{(\frac{1}{2}p-1)!}\int_0^1 u^s(1-u)^{\frac{1}{2}p-1}\,du.$$

Hence derive the integral

$$(p, x)! = 2^{p-1}p\,(\tfrac{1}{2}p-\tfrac{1}{2})!\,x^{\frac{1}{2}p+\frac{1}{2}}\,e^{-x}$$
$$\times\int_0^1 u^{-\frac{1}{2}p-\frac{1}{2}}(1-u)^{\frac{1}{2}p-1}\,e^{\frac{1}{2}ux}\,I_{\frac{1}{2}p+\frac{1}{2}}(\tfrac{1}{2}ux)\,du.$$

3. In $S(x) = \sum_0^\infty \Psi(s)\,x^s/s!$ choose

$$S^0(x) = \sum_0^\infty [\Psi(s) + C]\,x^s = -(1-x)^{-1}\ln(1-x), \qquad |x| < 1,$$

and deduce the integral

$$-Ei(-x) = -C - \ln x - \frac{e^{-x}}{2\pi i}\int_{\infty}^{(0+)} \frac{e^{-u}\ln(1 + x/u)}{u + x}\,du, \qquad x \text{ enclosed.}$$

4. In $S(x) = \sum_0^\infty \Psi(s)\, x^s/s!$ choose

$$S^0(x) = \sum_0^\infty \Psi(s)\, x^s/(s!)^2 = K_0(2x^{\frac{1}{2}}) + \tfrac{1}{2}I_0(2x^{\frac{1}{2}})\ln x,$$

and deduce the integral

$$-Ei(-x) = -\ln x + e^{-x}\int_0^\infty e^{-u}\{K_0[2(ux)^{\frac{1}{2}}] + \tfrac{1}{2}I_0[2(ux)^{\frac{1}{2}}]\ln ux\}\,du.$$

5. In

$$S(x) = \sum_0^\infty (s + a - 1)!\, x^s/s!\,(s + c - 1)!,$$

choose

$$S^0(x) = \sum_0^\infty (s + a - 1)!\, x^s/s! = (a - 1)!\,(1 - x)^{-a}, \qquad |x| < 1,$$

and deduce the integral

$$F(a, c, x) = -\frac{(c-1)!}{2\pi i}\int_{\infty}^{(0+)} \frac{e^{-u}\,du}{(-u)^c\,(1 + x/u)^a}, \qquad x \text{ enclosed.}$$

6. In

$$S(x) = \sum_0^\infty (s + a - 1)!\, x^s/s!\,(s + c - 1)!,$$

choose

$$S^0(x) = \sum_0^\infty (s + \tfrac{1}{2}c - 1)!\, x^s/s!\,(s + c - 1)! = \sqrt{\pi}\; x^{\frac{1}{4} - \frac{1}{2}c}\, e^{\frac{1}{2}x}\, I_{\frac{1}{2}c - \frac{1}{2}}(\tfrac{1}{2}x),$$

and deduce the integral

$$F(a, c, x) = \frac{\sqrt{\pi}\,(c-1)!\,x^{\frac{1}{2}-\frac{1}{2}c}}{(a-1)!\,(\frac{1}{2}c-a-1)!}\int_0^1 u^{a-\frac{1}{2}c-\frac{1}{2}}\,(1-u)^{\frac{1}{2}c-a-1}\,e^{\frac{1}{2}ux}\,I_{\frac{1}{2}c-\frac{1}{2}}(\tfrac{1}{2}ux)\,du.$$

Similarly establish the result

$$F(a, c, x) = \frac{\sqrt{\pi}\,(c-1)!\,x^{\frac{1}{2}-a}}{(a-1)!\,(c-2a-1)!}\int_0^1 u^{a-\frac{1}{2}}\,(1-u)^{c-2a-1}\,e^{\frac{1}{2}ux}\,I_{a-\frac{1}{2}}(\tfrac{1}{2}ux)\,du.$$

7. In

$$S(x) = \sum_0^\infty (s+c-a-1)!\,(-x)^s/s!\,(s+c-1)!,$$

choose

$$S^0(x) = \sum_0^\infty (-x)^s/s!\,(s+c-1) = x^{1-c}(c-2, x)!,$$

and deduce the integral

$$F(a, c, x) = \frac{(c-1)!}{(c-a-1)!\,(a-2)!}\,x^{1-c}\,e^x\int_0^1 u^{-a}\,(1-u)^{a-2}\,(c-2, ux)!\,du.$$

8. Specializing the results of questions (5) and (7), establish the following integral representations for the modified Bessel function:

$$I_p(x) = \begin{cases} -\dfrac{(p-\frac{1}{2})!}{\sqrt{\pi}\,(\frac{1}{2}x)^p}\dfrac{1}{2\pi i}\displaystyle\int_\infty^{(1+)} \frac{e^{-vx}\,dv}{(v^2-1)^{p+\frac{1}{2}}} \\[2ex] \dfrac{e^x}{\sqrt{\pi}\,(2x)^p\,(p-\frac{3}{2})!}\displaystyle\int_0^1 u^{-p-\frac{1}{2}}\,(1-u)^{p-\frac{3}{2}}\,(2p-1, 2ux)!\,du. \end{cases}$$

9. The function exponentially increasing when $\Re(x) > 0$, analogous to the then exponentially decreasing $K_p(x)$, is

$$\mathscr{K}_p(x) = \tfrac{1}{2}\pi\bigl(I_{-p}(x) + I_p(x)\bigr).$$

Show that its asymptotic expansion is

$$\mathscr{K}_p(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \frac{1}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!}\left[e^x \sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(2x)^r}\right.$$

$$\left. + \begin{pmatrix} i \\ -i \\ 0 \end{pmatrix} \cos \pi p \;\; e^{-x} \sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(-2x)^r}\right], \qquad \left.\begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 \end{matrix}\right\}.$$

10. Establish the representation

$$\mathscr{K}_p(x) = \int_0^\pi \cos p\theta \, e^{x\cos\theta}\, d\theta + \sin \pi p \int_0^\infty \sinh p\omega \;\; e^{-x\cosh\omega}\, d\omega, \qquad \Re(x) > 0.$$

11. In

$$S(x) = (\tfrac{1}{2}x)^{-p} J_p(x) = \sum_0^\infty (-\tfrac{1}{4}x^2)^s/s!\,(s+p)!,$$

choose

$$S^0(x) = \sum_0^\infty (-\tfrac{1}{4}x^2)^s/s! = e^{-\frac{1}{4}x^2},$$

and deduce the integral

$$J_p(x) = -\frac{1}{2\pi i}\int_\infty^{(0+)} \frac{e^{\frac{1}{2}x(v^{-1}-v)}\,dv}{(-v)^{p+1}}, \qquad \Re(x) > 0.$$

By taking a contour composed of a unit circle centred at $v = 0$, together with straight paths from $v = 1$ to ∞ above and below the cut along the real axis from 0 to ∞, derive Schläfli's representation

$$J_p(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - p\theta)\,d\theta - \frac{\sin \pi p}{\pi}\int_0^\infty e^{-p\omega - x\sinh\omega}\,d\omega,$$

$$\Re(x) > 0.$$

12. By writing $v = -e^{\omega}$ in the first integral representation of question (11), show that

$$J_p(x) = -\frac{i}{2\pi}\int_{\infty - i\pi}^{\infty + i\pi} e^{x\sinh\omega - p\omega}\, d\omega, \qquad \Re(x) > 0.$$

(This proves to be the most convenient integral representation to which to apply stationary-point methods, Chapter VIII, Section 5).

13. From Schläfli's representation for $J_p(x)$ (question 11), deduce the following representation for the Bessel function of the second kind:

$$\begin{aligned} Y_p(x) &= \cot \pi p \;\; J_p(x) - \operatorname{cosec} \pi p \;\; J_{-p}(x) \\ &= \frac{1}{\pi}\int_0^{\pi} \sin(x \sin\theta - p\theta)\, d\theta - \frac{1}{\pi}\int_0^{\infty} (e^{p\omega} + e^{-p\omega}\cos \pi p)\, e^{-x\sinh\omega}\, d\omega, \end{aligned}$$

$$\Re(x) > 0.$$

14. By dividing the path $\omega = -\infty$ to $\infty + i\pi$ into the three sections $-\infty$ to 0, 0 to $i\pi$, and $i\pi$ to $\infty + i\pi$, prove that

$$\int_{-\infty}^{\infty + i\pi} e^{x\sinh\omega - p\omega}\, d\omega = \int_0^{\infty} e^{p\mu - x\sinh\mu}\, d\mu + i\int_0^{\pi} e^{i(x\sinh\theta - p\theta)}\, d\theta$$

$$+ e^{-i\pi p}\int_0^{\infty} e^{-x\sinh v - pv}\, dv.$$

Comparing with the result of question (13), deduce the integral representation

$$Y_p(x) = -\frac{1}{2\pi}\left(\int_{-\infty}^{\infty + i\pi} + \int_{-\infty}^{\infty - i\pi}\right) e^{x\sinh\omega - p\omega}\, d\omega, \qquad \Re(x) > 0.$$

(This proves to be the most convenient integral representation to which to apply stationary-point methods, Chapter VIII, Section 6).

15. In

$$S(x) = \sum_1^{\infty} (2s-2)!\, \zeta(2s)/(-4\pi^2 x^2)^s,$$

choose

$$S^0(x) = \sum_1^\infty \zeta(2s)/(-4\pi^2x^2)^s = \tfrac{1}{2} - \frac{1}{4x}\coth\left(\frac{1}{2x}\right).$$

Hence establish Binet's "first representation"

$$\ln(x!) = \tfrac{1}{2}\ln 2\pi x + x(\ln x - 1) - x\int_0^\infty \left\{1 - \frac{u}{2x}\coth\left(\frac{u}{2x}\right)\right\}\frac{e^{-u}\,du}{u^2}.$$

16. Derive the Bose–Einstein integral

$$\mathscr{B}_p(x) = \frac{1}{p!}\int_0^\infty \frac{u^p\,du}{e^{u-x}-1}$$

(taking the principal value when x is positive)
from its convergent series

$$\mathscr{B}_p(x) = \sum_1^\infty e^{xs}/s^{p+1}, \qquad \Re(x) < 0.$$

17. In

$$S(x) = 2\sum_0^\infty \zeta(2r)/x^{2r-1}(-2r+p+1)!,$$

choose

$$S^0(x) = 2\sum_1^\infty \frac{1}{x^{2r-1}(-2r+p+1)!\,(2r-1)!}$$

$$= \frac{1}{p!}\left\{\left(1+\frac{1}{x}\right)^p - \left(1-\frac{1}{x}\right)^p\right\}, \qquad (2r-1)!\,\zeta(2r) = \int_0^\infty \frac{u^{2r-1}\,du}{e^u-1},$$

and derive the representation

$$\mathscr{B}_p(x) = \frac{x^{p+1}}{(p+1)!}\int_{-\infty}^\infty \left(1+\frac{u}{x}\right)^{p+1}\frac{e^u}{(e^u-1)^2}\,du + \begin{pmatrix} e^{i\pi p} \\ e^{-i\pi p} \\ \cos\pi p \end{pmatrix}\mathscr{B}_p(-x),$$

$$\left.\begin{array}{c} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 \end{array}\right\}.$$

Further exercises on deriving integral representations: *Chapter* IX, *question* 2; *Chapter* XVIII, *questions* 3–6, 9–12, 14–16.

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Chapter IV

Derivation of Asymptotic Power Series from Integral Representations of the Types

$$\int_0^\infty e^{-u} u^\sigma A(u/x)\, du, \quad \int_0^\infty (e^u \mp 1)^{-1} u^\sigma A(u/x)\, du, \text{ etc.}$$

1. In this chapter we suppose the function to have been expressed as an integral representation in which the integrand can be resolved into a product of two factors: one which is fast-varying and can be integrated exactly; and one, $A(u/x)$, which is slowly varying and expansible as a Taylor power series

$$A(u) = \sum_0^\infty a_r u^r \tag{1}$$

convergent within a certain circle.

The most frequently encountered case is that of the Laplace-type representation,

$$\int_0^\infty e^{-u} u^\sigma A(u/x)\, du, \qquad \sigma > -1, \tag{2}$$

where $e^{-u} A(u/x)$ is exponentially decreasing for large u whatever the phase of x. Inserting (1) into (2) and integrating term by term,

$$\int_0^\infty e^{-u} u^\sigma A(u/x)\, du = \sum_0^\infty (r+\sigma)!\, a_r/x^r. \tag{3}$$

Likewise,

$$\frac{i}{2\pi}\int_\infty^{(0+)} e^{-u}(-u)^\sigma A(u/x)\, du = \sum_0^\infty \frac{(-1)^r a_r}{(-r-\sigma-1)!\, x^r}$$

$$= \frac{1}{(-\sigma-1)!\,\sigma!}\sum_0^\infty \frac{(r+\sigma)!\, a_r}{x^r}. \tag{4}$$

The remainder after a *finite* number of terms, needed in the "Poincaré–Watson" definition of a complete asymptotic expansion, may be found by introducing the terminated Taylor series for $A(u)$. (The well known "Watson lemma" (1918) suffices only in the Poincaré approximation).

These power series are in general asymptotic rather than convergent, since the range of integration over u extends beyond any finite circle of convergence of the Taylor series for $A(u)$. Each such asymptotic power series represents a continuous function only within a phase sector bounded by its Stokes rays.

If σ is large or $A(u)$ fast-varying, the terms in (3) and (4) start increasing in magnitude inconveniently early even for moderately large $|x|$. This practical nuisance can be overcome only by including *all* fast-varying factors in the exponent, i.e. by rewriting (2) as $\int e^{-F(u)}G(u)\,du$, a form examined in Chapter V.

Extension of the method to cover other fast-varying factors is straightforward. For example,

$$\int_0^\infty (e^u - 1)^{-1} u^\sigma A(u/x)\,du = \sum_0^\infty (r+\sigma)!\,\zeta(r+\sigma+1)\,a_r/x^r \tag{5}$$

$$\int_0^\infty (e^u + 1)^{-1} u^\sigma A(u/x)\,du = \sum_0^\infty (r+\sigma)!\,t_{r+\sigma+1}\,a_r/x^r \tag{6}$$

$$\int_0^\infty e^u(e^u - 1)^{-2} u^\sigma A(u/x)\,du = \sum_0^\infty (r+\sigma)!\,\zeta(r+\sigma)\,a_r/x^r \tag{7}$$

$$\int_0^\infty e^u(e^u + 1)^{-2} u^\sigma A(u/x)\,du = \sum_0^\infty (r+\sigma)!\,t_{r+\sigma}\,a_r/x^r \tag{8}$$

$$\int_0^\infty [p,u]!\,u^\sigma A(u/x)\,du = \sum_0^\infty (r+\sigma+1)^{-1}(r+\sigma+p+1)!\,a_r/x^r \tag{9}$$

$$\int_0^\infty K_p(u)\,u^\sigma A(u/x)\,du$$

$$= 2^{\sigma-1}\sum_0^\infty \{\tfrac{1}{2}(r+\sigma+p-1)\}!\,\{\tfrac{1}{2}(r+\sigma-p-1)\}!\,a_r/(\tfrac{1}{2}x)^r, \tag{10}$$

where $t_r = (1 - 2^{1-r})\,\zeta(r)$, and $K_p(u)$ is the Bessel function of the third kind.

2. Incomplete Factorial Function

This function is normally defined by the integral representation

$$(p, x)! = \int_0^x t^p e^{-t}\, dt, \qquad p > -1, \tag{11}$$

or by the variant obtained on writing $\int_0^x = \int_0^\infty - \int_x^\infty$, i.e.

$$[p, x]! = \int_x^\infty t^p e^{-t}\, dt = p! - (p, x)!. \tag{12†}$$

To reduce (12) to the required standard form we take $t = x + u$; then

$$[p, x]! = x^p e^{-x} \int_0^\infty e^{-u}(1 + u/x)^p\, du. \tag{13}$$

(Note the singularity—branch point or pole—on the path of integration at $|\mathrm{ph}\, x| = \pi$). Since

$$A(u) = (1 + u)^p = p! \sum_0^\infty u^r/r!\,(-r + p)!,$$

the relevant quantities in (3) are

$$\sigma = 0, \qquad a_r = p!/r!(-r + p)!.$$

Hence

$$[p, x]! = p!\, x^p e^{-x} \sum_0^\infty \frac{1}{(-r + p)!\, x^r} = \frac{x^p e^{-x}}{(-p - 1)!} \sum_0^\infty \frac{(r - p - 1)!}{(-x)^r}, \qquad |\mathrm{ph}\, x| < \pi. \tag{14}$$

Alternatively, from III (11), namely

$$[p, x]! = \frac{x^p e^{-x}}{(-p - 1)!} \int_0^\infty \frac{u^{-p-1} e^{-u}\, du}{1 + u/x}, \qquad p < 0, \qquad |\mathrm{ph}\, x| < \pi, \tag{15}$$

we set $\sigma = -p - 1$ and expand $(1 + u)^{-1}$ to give $a_r = (-1)^r$. Application of (3) at once produces (14).

† The "general exponential integral" defined for $\Re(x) > 0$ by

$$E_p(x) = \int_1^\infty u^{-p} e^{-ux}\, dx = x^{p-1}[-p, x]!$$

is frequently introduced in place of $[p, x]!$

3. Exponential Integral

This is normally defined by the integral representation

$$-Ei(-x) = \int_x^\infty u^{-1} e^{-u}\, du. \tag{16}$$

Since by (12) this is just $[-1, x]!$, the asymptotic power series can be written down at once from (14) as

$$-Ei(-x) = x^{-1} e^{-x} \sum_0^\infty r!/(-x)^r, \qquad |\mathrm{ph}\, x| < \pi. \tag{17}$$

4. Confluent Hypergeometric Function

While usually introduced via its convergent series II (40), this function is sometimes defined by an integral representation (III, (19)) which can be written

$$F(a, c, x) = \frac{(c-1)!\, x^{a-c} e^x}{(a-1)!\,(c-a-1)!} \int_0^x v^{c-a-1} \left(1 + \frac{v}{(-x)}\right)^{a-1} e^{-v}\, dv. \tag{18}$$

Dissecting this into $\int_0^\infty - \int_x^\infty$, setting $v = x + u$ in the second integral and noting that by our phase convention making fractional powers definite

$$\left\{1 + \frac{v}{(-x)}\right\}^{a-1} = \left\{\frac{u}{(-x)}\right\}^{a-1} = -\begin{pmatrix} e^{i\pi a} \\ e^{-i\pi a} \\ \cos \pi a \end{pmatrix} x^{1-a} u^{a-1}, \qquad \left.\begin{matrix} 0 < \mathrm{ph}\, x < \pi \\ 0 > \mathrm{ph}\, x > -\pi \\ \mathrm{ph}\, x = 0 \end{matrix}\right\},$$

we arrive at the result

$$F(a, c, x) = \frac{(c-1)!}{(a-1)!} x^{a-c} e^x \psi(c-a, c, -x)$$
$$+ \begin{pmatrix} e^{i\pi a} \\ e^{-i\pi a} \\ \cos \pi a \end{pmatrix} \frac{(c-1)!}{(c-a-1)!} x^{-a} \psi(a, c, x), \qquad \left.\begin{matrix} 0 < \mathrm{ph}\, x < \pi \\ 0 > \mathrm{ph}\, x > -\pi \\ \mathrm{ph}\, x = 0 \end{matrix}\right\}, \tag{19}$$

where

$$\psi(a, c, x) = \frac{1}{(a-1)!} \int_0^\infty u^{a-1} \left(1 + \frac{u}{x}\right)^{c-a-1} e^{-u}\, du, \qquad |\mathrm{ph}\, x| < \pi. \tag{20}$$

This representation has already been derived by other means in the preceding chapter, together with the relation $\psi(a, c, x) = \psi(a - c + 1, 2 - c, x)$. At $\mathrm{ph}\, x = 0$ the principal value of the integral $\psi(c - a, c, -x)$ is to be understood.

Taking in (3) $\sigma = a - 1$ and noting that expansion of $(1 + u)^{c-a-1}$ gives

$$a_r = (c - a - 1)!/r!(-r + c - a - 1)!,$$

we have

$$\psi(a, c, x) = \frac{1}{(a - 1)!\,(a - c)!}\sum_0^\infty \frac{(r + a - 1)!\,(r + a - c)!}{r!(-x)^r}. \tag{21}$$

5. Modified Bessel Function

Specialization of the results of the preceding section to

$$p!\,I_p(x) = (\tfrac{1}{2}x)^p\, e^{\pm x}\, F(p + \tfrac{1}{2}, 2p + 1, \mp 2x) \tag{22}$$

leads to the asymptotic expansions

$$I_p(x) = (2\pi x)^{-\frac{1}{2}}\left[e^x\psi_p(-x) - \begin{pmatrix}\sin\pi p - i\cos\pi p\\ \sin\pi p + i\cos\pi p\\ \sin\pi p\end{pmatrix} e^{-x}\psi_p(x)\right]$$

$$\left.\begin{array}{l} 0 < \mathrm{ph}\, x < \pi \\ 0 > \mathrm{ph}\, x > -\pi \\ \mathrm{ph}\, x = 0 \end{array}\right\}, \tag{23}$$

where

$$\psi_p(x) = \psi_{-p}(x) = \psi(p + \tfrac{1}{2}, 2p + 1, 2x) = \frac{1}{(p - \frac{1}{2})!\,(-p - \frac{1}{2})!}$$

$$\times \sum_0^\infty \frac{(r + p - \frac{1}{2})!\,(r - p - \frac{1}{2})!}{r!\,(-2x)^r}. \tag{24}$$

Alternatively, we can start from the integral representation III (36) for the Bessel function of the third kind,

$$K_p(x) = \int_0^\infty e^{-x\cosh\omega}\cosh p\omega\, d\omega, \qquad \mathfrak{R}(x) > 0. \tag{25}$$

Putting $\cosh \omega = 1 + 2u^2$ where $u = \sinh \frac{1}{2}\omega$, we have

$$K_p(x) = 2e^{-x} \int_0^\infty e^{-2xu^2} \frac{\cosh(2p \sinh^{-1} u)}{\cosh(\sinh^{-1} u)} du, \qquad \Re(x) > 0.$$

Substitution of the Taylor series

$$\frac{\cosh(2p \sinh^{-1} u)}{\cosh(\sinh^{-1} u)} = \frac{(-\frac{1}{2})!}{(p - \frac{1}{2})!\,(-p - \frac{1}{2})!} \times \sum_0^\infty \frac{(r + p - \frac{1}{2})!\,(r - p - \frac{1}{2})!\,(-u^2)^r}{r!\,(r - \frac{1}{2})!} \tag{26}$$

followed by term by term integration leads to the asymptotic expansion

$$K_p(x) = (\pi/2x)^{\frac{1}{2}} e^{-x} \psi_p(x), \qquad |\mathrm{ph}\, x| < \pi. \tag{27}$$

6. Logarithm of Factorial

Starting from Binet's "second representation" III (40),

$$\ln(x!) = \tfrac{1}{2} \ln 2\pi x + x(\ln x - 1) + \frac{1}{\pi} \int_0^\infty \frac{\tan^{-1}(u/2\pi x)}{e^u - 1} du, \tag{28}$$

substitution of the Taylor series

$$\tan^{-1}\left(\frac{u}{2\pi x}\right) = \frac{u}{2\pi x} \sum_0^\infty \frac{1}{2r + 1}\left(-\frac{u^2}{4\pi^2 x^2}\right)^r, \qquad |u| < |2\pi x|, \tag{29}$$

followed by application of (5) with $\sigma = 0$ yields Stirling's asymptotic expansion

$$\ln(x!) = \tfrac{1}{2} \ln 2\pi x + x(\ln x - 1) + \frac{1}{\pi} \sum_{1,3,5,\ldots}^\infty \frac{(-1)^{\frac{1}{2}(r-1)}(r - 1)!\,\zeta(r + 1)}{(2\pi x)^r}, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{30}$$

Malmstén's representation III (39),

$$\ln(x!) = \int_0^\infty \left\{\frac{e^{-ux} - 1}{e^u - 1} + \frac{x}{e^u}\right\} \frac{du}{u}, \qquad \Re(x) > 0, \tag{31}$$

is not so easily treated directly, because of the logarithmic divergence of individual terms in the integrand at the lower limit $u = 0$. The corresponding representation for $\Psi(x)$, obtained from (31) by differentiation with respect to x, affords a more convenient starting point:

$$\Psi(x) = \int_0^\infty \left\{\frac{e^{-u}}{u} - \frac{e^{-ux}}{e^u - 1}\right\} du, \qquad \Re(x) > 0. \tag{32}$$

The function $\{(e^u - 1)^{-1} - u^{-1} + \frac{1}{2}\}$ can be expanded as a power series in u, the coefficients being expressible as Riemann zeta functions or Bernoulli numbers. We are therefore led to re-write (32) as

$$\Psi(x) = \int_0^\infty (e^{-u} - e^{-ux})\frac{du}{u} + \tfrac{1}{2}\int_0^\infty e^{-ux}\,du$$

$$- \int_0^\infty e^{-ux}\left(\frac{1}{e^u - 1} - \frac{1}{u} + \frac{1}{2}\right)du. \tag{33}$$

The first integral is a well known representation for $\ln x$, and the second is of course $1/2x$. Substitution in the third of the Taylor series

$$(e^u - 1)^{-1} - u^{-1} + \tfrac{1}{2} = -2\sum_1^\infty \zeta(2r)\,u^{2r-1}/(-4\pi^2)^r, \qquad |u| < 2\pi, \tag{34}$$

followed by term by term integration then leads to the asymptotic expansion

$$\Psi(x) = \ln x + 1/2x + 2\sum_1^\infty (2r-1)!\,\zeta(2r)/(-4\pi^2x^2)^r, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{35}$$

If this is integrated with respect to x, we regain (30) apart from the constant term. This constant can be found from the Malmstén integral by rather tricky limiting processes, but these will not be considered further here as there are much easier ways of establishing its value.

7. Fermi–Dirac Integral

This is defined by

$$\mathscr{F}_p(x) = \frac{1}{p!}\int_0^\infty \frac{v^p\,dv}{e^{v-x} + 1}. \tag{36}$$

For large positive values of x, $(e^{v-x} + 1)^{-1}$ is approximately unity for $v < x$, and then rapidly drops to almost nothing. This factor does not fulfil the requirements imposed in Section 1; for additional to the fast-varying region near $v = x$, the factor maintains a "large" and almost constant value over the broad range $0 < v < x$. Recalling the disappearance of such a constant on differentiation, we are led to integrate (36) by parts to obtain

$$\mathscr{F}_p(x) = \frac{1}{(p+1)!}\int_0^\infty \frac{e^{v-x}\,v^{p+1}\,dv}{(e^{v-x} + 1)^2}. \tag{37}$$

Dissecting this into $\int_{-\infty}^{\infty} - \int_{-\infty}^{0}$, placing $v = x + u$ in the first integral and introducing our phase convention making fractional powers definite,

$$\mathscr{F}_p(x) = \frac{x^{p+1}}{(p+1)!} \int_{-\infty}^{\infty} \frac{e^u}{(e^u+1)^2}\left(1+\frac{u}{x}\right)^{p+1} du + \begin{pmatrix} e^{i\pi p} \\ e^{-i\pi p} \\ \cos \pi p \end{pmatrix} \mathscr{F}_p(-x), \quad \left.\begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 \end{matrix}\right\}. \tag{38}$$

Taking in (8) $\sigma = 0$ and noting that expansion of $(1+u)^{p+1}$ gives

$$a_r = (p+1)!/r!\,(-r+p+1)!,$$

terms odd in u disappear on integration to leave the asymptotic expansion

$$\mathscr{F}_p(x) = 2x^{p+1} \sum_{0,2,4,\ldots}^{\infty} \frac{t_r}{(-r+p+1)!\,x^r} + \begin{pmatrix} e^{i\pi p} \\ e^{-i\pi p} \\ \cos \pi p \end{pmatrix} \mathscr{F}_p(-x), \quad \left.\begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 \end{matrix}\right\}, \tag{39}$$

which is equivalent to II (68).

8. Note on Alternative Expansion Points

It has from time to time been remarked, especially in semiconductor theory (Conwell and Weisskopf 1946, 1950; Dingle 1955), that in Laplace-type integrals expansion of $A(u/x)$ about the maximum of $e^{-u}u^{\sigma}$ would lead to a series with (initially) more rapidly decreasing terms than the customary one resulting from expansion about $u = 0$. An explicit development, starting from an initial approximation chosen this way, has been formulated by Franklin and Friedman (1957). Apart, however, from especially simple applications, the results usually turn out to be as complicated as those obtained by the more comprehensive approach of Chapters V–VIII, with the additional drawback that no explicit formula for general late terms has yet been derived. The objection of complexity applies still more to expansion of A about the median of $e^{-u}u^{\sigma}$, offsetting its even more rapid initial convergence.

Exercises

Derive the asymptotic power series for the following functions. (The answers can be checked from the terminated forms in the text and exercises of Chapter XXII).

1. $[p, x]! = -p!\, x^p e^{-x} \dfrac{1}{2\pi i} \displaystyle\int_{\infty}^{(0+)} \frac{e^{-u}\, du}{(-u)^{p+1}(1 + u/x)},$ x outwith contour.

2. $K_p(x) = \dfrac{\sqrt{\pi}\, (\frac{1}{2}x)^p}{(p - \frac{1}{2})!} \displaystyle\int_1^{\infty} e^{-vx}(v^2 - 1)^{p-\frac{1}{2}}\, dv,$ $\quad \Re(x) > 0.$

3. $\mathscr{K}_p(x) = \displaystyle\int_0^{\pi} e^{x \cos\theta} \cos p\theta\, d\theta + \sin \pi p \int_0^{\infty} e^{-x \cosh\omega} \sinh p\omega\, d\omega,$

$\Re(x) > 0.$

4. $\mathscr{C}_p(x) = \displaystyle\int_0^{\infty} e^{-x \sinh\omega} \cosh p\omega\, d\omega,$ $\quad \Re(x) > 0.$

Hint: write $u = \sinh\omega$ and insert the Taylor series for

$$\cosh(p \sinh^{-1} u)/\cosh(\sinh^{-1} u)$$

deduced from the analogous trigonometrical entry in Chapter III, Section 8.

5. $\mathscr{S}_p(x) = \displaystyle\int_0^{\infty} e^{-x \sinh\omega} \sinh p\omega\, d\omega,$ $\quad \Re(x) > 0.$

6. $\mathscr{A}(x) = \displaystyle\int_0^{\infty} e^{p\omega - x\sinh\omega}\, d\omega,$ $\quad \Re(x) > 0,$

essentially the Anger function of negative order.

Hint: add $\mathscr{C}_p$ and $\mathscr{S}_p$.

7. $\mathbf{H}_p(x) - Y_p(x) = \dfrac{(\frac{1}{2}x)^{p-1}}{\sqrt{\pi}\, (p - \frac{1}{2})!} \displaystyle\int_0^{\infty} e^{-u} \left(1 + \frac{u^2}{x^2}\right)^{p-\frac{1}{2}} du,$

where $\mathbf{H}$ is the Struve function.

8. $S_{qp}(x) = \dfrac{2^{q+1}\, x^{q-1}}{\{\frac{1}{2}(p - q - 1)\}!\, \{-\frac{1}{2}(p + q + 1)\}!} \displaystyle\int_0^{\infty} K_p(u) \frac{u^{-q}\, du}{1 + u^2/x^2},$

$|p| + q < 1,$

a Lommel function.

9. $\ln(x!) = \frac{1}{2}\ln 2\pi x + x(\ln x - 1) - x\int_0^\infty e^{-u}\left\{1 - \frac{u}{2x}\coth\left(\frac{u}{2x}\right)\right\}\frac{du}{u^2}.$

10. $\mathscr{B}_p(x) = \frac{1}{p!}\int_0^\infty \frac{v^p\,dv}{e^{v-x}-1}$ (principal value when x positive).

Hint: first deduce the representation

$$\mathscr{B}_p(x) = \frac{x^{p+1}}{(p+1)!}\int_{-\infty}^\infty \frac{e^u}{(e^u-1)^2}\left(1+\frac{u}{x}\right)^{p+1} du$$

$$+\begin{pmatrix} e^{i\pi p} \\ e^{-i\pi p} \\ \cos \pi p \end{pmatrix} \mathscr{B}_p(-x), \quad \left.\begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 \end{matrix}\right\}.$$

11. When q is negative, the major contribution to integrals containing the product $K_p(u)\,u^{-q}$ is centred on the region around $u_0 = (q^2 - p^2)^{\frac{1}{2}}$. Expanding $(x^2 + u^2)^{-1}$ about this point, derive the expansion

$$S_{qp}(x) = \frac{x^{q+1}}{x^2 + u_0^2}\left\{1 + \frac{2q-1}{x^2+u_0^2} + \frac{3(2q-1)(2q-3) - 4(q-2)u_0^2}{(x^2+u_0^2)^2}\right.$$

$$+ \frac{15(2q-1)(2q-3)(2q-5) - 2(68q^2 - 276q + 241)\,u_0^2 + 2(5q-4)\,u_0^4}{(x^2+u_0^2)^3}$$

$$\left.\ldots\right\},$$

and meditate on its limitations.

References

(*additional to those listed at end of Chapter II*).

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Chapter V

Derivation of Asymptotic Expansions from Integral Representations of the Forms

$$\int_{\text{limit}} e^{-F}\, G\, du \text{ and} \int_0 e^{-F}\, u^\sigma\, G\, du\,.$$

1. Plan of Chapters V–VIII

In the preceding chapter we confined attention to integral representations in which three conditions were fulfilled:

(i) One factor in the integrand was fast varying, e.g. e^{-u} in the simplest case.

(ii) This fast-varying factor decreased steadily as the integration variable moved away from one of the limits of integration, e.g. e^{-u} decreases steadily as u increases from the lower limit.

(iii) The product of the fast-varying factor and an arbitrary power of the variable of integration could be exactly integrated, e.g. $\int_0^\infty e^{-u}\, u^\sigma\, du = \sigma!$.

The generalization carried through in the present chapter is to remove condition (iii) and consider a more general fast-varying factor, written for convenience as $e^{-F(u)}$. As will be shown in the subsequent chapter, the further removal of condition (ii) leads to asymptotic expansions involving exactly the same algebraic expressions, so to obviate unnecessary repetition while at the same time introducing concepts methodically one at a time, we will lay out the exposition in the following manner:

Chapter V —Fast-varying factor decreasing steadily away from a limit of integration.

Chapter VI —Fast-varying factor having a stationary point part-way through the range of integration.

Chapter VII —Calculation of late terms in the theory of V and VI.

Chapter VIII—Applications of the theory in V–VII.

2. Linear Dependence of $F(u)$ at a Limit of Integration

Evaluation of $\int e^{-F} G\,du$. To reduce the number of symbols which need to be introduced we shall suppose in the derivation that the integral has already been reduced to a standard form in which the limit of integration is at $u = 0$. Let the value of F at this point be F_0. Then the assumptions of this section are

(A) $f = F - F_0 \to F_1 u$ as $u \to 0$,

(B) F increases steadily up to $+\infty$ towards the upper limit of integration, and

(C) the slowly-varying function $G(u)$ is expansible as a power series in zero and positive integer powers of u.

There are four alternative ways of proceeding, the first three facilitated by a preliminary change of variable from u to f, thus

$$\int_0 e^{-F} G\,du = e^{-F_0}\int_0^\infty e^{-f}(G/f')\,df, \qquad f' = df/du = dF/du. \tag{1}$$

In the *Taylor series method*, G/f' is expanded as a Taylor power series in f:

$$\frac{G}{f'} = \sum_0^\infty \left(\frac{d}{df}\right)^r \frac{G}{f'}\bigg|_{f=0} \frac{f^r}{r!}. \tag{2}$$

The integration over f in (1) is then effected term by term. The resultant series is asymptotic, rather than convergent, because the range of integration extends beyond the circle of convergence of (2), the radius of this circle being fixed by the zero of f' in the complex f-plane lying closest to the origin. Recalling the initial expression of G and F as functions of u and not of f, it is helpful to replace d/df by $d/F'du$. With this modification,

$$\int_0 e^{-F} G\,du = F_1{}^{-1} e^{-F_0} \sum_0^\infty L_r \tag{3}$$

where

$$L_r = F_1 \left(\frac{d}{F'du}\right)^r \frac{G}{F'}\bigg|_{u=0}, \tag{4}$$

F_1 signifying the first derivative at $u = 0$.

In the *method of repeated integration by parts*, the factor e^{-f} in the integrand of (1) is repeatedly integrated with respect to f, whereupon

$$\int_0^\infty e^{-f}\frac{G}{F'}\,df = -\left[e^{-f}\frac{G}{F'}\right]_0^\infty + \int_0^\infty e^{-f}\frac{d}{df}\frac{G}{F'}\,df$$

$$= -\left[e^{-f}\left\{\frac{G}{F'} + \frac{d}{df}\frac{G}{F'}\right\}\right]_0^\infty + \int_0^\infty e^{-f}\frac{d^2}{df^2}\frac{G}{F'}\,df,$$

and so on, leading again to (3) and (4). In this specific application, such repeated integration by parts is precisely equivalent to introduction in the integrand of a series for G/F' coinciding with the one and only Taylor series required in this application, (2) above. More generally, however, the deceptively simple-looking process of repeated integration by parts should be handled with the utmost circumspection, because it may be equivalent to introduction in the integrand of series which are

(a) invalid near one or other limit, or

(b) valid close to each of the limits, but which lead to different integration constants pertinent to the two end regions.

A warning example has already been noted in the footnote to Chapter I, Section 5.

The *reversion method* is based on Lagrange's (1770) reversion theorem. In the form in which it is required here (Section 5), if as supposed $f(u) \propto u$ as $u \to 0$, u can be expanded as a power series in f according to the formula

$$u = \sum_0^\infty \left(\frac{d}{du}\right)^r \left(\frac{u}{f}\right)^{r+1}\Bigg|_{u=0} \frac{f^{r+1}}{(r+1)!}, \tag{5}$$

and moreover the power series for a function of u is

$$H(u) = H(0) + \sum_0^\infty \left(\frac{d}{du}\right)^r \left(\frac{u}{f}\right)^{r+1} \frac{dH}{du}\Bigg|_{u=0} \frac{f^{r+1}}{(r+1)!}. \tag{6}$$

The most expeditious application of this reversion theorem to the given integral consists in setting in (6) $H(u) = \int_0^u G\,du$ and then differentiating both sides with respect to f, yielding

$$\frac{G}{f'} = \sum_0^\infty \left(\frac{d}{du}\right)^r \left(\frac{u}{f}\right)^{r+1} G\Bigg|_{u=0} \frac{f^r}{r!}. \tag{7}$$

Term by term integration over f, extending beyond the circle of convergence of (7), leads again to (3) but with an alternative form for the L_r:

$$L_r = F_1 \left(\frac{d}{du}\right)^r \left(\frac{u}{f}\right)^{r+1} G \bigg|_{u=0}. \tag{8}$$

That there is a one-to-one correspondence between (4) and (8) for each value of r can be established by denoting the rth differential coefficient at the origin as $r!$ times the coefficient of f^r in (4), and of u^r in (8), and expressing these coefficients as contour integrals. Then (4) and (8) lead to the same representation

$$L_r = \frac{F_1 \ r!}{2\pi i} \oint \frac{G\,du}{f^{r+1}}, \tag{9}$$

$\oint$ denoting integration round a small circle enclosing the point $f = 0$. Notwithstanding the basic equivalence of the terms, the straight r-fold differentiation with respect to u prescribed in (8) is frequently simpler to execute than the r-fold mixed algebraic and differential operation required in (4).

Fourth, and least satisfactory, is the *method of expanding most of the exponential.* Retaining in its original exponential form only the linear term in $F(u)$ and expanding the rest in rising powers of u,

$$\int_0^\infty e^{-F} G\,du = \sum_0^\infty (\text{coefficient of } u^s \text{ in } G\,e^{-F+F_1 u}) \times \int_0^\infty e^{-F_1 u} u^s\,du$$

$$= F_1^{-1} e^{-F_0} \sum_0^\infty U_s/F_1^{\ s}, \tag{10}$$

where

$$U_s = s! \times \text{coefficient of } u^s \text{ in } G(u)\,e^{-F(u)+F_0+F_1 u}. \tag{11}$$

The sum $\Sigma U_s/F_1^{\ s}$ is equivalent to the earlier ΣL_r, but with contributions more crudely ordered—simply according to inverse powers of F_1. The procedure is mentioned here because, though decidedly inferior to available alternatives in the present context, it does come into its own when additional complicated factors are introduced into the integrand, as in Chapter XI, Section 3; for then it affords an acceptable compromise between convenient ordering and complexity.

Only in the most elementary applications can the general term be evaluated exactly and explicitly. We therefore quote the first terms of the asymptotic expansion *in extenso*, adopting the notation

$$F_\nu = \left(\frac{d}{du}\right)^\nu F(u)\bigg|_{\text{limit}}, \qquad G_\nu = \left(\frac{d}{du}\right)^\nu G(u)\bigg|_{\text{limit}},$$

$$\int_{\text{limit}} e^{-F}\, G\, du = F_1^{-1} e^{-F_0} \sum_0^\infty L_r. \tag{12}$$

Then

$$L_0 = G_0, \qquad L_1 = -F_1^{-2}\{G_0F_2 - G_1F_1\},$$

$$L_2 = F_1^{-4}\{G_0(3F_2^2 - F_1F_3) - 3G_1F_1F_2 + G_2F_1^2\},$$

$$\begin{aligned} L_3 = -F_1^{-6}\{&G_0(15F_2^3 - 10F_1F_2F_3 + F_1^2F_4) - G_1F_1(15F_2^2 - 4F_1F_3) \\ &+ 6G_2F_1^2F_2 - G_3F_1^3\}, \end{aligned}$$

$$\begin{aligned} L_4 = F_1^{-8}\{&G_0(105F_2^4 - 105F_1F_2^2F_3 + 10F_1^2F_3^2 + 15F_1^2F_2F_4 - F_1^3F_5) \\ &- 5G_1F_1(21F_2^3 - 12F_1F_2F_3 + F_1^2F_4) + 5G_2F_1^2(9F_2^2 - 2F_1F_3) \\ &- 10G_3F_1^3F_2 + G_4F_1^4\}, \end{aligned}$$

$$\begin{aligned} L_5 = -F_1^{-10}\{&G_0(945F_2^5 - 1260F_1F_2^3F_3 + 280F_1^2F_2F_3^2 + 210F_1^2F_2^2F_4 \\ &- 35F_1^3F_3F_4 - 21F_1^3F_2F_5 + F_1^4F_6) - G_1F_1(945F_2^4 \\ &- 840F_1F_2^2F_3 + 70F_1^2F_3^2 + 105F_1^2F_2F_4 - 6F_1^3F_5) \\ &+ 15G_2F_1^2(28F_2^3 - 14F_1F_2F_3 + F_1^2F_4) - 5G_3F_1^3(21F_2^2 \\ &- 4F_1F_3) + 15G_4F_1^4F_2 - G_5F_1^5\}, \end{aligned}$$

$$\begin{aligned} L_6 = F_1^{-12}\{&G_0(10{,}395F_2^6 - 17{,}325F_1F_2^4F_3 + 6{,}300F_1^2F_2^2F_3^2 \\ &+ 3{,}150F_1^2F_2^3F_4 - 280F_1^3F_3^3 - 1{,}260F_1^3F_2F_3F_4 \\ &- 378F_1^3F_2^2F_5 + 35F_1^4F_4^2 + 56F_1^4F_3F_5 + 28F_1^4F_2F_6 \\ &- F_1^5F_7) - 7G_1F_1(1{,}485F_2^5 - 1{,}800F_1F_2^3F_3 + 360F_1^2F_2F_3^2 \\ &+ 270F_1^2F_2^2F_4 - 40F_1^3F_3F_4 - 24F_1^3F_2F_5 + F_1^4F_6) \\ &+ 7G_2F_1^2(675F_2^4 - 540F_1F_2^2F_3 + 40F_1^2F_3^2 + 60F_1^2F_2F_4 \\ &- 3F_1^3F_5) - 35G_3F_1^3(36F_2^3 - 16F_1F_2F_3 + F_1^2F_4) \\ &+ 35G_4F_1^4(6F_2^2 - F_1F_3) - 21G_5F_1^5F_2 + G_6F_1^6\}, \end{aligned}$$

$$
\begin{aligned}
L_7 = -F_1{}^{-14}\{&G_0(135{,}135F_2{}^7 - 270{,}270F_1F_2{}^5F_3 + 138{,}600F_1{}^2F_2{}^3F_3{}^2 \\
&- 15{,}400F_1{}^3F_2F_3{}^3 + 51{,}975F_1{}^2F_2{}^4F_4 - 34{,}650F_1{}^3F_2{}^2F_3F_4 \\
&+ 2{,}100F_1{}^4F_3{}^2F_4 + 1{,}575F_1{}^4F_2F_4{}^2 - 6{,}930F_1{}^3F_2{}^3F_5 \\
&+ 2{,}520F_1{}^4F_2F_3F_5 - 126F_1{}^5F_4F_5 + 630F_1{}^4F_2{}^2F_6 - 84F_1{}^5F_3F_6 \\
&- 36F_1{}^5F_2F_7 + F_1{}^6F_8) - G_1F_1(135{,}135F_2{}^6 - 207{,}900F_1F_2{}^4F_3 \\
&+ 69{,}300F_1{}^2F_2{}^2F_3{}^2 - 2{,}800F_1{}^3F_3{}^3 + 34{,}650F_1{}^2F_2{}^3F_4 \\
&- 12{,}600F_1{}^3F_2F_3F_4 + 315F_1{}^4F_4{}^2 - 3{,}780F_1{}^3F_2{}^2F_5 \\
&+ 504F_1{}^4F_3F_5 + 252F_1{}^4F_2F_6 - 8F_1{}^5F_7) + 14G_2F_1{}^2(4{,}455F_2{}^5 \\
&- 4{,}950F_1F_2{}^3F_3 + 900F_1{}^2F_2F_3{}^2 + 675F_1{}^2F_2{}^2F_4 - 90F_1{}^3F_3F_4 \\
&- 54F_1{}^3F_2F_5 + 2F_1{}^4F_6) - 7G_3F_1{}^3(2{,}475F_2{}^4 - 1{,}800F_1F_2{}^2F_3 \\
&+ 120F_1{}^2F_3{}^2 + 180F_1{}^2F_2F_4 - 8F_1{}^3F_5) + 70G_4F_1{}^4(45F_2{}^3 \\
&- 18F_1F_2F_3 + F_1{}^2F_4) - 14G_5F_1{}^5(27F_2{}^2 - 4F_1F_3) \\
&+ 28G_6F_1{}^6F_2 - G_7F_1{}^7\}.
\end{aligned}
$$

Calculation of yet higher terms is not only laborious, but also of little material assistance in practice. When $F_1/\sqrt{F_2}$ is large, high accuracy is already assured from the terms quoted. When it is not so large, higher terms will very soon start increasing in magnitude on account of the asymptotic nature of the expansion, and then require interpretation through a terminant (Chapters XXI onwards). While it is true that this terminant can be estimated from exact values of the first few increasing terms by a fitting and differencing procedure, its precise calculation would require knowledge of the *general* late term. Effort is therefore far more profitably expended on the asymptotic evaluation of the general late term, the subject of Chapter VII, than on individual calculations of the next few higher terms.

Evaluation of $\int e^{-F} u^{\sigma} G\,du$. So far it has been assumed that the slowly-varying factor in the integrand can be expanded as a power series in zero and positive integer powers of u. The corresponding asymptotic series for the more general integral

$$\int_0 e^{-F} u^{\sigma} G\,du, \qquad \sigma > -1,$$

where $G(u)$ is as before expansible in zero and positive integer powers of u, can be transcribed from the earlier result by introducing an artifice aptly described as "conversion by power identification" (further exemplified in Chapter X, questions 1 and 16).

Let us imagine for the moment the earlier simpler integral $\int e^{-F}\, G\, du$ to have been evaluated by the crude but effective fourth method above, i.e. by retaining only $e^{-F_0}e^{-F_1u}$ in exponential form and expanding the rest of the integrand in rising powers of u. Omitting the multipliers including e^{-F_0}, a typical integral in the result would have been

$$\int_0^\infty e^{-F_1u}\, u^n\, du = n!/F_1^{\,1+n}.$$

Thus in such a formulation the power of u embodied in an integrand is immediately identified by the inverse power of F_1 appearing in the contribution to the final asymptotic series. In our required extension the above integral stands to be replaced by

$$\int_0^\infty e^{-F_1u}\, u^{\sigma+n}\, du = (\sigma+n)!/F_1^{\,1+\sigma+n}.$$

Hence in the extension of

$$\int_{\text{limit}} e^{-F}\, G\, du = F_1^{\,-1}\, e^{-F_0} \sum_0^\infty L_r$$

to

$$\int_0 e^{-F}\, u^\sigma\, G\, du = \sigma!\, F_1^{\,-1-\sigma}\, e^{-F_0} \sum_0^\infty L_r^{(\sigma)}, \tag{13}$$

L_r can be converted into $L_r^{(\sigma)}$ by replacing $1/F_1^{\,n}$ with $(\sigma+n)!/\sigma!n!F_1^{\,n}$. This corresponds to the integral representation (question 3)

$$L_r^{(\sigma)} = F_1^{\,1+\sigma}\frac{(\sigma+r)!}{\sigma!}\,\frac{1}{2\pi i}\oint \frac{u^\sigma\, G\, du}{f^{\sigma+r+1}}. \tag{14}$$

The first few values are as follows:

$$L_0^{(\sigma)} = G_0, \qquad L_1^{(\sigma)} = -\tfrac{1}{2}(\sigma+1)\,F_1^{\,-2}\{(\sigma+2)G_0F_2 - 2G_1F_1\},$$

$$L_2^{(\sigma)} = \frac{1}{24}(\sigma+1)(\sigma+2)\,F_1^{\,-4}\{(\sigma+3)G_0[3(\sigma+4)F_2^{\,2} - 4F_1F_3]$$
$$- 12(\sigma+3)G_1F_1F_2 + 12G_2F_1^{\,2}\},$$

$$L_3^{(\sigma)} = -\frac{1}{48}(\sigma+1)(\sigma+2)(\sigma+3)\,F_1^{\,-6}\{(\sigma+4)G_0[(\sigma+5)(\sigma+6)F_2^{\,3}$$
$$- 4(\sigma+5)F_1F_2F_3 + 2F_1^{\,2}F_4] - 2(\sigma+4)G_1F_1[3(\sigma+5)F_2^{\,2}$$
$$- 4F_1F_3] + 12(\sigma+4)G_2F_1^{\,2}F_2 - 8G_3F_1^{\,3}\},$$

$$L_4^{(\sigma)} = \frac{1}{5760}(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)F_1^{-8}\{(\sigma+5)G_0[15(\sigma+6)$$
$$\times(\sigma+7)(\sigma+8)F_2^4 - 120(\sigma+6)(\sigma+7)F_1F_2^2F_3$$
$$+ 80(\sigma+6)F_1^2F_3^2 + 120(\sigma+6)F_1^2F_2F_4 - 48F_1^3F_5]$$
$$- 120(\sigma+5)G_1F_1[(\sigma+6)(\sigma+7)F_2^3 - 4(\sigma+6)F_1F_2F_3$$
$$+ 2F_1^2F_4] + 120(\sigma+5)G_2F_1^2[3(\sigma+6)F_2^2 - 4F_1F_3]$$
$$- 480(\sigma+5)G_3F_1^3F_2 + 240G_4F_1^4\},$$

$$L_5^{(\sigma)} = -\frac{1}{11{,}520}(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)F_1^{-10}\{(\sigma+6)$$
$$\times G_0[3(\sigma+7)(\sigma+8)(\sigma+9)(\sigma+10)F_2^5 - 40(\sigma+7)$$
$$\times(\sigma+8)(\sigma+9)F_1F_2^3F_3 + 80(\sigma+7)(\sigma+8)F_1^2F_2F_3^2$$
$$+ 60(\sigma+7)(\sigma+8)F_1^2F_2^2F_4 - 80(\sigma+7)F_1^3F_3F_4$$
$$- 48(\sigma+7)F_1^3F_2F_5 + 16F_1^4F_6] - 2(\sigma+6)G_1F_1[15(\sigma+7)$$
$$\times(\sigma+8)(\sigma+9)F_2^4 - 120(\sigma+7)(\sigma+8)F_1F_2^2F_3$$
$$+ 80(\sigma+7)F_1^2F_3^2 + 120(\sigma+7)F_1^2F_2F_4 - 48F_1^3F_5]$$
$$+ 120(\sigma+6)G_2F_1^2[(\sigma+7)(\sigma+8)F_2^3 - 4(\sigma+7)F_1F_2F_3$$
$$+ 2F_1^2F_4] - 80(\sigma+6)G_3F_1^3[3(\sigma+7)F_2^2 - 4F_1F_3]$$
$$+ 240(\sigma+6)G_4F_1^4F_2 - 96G_5F_1^5\},$$

$$L_6^{(\sigma)} = \frac{1}{2{,}903{,}040}(\sigma+1)\ldots(\sigma+6)F_1^{-12}\{(\sigma+7)G_0[63(\sigma+8)\ldots(\sigma+12)F_2^6$$
$$- 1{,}260(\sigma+8)\ldots(\sigma+11)F_1F_2^4F_3 + 5{,}040(\sigma+8)(\sigma+9)$$
$$\times(\sigma+10)F_1^2F_2^2F_3^2 + 2{,}520(\sigma+8)(\sigma+9)(\sigma+10)F_1^2F_2^3F_4$$
$$-2{,}240(\sigma+8)(\sigma+9)F_1^3F_3^3 - 10{,}080(\sigma+8)$$
$$\times(\sigma+9)F_1^3F_2F_3F_4 - 3{,}024(\sigma+8)(\sigma+9)F_1^3F_2^2F_5$$
$$+ 2{,}520(\sigma+8)F_1^4F_4^2 + 4{,}032(\sigma+8)F_1^4F_3F_5$$
$$+ 2{,}016(\sigma+8)F_1^4F_2F_6 - 576F_1^5F_7]$$
$$- 252(\sigma+7)G_1F_1[3(\sigma+8)\ldots(\sigma+11)F_2^5$$
$$- 40(\sigma+8)(\sigma+9)(\sigma+10)F_1F_2^3F_3 + 80(\sigma+8)$$
$$\times(\sigma+9)F_1^2F_2F_3^2 + 60(\sigma+8)(\sigma+9)F_1^2F_2^2F_4$$

$$- 80(\sigma + 8)F_1{}^3F_3F_4 - 48(\sigma + 8)F_1{}^3F_2F_5 + 16F_1{}^4F_6]$$
$$+ 252(\sigma + 7)G_2F_1{}^2[15(\sigma + 8)(\sigma + 9)(\sigma + 10)F_2{}^4$$
$$- 120(\sigma + 8)(\sigma + 9)F_1F_2{}^2F_3 + 80(\sigma + 8)F_1{}^2F_3{}^2$$
$$+ 120(\sigma + 8)F_1{}^2F_2F_4 - 48F_1{}^3F_5] - 10{,}080(\sigma + 7)G_3F_1{}^3$$
$$\times [(\sigma + 8)(\sigma + 9)F_2{}^3 - 4(\sigma + 8)F_1F_2F_3 + 2F_1{}^2F_4]$$
$$+ 5{,}040(\sigma + 7)G_4F_1{}^4[3(\sigma + 8)F_2{}^2 - 4F_1F_3]$$
$$- 12{,}096(\sigma + 7)G_5F_1{}^5F_2 + 4{,}032G_6F_1{}^6\}.$$

3. Quadratic Dependence of $F(u)$ at a Limit of Integration

Evaluation of $\int e^{-F}G\,du$. The integral will be supposed to have been reduced to a standard form in which the limit of integration is at $u = 0$, so $F - F_0 \to \frac{1}{2}F_2u^2$ as $u \to 0$; and as before F increases steadily up to ∞ towards the upper limit of integration.

The processes of reduction to exactly integrable terms can be made to follow lines closely similar to those of Section 2 by introducing a variable defined (when $F_2 > 0$) by

$$f = (F - F_0)^{\frac{1}{2}} \underset{u\to 0}{\to} (\tfrac{1}{2}F_2)^{\frac{1}{2}}u,$$

the positive root being selected. The integral to be investigated can then be written as

$$e^{-F_0}\int e^{-f^2}G\,du = e^{-F_0}\int_0^\infty e^{-f^2}(G/f')\,df.$$

When $F_2 > 0$ the path in the u-plane is of course similar to that in the f-plane at least throughout the region of significant contribution, i.e. the path starts at $u = 0$ and runs along the positive real axis.

Extension to $F_2 < 0$ can be effected by the replacement $F_2{}^{\frac{1}{2}} \equiv -i(-F_2)^{\frac{1}{2}}$ in both formulation and answer. Since this corresponds to

$$f = -i(F_0 - F)^{\frac{1}{2}} \underset{u\to 0}{\to} (-\tfrac{1}{2}F_2)^{\frac{1}{2}}(-iu),$$

i.e. $u \propto if$ for small u, the path now starts at $u = 0$ and runs up the imaginary axis. More generally, the path runs from the limit out into the first quadrant, the exact direction being immaterial.

In the Taylor series method G/f' is expanded in powers of f exactly as in (2), and integration effected term by term through

$$\int_0^\infty e^{-f^2}f^r\,df = \tfrac{1}{2}\,(\tfrac{1}{2}r - \tfrac{1}{2})! = \sqrt{\pi}\;r!/2^{r+1}(\tfrac{1}{2}r)!. \qquad (15)$$

The result is

$$\int_0 e^{-F} G\,du = (\pi/2F_2)^{\frac{1}{2}} e^{-F_0} \sum_0^\infty Q_r \tag{16}$$

where

$$Q_r = \frac{(\frac{1}{2}F_2)^{\frac{1}{2}}}{(\frac{1}{2}r)!} \left(\frac{d}{2f'\,du}\right)^r \frac{G}{f'}\bigg|_{u=0}. \tag{17}$$

In the reversion method, G/f' is expanded exactly as in (7), leading to the alternative form

$$Q_r = \frac{(\frac{1}{2}F_2)^{\frac{1}{2}}}{(\frac{1}{2}r)!} \left(\frac{d}{2\,du}\right)^r \left(\frac{u}{f}\right)^{r+1} G\bigg|_{u=0}. \tag{18}$$

Expressing (17) and (18) as contour integrals, both lead to the representation

$$Q_r = \left(\frac{F_2}{2\pi}\right)^{\frac{1}{2}} \frac{(\frac{1}{2}r - \frac{1}{2})!}{2\pi i} \oint \frac{G\,du}{f^{r+1}}. \tag{19}$$

The expansion (16) is asymptotic, rather than convergent, because the range of integration over f extends beyond the circle of convergence of the Taylor and Lagrange power series in f.

As in the linear case, only in the most elementary applications can the general term be evaluated exactly and explicitly, so we quote the first terms of the asymptotic expansion *in extenso*, adopting the notation

$$F_\nu = \left(\frac{d}{du}\right)^\nu F(u)\bigg|_{\text{limit}}, \qquad F_1 = 0, \qquad G_\nu = \left(\frac{d}{du}\right)^\nu G(u)\bigg|_{\text{limit}},$$

$$\int_{\text{limit}} e^{-F} G\,du = (\pi/2F_2)^{\frac{1}{2}} e^{-F_0} \sum_0^\infty Q_r. \tag{20}$$

Then

$$Q_0 = G_0, \qquad Q_1 = -\frac{\sqrt{2}}{3\sqrt{\pi}\,F_2^{\frac{3}{2}}}\{G_0F_3 - 3G_1F_2\},$$

$$Q_2 = \frac{1}{24F_2^{\,3}}\{G_0(5F_3^{\,2} - 3F_2F_4) - 12G_1F_2F_3 + 12G_2F_2^{\,2}\},$$

$$Q_3 = -\frac{\sqrt{2}}{135\sqrt{\pi}F_2^{\,9/2}}\{G_0(40F_3^{\,3} - 45F_2F_3F_4 + 9F_2^{\,2}F_5) - 45G_1F_2 \times (2F_3^{\,2} - F_2F_4) + 90G_2F_2^{\,2}F_3 - 45G_3F_2^{\,3}\},$$

$$Q_4 = \frac{1}{1{,}152F_2^6}\{G_0(385F_3^4 - 630F_2F_3^2F_4 + 105F_2^2F_4^2 + 168F_2^2F_3F_5 - 24F_2^3F_6) - 24G_1F_2(35F_3^3 - 35F_2F_3F_4 + 6F_2^2F_5) + 120G_2F_2^2(7F_3^2 - 3F_2F_4) - 480G_3F_2^3F_3 + 144G_4F_2^4\},$$

$$Q_5 = \frac{-\sqrt{2}}{2{,}835\sqrt{\pi}\,F_2^{15/2}}\{G_0(1{,}960F_3^5 - 4{,}200F_2F_3^3F_4 + 1{,}575F_2^2F_3F_4^2 + 1{,}260F_2^2F_3^2F_5 - 378F_2^3F_4F_5 - 252F_2^3F_3F_6 + 27F_2^4F_7) - 21G_1F_2(200F_3^4 - 300F_2F_3^2F_4 + 45F_2^2F_4^2 + 72F_2^2F_3F_5 - 9F_2^3F_6) + 21G_2F_2^2(200F_3^3 - 180F_2F_3F_4 + 27F_2^2F_5) - 315G_3F_2^3(8F_3^2 - 3F_2F_4) + 945G_4F_2^4F_3 - 189G_5F_2^5\},$$

$$Q_6 = \frac{1}{414{,}720F_2^9}\{G_0(425{,}425F_3^6 - 1{,}126{,}125F_2F_3^4F_4 + 675{,}675F_2^2F_3^2F_4^2 - 51{,}975F_2^3F_4^3 + 360{,}360F_2^2F_3^3F_5 - 249{,}480F_2^3F_3F_4F_5 + 13{,}608F_2^4F_5^2 - 83{,}160F_2^3F_3^2F_6 + 22{,}680F_2^4F_4F_6 + 12{,}960F_2^4F_3F_7 - 1{,}080F_2^5F_8) - 180G_1F_2(5{,}005F_3^5 - 10{,}010F_2F_3^3F_4 + 3{,}465F_2^2F_3F_4^2 + 2{,}772F_2^2F_3^2F_5 - 756F_2^3F_4F_5 - 504F_2^3F_3F_6 + 48F_2^4F_7) + 1{,}260G_2F_2^2(715F_3^4 - 990F_2F_3^2F_4 + 135F_2^2F_4^2 + 216F_2^2F_3F_5 - 24F_2^3F_6) - 10{,}080G_3F_2^3(55F_3^3 - 45F_2F_3F_4 + 6F_2^2F_5) + 75{,}600G_4F_2^4(3F_3^2 - F_2F_4) - 60{,}480G_5F_2^5F_3 + 8{,}640G_6F_2^6\},$$

$$Q_7 = -\frac{\sqrt{2}}{8{,}505\sqrt{\pi}\,F_2^{21/2}}[G_0\{22{,}400F_3^7 - 70{,}560F_2F_3^5F_4 + 11{,}760F_2^2F_3^3(5F_4^2 + 2F_3F_5) - 735F_2^3F_3(15F_4^3 + 36F_3F_4F_5 + 8F_3^2F_6) + 27F_2^4(105F_4^2F_5 + 140F_3F_4F_6 + 84F_3F_5^2 + 40F_3^2F_7) - 27F_2^5(10F_4F_7 + 14F_5F_6 + 5F_3F_8) + 9F_2^6F_9\} - 3G_1F_2\{15{,}680F_3^6 - 39{,}200F_2F_3^4F_4 + 1{,}470F_2^2F_3^2(15F_4^2 + 8F_3F_5) - 315F_2^3(5F_4^3 + 24F_3F_4F_5 + 8F_3^2F_6) + 18F_2^4(21F_5^2 + 35F_4F_6 + 20F_3F_7) - 27F_2^5F_8\}$$

$$+ 6G_2F_2^2\{7{,}840F_3^5 - 14{,}700F_2F_3^3F_4 + 945F_2^2F_3(5F_4^2 + 4F_3F_5) - 315F_2^3(3F_4F_5 + 2F_3F_6) + 54F_2^4F_7\} - 21G_3F_2^3\{1{,}400F_3^4 - 1{,}800F_2F_3^2F_4 + 45F_2^2(5F_4^2 + 8F_3F_5) - 36F_2^3F_6\} + 126G_4F_2^4\{100F_3^3 - 75F_2F_3F_4 + 9F_2^2F_5\} - 378G_5F_2^5\{10F_3^2 - 3F_2F_4\} + 756G_6F_2^6F_3 - 81G_7F_2^7],$$

$$Q_8 = \frac{1}{39{,}813{,}120F_2^{12}}[G_0\{185{,}910{,}725F_3^8 - 678{,}978{,}300F_2F_3^6F_4 + 29{,}099{,}070F_2^2F_3^4(25F_4^2 + 8F_3F_5) - 15{,}315{,}300F_2^3F_3^2 \times (15F_4^3 + 24F_3F_4F_5 + 4F_3^2F_6) + 19{,}305F_2^4(525F_4^4 + 5{,}040F_3F_4^2F_5 + 2{,}016F_3^2F_5^2 + 3{,}360F_3^2F_4F_6 + 640F_3^3F_7) - 185{,}328F_2^5(35F_4^2F_6 + 42F_4F_5^2 + 56F_3F_5F_6 + 40F_3F_4F_7 + 10F_3^2F_8) + 9{,}504F_2^6(42F_6^2 + 45F_4F_8 + 72F_5F_7 + 20F_3F_9) - 10{,}368F_2^7F_{10}\} - 48G_1F_2\{8{,}083{,}075F_3^7 - 24{,}249{,}225F_2F_3^5F_4 + 3{,}828{,}825F_2^2F_3^3(5F_4^2 + 2F_3F_5) - 225{,}225F_2^3F_3(15F_4^3 + 36F_3F_4F_5 + 8F_3^2F_6) + 7{,}722F_2^4 \times (105F_4^2F_5 + 140F_3F_4F_6 + 84F_3F_5^2 + 40F_3^2F_7) - 7{,}128F_2^5(10F_4F_7 + 14F_5F_6 + 5F_3F_8) + 2{,}160F_2^6F_9\} + 48G_2F_2^2\{8{,}083{,}075F_3^6 - 19{,}144{,}125F_2F_3^4F_4 + 675{,}675 \times F_2^2F_3^2(15F_4^2 + 8F_3F_5) - 135{,}135F_2^3(5F_4^3 + 24F_3F_4F_5 + 8F_3^2F_6) + 7{,}128F_2^4(21F_5^2 + 35F_4F_6 + 20F_3F_7) - 9{,}720F_2^5F_8\} - 2{,}880G_3F_2^3\{85{,}085F_3^5 - 150{,}150F_2F_3^3F_4 + 9{,}009F_2^2F_3(5F_4^2 + 4F_3F_5) - 2{,}772F_2^3(3F_4F_5 + 2F_3F_6) + 432F_2^4F_7\} + 30{,}240G_4F_2^4\{3{,}575F_3^4 - 4{,}290F_2F_3^2F_4 + 99F_2^2(5F_4^2 + 8F_3F_5) - 72F_2^3F_6\} - 48{,}384G_5F_2^5\{715F_3^3 - 495F_2F_3F_4 + 54F_2^2F_5\} + 725{,}760G_6F_2^6(11F_3^2 - 3F_2F_4) - 1{,}244{,}160G_7F_2^7F_3 + 103{,}680G_8F_2^8].$$

Just as for the linear case, calculation of still higher terms rarely repays the labour†. Effort is more profitably expended on the asymptotic evaluation of the general late term (Chapter VII).

† An exception to this general conclusion is discussed in Chapter XXIII, Section 9.

Evaluation of $\int e^{-F} u^\sigma G\,du$. Let us imagine the integral $\int e^{-F} G\,du$ had been evaluated by retaining only $e^{-F_0} e^{-\frac{1}{2}F_2 u^2}$ in exponential form and expanding the rest of the integrand in powers of u. A typical integral in the result would have been

$$\int_0^\infty e^{-\frac{1}{2}F_2 u^2} u^n\,du = 2^{\frac{1}{2}n-\frac{1}{2}}(\tfrac{1}{2}n - \tfrac{1}{2})!/F_2{}^{\frac{1}{2}n+\frac{1}{2}}.$$

Thus in such a formulation the power of u embodied in an integrand is identified by the inverse power of F_2 appearing in the contribution to the final expansion. In our required extension the above integral stands to be replaced by

$$\int_0^\infty e^{-\frac{1}{2}F_2 u^2} u^{\sigma+n}\,du = 2^{\frac{1}{2}\sigma+\frac{1}{2}n-\frac{1}{2}}(\tfrac{1}{2}\sigma + \tfrac{1}{2}n - \tfrac{1}{2})!/F_2{}^{\frac{1}{2}\sigma+\frac{1}{2}n+\frac{1}{2}}.$$

Hence in the extension of

$$\int_{\text{limit}} e^{-F} G\,du = \tfrac{1}{2}(-\tfrac{1}{2})!\,(2/F_2)^{\frac{1}{2}}\,e^{-F_0}\sum_0^\infty Q_r$$

to

$$\int_0 e^{-F} u^\sigma G\,du = \tfrac{1}{2}(\tfrac{1}{2}\sigma - \tfrac{1}{2})!\,(2/F_2)^{\frac{1}{2}\sigma+\frac{1}{2}}\,e^{-F_0}\sum_0^\infty Q_r^{(\sigma)}, \tag{21}$$

Q_r can be converted into $Q_r^{(\sigma)}$ by replacing $1/F_2{}^{\frac{1}{2}n}$ with

$$\sqrt{\pi}\,(\tfrac{1}{2}\sigma + \tfrac{1}{2}n - \tfrac{1}{2})!/(\tfrac{1}{2}\sigma - \tfrac{1}{2})!\,(\tfrac{1}{2}n - \tfrac{1}{2})!\,F_2{}^{\frac{1}{2}n}.$$

This corresponds to the integral representation (question 4)

$$Q_r^{(\sigma)} = (\tfrac{1}{2}F_2)^{\frac{1}{2}\sigma+\frac{1}{2}}\,\frac{(\frac{1}{2}\sigma + \frac{1}{2}r - \frac{1}{2})!}{(\frac{1}{2}\sigma - \frac{1}{2})!}\,\frac{1}{2\pi i}\oint \frac{u^\sigma\ G\,du}{f^{\sigma+r+1}}. \tag{22}$$

The first few values are as follows:

$$Q_0^{(\sigma)} = G_0, \qquad Q_1^{(\sigma)} = -\frac{(\frac{1}{2}\sigma)!}{3\sqrt{2}\,(\frac{1}{2}\sigma - \frac{1}{2})!\,F_2{}^{3/2}}\{(\sigma+2)G_0F_3 - 6G_1F_2\},$$

$$Q_2^{(\sigma)} = \frac{(\sigma+1)}{72F_2{}^3}\{(\sigma+3)G_0[(\sigma+5)F_3{}^2 - 3F_2F_4] - 12(\sigma+3)G_1F_2F_3$$
$$+ 36G_2F_2{}^2\},$$

$$Q_3^{(\sigma)} = -\frac{(\frac{1}{2}\sigma + 1)!}{1{,}620\sqrt{2}\,(\frac{1}{2}\sigma - \frac{1}{2})!\,F_2{}^{9/2}}\{(\sigma+4)G_0[5(\sigma+6)(\sigma+8)F_3{}^3$$
$$- 45(\sigma+6)F_2F_3F_4 + 54F_2{}^2F_5] - 90(\sigma+4)G_1F_2[(\sigma+6)F_3{}^2$$
$$- 3F_2F_4] + 540(\sigma+4)G_2F_2{}^2F_3 - 1{,}080G_3F_2{}^3\},$$

$$\begin{aligned}
Q_4^{(\sigma)} = \frac{(\sigma+1)(\sigma+3)}{155{,}520F_2^{\,6}} &\{(\sigma+5)G_0[5(\sigma+7)(\sigma+9)(\sigma+11)F_3^{\,4} \\
&- 90(\sigma+7)(\sigma+9)F_2F_3^{\,2}F_4 + 135(\sigma+7)F_2^{\,2}F_4^{\,2} \\
&+ 216(\sigma+7)F_2^{\,2}F_3F_5 - 216F_2^{\,3}F_6] - 24(\sigma+5) \\
&\times G_1F_2[5(\sigma+7)(\sigma+9)F_3^{\,3} - 45(\sigma+7)F_2F_3F_4 + 54F_2^{\,2}F_5] \\
&+ 1{,}080(\sigma+5)G_2F_2^{\,2}[(\sigma+7)F_3^{\,2} - 3F_2F_4] - 4{,}320(\sigma+5) \\
&\times G_3F_2^{\,3}F_3 + 6{,}480G_4F_2^{\,4}\},
\end{aligned}$$

$$\begin{aligned}
Q_5^{(\sigma)} = -\frac{(\tfrac{1}{2}\sigma+2)!}{816{,}480\sqrt{2}\,(\tfrac{1}{2}\sigma-\tfrac{1}{2})!\,F_2^{\,15/2}} &\{(\sigma+6)G_0[7(\sigma+8)\ldots(\sigma+14)F_3^{\,5} \\
&- 210(\sigma+8)(\sigma+10)(\sigma+12)F_2F_3^{\,3}F_4 + 945(\sigma+8) \\
&\times (\sigma+10)F_2^{\,2}F_3F_4^{\,2} + 756(\sigma+8)(\sigma+10)F_2^{\,2}F_3^{\,2}F_5 \\
&- 2{,}268(\sigma+8)F_2^{\,3}F_4F_5 - 1{,}512(\sigma+8)F_2^{\,3}F_3F_6 \\
&+ 1{,}296F_2^{\,4}F_7] - 42(\sigma+6)G_1F_2[5(\sigma+8)(\sigma+10)(\sigma+12)F_3^{\,4} \\
&- 90(\sigma+8)(\sigma+10)F_2F_3^{\,2}F_4 + 135(\sigma+8)F_2^{\,2}F_4^{\,2} \\
&+ 216(\sigma+8)F_2^{\,2}F_3F_5 - 216F_2^{\,3}F_6] + 504(\sigma+6)G_2F_2^{\,2} \\
&\times [5(\sigma+8)(\sigma+10)F_3^{\,3} - 45(\sigma+8)F_2F_3F_4 + 54F_2^{\,2}F_5] \\
&- 15{,}120(\sigma+6)G_3F_2^{\,3}[(\sigma+8)F_3^{\,2} - 3F_2F_4] \\
&+ 45{,}360(\sigma+6)G_4F_2^{\,4}F_3 - 54{,}432G_5F_2^{\,5}\}.
\end{aligned}$$

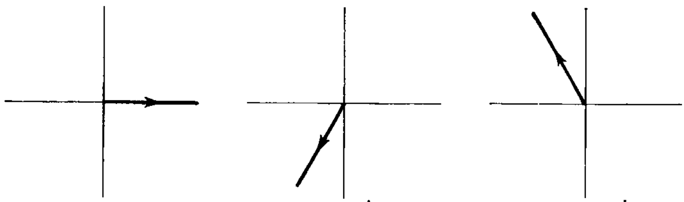
4. Cubic Dependence of $F(u)$ at a Limit of Integration

Evaluation of $\int e^{-F}G\,du$. The integral will be supposed to have been reduced to a standard form in which the limit of integration is at $u = 0$, so $F - F_0 \to \frac{1}{6}F_3u^3$ as $u \to 0$; and as before F increases steadily up to ∞ towards the upper limit of integration.

On introducing a variable f defined as one of the three cube roots of $F - F_0$, reduction to exactly integrable terms follows lines closely similar to those in the linear and quadratic cases. But since for the fixed path $f = 0$ to ∞ each of these roots is associated with a distinct path in the complex u-plane, the association is no longer purely a matter of convenience, as it was with the selection of the positive square root of F_2 in the quadratic case. When $F_3 > 0$ these roots may be written

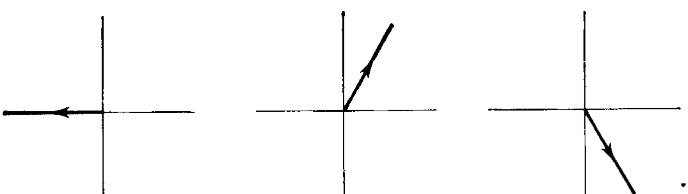
$$f = (F - F_0)^{\frac{1}{3}} \begin{pmatrix} 1 \\ e^{\frac{2}{3}i\pi} \\ e^{-\frac{2}{3}i\pi} \end{pmatrix} \to (\tfrac{1}{6}F_3)^{\frac{1}{3}} \begin{pmatrix} u \\ ue^{\frac{2}{3}i\pi} \\ ue^{-\frac{2}{3}i\pi} \end{pmatrix}, \tag{23}$$

where $(F - F_0)^{\frac{1}{3}}$ and $F_3^{\frac{1}{3}}$ refer here and henceforth to principal cube roots. Thus, at least for small u, the paths in the u-plane corresponding to $f = 0$ to ∞ are respectively $u = 0$ to ∞, $u = 0$ to $\infty\, e^{-\frac{2}{3}i\pi}$ and $u = 0$ to $\infty e^{\frac{2}{3}i\pi}$. Only the principal association $f \to (\frac{1}{6}F_3)^{\frac{1}{3}}u$ need be examined in detail, it being clear from (23) that the other two can be deduced therefrom simply by replacing $F_3^{\frac{1}{3}}$ throughout by $F_3^{\frac{1}{3}}\, e^{\frac{2}{3}i\pi}$ or $F_3^{\frac{1}{3}}\, e^{-\frac{2}{3}i\pi}$. The paths in the u-plane and associated cube roots are:—



Principal root $F_3^{1/3}$ $\qquad$ $F_3^{1/3}\, e^{2/3\, i\pi}$ $\qquad$ $F_3^{1/3}\, e^{-2/3\, i\pi}$

Extension to $F_3 < 0$ can now be effected unambiguously via the equivalence $F_3 \equiv (-1)^3(-F_3)$. The paths in the u-plane and associated cube roots are now:—



$-(-F_3)^{1/3}$ $\qquad$ $-(-F_3)^{1/3}e^{2/3\, i\pi}$ $\qquad$ $-(-F_3)^{1/3}e^{-2/3\, i\pi}$

Keeping henceforth in the exposition to the principal association, the integral can be expressed as

$$e^{-F_0}\int e^{-f^3}\, G\, du = e^{-F_0}\int_0^\infty e^{-f^3}(G/f')\, df.$$

In the Taylor series method G/f' is expanded in powers of f as in (2), and integration effected term by term through

$$\int_0^\infty e^{-f^3} f^r\, df = \tfrac{1}{3}\ (\tfrac{1}{3}r - \tfrac{2}{3})!. \tag{24}$$

The result is

$$\int_0 e^{-F}\, G\, du \propto F_3^{-\frac{1}{3}} e^{-F_0} \sum_0^\infty C_r \tag{25}$$

where

$$C_r = \left(\frac{F_3}{6}\right)^{\frac{1}{3}} \frac{(\frac{1}{3}r - \frac{2}{3})!}{(-\frac{2}{3})!\; r!} \left(\frac{d}{f'\, du}\right)^r \frac{G}{f'}\bigg|_{u=0}. \tag{26}$$

In the reversion method G/f' is expanded as in (7), leading to the alternative form

$$C_r = \left(\frac{F_3}{6}\right)^{\frac{1}{3}} \frac{(\frac{1}{3}r - \frac{2}{3})!}{(-\frac{2}{3})!\; r!} \left(\frac{d}{du}\right)^r \left(\frac{u}{f}\right)^{r+1} G\bigg|_{u=0}. \tag{27}$$

The two alternatives correspond to the same integral representation (question 4)

$$C_r = \left(\frac{F_3}{6}\right)^{\frac{1}{3}} \frac{(\frac{1}{3}r - \frac{2}{3})!}{(-\frac{2}{3})!} \frac{1}{2\pi i} \oint \frac{G\, du}{f^{r+1}}. \tag{28}$$

Adopting the notation

$$F_\nu = \left(\frac{d}{du}\right)^\nu F(u)\bigg|_{\text{limit}}, \qquad F_1 = F_2 = 0, \qquad G_\nu = \left(\frac{d}{du}\right)^\nu G(u)\bigg|_{\text{limit}},$$

$$\alpha = 6^{\frac{1}{3}}(-\tfrac{2}{3})!/3 = 2^{4/3}\pi/3^{7/6}(-\tfrac{1}{3})! = 1{\cdot}6226515\ldots,$$

$$\beta = (-\tfrac{1}{3})!/6^{\frac{2}{3}}(-\tfrac{2}{3})! = (-\tfrac{1}{6})!/2\pi^{\frac{1}{2}}3^{\frac{2}{3}} = \pi^{\frac{1}{2}}/2.3^{5/3}(\tfrac{1}{6})! = 0{\cdot}15308275\ldots,$$

our result for cubic dependence near a limit of integration can be written

$$\int_{\text{limit}} e^{-F}\, G\, du = \alpha\, F_3^{-\frac{1}{3}} e^{-F_0} \sum_0^\infty C_r, \tag{29}$$

$$C_0 = G_0, \qquad C_1 = -\frac{\beta}{F_3^{4/3}}\{G_0F_4 - 6G_1F_3\},$$

$$C_2 = \frac{1}{40\alpha F_3^{8/3}}\{G_0(5F_4^2 - 4F_3F_5) - 20G_1F_3F_4 + 40G_2F_3^2\},$$

$$C_3 = -\frac{1}{3{,}240F_3^4}\{G_0(175F_4^3 - 252F_3F_4F_5 + 72F_3^2F_6)$$
$$- 18G_1F_3(35F_4^2 - 24F_3F_5) + 1{,}080G_2F_3^2F_4 - 1{,}080G_3F_3^3\},$$

$$C_4 = \frac{\beta}{45{,}360F_3^{16/3}}\{G_0(13{,}475F_4^4 - 27{,}720F_3F_4^2F_5 + 2{,}016F_3^2[3F_5^2 + 5F_4F_6] - 2{,}160F_3^3F_7) - 840G_1F_3(55F_4^3 - 72F_3F_4F_5 + 18F_3^2F_6) + 15{,}120G_2F_3^2(5F_4^2 - 3F_3F_5) - 75{,}600G_3F_3^3F_4 + 45{,}360G_4F_3^4\},$$

$$C_5 = -\frac{1}{22{,}400\alpha F_3^{20/3}}\{G_0(1{,}575F_4^5 - 4{,}200F_3F_4^3F_5 + 336F_3^2[6F_4F_5^2 + 5F_4^2F_6] - 96F_3^3[7F_5F_6 + 5F_4F_7] + 80F_3^4F_8) - 2G_1F_3(2{,}625F_4^4 - 5{,}040F_3F_4^2F_5 + 336F_3^2[3F_5^2 + 5F_4F_6] - 320F_3^3F_7) + 560G_2F_3^2(15F_4^3 - 18F_3F_4F_5 + 4F_3^2F_6) - 560G_3F_3^3(15F_4^2 - 8F_3F_5) + 5{,}600G_4F_3^4F_4 - 2{,}240G_5F_3^5\},$$

$$C_6 = \frac{1}{10{,}497{,}600F_3^8}\{G_0(475{,}475F_4^6 - 1{,}556{,}100F_3F_4^4F_5 + 131{,}040F_3^2F_4^2[9F_5^2 + 5F_4F_6] - 8{,}424F_3^3[14F_5^3 + 70F_4F_5F_6 + 25F_4^2F_7] + 3{,}240F_3^4[14F_6^2 + 24F_5F_7 + 15F_4F_8] - 6{,}480F_3^5F_9) - 180G_1F_3(8{,}645F_4^5 - 21{,}840F_3F_4^3F_5 + 1{,}638F_3^2F_4[6F_5^2 + 5F_4F_6] - 432F_3^3[7F_5F_6 + 5F_4F_7] + 324F_3^4F_8) + 1{,}080G_2F_3^2(2{,}275F_4^4 - 4{,}095F_3F_4^2F_5 + 252F_3^2[3F_5^2 + 5F_4F_6] - 216F_3^3F_7) - 7{,}560G_3F_3^3(325F_4^3 - 360F_3F_4F_5 + 72F_3^2F_6) + 68{,}040G_4F_3^4(25F_4^2 - 12F_3F_5) - 816{,}480G_5F_3^5F_4 + 233{,}280G_6F_3^6\},$$

$$C_7 = -\frac{\beta}{8{,}377{,}084{,}800F_3^{28/3}}\{G_0(2{,}788{,}660{,}875F_4^7 - 10{,}811{,}423{,}700F_3F_4^5F_5 + 940{,}123{,}800F_3^2F_4^3[12F_5^2 + 5F_4F_6] - 64{,}465{,}632F_3^3F_4[42F_5^3 + 105F_4F_5F_6 + 25F_4^2F_7] + 5{,}688{,}144F_3^4[168F_5^2F_6 + 140F_4F_6^2 + 240F_4F_5F_7 + 75F_4^2F_8] - 16{,}251{,}840F_3^5[12F_6F_7 + 9F_5F_8 + 5F_4F_9] + 8{,}864{,}640F_3^6F_{10}) - 342G_1F_3(26{,}343{,}625F_4^6 - 82{,}467{,}000F_3F_4^4F_5 + 6{,}597{,}360F_3^2F_4^2[9F_5^2 + 5F_4F_6] - 399{,}168F_3^3[14F_5^3 + 70F_4F_5F_6 + 25F_4^2F_7] + 142{,}560F_3^4$$

$$\times[14F_6^2 + 24F_5F_7 + 15F_4F_8] - 259{,}200F_3^5F_9) + 20{,}520G_2F_3^2$$
$$\times\ (687{,}225F_4^5 - 1{,}649{,}340F_3F_4^3F_5 + 116{,}424F_3^2F_4[6F_5^2$$
$$+ 5F_4F_6] - 28{,}512F_3^3[7F_5F_6 + 5F_4F_7] + 19{,}440F_3^4F_8)$$
$$- 61{,}560G_3F_3^3(229{,}075F_4^4 - 388{,}080F_3F_4^2F_5 + 22{,}176F_3^2$$
$$\times\ [3F_5^2 + 5F_4F_6] - 17{,}280F_3^3F_7) + 25{,}855{,}200G_4F_3^4(385F_4^3$$
$$- 396F_3F_4F_5 + 72F_3^2F_6) - 93{,}078{,}720G_5F_3^5(55F_4^2 - 24F_3F_5)$$
$$+ 1{,}861{,}574{,}400G_6F_3^6F_4 - 398{,}908{,}800G_7F_3^7\}.$$

Evaluation of $\int e^{-F} u^\sigma G\,du$. If only $e^{-F_0}e^{-\frac{1}{6}F_3u^3}$ had been retained in exponential form and the rest of the integrand expanded, a typical integral would have been

$$\int_0^\infty e^{-\frac{1}{6}F_3u^3}u^n\,du = 6^{\frac{1}{3}n+\frac{1}{3}}(\tfrac{1}{3}n - \tfrac{2}{3})!/3\,F_3^{\frac{1}{3}n+\frac{1}{3}}.$$

This will need replacing by $\int_0^\infty e^{-\frac{1}{6}F_3u^3}u^{\sigma+n}\,du$. Hence in the extension of

$$\int_{\text{limit}} e^{-F}G\,du = \tfrac{1}{3}(-\tfrac{2}{3})!\left(\frac{6}{F_3}\right)^{\frac{1}{3}} e^{-F_0}\sum_0^\infty C_r$$

to

$$\int_0 e^{-F}u^\sigma G\,du = \tfrac{1}{3}(\tfrac{1}{3}\sigma - \tfrac{2}{3})!\left(\frac{6}{F_3}\right)^{\frac{1}{3}\sigma+\frac{1}{3}} e^{-F_0}\sum_0^\infty C_r^{(\sigma)}, \tag{30}$$

C_r can be converted into $C_r^{(\sigma)}$ by replacing $1/F_3^{\frac{1}{3}n}$ with

$$(-\tfrac{2}{3})!\,(\tfrac{1}{3}\sigma + \tfrac{1}{3}n - \tfrac{2}{3})!/(\tfrac{1}{3}\sigma - \tfrac{2}{3})!\,(\tfrac{1}{3}n - \tfrac{2}{3})!\,F_3^{\frac{1}{3}n}.$$

This corresponds to the integral representation (question 4)

$$C_r^{(\sigma)} = \left(\frac{F_3}{6}\right)^{\frac{1}{3}\sigma+\frac{1}{3}}\frac{(\frac{1}{3}\sigma + \frac{1}{3}r - \frac{2}{3})!}{(\frac{1}{3}\sigma - \frac{2}{3})!}\frac{1}{2\pi i}\oint\frac{u^\sigma G\,du}{f^{\sigma+r+1}}. \tag{31}$$

5. The Lagrange Reversion Theorem

Let $g(u)$ be a function of u analytic within a region surrounding the origin $u = 0$. Within this region let the equation

$$u = f\,g(u) \tag{32}$$

have one root, at $u = 0$.

By Cauchy's residue theorem (1825, 1846), a function $H(u)$ can then be expressed as the contour integral

$$H(u) = \frac{1}{2\pi i} \oint \frac{H(\omega)\{1 - f g'(\omega)\}\, d\omega}{\omega - fg(\omega)}. \tag{33}$$

Introducing into this integrand the expansion

$$(\omega - fg)^{-1} = \sum_0^\infty f^r g^r / \omega^{r+1}, \tag{34}$$

we have

$$\begin{aligned} H(u) &= \frac{1}{2\pi i} \sum_0^\infty \left\{ f^r \oint \frac{H g^r\, d\omega}{\omega^{r+1}} - f^{r+1} \oint \frac{H g'\, g^r\, d\omega}{\omega^{r+1}} \right\} \\ &= \frac{1}{2\pi i} \oint \frac{H\, d\omega}{\omega} + \sum_0^\infty \frac{f^{r+1}}{2\pi i} \left\{ \oint \frac{H\, g^{r+1}\, d\omega}{\omega^{r+2}} - \oint \frac{H g'\, g^r\, d\omega}{\omega^{r+1}} \right\}. \end{aligned}$$

The first term is simply $H(0)$. The others partly cancel on integration by parts, because

$$\oint \frac{H g^{r+1}\, d\omega}{\omega^{r+2}} = \frac{1}{r+1} \oint \frac{H' g^{r+1}\, d\omega}{\omega^{r+1}} + \oint \frac{H\, g'\, g^r\, d\omega}{\omega^{r+1}}.$$

Hence

$$H(u) = H(0) + \sum_0^\infty \{f^{r+1}/(r+1)\} \times \text{coefficient of } \omega^r \text{ in } H'(\omega)\, g^{r+1}(\omega). \tag{35}$$

Replacing the dummy variable ω by u and inserting $g(u) = u/f$ from (32),

$$H(u) = H(0) + \sum_0^\infty \left(\frac{d}{du}\right)^r \left(\frac{u}{f}\right)^{r+1} \frac{dH}{du}\bigg|_{u=0} \frac{f^{r+1}}{(r+1)!}. \tag{36}$$

Exercises

1. Substantiate the following tests of the formula for linear dependence of $F(u)$ near a limit of integration:

(a) If $G_i = F_{i+1}$, then $L_0 = F_1$ and $L_{r \neq 0} = 0$.

(b) If $G_0 = 1, G_{i \neq 0} = 0$, $F_i = (-1)^{i-1}(i-1)!$, then $L_r = 1$ for all r.

2. In the formula for quadratic dependence of $F(u)$ near a limit of integration, take

$$G_0 = 1, \qquad G_{i \neq 0} = 0, \qquad F_0 = F_1 = 0, \qquad F_{i>1} = (-1)^i (i-1)!$$

and obtain the relation

$$2 = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \left\{1 + \frac{1}{12} + \frac{1}{288} - \frac{139}{51{,}840} - \frac{571}{2{,}488{,}320} \cdots\right\} + \left\{\frac{2}{3} - \frac{4}{135} + \frac{8}{2835} + \frac{16}{8505} \cdots\right\}.$$

Recognising the first as essentially the Stirling–Laplace expansion of 1! (Chapter VIII, Section 2), derive the following asymptotic expansion for the base of Napierian logarithms:

$$e = \frac{8}{3}\left\{1 + \frac{1}{45} - \frac{2}{945} - \frac{4}{2835} \cdots\right\}.$$

3. Writing $f = F_1 u(1 + U/F_1)$ where $U = F_2 u/2! + F_3 u^2/3! + \ldots$, show that the integral representation for L_r can be expanded in inverse powers of F_1 as

$$L_r = \frac{1}{2\pi i} \sum_0^\infty \frac{(-1)^s (r+s)!}{s!\, F_1^{\,r+s}} \oint \frac{U^s\, G\, du}{u^{r+1}}.$$

Replacing $1/F_1^{\,r+s}$ by $(\sigma + r + s)!/\sigma!(r+s)!\, F_1^{\,r+s}$ and resumming by the binomial theorem, derive the integral representation

$$L_r^{(\sigma)} = F_1^{\,1+\sigma} \frac{(\sigma + r)!}{\sigma!} \frac{1}{2\pi i} \oint \frac{u^\sigma\, G\, du}{f^{\sigma+r+1}}.$$

4. Following the method outlined in the preceding question, find integral representations for $Q_r^{(\sigma)}$ and $C_r^{(\sigma)}$.

5. Show from the Lagrange reversion theorem that if $f = u(1+u)^{-\alpha}$,

$$u = \sum_0^\infty \frac{\{\alpha(r+1)\}!}{(r+1)!\,\{\alpha(r+1) - r\}!} f^{r+1}.$$

Check this by direct solution for $\alpha = 0$, 1 and 2.

6. Show that if $f = ue^u$,

$$u = f\sum_0^\infty (-f)^r (r+1)^{r-1}/r!, \qquad u^{-1} = \sum_0^\infty (-f)^{r-1}(r-1)^{r-1}/r!.$$

[*Hint*: e^u satisfies the conditions for Lagrange's theorem whereas u^{-1} does not].

7. Show that if $f = u/\cos u$,

$$u = \sum_{r=0}^{\infty} \frac{f^{2r+1}}{(-4)^r} \sum_{s=0}^{r} \frac{(1+2r-2s)^{2r}}{s!\,(1+2r-s)!} = \tfrac{1}{2}\sum_{r=0}^{\infty} \frac{f^{2r+1}}{(-4)^r} \sum_{s=0}^{2r+1} \frac{(1+2r-2s)^{2r}}{s!\,(1+2r-s)!}.$$

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Chapter VI

Derivation of Asymptotic Expansions from Integral Representations of the Form $\int_{\text{s.p.}} e^{-F}\,G\,du$ (Stationary-point methods).

1. Expansion about a Stationary Point

In the preceding chapter we were studying integrals in which the fast-varying factor $e^{-F(u)}$ decreased steadily as the integration variable moved away from one of the limits of integration, and accordingly we expanded the integrand *about that limit*.

Next in order of simplicity come integrals in which the fast varying factor starts at zero at one limit of integration, rises to a maximum somewhere along the path, and then falls to zero at the second limit. The integrand can be expanded *about this maximum* in just the same way, the one difference being that the earlier possibility of linear dependence of $F(u)$ is now automatically excluded by the stationary condition $dF/du = 0$. The same algebraic expressions Q_r and C_r result from quadratic and cubic dependences respectively; there are, however, now different numerical factors multiplying them, arising from the integration right through a stationary point instead of the earlier integration from a limit.

2. Multiple Critical Points

The technical processes of reduction to exactly integrable terms, developed in the preceding chapter for expansion about a single limit, thus apply equally to expansion about a single stationary point. In generalizing to more than one stationary point, or to one or more stationary points together with a limit where the integrand is still contributory, it must be recognised from the start that there is no *a priori* guarantee of contributions from expansions around these "critical points"† being exactly additive. For the expansion introduced around one point may extend to the region of a second, resulting

† A useful generic term introduced by van der Corput in 1948.

in overlapping and consequent spurious duplication if a new expansion of the same function is introduced around the second critical point. This complication should be avoided by bringing in Cauchy's theorem (1825), according to which the value of an integral is unaffected by deformations of the path in the complex plane provided no singularity is crossed in the process, and no divergences are introduced. Thus before the evaluation a given integral should be dissected into a set of simpler ones by diverting the path between each pair of critical points to run through some location (usually at infinity) where the fast-varying factor vanishes. Each new integral is to involve just one subscribing critical point, either one limit where the integrand is still contributing, or one stationary point which is not by-passed.†

A valuable side-product of this resolution is the clarification which results respecting the sort of path to be drawn near a stationary point $F'(u) = 0$. We draw through the point a short line in a direction making the quadratic term $\frac{1}{2}F_2u^2$ real and positive (corresponding to the deepest attainable minimum in F, i.e. sharpest maximum in the integrand), and extend this line each side to the nearest places where $F(u) \to \infty$ (where the fast-varying factor vanishes). If such a deformation is consistent—in the Cauchy sense—with the path signified in the given integral representation, the stationary point contributes; if not, it is by-passed.

(It is frequently argued that additivity of contributions from limits and stationary points follows from localization of significant contributions to small regions surrounding them. This assertion is conceptually and quantitatively deficient. Apart from providing no estimate of duplication due to overlap, it leaves obscure the path to be taken through each point, or even whether the path really should be taken through each. Indeed, unqualified assumption of additivity would lead to answers varying with the way an integral had been written down, because additional stationary points are introduced by transferring cubic or higher terms from the slowly-varying factor to the exponent in the integrand!)

Because of the liability of a Stokes discontinuity occurring when two critical points coalesce (Chapter I, Section 3), a separate investigation is needed for each range over which the total number of critical points in the complex plane is constant. Continuity in asymptotic form should not be presumed when this number changes.

In the rare cases where complete reduction to single critical points proves impossible—for instance if between two critical points $F(u)$ cannot be diverted out to $+\infty$ on account of an intervening branch cut—there is no straightforward procedure for rigorously identifying how far contributions

† A trickier alternative approach is to try and construct a "neutralizer" (van der Corput, 1948), an inserted function—with continuous derivatives all vanishing at each critical point—which serves to isolate the various critical points.

from one critical point are also included in late terms of the expansion about the other. The best we could do then would be to scrutinize the expressions for the general late term in the asymptotic series originating from each critical point, quash any portion which would sum to the alien exponential corresponding to the other, and only then add the contributions from the critical points. However, quite apart from objections centred on the intuitive content of this resolving procedure, asymptotic expansions tend in the circumstances envisaged to display poor "initial convergence". This weakness in form occurs more generally whenever important points in the complex plane approach each other, even where no path-separation difficulty exists—e.g. on the mutual approach of two critical points. Hence when critical points approach each other or venture near singularities, on practical grounds substitute treatments are needed, such as the more sophisticated uniform expansions of Chapters X and XI.

3. Orientation of Path Near a Contributing Stationary Point

Relying again on Cauchy's theorem (1825), it is expedient to deform the path in the complex plane (without crossing any singularity of the integrand) so as to pass through a stationary point even if in the initial problem this lay somewhat aside from the prescribed path of integration. For it is mathematically less complicated to expand a function about its stationary point, where one derivative vanishes, than about an arbitrary point.

If along one path there is a minimum, this would show up as a maximum at the same point along a path taken at right angles to the first; in the complex plane, stationary points are therefore frequently called "saddle-points" or "cols". We might expect the best choice of path through such a point would turn out to be that for which contributions to the integrand are most sharply localized. This condition is met in the *method of steepest descent*, introduced by Georg Riemann (1863) and Peter Debye (1909, 1910), where the direction prescribed is that for which $|e^{-F(u)}|$ changes with maximum rapidity. Since a modulus changes fastest when the phase is held constant, the condition imposed for the steepest descent is constancy of the imaginary part of $F(u)$. Thus $F-F_0$ is positive and real along this path of steepest descent.

In *Laplace's method* (1820) the condition is weakened to refer only to the quadratic term in $F(u)$, i.e. the path is such that only the term in u^2 is necessarily positive and real. Then, analogously to the fourth approach outlined in Chapter V, Section 2, $\int e^{-F}\,G\,du$ is evaluated by retaining the factors $e^{-F_0}\,e^{-\frac{1}{2}F_2u^2}$ in exponential form, and expanding the rest of the integrand in powers of u, a typical contributory integral in the result being $\int_{-\infty}^{\infty} e^{-\frac{1}{2}F_2u^2}\,u^r\,du$.

In a still further relaxation, it is only required that the real part of the quadratic term in $F(u)$ shall be positive. This and the methods of Laplace and steepest descent are known collectively as *saddle-point methods*†. The term by term integration involved may be justified by Watson's lemma (1918).

A different viewpoint is adopted in the *method of stationary phase* introduced by Stokes and Kelvin (1887). If a path can be chosen making $F - F_0$ a pure imaginary, say $F - F_0 = i\theta$ where θ denotes a real phase, for a given u the fast-varying part of the integrand $e^{-i\theta}$ may be viewed in physical terms as a wave of phase θ. The continuum of waves $\int du\, e^{-i\theta(u)}$ will interfere mutually destructively except where there is an exceptionally high concentration of waves with virtually the same phase, a situation occurring only in the immediate vicinity of a stationary point with respect to phase variation, i.e. where $d\theta/du = 0$. After expanding about this point, term by term integration may be justified by a modification of Watson's original lemma (Watson 1920).

When carried through systematically, these four approaches lead to one and the same asymptotic expansion, apart perhaps from minor rearrangements. In other words, provided there is *some* degree of localization, in amplitude or in phase, the fact that one path direction may cross a shorter and steeper pass than another is ultimately irrelevant. This comes about because, as noted in the previous chapter, the asymptotic nature of the final expansion really originates through extending the range of integration beyond a circle of convergence whose radius in the F-plane is fixed by the next zero of dF/du; a mere change in direction of the path crossing this fateful circle cannot materially affect the resulting series. To this extent, the distinctions emphasized in the past between the four methods are somewhat academic. Likewise, from the standpoint of strict rigour, progress is being made in unifying the approaches by establishing theorems of wider generality (van der Waerden 1951, Wyman 1964, Olver 1970, Ursell 1970).

4. Quadratic Dependence of $F(u)$ at a Stationary Point

The integral will be supposed to have been reduced to a standard form in which the single stationary point $dF/du = 0$ lies at $u = 0$, so $F - F_0 \to \frac{1}{2}F_2 u^2$ as $u \to 0$; and F increases steadily up to $+\infty$ towards both limits of integration.

The processes of reduction to exactly integrable terms are all very similar to those of Chapter V, Section 3. A variable f is defined (when $F_2 > 0$) by

$$f = (F - F_0)^{\frac{1}{2}} \to (\tfrac{1}{2}F_2)^{\frac{1}{2}}\, u,$$

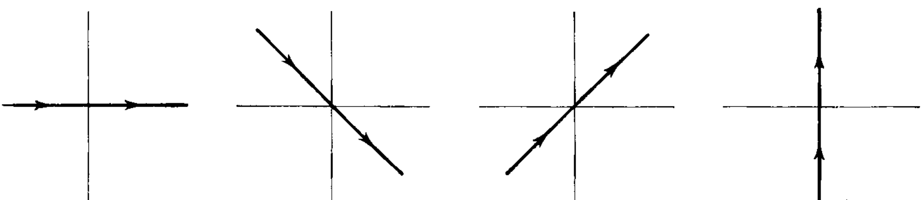
† A connotation sometimes extended to embrace also the method of stationary phase.

the positive root being selected. The integral to be investigated, $\int e^{-F}\, G\, du$, can then be written as

$$e^{-F_0} \int_{f=-\infty}^{\infty} e^{-f^2}\, G\, du.$$

$F - F_0 = f^2$ is thereby taken as real along the path through the stationary point at $f = 0$, corresponding to steepest descent in the f-plane. When F_2 is real and positive, the path in the u-plane is similar, i.e. from left to right along the real u-axis for small $|u|$. For complex F_2 the orientation is $\mathrm{ph}\, u = -\frac{1}{2}\, \mathrm{ph}\, F_2$.

When F_2 is real and negative, results are clarified by replacing $F_2^{\frac{1}{2}}$ with $-i(-F_2)^{\frac{1}{2}}$, selecting the positive root of $-F_2$. Since this corresponds to $u \propto if$ for small $|u|$, the path near the stationary point $u = 0$ is vertically up the imaginary u-axis. As seen from the following diagrams of paths in the u-plane, the effect of the convention on assuming positive roots is to make this substitution correspond to $\mathrm{ph}\, F_2 \to -\pi$:-



In the ensuing analysis the difference compared with Section 3 of the previous chapter is that the relevant basic integral replacing V (15) is

$$\int_{-\infty}^{\infty} e^{-f^2} f^r\, df = \begin{cases} (\frac{1}{2}r - \frac{1}{2})!, & r \text{ even}, \\ 0, & r \text{ odd}. \end{cases} \tag{1}$$

Hence

$$\int_{\rightarrow} e^{-F}\, G\, du = (2\pi/F_2)^{\frac{1}{2}}\, e^{-F_0} \sum_0^\infty Q_{2r}, \qquad |\mathrm{ph}\, F_2| < \pi, \tag{2}$$

$$\int_{\uparrow} e^{-F}\, G\, du = i(2\pi/(-F_2))^{\frac{1}{2}}\, e^{-F_0} \sum_0^\infty Q_{2r}, \quad |\mathrm{ph}(-F_2)| < \pi, \tag{3}$$

where the Q's are set out in Chapter V, Section 3. The distinction between

(2) and (3) is one of sign convention in extracting the square root, the respective conventions being $\Re(F_2^{\frac{1}{2}}) > 0$ and $\Re(-F_2)^{\frac{1}{2}} > 0$. From the diagrams,

$$\int_{\rightarrow} \equiv \int_{\uparrow}, \qquad 0 > \operatorname{ph} F_2 > -\pi, \tag{4}$$

$$\int_{\rightarrow} \equiv -\int_{\uparrow}, \qquad 0 < \operatorname{ph} F_2 < \pi. \tag{5}$$

5. Cubic Dependence of $F(u)$ at a Stationary Point

Here the standard form will have a single stationary point $dF/du = d^2F/du^2 = 0$ at $u = 0$, so $F - F_0 \to \frac{1}{6} F_3 u^3$ as $u \to 0$; and as before F is to increase steadily up to $+\infty$ at both limits of integration.

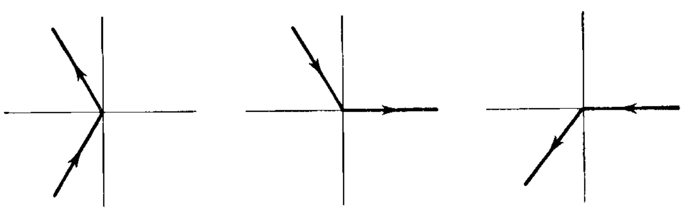
Following Chapter V, Section 4, we introduce a variable f defined as one of the three cube roots of $F - F_0$. When $F_3 > 0$ these roots may be written

$$f = (F - F_0)^{\frac{1}{3}} \begin{pmatrix} 1 \\ e^{\frac{2}{3}i\pi} \\ e^{-\frac{2}{3}i\pi} \end{pmatrix} \to \left(\tfrac{1}{6} F_3\right)^{\frac{1}{3}} \begin{pmatrix} u \\ ue^{\frac{2}{3}i\pi} \\ ue^{-\frac{2}{3}i\pi} \end{pmatrix}. \tag{6}$$

For reasons of symmetry and correlation with the case of quadratic dependence just treated, it is strategic to select the principal form in the f-plane to be

$$e^{-F_0} \int_{f = -\infty e^{\frac{1}{3}i\pi}}^{-\infty e^{-\frac{1}{3}i\pi}} e^{-f^3} G\, du.$$

Near the stationary point, when $F_3 > 0$ the relative paths in the u-plane are $u = -\infty e^{\frac{1}{3}i\pi}$ through 0 to $-\infty e^{-\frac{1}{3}i\pi}$, $-\infty e^{-\frac{1}{3}i\pi}$ through 0 to $+\infty$, and $+\infty$ through 0 to $-\infty e^{\frac{1}{3}i\pi}$ respectively. Of these, only the principal (first) association $f \to (\frac{1}{6} F_3)^{\frac{1}{3}} u$ need be examined in detail, since the other two can be deduced therefrom by replacing $F_3^{\frac{1}{3}}$ throughout by $F_3^{\frac{1}{3}} e^{\frac{2}{3}i\pi}$ or $F_3^{\frac{1}{3}} e^{-\frac{2}{3}i\pi}$. The contours in the u-plane (and associated cube roots) are displayed diagrammatically as follows when F_3 is real and positive:

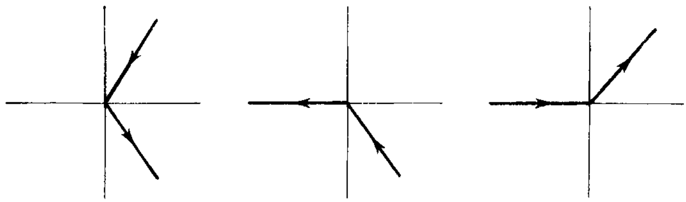


Contour $>$ Principal root $F_3^{1/3}$

Contour ∖_ $F_3^{1/3}\, e^{2/3\, i\pi}$

Contour /‾ $F_3^{1/3}\, e^{-2/3\, i\pi}$

The three contours possible when $F_3 < 0$ can likewise be derived from the principal association through introducing the equivalence $F_3 \equiv (-1)^3 (-F_3)$. The contours in the u-plane (and associated cube roots) when F_3 is real and negative appear as follows:



Contour ⟨ $-(-F_3)^{1/3}$ Contour ⌐\ $-(-F_3)^{1/3} e^{2/3\, i\pi}$ Contour —/ $-(-F_3)^{1/3} e^{-2/3\, i\pi}$

In the ensuing analysis the difference compared with Section 4 of the previous chapter is that the relevant basic integral replacing V (24) is

$$\int_{-\infty\, e^{\frac{1}{3}i\pi}}^{-\infty\, e^{-\frac{1}{3}i\pi}} e^{-f^3} f^r\, df = \tfrac{2}{3} i (-1)^r \left(\tfrac{1}{3}r - \tfrac{2}{3}\right)! \sin \tfrac{1}{3}(r+1)\pi$$

$$= \begin{cases} 3^{-\frac{1}{2}}\, i\left(\tfrac{1}{3}r - \tfrac{2}{3}\right)!, & r = 0, 3, 6, \ldots \\ -\,3^{-\frac{1}{2}}\, i\left(\tfrac{1}{3}r - \tfrac{2}{3}\right)!, & r = 1, 4, 7, \ldots \\ 0\quad , & r = 2, 5, 8, \ldots \end{cases} \tag{7}$$

Hence

$$\int_{>} e^{-F}\, G\, du = i\alpha\, 3^{\frac{1}{2}}\, F_3^{-\frac{1}{3}}\, e^{-F_0}$$

$$\times\, (C_0 - C_1 + C_3 - C_4 + C_6 - C_7 + C_9 - C_{10} \ldots), \tag{8}$$

where the C_r and numerical factor $\alpha = 6^{\frac{1}{3}}(-2/3)!/3 = 1{\cdot}6226515\ldots$ are as in Chapter V, Section 4.

6. Correspondences between Contours in Quadratic and Cubic Cases

Since a cubic term in F varies more rapidly in both magnitude and phase than a quadratic term, correspondences between contours are largely dictated by the alternatives set out in the preceding section for cubic dependence at a stationary point. It is therefore appropriate to note such correspondences here, though their main application will not come until Chapter X, Section 5, when we discuss uniform expansions which span the entire range F quadratic to cubic.

For negative F_2, there are evident equivalences in both shape of path and nature of resultant expansion between contours already noted:

F_2 negative, F_3 positive.

$$\text{Quadratic Contour} \uparrow \equiv \text{Cubic Contour} > . \tag{9}$$

F_2 negative, F_3 negative.

$$\text{Quadratic Contour} \uparrow \equiv -\ \text{Cubic Contour} < . \tag{10}$$

Cases where F_2 is positive require a little more investigation. When $|F_3|$ is large, the dominant terms in the expansion for quadratic dependence at a stationary point are

$$1 + 5F_3{}^2/24F_2{}^3 + 385F_3{}^4/1152F_2{}^6 + 425{,}425F_3{}^6/414{,}720F_2{}^9$$
$$+ 185{,}910{,}725F_3{}^8/39{,}813{,}120F_2{}^{12} + \dots,$$

and are therefore all of the same sign and phase when F_2 is positive. The quadratic contour $\rightarrow$ then specifies an expansion on a Stokes ray $\operatorname{ph} u = 0$. By Rule E.D of Chapter I, Section 2, on such a Stokes ray the factor multiplying any associated series is the mean of the factors to either side of the ray, leading to the following equivalences with combinations of cubic contours:—

F_2 positive, F_3 positive.

$$\text{Quadratic Contour} \rightarrow \equiv \text{Mean of cubic contours} \setminus\!\_ \text{ and } - /\overline{\ }.$$

Explicitly,

$$\int_{\frac{1}{2}(\setminus\_ - /\overline{\ })} e^{-F}\, G\, du = \tfrac{1}{2}\,\alpha\, F_3{}^{-\frac{1}{3}}\, e^{-F_0}$$
$$\times\ (C_0 + C_1 + C_3 + C_4 + C_6 + C_7 + \dots), \tag{12}$$

in which principal cube roots are to be understood.

F_2 positive, F_3 negative.

$$\text{Quadratic Contour} \rightarrow \equiv \text{Mean of cubic contours} \quad -\overline{\ }\!\setminus \text{ and } \_/ . \tag{13}$$

EXERCISES

1. Show that for quadratic dependence of $F(u)$ at a stationary point $u = 0$,

$$\int_{\uparrow} e^{-F} u^{\sigma} G\, du = (\tfrac{1}{2}\sigma - \tfrac{1}{2})!\left(\frac{2}{F_2}\right)^{\frac{1}{2}\sigma+\frac{1}{2}} e^{-F_0} \begin{cases} \sum_0^\infty Q_{2r}^{(\sigma)}, & \sigma \text{ even} \\ \sum_0^\infty Q_{2r+1}^{(\sigma)}, & \sigma \text{ odd}, \end{cases}$$

where the $Q^{(\sigma)}$ are as in Chapter V, Section 3.

2. Show that for cubic dependence of $F(u)$ at a stationary point $u = 0$,

$$\int_{>} e^{-F} u^{\sigma} G\, du$$

$$= \frac{i}{\sqrt{3}} (\tfrac{1}{3}\sigma - \tfrac{2}{3})! \left(\frac{6}{F_3}\right)^{\frac{1}{3}\sigma + \frac{1}{3}} e^{-F_0} \begin{cases} C_0^{(\sigma)} - C_1^{(\sigma)} + C_3^{(\sigma)} - C_4^{(\sigma)} & \\ \quad + C_6^{(\sigma)} - C_7^{(\sigma)} + \ldots, & \sigma = 0, 3, 6, \ldots \\ -C_0^{(\sigma)} + C_2^{(\sigma)} - C_3^{(\sigma)} + C_5^{(\sigma)} - C_6^{(\sigma)} & \\ \quad + \ldots, & \sigma = 1, 4, 7, \ldots \\ C_1^{(\sigma)} - C_2^{(\sigma)} + C_4^{(\sigma)} & \\ \quad - C_5^{(\sigma)} + C_7^{(\sigma)} - \ldots, & \sigma = 2, 5, 8, \ldots, \end{cases}$$

where the $C^{(\sigma)}$ are as in Chapter V, Section 4.

3. Assuming $F_1 = F_2 = 0$, $F_3 < 0$, write down an explicit result in terms of the principal cube root $(-F_3)^{\frac{1}{3}}$ for the combination contour

$$\int_{\frac{1}{2}(-\overline{\ }\backslash + \underline{\ }/)} e^{-F} G\, du.$$

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Chapter VII

Calculation of Late Terms; L_r, Q_r, C_r, etc. for $r \gg 1$

1. Need for Late-term Formulae

As we have several times remarked, only in a few of the most elementary applications of the foregoing theory can the *general* term in the asymptotic expansion be found directly, and we therefore quoted the first few terms *in extenso* for each class treated. As a rule, calculation of higher terms by the preceding methods is exceedingly laborious. But knowledge of these is not only required to gain greater accuracy, with or without the help of terminants (Chapters XXI onwards): until we establish at least the form of the general term, we cannot even guarantee that the expansion really is of the expected asymptotic form. Fortunately it proves possible to derive formulae for evaluating late terms, the general late term being itself first found as a complete asymptotic expansion.

2. Review and Illustrations of the Darboux Theorem

The most convenient method for our purposes is based on a theorem due to Darboux (1878) according to which late terms in the development of a function as a Taylor series are determined by the behaviour of the function in the vicinity of singular points on the circle of convergence. Flamme (1887), Hamy (1908) and Haar (1926) developed the theorem in various directions. However, apart from a few specialized applications by Watson (1944) and Ninham (1963), the topic has received scant attention in recent literature. Moreover the few brief accounts, notably Courant and Hilbert (1953), Szegö (1959) and Davis (1963), all refer to the same example of expanding a Legendre polynomial of large order, an example which shrouds certain difficulties because the singularities in the function expanded happen to lie on a single circle in the complex plane. We shall therefore develop the method to the extent required for our purposes, and include a number of exercises at the end of the chapter.

Suppose a function $\phi(f)$ converges within a circle, and can be expanded within as a Taylor–Maclaurin series $\sum_0^\infty a_r f^r$. Let f_i represent a singular point

—pole or branch point, not essential singularity—on or outwith this circle of convergence. In the vicinity it is permissible to write

$$\phi(f) = (f_i - f)^{-p_i}\,\phi_i(f), \tag{1}$$

where p_i is a positive integer for a pole and fractional (positive or negative) for a branch point, and $\phi_i(f)$ is expansible as a Taylor series in $f_i - f$, thus

$$\phi_i(f) = \phi_i(f_i) - (f_i - f)\,\phi_i'(f_i) + (f_i - f)^2\,\phi_i''(f_i)/2! - \dots \tag{2}$$

Now the coefficient of f^r in $(f_i - f)^{-p_i}$ is $(r + p_i - 1)!/(p_i - 1)!r!f_i^{r+p_i}$, so the contribution to a_r from the singularity at $f = f_i$ is

$$\frac{(r + p_i - 1)!}{r!\,(p_i - 1)!f_i^{r+p_i}}\left\{\phi_i - \frac{p_i - 1}{r + p_i - 1}\,f_i\,\phi_i' + \frac{(p_i - 1)(p_i - 2)}{2!\,(r + p_i - 1)(r + p_i - 2)}\,f_i^2\,\phi_i'' - \dots\right\}, \tag{3}$$

an expansion suitable for late coefficients $r \gg p_i$.

For such large r the factor f_i^r in the denominator ensures dominant contributions from those f_i with smallest modulus, i.e. from singularities lying *on* the circle of convergence. It is indeed frequently categorically stated that a_r is the sum of (3) over singular points on the circle of convergence. From a strict standpoint, however, this is in general logically unacceptable as well as imprecise; first, because a function cannot be expanded about several points and the results added without due regard to overlapping; and second, because a contribution cannot abruptly disappear when its originating singularity is moved slightly outside the circle of convergence. A clue towards a more satisfactory statement comes from studying the consequences of a mutual approach of two singularities i and j. Since ϕ_i would contain a factor $(f_j - f)^{-p_j}$, the later part of the expansion (3) around the singularity f_i would incorporate a series in rising powers of the large quantity $f_i/(f_j - f_i)$. Without summing, these terms would be useless, there being too many of comparable magnitude to calculate; if summed, their combined contribution would be approximately $(r + p_j - 1)!\,\phi_j/r!\,(p_j - 1)!f_j^{r+p_j}$, the expected main contribution from the other singularity! The conclusion is that we should *either* expand about a single chosen singularity and sum the late terms in (3), *or* (far more conveniently) expand about every singularity—both on and outside the circle of convergence—retain only the first few terms of (3) for each, and add the contributions. These options are admirably illustrated by working out late coefficients in the easy example

$\phi = [(f_1 - f)(f_2 - f)]^{-1}$. Expanding only about the pole $f = f_1$ but retaining every term in this single expansion, we find from (3)

$$a_r = \frac{1}{(f_2 - f_1)f_1^{r+1}} \left\{1 + \frac{1}{r!} \sum_{s=r+1}^{\infty} \left(-\frac{f_1}{f_2 - f_1}\right)^s \frac{(r-s)!}{(-s)!}\right\}$$

$$= \frac{1}{f_2 - f_1}\left(\frac{1}{f_1^{r+1}} - \frac{1}{f_2^{r+1}}\right).$$

Much more simply, the leading term in expanding about the pole $f = f_1$ is obviously

$$\frac{1}{f_2 - f_1} \times \text{coefficient of } f^r \text{ in } \frac{1}{f_1 - f} \qquad = \frac{1}{f_2 - f_1} \frac{1}{f_1^{r+1}},$$

and on adding the similar result for the pole $f = f_2$ we regain the previous answer. This particular example can in fact be completed exactly on taking partial fractions:

$$a_r = \text{coefficient of } f^r \text{ in } \frac{1}{f_2 - f_1}\left\{\frac{1}{f_1 - f} - \frac{1}{f_2 - f}\right\},$$

verifying our two options.

To provide a fairer idea of the method's accuracy than could be gleaned from the preceding fortuitously exact answer, we will determine the coefficient of f^r in

$$\phi = \frac{(1+f)^{\alpha-1}}{\beta + \ln(1+f)}.$$

The denominator has a zero at $f_0 = e^{-\beta} - 1$, near which it behaves as $(f - f_0)/(1 + f_0)$. The required coefficient is therefore approximately

$$\text{coefficient of } f^r \text{ in } -\frac{(1+f_0)^{\alpha}}{f_0 - f} \qquad = (-1)^r e^{-\alpha\beta}/(1 - e^{-\beta})^{r+1}.$$

In the special case $\alpha = \beta = 1, r = 8$, this first Darboux approximation equals 22·830, an error of only 0·04% compared to the exact coefficient 22·839881....

3. Application of the Darboux Theorem to the Relevant Contour Integral

In every variety of asymptotic expansion encountered in the two preceding chapters, a typical term can be reduced to substantially the same integral representation. For instance L_r, Q_r and C_r can all be couched in terms of

$$\frac{1}{2\pi i}\oint_{f=0}\frac{G du}{f^{r+1}} = \text{coefficient of } f^r \text{ in } G du/df, \tag{4}$$

where f stands in turn for $F - F_0$, $(F - F_0)^{\frac{1}{2}}$ and $(F - F_0)^{\frac{1}{3}}$ in the cases of linear, quadratic and cubic dependence respectively. Singular points of $G\,du/df$ occur whenever $du/df \to \infty$ or $G \to \infty$. We shall suppose throughout that the circle of convergence is dictated by a singular point of the first set; i.e. any singularities of G will be assumed to lie further out from the origin $u = 0$ than one of du/df. This condition is almost invariably fulfilled in practice because of its correlation with the concept of G as the slowly-varying factor in an integrand. (A minor change of integration variable may occasionally be required; for example the branch point in a factor $(1 + u)^{-\frac{1}{2}}$ at $u = -1$ could if necessary be suppressed by transforming to a new variable $v = (1 + u)^{\frac{1}{2}}$, for then $(1 + u)^{-\frac{1}{2}}\,du = 2\,dv$). With this proviso the circle of convergence corresponds to that solution of $df/du = 0$ for which $|f|$ is least but non-zero. Let f_0, $f_1 = 0$, f_2, f_3 etc. denote the values of f and its successive derivatives with respect to u at this special point $u = u_0$ in the u-plane.

At this point the theory divides, because in problems on linear and quadratic dependence the second derivative f_2 is dominant, whereas in the cubic case it is identically zero.

4. Asymptotic Evaluation of L_r and Q_r

When $f_2 \neq 0$ the expansion in the vicinity of the singular point is

$$f = f_0 + (u - u_0)^2 f_2/2! + (u - u_0)^3 f_3/3! + (u - u_0)^4 f_4/4! \dots. \tag{5}$$

Solving for u,

$$u - u_0 = \left(\frac{2}{f_2}\right)^{\frac{1}{2}}(f - f_0)^{\frac{1}{2}} - \frac{f_3}{3f_2^{\,2}}(f - f_0)$$
$$+ \left(\frac{2}{f_2}\right)^{\frac{1}{2}}\frac{5f_3^{\,2} - 3f_2 f_4}{36 f_2^{\,3}}(f - f_0)^{3/2}\dots. \tag{6}$$

Let $\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2, \dots$ denote the values of G and its derivatives with respect to u at the singular point u_0. Expressing

$$\int_{u_0}^{u} G\,du = (u - u_0)\mathscr{G}_0 + \tfrac{1}{2}(u - u_0)^2\mathscr{G}_1 + \dots$$

in terms of $f - f_0$ and then differentiating with respect to f,

$$G\frac{du}{df} = \frac{\mathscr{G}_0}{(-2f_0f_2)^{\frac{1}{2}}}\left(1 - \frac{f}{f_0}\right)^{-\frac{1}{2}} - \frac{\mathscr{G}_0f_3 - 3\mathscr{G}_1f_2}{3f_2^{\,2}}$$

$$+ \frac{(-2f_0f_2)^{\frac{1}{2}}}{24f_2^{\,4}}\left(1 - \frac{f}{f_0}\right)^{\frac{1}{2}}\{\mathscr{G}_0(5f_3^{\,2} - 3f_2f_4) - 12\mathscr{G}_1f_2f_3 + 12\mathscr{G}_2f_2^{\,2}\}\ldots \tag{7}$$

Extracting coefficients of f^r throughout as required in (4),

$$\frac{1}{2\pi i}\oint\frac{G\,du}{f^{r+1}} = \frac{(r-\frac{1}{2})!}{r!(-2\pi f_2f_0)^{\frac{1}{2}}f_0^{\,r}}\left[\mathscr{G}_0 + \frac{f_0}{24f_2^{\,3}(r-\frac{1}{2})}\{\mathscr{G}_0(5f_3^{\,2} - 3f_2f_4)\right.$$

$$\left. - 12\mathscr{G}_1f_2f_3 + 12\mathscr{G}_2f_2^{\,2}\}\ldots\right]. \tag{8}$$

This expansion for late terms applies equally to the cases of linear or quadratic dependence of F on u, but for the latter it proves more convenient to rewrite the derivatives of f as derivatives of $\mathscr{F} = F - F_0 = f^2$. The initial result is

$$\frac{(r-\frac{1}{2})!}{r!(-\pi\mathscr{F}_2)^{\frac{1}{2}}\mathscr{F}_0^{\,\frac{1}{2}r}}\left[\mathscr{G}_0\left(1 + \frac{3}{8(r-\frac{1}{2})}\right) + \frac{\mathscr{F}_0}{12\mathscr{F}_2^{\,3}(r-\frac{1}{2})}\times\right.$$

$$\left.\{\mathscr{G}_0(5\mathscr{F}_3^{\,2} - 3\mathscr{F}_2\mathscr{F}_4) - 12\mathscr{G}_1\mathscr{F}_2\mathscr{F}_3 + 12\mathscr{G}_2\mathscr{F}_2^{\,2}\}\ldots\right],$$

but to the same order this can be recast as

$$\frac{1}{2\pi i}\oint\frac{G\,du}{\mathscr{F}^{\frac{1}{2}r+\frac{1}{2}}} = \frac{(\frac{1}{2}r-1)!}{(\frac{1}{2}r-\frac{1}{2})!\,(-2\pi\mathscr{F}_2)^{\frac{1}{2}}\mathscr{F}_0^{\,\frac{1}{2}r}}\left[\mathscr{G}_0 + \frac{\mathscr{F}_0}{24\mathscr{F}_2^{\,3}(\frac{1}{2}r-1)}\times\right.$$

$$\left.\{\mathscr{G}_0(5\mathscr{F}_3^{\,2} - 3\mathscr{F}_2\mathscr{F}_4) - 12\mathscr{G}_1\mathscr{F}_2\mathscr{F}_3 + 12\mathscr{G}_2\mathscr{F}_2^{\,2}\}\ldots\right]. \tag{9}$$

This is the same as (8) apart from replacement of $r + 1$ in (8) by $\frac{1}{2}r + \frac{1}{2}$ in (9), an accord which persists in later terms, thereby showing that calculations can be shortened by formally regarding $(2\pi i)^{-1}\oint\mathscr{F}^{-\frac{1}{2}r-\frac{1}{2}}G\,du$ as the coefficient of $\mathscr{F}^{\frac{1}{2}r-\frac{1}{2}}$ in $G\,du/d\mathscr{F}$ *with $\frac{1}{2}r - \frac{1}{2}$ treated as if it were always an integer.* We can now exploit this observation to relate the coefficients in (9) to the algebraic formulae for Q_r quoted in Chapter V, Section 3. For since the only difference here is that all derivatives are to be taken at the singular point u_0, we may write

$$G\frac{du}{d\mathscr{F}} = \left(\frac{\pi}{2\mathscr{F}_2}\right)^{\frac{1}{2}} \sum_0^\infty \frac{\mathscr{Q}_s}{(\frac{1}{2}s-\frac{1}{2})!}(\mathscr{F}-\mathscr{F}_0)^{\frac{1}{2}s-\frac{1}{2}}, \tag{10}$$

where the $\mathscr{Q}_s$ are formed from derivatives at u_0. In extracting the coefficient of $\mathscr{F}^{\frac{1}{2}r-\frac{1}{2}}$, treating $\frac{1}{2}r-\frac{1}{2}$ as if it were an integer, the terms for which s is odd ($\frac{1}{2}s-\frac{1}{2}$ = integer)—which could not subscribe unless $s \geqslant r$—are to be omitted. For, by the argument on options in the Darboux method (Section 2), if such terms were to be included we should be forced to sum this infinite series to a closed form and concomitantly disallow addition of contributions from any other singular point. On the understsnding that we are adopting the simpler option of allowing additions from other singular points, the full version of (9) for one singular point is

$$\frac{1}{2\pi i}\oint \frac{G du}{\mathscr{F}^{\frac{1}{2}r+\frac{1}{2}}} = \frac{1}{(\frac{1}{2}r-\frac{1}{2})!\,(-2\pi\mathscr{F}_2)^{\frac{1}{2}}\,\mathscr{F}_0^{\frac{1}{2}r}} \sum_{s=0} (\tfrac{1}{2}r-s-1)!\,\mathscr{Q}_{2s}\mathscr{F}_0^{\,s}. \tag{11}$$

Referring to the integral representations for Q_r and L_r, Chapter V (19) and (9) respectively, (11) above gives

$$Q_r = \frac{1}{2\pi}\left(\frac{F_2}{-\mathscr{F}_2}\right)^{\frac{1}{2}} \frac{1}{\mathscr{F}_0^{\frac{1}{2}r}} \sum_{s=0} (\tfrac{1}{2}r-s-1)!\,\mathscr{Q}_{2s}\mathscr{F}_0^{\,s}, \tag{12}$$

$$L_r = \frac{F_1}{(-2\pi\mathscr{F}_2\mathscr{F}_0)^{\frac{1}{2}}\,\mathscr{F}_0^{\,r}} \sum_{s=0} (r-s-\tfrac{1}{2})!\,\mathscr{Q}_{2s}\mathscr{F}_0^{\,s}. \tag{13}$$

The root is to be interpreted as $(-\mathscr{F}_2)^{\frac{1}{2}} \equiv i\mathscr{F}_2^{\frac{1}{2}}$ or $-i\mathscr{F}_2^{\frac{1}{2}}$ depending respectively on whether the singular point lies below or above the real axis in the $\mathscr{F}$-plane.

5. Asymptotic Evaluation of C_r

The derivative f_2 is almost invariably identically zero in the application to cubic dependence of $F(u)$ at a stationary point. In these circumstances (6) must be replaced by

$$u-u_0 = \left(\frac{6}{f_3}\right)^{\frac{1}{3}}(f-f_0)^{\frac{1}{3}} - \frac{f_4}{72}\left(\frac{6}{f_3}\right)^{\frac{4}{3}}(f-f_0)^{\frac{2}{3}} + \ldots \tag{14}$$

Then

$$G\frac{du}{df} = \frac{6^{\frac{1}{3}}\mathscr{G}_0}{3f_3^{\frac{1}{3}}f_0^{\frac{2}{3}}}\left(1 - \frac{f}{f_0}\right)^{-\frac{2}{3}} + \frac{\mathscr{G}_0 f_4 - 6\mathscr{G}_1 f_3}{3.6^{\frac{1}{3}} f_3^{\frac{5}{3}} f_0^{\frac{1}{3}}}\left(1 - \frac{f}{f_0}\right)^{-\frac{1}{3}} \dots \tag{15}$$

Extracting coefficients of f^r throughout,

$$\frac{1}{2\pi i}\oint \frac{G du}{f^{r+1}} = \frac{\sqrt{3}\ \alpha}{2\pi\ r!\, f_3^{\frac{1}{3}} f_0^{r+\frac{2}{3}}}\left[(r - \tfrac{1}{3})!\,\mathscr{G}_0 \right.$$

$$\left. + \frac{\beta\ (r - \frac{2}{3})!\, f_0^{1/3}(\mathscr{G}_0 f_4 - 6\mathscr{G}_1 f_3)}{f_3^{4/3}} \dots \right] \tag{16}$$

where α and β are the numerical constants defined in Chapter V, Section 4. Rewriting the derivatives of f as derivatives of $\mathscr{F} = F - F_0 = f^3$, the initial result is

$$\frac{3^{5/6}\alpha}{2\pi\ r!\, \mathscr{F}_3^{\frac{1}{9}}\, \mathscr{F}_0^{\frac{1}{3}r}}\left[(r - \tfrac{1}{3})!\,\mathscr{G}_0 + \frac{3^{\frac{1}{3}}\beta(r - \frac{2}{3})!\, \mathscr{F}_0^{\frac{1}{3}}(\mathscr{G}_0\mathscr{F}_4 - 6\mathscr{G}_1\mathscr{F}_3)}{\mathscr{F}_3^{\frac{4}{3}}} \dots \right],$$

but to the same order this can be recast as

$$\frac{1}{2\pi i}\oint \frac{G du}{\mathscr{F}^{\frac{1}{3}r+\frac{1}{3}}} = \frac{\sqrt{3}\ \alpha}{2\pi(\frac{1}{3}r - \frac{2}{3})!\, \mathscr{F}_3^{\frac{1}{3}}\mathscr{F}_0^{\frac{1}{3}r}}$$

$$\times \left[(\tfrac{1}{3}r - 1)!\,\mathscr{G}_0 + \frac{\beta\ (\frac{1}{3}r - \frac{4}{3})!\, \mathscr{F}_0^{\frac{1}{3}}(\mathscr{G}_0\mathscr{F}_4 - 6\mathscr{G}_1\mathscr{F}_3)}{\mathscr{F}_3^{\frac{4}{3}}} \dots \right]. \tag{17}$$

This is the same as (16) apart from replacement of $r + 1$ in (16) by $\frac{1}{3}r + \frac{1}{3}$ in (17), indicating that calculations can be shortened by formally regarding $(2\pi i)^{-1}\oint \mathscr{F}^{-\frac{1}{3}r-\frac{1}{3}}\, G\, du$ as the coefficient of $\mathscr{F}^{\frac{1}{3}r-\frac{2}{3}}$ in $G\, du/d\mathscr{F}$ *with $\frac{1}{3}r - \frac{2}{3}$ treated as if it were always an integer*. Now

$$G\frac{du}{d\mathscr{F}} = \frac{\alpha}{\mathscr{F}_3^{\frac{1}{3}}}\sum_0^\infty \frac{\mathscr{C}_s}{(\frac{1}{3}s - \frac{2}{3})!}(\mathscr{F} - \mathscr{F}_0)^{\frac{1}{3}s-\frac{2}{3}}, \tag{18}$$

where the $\mathscr{C}_s$ are formed from derivatives at u_0. To extract the required coefficient, we first note that when R is an integer the coefficient of x^R in

$(1-x)^S$ is

$$\frac{(-1)^R S!}{R!\,(S-R)!} = \frac{(-1)^R S!\,(R-S-1)!\sin\pi(R-S)}{\pi\,R!}$$

$$= -\frac{S!\,(R-S-1)!\sin\pi S}{\pi\,R!},$$

on introducing successively the reflection formula and the integer condition on R. Thus, setting $R = \frac{1}{3}r - \frac{2}{3}$ (treated as an integer) and $S = \frac{1}{3}s - \frac{2}{3}$, the full version of (17) is

$$\frac{1}{2\pi i}\oint \frac{G\,du}{\mathscr{F}^{\frac{1}{3}r+\frac{1}{3}}} = -\frac{\alpha}{\pi(\frac{1}{3}r-\frac{2}{3})!\,\mathscr{F}_3{}^{\frac{1}{3}}\mathscr{F}_0{}^{\frac{1}{3}r}}$$
$$\times \sum_{s=0} (-1)^s\,(\tfrac{1}{3}r-\tfrac{1}{3}s-1)!\sin\tfrac{1}{3}\pi(s-2)\;\mathscr{C}_s\mathscr{F}_0{}^{\frac{1}{3}s}. \tag{19}$$

Referring to the integral representation for C_r, Chapter V (28), equation (19) gives

$$C_r = \frac{1}{2\sqrt{3}\,\pi}\left(\frac{F_3}{\mathscr{F}_3}\right)^{\frac{1}{3}}\frac{1}{\mathscr{F}_0{}^{\frac{1}{3}r}}\{(\tfrac{1}{3}r-1)!\,\mathscr{C}_0 - (\tfrac{1}{3}r-\tfrac{4}{3})!\,\mathscr{C}_1\mathscr{F}_0{}^{\frac{1}{3}}$$
$$+ (\tfrac{1}{3}r-2)!\,\mathscr{C}_3\mathscr{F}_0 - (\tfrac{1}{3}r-\tfrac{7}{3})!\,\mathscr{C}_4\mathscr{F}_0{}^{\frac{4}{3}} + (\tfrac{1}{3}r-3)!\,\mathscr{C}_6\mathscr{F}_0{}^2$$
$$- (\tfrac{1}{3}r-\tfrac{10}{3})!\,\mathscr{C}_7\mathscr{F}_0{}^{\frac{7}{3}} + \dots\}. \tag{20}$$

The conventions to be observed in extracting the cube roots are $(-1)^{\frac{1}{3}} = -1$ and $(\pm i)^{\frac{1}{3}} = \mp i$.

6. Meaning and Magnitude of $\mathscr{F}_0$, a "Singulant"

As we shall see in Chapters XXI onwards, the functional dependence on r displayed in (12), (13) and (20), namely factorials of r divided by an rth power, is ideally expressed for subsequent evaluation of terminants.

The close relationship implicit in the preceding sections between application of the Darboux theorem and stationary-point methods extends beyond their conjoint application to the technical calculation of late terms. Formulae for late terms can in fact alternatively be derived wholly from stationary-point methods, treating $f^{-r-1} = e^{-(r+1)\ln f}$ as fast-varying (questions 11, 12); though in execution this involves heavy algebra and initially yields results which, while equivalent to those quoted, emerge couched in unsuitable and complicated form. This equivalence permits us to identify each Darboux "singular point" u_0 with a certain stationary point of $F(u)$, thereby illuminating the meaning of the extremely important quantity $\mathscr{F}_0$ appearing in all our formulae for late terms, which we shall henceforth call a "singulant".

Let F_0 be the value of F at the critical expansion point concerned in deriving some asymptotic series. The chief stationary point involved in the evaluation of late terms in that series is specified by three conditions: $dF/du = 0$, $\mathscr{F} = F - F_0 \neq 0$, $|\mathscr{F}|$ otherwise least. The chief singulant $\mathscr{F}_0$ is the value of $\mathscr{F}$ at this stationary point.

In the case of linear dependence at a limit, the second condition is automatically satisfied by virtue of the initial premise $F_1 \neq 0$; for $F - F_0$, being zero at the limit, cannot then be zero again anywhere near. The chief singulant is here the change in value of F in going from the limit of integration to the nearest stationary point in the F-plane.

Turning to the cases of quadratic or cubic dependence, sometimes the required smallest $|\mathscr{F}|$ corresponds to the change in value of F in going from the original stationary point, that concerned in deriving the asymptotic expansion, to another distinct stationary point. But this may be superseded by a pair of values still smaller in modulus if $F(u)$ includes some many-valued function such as a logarithm or an inverse trigonometrical or hyperbolic function. In deriving the original asymptotic expansion, the value of F at the stationary point, F_0, will have been taken as the principal value; the chief singulant pair would then correspond to the changes in F in switching from this principal value to the nearest pair of non-principal values in the F-plane, which will be conjugate complexes. *In the u-plane* the locations of principal and non-principal values coincide, since the u-values differ only by phase factors $e^{\pm 2\pi i m}$, $m = 1, 2, \ldots$; thus if the principal stationary point had been at $u = 0$, the non-principal stationary points would be at $u_0 = 0$.

Values of G and of derivatives of F at stationary points controlling late terms frequently differ only by sign changes from those at the stationary point concerned in deriving an original asymptotic expansion. In such instances $\mathscr{Q}_r = \pm Q_r$ and $\mathscr{C}_r = \pm C_r$. Then $L_{r \gg 1}$ and $Q_{r \gg 1}$ are expressed as asymptotic developments in which successive coefficients are essentially Q_{2r}; and $C_{r \gg 1}$ is expressed as an asymptotic development in which successive coefficients are essentially C_0, C_1, C_3, C_4, etc. Formulae for late terms $L_{r \gg 1}$ and $Q_{r \gg 1}$ can then be written down by inspection from the first few terms of an original asymptotic series obtained on the assumption of quadratic dependence at the critical point; and $C_{r \gg 1}$ surmised from initial terms of an asymptotic series assuming cubic dependence. This remarkable interdependence of late and early terms occurs whenever there is similarity between coefficients in a pair of associated asymptotic series (as in phase-integral expansions, Chapter XIII), because late terms in the one dictate early terms in the other; to anticipate our interpretative theory of Chapters

XXI onwards, the intermediate steps may be sketched as follows: late terms of one series → terminant of that series → discontinuity on crossing Stokes ray → early terms of associated series.

To summarize so far, in the evaluation of asymptotic expansions from integral representations, the chief singulant $\mathscr{F}_0$ or singulant pair equals the least change in value of F in going *either*

(a) from its value at the critical point determining the original asymptotic expansion, to a neighbouring stationary point, *or*

(b) from its principal value at the stationary point determining the original asymptotic expansion, to a conjugate pair of non-principal values.

Now an asymptotic series for a function satisfying a homogeneous second-order differential equation can have only one associated series, since such an equation can have only two independent solutions; correspondingly, there is either a single singulant, or one singulant pair. In asymptotic series derived from other sources, there can in principle be subsidiary singulants also. Because of their larger modulus, their numerical influence on late terms is slight, but they are relevant to the continuation of asymptotic expansions across Stokes rays—indeed, comparison with independently-derived Stokes discontinuities frequently affords the simplest confirmation of their presence. Evidence of contributions from subsidiary singulants may also come from calculations starting from known late terms in the asymptotic expansion of a *function* of the quantity to be expanded—e.g. those in $p!$ (Chapter VIII, Section 2) can be found from the known late terms in the expansion of $\ln(p!)$—or empirically from numerical computations of late terms, as for the Bessel functions $J_p(p)$, $Y_p(p)$ and their derivatives (Chapter VIII, Section 5).

In Chapters XIII, XVI and XIX we shall show how singulants are found for asymptotic expansions derived from homogeneous and inhomogeneous differential equations.

7. Asymptotic Evaluation of $L_r^{(\sigma)}$, $Q_r^{(\sigma)}$ and $C_r^{(\sigma)}$ when $u_o = 0$

These can all be expressed in terms of

$$\frac{1}{2\pi i}\oint \frac{u^\sigma G du}{f^{r+\sigma+1}} = \text{coefficient of } f^{r+\sigma} \text{ in } u^\sigma G du/df. \tag{21}$$

The singular point of $u^\sigma G du/df$, regarded as a function of f, again occurs when $df/du = 0$, i.e. at $u = u_0$ as before, but on account of the explicit appearance of u^σ as a factor in the integrand it is necessary here to proceed differently according to whether $u_0 = 0$ or $u_0 \neq 0$. The former corresponds to a chief singulant pair coming from the changes in F in switching at the stationary point from its principal to the nearest pair of non-principal values.

In either event, the generalization of (10) is

$$(u-u_0)^\sigma G\frac{du}{d\mathscr{F}}=\tfrac{1}{2}\left(\frac{2}{\mathscr{F}_2}\right)^{\frac{1}{2}\sigma+\frac{1}{2}}(\tfrac{1}{2}\sigma-\tfrac{1}{2})!$$

$$\times\sum_0^\infty\frac{\mathscr{Q}_s^{(\sigma)}}{(\frac{1}{2}s+\frac{1}{2}\sigma-\frac{1}{2})!}(\mathscr{F}-\mathscr{F}_0)^{\frac{1}{2}s+\frac{1}{2}\sigma-\frac{1}{2}}. \tag{22}$$

When $u_0=0$ the required coefficient can be extracted directly. Reference to the full integral representations for $Q_r^{(\sigma)}$ and $L_r^{(\sigma)}$, respectively V (22) and V (14), then yields their complete asymptotic expansions for large r:

$$Q_r^{(\sigma)}=\frac{1}{2\pi}\left(\frac{F_2}{-\mathscr{F}_2}\right)^{\frac{1}{2}\sigma+\frac{1}{2}}\frac{1}{\mathscr{F}_0^{\frac{1}{2}r}}\begin{cases}\sum\limits_{s=0}(\frac{1}{2}r-s-1)!\,\mathscr{Q}_{2s}^{(\sigma)}\mathscr{F}_0^{\,s}, & \sigma\text{ even},\\ \sum\limits_{s=0}(\frac{1}{2}r-s-\frac{3}{2})!\,\mathscr{Q}_{2s+1}^{(\sigma)}\mathscr{F}_0^{\,s+\frac{1}{2}}, & \sigma\text{ odd},\end{cases} \tag{23}$$

$$L_r^{(\sigma)}=\frac{(\frac{1}{2}\sigma-\frac{1}{2})!}{2\pi\cdot\sigma!}\left(\frac{2F_1^{\,2}}{-\mathscr{F}_2\mathscr{F}_0}\right)^{\frac{1}{2}\sigma+\frac{1}{2}}\frac{1}{\mathscr{F}_0^{\,r}}$$

$$\times\begin{cases}\sum\limits_{s=0}(r-s+\frac{1}{2}\sigma-\frac{1}{2})!\,\mathscr{Q}_{2s}^{(\sigma)}\mathscr{F}_0^{\,s}, & \sigma\text{ even},\\ \sum\limits_{s=0}(r-s+\frac{1}{2}\sigma-1)!\,\mathscr{Q}_{2s+1}^{(\sigma)}\mathscr{F}_0^{\,s+\frac{1}{2}}, & \sigma\text{ odd}.\end{cases} \tag{24}$$

The square root is to be interpreted as $(-\mathscr{F}_2)^{\frac{1}{2}}\equiv i\mathscr{F}_2^{\,\frac{1}{2}}$ or $-i\mathscr{F}_2^{\,\frac{1}{2}}$ depending respectively on whether the singular point lies below or above the real axis in the F-plane.

Turning to cubic dependence, independently of whether u_0 vanishes or not the generalization of (18) is

$$(u-u_0)^\sigma G\frac{du}{d\mathscr{F}}$$

$$=\tfrac{1}{3}\left(\frac{6}{\mathscr{F}_3}\right)^{\frac{1}{3}\sigma+\frac{1}{3}}(\tfrac{1}{3}\sigma-\tfrac{2}{3})!\sum_0^\infty\frac{\mathscr{C}_s^{(\sigma)}}{(\frac{1}{3}s+\frac{1}{3}\sigma-\frac{2}{3})!}(\mathscr{F}-\mathscr{F}_0)^{\frac{1}{3}s+\frac{1}{3}\sigma-\frac{2}{3}}. \tag{25}$$

When $u_0 = 0$, direct extraction of the required coefficient via V (31) gives

$$C_r^{(\sigma)} = \frac{1}{2\sqrt{3}\ \pi}\left(\frac{F_3}{\mathscr{F}_3}\right)^{\frac{1}{3}\sigma+\frac{1}{3}}\frac{1}{\mathscr{F}_0^{\frac{1}{3}r}}$$

$$\times\begin{cases}(\tfrac{1}{3}r-1)!\,\mathscr{C}_0^{(\sigma)}-(\tfrac{1}{3}r-\tfrac{4}{3})!\,\mathscr{C}_1^{(\sigma)}\mathscr{F}_0^{\frac{1}{3}}+(\tfrac{1}{3}r-2)!\,\mathscr{C}_3^{(\sigma)}\mathscr{F}_0\ldots; \\ \qquad\qquad\qquad\qquad\qquad\qquad\sigma=0,3,6,\ldots \\ -(\tfrac{1}{3}r-1)!\,\mathscr{C}_0^{(\sigma)}+(\tfrac{1}{3}r-\tfrac{5}{3})!\,\mathscr{C}_2^{(\sigma)}\mathscr{F}_0^{\frac{2}{3}}-(\tfrac{1}{3}r-2)!\,\mathscr{C}_3^{(\sigma)}\mathscr{F}_0\ldots; \\ \qquad\qquad\qquad\qquad\qquad\qquad\sigma=1,4,7,\ldots \\ (\tfrac{1}{3}r-\tfrac{4}{3})!\,\mathscr{C}_1^{(\sigma)}\mathscr{F}_0^{\frac{1}{3}}-(\tfrac{1}{3}r-\tfrac{5}{3})!\,\mathscr{C}_2^{(\sigma)}\mathscr{F}_0^{\frac{2}{3}}+(\tfrac{1}{3}r-\tfrac{7}{3})!\,\mathscr{C}_4^{(\sigma)}\mathscr{F}_0^{\frac{4}{3}}\ldots; \\ \qquad\qquad\qquad\qquad\qquad\qquad\sigma=2,5,8,\ldots\end{cases} \tag{26}$$

The likelihood of inter-dependence between late and early terms is preserved in these generalizations. Thus whenever in the u-plane the singular points u_0 coincide with the stationary point $u = 0$, formulae for $L_{r\geqslant 1}^{(\sigma)}$ and $Q_{r\geqslant 1}^{(\sigma)}$ can frequently be written down by inspection from the first few terms of an original asymptotic series obtained on the assumption of quadratic dependence at the critical point; and $C_{r\geqslant 1}$ surmised from initial terms of an asymptotic series assuming cubic dependence.

8. Asymptotic Evaluation of $L_r^{(\sigma)}$, $Q_r^{(\sigma)}$ and $C_r^{(\sigma)}$ when $u_0 \neq 0$

When $u_0 \neq 0$, u^σ is slowly varying near $u = u_0$ just like $G(u)$. The integral representations can then be evaluated exactly as in Sections 4 and 5 apart from the replacements $G \to u^\sigma G$ and $r \to r + \sigma$. On taking a factor u_0^σ outside each representation, the first replacement is more conveniently expressed as $G \to u_0^{-\sigma}u^\sigma G$, corresponding to a change of derivative values from $\mathscr{G}_s$ to

$$\bar{\mathscr{G}}_s = u_0^{-\sigma}\left(\frac{d}{du}\right)^s u^\sigma G\bigg|_{u=u_0} = \sigma!\,s!\sum_{t=0}^{s}\frac{\mathscr{G}_{s-t}}{t!\,(s-t)!\,(\sigma-t)!\,u_0^{\,t}}. \tag{27}$$

Reference to the full integral representations V (14), (22) and (31), and comparison with (12), (13) and (20) of the present chapter, then leads to the following complete asymptotic expansions for large r:

$$L_r^{(\sigma)} = \frac{F_1}{\sigma!\,(-2\pi\mathscr{F}_2\mathscr{F}_0)^{\frac{1}{2}}}\left(\frac{F_1u_0}{\mathscr{F}_0}\right)^\sigma\frac{1}{\mathscr{F}_0^{\,r}}\sum_{s=0}(r-s+\sigma-\tfrac{1}{2})!\,\bar{\mathscr{Q}}_{2s}\mathscr{F}_0^{\,s}, \tag{28}$$

$$Q_r^{(\sigma)} = \frac{1}{2(\frac{1}{2}\sigma-\frac{1}{2})!}\left(\frac{F_2}{-\pi\,\mathscr{F}_2}\right)^{\frac{1}{2}}\left(\frac{F_2u_0^{\,2}}{2\mathscr{F}_0}\right)^{\frac{1}{2}\sigma}\frac{1}{\mathscr{F}_0^{\frac{1}{2}r}}\sum_{s=0}(\tfrac{1}{2}r-s+\tfrac{1}{2}\sigma-1)!\,\bar{\mathscr{Q}}_{2s}\mathscr{F}_0^{\,s}, \tag{29}$$

$$
\begin{aligned}
C_r^{(\sigma)} = {} & \frac{1}{2\sqrt{3}\,\pi} \frac{(-\frac{2}{3})!}{(\frac{1}{3}\sigma - \frac{2}{3})!} \left(\frac{F_3}{\mathscr{F}_3}\right)^{\frac{1}{3}} \left(\frac{F_3 u_0^{\,3}}{6\mathscr{F}_0}\right)^{\frac{1}{3}\sigma} \frac{1}{\mathscr{F}_0^{\,\frac{1}{3}r}} \{(\tfrac{1}{3}r + \tfrac{1}{3}\sigma - 1)!\,\bar{\mathscr{C}}_0 \\
& - (\tfrac{1}{3}r + \tfrac{1}{3}\sigma - \tfrac{4}{3})!\,\bar{\mathscr{C}}_1 \mathscr{F}_0^{\,\frac{1}{3}} + (\tfrac{1}{3}r + \tfrac{1}{3}\sigma - 2)!\,\bar{\mathscr{C}}_3 \mathscr{F}_0 \\
& - (\tfrac{1}{3}r + \tfrac{1}{3}\sigma - \tfrac{7}{3})!\,\bar{\mathscr{C}}_4 \mathscr{F}_0^{\,\frac{4}{3}} + (\tfrac{1}{3}r + \tfrac{1}{3}\sigma - 3)!\,\bar{\mathscr{C}}_6 \mathscr{F}_0^{\,2} \\
& - (\tfrac{1}{3}r + \tfrac{1}{3}\sigma - \tfrac{10}{3})!\,\bar{\mathscr{C}}_7 \mathscr{F}_0^{\,\frac{7}{3}} \ldots \}.
\end{aligned} \tag{30}
$$

Here, $\bar{\mathscr{Q}}$ and $\bar{\mathscr{C}}$ differ from $\mathscr{Q}$ and $\mathscr{C}$ (*not* $\mathscr{Q}^{(\sigma)}$ and $\mathscr{C}^{(\sigma)}$) only by the replacement throughout of $\mathscr{G}_s$ by $\bar{\mathscr{G}}_s$ as defined in (27). Note the disappearance when $u_0 \neq 0$ of all resemblance between coefficients involved in late and early terms.

Exercises

1. Prove that when r is large and $|a| < |b|$, the coefficient of f^r in $(1 - f/a)^{-p}(1 - f/b)^{-q}$ is

$$\left(\frac{b}{b-a}\right)^q \frac{(r+p-1)!}{a^r\, r!\,(p-1)!} \left\{1 - \frac{qa}{b-a}\,\frac{p-1}{r+p-1} \cdots\right\} + O(b^{-r}).$$

2. By differentiation with respect to p deduce late terms in the expansion of $(1 - f/a)^{-p}(1 - f/b)^{-q}\ln(1 - f/a)$.

3. By differentiation with respect to q deduce late terms in the expansion of $(1 - f/a)^{-p}(1 - f/b)^{-q}\ln(1 - f/b)$.

4. Devise a series of excuses for a term being late.

5. Prove that when r is large and even the coefficient of f^r in $\{f^3/(\tan f - f)\}^m$ is

$$\frac{2(-1)^m(r+m-1)!}{r!\,(m-1)!\,(4{\cdot}49)^r}\left\{1 + \frac{18{\cdot}2\,m(m-1)}{r+m-1} \cdots\right\} + O(7{\cdot}73^{-r}).$$

6. Find that value of α for which the coefficients of high even powers of f in the expansion of $f/(\sinh f - \alpha f)$ tend to a limiting value, and show that this limiting coefficient is $-2e$.

7. Show that for large r,

$$\left(\frac{d}{df}\right)^r (1-f^2)^{-p} \doteq 2^{-p}\frac{(r+p-1)!}{(p-1)!}\left\{\frac{1}{(1-f)^{r+p}}+\frac{(-1)^r}{(1+f)^{r+p}}\right\}.$$

8. Prove that when r is large the coefficient of f^r in $[(1+f)(1+\frac{1}{2}f)(1+\frac{1}{3}f)]^{-\frac{1}{2}}$ is

$$\left(\frac{3}{\pi}\right)^{\frac{1}{2}}(-1)^r\frac{(r-\frac{1}{2})!}{r!}\left\{1+\frac{3}{8(r-\frac{1}{2})}+\frac{57}{128(r-\frac{1}{2})(r-\frac{3}{2})}\cdots\right\}+O(2^{-r}).$$

9. For the case of linear dependence at a limit, to cover the rare instances in which the second derivative vanishes at the singular point derive the following formula applicable when $\mathscr{F}_2 = 0$:

$$L_{r\gg 1} = \frac{(-\frac{2}{3})!\,F_1}{(48)^{\frac{1}{6}}\,\pi(\mathscr{F}_3\mathscr{F}_0{}^2)^{\frac{1}{3}}\,\mathscr{F}_0{}^r}\{(\tfrac{1}{3}r-\tfrac{1}{3})!\,\mathscr{C}_0-(\tfrac{1}{3}r-\tfrac{2}{3})!\,\mathscr{C}_1\mathscr{F}_0{}^{1/3}$$

$$+(\tfrac{1}{3}r-\tfrac{4}{3})!\,\mathscr{C}_3\mathscr{F}_0-(\tfrac{1}{3}r-\tfrac{5}{3})!\,\mathscr{C}_4\mathscr{F}_0{}^{\frac{4}{3}}+(\tfrac{1}{3}r-\tfrac{7}{3})!\,\mathscr{C}_6\mathscr{F}_0{}^2$$

$$-(\tfrac{1}{3}r-\tfrac{8}{3})!\,\mathscr{C}_7\mathscr{F}_0{}^{\frac{7}{3}}\ldots\}.$$

10. For the case of quadratic dependence at a limit, to cover the rare instances in which the second derivative vanishes at the singular point derive the following formula applicable when $\mathscr{F}_2 = 0$:

$$Q_{r\gg 1} = \frac{(-\frac{2}{3})!\,F_2{}^{\frac{1}{2}}}{(384)^{\frac{1}{6}}\,\pi^{\frac{3}{2}}\mathscr{F}_3{}^{\frac{1}{3}}\mathscr{F}_0{}^{\frac{1}{2}r+\frac{1}{6}}}\{(\tfrac{1}{2}r-\tfrac{5}{6})!\,\mathscr{C}_0-(\tfrac{1}{2}r-\tfrac{7}{6})!\,\mathscr{C}_1\mathscr{F}_0{}^{1/3}$$

$$+(\tfrac{1}{2}r-\tfrac{11}{6})!\,\mathscr{C}_3\mathscr{F}_0-(\tfrac{1}{2}r-\tfrac{13}{6})!\,\mathscr{C}_4\mathscr{F}_0{}^{4/3}+(\tfrac{1}{2}r-\tfrac{17}{6})!\,\mathscr{C}_6\mathscr{F}_0{}^2$$

$$-(\tfrac{1}{2}r-\tfrac{19}{6})!\,\mathscr{C}_7\mathscr{F}_0{}^{\frac{7}{3}}\ldots\}.$$

11. Treating $f^{-r-1} = e^{-(r+1)\ln f}$ as fast-varying, apply stationary-point methods to derive the following formula for the integral representation involved in the calculation of late terms, in which f_0, f_2, f_3 etc. denote the values of f and its second and higher derivatives at the point for which the

first derivative vanishes:

$$\frac{1}{2\pi i}\oint \frac{G\,du}{f^{r+1}} = \frac{1}{[-2\pi(r+1)f_2 f_0]^{\frac{1}{2}} f_0^r}\left\{\mathscr{G}_0 + \frac{f_0}{24(r+1)f_2^3}\{\mathscr{G}_0(5f_3^2\right.$$

$$- 3f_2 f_4 + 9f_2^3/f_0) - 12\mathscr{G}_1 f_2 f_3 + 12\mathscr{G}_2 f_2^2\} + \frac{f_0^2}{1152(r+1)^2 f_2^6}$$

$$\times\{\mathscr{G}_0(385f_3^4 - 630f_2 f_3^2 f_4 + 105f_2^2 f_4^2 + 168f_2^2 f_3 f_5 - 24f_2^2 f_6$$

$$+ 450f_2^3 f_3^2/f_0 - 270f_2^4 f_4/f_0 + 225f_2^6/f_0^2) - 24\mathscr{G}_1 f_2(35f_3^3$$

$$- 35f_2 f_3 f_4 + 6f_2^2 f_5 + 45f_2^3 f_3/f_0) + 120\mathscr{G}_2 f_2^2(7f_3^2 - 3f_2 f_4$$

$$\left. + 9f_2^3/f_0) - 480\mathscr{G}_3 f_2^3 f_3 + 144\mathscr{G}_4 f_2^4\}\ldots\right\}.$$

12. In the preceding answer, make the outer factor correspond to the one in equation (8) by introducing the expansion

$$r!/(r-\tfrac{1}{2})!\,(r+1)^{\frac{1}{2}} = 1 - 3/8(r+1) - 7/128(r+1)^2 \ldots,$$

and hence verify that the expansion reduces to (8) and (13).

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Chapter VIII

Applications of Preceding Theory on $\int e^{-F}\, G\, du$ and $\int e^{-F}\, u^{\sigma}\, G\, du$

1. Calculation of High Derivatives.

To find L_r we require derivatives of F up to F_{r+1} and of G up to G_r; while Q_r requires derivatives up to F_{r+2} and G_r, and C_r requires them up to F_{r+3} and G_r. We shall therefore begin this chapter on applications of the theory of the preceding three chapters by reviewing methods of calculating high derivatives.

Leibnitz formula. If the function to be multiply-differentiated can be split into the product of a polynomial and a function which does possess simple multiple derivatives, the easiest approach is through the well-known Leibnitz formula

$$\left(\frac{d}{du}\right)^r PQ = r!\sum_0^r \frac{1}{s!\,(r-s)!}\left\{\left(\frac{d}{du}\right)^s P\right\}\left\{\left(\frac{d}{du}\right)^{r-s} Q\right\}. \tag{1}$$

For if the polynomial $P(u)$ is of degree p, all its derivatives past the pth vanish, reducing the number of terms in the summation (1) to $p+1$.

Schlömilch formula. If the function can more easily be multiply-differentiated with respect to a function $U(u)$ than with respect to u itself, the little-known Schlömilch formula (1895) may prove effective:

$$\left(\frac{d}{du}\right)^r F = \sum_0^r S_{rs}\left(\frac{d}{dU}\right)^s F, \qquad S_{rs} = \frac{1}{s!}\left(\frac{d}{d\varepsilon}\right)^r\left\{U(u+\varepsilon) - U(u)\right\}^s\Big|_{\varepsilon=0}. \tag{2}$$

To verify this formula, it is easiest to start from the observation that the coefficients S_{rs} are independent of F and may therefore be calculated from *any* convenient choice for F. The simplest is $F = e^{\omega U}$, for which the Schlömilch form reads

$$(d/du)^r\, e^{\omega U} = \sum_0^r \omega^s S_{rs}\, e^{\omega U},$$

so

S_{rs} = coefficient of ω^s in

$$e^{-\omega U}\left(\frac{d}{du}\right)^r e^{\omega U} = e^{-\omega U}\left(\frac{d}{d\varepsilon}\right)^r e^{\omega U(u+\varepsilon)}\bigg|_{\varepsilon=0} = \left(\frac{d}{d\varepsilon}\right)^r e^{\omega[U(u+\varepsilon)-U(u)]}\bigg|_{\varepsilon=0}.$$

Unfortunately, only in a limited number of instances can a closed form for S_{rs} be found in terms of known functions. The most important are as follows:

$$U = u^{-1}, \qquad S_{rs} = (-1)^r r!\,(r-1)\,!/s\,!(s-1)\,!\,(r-s)\,!\,u^{r+s} \tag{3}$$

$$U = u^2, \qquad S_{rs} = r!/(r-s)\,!\,(2s-r)\,!\,(2u)^{r-2s} \tag{4}$$

$$U = \ln u, \qquad S_{rs} = S_r^{\,s}/u^r \tag{5}$$

$$U = e^u, \qquad S_{rs} = C_r^{\,s}\,e^{us}, \tag{6}$$

where $S_r^{\,s}$ and $C_r^{\,s}$ are respectively Stirling numbers of the first and second kinds (cf. Jordan 1960, Ch. 4).

Recurrence relation. If $F(u)$ can be expressed as a function of a power series $P(u)$ which has known coefficients, a recurrence relation between successive derivatives can often be established by forming a differential equation in which there occur only small integral powers of F, P and their derivatives. The resultant recurrence relation is especially simple if P is a polynomial of low degree.

For instance, logarithmic differentiation of $F = P^\alpha$ (where α may be negative or irrational) provides such a differential equation, $F'P = \alpha FP'$. Substituting $F = \Sigma f_r u^r$, $P = \Sigma p_s u^s$, and equating throughout the coefficient of each power of u,

$$(r+1)p_0 f_{r+1} + (r-\alpha)p_1 f_r + (r-2\alpha-1)p_2 f_{r-1} \\ + (r-3\alpha-2)p_3 f_{r-2} + \ldots = 0 \tag{7}$$

The required derivatives are then $F_r = r!f_r$. If derivatives are required at $u = u_0$ instead of at $u = 0$, the only difference is that the coefficients p_s must be defined by $P = \Sigma p_s(u-u_0)^s$.

As further examples, suppose next that $F = e^P$. Then equating coefficients throughout $F' = FP'$,

$$rf_r = p_1 f_{r-1} + 2p_2 f_{r-2} + 3p_3 f_{r-3} + \ \ldots. \tag{8}$$

Further, if $F = \ln P$, equation of coefficients throughout $F'P = P'$ leads to the recurrence relation

$$(r+1)p_0 f_{r+1} + rp_1 f_r + (r-1)p_2 f_{r-1} + \ldots = (r+1)p_{r+1}. \tag{9}$$

In constructing a differential equation of the required form, higher derivatives than the first may have to be introduced. For instance, if $F = \sin P$ or $\cos P$ the simplest such equation is

$$F''P' - F'P'' + F(P')^3 = 0, \tag{10}$$

while if $F = \sinh P$ or $\cosh P$ the simplest is

$$F''P' - F'P'' - F(P')^3 = 0. \tag{11}$$

Darboux method. Connoting a (high) derivative as a coefficient, the Darboux method discussed in VII §2 is applicable to functions in which the singularities closest to the origin are poles or branch points.† Relative to the Schlömilch formula—which gives the rth derivative as a finite sum of $r + 1$ terms—it has the drawback of expressing the result as an infinite series; but in compensation terms initially decrease rapidly when r is large, and in any case the method is more widely efficacious on account of the few Schlömilch S_{rs} known in closed form.

Method of Steepest Descents. Connoting a (high) derivative as a contour integral, the method of steepest descents detailed in VI is applicable to functions in which fast variation is due to the presence of an essential singularity.‡ The result is expressed as an asymptotic expansion in which terms initially decrease rapidly when r is large. As we shall see in XI, the idea can be developed to cover functions in which fast variation is due also to poles and branch points; but the results are complicated and should only be called upon if the Darboux method fails to cope through the dominance of an essential singularity.

2. Factorial Function

For $\Re(p) > -1$ this is defined by the integral representation

$$p! = \int_0^\infty t^p e^{-t}\, dt = \int_0^\infty e^{-F}\, dt, \qquad F(t) = t - p \ln t. \tag{12}$$

(The definition can be extended to cover other ranges of p through

$$p! = (2i \sin \pi p)^{-1} \int_\infty^{(0+)} (-t)^p e^{-t}\, dt,$$

which is analytic for all p apart from simple poles.)

† i.e. it is applicable to functions expansible as conditionally convergent series, those of basically hypergeometric type.

‡ i.e. it is applicable to functions expansible as absolutely convergent series, those of basically exponential type.

For brevity of exposition, let us suppose p to be real and positive. Then as t runs along the real axis from 0 to ∞, $F(t)$ starts at ∞, reaches a minimum value when $t = p$, and ends at ∞. The conditions laid down in VI §4 for quadratic dependence at a stationary point are all fulfilled, with

$$p! = \int_{\rightarrow} e^{-F}\,dt, \tag{13}$$

$$F_0 = p - p\ln p, \quad F_1 = 0, \quad F_{j\geqslant 2} = (-1)^j(j-1)!/p^{j-1}. \tag{14}$$

From VI (2) the asymptotic expansion is

$$p! = (2\pi p)^{\frac{1}{2}}\, p^p\, e^{-p}\left(1 + \frac{1}{12p} + \frac{1}{288p^2} - \frac{139}{51{,}840p^3} - \frac{571}{2{,}488{,}320p^4} \cdots\right). \tag{15}†$$

Late terms. Let

$$\mathscr{F} = F - F_0 = t - p\ln t - p + p\ln p. \tag{16}$$

The stationary point $t = p$, $\mathscr{F} = 0$ relevant to the derivation of (15) corresponds to registering the principal value of the logarithm of t. There is no second distinct stationary point, so the chief singulant pair must correspond to the values of $\mathscr{F}$ at the nearest non-principal readings $\ln t \pm 2\pi i$, i.e. to $\mathscr{F}_0 = \mp 2\pi i p$. Since the derivatives are identical with those at the principal stationary point, in VII (12) $\mathscr{Q}_{2s} = Q_2$ and the chief contributions to late terms Q_{2r} are

$$-\frac{1}{\mp 2\pi i}\frac{(r-1)!}{(\mp 2\pi i p)^r}\left\{1 + \frac{\mp 2\pi i}{12(r-1)} + \frac{(\mp 2\pi i)^2}{288(r-1)(r-2)} - \frac{139\,(\mp 2\pi i)^3}{51{,}840(r-1)(r-2)(r-3)} \cdots\right\}. \tag{17}$$

Thus far is plain sailing; but, short of recourse to later interpretative theory or alternative evidence (e.g. numerical), assessment of the multiplying factors to be associated with the subsidiary singulant pairs $\mathscr{F}_0 = \mp 2\pi i p m$ ($m = 2, 3, \ldots$) is speculative. As we shall shortly confirm, the right generalization is the intuitively expected one in which m is attached to *every* $\mp 2\pi i$

† To avoid reprinting long expressions in different sections of the book, only those few terms needed for expository purposes within a section will be retained. Consequently, the fullest version of a series will nearly always appear in the final chapters dealing with the evaluation of asymptotic expansions.

in (17), corresponding to subsidiary singulant pairs contributing with diminishing force m^{-1}. Summing all contributions on this assumption,

$$Q_{2r} = \begin{cases} \dfrac{(-1)^{\frac{1}{2}(r-1)}(r-1)!}{\pi(2\pi p)^r}\left\{\zeta(r+1) + \dfrac{\pi^2\zeta(r-1)}{72(r-1)(r-2)}\right. \\ \qquad \left. - \dfrac{571\pi^4\zeta(r-3)}{155{,}520(r-1)(r-2)(r-3)(r-4)}\cdots\right\}, & r \text{ odd}, \\ \dfrac{(-1)^{\frac{1}{2}r-1}(r-2)!}{6(2\pi p)^r}\left\{\zeta(r) + \dfrac{139\pi^2\zeta(r-2)}{1080(r-2)(r-3)}\right. \\ \qquad \left. + \dfrac{163{,}879\pi^4\zeta(r-4)}{1{,}088{,}740(r-2)(r-3)(r-4)(r-5)}\cdots\right\}, & r \text{ even}. \end{cases} \tag{18}$$

Direct confirmation is available from the exact calculations of all terms in the asymptotic expansion of $\ln p!$ carried through in II §7 and IV §6. According to these, the asymptotic power series involved in $p!$ must be

$$\begin{aligned}\sum_0^\infty Q_{2r} &\equiv \exp\left[\frac{1}{\pi}\sum_{1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{1}{2}(s-1)}(s-1)!\,\zeta(s+1)}{(2\pi p)^s}\right] \\ &= \exp\left(\frac{1}{12p} - \frac{1}{360p^3}\cdots\right)\exp\left\{\cdots\frac{(-1)^{\frac{1}{2}(s-1)}(s-1)!\,\zeta(s+1)}{\pi(2\pi p)^s}\right. \\ &\qquad \left. + \frac{(-1)^{\frac{1}{2}(s-3)}(s-3)!\,\zeta(s-1)}{\pi(2\pi p)^{s-2}}\cdots\right\} \\ &= \frac{(-1)^{\frac{1}{2}(s-1)}(s-1)!}{\pi(2\pi p)^s}\left\{1 + \frac{1}{12p} + \frac{1}{288p^2} - \frac{139}{51{,}840p^3}\cdots\right\} \\ &\qquad \times\left\{\cdots + \zeta(s+1) - \frac{4\pi^2p^2\,\zeta(s-1)}{(s-1)(s-2)} + \cdots\right\}.\end{aligned}$$

Extracting the set of terms of order p^{-r}, which is by definition Q_{2r}, reproduces (18).

To indicate the accuracy obtained from these asymptotic formulae (18) for late terms before introducing sophistications of interpretative theory or numerical analysis, we remark that if the first is crudely broken off at its second entry it gives the next contribution to (15) as $Q_{10} = {\cdot}00078394/p^5$, whereas the exact value is ${\cdot}000784039..{}/p^5$. The fractional error in a similar calculation from the formula for even r would admittedly be much bigger on account of the sizeable second coefficient, but the adverse effect is largely offset by the reduced magnitude of Q's for even r.

3. Incomplete Factorial Function

The problem here is to find an asymptotic expansion for

$$(p, x)! = \int_0^x t^p e^{-t}\, dt = \int_0^x e^{-F}\, dt, \qquad F(t) = t - p \ln t, \tag{19}$$

when both p and x are large, our earlier series II (29) or IV (14) being adequate only when $|x| >> |p|$. The mode of treatment will clearly depend on whether or not the stationary point at $t = p$ considered in the preceding section lies within the range of integration 0 to x. Unless otherwise stated, x and p will for brevity be regarded as real and positive in the following argument.

$x < p$.

When $x < p$, $F(t)$ increases roughly linearly as t moves down from the upper limit $t = x$, and ends at ∞ at the lower limit of integration $t = 0$. The problem is therefore one of linear dependence at a limit of integration, with the conditions laid down in V §2 all fulfilled; and V (12) is immediately applicable apart from a change in outer sign since the integration is from the upper limit downwards. With

$$F_0 = x - p \ln x, \qquad F_1 = 1 - p/x, \qquad F_{j \geqslant 2} = (-1)^j (j-1)!\, p/x^j, \tag{20}$$

the resultant expansion is

$$\begin{aligned}(p, x)! = \frac{x^{p+1} e^{-x}}{p - x} \Biggl\{ 1 &- \frac{p}{(p-x)^2} + \frac{p(p+2x)}{(p-x)^4} - \frac{p(p^2 + 8px + 6x^2)}{(p-x)^6} \\ &+ \frac{p(p^3 + 22p^2x + 58px^2 + 24x^3)}{(p-x)^8} \\ &- \frac{p(p^4 + 52p^3x + 328p^2x^2 + 444px^3 + 120x^4)}{(p-x)^{10}} \\ &+ \frac{p(p^5 + 114p^4x + 1452p^3x^2 + 4400p^2x^3 + 3708px^4 + 720x^5)}{(p-x)^{12}} \\ &- \frac{p(p^6 + 240p^5x + 5610p^4x^2 + 32{,}120p^3x^3 + 58{,}140p^2x^4 + 33{,}984px^5 + 5040x^6)}{(p-x)^{14}} \cdots \Biggr\}. \end{aligned} \tag{21}$$

Late terms. Here the singulant is the change in value of F in going from the limit of integration, where $F = x - p \ln x$, to the stationary point, where $F = p - p \ln p$, so

$$\mathscr{F}_0 = -p\{\ln(p/x) + x/p - 1\}. \tag{22}$$

Since $(p, x)!$ satisfies a homogeneous second-order differential equation, this is the sole singulant (VII §6); non-principal readings of the logarithm in (22) do not play any role in late terms of the asymptotic expansion.

As we saw in §2, the derivatives at this stationary point determined the asymptotic expansion of the factorial function $p!$. Consequently there is an intimate connection between the coefficients in (15) and those in the asymptotic formula for late terms in the expansion (21) for $(p, x)!$. Explicitly,

$$L_r = \frac{(-1)^r(r-\frac{1}{2})!\,(p/x-1)}{(2\pi\Xi)^{\frac{1}{2}}(p\Xi)^r}\left\{1 - \frac{\Xi}{12(r-\frac{1}{2})}\right.$$
$$\left. + \frac{\Xi^2}{288(r-\frac{1}{2})(r-\frac{3}{2})} + \frac{139\,\Xi^3}{51{,}840(r-\frac{1}{2})(r-\frac{3}{2})(r-\frac{5}{2})}\cdots\right\}, \tag{23}$$

where

$$\Xi = \ln(p/x) + x/p - 1.$$

The accuracy of (23) is extremely high, even before refinement by interpretative theory or numerical analysis. For instance when $x = \frac{1}{2}p$ the first three terms beget $L_7 = -1{\cdot}68974 \times 10^8/p^7$ compared to the exact value $-1{\cdot}68968\ldots \times 10^8/p^7$.

$x > p$.

When $x > p$ the integral involves two critical points, namely the limit $t = x$ where the integrand is still contributing, and the stationary point $t = p$. The obligatory separation into independent integrals containing one critical point each (VI §2) is easily accomplished by writing

$$(p, x)! = p! - [p, x]!, \tag{24}$$

$$[p, x]! = \int_x^\infty t^p e^{-t}\,dt = \int_x^\infty e^{-F}\,dt, \qquad F(t) = t - p \ln t. \tag{25}$$

$F(t)$ increases roughly linearly as t increases from its lower limit, and ends at ∞ at the upper limit. The problem is again one of linear dependence at a limit and the derivatives are the same as in (20). The only difference in the result is one of outer sign since we are now integrating from the lower limit upwards. Hence [cf. (21)]

$$[p, x]! = \frac{x^{p+1}e^{-x}}{x-p}\left\{1 - \frac{p}{(x-p)^2} + \dots\right\}. \tag{26}$$

This is still valid when p is negative; the notation of the "general exponential integral"

$$Ei_\rho(x) = \int_1^\infty u^{-\rho}\, e^{-xu}\, du = x^{\rho-1}\,[-\rho, x]! \tag{27}$$

is then commoner.

Late terms. It is easily shown that the asymptotic formula (23) for late terms is applicable to (26) so long as p is positive. The accuracy is superlative in this range; when $x = 2p$ the formula gives $L_6 = 1{\cdot}236050031 \times 10^5/p^6$ compared to the exact value $1{\cdot}236050000 \times 10^5/p^6$.

When p is negative (23) has to be revised, because the previous sole singulant is replaced by the singulant pair

$$\mathscr{F}_0 = p\{1 - x/p - \ln(-p/x) \pm i\pi\}. \tag{28}$$

Combining their contributions, the revised formula is

$$L_r = -\frac{(r-\tfrac{1}{2})!\,(1-p/x)}{(\tfrac{1}{2}\pi|\Xi|)^{\frac{1}{2}}(-p|\Xi|)^r}\left\{\sin(r+\tfrac{1}{2})\theta + \frac{|\Xi|}{12(r-\tfrac{1}{2})}\sin(r-\tfrac{1}{2})\theta\right.$$
$$+ \frac{|\Xi|^2}{288(r-\tfrac{1}{2})(r-\tfrac{3}{2})}\sin(r-\tfrac{3}{2})\theta$$
$$\left. - \frac{139\,|\Xi|^3}{51{,}840(r-\tfrac{1}{2})(r-\tfrac{3}{2})(r-\tfrac{5}{2})}\sin(r-\tfrac{5}{2})\theta \dots\right\} \tag{29}$$

where

$$|\Xi| = [\{1 - x/p - \ln(-p/x)\}^2 + \pi^2]^{\frac{1}{2}}, \qquad \text{positive root,}$$
$$\theta = \tan^{-1}[\pi/\{1 - x/p - \ln(-p/x)\}].$$

To indicate the accuracy, when $p = -x$ this predicts $L_6 = -0{\cdot}01779/(-p)^6$ compared to the exact value $-0{\cdot}017822../(-p)^6$.

$x \sim p$.

The asymptotic expansions (21) and (26) fail as $x \to p$, because the linear term in $F(u)$ at the limit of integration—hitherto supposed dominant—then vanishes. A simple solution covering a wide range $x \sim p$ is obtained by expanding $(p, x)!$ as a Taylor series in $(x - p)$. Applying the Leibnitz differentiation formula to the integral representation (19),

$$(p,x)! = (p,p)! + p^{p+1}e^{-p}p! \sum_{r=0}^{\infty} \frac{X^{r+1}}{r+1} \sum_{s=0}^{r} \frac{(-p)^s}{s!\,(r-s)!\,(p-r+s)!} \tag{30†}$$

$$= (p,p)! + p^{p+1}e^{-p}X\,\{1 - pX^2/6 + pX^3/12 + p(p-2)\,X^4/40\ldots\} \tag{31}$$

where $X = (x - p)/p$. When r is large, the sum over s in (30) can be approximated by

$$\frac{1}{r!\,(p-r)!}\left\{1 + \frac{p}{1!}\left(1 + \frac{p+1}{r}\cdots\right) + \frac{p^2}{2!}\left(1 + \frac{(p+1)+(p+2)-1}{r}\cdots\right) + \cdots\right\}$$

$$= \frac{e^p}{r!\,(p-r)!}\left\{1 + \frac{p(p+1)}{r}\ldots\right\},$$

and then the sum over r is essentially the binomial expansion of $(1 + X)^{p+1}$. Hence the power series in $(x - p)$ is convergent over the wide range $|X| < 1$.

To find the asymptotic expansion of $(p, p)!$, we note that when $x = p$ (20) reduces to

$$F_0 = p - p\ln p, \qquad F_1 = 0, \qquad F_{j\geqslant 2} = (-1)^j(j-1)!/p^{j-1}, \tag{32}$$

so the problem is one of quadratic dependence at a limit of integration. The conditions laid down in V §3 are all fulfilled, and V(20) is immediately applicable apart from a change in outer sign since the integration is from the upper limit downwards. The result is

$$(p,p)! = (\tfrac{1}{2}\pi p)^{\frac{1}{2}}\, p^p\, e^{-p}$$
$$\times \left\{1 - \frac{2\sqrt{2}}{3\sqrt{\pi}\,p^{\frac{1}{2}}} + \frac{1}{12p} + \frac{4\sqrt{2}}{135\sqrt{\pi}\,p^{\frac{3}{2}}} + \frac{1}{288p^2} - \frac{8\sqrt{2}}{2835\sqrt{\pi}\,p^{\frac{5}{2}}} - \frac{139}{51{,}840p^3} - \frac{16\sqrt{2}}{8505\sqrt{\pi}\,p^{\frac{7}{2}}} - \frac{571}{2{,}488{,}320p^4}\cdots\right\}. \tag{33}$$

The analogous series for $[p, p]!$, found either directly by applying V (20) to $\int_p^\infty t^p e^{-t}\,dt$ or by combining (15), (24) and (33), differs only in reversed signs of Q_{odd}.

† $p!\sum_{s=0}^{r}(-p)^s/s!(r-s)!(p-r+s)! = L_r^{(p-r)}(p)$, an associated Laguerre polynomial.

Late terms. The calculation is an extension of that in §2 relating to late terms in $p!$. The asymptotic formulae for Q_{2r} given in (18) still apply, but these have now to be supplemented by

$$Q_{2r+1} = \begin{cases} \dfrac{(-1)^{\frac{1}{2}r}(r-\frac{1}{2})!}{\pi\sqrt{2}\,(2\pi p)^{r+\frac{1}{2}}}\left\{\zeta(r+\tfrac{3}{2}) - \dfrac{\pi\zeta(r+\frac{1}{2})}{6(r-\frac{1}{2})} \right. \\ \qquad\qquad \left. - \dfrac{\pi^2\,\zeta(r-\frac{1}{2})}{72(r-\frac{1}{2})(r-\frac{3}{2})}\cdots\right\}, & r \text{ even}, \\ \dfrac{(-1)^{\frac{1}{2}(r-1)}(r-\frac{1}{2})!}{\pi\sqrt{2}\,(2\pi p)^{r+\frac{1}{2}}}\left\{\zeta(r+\tfrac{3}{2}) + \dfrac{\pi\zeta(r+\frac{1}{2})}{6(r-\frac{1}{2})} \right. \\ \qquad\qquad \left. - \dfrac{\pi^2\,\zeta(r-\frac{1}{2})}{72(r-\frac{1}{2})(r-\frac{3}{2})}\cdots\right\}, & r \text{ odd}. \end{cases} \tag{34}$$

As noted more generally above, these are reversed in sign in $[p, p]!$.

4. Fermi–Dirac Integral

Our earlier expansion for $\mathscr{F}_p(x)$, II (68) or IV (39), is adequate only when $|x| \gg |p|$. One possible way of proceeding when p is large would be to start from IV (38) cast in the form

$$\mathscr{F}_{p-1}(x) = \frac{x^p}{p!}\int_0^\infty \frac{e^{-v}}{(1+e^{-v})^2}\left\{\left(1+\frac{v}{x}\right)^p + \left(1-\frac{v}{x}\right)^p\right\} dv$$
$$- \begin{pmatrix} e^{i\pi p} \\ e^{-i\pi p} \\ \cos \pi p \end{pmatrix} \mathscr{F}_{p-1}(-x), \quad \left.\begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 \end{matrix}\right\}, \tag{35}$$

and by expanding $(1+e^{-v})^{-2}$ express the integral as an infinite series of incomplete factorial functions $[p, \pm rx]!, r = 1, 2, 3, \ldots$ However, except where $|x| \gg |p|$ – the range already well covered—the parameters in one or other of these might happen to lie in a borderline region where no series excels.

Thus when p is large it is better to make a direct expansion about the stationary point in the integrand of

$$\mathscr{F}_{p-1}(x) = \frac{1}{p!}\int_0^\infty \frac{t^p e^{x-t}\,dt}{(e^{x-t}+1)^2}, \tag{36}$$

notwithstanding the inconvenience of having to determine its location from a transcendental equation. (Similar situations arise in the evaluation of Stirling numbers of first and second kinds, cf. Q.4, Q.5). Let (36) be rewritten

$$p!\,\mathscr{F}_{p-1}(x) = \int_0^\infty e^{-F}\,dt,$$

$$F(t) = t - x + 2\ln(e^{x-t} + 1) - p\ln t, \tag{37}$$

and for brevity of exposition suppose p to be real and positive. Then as t runs along the real axis from 0 to ∞, $F(t)$ starts at ∞, reaches a minimum value at $t = t_0$ where

$$0 = F'(t_0) = 1 - \frac{2}{e^{t_0 - x} + 1} - \frac{p}{t_0}, \tag{38}$$

and ends at ∞. The conditions laid down in VI §4 for quadratic dependence at a stationary point are all fulfilled, and the main task is to organize the algebra amenably for calculation.

In terms of the parameter $\tau = p/t_0$, (38) can be written

$$\tau = \frac{p}{x + 2\tanh^{-1}\tau}, \tag{39}$$

a form admirably adapted to iterative solution on two counts: the inverse hyperbolic tangent is slowly varying except when τ is close to unity, and it has been extensively tabulated (Harvard 1949). Schlömilch's formula (2) + (6) develops the general derivative as the series

$$F_r = \tfrac{1}{2}(1 - \tau^2)\sum_0^{r-2}(-2)^{-s}(s+1)!\,C_{r-1}^{s+1}(1+\tau)^s$$
$$+ (r-1)!\,p\,(-\tau/p)^r. \tag{40}$$

The first few are as follows:

$$F_2 = \tfrac{1}{2}\{1 - \tau^2(1 - 2/p)\},$$
$$F_3 = -\tfrac{1}{2}\tau\{1 - \tau^2(1 - 4/p^2)\},$$
$$F_4 = -\tfrac{1}{4}\{1 - 4\tau^2 + 3\tau^4(1 - 8/p^3)\},$$
$$F_5 = \tfrac{1}{2}\tau\{2 - 5\tau^2 + 3\tau^4(1 - 16/p^4)\},$$
$$F_6 = \tfrac{1}{4}\{2 - 17\tau^2 + 30\tau^4 - 15\tau^6(1 - 32/p^5)\},$$
$$F_7 = -\tfrac{1}{4}\tau\{17 - 77\tau^2 + 105\tau^4 - 45\tau^6(1 - 64/p^6)\},$$

$$F_8 = -\tfrac{1}{8}\{17 - 248\tau^2 + 756\tau^4 - 840\tau^6 + 315\tau^8(1 - 128/p^7)\},$$
$$F_9 = \tfrac{1}{2}\tau\{62 - 440\tau^2 + 1008\tau^4 - 945\tau^6 + 315\tau^8(1 - 256/p^8)\},$$
$$F_{10} = \tfrac{1}{4}\{62 - 1{,}382\tau^2 + 6{,}360\tau^4 - 11{,}655\tau^6 + 9{,}450\tau^8 - 2{,}835\tau^{10} \times (1 - 512/p^9)\}.$$

No dramatic simplification results when these polynomials in τ are inserted in

$$\mathscr{F}_{p-1}(x) = \frac{1}{p!}\int_{\rightarrow} e^{-F}\,dt = \frac{(1-\tau^2)(p/\tau)^p}{2\,p!}\left\{\frac{\pi}{1-\tau^2(1-2/p)}\right\}^{\frac{1}{2}} \sum_0^\infty Q_{2r}, \tag{41}$$

so the derivatives might just as well be calculated numerically as required and the numbers introduced into the Q's cited in V §3.

Late terms. The theory turns out to require only a minor generalization of that detailed in §2 in relation to $p!$. In the present problem $\mathscr{F} = F - F_0$ assumes the fearsome appearance

$$\mathscr{F} = t - t_0 - p\ln(t/t_0) + 2\ln\left[(e^{x-t}+1)/(e^{x-t_0}+1)\right], \tag{42}$$

but on switching to non-principal readings by writing $t = t_0\,e^{\pm 2\pi i m}$ the singulants are seen to be simply $\mathscr{F}_0 = \mp 2\pi i p m\ (m = 1, 2, 3, ..)$ as before. Derivatives are identical with those at the principal reading, so again $\mathscr{Q}_{2s} = Q_{2s}$. On summing contributions from singulant pairs with diminishing factors m^{-1} by analogy with §2,

$$Q_{2r} = \begin{cases} \dfrac{(-1)^{\frac{1}{2}(r-1)}(r-1)!}{\pi\,(2\pi p)^r}\left\{\zeta(r+1) - \dfrac{(2\pi p)^2\,\zeta(r-1)\,Q_4}{(r-1)(r-2)}\cdots\right\}, & r \text{ odd}. \\[2ex] \dfrac{(-1)^{\frac{1}{2}r-1}(r-2)!}{\pi(2\pi p)^{r-1}}\left\{\zeta(r)Q_2 - \dfrac{(2\pi p)^2\,\zeta(r-2)\,Q_6}{(r-2)(r-3)}\cdots\right\}, & r \text{ even}. \end{cases} \tag{43}$$

5. Bessel Function $J_p(x)$

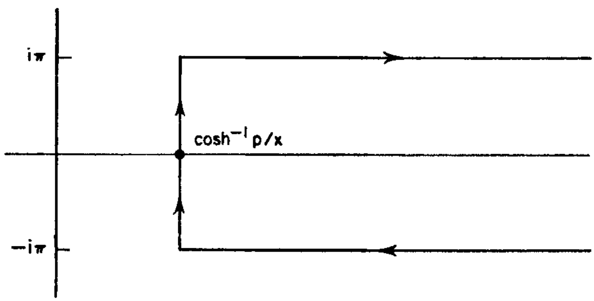
The most convenient starting point is the integral representation

$$J_p(x) = -\frac{i}{2\pi}\int_{\infty - i\pi}^{\infty + i\pi} e^{-F}\,d\omega, \qquad F(\omega) = p\omega - x\sinh\omega, \qquad |\operatorname{ph} x| < \tfrac{1}{2}\pi, \tag{44}$$

derived in III Q.12. Since sinh $(\omega \pm i\pi) = -\sinh \omega$, $F(\omega) = \infty$ at both limits in accord with the suppositions of VI, but whether the path is to pass through one stationary point or two depends on the range.

$p > x > 0$.

The two stationary points of $F(\omega)$ are at $\omega = \pm \cosh^{-1}p/x$, where $F_2 = \mp (p^2 - x^2)^{\frac{1}{2}}$ respectively. The negative F_2 corresponds to a minimum along a path parallel to the imaginary axis, as needed to change the imaginary part of ω from $-i\pi$ to $i\pi$; while the positive F_2 would correspond to a minimum along a path following the negative real axis, which would require a deformation out to $\omega = -\infty$ incompatible with the specification (44) unless accompanied by a cancelling return.



Hence when $p > x$ only the stationary point at $\omega = \cosh^{-1}p/x$ contributes, and in the notation of VI §4

$$J_p(x) = -\frac{i}{2\pi}\int_{\uparrow} e^{-F}\, d\omega \tag{45}$$

with

$$F_0 = p\cosh^{-1}p/x - (p^2 - x^2)^{\frac{1}{2}}, \qquad F_1 = 0,$$

$$F_{2j\geqslant 2} = -(p^2 - x^2)^{\frac{1}{2}}, \qquad F_{2j+1\geqslant 3} = -p. \tag{46}$$

It is advantageous to express these derivatives in terms of the new parameter

$$q = p/(p^2 - x^2)^{\frac{1}{2}}, \tag{47}$$

since this removes the surds and thereby reduces the expansion terms to polynomials. The result from VI (3) is

$$J_p(x) = \left(\frac{q}{2\pi p}\right)^{\frac{1}{2}} e^{-p(\tanh^{-1}q^{-1}-q^{-1})} \left\{1 - \frac{q}{24p}(5q^2-3) + \frac{q^2}{1152p^2}\right.$$

$$\times (385q^4 - 462q^2 + 81) - \frac{q^3}{414{,}720p^3}(425{,}425q^6 - 765{,}765q^4$$

$$+ 369{,}603q^2 - 30{,}375) + \frac{q^4}{39{,}813{,}120p^4}(185{,}910{,}725q^8$$

$$-446{,}185{,}740q^6 + 349{,}922{,}430q^4 - 94{,}121{,}676q^2$$

$$\left. + 4{,}465{,}125)\dots\right\}. \tag{48}$$

Late terms. Here the singulant is the change in value of F in going across to the other stationary point, i.e.

$$\mathscr{F}_0 = -2p(\tanh^{-1}q^{-1} - q^{-1}), \qquad \mathscr{F}_{2j\geqslant 2} = p/q, \qquad \mathscr{F}_{2j+1\geqslant 3} = -p. \tag{49}$$

Since $J_p(x)$ satisfies a homogeneous second-order differential equation, this is the sole singulant (VII §6); non-principal readings of $\tanh^{-1}$ do not play any role in late terms of the asymptotic expansion.

The derivatives at the two stationary points differ only by reversal in sign of those of even order. The expected consequential connection between late and early terms is confirmed by VII (12), which gives

$$Q_{2r} = \frac{(r-1)!}{2\pi(-2p\Xi)^r}\left\{1 - \frac{\Xi q(5q^2-3)}{12(r-1)} + \frac{\Xi^2 q^2(385q^4-462q^2+81)}{288(r-1)(r-2)}\dots\right\} \tag{50}$$

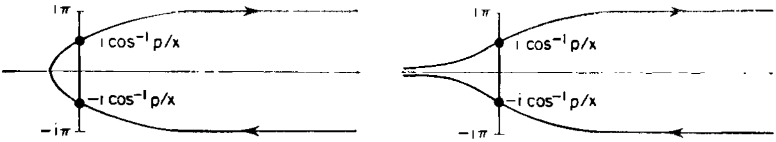
where

$$\Xi = \tanh^{-1}q^{-1} - q^{-1}.$$

To indicate the accuracy, when $q = 2$ this predicts $Q_8 = \cdot 97528 \times 10^4/p^4$ (breaking off the alternating series and retaining half the last term), compared to the exact value $\cdot 97512.. \times 10^4/p^4$.

$x > p > 0.$

To clarify the exposition for this range, we rewrite the locations of the two stationary points as $\omega^{\pm} = \pm i\cos^{-1}p/x$, where $F_2 = \mp i(x^2 - p^2)^{\frac{1}{2}}$ respectively. These correspond to minima along paths at phase angles $\pm 45°$ respectively, and passage through both stationary points will be needed to change the imaginary part of ω from $-i\pi$ to $+i\pi$ as required by (44).



The obligatory separation into integrals containing one stationary point each (VI §2) is easily achieved by diverting the path between the points out to $\omega \to -\infty$, $F(\omega) \to \infty$. The conditions laid down in VI §4 are then fulfilled for each of the integrals in

$$J_p(x) = -\frac{i}{2\pi}\left\{\int_{-\infty}^{\infty + i\pi} e^{-F}\, d\omega + \int_{\infty - i\pi}^{-\infty} e^{-F}\, d\omega\right\}, \tag{51}$$

so there are now contributions $(-i/2\pi)\int_\uparrow$ from both stationary points†, as compared with the lone contribution calculated in (45)–(48). Accordingly, we replace q in (48) by $\pm i\mathfrak{q}$, where

$$\mathfrak{q} = p/(x^2 - p^2)^{\frac{1}{2}}, \tag{52}$$

and add the pair of expansions. Since $\tanh^{-1}(i\mathfrak{q})^{-1} = -i\tan^{-1}\mathfrak{q}^{-1}$ is the correct analytical continuation through‡ $|q^{-1}| \to 0$, this procedure gives

$$\begin{aligned} J_p(x) = \left(\frac{2\mathfrak{q}}{\pi p}\right)^{\frac{1}{2}} \Bigg[\Bigg\{ 1 &- \frac{\mathfrak{q}^2}{1152p^2}(385\mathfrak{q}^4 + 462\mathfrak{q}^2 + 81) + \frac{\mathfrak{q}^4}{39{,}813{,}120p^4} \\ &\times (185{,}910{,}725\mathfrak{q}^8 + 446{,}185{,}740\mathfrak{q}^6 + 349{,}922{,}430\mathfrak{q}^4 \\ &+ 94{,}121{,}676\mathfrak{q}^2 + 4{,}465{,}125)\ldots\Bigg\} \sin(p\Upsilon + \tfrac{1}{4}\pi) - \Bigg\{\frac{\mathfrak{q}}{24p}(5\mathfrak{q}^2 + 3) \\ &- \frac{\mathfrak{q}^3}{414{,}720p^3}(425{,}425\mathfrak{q}^6 + 765{,}765\mathfrak{q}^4 + 369{,}603\mathfrak{q}^2 + 30{,}375)\ldots\Bigg\} \\ &\times \cos(p\Upsilon + \tfrac{1}{4}\pi)\Bigg] \end{aligned} \tag{53}$$

† Or equivalently $(-i/2\pi)\{\int_\to(\omega^+) - \int_\to(\omega^-)\}$, cf. VI (4), (5).

‡ $\tanh^{-1}u = u + \frac{1}{3}u^3 + \ldots$, $\tan^{-1}v = v - \frac{1}{3}v^3 + \ldots$, for small u and v.

where

$$\Upsilon = \varphi^{-1} - \tan^{-1}\varphi^{-1}.$$

Late terms. On making the requisite notational changes in (50), the asymptotic formula for late terms becomes

$$Q_{2r} = \begin{pmatrix} (-1)^{\frac{1}{2}r} \\ (-1)^{\frac{1}{2}(r-1)} \end{pmatrix} \frac{(r-1)!}{2\pi(2p\Upsilon)^r}\left[1 - \frac{\Upsilon\varphi(5\varphi^2+3)}{12(r-1)}\right.$$

$$\left. + \frac{\Upsilon^2\varphi^2(385\varphi^4 + 462\varphi^2 + 81)}{288(r-1)(r-2)} \cdots\right] \quad \left.\begin{matrix} r\text{ odd.} \\ r\text{ even.}\end{matrix}\right\}. \tag{54}$$

$x \sim p$

The asymptotic expansions (48) and (53) fail as $x \to p$ $(q \to \infty)$, because the quadratic term in $F(u)$ at the stationary points—hitherto supposed dominant—then vanishes. A solution for $x \sim p$ is obtained by expanding $J_p(x)$ as a Taylor series in $x - p$, in which all derivatives can be expressed in terms of J and $J^{(1)}$ since the function satisfies a second-order differential equation. Applying the Leibnitz differentiation formula to the right-hand side of

$$J_p''(x) = (p^2/x^2 - 1)\, J_p(x) - x^{-1}\, J_p'(x), \tag{55}$$

the reduction relation for higher derivatives at $x = p$ is found to be

$$J_p^{(v+2)} = -\frac{1}{p} J_p^{(v+1)} + \frac{v}{p^2} J_p^{(v)} + v!\sum_1^v \frac{(-1)^s\{p^2(s+1) + v - s\}}{p^{s+2}\,(v-s)!} J_p^{(v-s)}. \tag{56}$$

This reduces the Taylor series to

$$\begin{aligned} J_p(x) = J_p\,&\{1 - p^2X^3/3 + p^2X^4/3 - 19p^2X^5/60\ldots\} \\ &+ J_p^{(1)}\,pX\{1 - X/2 + X^2/3 \\ &- (2p^2 + 3)X^3/12 + (7p^2 + 6)X^4/30\ldots\}, \end{aligned} \tag{57}$$

where $X = (x - p)/p$.

On estimating $J_p^{(v)}$ for large v and p (Q.9), an approximation covering dominant late terms in these series is found to be

$$J_p(x) \sim J_p \sum \frac{(s - 2/3)!}{(-2/3)!\,(3s)!}\left[-\frac{6(x-p)^3}{p}\right]^s$$

$$+ J_p^{(1)}\,(x - p)\sum \frac{(s - 1/3)!}{(-1/3)!\,(3s+1)!}\left[\frac{-6(x-p)^3}{p}\right]^s. \tag{58}$$

Thus the series in (57) are absolutely convergent.

Asymptotic expansion of $J_p(p)$. When $x = p$ the problem is one of cubic dependence at the sole stationary point $\omega = \cosh^{-1} 1 = 0$, where (46) reduces to

$$F_0 = F_1 = 0, \qquad F_{2j} \to -0, \qquad F_{2j+1 \geq 3} = -p. \tag{59}$$

From the contour equivalence VI (10), (45) becomes—omitting zero C's—

$$J_p = \frac{i}{2\pi}\int_< e^{-F}\,d\omega = \frac{\alpha\sqrt{3}}{2\pi p^{1/3}}(C_0 - C_4 + C_6 - C_{10}\ldots)$$

$$= \frac{(-2/3)!\,6^{1/3}}{2\pi\sqrt{3}\,p^{1/3}}\left(1 - \frac{3\beta}{35p^{4/3}} - \frac{1}{225p^2}\cdots\right), \tag{60}$$

where $\beta = 0{\cdot}15308275\ldots.$

Late terms in $J_p(p)$. When $x = p$ there is no longer a second distinct stationary point, so the chief singulant pair must correspond to switching in

$$\mathscr{F} = F - F_0 = p(\omega - \sinh\omega), \qquad \mathscr{F}' = 0 = p(1 - \cosh\omega), \tag{61}$$

from the principal solution $\omega = 0$ to the nearest non-principal readings $\omega = \pm 2\pi i$, $\mathscr{F}_0 = \pm 2\pi i p$. Since the derivatives are identical with those at the principal point, in VII (20) $\mathscr{C}_s = C_s$ and the chief contributions to late terms C_r are

$$\frac{1}{2\sqrt{3}\,\pi\,(\pm 2\pi i p)^{\frac{1}{3}r}}\left\{(\tfrac{1}{3}r - 1)! - \frac{3\beta(\pm 2\pi i p)^{4/3}}{35p^{4/3}}(\tfrac{1}{3}r - \tfrac{7}{3})! - \frac{(\pm 2\pi i p)^2}{225p^2}(\tfrac{1}{3}r - 3)!\ldots\right\}.$$

If these are the sole contributions, $C_r = 0$ when r is odd, while for even r

$$C_r = \frac{(-1)^{\frac{1}{2}r}}{\sqrt{3}\,\pi\,(2\pi p)^{\frac{1}{3}r}}\left\{(\tfrac{1}{3}r - 1)! - \frac{3\beta(2\pi)^{4/3}}{35}(\tfrac{1}{3}r - \tfrac{7}{3})! + \frac{4\pi^2}{225}(\tfrac{1}{3}r - 3)!\ldots\right\}$$

$$= \frac{.1837763(-1)^{\frac{1}{2}r}}{(2\pi p)^{\frac{1}{3}r}}\left\{(\tfrac{1}{3}r - 1)! - 0{\cdot}1521315(\tfrac{1}{3}r - \tfrac{7}{3})! + 0{\cdot}1754596(\tfrac{1}{3}r - 3)!\ldots\right\}. \tag{62}$$

Numerical evidence shows conclusively that subsidiary singulants do not contribute. For the exact value of $-p^{10/3}\,C_{10}$ is known to be $1213\beta/170{,}625 = 0{\cdot}00108829$; retaining half the last quoted entry in the alternating series (62) gives the close approximation ·001086, an agreement utterly destroyed on adding in any plausible contributions from subsidiary singulants; addition with equal force and like signs leads to the value ·00123, with equal force but alternating signs ·00100, with force m^{-1} and like signs ·00115, and with force m^{-1} and alternating signs ·00104.

Asymptotic expansion of $J_p^{(1)}(p)$. Differentiating (44) with respect to x and then placing $x = p$,

$$J_p^{(1)} = -\frac{i}{2\pi}\int_{\infty - i\pi}^{\infty + i\pi} e^{-F}\, G\, d\omega, \qquad F(\omega) = p(\omega - \sinh\omega),$$

$$G(\omega) = \sinh\omega, \tag{63}$$

so the derivatives in (59) have now to be supplemented by

$$G_{2j} = 0, \qquad G_{2j+1} = 1. \tag{64}$$

Omitting zero $\bar{C}$'s, the result is

$$J_p^{(1)} = \frac{i}{2\pi}\int_{<} e^{-F}\, G\, d\omega = \frac{\alpha\sqrt{3}}{2\pi p^{1/3}}(-\bar{C}_1 + \bar{C}_3 - \bar{C}_7 + \bar{C}_9 \dots)$$

$$= \frac{(-2/3)!\, 6^{1/3}}{2\pi\sqrt{3}\, p^{1/3}}\left(\frac{6\beta}{p^{1/3}} - \frac{1}{5p} + \frac{23\beta}{525 p^{7/3}} \cdots\right). \tag{65}$$

Late terms in $J_p^{(1)}(p)$. The argument is the same as for J_p, except that now the coefficients are $\mathscr{C}_s = \bar{C}_s$. When r is even $\bar{C}_r = 0$, while for odd r

$$\bar{C}_r = \left(\frac{16}{\pi^2}\right)^{1/3} \beta\sqrt{3}\, \frac{(-1)^{\frac{1}{2}(r-1)}}{(2\pi p)^{\frac{1}{3}r}} \left\{(\tfrac{1}{3}r - \tfrac{4}{3})! + \frac{(2\pi)^{2/3}}{30\beta}(\tfrac{1}{3}r - 2)! \right.$$

$$\left. - \frac{46\pi^2}{1575}(\tfrac{1}{3}r - \tfrac{10}{3})! \dots\right\}$$

$$= \frac{.3114777\,(-1)^{\frac{1}{2}(r-1)}}{(2\pi p)^{\frac{1}{3}r}} \{(\tfrac{1}{3}r - \tfrac{4}{3})! + .7414338(\tfrac{1}{3}r - 2)!$$

$$- .2882551(\tfrac{1}{3}r - \tfrac{10}{3})! \dots\}. \tag{66}$$

6. Bessel Function $Y_p(x)$.

The most convenient starting point is the companion integral representation (III Q.14) to (44) for $J_p(x)$:

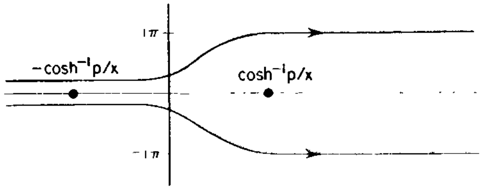
$$Y_p(x) = -\frac{1}{2\pi}\left\{\int_{-\infty}^{\infty + i\pi} e^{-F}\, d\omega + \int_{-\infty}^{\infty - i\pi} e^{-F}\, d\omega\right\},$$

$$F(\omega) = p\omega - x\sinh\omega, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{67}$$

The suppositions of VI are then fulfilled, since $F(\omega) = \infty$ at each limit and (as we shall confirm) each integral includes only one stationary point.

$p > x > 0.$

As seen from the diagram, the contribution from the stationary point at $\omega = -\cosh^{-1}p/x$, $F_2 = (p^2 - x^2)^{\frac{1}{2}}$, comes in with the same sign for the two integrals. The other stationary point at $\omega = \cosh^{-1}p/x$, $F_2 = -(p^2 - x^2)^{\frac{1}{2}}$, corresponding to a minimum for a path parallel to the i-axis, is to be by-passed because the $\pm i$ symmetry of (67) disallows a contribution.



Hence

$$Y_p(x) = -\frac{1}{\pi}\int_{\rightarrow} e^{-F}\,d\omega, \tag{68}$$

$$F_0 = p(q^{-1} - \tanh^{-1}q^{-1}), \qquad F_1 = 0,$$

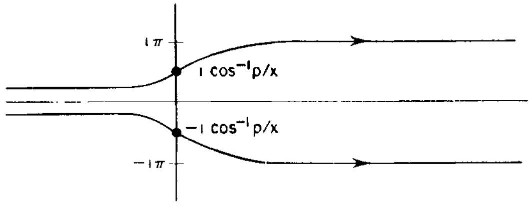
$$F_{2j\geqslant 2} = p/q, \qquad F_{2j+1\geqslant 3} = -p. \tag{69}$$

The resultant asymptotic expansion is

$$Y_p(x) = -\left(\frac{2q}{\pi p}\right)^{1/2} e^{p(\tanh^{-1}q^{-1}-q^{-1})}\left\{1 + \frac{q}{24p}(5q^2 - 3) + \frac{q^2}{1152p^2}(385q^4 - 462q^2 + 81) + \dots\right\}. \tag{70}$$

Late terms are as in (50) with the $(-1)^r$ factor omitted.

$x > p > 0.$



Here we have to evaluate $(-1/2\pi)\int_{\rightarrow} e^{-F}\,d\omega$ through each of the stationary points $\omega = \pm i\cos^{-1}p/x$, and add the results. This is most easily done by replacing q in (69) and (70) by $\pm i\mathscr{q}$ and halving the sum of the two expansions, leading to

$$Y_p(x) = -\left(\frac{2\mathscr{q}}{\pi p}\right)^{1/2}\left[\left\{1 - \frac{\mathscr{q}^2}{1152p^2}(385\mathscr{q}^4 + 462\mathscr{q}^2 + 81)\ldots\right\}\cos(p\Upsilon + \tfrac{1}{4}\pi)\right.$$

$$\left. + \left\{\frac{\mathscr{q}}{24p}(5\mathscr{q}^2 + 3) - \ldots\right\}\sin(p\Upsilon + \tfrac{1}{4}\pi)\right]. \tag{71}$$

Late terms are exactly as in (54).

$x \sim p$.

Since Y satisfies the same differential equation as J, the series expressing $Y_p(x)$ in terms of $Y_p(p)$ and $Y_p^{(1)}(p)$ are the same as those in (57).

Asymptotic expansion of $Y_p(p)$. When $x = p$ the problem is one of cubic dependence at the stationary point $\omega = \cosh^{-1} 1 = 0$, where by (69)

$$F_0 = F_1 = 0, \qquad F_{2j} \to +0, \qquad F_{2j+1\geq 3} = -p. \tag{72}$$

From the contour equivalence VI (13), (68) becomes

$$Y_p = \frac{1}{2\pi}\left\{\int_{\neg\backslash} - \int_{\_/}\right\}. \tag{73}$$

Referring to VI §5, these integrals can be deduced from $\int_<$ by attaching respective factors $e^{\frac{2}{3}i\pi}$ and $e^{-\frac{2}{3}i\pi}$ to $(-F_3)^{1/3} = p^{1/3}$. Thus from (60),

$$Y_p = -\frac{(-2/3)!\,6^{1/3}}{2\pi\,p^{1/3}}(C_0 + C_4 + C_6 + C_{10}\ldots)$$

$$= -\frac{(-2/3)!\,6^{1/3}}{2\pi\,p^{1/3}}\left(1 + \frac{3\beta}{35p^{4/3}} - \frac{1}{225p^2}\cdots\right), \tag{74}$$

with late terms as in (62).

Asymptotic expansion of $Y_p^{(1)}(p)$. This is deduced from (65) in exactly the same way, the result being

$$Y_p^{(1)} = -\frac{(-2/3)!\,6^{1/3}}{2\pi\,p^{1/3}}(\bar{C}_1 + \bar{C}_3 + \bar{C}_7 + \bar{C}_9\ldots)$$

$$= \frac{(-2/3)!\,6^{1/3}}{2\pi\,p^{1/3}}\left(\frac{6\beta}{p^{1/3}} + \frac{1}{5p} + \frac{23\beta}{525\,p^{7/3}}\cdots\right), \tag{75}$$

with late terms as in (66).

7. Confluent Hypergeometric Function $F(a, c, x)$ for large $|c|$

Our earlier asymptotic expansion for $F(a, c, x)$, II (47) or IV (19)–(21), is adequate only when $|x| \gg |a(c-a)|$. In the present section we examine the case where $|c|$ is large but $|a|$ small; this constitutes a natural generalization of our work in §3 on the incomplete factorial function. The remaining cases, for which a change to the Whittaker notation proves advantageous, will be treated in subsequent sections. As remarked in the prologue, these are perhaps the most important and versatile of all the higher transcendental functions, but their asymptotic behaviour has hitherto been difficult to elicit accurately. Accordingly, our investigations will be recorded in considerable detail.

Starting from the integral representation III (19),

$$(a-1)!\,(c-a-1)!\,F(a,c,x) = (c-1)!\int_0^1 u^{a-1}(1-u)^{c-a-1}e^{ux}\,du, \tag{76}$$

we are at once faced with a degree of arbitrariness in splitting up the integrand into "fast-varying" and "slowly-varying" factors; for if $|c|$ is large and $|a|$ small, the fast-varying factor e^{-f} can be chosen with any f of the type $f = -ux - (c + \alpha a + \beta)\ln(1-u)$ where α and β are small numerical factors. The most advantageous choice is that which makes the cardinal quantity $|f|$ an invariant under major transformations underlying the theory of the confluent hypergeometric function, namely $(a, c, x) \to (c-a, c, -x)$ in the Kummer relation, $(a, c, x) \to (a-c+1, 2-c, x)$ in converting to the second solution, and their amalgamation $(a, c, x) \to (1-a, 2-c, -x)$. This is fulfilled by

$$f = -ux - C\ln(1-u), \qquad C = c - 2a. \tag{77}$$

The mode of treatment will clearly depend on whether or not the stationary point at $u = 1 - C/x$ lies within the range of integration 0 to 1. Unless otherwise stated, C will for brevity be regarded as real and positive in the ensuing exposition.

$x < C$.

When $x < C$, including here negative x, $f(u)$ increases roughly linearly as u increases from the lower limit zero, and ends at ∞ at the upper limit of integration $u = 1$. Near $u = 0$ the remaining factors of the integrand behave as u^{a-1}, so V (13) is directly applicable when the integral in (76) is expressed as

$$\int_{u=0} e^{-f}u^{a-1}G\,du, \qquad f = (77), \qquad G = (1-u)^{a-1}. \tag{78}$$

Excessive algebra can be avoided, and results shortened, if the derivatives of f and G at $u = 0$ are rewritten in terms of the parameter

$$q = C/(C - x). \tag{79†}$$

Then

$$f_0 = 0, \qquad f_1 = C/q, \qquad f_{j\geqslant 2} = (j-1)!\, C,$$
$$G_j = (-1)^j\,(a-1)!\,/(a-j-1)!, \tag{80}$$

leading to an expansion of which the first few terms are

$$F(a, c, x) = \frac{(c-1)!}{(c-a-1)!}\left(\frac{q}{C}\right)^a \left[1 - \frac{aq}{2C}\{(a+1)q + 2(a-1)\}\right.$$
$$+ \frac{a(a+1)q^2}{24C^2}\{3(a+2)(a+3)q^2 + 4(a+2)(3a-5)q$$
$$\left. + 12(a-1)(a-2)\}\ldots\right]. \tag{81}$$

Late terms. Here the singulant is the change in value of f in going from the limit of integration, where $f = 0$, to the stationary point at $u = 1 - C/x = -(q-1)^{-1}$, so

$$\mathscr{F}_0 = -C\Xi, \qquad \Xi = \ln[q/(q-1)] - 1/q. \tag{82}$$

Since the confluent hypergeometric function satisfies a homogeneous second-order differential equation, this is the sole singulant (VII §6); the non-principal readings of the logarithm in (82) do not play any role in late terms of the asymptotic expansion. A straightforward though somewhat tedious calculation from VII (28) leads to the formula for late terms in the series (81):

$$L_r^{(a-1)} = \frac{(r+a-3/2)!}{(2\pi)^{1/2}(a-1)!\,(q-1)^{2a-1}(-C)^r\,\Xi^{r+a-1/2}}$$
$$\times\left[1 - \frac{\Xi\{(a-1)(a-2)q^2 + 2a(a-1)q + a^2 - a + 1/6\}}{2(r+a-3/2)}\right.$$
$$+ \frac{\Xi^2}{24(r+a-3/2)(r+a-5/2)}\{3(a-1)(a-2)(a-3)(a-4)q^4$$
$$+ 4(a-1)(a-2)(a-3)(3a+2)q^3$$
$$+ (a-1)(a-2)(18a^2 + 6a + 1)q^2 + 2a(a-1)(6a^2 - 6a + 1)q$$
$$\left. + 3a^4 - 10a^3 + 10a^2 - 3a + 1/12\}\ldots\right]. \tag{83}$$

† In the alternative Whittaker notation involving solutions $W_{km}(x)$ and $W_{-k,m}(-x)$, $q = k/(k - \frac{1}{2}x)$.

$x > C$.

When $x > C$ the integral (76) does not fulfil the conditions prescribed in VI §2 because it involves two critical points, namely the limit $u = 0$ where the integrand is still contributing, and the stationary point $u = 1 - C/x$. Separation of the critical points has already been effected for other reasons in IV §4, the result being

$$F(a, c, x) = \frac{(c-1)!}{(a-1)!} x^{a-c} e^x \psi(1-a, 2-c, -x) + \begin{pmatrix} e^{i\pi a} \\ e^{-i\pi a} \\ \cos \pi a \end{pmatrix} \frac{(c-1)!}{(c-a-1)!}$$

$$\times\, x^{-a} \psi(a, c, x), \qquad \left.\begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \\ \operatorname{ph} x = 0 \end{matrix}\right\}, \tag{84}$$

where

$$\psi(a, c, x) = \frac{1}{(a-1)!} \int_0^\infty v^{a-1} \left(1 + \frac{v}{x}\right)^{c-a-1} e^{-v}\, dv. \tag{85}$$

At $\operatorname{ph} x = 0$ the principal value of the integral $\psi(1-a, 2-c, -x)$ is to be understood.

Expressing (85) as

$$(a-1)!\, \psi(a, c, x) = \int_{v=0} e^{-f} v^{a-1} G\, dv, \qquad f = v - C \ln(1 + v/x),$$

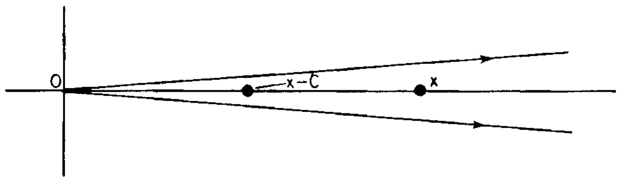
$$G = (1 + v/x)^{a-1}, \tag{86}$$

it is seen that $f(v)$ increases roughly linearly as v increases from the lower limit zero, and ends at ∞ at the upper limit of integration $v = \infty$, so V (13) is directly applicable. The resultant expansion is

$$\psi(a, c, x) = \left(-\frac{xq}{C}\right)^a \left[1 - \frac{aq}{2C}\{(a+1)q + 2(a-1)\} + \dots\right], \tag{87}$$

the asymptotic series within being formally identical with that in (81), though the parameter $q = C/(C - x)$ is now of course negative.

The other integral $\psi(1-a, 2-c, -x)$ needs closer attention. Here $f = v + C \ln(1 - v/x)$, so $f(v)$ again starts by increasing roughly linearly as v increases from the lower limit zero. There is a stationary point at $v = x - C$, but since $f_2 = -C^{-1}$ is negative this would correspond to a minimum along a path parallel to the imaginary axis; to pass through it would entail deformations out to $v = \pm i\infty$ which would be incompatible with the specified path $v = 0$ to ∞ unless accompanied by cancelling returns. Both this stationary point at $v = x - C$, and the later singular point at $v = x$, have therefore to be by-passed. As is also evident from the earlier remark on its interpretation as a principal value, the integral is specifying an expansion on a Stokes ray, so the mean of the paths depicted must be taken in the complex v-plane:



Since the only critical point contributing is the limit of integration $v = 0$ just as before, the expansion can be deduced from (87) by replacing a by $1 - a$ and reversing the signs prefacing x and C, thus

$$\psi(1 - a, 2 - c, -x) = \left(-\frac{xq}{C}\right)^{1-a}\left[1 + \frac{(1-a)q}{2C}\{(2-a)q - 2a\}\ldots\right]. \tag{88}$$

When (87) and (88) are further expanded in rising powers of C/x they reproduce as expected the first few terms of the respective asymptotic power series, e.g. IV (21). Thus (88) could alternatively have been derived by combining this requirement with the known relationship between the two power series.

Late terms. So long as $C = c - 2a$ is positive the asymptotic formula (83) for late terms is applicable to (87); and likewise to (88) on replacing a by $1 - a$ and C by $-C$, leaving q and Ξ unchanged.

When C is negative, (83) needs revision, because then $q < 1$ and the previous sole singulant (82) is replaced by the pair

$$\mathscr{F}_0 = C\left\{\frac{1}{q} - \ln\left(\frac{q}{1-q}\right) \pm i\pi\right\}. \tag{89}$$

Derivation of the revised formula is left as an exercise (Q.14).

$x \sim C$.

The asymptotic expansions (81), (87) and (88) fail as $x \to C$, because the linear term in $f(u)$ at the limit of integration—hitherto supposed dominant—then vanishes. A solution for $x \sim C$ is obtained by expanding $F(a, c, x)$ as a Taylor series in $x - C$, in which all derivatives can be expressed in terms of

F and $F^{(1)}$ since the function satisfies a second-order differential equation. Applying the Leibnitz differentiation formula to the right-hand side of

$$F'' = \frac{a}{x} F + \left(1 - \frac{c}{x}\right) F', \tag{90}$$

the reduction relation for higher derivatives at $x = C$ is found to be

$$F^{(\nu+2)} = -\frac{2a}{C} F^{(\nu+1)} + \nu! \sum_{0}^{\nu} \frac{(-1)^s \{aC + c(\nu - s)\}}{C^{s+2}\,(\nu - s)!} F^{(\nu-s)}. \tag{91}$$

This reduces the Taylor expansion to

$$\begin{aligned} F(a, c, x) = {} & F(a, c, C)\{1 + aCX^2/2 - a(2a + 1)CX^3/6 + aC[(a + 2)C \\ & + 2(a + 1)(2a + 1)]\, X^4/24 \ldots\} + F^{(1)}(a, c, C)\, CX\{1 - aX \\ & + [(a + 1)C + 2a(2a + 1)]\, X^2/6 - [(2a^2 + 4a + 1)C \\ & + 2a(a + 1)(2a + 1)]\, X^3/12 \ldots\}, \end{aligned} \tag{92}$$

where $X = (x - C)/C$. These power series in $(x - C)$ are convergent over the wide range $|\, aX \,| < 1$.

The find the asymptotic expansion of $F(a, c, C)$, we note that when $x = C$ the first derivative f_1 in (80) vanishes, so the problem becomes one of quadratic dependence at a limit of integration. From V (20).

$$\begin{aligned} F(a, c, C) = {} & \frac{(\frac{1}{2}a - 1)!\,(c - 1)!}{2(a - 1)!\,(c - a - 1)!} \left(\frac{2}{C}\right)^{\frac{1}{2}a} \left[1 - \frac{2(2a - 1)(\frac{1}{2}a - \frac{1}{2})!}{3(\frac{1}{2}a - 1)!} \left(\frac{2}{C}\right)^{\frac{1}{2}} \right. \\ & + \frac{a(32a^2 - 39\,a + 10)}{72} \left(\frac{2}{C}\right) \\ & - \frac{(320a^3 - 690a^2 + 427a - 69)(\frac{1}{2}a + \frac{1}{2})!}{810\,(\frac{1}{2}a - 1)!} \left(\frac{2}{C}\right)^{3/2} \\ & + \frac{a(a + 2)(5{,}120a^4 - 16{,}960a^3 + 18{,}463a^2 - 7{,}322a + 744)}{155{,}520} \left(\frac{2}{C}\right)^2 \\ & \left. - \frac{\begin{gathered}(3{,}766a^5 + 17{,}430a^4 - 233{,}177a^3 + 725{,}907a^2 \\ -922{,}049a + 408{,}195)(\frac{1}{2}a + \frac{3}{2})!\end{gathered}}{204{,}120\,(\frac{1}{2}a - 1)!} \left(\frac{2}{C}\right)^{5/2} \cdots \right]. \end{aligned} \tag{93}$$

Reference to (76) shows that the derivative is obtained by inserting an extra factor u in the integrand, retaining the same values for the f's and G's in (78) and (80). From V (20),

$$F^{(1)}(a, c, C) = \frac{(\frac{1}{2}a - \frac{1}{2})!\,(c-1)!}{2(a-1)!\,(c-a-1)!}\left(\frac{2}{C}\right)^{\frac{1}{2}a+\frac{1}{2}}$$

$$\times\left[1 - \frac{(4a-1)(\frac{1}{2}a)!}{3\,(\frac{1}{2}a-\frac{1}{2})!}\left(\frac{2}{C}\right)^{1/2} + \frac{(a+1)(32a^2-23a+3)}{72}\left(\frac{2}{C}\right)\right.$$

$$-\frac{(640a^3-900a^2+329a-24)(\frac{1}{2}a+1)!}{1620\,(\frac{1}{2}a-\frac{1}{2})!}\left(\frac{2}{C}\right)^{3/2}$$

$$\left.+\frac{(a+1)(a+3)(5{,}120a^4-11{,}840a^3+8{,}383a^2-1{,}852a+45)}{155{,}520}\left(\frac{2}{C}\right)^2\cdots\right]. \tag{94}$$

Derivations of formulae for late terms in (93) and (94) are analogous to those undertaken in §2 and §3 for $p!$ and $(p, p)!$, and are left as exercises (Q.15).

8. Whittaker Function for large k. (Confluent Hypergeometric Function $F(a, c, x)$ for large a).

When $|a|$ is large, it is easiest to find the asymptotic expansion for $\psi(a, c, x)$ first and deduce that for $F(a, c, x)$ from the connection (84). A further advantage is gained by going over at this point to Whittaker's notation

$$W_{km}(x) = W_{k,-m}(x) = x^k e^{-\frac{1}{2}x}\psi(\tfrac{1}{2}-k+m,\ 2m+1,\ x). \tag{95}$$

We shall therefore be dealing in this section with the Whittaker function when k is large and m is small. (This notational change was not made in the previous section, since the situation c large, a small considered there corresponds in the Whittaker notation to the more involved set of conditions k large, m large, $\kappa = (k^2 - m^2)^{\frac{1}{2}}$ small).

Translating the integral representations III (23), (27), (28) into the Whittaker notation,

$$W_{km}(x) = \begin{cases} \dfrac{x^k e^{-\frac{1}{2}x}}{(-k+m-\frac{1}{2})!}\displaystyle\int_0^\infty e^{-u}\, u^{-k+m-\frac{1}{2}} \left(1+\frac{u}{x}\right)^{k+m-\frac{1}{2}} du, \\ \qquad\qquad \Re(-k+m+\tfrac{1}{2}) > 0 \qquad (96) \\ \dfrac{4\, x^{\frac{1}{2}} e^{-\frac{1}{2}x}}{(-k+m-\frac{1}{2})!\,(-k-m-\frac{1}{2})!}\displaystyle\int_0^\infty e^{-v^2}\, v^{-2k} K_{2m}(2v\sqrt{x})\, dv, \\ \qquad\qquad \Re(-k-m+\tfrac{1}{2}) > 0 \qquad (97) \\ \dfrac{-2i\, x^{\frac{1}{2}} e^{\frac{1}{2}x}}{\pi}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} e^{v^2}\, v^{2k} K_{2m}(2v\sqrt{x})\, dv. \qquad (98) \end{cases}$$

Unfortunately none of these is a paragon starting point. In the first, logically superfluous algebra—expanding prohibitively for successive terms—is caused by the integrand and outer factor not being separately even in m. This drawback moreover makes the representation totally unsuitable for the case of large m to be developed in the next section, because when $[u(1 + u/x)]^m$ has to be included in the fast-varying portion of the integrand the stationary points and derivatives at them are different for $+m$ and $-m$, so calculations cannot be carried through in terms of a single simplifying parameter. The other two representations are free from this shortcoming, but entail substituting therein the cumbersome asymptotic expansions of the modified Bessel function before expansions expressed in elementary functions can be evolved. A further drawback to all three starting points is that it transpires—on comparison with results from the phase-integral method (XIII §10, 11)—that substantially simpler expansions for W_{km} would be directly derivable from a representation which had outer multiplying factor

$$[(\mp k + m - \tfrac{1}{2})!\,(\mp k - m - \tfrac{1}{2})!\,]^{\mp\frac{1}{2}}.$$

But of known representations, the one most nearly fulfilling this condition is for† W_{km}^2, and its adoption would still not facilitate intercomparison between results from integral representations and phase-integral methods applied to the differential equation. The third representation (98) can involve two stationary points as against the single one of the second, but we shall nevertheless select it as the least disagreeable choice because of its immediate application to the commonest range $k > 0$, $x > 0$.

† $W_{km}{}^2(x) = \dfrac{2x}{(-k+m-\frac{1}{2})!(-k-m-\frac{1}{2})!}\displaystyle\int_0^\infty (\coth \tfrac{1}{2}v)^{2k}\, e^{-x\coth v}\, K_{2m}(x \sinh v)\, dv.$

From IV (27) the fast-varying factor in $K_{2m}(2v\sqrt{x})$ is $e^{-2v\sqrt{x}}$ when $|v\sqrt{x}|$ is large. The stationary points of $e^{v^2}v^{2k}e^{-2v\sqrt{x}}$ lie at $v_0{}^{\pm} = \frac{1}{2}\{x^{\frac{1}{2}} \pm (x-4k)^{\frac{1}{2}}\}$, which indeed make $|v_0\sqrt{x}|$ large when k or x is large, justifying (*a posteriori*) replacement of K by its asymptotic power series. We are therefore led to express (98) in the form

$$W_{km}(x) = -i\pi^{-\frac{1}{2}}x^{\frac{1}{4}}e^{\frac{1}{2}x}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-F}\,G\,dv,$$

$$F = -v^2 - 2k\ln v + 2v\sqrt{x},$$

$$G = v^{\frac{1}{2}}\sum_0^{\infty} \eta(\eta-8)(\eta-24)\ldots(\eta-4\mu[\mu-1])/\mu!\,(16v\sqrt{x})^{\mu}. \qquad (99)$$

Here $\eta = 16m^2 - 1$ and the first term in the summation for G is to be taken as unity. $F(v) = \infty$ at both limits $v = \gamma \pm i\infty$ in accord with the suppositions of VI, but whether the path is to pass through one stationary point or two depends on the range.

$x > 4k > 0.$

In this range it is advantageous to express the F's and G's at the stationary points in terms of the parameter

$$q = \left(\frac{x}{x-4k}\right)^{\frac{1}{2}}. \qquad (100)$$

In this notation,

$$v_0{}^{\pm} = \left(\frac{k(q\pm 1)}{q\mp 1}\right)^{\frac{1}{2}},$$

$$F_0 = k\left[3 \pm \frac{2}{q\mp 1} - \ln\left\{\frac{k(q\pm 1)}{q\mp 1}\right\}\right], \quad F_1 = 0, \quad F_2 = \mp\frac{4}{q\pm 1},$$

$$F_{j\geq 3} = 2k(-1)^j(j-1)!\left\{\frac{q\mp 1}{k(q\pm 1)}\right\}^{\frac{1}{2}j}, \qquad G_j = (-1)^j\left\{\frac{q\mp 1}{k(q\pm 1)}\right\}^{\frac{1}{2}j+\frac{1}{4}}$$

$$\times\sum_0^{\infty}\frac{(\mu+j-\frac{1}{2})!}{\mu!\,(\mu-\frac{1}{2})!}\,\eta(\eta-8)(\eta-24)\ldots(\eta-4\mu[\mu-1])\left(\frac{q\mp 1}{32kq}\right)^{\mu}. \qquad (101)$$

The negative F_2 (top signs) corresponds to a minimum along a path parallel to the imaginary axis; to pass through it we choose $\gamma = v_0{}^+$; while the positive F_2 (lower signs) corresponds to a minimum along a path following the positive real axis, which would be incompatible with the $\pm i$ symmetry of the integral specified in (99). Hence when $x > 4k$ only the stationary point $v_0{}^+$ contributes, and in the notation of VI §4

$$W_{km}(x) = -i\pi^{-\frac{1}{2}}x^{\frac{1}{2}}e^{\frac{1}{2}x}\int_{\uparrow} e^{-F}G\,dv. \tag{102}$$

This yields the expansion

$$\begin{aligned}W_{km}(x) = k^k e^{-k} q^{\frac{1}{2}} e^{-2k\Xi}\Big[1 &- \frac{1}{96k}\{5q^3 - 6q - (3\eta - 1) + 3\eta/q\}\\ &+ \frac{1}{18{,}432k^2}\{385q^6 - 924q^4 - 10(3\eta - 1)q^3 - 6(7\eta - 114)q^2\\ &+ 12(3\eta - 1)q + (9\eta^2 + 102\eta - 143) - 6\eta(3\eta - 1)/q\\ &+ 9\eta(\eta - 8)/q^2\}\ldots\Big]\end{aligned} \tag{103}$$

where

$$\Xi = q/(q^2 - 1) - \coth^{-1}q. \tag{104}$$

As intimated earlier in this section, a more compact expansion results if an outer factor $[(k + m - \frac{1}{2})!\,(k - m - \frac{1}{2})!]^{\frac{1}{2}}$ is imposed. The compensating reciprocal is most easily expanded in rising powers of k^{-1} via Barnes' (1899) generalization of the Stirling series,

$$(k + \varepsilon)! = (2\pi)^{\frac{1}{2}} k^{k+\varepsilon+\frac{1}{2}} e^{-k} \exp\left(\sum_{r=1}^{\infty} (-1)^{r-1} \frac{B_{r+1}(\varepsilon + 1)}{r(r + 1)k^r}\right). \tag{105}$$

Introducing the resulting series

$$[(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!\,]^{-\frac{1}{2}} = (2\pi)^{-\frac{1}{2}}k^{-k}\,e^{k}\left\{1-\frac{3\eta-1}{96k}+\frac{9\eta^2-6\eta+1}{18{,}432k^2}\right.$$

$$-\frac{135\eta^3+8{,}505\eta^2-51{,}795\eta+4027}{26{,}542{,}080k^3}$$

$$\left.+\frac{405\eta^4+103{,}140\eta^3-656{,}370\eta^2+255{,}684\eta-16{,}123}{10{,}192{,}158{,}720k^4}\cdots\right\}, \tag{106}$$

(103) is simplified to

$$W_{km}(x) = [(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!\,]^{\frac{1}{2}}\left(\frac{q}{2\pi}\right)^{\frac{1}{2}} e^{-2k\Xi}\left[1-\frac{1}{96k}\right.$$

$$\times\{5q^3-6q+3\eta/q\}+\frac{1}{18{,}432k^2}\{385q^6-924q^4$$

$$\left.-6(7\eta-114)q^2+36(3\eta-4)+9\eta(\eta-8)/q^2\}\cdots\right]. \tag{107}$$

Late terms. Here the singulant is the change in value of F in going across to the other stationary point, i.e.

$$\mathscr{F}_0 = -4k\,\Xi. \tag{108}$$

Since $W_{km}(x)$ satisfies a homogeneous second-order differential equation, this is the sole singulant (VII §6); non-principal readings of $\coth^{-1}q$ do not play any rôle in late terms of the asymptotic expansion.

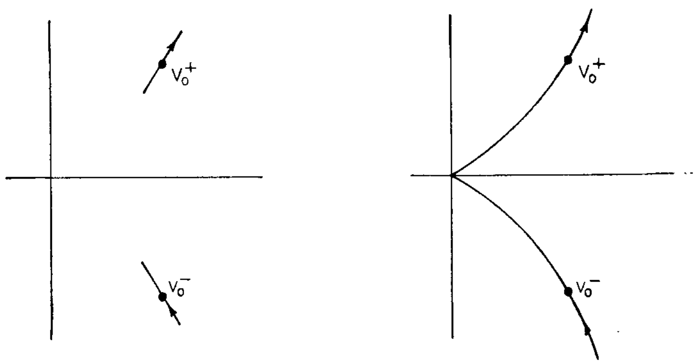
The derivatives at the two stationary points differ only by reversal in sign of q, so in VII (12) $\mathscr{Q}_s(q) = Q_s(-q)$ and formulae for late terms can be written down without further calculation. For instance, in the neater expansion (107),

$$Q_{2r} = \frac{(r-1)!}{2\pi(-4k\,\Xi)^r}\left[1-\frac{\Xi\,(5q^3-6q+3\eta/q)}{24(r-1)}\right.$$

$$\left.+\frac{\Xi^2\{385q^6-924q^4-6(7\eta-114)q^2+36(3\eta-4)+9\eta(\eta-8)/q^2\}}{1152\,(r-1)(r-2)}\cdots\right]. \tag{109}$$

$4k > x > 0.$

To clarify the exposition for this range, we rewrite the locations of the two stationary points as $v_0{}^{\pm} = \frac{1}{2}\{x^{\frac{1}{2}} \pm i(4k-x)^{\frac{1}{2}}\}$, where $F_2 = -(4k-x)^{\frac{1}{2}} \times\{(4k-x)^{\frac{1}{2}} \pm ix^{\frac{1}{2}}\}/k$ respectively. These correspond to minima along paths directed as shown in the diagram, and by the specification (99) both points have now to be included on an equal footing.



The obligatory separation into integrals containing one stationary point each (VI §2) is accomplished by diverting the path between the points to $v = 0$, where $F(\omega) \to \infty$. The conditions laid down in VI §4 are then fulfilled for each of the integrals in

$$i\pi^{\frac{1}{2}}\, x^{-\frac{1}{4}}\, e^{-\frac{1}{2}x}\, W_{km}(x) = \int_0^{\gamma + i\infty} e^{-F}\, G\, d\omega + \int_{\gamma - i\infty}^{0} e^{-F}\, G\, d\omega, \qquad (110)$$

so there are now contributions $\int_{\uparrow}$ from both stationary points. Accordingly, we replace q in (103) or (107) by $\pm\, i\mathscr{q}$, where

$$\mathscr{q} = \left(\frac{x}{4k - x}\right)^{\frac{1}{2}}, \qquad (111)$$

and add the pair of expansions. This procedure applied to (107) gives

$$W_{km}(x) = [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}} \left(\frac{2\mathscr{q}}{\pi}\right)^{\frac{1}{2}}$$

$$\times \left[\left\{1 - \frac{1}{18{,}432k^2}\,[385\mathscr{q}^6 + 924\mathscr{q}^4 - 6(7\eta - 114)\mathscr{q}^2\right.\right.$$

$$\left. - 36(3\eta - 4) + 9\eta(\eta - 8)/\mathscr{q}^2] \ldots\right\} \sin(2k\Upsilon + \tfrac{1}{4}\pi)$$

$$\left. - \left\{\frac{1}{96k}\,(5\mathscr{q}^3 + 6\mathscr{q} + 3\eta/\mathscr{q}) - \ldots\right\} \cos(2k\Upsilon + \tfrac{1}{4}\pi)\right], \qquad (112)$$

where

$$\Upsilon = \cot^{-1}\varphi - \varphi/(\varphi^2 + 1).$$

Late terms. On making the requisite notational changes in (109), the asymptotic formula for late terms becomes

$$Q_{2r} = \begin{pmatrix} (-1)^{\frac{1}{2}r} \\ (-1)^{\frac{1}{2}(r-1)} \end{pmatrix} \frac{(r-1)!}{2\pi(4k\,\Upsilon)^r}\left[1 - \frac{\Upsilon(5\varphi^3 + 6\varphi + 3\eta/\varphi)}{24(r-1)}\right.$$
$$+ \frac{\Upsilon^2\{385\varphi^6 + 924\varphi^4 - 6(7\eta - 114)\varphi^2 - 36(3\eta - 4) + 9\eta(\eta - 8)/\varphi^2\}}{1152\,(r-1)(r-2)}$$
$$\left.\ldots\right] \quad \begin{matrix} r \text{ even} \\ r \text{ odd} \end{matrix}\Bigg\}. \qquad (113)$$

$x \sim 4k$.

The asymptotic expansions (107) and (112) fail as $x \to 4k$ ($q \to \infty$), because the quadratic term in $F(u)$ at the stationary points vanishes. A solution for $x \sim 4k$ is obtained by expanding $W_{km}(x)$ as a Taylor series in $x - 4k$. Applying the Leibnitz differentiation formula to

$$W''(x) = \left(\frac{1}{4} - \frac{k}{x} + \frac{\eta - 3}{16x^2}\right) W(x), \qquad (114)$$

the reduction relation for higher derivatives at $x = 4k$ is found to be

$$W^{(\nu+2)} = \frac{\eta - 3}{256k^2} W^{(\nu)} + \tfrac{1}{4}(\nu!)\sum_1^{\nu} \frac{(-1)^{s-1}}{(4k)^s(\nu - s)!}\left\{1 - \frac{(\eta - 3)(s + 1)}{64k^2}\right\} W^{(\nu - s)}. \qquad (115)$$

This reduces the Taylor series to

$$W_{km}(x) = W_{km}\left[1 + \frac{\eta - 3}{8}X^2 + \frac{32k^2 - \eta + 3}{6}X^3\right.$$
$$\left. - \frac{2048k^2 - (\eta - 3)(\eta + 93)}{384}X^4 + \frac{32(\eta+21)k^2 - (\eta-3)(\eta+45)}{120}X^5 \ldots\right]$$
$$+ W_{km}^{(1)}\,8k\,X\left[1 + \frac{\eta - 3}{24}X^2 + \frac{32k^2 - \eta + 3}{12}X^3\right.$$
$$\left. - \frac{6144k^2 - (\eta - 3)(\eta + 285)}{1920}X^4 \ldots\right], \qquad (116)$$

where $X = (x - 4k)/8k$. By analogy with the parallel case of the Bessel functions, we believe the series in (116) to be absolutely convergent.

Asymptotic expansion of $W_{km}(4k)$. When $x = 4k$ $(q \to \infty)$, the problem is one of cubic dependence at the sole stationary point $v = \sqrt{k}$, where by (101)

$$F_0 = k(3 - \ln k), \quad F_1 = 0, \quad F_2 \to -0, \quad F_{j\geq 3} = 2(-1)^j (j-1)!\, k^{1-\frac{1}{2}j},$$

$$G_j = \frac{(-1)^j}{k^{\frac{1}{2}j+\frac{1}{4}}} \sum_0^\infty \frac{(\mu + j - \frac{1}{2})!}{\mu!\,(\mu - \frac{1}{2})!}\, \eta(\eta - 8)(\eta - 24)\ldots(\eta - 4\mu[\mu - 1])\,(32k)^{-\mu}. \tag{117}$$

From the contour equivalence VI (10), (102) becomes

$$W_{km} = i(2/\pi)^{\frac{1}{2}}\, k^{\frac{1}{4}}\, e^{2k} \int_< e^{-F}\, G\, dv = (-2/3)!\,(2k/3\pi^3)^{\frac{1}{6}}\, e^{-k}\, k^k$$
$$\times (C_0 - C_1 + C_3 - C_4 + C_6 - C_7 \ldots), \tag{118}$$

where

$$C_0 = 1 + \frac{\eta}{32k} + \frac{\eta(\eta - 8)}{2048k^2} \ldots, \qquad C_1 = \frac{3\beta\eta}{16 . 4^{\frac{1}{3}}\, k^{\frac{4}{3}}} \left(1 + \frac{\eta - 8}{32k} \ldots\right),$$

$$C_3 = -\frac{1}{96k}\left(1 - \frac{103\,\eta}{160\,k} \ldots\right), \quad C_4 = -\frac{27\beta}{560 . 4^{\frac{1}{3}}\, k^{\frac{4}{3}}}\left(1 - \frac{389\,\eta}{432\,k} \ldots\right),$$

$$C_6 = -\frac{199}{460{,}800k^2} \ldots, \quad C_7 = \frac{9\beta}{17{,}920 . 4^{\frac{1}{3}}\, k^{\frac{7}{3}}} \ldots.$$

(To avoid repetitious printing, a factor $k^{-\frac{1}{3}}$ originally in the C's has been transferred to the outer factor in (118)). Regrouping according to inverse powers of k and then introducing the simplifying factorial multiplier as in (106),

$$W_{km} = \left\{\pi^{-1}(-2/3)!\,[(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!\,]^{\frac{1}{2}}\,(k/12)^{\frac{1}{6}}\right\}$$
$$\times \{\mathbf{C}_0 - \mathbf{C}_4 + \mathbf{C}_6 - \mathbf{C}_{10} \ldots\}$$
$$= \{ \qquad \text{ditto} \qquad \} \times \left\{1 - \frac{3\beta\,(35\eta - 9)}{560.\,4^{\frac{1}{3}}\, k^{\frac{4}{3}}} + \frac{45\,\eta - 7}{14{,}400\,k^2} \ldots\right\}, \tag{119}$$

where $\beta = 0.15308275\ldots$ and zero **C**'s have been omitted.

Late terms in $W_{km}(4k)$. As in the parallel case of the expansion for $J_p(p)$ investigated in detail in §5, the sole contributing singulant pair corresponds to switching in

$$\mathscr{F} = F - F_0 = -v^2 - 2k \ln v + 4v\sqrt{k} - 3k + 3 \ln k \tag{120}$$

to the nearest non-principal readings of the logarithm, i.e. $\mathscr{F}_0 = \pm 4\pi ik$. Since the $F_{j>0}$ and G_j are identical with those at the principal point, $\mathscr{C}_s = C_s$ in VII (20) and formulae for late terms can be written down by inspection. For instance, in (119) $\mathbf{C}_r = 0$ when r is odd, while for even r

$$\mathbf{C}_r = \frac{(-1)^{\frac{1}{2}r}}{\sqrt{3}\,\pi(4\pi k)^{\frac{1}{3}r}}\left\{(\tfrac{1}{3}r-1)! - \frac{3\beta\,\pi^{\frac{4}{3}}(35\eta-9)}{140}(\tfrac{1}{3}r-\tfrac{7}{3})!\right.$$
$$\left. - \frac{\pi^2(45\eta-7)}{900}(\tfrac{1}{3}r-3)!\ldots\right\}. \tag{121}$$

Asymptotic expansion of $W_{km}^{(1)}(4k)$. Changing the argument of the Bessel function in (98) from $2v\sqrt{x}$ to $2u$, differentiating with respect to x and then reverting to the original variable, it is easily proved that

$$W_{km}^{(1)} = -\frac{i\,k^{\frac{1}{6}}\,e^{2k}}{2^{\frac{2}{3}}\sqrt{\pi}}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-F}\,\bar{G}\,dv\,, \tag{122}$$

where $\bar{G} = (1 - v^2/k)G$. The calculation therefore differs from that for W in replacing the G_j by

$$\bar{G}_j = -j[2k^{-\frac{1}{2}}G_{j-1} + (j-1)\,k^{-1}G_{j-2}]. \tag{123}$$

The ensuing replacements for the C's are

$$\bar{C}_0 = 0, \qquad \bar{C}_1 = \frac{12\beta}{4^{\frac{1}{3}}\,k^{\frac{1}{3}}}\left(1 + \frac{\eta}{32k} + \frac{\eta(\eta-8)}{2048k^2}\ldots\right),$$

$$\bar{C}_3 = \frac{1}{5k}\left(1 + \frac{\eta}{32k}\ldots\right), \qquad \bar{C}_4 = -\frac{\beta}{8\,.\,4^{\frac{1}{3}}\,k^{\frac{4}{3}}}\left(1 - \frac{11\eta}{32k}\ldots\right),$$

$$\bar{C}_6 = -\frac{1}{480k^2}\ldots, \qquad \bar{C}_7 = -\frac{283\beta}{89{,}600\,.\,4^{\frac{1}{3}}\,k^{\frac{7}{3}}}\ldots.$$

Regrouping in inverse powers of k and introducing the factorial multiplier,

$$W_{km}^{(1)} = \left\{(4\pi)^{-1}(-2/3)!\,[(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!\,]^{\frac{1}{2}}\,(k/12)^{\frac{1}{6}}\right\}$$
$$\times\{-\bar{\mathbf{C}}_1 + \bar{\mathbf{C}}_3 - \bar{\mathbf{C}}_7 + \bar{\mathbf{C}}_9\ldots\}$$

$$= \left\{\qquad \text{ditto} \qquad\right\} \times \left\{-\frac{12\beta}{4^{\frac{1}{3}}\,k^{\frac{1}{3}}} + \frac{1}{5k} + \frac{2\beta}{525\,.\,4^{\frac{1}{3}}\,k^{\frac{7}{3}}}\ldots\right\}, \tag{124}$$

where zero $\bar{\mathbf{C}}$'s have been omitted.

Late terms in $W_{km}^{(1)}(4k)$. The argument is the same as for W except in respect of replacement by barred C's and **C**'s. When r is even $\overline{\mathbf{C}}_r = 0$, while for odd r

$$\overline{\mathbf{C}}_r = \frac{12\beta(-1)^{\frac{1}{2}(r+1)}}{\sqrt{3}\,\pi^{\frac{3}{2}}\,(4\pi k)^{\frac{1}{3}r}}\left\{(\tfrac{1}{3}r - \tfrac{4}{3})! + \frac{\pi^{\frac{3}{2}}}{15\,\beta}(\tfrac{1}{3}r - 2)! + \frac{8\pi^2}{1575}(\tfrac{1}{3}r - \tfrac{10}{3})! \ldots\right\}. \tag{125}$$

9. Whittaker Function for large $\kappa = (k^2 - m^2)^{\frac{1}{2}}$. (Confluent Hypergeometric Function $F(a, c, x)$ for large a and c)

As in the previous section we start from the integral representation (98). In the Bessel function $K_{2m}(2v\sqrt{x})$ the order $2m$ is now large as well as the argument, so K must be replaced by the more sophisticated asymptotic expansion (Q.10)

$$K_{2m}(2v\sqrt{x}) = \tfrac{1}{2}\left(\frac{\pi\rho}{m}\right)^{\frac{1}{2}} e^{-2m(\rho^{-1}-\tanh^{-1}\rho)}\left\{1 + \frac{\rho}{48m}(5\rho^2 - 3)\right.$$
$$\left. + \frac{\rho^2}{4608m^2}(385\rho^4 - 462\rho^2 + 81) + \ldots\right\}, \tag{126}$$

where $\rho = m/(v^2x + m^2)^{\frac{1}{2}}$. We therefore express (98) as

$$W_{km}(x) = -i\left(\frac{x}{\pi m}\right)^{\frac{1}{2}} e^{\frac{1}{2}x}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-F}\, G\, dv,$$
$$F = -v^2 - 2k\ln v + 2m/\rho - 2m\tanh^{-1}\rho, \tag{127}$$
$$G = \rho^{\frac{1}{2}}[1 + \rho(5\rho^2 - 3)/48m + \ldots].$$

As before, $F(v) = \infty$ at both limits $v = \gamma \pm i\infty$ in accord with the suppositions of Chapter VI, but whether the path is to pass through one or both the stationary points

$$v_0{}^{\pm} = \tfrac{1}{2}\{(x - 2k + 2\kappa)^{\frac{1}{2}} \pm (x - 2k - 2\kappa)^{\frac{1}{2}}\}, \qquad \kappa = (k^2 - m^2)^{\frac{1}{2}}, \tag{128}$$

depends on the range.

$x > 2(k + \kappa) > 0.$

In this range it is advantageous to express the F's and G's at the stationary points in terms of the parameter

$$q = \left(\frac{x - 2k + 2\kappa}{x - 2k - 2\kappa}\right)^{\frac{1}{2}}. \tag{129}$$

In this notation,

$$v_0{}^{\pm} = \left(\frac{\kappa(q \pm 1)}{q \mp 1}\right)^{\frac{1}{2}}, \qquad x = \frac{2\{(k+\kappa)q^2 - k + \kappa\}}{q^2 - 1},$$

$$(v^2 x + m^2)^{\frac{1}{2}} = \frac{(k+\kappa)q \mp (k-\kappa)}{q \mp 1},$$

$$F_0 - \tfrac{1}{2}x = -\tfrac{1}{2}(k+m)\ln(k+m) - \tfrac{1}{2}(k-m)\ln(k-m) + k$$

$$\pm \frac{2\kappa q}{q^2 - 1} \mp 2k \coth^{-1} q \pm 2|m| \tanh^{-1} \frac{1}{q}\left(\frac{k-\kappa}{k+\kappa}\right)^{\frac{1}{2}},$$

$$F_1 = 0, \qquad F_2 = \mp \frac{8\kappa q}{(q \pm 1)\{(k+\kappa)q \mp (k-\kappa)\}},$$

$$F_3 = -\left(\frac{q \mp 1}{\kappa(q \pm 1)}\right)^{\frac{1}{2}} \frac{8\kappa^2\{(k+\kappa)^2 q^3 \mp (k-\kappa)^2\}}{\{(k+\kappa)q \mp (k-\kappa)\}^3},$$

$$F_4 = \frac{24(q \mp 1)^2}{(q \pm 1)^2\{(k+\kappa)q \mp (k-\kappa)\}^5}[2\kappa(k+\kappa)q^5$$
$$\mp (k-\kappa)(k+\kappa)^3 q^4 + 3(k^2-\kappa^2)^2 q^3 \mp 3(k^2-\kappa^2)^2 q^2$$
$$+ (k+\kappa)(k-\kappa)^3 q \pm 2\kappa(k-\kappa)^3],$$

$$G_0 = \left(\frac{m(q \mp 1)}{(k+\kappa)q \mp (k-\kappa)}\right)^{\frac{1}{2}} \left\{1 + \frac{q \mp 1}{24\{(k+\kappa)q \mp (k-\kappa)\}^3}\right.$$
$$\left.\times [(k+\kappa)(k-4\kappa)q^2 \mp 2(k^2-\kappa^2)q + (k-\kappa)(k+4\kappa)] \ldots\right\},$$

$$G_1 = -\left(\frac{m(q \mp 1)}{(k+\kappa)q \mp (k-\kappa)}\right)^{\frac{1}{2}} \left(\frac{\kappa(q \mp 1)}{q \pm 1}\right)^{\frac{1}{2}}$$
$$\times \frac{(k+\kappa)q^2 - k + \kappa}{\{(k+\kappa)q \mp (k-\kappa)\}^2} \ldots$$

$$G_2 = -\left(\frac{m(q \mp 1)}{(k+\kappa)q \mp (k-\kappa)}\right)^{\frac{1}{2}} \left(\frac{q \mp 1}{q \pm 1}\right) \frac{(k+\kappa)q^2 - k + \kappa}{\{(k+\kappa)q \mp (k-\kappa)\}^4}$$
$$\times [(k+\kappa)(k-4\kappa)q^2 \mp 2(k^2-\kappa^2)q + (k-\kappa)(k+4\kappa)] \ldots \tag{130}$$

As in the previous section, the negative F_2 (top signs) corresponds to a minimum along a path parallel to the imaginary axis, and to pass through it we choose $\gamma = v_0{}^+$; while the positive F_2 (lower signs) corresponds to a minimum along a path following the positive real axis, which is incompatible with the $\pm i$ symmetry of the integral specified in (127). Hence, when $x > 2(k+\kappa)$ only the stationary point $v_0{}^+$ contributes, leading to the expansion

$$W_{km}(x) = (k+m)^{\frac{1}{2}(k+m)}(k-m)^{\frac{1}{2}(k-m)}\, e^{-k}\left(\frac{(k+\kappa)q^2 - k + \kappa}{2\kappa q}\right)^{\frac{1}{2}}$$

$$\times\, e^{-2\kappa\Xi}\left[1 - \frac{1}{192\kappa^2}\{5(k+\kappa)q^3 - 3(3k+\kappa)q\right.$$

$$\left. + 8k - 3(3k-\kappa)/q + 5(k-\kappa)/q^3\}\ldots\right], \tag{131}$$

where

$$\Xi = \frac{q}{q^2-1} - \left(\frac{k}{\kappa}\right)\coth^{-1}q + \left(\frac{m}{\kappa}\right)\tanh^{-1}\frac{1}{q}\left(\frac{k-\kappa}{k+\kappa}\right)^{\frac{1}{2}}, \tag{132}$$

with positive signs allotted to k/κ and m/κ.

In common with the case of large k, a more compact expansion results if an outer factor $[(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!]^{\frac{1}{2}}$ is implanted. Counterbalancing this by multiplying by the series

$$[(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!]^{-\frac{1}{2}} = \frac{(k+m)^{-\frac{1}{2}(k+m)}(k-m)^{-\frac{1}{2}(k-m)}\,e^k}{(2\pi)^{\frac{1}{2}}}$$

$$\times\left\{1 + \frac{k}{24\kappa^2} + \frac{k^2}{1152\kappa^4} - \frac{k(4027k^2 - 3024\kappa^2)}{414{,}720\kappa^6}\right.$$

$$\left. - \frac{k^2(16{,}123k^2 - 12{,}096\kappa^2)}{39{,}813{,}120\kappa^8}\ldots\right\}, \tag{133}$$

(131) is simplified to

$$W_{km}(x) = [(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!]^{\frac{1}{2}}\left(\frac{(k+\kappa)q^2 - k + \kappa}{4\pi\kappa q}\right)^{\frac{1}{2}}$$

$$\times\, e^{-2\kappa\Xi}\left[1 - \frac{1}{192\kappa^2}\{5(k+\kappa)q^3 - 3(3k+\kappa)q - 3(3k-\kappa)/q\right.$$

$$\left. + 5(k-\kappa)/q^3\}\ldots\right]. \tag{134}$$

Late terms. The theory is similar to that for the case of large k, small m. The singulant is again the change in value of F in going across to the other stationary point, i.e. now

$$\mathscr{F}_0 = -4\kappa\Xi. \tag{135}$$

Derivatives at the two stationary points still differ only by reversal in sign of q, so $\mathscr{Q}_s(q) = Q_s(-q)$ and formulae for late terms can be written down without further calculation. For instance, in the neater expansion (134),

$$Q_{2r} = \frac{(r-1)!}{2\pi(-4\kappa\Xi)^r}\left[1 - \frac{\Xi}{48(r-1)\kappa}\{5(k+\kappa)q^3 - 3(3k+\kappa)q - 3(3k-\kappa)/q + 5(k-\kappa)/q^3\}\ldots\right]. \tag{136}$$

$2(k + \kappa) > x > 2(k - \kappa)$.

As for the closely parallel case of large k, expansions for this range can be deduced from those for $x > 2(k + \kappa)$ by replacing q in (131) or (134) by $\pm i\mathscr{q}$, where

$$\mathscr{q} = \left(\frac{2\kappa - 2k + x}{2\kappa + 2k - x}\right)^{\frac{1}{2}}, \tag{137}$$

and adding the pair of expansions. Applied to (134), this procedure gives

$$W_{km}(x) = [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}}\left(\frac{(k+\kappa)\mathscr{q}^2 + k - \kappa}{\pi\kappa\mathscr{q}}\right)^{\frac{1}{2}}$$
$$\times\left[\{1 - \ldots\}\sin(2\kappa\Upsilon + \tfrac{1}{4}\pi) - \left\{\frac{1}{192\kappa^2}[5(k+\kappa)\mathscr{q}^3 + 3(3k+\kappa)\mathscr{q} - 3(3k-\kappa)/\mathscr{q} - 5(k-\kappa)/\mathscr{q}^3]\ldots\right\}\cos(2\kappa\Upsilon + \tfrac{1}{4}\pi)\right], \tag{138}$$

where

$$\Upsilon = \left(\frac{k}{\kappa}\right)\cot^{-1}\mathscr{q} - \frac{\mathscr{q}}{\mathscr{q}^2+1} - \left(\frac{m}{\kappa}\right)\tan^{-1}\frac{1}{\mathscr{q}}\left(\frac{k-\kappa}{k+\kappa}\right)^{\frac{1}{2}},$$

with positive signs allotted to k/κ and m/κ.

Late terms. On making the requisite notational changes in (136), the asymptotic formula for late terms becomes

$$Q_{2r} = \begin{pmatrix} (-1)^{\frac{1}{2}r} \\ (-1)^{\frac{1}{2}(r-1)} \end{pmatrix} \frac{(r-1)!}{2\pi(4\kappa\Upsilon)^r}\left[1 - \frac{\Upsilon}{48(r-1)\kappa}\{5(k+\kappa)q^3 + 3(3k+\kappa)q - 3(3k-\kappa)/q - 5(k-\kappa)/q^3)\ldots\right] \left.\begin{matrix} r \text{ even} \\ r \text{ odd} \end{matrix}\right\}. \tag{139}$$

$x \sim 2(k + \kappa)$.

Expansions (134) and (138) fail as $x \to 2(k + \kappa)$ through disappearance of the quadratic term in $F(u)$ at the stationary points, and need replacing by a Taylor series in powers of $x - 2(k + \kappa)$. From the Leibnitz formula, the reduction relation for derivatives at $x = 2(k + \kappa)$ is

$$W^{(\nu+2)} = -\frac{1}{16(k+\kappa)^2} W^{(\nu)} + k\,\nu!\sum_1^{\nu} \frac{(-1)^{s-1}}{[2(k+\kappa)]^{s+1}(\nu-s)!} \times \left\{1 - \frac{(k^2-\kappa^2-\frac{1}{4})(s+1)}{2k(k+\kappa)}\right\} W^{(\nu-s)}, \tag{140}$$

which reduces the Taylor series to

$$\begin{aligned} W_{km}(x) = {} & W_{km}[1 - X^2/2 + 2(4k\kappa + 4\kappa^2 + 1)X^3/3 + (32k^2 - 64k\kappa \\ & - 96\kappa^2 - 23)X^4/24 - 2(24k^2 - 20k\kappa - 44\kappa^2 - 11)X^5/15\ldots] \\ & + W_{km}{}^{(1)}\, 4(k+\kappa)X[1 - X^2/6 + (4k\kappa + 4\kappa^2 + 1)X^3/3 + (96k^2 \\ & - 192k\kappa - 288\kappa^2 - 71)X^4/120\ldots], \end{aligned} \tag{141}$$

where $X = (x - 2k - 2\kappa)/4(k + \kappa)$. We believe the series in (141) are absolutely convergent.

Asymptotic expansion of $W_{km}(2k + 2\kappa)$. In this limit $q \to \infty$ the problem becomes one of cubic dependence at the sole stationary point $v = \sqrt{\kappa}$, where [cf. (130)]

$$F_0 = 2k + \kappa - \tfrac{1}{2}(k+m)\ln(k+m) - \tfrac{1}{2}(k-m)\ln(k-m),$$

$$F_1 = F_2 = 0, \qquad F_3 = -8\kappa^{\frac{1}{2}}/(k+\kappa), \qquad F_4 = 48\kappa/(k+\kappa)^2,$$

$$F_5 = -24(k^2 - 2k\kappa + 17\kappa^2)/\kappa^{\frac{1}{2}}(k+\kappa)^3,$$

$$F_6 = 120(k^3 + 5k^2\kappa - 13k\kappa^2 + 39\kappa^3)/\kappa(k+\kappa)^4,$$

$$F_7 = -720(k^4 + 5k^3\kappa + 19k^2\kappa^2 - 57k\kappa^3 + 96\kappa^4)/\kappa^{\frac{1}{2}}(k+\kappa)^5,$$

$$F_8 = 720(7k^5 + 42k^4\kappa + 98k^3\kappa^2 + 464k^2\kappa^3 - 1505k\kappa^4 + $$
$$+ 1790\kappa^5)/\kappa^2(k+\kappa)^6,$$

$$F_9 = -1440(28k^6 + 196k^5\kappa + 595k^4\kappa^2 + 628k^3\kappa^3 + 7566k^2\kappa^4$$
$$- 21{,}264k\kappa^5 + 19{,}419\kappa^6)/\kappa^{\frac{5}{2}}(k+\kappa)^7,$$

$$F_{10} = 1440(252k^7 + 2016k^6\kappa + 7119k^5\kappa^2 + 15{,}017k^4\kappa^3 - 9094k^3\kappa^4$$
$$+ 278{,}922k^2\kappa^5 - 654{,}485k\kappa^6 + 489{,}277\kappa^7)/\kappa^3(k+\kappa)^8,$$

$$G_0 = 1 + \frac{k - 4\kappa}{24(k+\kappa)^2} + \frac{k^2 - 152k\kappa + 232\kappa^2}{1152(k+\kappa)^4} \cdots$$

$$G_1 = -\frac{\kappa^{\frac{1}{2}}}{k+\kappa} - \frac{\kappa^{\frac{1}{2}}(13k - 22\kappa)}{24(k+\kappa)^3} - \frac{\kappa^{\frac{1}{2}}(313k^2 - 2300k\kappa + 2392\kappa^2)}{1152(k+\kappa)^5} \cdots$$

$$G_2 = -\frac{k - 4\kappa}{(k+\kappa)^2} - \frac{13k^2 - 170k\kappa + 202\kappa^2}{24(k+\kappa)^4} \cdots$$

$$G_3 = \frac{15\kappa^{\frac{1}{2}}(k - 2\kappa)}{(k+\kappa)^3} + \frac{7\kappa^{\frac{1}{2}}(23k^2 - 130k\kappa + 122\kappa^2)}{8(k+\kappa)^5} \cdots$$

$$G_4 = \frac{15(k^2 - 16k\kappa + 22\kappa^2)}{(k+\kappa)^4} + \frac{7(23k^3 - 712k^2\kappa + 2522k\kappa^2 - 1968\kappa^3)}{8(k+\kappa)^6} \cdots$$

$$G_5 = -\frac{45\kappa^{\frac{1}{2}}(15k^2 - 100k\kappa + 106\kappa^2)}{(k+\kappa)^5} \cdots,$$

$$G_6 = -\frac{135(10k^3 - 63k^2\kappa + 866k\kappa^2 - 967\kappa^3)}{2(k+\kappa)^6} \cdots,$$

$$G_7 = \frac{135\kappa^{\frac{1}{2}}(316k^3 - 4721k^2\kappa + 22{,}508k\kappa^2 - 20{,}105\kappa^3)}{2(k+\kappa)^7} \cdots \tag{142}$$

(To avoid repetitious printing, a factor $[(k-\kappa)/(k+\kappa)]^{\frac{1}{4}}$ originally in the G's has been omitted here, to be reinstated in the outer factor). From the contour equivalence VI (10), (127) becomes

$$W_{km} = (-\tfrac{2}{3})!\left(\frac{(k+\kappa)^2}{6\kappa\pi^3}\right)^{\frac{1}{6}} (k+m)^{\frac{1}{2}(k+m)}(k-m)^{\frac{1}{2}(k-m)} e^{-k}(C_0 - C_1 + C_3$$
$$- C_4 + C_6 - C_7 \ldots), \tag{143}$$

where

$$C_0 = 1 + \frac{k - 4\kappa}{24(k+\kappa)^2} + \frac{k^2 - 152k\kappa + 232\kappa^2}{1152(k+\kappa)^4} \dots$$

$$C_1 = \frac{3\beta\kappa^{\frac{1}{3}}}{4(k+\kappa)^{8/3}} \left\{(2k - 3\kappa) + \frac{26k^2 - 179k\kappa + 180\kappa^2}{24(k+\kappa)^2} \dots\right\}$$

$$C_3 = -\frac{1}{24\kappa^2(k+\kappa)^2} \left\{(k^3 + 2k^2\kappa + 2k\kappa^2 - 4\kappa^3) + \frac{1}{120(k+\kappa)^2} \times (5k^4 + 62k^3\kappa + 1698k^2\kappa^2 - 8220k\kappa^3 + 6764\kappa^4) \dots\right\}$$

$$C_4 = -\frac{3\beta}{140\kappa^{8/3}(k+\kappa)^{8/3}} \left\{(9k^4 + 24k^3\kappa + 17k^2\kappa^2 + 68k\kappa^3 - 109\kappa^4) + \frac{1}{24(k+\kappa)^2} (9k^5 + 58k^4\kappa - 44k^3\kappa^2 + 24{,}290k^2\kappa^3 - 73{,}951k\kappa^4 + 50{,}416\kappa^5) \dots\right\}$$

$$C_6 = -\frac{1}{28{,}800\kappa^4(k+\kappa)^4} \{199k^6 + 796k^5\kappa + 1032k^4\kappa^2 - 288k^3\kappa^3 - 17{,}476k^2\kappa^4 + 77{,}904k\kappa^5 - 61{,}968\kappa^6\} \dots$$

$$C_7 = \frac{\beta}{1120\kappa^{14/3}(k+\kappa)^{14/3}} \{9k^7 + 42k^6\kappa + 83k^5\kappa^2 + 114k^4\kappa^3 - 35k^3\kappa^4 + 23{,}370k^2\kappa^5 - 67{,}690k\kappa^6 + 44{,}116\kappa^7\} \dots$$

Regrouping according to inverse powers of κ and then introducing the simplifying factorial multiplier as in (134),

$$W_{km} = \frac{(-\frac{2}{3})!}{\pi}\left\{ [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}} \left(\frac{(k+\kappa)^2}{48\kappa}\right)^{\frac{1}{6}} \right\} \times \{\mathbf{C}_0 - \mathbf{C}_4 + \mathbf{C}_6 - \mathbf{C}_{10} \dots\}$$

$$= \left\{ \qquad \text{ditto} \qquad \right\} \times \left\{1 + \frac{3\beta(9k^2 + 6k\kappa - 4\kappa^2)}{140\kappa^{\frac{8}{3}}(k+\kappa)^{\frac{2}{3}}} - \frac{14k^3 + 14k^2\kappa - 7k\kappa^2 - 8\kappa^3}{1800\kappa^4(k+\kappa)} \dots\right\}, \tag{144}$$

where $\beta = 0{\cdot}15308275..$ and zero **C**'s have been omitted.

Late terms in $W_{km}(2k+2\kappa)$. The theory is similar to that for large k, small m. The sole contributing singulant pair corresponds to switching in (127) to the nearest non-principal readings of $\ln v$, i.e. $\mathscr{F}_0 = \pm 4\pi i k$ as before. Likewise the $F_{j>0}$ and G_j are identical with those at the principal point, so $\mathscr{C}_s = C_s$ in VII (20) and formulae for late terms can be written down by inspection. For instance, in (144) $\mathbf{C}_r = 0$ when r is odd, while for even r

$$\mathbf{C}_r = \frac{(-1)^{\frac{1}{2}r}}{\sqrt{3}\,\pi(4\pi k)^{\frac{1}{3}r}}\left\{(\tfrac{1}{3}r-1)! + \frac{3\beta(4\pi k)^{\frac{4}{3}}(9k^2+6k\kappa-4\kappa^2)}{140\kappa^{8/3}(k+\kappa)^{\frac{2}{3}}}(\tfrac{1}{3}r-\tfrac{7}{3})!\right.$$
$$\left. + \frac{2\pi^2k^2(14k^3+14k^2\kappa-7k\kappa^2-8\kappa^3)}{225\kappa^4(k+\kappa)}(\tfrac{1}{3}r-3)!\ldots\right\}. \tag{145}$$

Asymptotic expansion of $W^{(1)}_{km}(2k+2\kappa)$. The representation analogous to (122) is found to be

$$W^{(1)}_{km} = -\frac{i\,e^{k+\kappa}}{(2\pi m)^{\frac{1}{2}}(k+\kappa)^{\frac{1}{2}}}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-F}\,\bar{G}\,dv, \tag{146}$$

where $\bar{G} = (1 - v^2/\kappa)G$, so to form $W^{(1)}$ we have to replace the G_j in the integral for W by

$$\bar{G}_j = -j[2\kappa^{-\frac{1}{2}}G_{j-1} + (j-1)\kappa^{-1}G_{j-2}]. \tag{147}$$

The replacements for the C's are

$$\bar{C}_0 = 0, \qquad \bar{C}_1 = \frac{6\beta(k+\kappa)^{\frac{1}{3}}}{\kappa^{\frac{4}{3}}}\left\{1 + \frac{k-4\kappa}{24(k+\kappa)^2} + \frac{k^2-152k\kappa+232\kappa^2}{1152(k+\kappa)^4}\ldots\right\}$$

$$\bar{C}_3 = -\frac{1}{10\kappa^2(k+\kappa)}\left\{(k^2-2k\kappa-3\kappa^2)\right.$$
$$\left. + \frac{k^3+54k^2\kappa-295k\kappa^2+252\kappa^3}{24(k+\kappa)^2}\ldots\right\}$$

$$\bar{C}_4 = -\frac{\beta}{4\,\kappa^{\frac{8}{3}}(k+\kappa)^{\frac{4}{3}}}\left\{(k^3+2k^2\kappa+2k\kappa^2-4\kappa^3)\right.$$
$$\left. + \frac{k^4-2k^3\kappa+894k^2\kappa^2-2844k\kappa^3+1984\kappa^4}{24(k+\kappa)^2}\ldots\right\}$$

$$\bar{C}_6 = \frac{1}{240\kappa^4(k+\kappa)^3}(k^5-5k^3\kappa^2+46k^2\kappa^3-298k\kappa^4+252\kappa^5)\ldots$$

$$\bar{C}_7 = \frac{\beta}{33{,}600\kappa^{14/3}(k+\kappa)^{11/3}}(2391k^6 + 7404k^5\kappa + 5848k^4\kappa^2 - 4752k^3\kappa^3$$
$$+ 305{,}756k^2\kappa^4 - 971{,}264k\kappa^5 + 653{,}768\kappa^6)\ldots$$

Regrouping in inverse powers of κ and introducing the simplifying factorial multiplier,

$$W_{km}{}^{(1)} = \frac{(-2/3)!}{2\pi}\left\{[(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!\,]^{\frac{1}{2}}\left(\frac{\kappa^5}{48(k+\kappa)^4}\right)^{\frac{1}{6}}\right\}$$
$$\times\{-\bar{\mathbf{C}}_1 + \bar{\mathbf{C}}_3 - \bar{\mathbf{C}}_7 + \bar{\mathbf{C}}_9\ldots\}$$
$$= \left\{\qquad \text{ditto} \qquad\right\} \times \left\{-\frac{6\beta(k+\kappa)^{\frac{1}{3}}}{\kappa^{\frac{2}{3}}} - \frac{k-3\kappa}{10\kappa^2}\right.$$
$$\left. - \frac{\beta(277k^3 + 7k^2\kappa - 296k\kappa^2 - 4\kappa^3)}{4200\kappa^{14/3}(k+\kappa)^{\frac{2}{3}}}\ldots\right\}, \tag{148}$$

where zero $\bar{\mathbf{C}}$'s have been omitted.

Late terms in $W_{km}^{(1)}(2k+2\kappa)$. The argument is the same as for W except for replacement by barred C's and $\mathbf{C}$'s. When r is even $\bar{\mathbf{C}}_r = 0$, while for odd r

$$\bar{\mathbf{C}}_r = \frac{12\beta(-1)^{\frac{1}{2}(r+1)}}{\sqrt{3}\,\pi^{\frac{2}{3}}(4\pi k)^{\frac{1}{3}r}}\left(\frac{k(k+\kappa)}{2\kappa^2}\right)^{\frac{1}{3}}\left\{(\tfrac{1}{3}r - \tfrac{4}{3})! - \frac{(4\pi k)^{\frac{2}{3}}(k-3\kappa)}{60\beta\,\kappa^{\frac{4}{3}}(k+\kappa)^{\frac{1}{3}}}(\tfrac{1}{3}r - 2)!\right.$$
$$\left. - \frac{\pi^2k^2(277k^3 + 7k^2\kappa - 296k\kappa^2 - 4\kappa^3)}{1575\kappa^4(k+\kappa)}(\tfrac{1}{3}r - \tfrac{10}{3})!\ldots\right\}. \tag{149}$$

EXERCISES

1. By successive integration by parts derive the following expansion for large integer n:

$$\int_{m\pi}^{\infty}\frac{\cos nu}{u^2+1}\,du = -(-1)^{mn}\sum_{r=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{1}{2}(r-1)}}{n^{r+1}}\left(\frac{d}{du}\right)^r\frac{1}{u^2+1}\bigg|_{m\pi},$$

where $m = 1, 2, 3, \ldots$. Expressing the derivatives through the Schlömilch formula, verify the expansion for

$$I(n) = \int_0^{m\pi}\frac{\cos nu}{u^2+1}\,du$$

quoted in I (27).

2. As we shall find in Chapter XXI, the most important range over which basic terminants have to be evaluated coincides with the most difficult, namely where order and argument are comparable. Starting from the representation

$$\Lambda_{s-1}(z) = z \int_0^\infty (1+u)^{-s} e^{-uz}\, du = z^s e^z [-s, z]!\,,$$

deduce the asymptotic power series

$$\Lambda_{s-1}(s) = \tfrac{1}{2}\left(1 + \frac{1}{4s} - \frac{1}{16s^2} - \frac{1}{64s^3} + \frac{13}{256s^4} - \frac{47}{1024s^5} - \frac{73}{4096s^6} + \frac{2447}{16{,}384s^7} \cdots\right),$$

and the formula for late terms within the brackets:

$$L_r = -\frac{2\,(r-\frac{1}{2})!}{(\frac{1}{2}\pi\Xi)^{\frac{1}{2}}\,(s\Xi)^r}\left\{\sin\,(r+\tfrac{1}{2})\theta + \frac{\Xi}{12(r-\frac{1}{2})}\sin\,(r-\tfrac{1}{2})\theta + \frac{\Xi^2}{288(r-\frac{1}{2})(r-\frac{3}{2})}\sin\,(r-\tfrac{3}{2})\theta \ldots\right\}$$

where

$$\Xi = (\pi^2+4)^{\frac{1}{2}}, \qquad \theta = \tan^{-1}\tfrac{1}{2}\pi.$$

3. Starting from the representation

$$\overline{\Lambda}_{s-1}(-z) = z\,P\int_0^\infty (1-u)^{-s} e^{-uz}\, du = P(-z)^s e^{-z}[-s, -z]!,$$

deduce the asymptotic power series

$$\overline{\Lambda}_{s-1}(-s) = \tfrac{2}{3} + \frac{4}{135s} + \frac{8}{2835s^2} - \frac{16}{8505s^3} - \frac{8{,}992}{12{,}629{,}925s^4} + \frac{334{,}144}{492{,}567{,}075s^5} + \frac{698{,}752}{1{,}477{,}701{,}225s^6} - \ldots,\dagger$$

† The three last terms are taken from Bowen (1961) Appendix I, Table 10, where the δ_n' in $\overline{\Lambda}_{s-1}(-s) = \Sigma\,(-1)^n\,\delta_n'/s^n$ are quoted up to $n = 9$.

and the formulae for late terms:

$$Q_r = \begin{cases} \dfrac{(-1)^{\frac{1}{2}r+1}(r-\frac{1}{2})!}{2^{\frac{1}{2}}\pi\,(2\pi s)^r}\left\{\zeta(r+\tfrac{3}{2}) - \dfrac{\pi\,\zeta(r+\frac{1}{2})}{6(r-\frac{1}{2})} - \dfrac{\pi^2\,\zeta(r-\frac{1}{2})}{72(r-\frac{1}{2})(r-\frac{3}{2})}\ \cdots\right\}, \\ \hfill r = 0, 2, 4, \ldots \\ \dfrac{(-1)^{\frac{1}{2}(r-1)}(r-\frac{1}{2})!}{2^{\frac{1}{2}}\pi\,(2\pi s)^r}\left\{\zeta(r+\tfrac{3}{2}) + \dfrac{\pi\,\zeta(r+\frac{1}{2})}{6\,(r-\frac{1}{2})} - \dfrac{\pi^2\,\zeta(r-\frac{1}{2})}{72(r-\frac{1}{2})(r-\frac{3}{2})}\ \cdots\right\}, \\ \hfill r = 1, 3, 5, \ldots. \end{cases}$$

4. The Stirling numbers of the second kind (e.g. Jordan 1960, p.168 onwards) can be expressed as

$$C_n^m = \frac{\Delta^m O^n}{m!} = \frac{n!}{m!} \times \text{coefficient of } u^n \text{ in } (e^u - 1)^m = \frac{1}{2\pi i}\frac{n!}{m!}\oint \frac{(e^u-1)^m\,du}{u^{n+1}}\ .$$

Show that the stationary point u_0 of the integrand is determined by

$$\frac{u_0}{1 - e^{-u_0}} = \frac{n+1}{m} \qquad (>1),$$

a transcendental equation admirably adapted to iterative solution, and that the derivatives at this point are

$$F_2 = -\,m(1+\Gamma)(1/u_0 - \Gamma), \qquad F_3 = m(1+\Gamma)(2/u_0{}^2 - 2\Gamma^2 - \Gamma),$$

$$F_4 = -\,m(1+\Gamma)(6/u_0{}^3 - 6\Gamma^3 - 6\Gamma^2 - \Gamma),$$

$$F_r = (-1)^{r-1}\,m(1+\Gamma)\left\{\frac{(r-1)!}{u_0{}^{r-1}} - \sum_{s=0}^{r-2}(s+1)!\,C_{r-1}^{s+1}\,\Gamma^{s+1}\right\},$$

where

$$\Gamma = \frac{1}{e^{u_0}-1} = \frac{n+1}{m\,u_0} - 1\,.$$

Hence derive the expansion

$$C_n^m = \frac{n!}{m!\,(-2\pi F_2)^{\frac{1}{2}}\,u_0{}^{n+1}\,\Gamma^m}\,[1 + Q_2 + \ldots]\,.$$

5. The Stirling numbers of the first kind (e.g. Jordan 1960, p. 142 onwards) can be expressed as

$$S_n^m = \frac{n!}{m!} \times \text{coefficient of } u^n \text{ in } [\ln(1+u)]^m = \frac{1}{2\pi i}\frac{(n-1)!}{(m-1)!}\oint \frac{u^{m-1}\,du}{(e^u-1)^n}\,.$$

Show that the stationary point $u = -u_0$ of the integrand is determined by

$$\frac{u_0}{e^{u_0} - 1} = \frac{m-1}{n} \quad (<1),$$

and that the derivatives at this point are

$$F_2 = -n(\Gamma - 1)(\Gamma - 1/u_0), \qquad F_3 = -n(\Gamma - 1)(2\Gamma^2 - \Gamma - 2/u_0^2),$$

$$F_4 = -n(\Gamma - 1)(6\Gamma^3 - 6\Gamma^2 + \Gamma - 6/u_0^3),$$

$$F_r = -n(\Gamma - 1)\left\{(-1)^r \sum_{s=0}^{r-2} (-1)^s (s+1)!\, C_{r-1}^{s+1}\, \Gamma^{s+1} - \frac{(r-1)!}{u_0^{\,r-1}}\right\},$$

where

$$\Gamma = \frac{1}{1 - e^{-u_0}} = \frac{m-1}{n\,u_0} + 1.$$

Hence derive the expansion

$$S_n^{\,m} = \frac{(n-1)!\,(-1)^{m+n} u_0^{\,m-1}\, \Gamma^n}{(m-1)!\,(-2\pi F_2)^{\frac{1}{2}}}\,[1 + Q_2 + \ldots].$$

6. Show how an alternative expansion for Bessel functions when $x \sim p$ could be based on Bessel's addition theorems, e.g.

$$J_p(x) = \left(\frac{x}{p}\right)^p \left\{J_p(p) - \frac{1}{1!}\left(\frac{x^2 - p^2}{2p}\right) J_{p+1}(p) + \frac{1}{2!}\left(\frac{x^2 - p^2}{2p}\right)^2 J_{p+2}(p) - \ldots\right\},$$

coupled with a reduction formula expressing these coefficients in terms of J_p and J_{p-1}.

7. Derive the expansion

$$J_{p-1}(p) = -\frac{i}{2\pi}\int_{\infty - i\pi}^{\infty + i\pi} e^{-p(\omega - \sinh\omega)}\, e^{\omega}\, d\omega = \frac{(-2/3)!\, 6^{\frac{1}{3}}}{2\pi\sqrt{3}\, p^{\frac{1}{3}}}$$

$$\times \{C_0 - C_1 + C_3 - C_4 + C_6 - C_7 \ldots\}$$

$$= \frac{(-2/3)!\, 6^{\frac{1}{3}}}{2\pi\sqrt{3}\, p^{\frac{1}{3}}}\left\{1 + \frac{6\beta}{p^{\frac{1}{3}}} - \frac{1}{5p} - \frac{3\beta}{35 p^{\frac{4}{3}}} - \frac{1}{225 p^2} + \frac{23\beta}{525 p^{\frac{7}{3}}} \cdots\right\},$$

and verify its equality with $J_p + J_p^{(1)}$.

8. Write down the corresponding representation for $Y_{p-1}(p)$ and show how it leads to the expansion

$$Y_{p-1}(p) = -\frac{(-2/3)!\,6^{\frac{1}{3}}}{2\pi\, p^{\frac{1}{3}}}\left\{C_0 + C_1 + C_3 + C_4 + C_6 + C_7 \ldots\right\}.$$

Verify equality with $Y_p + Y_p^{(1)}$.

9. In assessing convergence properties of the Taylor series for $J_p(x)$ in powers of $(x - p)$, an estimate is needed of high derivatives $J_p^{(\sigma)}(p)$. Establish the appropriate integral representation

$$J_p^{(\sigma)} = -\frac{i}{2\pi}\int_{\infty - i\pi}^{\infty + i\pi} e^{-F}\,\omega^\sigma\, G\, d\omega, \qquad F = p(\omega - \sinh\omega), \qquad G = \left(\frac{\sinh\omega}{\omega}\right)^\sigma,$$

and show from VI Q.2 that leading terms are

$$J_p^{(\sigma)} \sim \frac{(\frac{1}{3}\sigma - \frac{2}{3})!}{2\pi\sqrt{3}}\left(\frac{6}{p}\right)^{\frac{1}{3}\sigma + \frac{1}{3}}\left.\begin{pmatrix}(-1)^\sigma \\ (-1)^{\sigma-1} \\ 0\end{pmatrix}\qquad \begin{matrix}\sigma = 0, 3, 6, \ldots \\ \sigma = 1, 4, 7, \ldots \\ \sigma = 2, 5, 8, \ldots\end{matrix}\right\}.$$

Inserting these derivatives in the Taylor series, deduce the approximation (58) covering dominant late terms.

10. Starting from the representation

$$K_p(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{p\omega - x\cosh\omega}\, d\omega, \qquad |\operatorname{ph} x| < \tfrac{1}{2}\pi,$$

for the modified Bessel function, derive the expansion

$$K_p(x) = \left(\frac{\pi q}{2p}\right)^{\frac{1}{2}} e^{-p(q^{-1} - \tanh^{-1} q)}\left\{1 + \frac{q}{24p}(5q^2 - 3) + \frac{q^2}{1152p^2}\right.$$

$$\left. \times\,(385q^4 - 462q^2 + 81)\ldots\right\},$$

where $q = p/(p^2 + x^2)^{\frac{1}{2}}$.

Comparing with results in the text for $J_p(x)$, write down the formula for late terms when $q > 1$, and deduce the following modification appropriate to the more usual case $q < 1$:-

$$Q_{2r} = \frac{(r-1)!}{\pi(2p\Xi)^r}\left\{\cos r\theta + \frac{\Xi q(5q^2-3)}{12(r-1)}\cos\,(r-1)\theta\right.$$
$$\left. + \frac{\Xi^2 q^2(385q^4 - 462q^2 + 81)}{288(r-1)(r-2)}\cos\,(r-2)\theta \dots\right\},$$

where

$$\Xi = [(q^{-1} - \tanh^{-1}q)^2 + \tfrac{1}{4}\pi^2]^{\frac{1}{2}}, \quad \theta = \tan^{-1}\frac{\pi}{2(q^{-1} - \tanh^{-1}q)}.$$

(The root is to be taken with the same sign as q). Verify that for $q = \frac{1}{2}$ this gives $Q_8 = -5.4054 \times 10^{-3}/p^4$ compared with the actual value $-5{\cdot}4011.. \times 10^{-3}/p^4$.

11. Derive the following asymptotic expansion for the range $x > p > 0$:

$$A_p(x) = \frac{\pi}{\sin \pi p}(J_{-p}(x) - \mathbf{J}_{-p}(x)) = \int_0^\infty e^{p\omega - x\sinh\omega}\,d\omega$$
$$= \frac{1}{x-p}\left\{1 - \frac{x}{(x-p)^3} + \frac{x(p+9x)}{(x-p)^6}\right.$$
$$\left. - \frac{x(p^2 + 54px + 225x^2)}{(x-p)^9}\dots\right\}.$$

($\mathbf{J}$ is the Anger function). Show that $F'(\omega) = 0$ when $\omega = \pm i\cos^{-1}p/x = \pm i\tan^{-1}\varphi^{-1}$, where $\varphi = p/(x^2 - p^2)^{\frac{1}{2}}$, leading to the singulant pair

$$\mathscr{F}_0 = \pm ip\Upsilon, \qquad \Upsilon = \varphi^{-1} - \tan^{-1}\varphi^{-1},$$

and thence to the late-term formulae $L_{2r+1} = 0$ and

$$L_{2r} = \frac{x-p}{p}\left(\frac{2\varphi}{\pi\Upsilon}\right)^{\frac{1}{2}}\frac{(-1)^r(2r-\frac{1}{2})!}{(p\Upsilon)^{2r}}\left\{1 - \frac{\Upsilon\varphi(5\varphi^2+3)}{24(2r-\frac{1}{2})}\right.$$
$$\left. + \frac{\Upsilon^2\varphi^2(385\varphi^4 + 462\varphi^2 + 81)}{1152(2r-\frac{1}{2})(2r-\frac{3}{2})}\dots\right\}.$$

12. Derive the expansions

$$A_p(p) = \frac{\alpha}{p^{\frac{1}{3}}}\left(1 - \frac{1}{10\alpha p^{\frac{2}{3}}} + \frac{3\beta}{35 p^{\frac{4}{3}}} - \frac{1}{225p^2} \cdots\right),$$

$$A_p{}^{(1)}(p) = \frac{\alpha}{p^{\frac{1}{3}}}\left(- \frac{6\beta}{p^{\frac{1}{3}}} - \frac{1}{5p} + \frac{27}{700\alpha p^{\frac{5}{3}}} \cdots\right),$$

and relate the terms to those in $J_p(p)$ and $J_p{}^{(1)}(p)$.

13. Derive the following asymptotic expansion for the Struve function $\mathbf{H}_p(x)$:

$$\mathbf{H}_p(x) - Y_p(x) = \frac{(\frac{1}{2}x)^{p-1}}{\sqrt{\pi}\,(p-\frac{1}{2})!}\int_0^\infty e^{-u}\left(1 + \frac{u^2}{x^2}\right)^{p-\frac{1}{2}} du$$

$$= \frac{(\frac{1}{2}x)^{p-1}}{\sqrt{\pi}\,(p-\frac{1}{2})!}\left(1 + \frac{2p}{x^2} + \frac{12p^2 - x^2}{x^4} + \frac{24p(5p^2 - x^2)}{x^6} \cdots\right).$$

(There is no point in finding a formula for late terms, because the series can be re-ordered to read

$$1 - \frac{1-2p}{x^2} + \frac{(1-2p)\,3(3-2p)}{x^4} - \frac{(1-2p)\,3(3-2p)\,5(5-2p)}{x^6} \cdots,$$

a transposition in which the general term can be written down exactly).

14. Show that when C is negative, the formula (83) for late terms in $F(a, c, x)$ and $\psi(a, c, x)$ is replaced by

$$L_r{}^{(a-1)} = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{(r + a - \frac{3}{2})!}{(a-1)!\,(1-q)^{2a-1}(-C)^r|\Xi|^{r+a-\frac{1}{2}}}$$
$$\times\ [\sin\{(r + a - \tfrac{1}{2})\theta + 2a\pi\} + \ldots],$$

where

$$|\Xi| = \left[\left\{\frac{1}{q} - \ln\left(\frac{q}{1-q}\right)\right\}^2 + \pi^2\right]^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left[\pi\Big/\left\{\frac{1}{q} - \ln\left(\frac{q}{1-q}\right)\right\}\right].$$

15. Following procedures analogous to those in §2 and §3 for $p!$ and $(p, p)!$, find formulae for late terms in the expansions of $F(a, c, C)$ and $F^{(1)}(a, c, C)$.

16. Specializing results derived in the text, write down asymptotic expansions for the parabolic cylinder function

$$D_p(x) = 2^{\frac{1}{2}p+\frac{1}{4}}x^{-\frac{1}{2}}\, W_{\frac{1}{2}p+\frac{1}{4},-\frac{1}{4}}\,(\tfrac{1}{2}x^2)$$

in the three ranges $x > (4p + 2)^{\frac{1}{2}}$, $\quad 0 < x < (4p + 2)^{\frac{1}{2}}$, $\quad x \sim (4p + 2)^{\frac{1}{2}}$.

17. Starting from the representation

$$D_p(x) = -\,i\,(2\pi)^{-\frac{1}{2}}\,e^{\frac{1}{4}x^2}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{1}{2}u^2-ux}\,u^p\,du,$$

obtain the expansions

$$D_p(2\surd p) = \left(\frac{8}{3\pi^3}\right)^{\frac{1}{6}}\left(-\frac{2}{3}\right)!\,p^{\frac{1}{2}p+\frac{1}{6}}\,e^{-\frac{1}{2}p}\left\{1+\frac{3\beta}{(2p)^{\frac{1}{3}}}+\frac{1}{240p}\right.$$

$$\left.-\frac{67\beta}{560(2p)^{\frac{4}{3}}}\cdots\right\}$$

$$D_{p+1}(2\surd p) = \left(\frac{8}{3\pi^3}\right)^{\frac{1}{6}}\left(-\frac{2}{3}\right)!\,p^{\frac{1}{2}p+\frac{2}{3}}\,e^{-\frac{1}{2}p}\left\{1+\frac{9\beta}{(2p)^{\frac{1}{3}}}-\frac{17}{240p}\right.$$

$$\left.+\frac{213\beta}{560(2p)^{\frac{4}{3}}}\cdots\right\}.$$

18. In the theory of uniform expansions it is of interest to evaluate

$$r!\,\mathscr{T}_r(z) = \int_0^\infty e^{-v-v^2/2z^2}\,v^r\,dv$$

for large z and r (X §2). Derive the result

$$r!\,\mathscr{T}_r(z) \sim (2\pi)^{\frac{1}{2}}(1+4r/z^2)^{-\frac{1}{4}}\,[\tfrac{1}{2}z^2\{(1+4r/z^2)^{\frac{1}{2}}-1\}]^{r+\frac{1}{2}}\,e^{-\frac{1}{2}r}$$

$$\times\exp[-\tfrac{1}{4}z^2\{(1+4r/z^2)^{\frac{1}{2}}-1\}].$$

19. Starting from the representation

$$W_{-k,m}(x) = \frac{4x^{\frac{1}{2}}\,e^{-\frac{1}{2}x}}{(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!}\int_0^\infty e^{-v^2}\,v^{2k}\,K_{2m}(2v\surd x)\,dv,$$

$$\Re(k-m-\tfrac{1}{2})>0,$$

and introducing the parameters $\eta = 16m^2-1$ and $q = \{x/(x+4k)\}^{\frac{1}{2}}$, derive the expansion

$$W_{-k,m}(x) = \frac{2\pi\,k^k\,e^{-k}\,q^{\frac{1}{2}}}{(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!}\,e^{-2k\Pi}\left[1+\frac{1}{96k}\{5q^3-6q+3\eta\right.$$

$$-1+3\eta/q\}+\frac{1}{18{,}432k^2}\{385q^6-924q^4+10(3\eta-1)q^3$$

$$-6(7\eta-114)q^2-12(3\eta-1)q+(9\eta^2+102\eta-143)$$

$$\left.+6\eta(3\eta-1)/q+9\eta(\eta-8)/q^2\}\ldots\right],$$

where

$$\Pi = q/(1 - q^2) + \tanh^{-1} q.$$

Derive also the following formula for late terms:

$$Q_{2r} = \frac{(r-1)!}{\pi\,(-4k\Xi)^r}\left[\cos r\theta + \frac{\Xi\{5q^3 - 6q - (3\eta - 1) + 3\eta/q\}}{24(r-1)}\cos(r-1)\theta \ldots\right],$$

where

$$\Xi = (\Pi^2 + \tfrac{1}{4}\pi^2)^{\frac{1}{2}}, \qquad \theta = \tan^{-1}(\pi/2\Pi),$$

and verify that when $\eta = 0$, $q = \frac{1}{2}$, this gives $Q_8 = -0.0^3 41201/k^4$ as compared with the actual value $-0.0^3 41198\ldots/k^4$.

20. Multiplying the expansion of $W_{-k,m}(x)$ obtained in the preceding question by the series (106), deduce the more compact result

$$W_{-k,m}(x) = \frac{(2\pi q)^{\frac{1}{2}}}{[(k + m - \frac{1}{2})!\,(k - m - \frac{1}{2})!]^{\frac{1}{2}}} e^{-2k\Pi}\left[1 + \frac{1}{96k}\{5q^3 - 6q + 3\eta/q\}\right.$$

$$+ \frac{1}{18{,}432k^2}\{385q^6 - 924q^4 - 6(7\eta - 114)q^2 + 36(3\eta - 4)$$

$$\left. + 9\eta(\eta - 8)/q^2\} + \ldots\right],$$

with late terms

$$Q_{2r} = \frac{(r-1)!}{\pi\,(-4k\Xi)^r}\left[\cos r\theta + \frac{\Xi\{5q^3 - 6q + 3\eta/q\}}{24(r-1)}\cos(r-1)\theta \ldots\right].$$

21. Starting from the integral representation quoted in Q.19, derive the expansion

$$W_{-k,m}(x) = \frac{\pi\sqrt{2}(k + m)^{\frac{1}{2}(k+m)}(k - m)^{\frac{1}{2}(k-m)}e^{-k}}{(k + m - \frac{1}{2})!\,(k - m - \frac{1}{2})!}\left(\frac{(k+\kappa)q^2 - k + \kappa}{\kappa q}\right)^{\frac{1}{2}} e^{-2\kappa\Pi}$$

$$\times \left[1 + \frac{1}{192\kappa^2}\{5(k + \kappa)q^3 - 3(3k + \kappa)q - 8k - 3(3k - \kappa)/q\right.$$

$$\left. + 5(k - \kappa)/q^3\}\ldots\right],$$

where $\kappa = (k^2 - m^2)^{\frac{1}{2}}$, $q = [(x + 2k - 2\kappa)/(x + 2k + 2\kappa)]^{\frac{1}{2}}$ and

$$\Pi = \frac{q}{1 - q^2} + \left(\frac{k}{\kappa}\right) \tanh^{-1} q + \left(\frac{m}{\kappa}\right) \tanh^{-1} \frac{1}{q} \left(\frac{k - \kappa}{k + \kappa}\right)^{\frac{1}{2}},$$

with positive signs allotted to k/κ and m/κ.

Derive also the following formula for late terms:

$$Q_{2r} = \frac{(r-1)!}{\pi\,(-4\kappa\Xi)^r} \left[\cos r\theta + \frac{\Xi\{5(k+\kappa)q^3 - 3(3k+\kappa)q + 8k - 3(3k-\kappa)/q + 5(k-\kappa)/q^3\}}{48(r-1)\kappa} \cos(r-1)\theta \ldots \right],$$

where

$$\Xi = (\Pi^2 + \tfrac{1}{4}\pi^2 k^2/\kappa^2)^{\frac{1}{2}}, \qquad \theta = \tan^{-1}(\pi k/2\kappa\Pi).$$

22. Multiplying the expansion of $W_{-k,m}(x)$ obtained in the preceding question by the series (133), deduce the more compact result

$$W_{-k,m}(x) = \frac{\sqrt{\pi}}{[(k + m - \frac{1}{2})!\,(k - m - \frac{1}{2})!]^{\frac{1}{2}}} \left(\frac{(k+\kappa)q^2 - k + \kappa}{\kappa q}\right)^{\frac{1}{2}} e^{-2\kappa\Pi}$$
$$\times \left[1 + \frac{1}{192\kappa^2}\{5(k+\kappa)q^3 - 3(3k+\kappa)q - 3(3k-\kappa)/q + 5(k-\kappa)/q^3\} \ldots \right],$$

with late terms

$$Q_{2r} = \frac{(r-1)!}{\pi\,(-4\kappa\Xi)^r} \left[\cos r\theta + \frac{\Xi\{5(k+\kappa)q^3 - 3(3k+\kappa)q - 3(3k-\kappa)/q + 5(k-\kappa)/q^3\}}{48(r-1)\kappa} \cos(r-1)\theta \ldots \right].$$

23. Adopting the reversion method (V §2), derive the expansion

$$\int_0^\infty e^{-xu(1+u)^{\frac{1}{3}}}\,du = x^{-1}\sum_0^\infty L_r, \qquad L_r = (\tfrac{4}{3}r - \tfrac{2}{3})!\,/(\tfrac{1}{3}r - \tfrac{2}{3})!\,(-x)^r.$$

Show that the stationary point of the exponent lies at $u = -3/4$, where the derivatives are

$$\mathscr{F}_j = 4^{j-\frac{1}{3}}(j-1)(\tfrac{1}{3})!\,x/(\tfrac{4}{3}-j)!\,.$$

Hence find the following alternative formula for late terms (which for calculating terminants proves far more convenient than the exact result):

$$L_r = .31664071\left(-\frac{2{\cdot}1165348}{x}\right)^r(r-\tfrac{1}{2})!\left\{1 + \frac{{\cdot}1041\dot{6}}{r-\frac{1}{2}} + \frac{{\cdot}00542535}{(r-\frac{1}{2})(r-\frac{3}{2})}\right.$$

$$\left. - \frac{{\cdot}0692561}{(r-\frac{1}{2})(r-\frac{3}{2})(r-\frac{5}{2})} + \frac{{\cdot}1956791}{(r-\frac{1}{2})(r-\frac{3}{2})(r-\frac{5}{2})(r-\frac{7}{2})}\cdots\right\}.$$

Verify that this gives L_8 with an error ·0015%.

24. Derive the series

$$\int_{-\infty}^\infty e^{-xu(1+u)^{\frac{1}{3}}}\,du = \frac{1}{4^{\frac{1}{3}}}\left(\frac{3\pi}{2x}\right)^{\frac{1}{2}} e^{3x/4^{\frac{4}{3}}}\left(1 - \frac{5}{9.4^{\frac{2}{3}}x} + \frac{25.4^{\frac{2}{3}}}{2592x^2} + \frac{45{,}955}{69{,}984x^3}\right.$$

$$\left. + \frac{24{,}929{,}905\,.4^{\frac{1}{3}}}{10{,}077{,}696x^4}\cdots\right).$$

Show that the non-principal readings of the stationary point are $u = -1 + \frac{1}{4}e^{\pm 2\pi i}$, and the consequent singulant pair $\mathscr{F}_0 = -F_0\sqrt{3}\,e^{\mp\frac{1}{6}\pi i}$. Noting the relation $\mathscr{Q}_j = Q_j\,e^{\mp\frac{1}{3}\pi i j}$, find the following formula for late terms:

$$Q_{2r} = -{\cdot}31830989\left(\frac{1{\cdot}22198194}{x}\right)^r(r-1)!\left\{\cos\tfrac{1}{6}\pi(r+1) + \frac{{\cdot}18042196}{r-1}\right.$$

$$\times\cos\tfrac{1}{6}\pi(r+2) + \frac{{\cdot}01627604}{(r-1)(r-2)}\cos\tfrac{1}{6}\pi(r+3) - \frac{{\cdot}3598651}{(r-1)(r-2)(r-3)}$$

$$\left.\times\cos\tfrac{1}{6}\pi(r+4) + \frac{1{\cdot}761112}{(r-1)(r-2)(r-3)(r-4)}\cos\tfrac{1}{6}\pi(r+5)\cdots\right\}.$$

25. The integral

$$I(x) = \int_0^\infty e^{-xu(1+u)^{\frac{1}{3}}} \{(1+u)(1+\tfrac{1}{2}u)(1+\tfrac{1}{3}u)\}^{-\frac{1}{2}}\, du$$

is used by van der Corput (1954, pages 22 onwards) to illustrate upper bounds. Show that the derivatives of $F(u) = x\,u(1+u)^{\frac{1}{3}}$ at the lower limit are $F_j = j(\frac{1}{3})!\,x/(\frac{4}{3} - j)!$, while the $G_j = j!\,g_j$ can most easily be found from the recurrence relation (§1)

$$12(j+1)\,g_{j+1} + 11(2j+1)g_j + 12j\,g_{j-1} + (2j-1)g_{j-2} = 0.$$

Hence derive the asymptotic power series

$$x\,I(x) = 1 - \frac{19}{12x} + \frac{257}{48x^2} - \frac{49{,}789}{1{,}728x^3} + \frac{4{,}452{,}617}{20{,}736x^4} - \frac{56{,}691{,}755}{27{,}648x^5}$$

$$+ \frac{71{,}366{,}216{,}635}{2{,}985{,}984x^6} - \ldots .$$

Calculate the $\mathscr{G}_j = j!\,\mathscr{g}_j$ at the stationary point $u = -3/4$ from the recurrence relation

$$45(j+1)\mathscr{g}_{j+1} + 118(2j+1)\mathscr{g}_j + 240j\,\mathscr{g}_{j-1} + 32(2j-1)\mathscr{g}_{j-2} = 0,$$

and find the following formula for late terms:

$$L_r = .92496673\left(-\frac{2{\cdot}1165348}{x}\right)^r (r-\tfrac{1}{2})!\left\{1 + \frac{{\cdot}01208\dot{3}}{r-\frac{1}{2}} - \frac{{\cdot}12377792}{(r-\frac{1}{2})(r-\frac{3}{2})}\right.$$

$$\left. + \frac{{\cdot}1488314}{(r-\frac{1}{2})(r-\frac{3}{2})(r-\frac{5}{2})}\ldots\right\}.$$

Verify that this gives L_6 with an error ·051 %.

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Chapter IX

Derivation of Asymptotic Expansions from Integral Representations of the Form $\iint_{\text{s.p.}} e^{-F(u,v)}\, G(u,v)\, du\, dv.$

1. Origin of Double Integral Representations

In Chapter III it was shown how a convergent series $S(x) = \Sigma a_s x^s$ could be converted into an integral representation containing in its integrand some more familiar series $S^0(x) = \Sigma\, a_s{}^0 x^s$. The essence of the method consisted in representing $a_s/a_s{}^0$ as an integral operator $U(u)$ acting upon u^s. There are two proximate causes why such a procedure might fail to produce an integral representation $\int e^{-F}\, G\, du$ fully amenable to the treatments of Chapters V–VIII:

(i) It might not prove possible to find a representation of $a_s/a_s{}^0$ as an integral operator $U(u)$ of a single variable u. The resultant representation for $S(x)$ would then be at least a double integral.

(ii) Even though $S(x)$ might successfully be converted into an integral representation involving only a single integration over the "familiar function" S^0, it might transpire that the subsequent locating of a stationary point required solution of a *transcendental* equation, a situation indeed already found for large orders in the Fermi–Dirac integral and Stirling numbers, (Chapter VIII, Section 4 and Questions 4 and 5). Since such a transcendental equation would have originated essentially from the complexity of the derivative of $\ln S^0$, i.e. the quotient $(S^0)'/S^0$, one available cure would be to re-express the over-complicated familiar function S^0 as an integral representation admitting only elementary functions. As in (i), the resultant integral representation for $S(x)$ would then be at least a double integral.

Evidently, therefore, some attention needs to be paid to the derivation of asymptotic expansions from integral representations of the form $\iint e^{-F(u,v)} G(u,v)\, du\, dv$, above all those in which $F(u,v)$ exhibits a stationary point. To anticipate a little, it turns out that such derivation—though presenting no major difficulties over and above those disposed of in the preceding chapters—becomes in general intractably cumbersome after the first two surviving terms, on account of the overwhelmingly rapid increase in the number of partial derivatives entering successive correction terms. Thus instead of continuing to expend labour on a derivation from a double integral, it is in practice more profitable to seek ways of avoiding such a double integration. For instance, its appearance by mechanism (i) above might be obviated by choosing a different familiar series for which a_s/a_s^0 *can* be represented as an integral operator in one variable only; and in (ii) the transcendental equation might be evaded by retaining in e^{-F} only the fastest-varying factors of S^0, so that only these influence the location of the stationary point, and correspondingly transferring the remainder of S^0 to the slowly-varying weighting factor G.

For these reasons our treatment of double integrals will be cursory and restricted to the most important case, that of deriving a general formula for the leading and first correction terms where $F(u, v)$ is quadratic in u and v in the immediate neighbourhood of a stationary point and tends to infinity at the integration limits of u and v.

2. Quadratic Dependence at a Stationary Point

For brevity of notation in the derivation we shall suppose the variables of integration to have been chosen such that the stationary point $\partial F/\partial u = \partial F/\partial v = 0$ lies at $u = v = 0$. Then by Taylor's theorem for two independent variables we are entitled to assume in this neighbourhood:

$$\begin{aligned} F(u,v) = F_{00} &+ \tfrac{1}{2}(u^2F_{20} + 2uvF_{11} + v^2F_{02}) \\ &+ \tfrac{1}{6}(u^3F_{30} + 3u^2vF_{21} + 3uv^2F_{12} + v^3F_{03}) \\ &+ \tfrac{1}{24}(u^4F_{40} + 4u^3vF_{31} + 6u^2v^2F_{22} + 4uv^3F_{13} + v^4F_{04}) + \dots \end{aligned} \tag{1}$$

where

$$F_{ij} = \left(\frac{\partial}{\partial u}\right)^i \left(\frac{\partial}{\partial v}\right)^j F(u, v)\bigg|_{u=v=0}. \tag{2}$$

Of the terms in e^{-F}, apart from the constant $e^{-F_{00}}$ only the quadratic group in (1) needs to be kept in exponential form in the subsequent integration. The calculation can therefore be simplified by changing the variables such that this quadratic group reduces to two perfect squares. The least troublesome substitution here is

$$u = u, \qquad v = w - uF_{11}/F_{02}, \tag{3}$$

leading to

$$\tfrac{1}{2}(u^2F_{20} + 2uvF_{11} + v^2F_{02}) = \tfrac{1}{2}u^2(F_{20} - F_{11}{}^2/F_{02}) + \tfrac{1}{2}w^2F_{02}. \tag{4}$$

The Jacobian is unity, i.e. $du\,dv$ can be replaced directly by $du\,dw$.

To attain an accuracy equivalent to the terms retained in (1), the slowly-varying factor $G(u,v)$ need be expanded by Taylor's theorem for two variables only up to second derivatives; the notation

$$G_{ij} = \left(\frac{\partial}{\partial u}\right)^i \left(\frac{\partial}{\partial v}\right)^j G(u, v)\bigg|_{u=v=0} \tag{5}$$

is naturally adopted in analogy with (2). Changing to the variables u and w, multiplying the expansions of e^{-F} and G (discarding in the product all terms odd in either u or w because these will vanish on integration), and then integrating successively over u and w each from $-\infty$ to ∞, a tedious calculation punctuated by extensive cancellation of laboriously computed terms ultimately leads to the following result:

$$\iint_{\to} e^{-F}\, G\, du\, dv = \frac{2\pi\, e^{-F_{00}}}{(F_{02}F_{20} - F_{11}{}^2)^{\frac{1}{2}}}(Q_0 + Q_2 + Q_4 + \dots), \tag{6}$$

where

$$Q_0 = G_{00},$$

$$\begin{aligned}
Q_2 = \frac{1}{24(F_{02}F_{20}-F_{11}{}^2)^3}\Big[\,G_{00}\Big\{&F_{02}{}^3(5F_{30}{}^2 - 3F_{20}F_{40}) + F_{20}{}^3(5F_{03}{}^2\\
&- 3F_{02}F_{04}) + 3F_{02}F_{20}(F_{02}[2F_{12}F_{30} + 3F_{21}{}^2] + F_{20}[2F_{21}F_{03}+3F_{12}{}^2]\\
&- 2F_{02}F_{20}F_{22}) - 6F_{11}(F_{02}{}^2[5F_{21}F_{30} - 2F_{20}F_{31}] + F_{20}{}^2[5F_{12}F_{03}\\
&- 2F_{02}F_{13}] + F_{02}F_{20}[9F_{12}F_{21} + F_{03}F_{30}]) + 3F_{11}{}^2(F_{02}[8F_{12}F_{30}\\
&+ 12F_{21}{}^2 + F_{02}F_{40}] + F_{20}[8F_{21}F_{03} + 12F_{12}{}^2 + F_{20}F_{04}]\\
&- 2F_{02}F_{20}F_{22}) - 4F_{11}{}^3(3F_{02}F_{31} + 3F_{20}F_{13} + 9F_{12}F_{21} + F_{03}F_{30})\\
&+ 12F_{11}{}^4F_{22}\Big\} - 12(F_{02}F_{20}-F_{11}{}^2)\Big\{G_{10}(F_{02}[F_{02}F_{30} + F_{20}F_{12}
\end{aligned}$$

$$- 3F_{11}F_{21}] + F_{11}[2F_{11}F_{12} - F_{20}F_{03}]) + G_{01}(F_{20}[F_{20}F_{03} + F_{02}F_{21}$$
$$- 3F_{11}F_{12}] + F_{11}[2F_{11}F_{21} - F_{02}F_{30}])\Big\} + 12(F_{02}F_{20} - F_{11}{}^2)^2$$
$$\times \Big\{G_{02}F_{20} + G_{20}F_{02} - 2G_{11}F_{11}\Big\}\Big].$$

3. Asymptotic Evaluation of Q_r.

An analogous argument to that of Chapter VII Section 4 leads to the late-term contribution per singular point

$$Q_r = \frac{1}{2\pi}\left(\frac{F_{02}F_{20} - F_{11}{}^2}{-(\mathscr{F}_{02}\mathscr{F}_{20} - \mathscr{F}_{11}{}^2)}\right)^{\frac{1}{2}} \frac{(\frac{1}{2}r - 1)!}{\mathscr{F}_{00}{}^{\frac{1}{2}r}} \sum_{s=0} (\tfrac{1}{2}r - s - 1)!\, \mathscr{Q}_{2s}\mathscr{F}_{00}{}^s, \quad (7)$$

where the singulant $\mathscr{F}_{00}$ represents the change in value of F in going *either* from its value at the stationary point determining an original asymptotic expansion, to a neighbouring stationary point; *or* from its principal value at the stationary point determining an original asymptotic expansion, to a conjugate pair of non-principal values: and the $\mathscr{Q}$ differ from the Q only through involving derivatives at the new stationary point(s).

4. Whittaker Function for large $\kappa = (k^2 - m^2)^{\frac{1}{2}}$

In the integral representation VIII (98), i.e.

$$W_{km}(x) = -\frac{2i\, x^{\frac{1}{2}}\, e^{\frac{1}{2}x}}{\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{v^2}\, v^{2k}\, K_{2m}(2v\sqrt{x})\, dv, \quad (8)$$

let us introduce the following representation for the modified Bessel function, essentially III (36):

$$K_p(z) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-z\cosh u}\, e^{pu}\, du, \qquad \Re(z) > 0. \quad (9)$$

Then

$$W_{km}(x) = -\frac{i\, x^{\frac{1}{2}}\, e^{\frac{1}{2}x}}{\pi}\int_{u=-\infty}^{\infty}\int_{v=\gamma-i\infty}^{\gamma+i\infty} e^{-F(u,v)}\, du\, dv, \quad (10)$$

where

$$F(u, v) = -v^2 - 2k\ln v + 2v\, x^{\frac{1}{2}}\cosh u - 2mu.$$

As u runs along the real axis from $-\infty$ to ∞, F starts at ∞, has a stationary point when

$$0 = \partial F/\partial u = 2(vx^{\frac{1}{2}}\sinh u - m), \tag{11}$$

and finally ends at ∞. Moreover, as v runs parallel to the imaginary v-axis from $\gamma - i\infty$ to $\gamma + i\infty$, F starts at ∞, has stationary points when

$$0 = \partial F/\partial v = 2(-v - k/v + x^{\frac{1}{2}}\cosh u), \tag{12}$$

and finally ends at ∞. There are absolute stationary points—points stationary with respect to changes in both independent variables u and v—when (11) and (12) are simultaneously satisfied, i.e. at

$$u = \sinh^{-1}(m/v\sqrt{x}), \qquad v = \tfrac{1}{2}\{(x - 2k + 2\kappa)^{\frac{1}{2}} \pm (x - 2k - 2\kappa)^{\frac{1}{2}}\}$$

$$= [\kappa(q \pm 1)/(q \mp 1)]^{\frac{1}{2}} \tag{13}$$

where $q = [(x - 2k + 2\kappa)/(x - 2k - 2\kappa)]^{\frac{1}{2}}$ as in the last chapter. At these absolute stationary points,

$$F_{00} - \tfrac{1}{2}x = -\tfrac{1}{2}(k + m)\ln(k + m) - \tfrac{1}{2}(k - m)\ln(k - m) + k \pm \frac{2\kappa q}{q^2 - 1}$$

$$\mp 2k\coth^{-1}q \pm 2|m|\tanh^{-1}\frac{1}{q}\left(\frac{k - \kappa}{k + \kappa}\right)^{\frac{1}{2}},$$

$$F_{10} = F_{01} = F_{12} = F_{22} = F_{13} = 0,$$

$$F_{20} = F_{40} = \frac{2\{(k + \kappa)q \mp (k - \kappa)\}}{q \mp 1}, \qquad F_{02} = \frac{2\{(k - \kappa)q \mp (k + \kappa)\}}{\kappa(q \pm 1)},$$

$$F_{30} = -2m, \quad F_{03} = -4k\left(\frac{q \mp 1}{\kappa(q \pm 1)}\right)^{\frac{1}{2}}, \qquad F_{04} = \frac{12k}{\kappa^2}\left(\frac{q \mp 1}{q \pm 1}\right)^2,$$

$$F_{11} = F_{31} = -2m\left(\frac{q \mp 1}{\kappa(q \pm 1)}\right)^{\frac{1}{2}}, \quad F_{21} = \mp\frac{2\{(k + \kappa)q \mp (k - \kappa)\}}{[\kappa(q^2 - 1)]^{\frac{1}{2}}}. \tag{14}$$

If $x > 2(k + \kappa)$, q is real and greater than unity and the curvature

$$F_{02}F_{20} - F_{11}{}^2 = \mp \frac{16\kappa q}{q^2 - 1}$$

is negative if the upper signs are taken. There is then a maximum in F for paths through this point parallel to the real v-axis, and thus the desired minimum in F for the path specified parallel to the imaginary v-axis. In this range $x > 2(k + \kappa)$ we are not concerned with the other stationary point (lower signs), since this corresponds to a minimum in the integrand for the given path. After a large amount of algebra the result from (6) reduces to the first two terms of VIII (131).

If $2(k + \kappa) > x > 2(k - \kappa)$, q is complex and the real part of the curvature is negative with both upper and lower signs. Integration paths have now to be taken through both stationary points as in Chapter VIII, Sections 8 and 9, by replacing q by $\pm i\varphi$ and adding the pair of expansions.

Exercises

1. Starting from

$$W_{-k,m}(x) = \frac{4x^{\frac{1}{2}}\, e^{-\frac{1}{2}x}}{(k + m - \frac{1}{2})!\,(k - m - \frac{1}{2})!}\int_0^{\infty} e^{-v^2}\, v^{2k}\, K_{2m}(2v\sqrt{x})\, dv,$$

$$\Re(k - m + \tfrac{1}{2}) > 0,$$

and introducing the representation

$$K_p(z) = \tfrac{1}{2}\int_{u=-\infty}^{\infty} e^{-z\cosh u}\, e^{pu}\, du, \qquad \Re(z) > 0,$$

derive the expansion

$$W_{-k,m}(x) = \frac{\pi\sqrt{2}\,(k + m)^{\frac{1}{2}(k+m)}(k - m)^{\frac{1}{2}(k-m)}e^{-k}}{(k + m - \frac{1}{2})!\,(k - m - \frac{1}{2})!}\left(\frac{(k + \kappa)q^2 - k + \kappa}{\kappa q}\right)^{\frac{1}{2}}$$

$$\times\, e^{-2\kappa\Pi}\left[1 + \frac{1}{192\kappa^2}\{5(k + \kappa)q^3 - 3(3k + \kappa)q - 8k\right.$$

$$\left. - 3(3k - \kappa)/q + 5(k - \kappa)/q^3\} \ldots\right],$$

where

$$q = \left(\frac{x + 2k - 2\kappa}{x + 2k + 2\kappa}\right)^{\frac{1}{2}}, \quad \Pi = \frac{q}{1 - q^2} + \left(\frac{k}{\kappa}\right)\tanh^{-1} q$$

$$+ \left(\frac{m}{\kappa}\right)\tanh^{-1}\frac{1}{q}\left(\frac{k - \kappa}{k + \kappa}\right)^{\frac{1}{2}},$$

with positive signs alloted to k/κ and m/κ.

2. Establish the integral representation

$$S_{qp}(x) = \frac{x^{q+1}}{4\{\frac{1}{2}(p - q - 1)\}!}\int_0^\infty\int_0^\infty e^{-u-\frac{1}{4}x^2 v}\, u^{\frac{1}{2}(p-q-1)}(1 + uv)^{\frac{1}{2}(p+q-1)}\, du\, dv$$

for the Lommel function of two variables. Show that stationary points occur at

$$u = \tfrac{1}{2}[p - 1 \pm \{(p - 1)^2 - x^2\}^{\frac{1}{2}}], \qquad v = 2[q \pm \{(p - 1)^2 - x^2\}^{\frac{1}{2}}]/x^2,$$

and at these

$$F_{02}F_{20} - F_{11}{}^2 = \pm\{(p - 1)^2 - x^2\}^{\frac{1}{2}}[(p - 1)^2 - \tfrac{1}{2}x^2 \mp (p - 1)$$
$$\times \{(p - 1)^2 - x^2\}^{\frac{1}{2}}]/(p + q - 1).$$

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Chapter X

Derivation of Uniform Asymptotic Expansions from Integral Representations of the Form $\int e^{-F} G\, du$

1. The Status of Uniform Expansions in the Asymptotics of Integral Representations

The asymptotic expansion in Chapter V (12), i.e.

$$\int_{\text{limit}} e^{-F}\, G\, du = F_1^{-1} e^{-F_0} \sum_0^\infty L_r, \qquad L_0 = G_0,$$

$$L_1 = -F_1^{-2}(G_0 F_2 - G_1 F_1), \qquad L_2 = F_1^{-4}\{G_0(3F_2^2 - F_1 F_3) \\ -3G_1 F_1 F_2 + G_2 F_1^2\}, \text{ etc.,} \quad (1)$$

evidently contains series in rising powers of F_2/F_1^2, and thus fails as it stands if the first derivative becomes small. By calculating late terms (Chapter VII) and thence the terminant (Chapter XXIII), the range for a given minimum accuracy can be greatly extended, but the expansion ceases to be propitious when F_2/F_1^2 exceeds unity. The best plan to adopt then is to expand the given function as a Taylor (or Newton) series about that value of the variable for which F_1 vanishes identically, calculating the value of the function and its derivatives (or differences) at that point, as e.g. in Chapter VIII, Section 3 when dealing with $(p, x)!$ for $x \sim p$. Very commonly the function under investigation will be one obeying a second-order differential or difference equation, and it will be necessary to find asymptotic expansions only for the function itself at that point, and for its first derivative or difference.

Similarly, the asymptotic expansion in Chapter VI (2), i.e.

$$\int_{\text{s.p.}} e^{-F}\, G\, du = (2\pi/F_2)^{\frac{1}{2}}\, e^{-F_0} \sum_0^\infty Q_{2r}, \qquad Q_0 = G_0,$$

$$Q_2 = (24F_2^3)^{-1}\{G_0(5F_3^2 - 3F_2F_4) - 12G_1F_2F_3 + 12G_2F_2^2\}, \text{etc.,} \qquad (2)$$

contains series in rising powers of F_3^2/F_2^3, and ceases to be propitious when this ratio exceeds unity. The best procedure then is to expand the given function as a Taylor or Newton series about that value of the variable for which F_2 vanishes identically.

The alternative approach to these problems, with which we shall be concerned in the next two chapters, is to develop more complicated "uniform" expansions based on higher transcendental functions. We shall treat in detail only commonly encountered cases, because uniform expansions are not quite as valuable in the asymptotics of integral representations as has frequently been supposed. However, as we shall see in Chapters XV, XVII and XX, they do play a vital role in the asymptotics of solutions to differential equations, so their study in the present context is opportune as an introduction and comparison.

One of the reasons commonly advanced to attest the need for uniform expansions of integral representations is indeed based on a misapprehension, namely that an expansion about a point $F_1 = 0$, or $F_1 = F_2 = 0$, is necessarily narrowly limited in range. To take a concrete example, the well-known expansion (Airey 1916, p. 524)

$$J_p(x) = \frac{1}{2\pi\sqrt{3}}\left(\frac{6}{x}\right)^{\frac{1}{3}}\left\{B_0(-\tfrac{2}{3})! + B_1(-\tfrac{1}{3})!\left(\frac{6}{x}\right)^{\frac{1}{3}} - \tfrac{1}{3}B_3(-\tfrac{2}{3})!\left(\frac{6}{x}\right)\right.$$

$$\left. - \tfrac{2}{3}B_4(-\tfrac{1}{3})!\left(\frac{6}{x}\right)^{\frac{4}{3}}\cdots\right\}, \qquad B_0 = 1\,, \qquad B_1 = x - p\,,$$

$$B_3 = (x-p)^3/6 - (x-p)/15\,, \qquad B_4 = (x-p)^4/24 - (x-p)^2/24 + 1/280\,, \ldots,$$

assuredly *appears* to be an asymptotic series suitable only in the narrow range $|(x-p)/x^{\frac{1}{3}}| \ll 1$. But comparison with Chapter VIII, Section 5, case $x \sim p$, shows it to be in reality an injudicious amalgam between

(a) an easily calculable ***absolutely convergent*** series in rising powers of $(x-p)/p^{\frac{1}{3}}$, and

(b) asymptotic series for $J_p(p)$ and $J_p^{(1)}(p)$ which are excellent for all p down to unity (Chapter XXIII, Section 10).

Weighing against any superfluous recourse to uniform expansions are the following factors: (i) their dependence on higher transcendental functions scarcely tabulated in the complex plane, (ii) the difficulty of evaluating terminants for asymptotic expansions involving higher transcendental functions, and (iii) the heavy labour of deriving them from integral representations, since to cover at a predetermined standard of accuracy the whole range *in a single expansion* perforce involves many more derivatives and their cross-products than are needed in the corresponding coverage by two algebraic asymptotic expansions (i.e. "linear case" + "quadratic case", or "quadratic" + "cubic").

In the evaluation of integral representations, there remain two types of problem for which uniform expansions really are valuable: those where subsequent operations have to be performed over a very broad range, e.g. integrations; and those where the function expanded does not satisfy a differential or difference equation of low order, i.e. those for which the Taylor (or Newton) series would require asymptotic evaluation of a large number of derivatives (or differences).

2. Uniform Expansion for $\int_{\text{limit}} e^{-F}\, G\, du$, F Linear to Quadratic in u

As in Chapter V, Section 2, we shall suppose the integral to have been reduced to a standard form in which the limit of integration is set at $u = 0$, and F increases steadily up to $+\infty$ at the upper limit of integration. (Otherwise there would be a stationary point along the path of integration, contrary to our stipulation in Chapter VII, Section 2 that any given integral has been dissected into a set of simpler integrals each involving just one critical point—i.e. one finite limit to $F(u)$ *or* one stationary point $dF/du = 0$). Retaining in their original exponential forms both linear and quadratic terms in F, and expanding the rest of the integrand in rising powers of u, we obtain the required uniform expansion:

$$\int_{u=0} e^{-F}\, G\, du = \sum_{r=0}^{\infty} \left(\int_0^{\infty} e^{-F_1 u - \frac{1}{2} F_2 u^2} u^r\, du \right)$$

$$\times \left(\text{coefficient of } u^r \text{ in } G e^{-F + F_1 u + \frac{1}{2} F_2 u^2} \right), \qquad \Re\,(F_2) > 0. \tag{3}$$

To normalize the integral such that it tends to unity as $F_1 \to \infty$, we introduce as basic functions

$$\mathscr{T}_r(z) = \frac{1}{r!} \int_0^{\infty} e^{-v - v^2/2z^2} v^r\, dv = \frac{z^{r+1}}{r!} \int_0^{\infty} e^{-vz - \frac{1}{2} v^2}\, v^r\, dv. \tag{4}$$

In terms of these the uniform expansion (3) becomes

$$\int_{\text{limit}} e^{-F}\, G\, du = F_1^{-1} e^{-F_0} \sum_0^{\infty} F_1^{-r}\, \mathscr{U}_r\, \mathscr{T}_r(z), \qquad z = F_1/\sqrt{F_2}, \qquad |\mathrm{ph}\, z| < \tfrac{1}{2}\pi, \tag{5}$$

where

$$\mathscr{U}_r = r! \times \text{coefficient of } u^r \text{ in } G(u)e^{-F(u)+F_0+F_1u+\frac{1}{2}F_2u^2}. \tag{6}$$

The first thirteen coefficients are as follows:

$$\mathscr{U}_0 = G_0, \qquad \mathscr{U}_1 = G_1, \qquad \mathscr{U}_2 = G_2, \qquad \mathscr{U}_3 = G_3 - G_0F_3,$$

$$\mathscr{U}_4 = G_4 - 4G_1F_3 - G_0F_4, \qquad \mathscr{U}_5 = G_5 - 10G_2F_3 - 5G_1F_4 - G_0F_5,$$

$$\mathscr{U}_6 = G_6 - 20G_3F_3 - 15G_2F_4 - 6G_1F_5 - G_0F_6^\dagger,$$

$$\mathscr{U}_7 = G_7 - 35G_4F_3 - 35G_3F_4 - 21G_2F_5 - 7G_1F_6^\dagger - G_0F_7^\dagger,$$

$$\mathscr{U}_8 = G_8 - 56G_5F_3 - 70G_4F_4 - 56G_3F_5 - 28G_2F_6^\dagger - 8G_1F_7^\dagger - G_0F_8^\dagger,$$

$$\mathscr{U}_9 = G_9 - 84G_6F_3 - 126G_5F_4 - 126G_4F_5 - 84G_3F_6^\dagger - 36G_2F_7^\dagger$$
$$- 9G_1F_8^\dagger - G_0F_9^\dagger,$$

$$\mathscr{U}_{10} = G_{10} - 120G_7F_3 - 210G_6F_4 - 252G_5F_5 - 210G_4F_6^\dagger - 120G_3F_7^\dagger$$
$$- 45G_2F_8^\dagger - 10G_1F_9^\dagger - G_0F_{10}^\dagger,$$

$$\mathscr{U}_{11} = G_{11} - 165G_8F_3 - 330G_7F_4 - 462G_6F_5 - 462G_5F_6^\dagger - 330G_4F_7^\dagger$$
$$- 165G_3F_8^\dagger - 55G_2F_9^\dagger - 11G_1F_{10}^\dagger - G_0F_{11}^\dagger,$$

$$\mathscr{U}_{12} = G_{12} - 220G_9F_3 - 495G_8F_4 - 792G_7F_5 - 924G_6F_6^\dagger - 792G_5F_7^\dagger$$
$$- 495G_4F_8^\dagger - 220G_3F_9^\dagger - 66G_2F_{10}^\dagger - 12G_1F_{11}^\dagger - G_0F_{12}^\dagger, \tag{7}$$

where

$$F_6^\dagger = F_6 - 10F_3^2, \quad F_7^\dagger = F_7 - 35F_3F_4, \quad F_8^\dagger = F_8 - 35F_4^2 - 56F_3F_5,$$

$$F_9^\dagger = F_9 - 84F_3F_6 - 126F_4F_5 + 280F_3^3,$$

$$F_{10}^\dagger = F_{10} - 120F_3F_7 - 126F_5^2 - 210F_4F_6 + 2{,}100F_3^2F_4,$$

$$F_{11}^\dagger = F_{11} - 165F_3F_8 - 330F_4F_7 - 462F_5F_6 + 4{,}620F_3^2F_5 + 5{,}775F_3F_4^2,$$

$$F_{12}^\dagger = F_{12} - 220F_3F_9 - 462F_6^2 - 495F_4F_8 - 792F_5F_7 + 5{,}775F_4^3$$
$$+ 9{,}240F_3^2F_6 + 27{,}720F_3F_4F_5 - 15{,}400F_3^4. \tag{8}$$

Throughout the discipline of asymptotics, separate attention has to be paid to a function initially defined at a phase for which its asymptotic expansion exhibits a Stokes discontinuity, for example a function defined by the principal value of an integral. In the current context this would apply to an initial definition where z is a pure imaginary; for when $|F_2|$ is large—otherwise the uniform expansion is pointless—the contributions L_r are

$$L_r \sim \frac{(2r)!}{r!}\left(-\frac{1}{2z^2}\right)^r, \tag{9}$$

i.e. they are all of the same sign and phase where $\operatorname{ph} z = \pm\frac{1}{2}\pi$. When initially defined on these Stokes rays, the required functions are the means of those obtained by assigning the phases $\operatorname{ph} z = \pm\frac{1}{2}\pi$. The appropriate basic functions are therefore

$$\mathscr{T}_r(z) = \begin{cases} \dfrac{(-1)^{\frac{1}{2}r}}{r!}\displaystyle\int_0^\infty \sin v\, e^{-v^2/(-2z^2)} v^r\, dv\,, & r \text{ even}, \\[2ex] \dfrac{(-1)^{\frac{1}{2}(r+1)}}{r!}\displaystyle\int_0^\infty \cos v\, e^{-v^2/(-2z^2)} v^r\, dv\,, & r \text{ odd}, \end{cases} \tag{10}$$

and the uniform expansion is

$$\int_{\text{limit}} e^{-F} G\, du = F_1^{-1} e^{-F_0} \sum_0^\infty F_1^{-r}\, \mathscr{U}_r \mathscr{T}_r(z)\,, \quad z = F_1/\sqrt{F_2} \text{ pure imaginary.} \tag{11}$$

Application to the functions treated in Chapter VIII of these (and later) general formulae is straightforward though laborious, and is relegated to the exercises concluding this chapter.

Properties and evaluation of the integrals $\mathscr{T}_r$ and $\bar{\mathscr{T}}_r$.

Recurrence and differential relations. Proofs of the following results are straightforward:

$$rz^{-2}\mathscr{T}_r = \mathscr{T}_{r-2} - \mathscr{T}_{r-1}\,, \qquad \mathscr{T}_{-1} = 1, \tag{12}$$

$$\mathscr{T}_r{}' = (r+1)(r+2)z^{-3}\mathscr{T}_{r+2}\,, \qquad r \geqslant 0, \tag{13}$$

$$\mathscr{T}_r{}'' - \frac{z^2 + 2r + 2}{z}\mathscr{T}_r{}' + \frac{(r+1)(r+2)}{z^2}\mathscr{T}_r = 0, \tag{14}$$

with identical relations holding for the barred functions.

Asymptotic series for large $|z|$. Expanding the factor

$$e^{-\frac{1}{2}v^2/z^2} = \sum_0^{\infty} v^{2s}/s!(-2z^2)^s$$

and integrating (4) term by term,

$$\mathscr{T}_r(z) = \frac{1}{r!}\sum_0^{\infty}\frac{(2s+r)!}{s!}\left(-\frac{1}{2z^2}\right)^s = (\tfrac{1}{2}z^2)^{\frac{1}{2}r+\frac{1}{2}}\,\psi(\tfrac{1}{2}r+\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}z^2),$$

$$|\mathrm{ph}\,z| < \tfrac{1}{2}\pi, \tag{15}$$

in the notation of Chapter III, Section 4. From (10) a formally identical series is found for $\mathscr{T}_r(z)$ when z is purely imaginary. (At these phases $\mathrm{ph}\,z = \pm\frac{1}{2}\pi$ the asymptotic expansion for $\mathscr{T}_r$ abruptly acquires extra terms because of the Stokes discontinuity).

Placing $z = F_1/\sqrt{F_2}$ and substituting (15), (5) and (11) reduce to the expansion in Chapter V (12) for linear dependence at a limit of integration.

Absolutely convergent series. Similarly expanding the factor e^{-v} and integrating term by term, we obtain the absolutely convergent series

$$\mathscr{T}_r(z) = \frac{(z\sqrt{2})^{r+1}}{2(r!)}\sum_0^{\infty}\frac{(\frac{1}{2}s+\frac{1}{2}r-\frac{1}{2})!}{s!}(-z\sqrt{2})^s$$

$$= \sqrt{\pi}\left(\frac{z}{\sqrt{2}}\right)^{r+1}\left\{\frac{1}{(\frac{1}{2}r)!}F(\tfrac{1}{2}r+\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}z^2) - \frac{z\sqrt{2}}{(\frac{1}{2}r-\frac{1}{2})!}F(\tfrac{1}{2}r+1,\tfrac{3}{2},\tfrac{1}{2}z^2)\right\}. \tag{16}$$

Passage to the limit

$$\frac{\mathscr{T}_r(F_1/\sqrt{F_2})}{F_1^{\,r+1}} \underset{F_1\to 0}{\to} \frac{(\frac{1}{2}r-\frac{1}{2})!}{2(r!)}\left(\frac{2}{F_2}\right)^{\frac{1}{2}r+\frac{1}{2}}$$

reduces (5) to the expansion in Chapter V (20) for quadratic dependence at a limit of integration,

$$\int_{\text{limit}} e^{-F}G\,du = \left(\frac{\pi}{2F_2}\right)^{\frac{1}{2}} e^{-F_0}[Q_0 + Q_1 + \ldots].$$

On the Stokes rays $\mathrm{ph}\,z = \pm\frac{1}{2}\pi$, the relevant convergent series is

$$\mathscr{T}_r(z) = \frac{(-2z^2)^{\frac{1}{2}r+\frac{1}{2}}}{2(r!)}\sum_0^{\infty}\frac{(\frac{1}{2}s+\frac{1}{2}r-\frac{1}{2})!}{s!}\sin[\tfrac{1}{2}\pi(s-r)]\,(-2z^2)^{\frac{1}{2}s}$$

$$= \sqrt{\pi}\left\{-\frac{\sin\frac{1}{2}\pi r}{(\frac{1}{2}r)!}(-\tfrac{1}{2}z^2)^{\frac{1}{2}r+\frac{1}{2}}F(\tfrac{1}{2}r+\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}z^2) + \frac{2\cos\frac{1}{2}\pi r}{(\frac{1}{2}r-\frac{1}{2})!}\right.$$

$$\left.\times\,(-\tfrac{1}{2}z^2)^{\frac{1}{2}r+1}F(\tfrac{1}{2}r+1,\tfrac{3}{2},\tfrac{1}{2}z^2)\right\}. \tag{17}$$

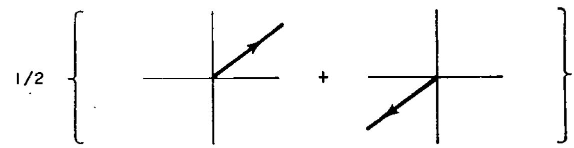
Substitution of the limits

$$\frac{\mathscr{T}_r(F_1/\sqrt{F_2})}{F_1^{\,r+1}} \underset{F_1\to 0}{\longrightarrow} \begin{cases} 0, & r \text{ even}, \\ \dfrac{(-1)^{\frac{1}{2}(r-1)}(\frac{1}{2}r-\frac{1}{2})!}{2(r!)}\left(-\dfrac{2}{F_2}\right)^{\frac{1}{2}r+\frac{1}{2}}, & r \text{ odd}, \end{cases}$$

reduces (11) to

$$\int_{\text{limit}} e^{-F}\, G\, du = \left(\frac{\pi}{2F_2}\right)^{\frac{1}{2}} e^{-F_0}[Q_1 + Q_3 + Q_5 + \dots],$$

the expansion for quadratic dependence along the "combination contour"



in the u-plane. (Cf. the somewhat similar problem at the end of Chapter VI, Section 6).

Continuation formula. The relation

$$\mathscr{T}_r(-z) = \mathscr{T}_r(z) - (-1)^r(2\pi)^{\frac{1}{2}}(r!)^{-1}z^{2r+1}e^{\frac{1}{2}z^2}\mathscr{T}_{-r-1}(iz), \tag{18}$$

or alternatively the absolutely convergent series (16) and (17), would enable us to proceed right through $z = 0$ to $\Re(z) < 0$; i.e. starting from the original region where F is linear to quadratic in u, going past the stationary point $F_1 = 0$ and entering the next region where F is again linear to quadratic in u. In practice, however, it is simpler to deal with each linear-to-quadratic region separately, redefining z in each range such that it is restricted to $\Re(z) \geqslant 0$ (e.g. question 2).

Asymptotic expansion for large r. Applying the methods of Chapter VI, this is found to be (Chapter VIII, question 18)

$$\mathscr{T}_r(z) \sim \frac{(2\pi)^{\frac{1}{2}}}{r!}\left(1+\frac{4r}{z^2}\right)^{-\frac{1}{4}}\left[\frac{z^2}{2}\left\{\left(1+\frac{4r}{z^2}\right)^{\frac{1}{2}}-1\right\}\right]^{r+\frac{1}{2}} e^{-\frac{1}{2}r}$$
$$\times \exp\left[-\tfrac{1}{4}z^2\{(1+4r/z^2)^{\frac{1}{2}}-1\}\right]. \tag{19}$$

When $r \gg \frac{1}{4}z^2$ the dominant variations with r are therefore

$$\mathscr{T}_r(z) \propto \frac{r^{\frac{1}{2}r}e^{-\frac{1}{2}r}z^r e^{-\frac{1}{2}z\sqrt{r}}}{r!} \propto \frac{(\frac{1}{2}r-\frac{1}{2})!\, z^r e^{-\frac{1}{2}z\sqrt{r}}}{r!} \propto \frac{(\frac{1}{2}z)^r e^{-\frac{1}{2}z\sqrt{r}}}{(\frac{1}{2}r)!}. \tag{20}$$

This result enables us to ascertain the form of late terms in the uniform expansion (5). Rather than mount a complete investigation along the lines of Chapter VII, we will for simplicity suppose the only significant derivatives of F and G to be F_1, F_2, F_3, G_0. Then, by (6),

$$\mathscr{U}_r = r! \times \text{coefficient of } u^r \text{ in } G_0 e^{-(1/6)F_3 u^3},$$

$$\text{i.e.} \quad \mathscr{U}_{3r} = G_0 (3r)!\,(-\tfrac{1}{6}F_3)^r / r!\,.$$

Accordingly, sufficiently late terms in (5) are proportional to

$$\frac{(\frac{3}{2}r - \frac{1}{2})!}{r!}\left(-\frac{F_3}{6F_2{}^{3/2}}\right)^r \exp\left[-\frac{F_1}{2}\left(\frac{r}{F_2}\right)^{\frac{1}{2}}\right]. \tag{21}$$

By considering even and odd values of r separately and applying the duplication and triplication formulae for factorials, (21) can be broken into two series in which sufficiently late terms are proportional to

$$(r' - \tfrac{3}{2})!\left(\frac{3F_3{}^2}{16F_2{}^3}\right)^{r'} \exp\left[-F_1\left(\frac{r'}{2F_2}\right)^{\frac{1}{2}}\right], \qquad (r' = \tfrac{1}{2}r). \tag{22}$$

The uniform expansion therefore follows the standard pattern for asymptotic power series (Chapter I, Sections 1, 4) apart from a relatively slowly-varying damping factor.

Mellin transform. In terms of the parameter $x = 1/2z^2$, the Mellin transform of $\mathscr{T}_r$ is

$$M(m) = \int_0^\infty \mathscr{T}_r x^{m-1} dx = \frac{1}{r!}\int_{v=0}^\infty e^{-v} v^r dv \int_{x=0}^\infty e^{-v^2 x} x^{m-1} dx$$

$$= \frac{(m-1)!}{r!}\int_0^\infty e^{-v} v^{r-2m} dv = \frac{(m-1)!\,(-2m+r)!}{r!}.$$

Hence

$$\mathscr{T}_r(z) = \frac{1}{2\pi i (r!)}\int_{\gamma - i\infty}^{\gamma + i\infty} (m-1)!\,(-2m+r)!\,(2z^2)^m dm,$$

$$|\mathrm{ph}\, z| < \tfrac{1}{4}\pi, \tag{23}$$

where the phase sector has been ascertained as in Chapter II, Section 2.

Representations for further phase ranges and for $\overline{\mathscr{T}}_r$ can be developed, but these are superfluous to our requirements since we shall call upon (23) only as part of a convenient technique in Chapter XVII for analysing into its component $\mathscr{T}_r$'s a uniform expansion derived from a differential equation in which the second derivative is relatively insignificant.

Tabulation. The integrals $\mathscr{T}_r$ and $\bar{\mathscr{T}}_r$ are essentially parabolic cylinder functions:

$$\mathscr{T}_r(z) = z^{r+1}e^{\frac{1}{4}z^2}D_{-r-1}(z) = z^{r+1}e^{\frac{1}{4}z^2}U(r+\tfrac{1}{2}, z)$$
$$\bar{\mathscr{T}}_r(iz) = z^{r+1}e^{-\frac{1}{4}z^2}\mathscr{D}_r(z) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}z^{r+1}e^{-\frac{1}{4}z^2}V(-r-\tfrac{1}{2}, z). \tag{24}$$

For computational purposes, the $\mathscr{T}_r$ can more conveniently be expressed in terms of either of two well tabulated functions, Hh_r in BA 1931, 1939, or i^rerfc in the Handbook of Mathematical Functions (1964). The connections are

$$\mathscr{T}_r(z) = z^{r+1}e^{\frac{1}{4}z^2}Hh_r(z) = \tfrac{1}{2}\sqrt{\pi}(z/\sqrt{2})^{r+1}e^{\frac{1}{4}z^2}i^r erfc(z/\sqrt{2}). \tag{25}$$

The table *infra* was constructed from the former.

There is much less to draw upon for the $\bar{\mathscr{T}}_r$. The short table below was calculated from entries for V on pages 703 and 705 of the Handbook of Mathematical Functions (1964). Noting the integral

$$\bar{\mathscr{T}}_0(iz) = \sqrt{2}\, ze^{-\frac{1}{4}z^2}\int_0^{z/\sqrt{2}} e^{v^2}dv, \tag{26}$$

a more comprehensive tabulation could be based on the ten-place listing of Dawson's integral in Lohmander and Rittsten (1958) plus the recurrence relation (12).

3. Uniform Expansion for $\int_{\text{limit}} e^{-F}\, G\, u^\sigma\, du$, F Linear to Quadratic in u

By an easy extension of the work of Section 2,

$$\int_{u=0} e^{-F}\, G\, u^\sigma du = \frac{e^{-F_0}}{F_1^{\sigma+1}}\sum_0^\infty \frac{(r+\sigma)!}{r!}F_1^{-r}\,\mathscr{U}_r\begin{pmatrix}\mathscr{T}_{r+\sigma}(z)\\ \bar{\mathscr{T}}_{r+\sigma}(z)\end{pmatrix}, \quad |\text{ph}\, z| \begin{Bmatrix} < \frac{1}{2}\pi \\ = \frac{1}{2}\pi\end{Bmatrix}, \tag{27}$$

where z, $\mathscr{U}_r$, $\mathscr{T}_r$ and $\bar{\mathscr{T}}_r$ denote exactly the same quantities as in Section 2. The only new feature is the requirement of tabulation over non-integer orders of $\mathscr{T}$ and $\bar{\mathscr{T}}$. Pertinent tables are as follows: Handbook of Mathematical Functions (1964) 703, 705–708, 710; BA (1927) 222–228; MacDonald (1949) 183–191; Rushton and Lang (1954) 377–411; Slater (1960), Appendix II.

z	$\mathscr{T}_0$	$\mathscr{T}_1$	$\mathscr{T}_2$	$\mathscr{T}_3$	$\mathscr{T}_4$	$\mathscr{T}_5$	$\mathscr{T}_6$	$\mathscr{T}_7$	$\mathscr{T}_8$	$\mathscr{T}_9$
·1	·1159262	·0088407	·0005354	·0000277	·0000013	·0000001	—	—	—	—
·2	·2151889	·0313924	·0036759	·0003696	·0000331	·0000027	·0000002	—	—	—
·3	·3005512	·0629504	·0106920	·0015678	·0002053	·0000245	·0000027	—	—	—
·4	·3742668	·1001173	·0219320	·0041699	·0007105	·0001107	·0000160	·000002	—	—
·5	·4381822	·1404544	·0372160	·0086032	·0017883	·0003408	·0000603	·000010	·000002	—
·6	·4938167	·1822260	·0560863	·0151368	·0036855	·0008245	·0001717	·000034	·000006	—
·7	·5424257	·2242114	·0779625	·0238873	·0066242	·0016918	·0004028	·000090	·000019	—
·8	·5850510	·2655674	·1022347	·0348443	·0107825	·0030799	·0008216	·000207	·000049	·00001
·9	·6225603	·3057262	·1283178	·0479003	·0162846	·0051217	·0015070	·000418	·000110	·00003
1·0	·6556795	·3443205	·1556795	·0628803	·0231998	·0079361	·0025440	·000770	·000222	·00006
1·1	·6850180	·3811282	·1838533	·0795675	·0315465	·0116211	·0040183	·001314	·000409	·00012
1·2	·7110892	·4160316	·2124415	·0977232	·0412986	·0162503	·0060116	·002106	·000703	·00023
1·3	·7343273	·4489869	·2411126	·1171025	·0523943	·0218714	·0085973	·003205	·001139	·00039
1·4	·7551008	·4800025	·2695963	·1374654	·0647441	·0285067	·0118376	·004667	·001757	·00063
1·5	·7737235	·5091222	·2976764	·1585844	·0782393	·0361553	·0157815	·006549	·002597	·00099
1·6	·7904635	·5364135	·3251839	·1802493	·0927581	·0447955	·0204641	·008898	·003701	·00148
1·7	·8055506	·5619589	·3519900	·2022701	·1081726	·0543883	·0259061	·011759	·005111	·00214
1·8	·8191823	·5858493	·3779996	·2244777	·1243527	·0648810	·0321148	·015166	·006864	·00299
1·9	·8315292	·6081797	·4031458	·2467242	·1411705	·0762098	·0390847	·019146	·008997	·00407
2·0	·8427385	·6290462	·4273846	·2688821	·1585025	·0883037	·0467992	·023717	·011541	·00541
2·1	·8529381	·6485431	·4506908	·2908429	·1762323	·1010866	·0552321	·028888	·014522	·00704
2·2	·8622392	·6667623	·4730542	·3125158	·1942515	·1144799	·0643491	·034662	·017961	·00898
2·3	·8707389	·6837913	·4944762	·3338256	·2124605	·1284043	·0741095	·041031	·021873	·01126
2·4	·8785219	·6997136	·5149680	·3547117	·2307691	·1427819	·0844677	·047984	·026268	·01390
2·5	·8856628	·7146076	·5345473	·3751257	·2490964	·1575366	·0953747	·055502	·031151	·01691
2·6	·8922268	·7285470	·5532377	·3950303	·2673705	·1725960	·1067793	·063560	·036520	·02031
2·7	·8982716	·7416004	·5710663	·4143978	·2855285	·1878914	·1186290	·072132	·042371	·02411
2·8	·9038480	·7538318	·5880633	·4332085	·3035154	·2033587	·1308715	·081186	·048692	·02831
2·9	·9090011	·7653007	·6042603	·4514498	·3212843	·2189384	·1434548	·090688	·055471	·03291
3·0	·9137709	·7760619	·6196903	·4691148	·3387949	·2345759	·1563284	·100604	·062690	·03791

z	$\mathscr{T}_0$	$\mathscr{T}_1$	$\mathscr{T}_2$	$\mathscr{T}_3$	$\mathscr{T}_4$	$\mathscr{T}_5$	$\mathscr{T}_6$	$\mathscr{T}_7$	$\mathscr{T}_8$	$\mathscr{T}_9$
3·1	·9181929	·7861665	·6343866	·4862019	·3560137	·2502216	·1694437	·110897	·070330	·04332
3·2	·9222987	·7956615	·6483823	·5027131	·3729130	·2658306	·1827539	·121529	·078367	·04911
3·3	·9261166	·8045904	·6617104	·5186543	·3894703	·2813628	·1962151	·132465	·086779	·05528
3·4	·9296719	·8129931	·6744031	·5340337	·4056677	·2967822	·2097860	·143668	·095540	·06182
3·5	·9329872	·8209069	·6864918	·5488619	·4214915	·3120575	·2234278	·155102	·104624	·06871
3·6	·9360829	·8283658	·6980067	·5631512	·4369317	·3271610	·2371048	·166733	·114003	·07593
3·7	·9389773	·8354013	·7089772	·5769154	·4519816	·3420688	·2507842	·178527	·123651	·08347
3·8	·9416868	·8420426	·7194313	·5901690	·4666370	·3567605	·2644360	·190452	·133540	·09131
3·9	·9442264	·8483164	·7293958	·6029274	·4808962	·3712187	·2780327	·202480	·143645	·09943
4·0	·9466095	·8542475	·7388963	·6152063	·4947599	·3854287	·2915497	·214581	·153938	·10781
4·1	·9488484	·8598588	·7479572	·6270220	·5082300	·3993788	·3049649	·226728	·164395	·11642
4·2	·9509540	·8651715	·7566017	·6383907	·5213104	·4130592	·3182585	·238898	·174991	·12526
4·3	·9529365	·8702051	·7648516	·6493284	·5340060	·4257879	·3314130	·251066	·185701	·13429
4·4	·9548049	·8749776	·7727279	·6598513	·5463227	·4395831	·3444131	·263213	·196504	·14350
4·5	·9565676	·8795059	·7802503	·6699752	·5582674	·4524169	·3572453	·275318	·207379	·15286
4·6	·9582323	·8838053	·7874373	·6797156	·5698476	·4649616	·3698980	·287364	·218304	·16237
4·7	·9598058	·8878903	·7943066	·6890877	·5810714	·4772159	·3823615	·299334	·229261	·17199
4·8	·9612945	·8917741	·8008749	·6981062	·5919474	·4891799	·3946272	·311214	·240232	·18171
4·9	·9627043	·8954692	·8071578	·7067856	·6024843	·5008546	·4066884	·322990	·251199	·19152
5·0	·9640405	·8989869	·8131702	·7151396	·6126912	·5122418	·4185393	·334652	·262149	·20140
5·1	·9653080	·9023380	·8189260	·7231817	·6225773	·5233442	·4301756	·346188	·273066	·21132
5·2	·9665114	·9055322	·8244385	·7309249	·6321518	·5341650	·4415939	·357589	·283937	·22128
5·3	·9676547	·9085789	·8297201	·7383816	·6414240	·5447081	·4527917	·368847	·294749	·23127
5·4	·9687419	·9114866	·8347825	·7455639	·6504031	·5549777	·4637675	·379956	·305493	·24126
5·5	·9697764	·9142632	·8396368	·7524833	·6590984	·5649786	·4745204	·390909	·316157	·25125
5·6	·9707616	·9169164	·8442935	·7591508	·6675188	·5747158	·4850505	·401700	·326732	·26122
5·7	·9717004	·9194528	·8487624	·7655770	·6756733	·5841944	·4953582	·412327	·337211	·27117
5·8	·9725957	·9218792	·8530529	·7717722	·6835706	·5934200	·5054446	·422785	·347585	·28108
5·9	·9734501	·9242014	·8571736	·7777459	·6912194	·6023981	·5153112	·433071	·357849	·29094
6·0	·9742660	·9264253	·8611329	·7835076	·6986278	·6111345	·5249600	·443183	·367995	·30075

z	$\mathscr{T}_0$	$\mathscr{T}_1$	$\mathscr{T}_2$	$\mathscr{T}_3$	$\mathscr{T}_4$	$\mathscr{T}_5$	$\mathscr{T}_6$	$\mathscr{T}_7$	$\mathscr{T}_8$	$\mathscr{T}_9$
6·1	·9750455	·9285559	·8649387	·7890661	·7058043	·6196350	·5343933	·453120	·378020	·31050
6·2	·9757909	·9305985	·8685982	·7944300	·7127566	·6279053	·5436137	·462881	·387919	·32017
6·3	·9765040	·9325575	·8721185	·7996073	·7194925	·6359514	·5526242	·472466	·397687	·32977
6·4	·9771866	·9344373	·8755062	·8046058	·7260195	·6437791	·5614278	·481873	·407322	·33929
6·5	·9778404	·9362420	·8787674	·8094330	·7323449	·6513942	·5700280	·491103	·416820	·34872
6·6	·9784671	·9379754	·8819082	·8140959	·7384758	·6588025	·5784281	·500158	·426180	·35805
6·7	·9790679	·9396411	·8849340	·8186013	·7444189	·6660096	·5866319	·509038	·435399	·36729
6·8	·9796444	·9412426	·8878501	·8229556	·7501808	·6730213	·5946431	·517744	·444477	·37643
6·9	·9801978	·9427829	·8906616	·8271650	·7557680	·6798429	·6024654	·526278	·453412	·38546
7·0	·9807293	·9442651	·8933731	·8312354	·7611865	·6864800	·6101027	·534641	·462203	·39439
7·1	·9812400	·9456919	·8959891	·8351725	·7664423	·6929379	·6175590	·542836	·470850	·40320
7·2	·9817310	·9470661	·8985139	·8389814	·7715411	·6992218	·6248382	·550864	·479353	·41190
7·3	·9822032	·9483901	·9009515	·8426675	·7764884	·7053369	·6319442	·558728	·487712	·42049
7·4	·9826577	·9496662	·9033056	·8462355	·7812896	·7112880	·6388810	·566430	·495928	·42896
7·5	·9830952	·9508968	·9055800	·8496902	·7859497	·7170801	·6456525	·573972	·504002	·43732
7·6	·9835166	·9520838	·9077779	·8530360	·7904738	·7227180	·6522626	·581358	·511933	·44555
7·7	·9839226	·9532293	·9099028	·8562771	·7948665	·7282062	·6587152	·588589	·519723	·45367
7·8	·9843140	·9543351	·9119575	·8594176	·7991325	·7335493	·6650141	·595668	·527374	·46167
7·9	·9846915	·9554031	·9139452	·8624614	·8032761	·7387515	·6711631	·602599	·534887	·46955
8·0	·9850557	·9564348	·9158686	·8654123	·8073015	·7438172	·6771660	·609383	·542262	·47731
8·1	·9854072	·9574319	·9177303	·8682737	·8112130	·7487506	·6830263	·616085	·549502	·48495
8·2	·9857466	·9583959	·9195329	·8710490	·8150142	·7535555	·6887478	·622524	·556608	·49247
8·3	·9860745	·9593281	·9212787	·8737415	·8187092	·7582359	·6943339	·628887	·563582	·49987
8·4	·9863913	·9602300	·9229701	·8763543	·8223014	·7627956	·6997882	·635114	·570426	·50715
8·5	·9866975	·9611028	·9246091	·8788904	·8257943	·7672381	·7051141	·641209	·577142	·51432
8·6	·9869937	·9619478	·9261979	·8813525	·8291914	·7715672	·7103148	·647175	·583731	·52137
8·7	·9872801	·9627660	·9277385	·8837435	·8324959	·7757861	·7153938	·653014	·590195	·52830
8·8	·9875574	·9635586	·9292327	·8860658	·8357108	·7798982	·7203541	·658728	·596537	·53512
8·9	·9878257	·9643266	·9306823	·8883220	·8388392	·7839068	·7251989	·664322	·602759	·54183
9·0	·9880855	·9650710	·9320890	·8905145	·8418839	·7878149	·7299313	·669797	·608861	·54842

z	$\mathscr{T}_0$	$\mathscr{T}_1$	$\mathscr{T}_2$	$\mathscr{T}_3$	$\mathscr{T}_4$	$\mathscr{T}_5$	$\mathscr{T}_6$	$\mathscr{T}_7$	$\mathscr{T}_8$	$\mathscr{T}_9$
9·1	·9883372	·9657928	·9334544	·8926455	·8448478	·7916256	·7345541	·675156	·614848	·55490
9·2	·9885811	·9664928	·9347802	·8947172	·8477334	·7953418	·7390704	·680401	·620720	·56127
9·3	·9888175	·9671718	·9360677	·8967317	·8505433	·7989663	·7434830	·685536	·626479	·56753
9·4	·9890467	·9678308	·9373185	·8986910	·8532801	·8025018	·7477945	·690563	·632128	·57369
9·5	·9892690	·9684705	·9385338	·9005971	·8559461	·8059511	·7520078	·695483	·637669	·57974
9·6	·9894847	·9690916	·9397149	·9024517	·8585435	·8093166	·7561254	·700300	·643104	·58569
9·7	·9896940	·9696948	·9408631	·9042567	·8610746	·8126008	·7601498	·705017	·648434	·59154
9·8	·9898971	·9702807	·9419796	·9060137	·8635415	·8158071	·7640835	·709634	·653663	·59728
9·9	·9900944	·9708501	·9430655	·9077243	·8659462	·8189349	·7679290	·714156	·658791	·60292
10·0	·9902860	·9714035	·9441218	·9093902	·8682907	·8219894	·7716886	·718583	·663821	·60847

iz	$\mathscr{T}_0$	$\mathscr{T}_1$	$\mathscr{T}_2$	$\mathscr{T}_3$	$\mathscr{T}_4$	$\mathscr{T}_5$
·1	0·0100	−0·0099	−0·0001	0·0000	0·0000	—
·2	0·0395	−0·0384	−0·0016	0·0005	0·0000	—
·3	0·0873	−0·0821	−0·0076	0·0022	0·0002	—
·4	0·1517	−0·1357	−0·0230	0·0060	0·0012	—
·5	0·2302	−0·1925	−0·0528	0·0116	0·0040	—
·6	0·3198	−0·2449	−0·1016	0·0172	0·0107	—
·7	0·4173	−0·2855	−0·1722	0·0185	0·0234	—
·8	0·5195	−0·3076	−0·2646	0·0092	0·0438	0·004
·9	0·6230	−0·3054	−0·3760	−0·0191	0·0723	0·015
1·0	0·7248	−0·2752	−0·5000	−0·0749	0·1063	0·036
1·1	0·8221	−0·2152	−0·6276	−0·1663	0·1395	0·074
1·2	0·9127	−0·1257	−0·7477	−0·2985	0·1617	0·133
1·3	0·9947	−0·0090	−0·8481	−0·4727	0·1586	0·213
1·4	1·0668	+0·1309	−0·9172	−0·6847	0·1139	0·313
1·5	1·1282	0·2885	−0·9447	−0·9249	+0·0112	0·421
1·6	1·1788	0·4575	−0·9231	−1·1782	−0·1632	0·564
1·7	1·2184	0·6314	−0·8483	−1·4255	−0·4170	0·583
1·8	1·2481	0·8036	−0·7200	−1·6454	−0·7496	0·580
1·9	1·2682	0·9681	−0·5416	−1·8167	−1·1508	0·481
2·0	1·2800	1·1199	−0·3201	−1·9200	−1·5999	+0·256
2·1	1·2846	1·2550	−0·0652	−1·9407	−2·0677	−0·112
2·2	1·2832	1·3707	+0·2117	−1·8698	−2·5186	−0·628
2·3	1·2770	1·4654	0·4983	−1·7053	−2·9143	−1·279
2·4	1·2672	1·5389	0·7827	−1·4521	−3·2180	−2·034
2·5	1·2547	1·5919	1·0536	−1·1213	−3·3983	−2·846
2·6	1·2405	1·6257	1·3019	−0·7296	−3·4333	−3·655
2·7	1·2253	1·6424	1·5204	−0·2965	−3·3111	−4·395
2·8	1·2098	1·6446	1·7045	+0·1565	−3·0340	−5·003
2·9	1·1944	1·6348	1·8517	0·6083	−2·6144	−5·420
3·0	1·1795	1·6155	1·9620	1·0396	−2·0755	−5·607
3·1	1·1654	1·5893	2·0371	1·1443	−1·4483	−4·983
3·2	1·1522	1·5584	2·0799	1·7801	−0·7676	−5·218
3·3	1·1400	1·5247	2·0948	2·0693	−0·0695	−4·658
3·4	1·1289	1·4898	2·0862	2·2980	+0·6122	−3·898
3·5	1·1188	1·4550	2·0590	2·4665	1·2481	−2·985
3·6	1·1097	1·4211	2·0178	2·5780	1·8151	−1·977
3·7	1·1014	1·3888	1·9669	2·6383	2·2975	−0·933
3·8	1·0941	1·3586	1·9101	2·6543	2·6864	+0·093
3·9	1·0875	1·3308	1·8503	2·6341	2·9803	1·053
4·0	1·0816	1·3054	1·7903	2·5859	3·1827	1·910
4·1	1·0762	1·2823	1·7315	2·5172	3·3019	2·638
4·2	1·0715	1·2615	1·6756	2·4349	3·3486	3·224
4·3	1·0672	1·2428	1·6233	2·3449	3·3359	3·665
4·4	1·0633	1·2260	1·5750	2·2520	3·2765	3·967
4·5	1·0598	1·2110	1·5310	2·1597	3·1829	4·144
4·6	1·0566	1·1975	1·4911	2·0708	3·0664	4·213
4·7	1·0537	1·1854	1·4553	1·9870	2·9366	4·195
4·8	1·0510	1·1745	1·4232	1·9094	2·8010	4·108
4·9	1·0485	1·1647	1·3944	1·8384	2·6658	3·973
5·0	1·0462	1·1557	1·3686	1·7744	2·5354	3·805

4. Uniform Expansion for $\int_{\text{limit}} e^{-F}\,G\,du$, F Linear to Cubic in u

It was tacitly assumed in Section 2 that the second derivative at the limit, F_2, did not vanish identically. There are, however, several important instances in which it does, notably Anger and Weber functions, so we quote the modified results.

Retaining in their original exponential forms linear and cubic terms in F, and expanding the rest of the integrand in rising powers of u, we have

$$\int_{u=0} e^{-F}\,G\,du = \sum_{r=0}^{\infty}\left(\int_0^{\infty} e^{-F_1u-\frac{1}{6}F_3u^3}u^r\,du\right)$$

$$\times\ (\text{coefficient of } u^r \text{ in } Ge^{-F+F_1u+\frac{1}{6}F_3u^3})$$

$$= F_1^{-1}e^{-F_0}\sum_0^{\infty} F_1^{-r}U_r\Upsilon_r(z), \qquad z = F_1(2/F_3)^{\frac{1}{3}}, \qquad |\mathrm{ph}\,z| < \tfrac{1}{3}\pi, \tag{28}$$

where

$$U_r = r! \times \text{coefficient of } u^r \text{ in } G(u)e^{-F(u)+F_0+F_1u+\frac{1}{6}F_3u^3}, \tag{29}$$

and

$$\Upsilon_r(z) = \frac{1}{r!}\int_0^{\infty} e^{-v-v^3/3z^3}v^r\,dv = \frac{z^{r+1}}{r!}\int_0^{\infty} e^{-vz-\frac{1}{3}v^3}\,v^r\,dv. \tag{30}$$

The first thirteen coefficients U_r are listed in Section 5.

When a function has initially been defined at $\mathrm{ph}\,z = \pm\frac{1}{3}\pi$, which are Stokes rays of the asymptotic series for Υ, the uniform expansion is

$$\int_{u=0} e^{-F}\,G\,du = F_1^{-1}e^{-F_0}\sum_0^{\infty} F_1^{-r}U_r\bar{\Upsilon}_r(z), \tag{31}$$

where

$$\bar{\Upsilon}_0(z) = \int_0^{\infty}\sin\,(u - u^3/3z^3)\;du,$$

$$\bar{\Upsilon}_1(z) = -\int_0^{\infty}\cos\,(u - u^3/3z^3)\;u\,du. \tag{32}$$

Properties and evaluation of the integrals Υ_r and $\bar{\Upsilon}_r$.

Recurrence and differential relations. Proofs of the following are straightforward:

$$r(r-1)z^{-3}\Upsilon_r = \Upsilon_{r-3} - \Upsilon_{r-2}, \qquad \Upsilon_{-1} = 1, \tag{33}$$

$$\Upsilon'_r = (r+1)(r+2)(r+3)z^{-4}\Upsilon_{r+3}, \qquad r \geqslant 0, \tag{34}$$

with identical relations for the barred functions.

Asymptotic series for large $|z|$. Expanding the factor $e^{-v^3/3z^3}$ and integrating (30) term by term,

$$\Upsilon_r(z) = \frac{1}{r!}\sum_0^\infty \frac{(3s+r)!}{s!}\left(-\frac{1}{3z^3}\right)^s, \qquad |\mathrm{ph}\, z| < \tfrac{1}{3}\pi. \tag{35}$$

The series for $\overline{\Upsilon}_r(z)$ at $\mathrm{ph}\, z = \pm\frac{1}{3}\pi$ is formally identical. (At these phases the asymptotic expansion for Υ_r abruptly acquires extra terms because of the Stokes discontinuity.)

Absolutely convergent series. Similarly expanding the factor e^{-v} and integrating term by term, we obtain the absolutely convergent series

$$\Upsilon_r(z) = \frac{(3^{\frac{1}{3}}z)^{r+1}}{3(r!)}\sum_0^\infty \frac{(\frac{1}{3}s+\frac{1}{3}r-\frac{2}{3})!}{s!}(-3^{\frac{1}{3}}z)^s. \tag{36}$$

On the Stokes rays $\mathrm{ph}\, z = \pm\frac{1}{3}\pi$, the relevant convergent series is

$$\overline{\Upsilon}_r(z) = \frac{(-3z^3)^{\frac{1}{3}(r+1)}}{3(r!)}\sum_0^\infty \frac{(\frac{1}{3}s+\frac{1}{3}r-\frac{2}{3})!}{s!}(-1)^s\cos\tfrac{1}{3}\pi(s+r+1)\,(-3z^3)^{\frac{1}{3}s}. \tag{37}$$

Continuation formulae.

$$\Upsilon_0(-z) = \overline{\Upsilon}_0(ze^{\frac{1}{3}i\pi}) - \pi z Bi(z), \qquad \Upsilon_1(-z) = \overline{\Upsilon}_1(ze^{\frac{1}{3}i\pi}) + \pi z^2 Bi'(z),$$

$$\Upsilon_2(-z) = \overline{\Upsilon}_2(ze^{\frac{1}{3}i\pi}) - \tfrac{1}{2}\pi z^4 Bi(z). \tag{38}$$

The list can be extended via the recurrence relations (33) and (34).

Mellin transform. In terms of the variable $x = 1/3z^3$, the Mellin transform of Υ_r is

$$M_r(m) = \int_0^\infty \Upsilon_r x^{m-1}dx = \frac{1}{r!}\int_{v=0}^\infty e^{-v}v^r dv\int_{x=0}^\infty e^{-v^3x}x^{m-1}dx$$

$$= \frac{(m-1)!\,(-3m+r)!}{r!}.$$

Hence

$$\Upsilon_r(z) = \frac{1}{2\pi i\ (r!)} \int_{\gamma - i\infty}^{\gamma + i\infty} (m-1)!\,(-3m+r)!\,(3z^3)^m dm, \qquad |\text{ph}\, z| < \tfrac{1}{6}\pi. \tag{39}$$

Tabulation. The integrals Υ_0, Υ_1, $\overline{\Upsilon}_0$ and $\overline{\Upsilon}_1$ can most conveniently be expressed in terms of the respective quantities Hi, Hi', Gi and Gi' tabulated in Scorer (1950), Rothman (1954) and AMS **52**, (1958). The pertinent relations are

$$\begin{aligned} \Upsilon_0(z) &= \pi z\, Hi(-z), & \Upsilon_1(z) &= -\pi z^2\, d\,Hi(-z)/dz, \\ \overline{\Upsilon}_0(ze^{\frac{1}{3}i\pi}) &= \pi z\, Gi(z), & \overline{\Upsilon}_1(ze^{\frac{1}{3}i\pi}) &= -\pi z^2\, Gi'(z). \end{aligned} \tag{40}$$

5. Uniform Expansion for $\int_{\text{s.p.}} e^{-F}\, G\, du$, F Quadratic to Cubic in u

As in Chapter VI, Section 4, we shall suppose the integral to have been reduced to a standard form in which the single stationary point $dF/du = 0$ lies at $u = 0$, and F increases steadily up to $+\infty$ at both limits of integration. By supposition, $F_1 = 0$. Retaining in their original exponential form both quadratic and cubic terms in F, and expanding the rest of the integrand in rising powers of u, we obtain the required uniform expansion:

$$\int_{\text{s.p.}} e^{-F}\, G\, du = \sum_{r=0}^{\infty} \left(\int_{\text{s.p.}} e^{-\frac{1}{2}F_2 u^2 - \frac{1}{6}F_3 u^3} u^r du \right)$$

$$\times\ (\text{coefficient of } u^r \text{ in } Ge^{-F+\frac{1}{2}F_2u^2+\frac{1}{6}F_3u^3}). \tag{41}$$

As remarked in Chapter VI, Section 6, there is a direct correspondence, in both path in the u-plane and nature of resulting expansion, between quadratic dependence when $F_2 < 0$ (Quadratic Contour $\uparrow$) and the principal association for cubic dependence when $F_3 > 0$ (Cubic Contour $>$). Now when $\Re F_2 < 0$ and $\Re F_3 > 0$ the Cubic Contour $>$ expresses the integral in (41) as

$$\left(\frac{2}{-F_2}\right)^{\frac{1}{2}r+\frac{1}{2}} \int_{-\infty e^{\frac{1}{3}i\pi}}^{-\infty e^{-\frac{1}{3}i\pi}} e^{v^2 - v^3(2/27z)^{\frac{1}{2}}}\, v^r dv, \qquad z = -F_2{}^3/3F_3{}^2,$$

$$0 \leqslant |\text{ph}\, z| < \pi.$$

To normalize this integral such that for integer r it tends to unity as

z	Υ_0	Υ_1	Υ_2	Υ_3	Υ_4	Υ_5
·1	·1198803	·008450	·000440	·000019	—	—
·2	·2237042	·030493	·003105	·000258	·00002	—
·3	·3138041	·062038	·009264	·001133	·00012	—
·4	·3921512	·099953	·019451	·003117	·00043	·0001
·5	·4604146	·141852	·033724	·006637	·00113	·0002
·6	·5200101	·185939	·051839	·012027	·00241	·0004
·7	·5721407	·230870	·073378	·019509	·00450	·0009
·8	·6178306	·275655	·097835	·029199	·00759	·0018
·9	·6579530	·319581	·124676	·041112	·01184	·0031
1·0	·6932540	·362142	·153373	·055185	·01740	·0049
1·1	·7243721	·402997	·183430	·071292	·02435	·0075
1·2	·7518546	·441929	·214398	·089259	·03276	·0108
1·3	·7761715	·478812	·245876	·108883	·04265	·0151
1·4	·7977270	·513593	·277519	·129944	·05398	·0203
1·5	·8168698	·546268	·309032	·152214	·06672	·0265
1·6	·8339003	·576872	·340172	·175465	·08079	·0337
1·7	·8490788	·605467	·370738	·199478	·09610	·0421
1·8	·8626303	·632131	·400570	·224044	·11254	·0515
1·9	·8747504	·656957	·429544	·248974	·12999	·0619
2·0	·8856087	·680041	·457565	·274090	·14832	·0734
2·1	·8953530	·701486	·484568	·299234	·16741	·0858
2·2	·9041122	·721393	·510506	·324266	·18713	·0992
2·3	·9119989	·739862	·535355	·349065	·20735	·1133
2·4	·9191113	·756991	·559103	·373525	·22797	·1283
2·5	·9255358	·772874	·581752	·397556	·24886	·1439
2·6	·9313479	·787600	·603315	·421086	·26992	·1601
2·7	·9366140	·801253	·623813	·444052	·29105	·1769
2·8	·9413929	·813913	·643272	·466407	·31216	·1941
2·9	·9457359	·825654	·661724	·488112	·33318	·2117
3·0	·9496886	·836546	·679204	·509140	·35402	·2296
3·1	·9532913	·846654	·695749	·529472	·37463	·2477
3·2	·9565796	·856037	·711400	·549096	·39496	·2659
3·3	·9595851	·864752	·726195	·568006	·41494	·2842
3·4	·9623359	·872848	·740174	·586203	·43455	·3026
3·5	·9648569	·880375	·753379	·603690	·45375	·3209
3·6	·9671704	·887376	·765848	·620479	·47250	·3391
3·7	·9692962	·893891	·777620	·636580	·49079	·3572
3·8	·9712519	·899958	·788732	·652006	·50860	·3751
3·9	·9730535	·905610	·799221	·666776	·52591	·3928
4·0	·9747150	·910880	·809120	·680907	·54272	·4103
4·1	·9762492	·915796	·818463	·694419	·55902	·4275
4·2	·9776676	·920384	·827283	·707334	·57481	·4443
4·3	·9789803	·924670	·835608	·719671	·59009	·4609
4·4	·9801965	·928676	·843469	·731448	·60486	·4771
4·5	·9813247	·932423	·850892	·742692	·61913	·4930
4·6	·9823723	·935930	·857903	·753423	·63290	·5085
4·7	·9833461	·939214	·864527	·763661	·64618	·5236
4·8	·9842522	·942291	·870788	·773427	·65897	·5384
4·9	·9850963	·945177	·876706	·782746	·67130	·5527
5·0	·9858832	·947885	·882300	·791629	·68318	·5667

$ze^{\frac{1}{2}i\pi}$	$\bar{\Upsilon}_0$	$\bar{\Upsilon}_1$	$\bar{\Upsilon}_2$	$\bar{\Upsilon}_3$	$\bar{\Upsilon}_4$	$\bar{\Upsilon}_5$
·1	·06860	−·00373	—	—	—	—
·2	·14375	−·01134	−·003	—	—	—
·3	·22288	−·01815	−·010	−·001	—	—
·4	·30374	−·02048	−·022	−·003	—	—
·5	·38441	−·01561	−·038	−·008	—	—
·6	·46327	−·00174	−·058	−·017	−·001	—
·7	·53903	+·02206	−·079	−·030	−·003	+·001
·8	·61068	·05600	−·100	−·047	−·007	+·001
·9	·67750	·09966	−·118	−·070	−·013	+·002
1·0	·73896	·15217	−·131	−·098	−·024	+·002
1·1	·79479	·21228	−·137	−·129	−·039	+·000
1·2	·84486	·27850	−·134	−·163	−·059	−·003
1·3	·88921	·34925	−·122	−·198	−·086	−·008
1·4	·92800	·42287	−·099	−·231	−·119	−·018
1·5	·96147	·49777	−·065	−·261	−·158	−·033
1·6	·98994	·57247	−·021	−·285	−·202	−·054
1·7	1·01378	·64562	+·034	−·301	−·250	−·082
1·8	1·03339	·71606	·097	−·308	−·301	−·118
1·9	1·04918	·78282	·169	−·304	−·351	−·162
2·0	1·06157	·84514	·246	−·289	−·399	−·214
2·1	1·07096	·90244	·329	−·260	−·443	−·273
2·2	1·07774	·95435	·414	−·219	−·480	−·337
2·3	1·08229	1·00066	·501	−·166	−·507	−·405
2·4	1·08495	1·04132	·587	−·101	−·523	−·475
2·5	1·08603	1·07639	·672	−0·025	−·526	−·545
2·6	1·08581	1·10608	·754	+0·059	−·516	−·611
2·7	1·08455	1·13065	·832	0·151	−·490	−·670
2·8	1·08247	1·15043	·905	0·249	−·449	−·721
2·9	1·07976	1·16582	0·973	0·350	−·393	−·759
3·0	1·07658	1·17723	1·034	0·453	−·32	−·78
3·1	1·07309	1·18508	1·089	0·556	−·24	−·79
3·2	1·06939	1·18981	1·137	0·658	−·14	−·79
3·3	1·06559	1·19183	1·179	0·756	−·04	−·76
3·4	1·06177	1·19154	1·214	0·850	+·07	−·71
3·5	1·05798	1·18933	1·243	0·939	·19	−·65
3·6	1·05427	1·18553	1·266	1·021	·31	−·57
3·7	1·05069	1·18048	1·284	1·096	·44	−·48
3·8	1·04727	1·17444	1·297	1·163	·56	−·37
3·9	1·04401	1·16768	1·305	1·222	·68	−·25
4·0	1·04093	1·16040	1·310	1·274	·80	−·11
4·1	1·03803	1·15281	1·311	1·318	·91	+·03
4·2	1·03533	1·14505	1·309	1·355	1·01	·17
4·3	1·03281	1·13727	1·304	1·384	1·11	·32
4·4	1·03047	1·12956	1·298	1·407	1·19	·47
4·5	1·02830	1·12201	1·290	1·423	1·27	·61
4·6	1·02630	1·11470	1·280	1·434	1·34	·75
4·7	1·02445	1·10766	1·270	1·440	1·40	·88
4·8	1·02276	1·10094	1·258	1·441	1·45	1·01
4·9	1·02119	1·09455	1·247	1·438	1·49	1·13
5·0	1·01976	1·08852	1·235	1·433	1·52	1·24

$|z| \to \infty$, we introduce as basic functions

$$\left.\begin{aligned} T_{2r}{}^{e}(z) &= -\frac{i(-1)^r}{(r-\frac{1}{2})!}\int_{-\infty e^{(1/3)i\pi}}^{-\infty e^{-(1/3)i\pi}} e^{v^2 - v^3(2/27z)^{1/2}}\, v^{2r}\,dv, \\ T_{2r+1}^{0}(z) &= \frac{i(-1)^r}{(r+\frac{3}{2})!}\left(\frac{27z}{2}\right)^{\frac{1}{2}}\int_{-\infty e^{(1/3)i\pi}}^{-\infty e^{-(1/3)i\pi}} e^{v^2 - v^3(2/27z)^{1/2}}\, v^{2r+1}\,dv. \end{aligned}\right\} \quad (42)$$

In terms of these the uniform expansion (41) becomes

$$\int_{>} e^{-F} G\,du = i\left(\frac{2\pi}{-F_2}\right)^{\frac{1}{2}} e^{-F_0}\sum_{0}^{\infty}\frac{1}{r!\,(2F_2)^r}$$

$$\times\left\{U_{2r}T_{2r}{}^{e}(z) - (\tfrac{1}{3}r+\tfrac{1}{2})\frac{F_3}{F_2{}^2}U_{2r+1}T_{2r+1}{}^{0}(z)\right\}, \qquad z = -F_2{}^3/3F_3{}^2,$$

$$|\mathrm{ph}\, z| < \pi, \qquad (43)$$

where

$$U_r = r! \times \text{coefficient of } u^r \text{ in } G(u)e^{-F(u)+F_0+\frac{1}{2}F_2u^2+\frac{1}{6}F_3u^3}. \qquad (44)$$

The first thirteen coefficients are as follows:

$$U_0 = G_0, \qquad U_1 = G_1, \qquad U_2 = G_2, \qquad U_3 = G_3,$$

$$U_4 = G_4 - G_0F_4, \qquad U_5 = G_5 - 5G_1F_4 - G_0F_5,$$

$$U_6 = G_6 - 15G_2F_4 - 6G_1F_5 - G_0F_6,$$

$$U_7 = G_7 - 35G_3F_4 - 21G_2F_5 - 7G_1F_6 - G_0F_7,$$

$$U_8 = G_8 - 70G_4F_4 - 56G_3F_5 - 28G_2F_6 - 8G_1F_7 - G_0F_8{}^*,$$

$$U_9 = G_9 - 126G_5F_4 - 126G_4F_5 - 84G_3F_6 - 36G_2F_7 - 9G_1F_8{}^* - G_0F_9{}^*,$$

$$U_{10} = G_{10} - 210G_6F_4 - 252G_5F_5 - 210G_4F_6 - 120G_3F_7 - 45G_2F_8{}^*$$
$$- 10G_1F_9{}^* - G_0F_{10}{}^*,$$

$$U_{11} = G_{11} - 330G_7F_4 - 462G_6F_5 - 462G_5F_6 - 330G_4F_7 - 165G_3F_8{}^*$$
$$- 55G_2F_9{}^* - 11G_1F_{10}{}^* - G_0F_{11}{}^*,$$

$$U_{12} = G_{12} - 495G_8F_4 - 792G_7F_5 - 924G_6F_6 - 792G_5F_7 - 495G_4F_8{}^*$$
$$- 220G_3F_9{}^* - 66G_2F_{10}{}^* - 12G_1F_{11}{}^* - G_0F_{12}{}^*, \qquad (45)$$

where

$$F_8^* = F_8 - 35F_4^2, \qquad F_9^* = F_9 - 126F_4F_5,$$

$$F_{10}^* = F_{10} - 126F_5^2 - 210F_4F_6, \qquad F_{11}^* = F_{11} - 330F_4F_7 - 462F_5F_6,$$

$$F_{12}^* = F_{12} - 462F_6^2 - 495F_4F_8 - 792F_5F_7 + 5{,}775F_4^3. \tag{46}$$

Special attention has to be bestowed on a function initially defined at $|\mathrm{ph}\, z| = \pi$, since these phases correspond to a Stokes ray in the asymptotic expansions. For when $|F_3|$ is large—otherwise the uniform expansion is pointless—the contributions Q_{2r} are

$$Q_{2r} \sim \frac{(3r-\frac{1}{2})!}{(-\frac{1}{2})!\,(2r)!}\left(-\frac{2}{27z}\right)^r, \tag{47}$$

i.e. they are all of the same sign and phase for real, negative z. When initially defined on this Stokes ray, the required functions are the means of those obtained by assigning the phases $\mathrm{ph}\, z = \pm\pi$, corresponding to the mean of Cubic Contours _ and −/‾. The appropriate basic functions are therefore

$$\bar{T}_{2r}{}^e(z) = -\frac{i(-1)^r}{(r-\frac{1}{2})!}\int_{-\infty e^{(1/3)i\pi}}^{-\infty e^{-(1/3)i\pi}} e^{v^2} \cos\left[v^3(-2/27z)^{\frac{1}{2}}\right] v^{2r}\,dv, \tag{48}$$

$$\bar{T}_{2r+1}{}^0(z) = -\frac{i(-1)^r}{(r+\frac{1}{2})!}\left(-\frac{27z}{2}\right)^{\frac{1}{2}}\int_{-\infty e^{(1/3)i\pi}}^{-\infty e^{-(1/3)i\pi}} e^{v^2} \sin\left[v^3(-2/27z)^{\frac{1}{2}}\right] v^{2r+1}\,dv, \tag{49}$$

and the uniform expansion is

$$\int_{\frac{1}{2}(\backslash\_\,-/\overline{\ })} e^{-F}\, G\, du = \left(\frac{2\pi}{F_2}\right)^{\frac{1}{2}} e^{-F_0} \sum_0^\infty \frac{1}{r!\,(2F_2)^r}$$

$$\times \left\{U_{2r}\bar{T}_{2r}{}^e(z) - (\tfrac{1}{3}r + \tfrac{1}{2})\frac{F_3}{F_2{}^2}\, U_{2r+1}\bar{T}_{2r+1}{}^0(z)\right\},$$

$$z = -F_2{}^3/3F_3{}^2 \text{ real and negative.} \tag{50}$$

While (43) and (50) record the simplest-looking general expressions of the uniform expansions, the recurrence relations (51)–(53) for the basic functions allow a number of variations. Of these, our own preference—for both theoretical and practical purposes—is for final expression of applications to be in basic functions of even order only, e.g. $T_0{}^e$, $T_2{}^e$, $T_4{}^e, \ldots,$ a point to which we shall return in Chapter XV, Section 2.

In the foregoing derivations we have adopted the attitude that uniform expansions are opportune when the cubic term in $F(u)$ is unusually large; thus in the integration through the stationary point $u = 0$ both quadratic and cubic components were left in their unexpanded exponential form $\exp(-\frac{1}{2}F_2u^2 - \frac{1}{6}F_3u^3)$. A different viewpoint is based on the observation that, in addition to the stationary point $u = 0$ on the chosen path of integration, $F_0 + \frac{1}{2}F_2u^2 + \frac{1}{6}F_3u^3$ possesses a second such point at $u = -2F_2/F_3$, which draws close to $u = 0$ under the same condition of unusually large $|F_3|$. We may therefore alternatively say a uniform expansion is opportune whenever a second stationary point approaches that through which the path of integration passes.

Properties and evaluation of the integrals T^e, T^0, $\bar{T}^e$ and $\bar{T}^0$.

Recurrence and differential relations. Proofs of the following are straightforward:

$$3T_{r+1}{}^e = (r+2)T_r{}^0 - (r-1)T_{r-2}{}^0, \tag{51}$$

$$(r+1)(r+3)T_{r+1}{}^0 = 36z(T_{r-2}{}^e - T_r{}^e), \tag{52}$$

$$r(r-2)T_{r+1}{}^e = -12z\{(r-2)T_{r-1}{}^e - (2r-3)T_{r-3}{}^e + (r-1)T_{r-5}{}^e\}, \tag{53}$$

$$\begin{aligned}(T_r{}^e)' &= (r+1)(r+3)(r+5)T_{r+3}{}^0/216z^2 \\ &= (r+1)(T_r{}^e - T_{r+2}{}^e)/6z,\end{aligned} \tag{54}$$

$$(T_r{}^0)' = (T_r{}^0 - T_{r+3}{}^e)/2z, \tag{55}$$

with identical relations for the barred functions.

Asymptotic series for large $|z|$. Expanding the second part of the exponential in (42), then integrating term by term,

$$\begin{aligned}T_{2r}{}^e(z) &= \frac{1}{(r-\frac{1}{2})!}\sum_0^\infty \frac{(3s+r-\frac{1}{2})!}{(2s)!}\left(-\frac{2}{27z}\right)^s = \frac{(-\frac{1}{2})!}{(\frac{1}{3}r-\frac{1}{6})!\,(\frac{1}{3}r-\frac{1}{2})!\,(\frac{1}{3}r-\frac{5}{6})!} \\ &\times \sum_0^\infty \frac{(s+\frac{1}{3}r-\frac{1}{6})!\,(s+\frac{1}{3}r-\frac{1}{2})!\,(s+\frac{1}{3}r-\frac{5}{6})!}{s!\,(s-\frac{1}{2})!}\left(-\frac{1}{2z}\right)^s, \\ &\quad |\mathrm{ph}\, z| < \pi,\end{aligned} \tag{56}$$

$$T_{2r+1}{}^{0}(z) = \frac{1}{(r+\frac{3}{2})!}\sum_{0}^{\infty}\frac{(3s+r+\frac{3}{2})!}{(2s+1)!}\left(-\frac{2}{27z}\right)^{s}$$

$$= \frac{(\frac{1}{2})!}{(\frac{1}{3}r+\frac{1}{2})!\,(\frac{1}{3}r+\frac{1}{6})!\,(\frac{1}{3}r-\frac{1}{6})!}$$

$$\times\sum_{0}^{\infty}\frac{(s+\frac{1}{3}r+\frac{1}{2})!\,(s+\frac{1}{3}r+\frac{1}{6})!\,(s+\frac{1}{3}r-\frac{1}{6})!}{s!\,(s+\frac{1}{2})!}\left(-\frac{1}{2z}\right)^{s}, \quad |\mathrm{ph}\, z| < \pi\,. \tag{57}$$

Expanding the trigonometrical functions in (48) and (49), then integrating term by term, asymptotic series formally identical to those quoted in (56) and (57) are found for $\bar{T}^{e}(z)$ and $\bar{T}^{0}(z)$ respectively when z is real and negative. (At this phase the asymptotic expansions for T^{e} and T^{0} abruptly acquire extra terms because of the Stokes discontinuity).

Placing $z = -F_2{}^3/3F_3{}^2$ and substituting (56) and (57), (43) and (50) reduce to the expansions in Chapter VI (3), (2) for quadratic dependence at a stationary point along contours $\uparrow$ and $\rightarrow$ respectively.

Absolutely convergent series. Expanding the first part of the exponential in (42), then integrating term by term,

$$T_{2r}{}^{e}(z) = \frac{2}{3(r-\frac{1}{2})!}\left(\frac{27z}{2}\right)^{\frac{1}{3}r+\frac{1}{6}}\sum_{0}^{\infty}\frac{\{\frac{2}{3}(s+r-1)\}!}{s!}$$

$$\times(-1)^{s}\cos\left[\tfrac{1}{3}\pi(s+r+\tfrac{1}{2})\right]\left(\frac{27z}{2}\right)^{\frac{1}{3}s}, \tag{58}$$

$$T_{2r+1}{}^{0}(z) = \frac{2}{3(r+\frac{3}{2})!}\left(\frac{27z}{2}\right)^{\frac{1}{3}r+\frac{5}{6}}\sum_{0}^{\infty}\frac{\{\frac{2}{3}(s+r-\frac{1}{2})\}!}{s!}$$

$$\times\,(-1)^{s}\cos\left[\tfrac{1}{3}\pi(s+r-\tfrac{1}{2})\right]\left(\frac{27z}{2}\right)^{\frac{1}{3}s}. \tag{59}$$

Substitution of the limits

$$\frac{T_{2r}{}^{e}(-F_2{}^3/3F_3{}^2)}{(-2F_2)^{r+\frac{1}{2}}}\underset{F_2\to 0}{\rightarrow}\tfrac{2}{3}\,\frac{(\frac{2}{3}r-\frac{2}{3})!}{(r-\frac{1}{2})!}\cos\left[\tfrac{1}{3}\pi(r+\tfrac{1}{2})\right]\left(\frac{3}{4F_3}\right)^{\frac{2}{3}r+\frac{1}{3}}$$

$$\frac{T^{0}_{2r+1}(-F_2{}^3/3F_3{}^2)}{(-2F_2)^{r+5/2}}\underset{F_2\to 0}{\rightarrow}\tfrac{2}{3}\,\frac{(\frac{2}{3}r-\frac{1}{3})!}{(r+\frac{3}{2})!}\cos\left[\tfrac{1}{3}\pi(r-\tfrac{1}{2})\right]\left(\frac{3}{4F_3}\right)^{\frac{2}{3}r+\frac{5}{3}}$$

reduces (43) to VI (8), the expansion for cubic dependence at a stationary point along contour > (for positive F_3) or along $-$ < (for negative F_3).

Expanding the exponential in (48) and (49), then integrating term by term,

$$\overline{T}_{2r}{}^{e}(z) = \frac{2}{3\,(r-\frac{1}{2})!}\left(-\frac{27z}{2}\right)^{\frac{1}{3}r+\frac{1}{6}} \sum_{0}^{\infty} \frac{\{\frac{2}{3}(s+r-1)\}!}{s!}$$

$$\times\ (-1)^{s}\cos^{2}\left[\tfrac{1}{3}\pi(s+r+\tfrac{1}{2})\right]\left(-\frac{27z}{2}\right)^{\frac{1}{3}s}, \tag{60}$$

$$\overline{T}^{0}_{2r+1}(z) = -\frac{2}{3\,(r+\frac{3}{2})!}\left(-\frac{27z}{2}\right)^{\frac{1}{3}r+\frac{5}{6}} \sum_{0}^{\infty} \frac{\{\frac{2}{3}(s+r-\frac{1}{2})\}!}{s!}$$

$$\times\ (-1)^{s}\cos^{2}\left[\tfrac{1}{3}\pi(s+r-\tfrac{1}{2})\right]\left(-\frac{27z}{2}\right)^{\frac{1}{3}s}. \tag{61}$$

Substitution of the limits

$$\frac{\overline{T}_{2r}{}^{e}(-F_2{}^{3}/3F_3{}^{2})}{(2F_2)^{r+\frac{1}{2}}} \underset{F_2\to 0}{\to} \begin{cases} 0, & r = 1, 4, 7, 10, \ldots \\ \dfrac{(\frac{2}{3}r-\frac{2}{3})!}{2\,(r-\frac{1}{2})!}\left(\dfrac{3}{4F_3}\right)^{\frac{2}{3}r+\frac{1}{3}}, & r = 0, 2, 3, 5, 6, 8, 9, \ldots \end{cases}$$

$$\frac{\overline{T}_{2r+1}{}^{0}(-F_2{}^{3}/3F_3{}^{2})}{(2F_2)^{r+\frac{3}{2}}} \underset{F_2\to 0}{\to} \begin{cases} 0, & r = 2, 5, 8, 11, \ldots \\ \dfrac{-(\frac{2}{3}r-\frac{1}{3})!}{2\,(r+\frac{3}{2})!}\left(\dfrac{3}{4F_3}\right)^{\frac{2}{3}r+\frac{5}{3}}, & r = 0, 1, 3, 4, 6, 7, \ldots \end{cases}$$

reduces (50) to VI (12) for positive F_3, the expansion for cubic dependence at a stationary point along the "combination contour" ½(__ − /‾); or, for negative F_3, along the combination contour ½(− ‾\\ + __/).

Mellin transform. In terms of the parameter $x = 2/27z$, the Mellin transform of $T_{2r}{}^{e}$ is

$$M_{2r}{}^{e}(m) = \int_{0}^{\infty} T_{2r}{}^{e}x^{m-1}dx = -\frac{i(-1)^{r}}{(r-\frac{1}{2})!}\int_{v=-\infty e^{(1/3)i\pi}}^{-\infty e^{-(1/3)i\pi}} e^{v^{2}}v^{2r}dv$$

$$\times \int_{x=0}^{\infty} e^{-v^{3}x^{1/2}}x^{m-1}dx = \frac{2}{(r-\frac{1}{2})!}(2m-1)!\,(-3m+r-\tfrac{1}{2})!\cos\pi m\,.$$

Hence

$$T_{2r}{}^{e}(z) = \frac{1}{\pi i(r-\frac{1}{2})!}\int_{\gamma-i\infty}^{\gamma+i\infty}(2m-1)!\,(-3m+r-\tfrac{1}{2})!\cos\pi m\left(\frac{27z}{2}\right)^{m}dm$$

$$= \frac{1}{2i\sqrt{\pi}(\frac{1}{3}r-\frac{1}{6})!\,(\frac{1}{3}r-\frac{1}{2})!\,(\frac{1}{3}r-\frac{5}{6})!}$$

$$\times\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(m-1)!(-m+\frac{1}{3}r-\frac{1}{6})!(-m+\frac{1}{3}r-\frac{1}{2})!(-m+\frac{1}{3}r-\frac{5}{6})!}{(-m-\frac{1}{2})!}(2z)^{m}\,dm$$

$$|\mathrm{ph}\,z| < \tfrac{1}{2}\pi. \tag{62}$$

Similarly,

$$T_{2r+1}{}^{0}(z) = -\frac{1}{\pi i(r+\frac{3}{2})!}\int_{\gamma-i\infty}^{\gamma+i\infty}(2m-2)!(-3m+r+\tfrac{3}{2})!\cos\pi m\left(\frac{27z}{2}\right)^{m}dm$$

$$= \frac{1}{4i\sqrt{\pi}(\frac{1}{3}r+\frac{1}{2})!\,(\frac{1}{3}r+\frac{1}{6})!\,(\frac{1}{3}r-\frac{1}{6})!}$$

$$\times\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(m-1)!(-m+\frac{1}{3}r+\frac{1}{2})!(-m+\frac{1}{3}r+\frac{1}{6})!(-m+\frac{1}{3}r-\frac{1}{6})!}{(-m+\frac{1}{2})!}(2z)^{m}\,dm,$$

$$|\mathrm{ph}\,z| < \tfrac{1}{2}\pi. \tag{63}$$

As will be seen in Chapter XV, these representations greatly facilitate resolution into their component T's of uniform expansions derived from homogeneous differential equations (uniform extension of "phase-integral" methods).

Tabulation. The integrals $T_0{}^e$, $T_1{}^0 = T_2{}^e$, $\overline{T}_0{}^e$ and $\overline{T}_1{}^0 = \overline{T}_2{}^e$ can be expressed in terms of modified Bessel functions of orders $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, or Airy functions and their first derivatives. Thus,

$$T_0{}^e(z) = (2z/\pi)^{\frac{1}{2}}e^{z}K_{\frac{1}{3}}(z) = (\tfrac{2}{3}\pi z)^{\frac{1}{2}}e^{z}(I_{-\frac{1}{3}}(z)-I_{\frac{1}{3}}(z)) = \sqrt{\pi}\,(96z)^{\frac{1}{6}}e^{z}Ai[(\tfrac{3}{2}z)^{\frac{2}{3}}], \tag{64}$$

$$T_1{}^0(z) = T_2{}^e(z) = (24\pi)^{\frac{1}{2}}z^{\frac{3}{2}}e^{z}(I_{-\frac{2}{3}} - I_{\frac{2}{3}} - I_{-\frac{1}{3}} + I_{\frac{1}{3}})$$

$$= -\sqrt{\pi}\,(1{,}990{,}656)^{\frac{1}{6}}z^{\frac{5}{6}}e^{z}\{(\tfrac{3}{2}z)^{\frac{1}{3}}Ai[(\tfrac{3}{2}z)^{\frac{2}{3}}] + Ai'[(\tfrac{3}{2}z)^{\frac{2}{3}}]\}, \tag{65}$$

$$\overline{T}_0{}^e(-z) = (2z/\pi)^{\frac{1}{2}}e^{-z}\mathscr{K}_{\frac{1}{3}}(z) = (\tfrac{1}{2}\pi z)^{\frac{1}{2}}e^{-z}(I_{-\frac{1}{3}}(z) + I_{\frac{1}{3}}(z))$$

$$= \sqrt{\pi}(\tfrac{3}{2}z)^{\frac{1}{6}}e^{-z}Bi[(\tfrac{3}{2}z)^{\frac{2}{3}}]\}, \tag{66}$$

$$\overline{T}_1{}^0(-z) = \overline{T}_2{}^e(-z) = (18\pi)^{\frac{1}{2}} z^{\frac{3}{2}} e^{-z}(I_{-\frac{1}{3}} + I_{\frac{1}{3}} - I_{-\frac{2}{3}} - I_{\frac{2}{3}})$$

$$= \sqrt{\pi}\,(31{,}104)^{\frac{1}{6}} z^{\frac{5}{6}} e^{-z}\{(\tfrac{3}{2}z)^{\frac{1}{3}} Bi[(\tfrac{3}{2}z)^{\frac{2}{3}}] - Bi'[(\tfrac{3}{2}z)^{\frac{2}{3}}]\}. \tag{67}$$

Regions of oscillatory variation are centred on the phases $|\mathrm{ph}\, z| = \frac{3}{2}\pi$. Alternative representations may then prove more convenient:

$$T_0{}^e(\zeta e^{\frac{3}{2}i\pi}) = (\tfrac{2}{3}\pi\zeta)^{\frac{1}{2}} e^{-i(\zeta - \frac{1}{4}\pi)}\{J_{-\frac{1}{3}}(\zeta) + J_{\frac{1}{3}}(\zeta)\}, \tag{68}$$

$$T_1{}^0(\zeta e^{\frac{3}{2}i\pi}) = (24\pi)^{\frac{1}{2}} \zeta^{\frac{3}{2}} e^{-i(\zeta - \frac{1}{4}\pi)}[J_{\frac{2}{3}} - J_{-\frac{2}{3}} + i\{J_{-\frac{1}{3}} + J_{\frac{1}{3}}\}], \tag{69}$$

$$\overline{T}_0{}^e(-\zeta e^{\frac{3}{2}i\pi}) = (\tfrac{1}{2}\pi\zeta)^{\frac{1}{2}} e^{i(\zeta + \frac{1}{4}\pi)}\{J_{-\frac{1}{3}} - J_{\frac{1}{3}}\}, \tag{70}$$

$$\overline{T}_1{}^0(-\zeta e^{\frac{3}{2}i\pi}) = (18\pi)^{\frac{1}{2}} \zeta^{\frac{3}{2}} e^{i(\zeta + \frac{1}{4}\pi)}[J_{\frac{2}{3}} + J_{-\frac{2}{3}} - i\{J_{-\frac{1}{3}} - J_{\frac{1}{3}}\}]. \tag{71}$$

We deliberately introduced z in (42), (48), (49) in such a way as to make it the argument in the Bessel functions of orders $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, rather than the argument in the alternative Airy functions and their derivatives. Airy functions undoubtedly possess descriptive and analytical merit in stationary and turning-point problems, largely resulting from the simple change in sign of their argument on passing straight through such a point. But Bessel functions of diverse orders have an enormously wider field of application, and interpolation between orders is straightforward (Salzer 1948). Considered numerically, the Airy functions are non-participating and redundant anomalies liable to be superseded for computational purposes.

The following tables, covering exponential-like variations, were constructed from the values of $I_{\pm\frac{1}{3}}$ and $I_{\pm\frac{2}{3}}$ published in "Tables of Bessel functions of fractional order" (1949), supplemented by the asymptotic series (56) and (57).

6. Alternative Derivations of Uniform Expansions, F Quadratic to Cubic in u

The uniform expansions (43) and (50) are simpler in both derivation and form (to a given order) than most published hitherto, though correspondingly not always as general. Earlier work has concentrated on expressing $\int_{\text{s.p.}} e^{-F}\, G\, du$ directly in terms of the well-known Airy function in the "linear/cubic" representation

$$Ai(\zeta) = \frac{1}{2\pi i}\int_{\infty e^{-(1/3)i\pi}}^{\infty e^{(1/3)i\pi}} e^{-\zeta v + \frac{1}{3}v^3}\, dv,$$

and its first derivative; (compare our "quadratic/cubic" representations (42)). This has been accomplished in either of two ways:

(i) Instead of expanding $F(u)$ about the stationary point $dF/du = 0$,

z	T_0^e	$T_1^0 = T_2^e$	T_3^0	T_4^e	T_5^0	T_6^e	T_7^0	T_8^e	T_9^0
·1	·80861491	·3100472	·1196563	− ·0072710	·032638	− ·083385	·00435	− ·05223	− ·0011
·2	·86264998	·4301345	·2076074	+ ·0592561	·076295	− ·098788	·01806	− ·09840	− ·0000
·3	·89024753	·5070380	·2759109	·1218261	·118865	− ·090529	·03640	− ·12852	+ ·0041
·4	·90771631	·5625633	·3313468	·1772025	·158548	− ·071849	·05693	− ·14632	·0108
·5	·91998817	·6052738	·3776572	·2259128	·195100	− ·048310	·07835	− ·15515	·0194
·6	·92917105	·6394810	·4171537	·2689355	·228679	− ·022620	·09996	− ·15747	·0294
·7	·93634325	·6676681	·4513742	·3071783	·259553	+ ·003791	·12136	− ·15504	·0404
·8	·94212240	·6913949	·4813967	·3413979	·287998	·030132	·14229	− ·14912	·0521
·9	·94689127	·7117030	·5080067	·3722091	·314274	·055965	·16264	− ·14063	·0643
1·0	·95090128	·7293202	·5317946	·4001109	·338615	·081043	·18232	− ·13026	·0768
1.1	·95432518	·7447740	·5532151	·4255092	·361225	·105239	·20131	− ·11851	·0895
1·2	·95728601	·7584577	·5726255	·4487373	·382283	·128494	·21960	− ·10578	·1022
1·3	·95987400	·7706717	·5903113	·4700711	·401946	·150792	·23718	− ·09236	·1149
1·4	·96215698	·7816498	·6065043	·4897496	·420349	·172142	·25408	− ·07846	·1276
1·5	·96418700	·7915774	·6213947	·5079396	·437613	·192570	·27032	− ·06427	·1401
1·6	·96600474	·8006035	·6351407	·5248321	·453841	·212108	·28592	− ·04992	·1525
1·7	·96764245	·8088496	·6478748	·5405582	·469127	·230796	·30091	− ·03552	·1646
1.8	·96912609	·8164157	·6597091	·5552380	·483552	·248675	·31532	− ·02114	·1766
1·9	·97047678	·8233848	·6707395	·5689759	·497188	·265785	·32918	− ·00684	·1884
2·0	·97171191	·8298269	·6810479	·5818619	·510100	·282168	·34251	+ ·00732	·1999
2·1	·97284592	·8358012	·6907053	·5939747	·522345	·297865	·35533	·02130	·2112
2·2	·97389092	·8413580	·6997740	·6053846	·533974	·312907	·36769	·03511	·2222
2·3	·97485713	·8465406	·7083070	·6161513	·545035	·327340	·37958	·04867	·2331
2·4	·97575325	·8513865	·7163524	·6263297	·555569	·341191	·39105	·06201	·2437
2·5	·97658674	·8559281	·7239517	·6359674	·565613	·354496	·40210	·07508	·2540
2·6	·97736402	·8601938	·7311425	·6451083	·575200	·367276	·41278	·08794	·2641
2·7	·97809065	·8642084	·7379569	·6537892	·584364	·379574	·42307	·10049	·2740
2·8	·97877148	·8679939	·7444252	·6620461	·593130	·391402	·43303	·11283	·2836
2·9	·97941075	·8715698	·7505731	·6699087	·601526	·402797	·44264	·12487	·2931
3·0	·98001220	·8749533	·7564244	·6774052	·609577	·413780	·45193	·13663	·3023

z	T_0^e	$T_1^0 = T_2^e$	T_3^0	T_4^e	T_5^0	T_6^e	T_7^0	T_8^e	T_9^0
3·1	·9805791	·878160	·76200	·68456	·61730	·4244	·4609	·1482	·3113
3·2	·9811144	·881203	·76732	·69140	·62472	·4346	·4696	·1594	·3202
3·3	·9816207	·884095	·77241	·69795	·63184	·4444	·4781	·1705	·3287
3·4	·9821002	·886847	·77727	·70421	·63871	·4540	·4862	·1812	·3372
3·5	·9825552	·889470	·78192	·71022	·64530	·4631	·4941	·1918	·3453
3·6	·9829874	·891972	·78638	·71598	·65167	·4721	·5018	·2020	·3535
3·7	·9833985	·894361	·79065	·72152	·65779	·4806	·5093	·2123	·3611
3·8	·9837900	·896645	·79476	·72684	·66370	·4890	·5166	·2223	·3685
3·9	·9841633	·898832	·79870	·73195	·66942	·4970	·5235	·2317	·3763
4·0	·9845197	·900926	·80250	·73688	·67494	·5049	·5303	·241	·384
4·1	·9848603	·902935	·80615	·74162	·68027	·5124	·5369	·250	·391
4·2	·9851861	·904863	·80966	·74619	·68548	·5199	·5431	·258	·399
4·3	·9854981	·906714	·81306	·75062	·69039	·5268	·5499	·268	·406
4·4	·9857971	·908494	·81632	·75487	·69527	·5339	·5557	·276	·413
4·5	·9860839	·910207	·81947	·75898	·69998	·5407	·561	·284	·420
4·6	·9863594	·911856	·82252	·76297	·70446	·5473	·567	·293	·426
4·7	·9866241	·913444	·82547	·76682	·70884	·5537	·573	·301	·432
4·8	·9868786	·914977	·82830	·77052	·71321	·5599	·578	·309	·439
4·9	·9871236	·916455	·83106	·77413	·71730	·5658	·584	·317	·445
5·0	·9873595	·91788	·8337	·7776	·7214	·572	·589	·324	·450
5·1	·9875869	·91926	·8363	·7810	·7253	·577	·594	·332	·455
5·2	·9878062	·92059	·8388	·7843	·7291	·582	·600	·340	·459
5·3	·9880180	·92188	·8413	·7875	·7328	·588	·605	·348	·463
5·4	·9882223	·92313	·8436	·7906	·7364	·593	·610	·355	·467
5·5	·9884198	·92434	·8459	·7936	·7398	·598	·614	·362	·471
5·6	·9886108	·92551	·8481	·7963	·7433	·604	·619	·370	·476
5·7	·9887955	·92665	·8502	·7993	·7467	·609	·623	·377	·480
5·8	·9889743	·92774	·8523	·8021	·7500	·613	·628	·385	·484
5·9	·9891475	·92881	·8543	·8047	·7531	·618	·632	·392	·488
6·0	·9893153	·92985	·8563	·8073	·7561	·623	·636	·399	·492

z	T_0^e	$T_1^0 = T_2^e$	T_3^0	T_4^e	T_5^0	T_6^e	T_7^0	T_8^e	T_9^0
6·1	·9894779	·93085	·8583	·8100	·760	·627	·64	·41	·50
6·2	·9896358	·93183	·8602	·8125	·763	·630	·65	·41	·50
6·3	·9897888	·93278	·8620	·8148	·766	·634	·65	·42	·51
6·4	·9899375	·93371	·8637	·8170	·769	·638	·65	·42	·51
6·5	·9900817	·93461	·8653	·8191	·772	·642	·66	·43	·51
6·6	·9902221	·93548	·8671	·8215	·775	·645	·66	·44	·52
6·7	·9903583	·93633	·8688	·8238	·777	·649	·66	·44	·52
6·8	·9904908	·93716	·8703	·8258	·780	·653	·67	·45	·53
6·9	·9906199	·93797	·8719	·8278	·782	·656	·67	·45	·53
7·0	·9907453	·93875	·8736	·8302	·784	·660	·67	·46	·53
7·1	·9908674	·93951	·8752	·8323	·786	·663	·68	·46	·54
7·2	·9909864	·94025	·8768	·8345	·788	·666	·68	·46	·54
7·3	·9911023	·94098	·8781	·8362	·790	·669	·68	·47	·55
7·4	·9912154	·94170	·8795	·8380	·792	·671	·68	·47	·55
7·5	·9913253	·94239	·8809	·8399	·794	·674	·69	·48	·56
7·6	·9914327	·94307	·8821	·8414	·796	·678	·69	·48	·56
7·7	·9915373	·94374	·8833	·8431	·798	·681	·69	·48	·57
7·8	·9916397	·94439	·8845	·8446	·800	·685	·70	·49	·57
7·9	·9917395	·94502	·8858	·8463	·802	·689	·70	·49	·57
8·0	·9918367	·94564	·8870	·8480	·804	·693	·70	·50	·58
8·1	·9919320	·94625	·8882	·8495	·806	·696	·70	·50	·58
8·2	·9920247	·94684	·8893	·8511	·808	·699	·71	·50	·59
8·3	·9921156	·94742	·8905	·8526	·810	·702	·71	·51	·59
8·4	·9922045	·94798	·8916	·8540	·812	·705	·71	·51	·59
8·5	·9922912	·94853	·8927	·8555	·814	·708	·71	·51	·60
8·6	·9923761	·94907	·8938	·8569	·816	·711	·72	·52	·60
8·7	·9924590	·94960	·8948	·8583	·817	·714	·72	·52	·60
8·8	·9925402	·95012	·8958	·8597	·819	·717	·72	·53	·61
8·9	·9926197	·95063	·8968	·8610	·821	·719	·72	·53	·61
9·0	·9926975	·95113	·8978	·8623	·822	·722	·73	·53	·61

z	T_0^e	$T_1^0 = T_2^e$	T_3^0	T_4^e	T_5^0	T_6^e	T_7^0	T_8^e	T_9^0
9·1	·9927737	·95162	·8988	·8636	·824	·724	·73	·54	·62
9·2	·9928483	·95210	·8997	·8649	·825	·726	·73	·54	·62
9·3	·9929213	·95257	·9006	·8662	·827	·728	·73	·55	·62
9·4	·9929929	·95303	·9016	·8674	·828	·730	·74	·55	·63
9·5	·9930630	·95348	·9025	·8686	·830	·732	·74	·55	·63
9·6	·9931319	·95392	·9033	·8698	·831	·735	·74	·56	·63
9·7	·9931993	·95436	·9042	·8709	·832	·737	·74	·56	·63
9·8	·9932652	·95478	·9051	·8721	·834	·739	·74	·56	·63
9·9	·9933302	.95520	·9059	·8732	·835	·741	·75	·56	·64
10·0	·9933936	·95561	·9068	·8743	·836	·743	·75	·57	·64

$-z$	$\overline{T}_0^{\,e}$	$\overline{T}_1^{\,0}=\overline{T}_2^{\,e}$	$\overline{T}_3^{\,0}$	$\overline{T}_4^{\,e}$	$\overline{T}_5^{\,0}$
·1	·86979494	−·1067598	−·23437	−·69269	−·0603
·2	·98142419	+·0626508	−·44101	−1·16518	−·2526
·3	1·03856276	·2865173	−·54147	−1·64020	−·5945
·4	1·07021588	·5123140	−·53559	−1·85128	−·9724
·5	1·08753306	·7188794	−·44238	−1·82484	−1·3082
·6	1·09616721	·8979739	−·28540	−1·61147	−1·5487
·7	1·09937315	1·0475834	−·08701	−1·26510	−1·6651
·8	1·09915892	1·1688062	+·13372	−·83450	−1·6484
·9	1·09681391	1·2642336	·36163	−·36017	−1·5037
1·0	1·09318525	1·3370654	·58531	+·08414	−1·2887
1·1	1·08883592	1·3906197	·79671	·40077	−1·1199
1·2	1·08414093	1·4280680	·99051	·69880	−·9001
1·3	1·07934857	1·4523010	1·16361	·97115	−·6434
1·4	1·07462059	1·4658722	1·31460	1·21376	−·3630
1·5	1·07005933	1·4709866	1·44334	1·42491	−·0711
1·6	1·06572619	1·4695144	1·55055	1·60457	+·2223
1·7	1·06165451	1·4630189	1·63757	1·75393	·5087
1·8	1·05785857	1·4527897	1·70610	1·87498	·7816
1·9	1·05433983	1·4398781	1·75805	1·97017	1·0076
2·0	1·05109140	1·4251309	1·79539	2·04223	1·2695
2·1	1·04810116	1·4092219	1·82005	2·09393	1·4790
2·2	1·04535392	1·3926805	1·83388	2·12802	1·6640
2·3	1·04283291	1·3759163	1·83862	2·14709	1·8244
2·4	1·04052085	1·3592407	1·83583	2·15355	1·9608
2·5	1·03840061	1·3428857	1·82691	2·14959	2·0744
2·6	1·03645567	1·3270190	1·81311	2·13718	2·1666
2·7	1·03467039	1·3117572	1·79552	2·11803	2·2391
2·8	1·03303016	1·2971766	1·77506	2·09366	2·2939
2·9	1·03152148	1·2833220	1·75253	2·06534	2·3326
3·0	1·03013196	1·2702140	1·72859	2·03417	2·3574
3·1	1·02885031	1·2578548	1·70379	2·00108	2·3698
3·2	1·02766630	1·2462331	1·67859	1·96683	2·3718
3·3	1·02657064	1·2353277	1·65336	1·93204	2·3648
3·4	1·02555500	1·2251103	1·62837	1·89721	2·3504
3·5	1·02461184	1·2155485	1·60387	1·86275	2·3299
3·6	1·02373442	1·2066068	1·58002	1·82896	2·3045
3·7	1·02291667	1·1982486	1·55695	1·79608	2·2752
3·8	1·02215314	1·1904369	1·53475	1·76429	2·2429
3·9	1·02143896	1·1831352	1·51348	1·73370	2·2086
4·0	1·02076974	1·1763082	1·49317	1·70441	2·1728
4·1	1·02014156	1·1699216	1·47383	1·67644	2·1361
4·2	1·01955090	1·1639435	1·45548	1·64983	2·0990
4·3	1·01899459	1·1583431	1·43808	1·62457	2·0620
4·4	1·01846978	1·1530919	1·42161	1·60062	2·0254
4·5	1·01797392	1·1481634	[illegible]·40605	1·57797	1·9894
4·6	1·01750469	1·1435328	1·39135	1·55656	1·9542
4·7	1·01706001	1·1391771	1·37748	1·53635	1·9200
4·8	1·01663801	1·1350755	1·36440	1·51728	1·8870
4·9	1·01623699	1·1312083	1·35206	1·49930	1·8552
5·0	1·01585541	1·1275577	1·34043	1·48234	1·8246

an expansion has been effected about the point $d^2F/du^2 = 0$; then the quadratic term vanishes, leaving linear, cubic and higher terms. The central drawback to this approach is that since the expansion point does not coincide with the one adopted in the simpler quadratic and cubic expansions about a stationary point (Chapter VI), an entirely new set of derivatives of F and G has to be found; moreover, arising out of this, the result does not pass over easily into these simpler expansions. Also, the expansion is only "transitional", not uniform.

(ii) $F(u)$ has been mapped as a cubic $(-\zeta v + \frac{1}{3}v^3)$ by a regular $(1, 1)$ transformation, an approach first carried through in this context by Chester, Friedman and Ursell (1957).

Once $F(u)$ has been expressed in approximate or exact cubic form by either of these methods, evaluation of the leading term in the expansion is straightforward. But higher terms present difficulty because the most natural development of the remaining factors in the integrand,

$$G(u)\, du/dv = \sum c_r v^r ,$$

leads to integrals belonging to the sequence

$$\int v^r e^{\lambda(-\zeta v + \frac{1}{3}v^3)} dv , \qquad r = 0, 1, 2, \ldots ,$$

where λ is a magnitude parameter. Successive contributions to this sequence fail to decrease uniformly for large λ, substantially because convergence of each integral would not be assured on retaining only the entry of *lowest* degree in the exponent, here linear in v; (compare our (42) in which convergence would be so assured by the entry of lowest degree, that quadratic in v). Consequently, this sequence does not constitute a valid asymptotic expansion. Chester, Friedman and Ursell ingeniously overcame the difficulty by stipulating a rearrangement of the series $\Sigma c_r v^r$ as follows:

$$G(u)\, du/dv = \sum p_r(v^2 - \zeta)^r + \sum q_r v(v^2 - \zeta)^r .$$

This successfully leads to a validly-asymptotic uniform expansion.

The one drawback to this powerful approach is that the intercalation of the two transformations—first mapping the exponent as a cubic, then rearranging the series for $G\, du/dv$ in the prescribed form—protracts the calculations to no mean extent. Whether instances can arise in which such mapping is actually unavoidable, only future experience will tell. The insensitivity of ordinary "saddle-point" asymptotic series to the path chosen through the stationary point and to the method of expansion adopted, and the underlying reason for this insensitivity based on the origin of asymptotic behaviour (Chapter VI, Section 3), indicate that once any good approximation to the cubic term is retained in the exponent, the resulting expansion will be uniform through *one* breakdown in the ordinary saddle-point asymptotic series. Mapping is therefore likely to be obligatory only when an expansion has to be uniform through two or more breakdowns in the ordinary series. While such super expansions are intrinsically of great interest, from a narrower pragmatic standpoint they are more general than is necessary for the one really vital commission of uniform expansions—finding from a differential equation values of the function and its derivatives at a turning point, the quantities required for the convergent Taylor series best in this region (Chapter XV, Section 1).

The particular case of Bessel functions in the transitional region has attracted a great deal of attention. The chief references are as follows: J. W. Nicholson, *Phil. Mag.* **19**, 228–249, 1910. G. N. Watson, *Proc. Camb. Phil. Soc.* **19**, 96–110, 1918. W. Schöbe, *Arch. Math.* **1**, 230–232, 1948. F. Tricomi, *Atti. Accad. Sci. Torino*; *Cl. Sci. Fis. Mat. Nat.*, **83**, 3–20, 1949. F. W. J. Olver, *Proc. Camb. Phil. Soc.* **48**, 414–427, 1952. A. Erdélyi and M. Wyman, *Arch. Rat. Mech. Anal.* **14**, 217–260, 1963. M. Wyman, *Trans. Roy. Soc. Can.* **2**, 227–256, 1964.

Our general theory developed in this chapter is most closely akin to Watson's calculation of the leading term for Bessel functions in the transitional region.

Exercises

1. Derive the uniform expansion for $\int_{\text{limit}} e^{-F}\, G\, du$, F linear to quadratic in u, via "conversion by power identification" (end part of Chapter V, Section 2).

Imagine the formula when $F_1 = 0$, i.e.

$$\int_{\text{limit}} e^{-F}\, G\, du = \left(\frac{\pi}{2F_2}\right)^{\frac{1}{2}} e^{-F_0} \sum_0^{\infty} Q_r,$$

to have been obtained through reduction to component integrals

$$\int_0^{\infty} e^{-\frac{1}{2}F_2 u^2} u^n du = \left(\frac{\pi}{2F_2}\right)^{\frac{1}{2}} \left[\left(\frac{2}{F_2}\right)^{\frac{1}{2}n} \frac{(\frac{1}{2}n-\frac{1}{2})!}{(-\frac{1}{2})!}\right].$$

When $F_1 \neq 0$, each such component integral stands to be replaced by

$$\int_0^{\infty} e^{-F_1 u - \frac{1}{2}F_2 u^2} u^n du = n!\, F_1^{\,-n-1} \mathscr{T}_n(F_1/\sqrt{F_2}).$$

Hence show that

$$\int_{\text{limit}} e^{-F}\, G\, du = F_1^{-1} e^{-F_0} \sum_0^\infty \bar{Q}_r,$$

where the $\bar{Q}_r$ are derived from the Q_r by replacing $1/F_2^{\frac{1}{2}n}$ with

$$(\tfrac{1}{2}n)!\,(2/F_2)^{\frac{1}{2}n} z^{-n} \mathscr{T}_n(z), \qquad z = F_1/\sqrt{F_2}.$$

2. Establish the following pair of uniform expansions for incomplete factorial functions, of which the leading term was found by Tricomi (1950):

$$(p, x)! = \frac{x^{p+1}e^{-x}}{p - x} \left\{ \mathscr{T}_0 - \frac{2p}{(p-x)^3} \mathscr{T}_3 - \frac{6p}{(p-x)^4} \mathscr{T}_4 - \frac{24p}{(p-x)^5} \mathscr{T}_5 \right.$$

$$+ \frac{40p(p-3)}{(p-x)^6} \mathscr{T}_6 + \frac{60p(7p-12)}{(p-x)^7} \mathscr{T}_7 + \frac{21p(143p-240)}{(p-x)^8} \mathscr{T}_8$$

$$\left. - \frac{224p(10p^2 - 171p + 180)}{(p-x)^9} \mathscr{T}_9 \ldots \right\}$$

where $\mathscr{T} = \mathscr{T}([p - x]/\sqrt{p})$; and

$$[p, x]! = \frac{x^{p+1}e^{-x}}{x - p} \left[\mathscr{T}_0 + \frac{2p}{(x-p)^3} \mathscr{T}_3 - \frac{6p}{(x-p)^4} \mathscr{T}_4 + \ldots \right]$$

where $\mathscr{T} = \mathscr{T}([x - p]/\sqrt{p})$.

3. Derive the following uniform expansions for confluent hypergeometric functions:

$$F(a, c, x) = \frac{(C + 2a - 1)!}{(C + a - 1)!} (x - C)^{-a} \left[\mathscr{T}_{a-1} - \frac{a}{3z\sqrt{C}} \right.$$

$$\times \{2(2a - 1)\mathscr{T}_a - (a + 1)\mathscr{T}_{a+1}\} + \frac{a(a+1)}{36(z\sqrt{C})^2} \{(32a^2 - 39a + 10)\, \mathscr{T}_{a+1}$$

$$- (a + 2)(16a - 7)\, \mathscr{T}_{a+2} + 2(a + 2)(a + 3)\, \mathscr{T}_{a+3}\}$$

$$- \frac{a(a+1)(a+2)}{1620(z\sqrt{C})^3} \{2(320a^3 - 690a^2 + 427a - 69)\, \mathscr{T}_{a+2}$$

$$- 3(a + 3)(160a^2 - 175a + 38)\, \mathscr{T}_{a+3} + 15(a + 3)(a + 4)(8a - 3)\, \mathscr{T}_{a+4}$$

$$\left. - 10(a + 3)(a + 4)(a + 5)\, \mathscr{T}_{a+5}\} \ldots \right],$$

where $C = c - 2a$, $\mathscr{T} = \mathscr{T}(z)$, $z = (C - x)/\sqrt{C}$; and

$$\psi(a, c, x) = \left(\frac{x}{C - x}\right)^a \left[\mathscr{T}_{a-1}(-z) - \frac{a}{3z\sqrt{C}}\right.$$
$$\left. \times \{2(2a - 1)\mathscr{T}_a(-z) - (a + 1)\mathscr{T}_{a+1}(-z)\} \ldots \right].$$

4. Establish the following uniform expansion for the Anger function of negative order:

$$A_p(x) = \int_0^\infty e^{p\omega - x\sinh\omega} d\omega = \frac{1}{x - p}\left\{\Upsilon_0 - \frac{x}{(x-p)^5}\Upsilon_5 - \frac{x}{(x-p)^7}\Upsilon_7\right.$$
$$\left. - \frac{x}{(x-p)^9}\Upsilon_9 + \frac{126x^2}{(x-p)^{10}}\Upsilon_{10} - \frac{x}{(x-p)^{11}}\Upsilon_{11} + \frac{792x^2}{(x-p)^{12}}\Upsilon_{12} \ldots\right\}$$

where

$$\Upsilon = \Upsilon([x - p](2/x)^{\frac{1}{3}}).$$

5. Establish the following uniform expansions for $J_p(x)$, of which the leading term was found by Watson (1944):

$$J_p(x) = \left(\frac{q}{2\pi p}\right)^{\frac{1}{2}} e^{-p\Xi}\left\{T_0{}^e + \frac{q}{8p}T_4{}^e + \frac{7q^4}{48p^2}T_5{}^0 - \frac{q^2}{48p^2}T_6{}^e - \frac{q^5}{32p^3}T_7{}^0\right.$$
$$\left. + \frac{q^2(35p + q)}{384p^3}T_8{}^e + \frac{11q^5(126p + q)}{2{,}304p^4}T_9{}^0 \ldots\right\}$$

$$= \left(\frac{q}{2\pi p}\right)^{\frac{1}{2}} e^{-p\Xi}\left\{T_0{}^e + \frac{1}{40(3p^2z)^{\frac{1}{3}}}(2T_2{}^e + 3T_4{}^e) + \frac{3}{4{,}480(3p^2z)^{\frac{2}{3}}}\right.$$
$$\times (16T_4{}^e + 40T_6{}^e + 49T_8{}^e) + \frac{1}{76{,}800(3p^2z)}(320T_6{}^e + 1{,}120T_8{}^e$$
$$+ 2{,}106T_{10}{}^e + 2{,}079T_{12}{}^e) + \frac{3}{445{,}018{,}112{,}000(3p^2z)^{\frac{4}{3}}}$$
$$\times (349{,}476{,}480T_8{}^e + 1{,}572{,}644{,}160T_{10}{}^e + 3{,}854{,}423{,}056T_{12}{}^e$$
$$\left. + 5{,}966{,}024{,}064T_{14}{}^e + 4{,}894{,}004{,}115T_{16}{}^e) \ldots\right\},$$

where

$$q = p/(p^2 - x^2)^{\frac{1}{2}}, \qquad \Xi = \tanh^{-1} q^{-1} - q^{-1}, \qquad T = T(z), \qquad z = p/3q^3.$$

6. Show that the corresponding uniform expansion for $Y_p(x)$ is

$$Y_p(x) = -\left(\frac{2q}{\pi p}\right)^{\frac{1}{2}} e^{p\Xi}\left\{\bar{T}_0{}^e - \frac{1}{40(3p^2z)^{\frac{1}{3}}}(2\bar{T}_2{}^e + 3\bar{T}_4{}^e) + \dots\right\},$$

where $\bar{T} = \bar{T}(-z)$; i.e. differing from that for $J_p(x)$ by having barred basic functions and reversed signs before z.

7. Derive the following uniform expansion for the Whittaker function when k is large and m small:

$$W_{km}(x) = \left(\frac{(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!}{2\pi}\right)^{\frac{1}{2}} q^{\frac{1}{2}} e^{-2k\Xi}\left[T_0{}^e + \frac{1}{40(6k^2z)^{\frac{1}{3}}}\right.$$

$$\times (2T_2{}^e + 3T_4{}^e) + \frac{1}{4{,}480(6k^2z)^{\frac{2}{3}}}\{-420\eta z T_0{}^e + 70(2-\eta)T_2{}^e + 183T_4{}^e$$

$$+ 195T_6{}^e + 147T_8{}^e\} - \frac{z}{134{,}400(6k^2z)}\{56(7-45\eta)\,T_0{}^e + 28(7-45\eta)\,T_2{}^e$$

$$\left. + 27(9-35\eta)\,T_4{}^e + 240T_6{}^e + 1{,}260T_8{}^e + 3{,}969T_{10}{}^e\}\dots\right],$$

where the notation is as in Chapter VIII, Section 8, with the additions $T = T(z)$, $z = 4k/3q^3$.

8. Derive the following uniform expansion for the Whittaker function when $\kappa = (k^2 - m^2)^{\frac{1}{2}}$ is large:

$$W_{km}(x) = \frac{1}{2}\left(\frac{(k+m-\frac{1}{2})!\,(k-m-\frac{1}{2})!}{\pi}\right)^{\frac{1}{2}}\left(\frac{Aq^2+a}{q}\right)^{\frac{1}{2}}$$

$$\times e^{-2\kappa\Xi}\left[T_0{}^e + \frac{1}{60}\left(\frac{9}{512A\kappa^2 z}\right)^{\frac{1}{3}} B(2T_2{}^e + 3T_4{}^e)\right.$$

$$+ \frac{1}{10{,}080}\left(\frac{9}{512A\kappa^2 z}\right)^{\frac{2}{3}}\{3B^2(16T_4{}^e + 40T_6{}^e + 49T_8{}^e)$$

$$- 40Ab(24zT_0^e + 28T_2^e + [24 + 18z]T_4^e + 15T_6^e)\}$$

$$- \frac{z}{453{,}600}\left(\frac{9}{512A\kappa^2 z}\right)\{B^3(1{,}792T_0^e + 896T_2^e - 432T_4^e - 2{,}760T_6^e$$

$$- 3{,}465T_8^e + 3{,}969T_{10}^e) + 20AbB(1{,}981T_0^e - 112T_2^e - 324T_4^e$$

$$- 600T_6^e - 945T_8^e) + 700A^2a([175 - 360z]T_0^e - 116T_2^e - 54T_4^e - 30T_6^e)$$

$$\left. - 1{,}400A(199 - 451k^2/\kappa^2)\, T_0^e\} \quad \ldots \right],$$

where the notation is as in Chapter XIII, Section 11, with the additions $T = T(z)$, $z = 8\kappa/3Aq^3$.

9. Establish the following uniform expansion covering the case F quadratic to cubic in u:

$$\int_{\text{limit}} e^{-F}\, G\, du = \left(\frac{\pi}{2F_2}\right)^{\frac{1}{2}} e^{-F_0} \sum_0^\infty \frac{1}{(\frac{1}{2}r)!\,(2F_2)^{\frac{1}{2}r}}\, U_r\, \mathfrak{T}_r(-F_2{}^3/3F_3{}^2)$$

where

$$\mathfrak{T}_r(z) = \frac{2}{(\frac{1}{2}r - \frac{1}{2})!} \int_0^\infty e^{-v^2 - v^3(-2/27z)^{1/2}} v^r\, dv,$$

and the U_r are as in (45)–(46).

10. Derive the recurrence relation

$$(\tfrac{1}{2}r)!\, \mathfrak{T}_{r+1} = (\tfrac{1}{2}r - \tfrac{1}{2})!\,(-6z)^{\frac{1}{2}}\,(\mathfrak{T}_{r-2} - \mathfrak{T}_r).$$

11. Derive the differentiation rule

$$(\tfrac{1}{2}r - \tfrac{1}{2})!\,(-54z^3)^{\frac{1}{2}}\, \mathfrak{T}_r' = (\tfrac{1}{2}r + 1)!\, \mathfrak{T}_{r+3}.$$

12. Obtain the asymptotic series

$$\mathfrak{T}_r(z) = \frac{1}{(\frac{1}{2}r - \frac{1}{2})!} \sum_0^\infty \frac{(-1)^s(\frac{3}{2}s + \frac{1}{2}r - \frac{1}{2})!}{s!} \left(-\frac{2}{27z}\right)^{\frac{1}{2}s}, \quad |\mathrm{ph}(-z)| < 2\pi,$$

and verify that its substitution reduces the uniform expansion of question 9 to the result in Chapter V (20) for quadratic dependence at a limit of integration.

13. Obtain the absolutely convergent series

$$\mathfrak{T}_r(z) = \tfrac{2}{3}\left(-\frac{27z}{2}\right)^{\frac{1}{3}r+\frac{1}{3}} \frac{1}{(\frac{1}{2}r-\frac{1}{2})!} \sum_0^\infty \frac{(-1)^s(\frac{2}{3}s+\frac{1}{3}r-\frac{2}{3})!}{s!}\left(-\frac{27z}{2}\right)^{\frac{1}{3}s},$$

and verify that passage to the limit $F_2 \to 0$ reduces the uniform expansion of question 9 to the result in Chapter V (29) for cubic dependence at a limit of integration.

14. Show that these basic functions appropriate to integration from a limit can be expressed in terms of (42), the functions appropriate to integration through a stationary point:

$$\mathfrak{T}_r(z) = T_r^e(z) - \left(-\frac{2}{27z}\right)^{\frac{1}{2}} \frac{(\frac{1}{2}r+1)!}{(\frac{1}{2}r-\frac{1}{2})!} T_r^0(z).$$

(Unfortunately $T_{\text{odd}}{}^e$ and $T_{\text{even}}{}^0$ do not appear to be reducible to tabulated functions).

15. Establish the following uniform expansion covering the case F quadratic to cubic in u:

$$\int_{\text{limit}} e^{-F} G u^\sigma \, du = \frac{1}{2}\left(\frac{2}{F_2}\right)^{\frac{1}{2}\sigma+\frac{1}{2}} e^{-F_0} \sum_0^\infty \frac{(\frac{1}{2}r+\frac{1}{2}\sigma-\frac{1}{2})!}{r!}$$

$$\times \left(\frac{2}{F_2}\right)^{\frac{1}{2}r} U_r \mathfrak{T}_{r+\sigma}(-F_2{}^3/3F_3{}^2),$$

where U_r and $\mathfrak{T}_r$ are the same as in question 9.

16. Derive the uniform expansion for $\int_{\text{limit}} e^{-F} G \, du$, F quadratic to cubic in u, via "conversion by power identification" (cf. question 1), showing that

$$\int_{\text{limit}} e^{-F} G \, du = (\pi/2F_2)^{\frac{1}{2}} e^{-F_0} \sum_0^\infty \bar{C}_r,$$

where the $\bar{C}_r$ are derived from the C_r by replacing $1/F_3{}^{\frac{1}{3}n}$ with

$$\frac{(\frac{1}{2}n-\frac{1}{2})!}{(-\frac{1}{2})!} \frac{(-\frac{2}{3})!}{(\frac{1}{3}n-\frac{2}{3})!} \left(\frac{2}{9F_2{}^3}\right)^{\frac{1}{3}n} \mathfrak{T}_n(z), \qquad z = -\frac{F_2{}^3}{3F_3{}^2}.$$

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Chapter XI

Derivation of Uniform Asymptotic Expansions from Integral Representations

$$\int_{u=0} e^{-F(u)}\,(u-u_0)^\sigma\,G(u)\,du \quad\text{and}\quad \int_{\mathrm{s.p.}} e^{-F(u)}\,(u-u_0)^\sigma\,G(u)\,du.$$

1. Behaviour of Earlier Asymptotic Expansions when a Singularity approaches the Critical Point

Linear dependence of $F(u)$ at a limit

We shall suppose the integral to have been reduced to a standard form in which the limit of integration is at $u = 0$ and the branch point or pole at $u = u_0$, normally negative. The required integral is thus written $\int_{u=0} e^{-F}(u - u_0)^\sigma G\,du$ where σ is a positive or negative fraction (branch point at u_0) or a negative integer (pole at u_0).

It is at once evident that when u_0 is small the factor $(u - u_0)^\sigma$ needs to be included in the fast-varying portion of the integrand, leading us to rewrite the integral as $\int_{u=0} e^{-\bar F}\,G(u)\,du$ where $\bar F = F - \sigma \ln(u - u_0)$. The modified exponent has for its derivatives at $u = 0$

$$\bar F_j = F_j + \sigma(j-1)!\,u_0^{-j}, \qquad j \geqslant 1, \tag{1}$$

so the additional contributions predominate when $|u_0|$ is sufficiently small; the criterion of smallness applicable in the first derivative, $|u_0| < |\sigma/F_1|$, almost invariably suffices in all. Substitution in V (12) yields

$$\int_{u=0} e^{-\bar F}\,G\,du = \frac{e^{-\bar F_0}}{\bar F_1}\left[G_0\left\{1 - \frac{\bar F_2}{\bar F_1{}^2} + \frac{3\bar F_2{}^2 - \bar F_1\bar F_3}{\bar F_1{}^4}\cdots\right\} + \frac{G_1}{\bar F_1}\left\{1 - \frac{3\bar F_2}{\bar F_1{}^2}\right.\right.$$
$$\left.\left. + \frac{15\bar F_2{}^2 - 4\bar F_1\bar F_3}{\bar F_1{}^4}\cdots\right\} + \frac{G_2}{\bar F_1{}^2}\left\{1 - \frac{6\bar F_2}{\bar F_1{}^2} + \frac{5(9\bar F_2{}^2 - 2\bar F_1\bar F_3)}{\bar F_1{}^4}\cdots\right\}\cdots\right]$$

$$\simeq \frac{e^{-F_0}}{\bar{F}_1}\left[G_0\left(1-\frac{1}{\sigma}+\frac{1}{\sigma^2}-\ldots\right)+\frac{G_1}{\bar{F}_1}\left(1-\frac{3}{\sigma}+\frac{7}{\sigma^2}-\ldots\right)\right.$$
$$\left.+\frac{G_2}{\bar{F}_1{}^2}\left(1-\frac{6}{\sigma}+\frac{25}{\sigma^2}-\ldots\right)+\ldots\right], \tag{2}$$

when $|u_0| < |\sigma/F_1|$. Since $|\sigma| = O(1)$ the normal asymptotic expansion thus fails, as it stands, to provide the required sequence of initially decreasing contributions.

The correction series in (2) can however all be recognized and summed by examining the limit $F \to 0$ in the original integral. The integral then reduces to $\int (u-u_0)^\sigma G\,du$, showing the σ-series multiplying G_j to originate from $\int_0^{u_0}(u-u_0)^\sigma\,(u^j/j!)\,du$, essentially a Beta-function. Adjusting the multiplying constant correctly,

$$\text{Factor of } G_j/\bar{F}_1{}^j \text{ in (2)} = \sigma^{j+1}\,\sigma!/(\sigma+j+1)!\,. \tag{3}$$

It is possible to reframe the entire expansion by proceeding further along this line of argument; but both derivation and result prove needlessly complicated.

Quadratic dependence of $F(u)$ at a stationary point

Again introducing the modified exponent $\bar{F}$ and working in a co-ordinate system such that $\bar{F}' = 0$ at $u = 0$, substitution in VI (2) yields for $|u_0| < |(\sigma/F_2)^{\frac{1}{2}}|$:

$$\int_{\text{s.p.}} e^{-F}\,G\,du \simeq \left(\frac{2\pi}{\bar{F}_2}\right)^{\frac{1}{2}} e^{-F_0}\left[\left\{G_0 - \frac{G_1}{(\sigma\bar{F}_2)^{\frac{1}{2}}}\right\}\left(1+\frac{1}{12\sigma}+\frac{1}{288\sigma^2}\right.\right.$$
$$\left.\left.-\frac{139}{51840\sigma^3}\cdots\right)+\frac{G_2}{2\bar{F}_2}\left(1+\frac{25}{12\sigma}\cdots\right)\cdots\right]. \tag{4}$$

The correction series in (4) can all be recognized by discarding all derivatives of F except $F_1 = -\sigma/u_0$, retention of this derivative being essential to exact the stationary condition $\bar{F}_1 = 0$. The integral then reduces to $\int e^{u\sigma/u_0}(u-u_0)^\sigma\,G du$, showing the σ-series multiplying G_j to originate from $\int_{\text{s.p.}} e^{u\sigma/u_0}(u-u_0)^\sigma\,(u^j/j!)du$. This integral can be reduced to elementary factorial functions once u^j has been expanded in powers of $u-u_0$ by the binomial theorem. Adjusting the multiplying constant correctly,

$$\text{Factor of } G_j/\bar{F}_2^{\frac{1}{2}j} \text{ in (4)} = \begin{cases} \dfrac{e^{\sigma}}{(2\pi)^{\frac{1}{2}}\,\sigma^{\sigma+\frac{1}{2}}}\,\sigma^{\frac{1}{2}j}\displaystyle\sum_0^j \frac{(s+\sigma)!}{s!(j-s)!\,(-\sigma)^s}, & \\ \qquad\qquad \Re(\sigma) > 0, & (5^+) \\ \dfrac{(2\pi)^{\frac{1}{2}}\,e^{\sigma}}{(-\sigma)^{\sigma+\frac{1}{2}}}\,\sigma^{\frac{1}{2}j}\displaystyle\sum_0^j \frac{1}{s!(j-s)!\,(-s-\sigma-1)!\,\sigma^s}, & \\ \qquad\qquad \Re(\sigma) < 0. & (5^-) \end{cases}$$

The approximate correction series in (4) are therefore essentially Stirling-type expansions for factorials.

As seen from (4), when $|\sigma| = O(1)$ the approximate correction series multiplying both G_0 and G_1 could have been roughly evaluated in its original form, whereas those multiplying G_2, G_3, .., could not. Hence the normal asymptotic expansion in Chapter VI (2) remains marginally serviceable provided derivatives from G_2 onwards can be discarded.

Cubic dependence of $F(u)$ at a stationary point

With the modified exponent $\bar{F}$ and a co-ordinate system such that $\bar{F}' = \bar{F}'' = 0$ at $u = 0$, substitution in VI (8) yields for $|u_0| < |(2\sigma/F_3)^{\frac{1}{3}}|$:

$$\int_{>} e^{-F}\,G\,du \simeq \frac{i\alpha\sqrt{3}\,e^{-F_0}}{\bar{F}_3^{\frac{1}{3}}}\left[\left\{G_0 - \frac{G_2}{(2\sigma\bar{F}_3^{\,2})^{\frac{1}{3}}}\right\}\left\{1 + \frac{3\beta}{(2\sigma)^{\frac{1}{3}}} + \frac{1}{240\sigma}\cdots\right\}\right.$$

$$\left. - \frac{6\beta G_1}{\bar{F}_3^{\frac{2}{3}}}\left\{1 - \frac{1}{40\,\beta\,(2\sigma)^{\frac{2}{3}}}\cdots\right\}\cdots\right]. \tag{6}$$

The initial convergence of the σ-series is much better than in the corresponding quadratic case; compare the coefficients of σ^{-1}, 1/240 here against 1/12 in the quadratic problem. The conclusion is that for most practical purposes the normal asymptotic expansion remains adequate when a branch point or pole approaches the stationary point. We will not bother to develop a uniform expansion for this cubic case.

The approximate correction series in (6) can be summed by discarding all derivatives of F except $F_1 = -\sigma/u_0$ and $F_2 = -\sigma/u_0^{\,2}$, retention of these two being necessary to attain the stationary conditions $\bar{F}_1 = \bar{F}_2 = 0$. The σ-series multiplying G_j thus originates from

$$\int_{\text{s.p.}} e^{u\sigma/u_0 + u^2\sigma/2u_0^2}(u - u_0)^{\sigma}(u^j/j!)\,du.$$

Adjusting the multiplying constant correctly,

$$\text{Factor of } G_j/\bar{F}_3{}^{\frac{1}{3}j} \text{ in (6)} = \frac{3^{\frac{1}{6}}\pi^{\frac{1}{2}}}{2^{\frac{1}{2}}(-\frac{2}{3})!}\frac{e^{\frac{1}{2}\sigma}}{\sigma^{\frac{1}{2}\sigma+\frac{1}{6}}}(2\sigma)^{\frac{1}{3}j}$$

$$\times \sum_0^j \frac{1}{s!(j-s)!\,(-\sqrt{\sigma})^s} D_{\sigma+s}(2\sqrt{\sigma}) \qquad (7)$$

when $\Re(\sigma) > 0$. (The method of integration and reason for the restriction on σ are discussed in Section 5 to follow.) The approximate correction series in (6) are therefore essentially expansions for parabolic cylinder functions near their turning points (Chapter VIII, question 17).

2. The Status of Uniform Expansions Covering the Region Where a Singularity Approaches the Critical Point

We have seen in the previous section that when a singularity (branch point or pole) approaches a critical point (limit of integration or stationary point), the usual asymptotic expansions are adversely affected to varying degrees. In the linear case the expansion fails as it stands; in the quadratic case the expansion remains marginally usable provided derivatives of G above the first are unimportant; while the cubic expansion is least impaired. By calculating late terms (Chapter VII) and thence the terminant (Chapters XXI onwards), the range for a given minimum accuracy can be greatly extended. Beyond this range, the best procedure is to expand the given function as a Taylor (or Newton) series about that value of the variable for which singularity and critical point coincide, calculating the value of the function and its derivatives (or differences) at that point. Very commonly the function under investigation will be one obeying a second-order differential (or difference) equation, and it will be necessary to find asymptotic expansions only for the function itself at that point, and for its first derivative (or difference).

The alternative approach, with which we shall be concerned in the remainder of this chapter, is to develop more complicated "uniform" expansions, equally applicable whether singularity and critical point are widely separated, close or coincident. However, because of their dependence on higher transcendental functions and the consequent difficulty of evaluating terminants, it appears likely that their future rôle will be virtually confined to two problems: those in which subsequent operations (e.g. integrations) have to be performed, and a single expansion spanning the whole region is therefore advantageous; and for investigating functions which do not satisfy a simple differential or difference equation, for which the Taylor (or Newton) series would require asymptotic evaluation of a number of derivatives (or differences).

3. Uniform Expansion, F Linear at Limit of Integration

We shall suppose the integral to have been reduced to a standard form in which the limit of integration is at $u = 0$ and the singularity is at $u = u_0$, normally negative. (If $u_0 > 0$ a cut must be made in the complex plane and appropriate phases assigned to $-1 \equiv e^{\pm i\pi}$.) Retaining in its original exponential form only the linear term in $F(u)$ and expanding the rest in rising powers of u:

$$\int_{u=0} e^{-F}(u - u_0)^\sigma G\,du = \sum_{r=0}^{\infty} (\text{coefficient of } u^r \text{ in } G\,e^{-F+F_1u})$$

$$\times \left(\int_{u=0} e^{-F_1u}(u - u_0)^\sigma u^r du \right)$$

$$= F_1^{-1} e^{-F_0}(-u_0)^\sigma \sum_0^{\infty} F_1^{-r} U_r \psi(-\sigma, -\sigma - r, -F_1u_0), \tag{8}$$

where ψ is one of the confluent hypergeometric functions introduced in Chapters III and IV, and

$$U_r = r! \times \text{coefficient of } u^r \text{ in } G(u)\,e^{-F(u)+F_0+F_1u}. \tag{9}$$

The first coefficients are as follows:

$$\begin{aligned}
&U_0 = G_0, \qquad U_1 = G_1, \qquad U_2 = G_2 - G_0F_2, \\
&U_3 = G_3 - 3G_1F_2 - G_0F_3, \qquad U_4 = G_4 - 6G_2F_2 - 4G_1F_3 - G_0\bar{F}_4, \\
&U_5 = G_5 - 10G_3F_2 - 10G_2F_3 - 5G_1\bar{F}_4 - G_0\bar{F}_5, \\
&U_6 = G_6 - 15G_4F_2 - 20G_3F_3 - 15G_2\bar{F}_4 - 6G_1\bar{F}_5 - G_0\bar{F}_6, \\
&U_7 = G_7 - 21G_5F_2 - 35G_4F_3 - 35G_3\bar{F}_4 - 21G_2\bar{F}_5 - 7G_1\bar{F}_6 - G_0\bar{F}_7, \\
&U_8 = G_8 - 28G_6F_2 - 56G_5F_3 - 70G_4\bar{F}_4 - 56G_3\bar{F}_5 - 28G_2\bar{F}_6 \\
&\qquad - 8G_1\bar{F}_7 - G_0\bar{F}_8, \\
&U_9 = G_9 - 36G_7F_2 - 84G_6F_3 - 126G_5\bar{F}_4 - 126G_4\bar{F}_5 - 84G_3\bar{F}_6 \\
&\qquad - 36G_2\bar{F}_7 - 9G_1\bar{F}_8 - G_0\bar{F}_9,
\end{aligned} \tag{10}$$

with

$$\bar{F}_4 = F_4 - 3F_2^2, \qquad \bar{F}_5 = F_5 - 10F_2F_3,$$

$$\tilde{F}_6 = F_6 - 15F_2F_4 - 10F_3^2 + 15F_2^3,$$

$$\tilde{F}_7 = F_7 - 21F_2F_5 - 35F_3F_4 + 105F_2^2F_3,$$

$$\tilde{F}_8 = F_8 - 28F_2F_6 - 56F_3F_5 - 35F_4^2 + 210F_2^2F_4 + 280F_2F_3^2 - 105F_2^4,$$

$$\tilde{F}_9 = F_9 - 36F_2F_7 - 84F_3F_6 - 126F_4F_5 + 378F_2^2F_5 + 1260F_2F_3F_4 + 280F_3^3 - 1260F_2^3F_3. \tag{11}$$

The confluent hypergeometric functions ψ can be reduced to incomplete factorial functions (Chapters II–IV), since repeated application of the recurrence relation

$$\mu^{-1}r\,\psi(-\sigma, -\sigma - r, \mu) = \psi(-\sigma - 1, -\sigma - r, \mu) - \psi(-\sigma, -\sigma - r + 1, \mu) \tag{12}$$

leads to the reduction formula

$$\psi(-\sigma, -\sigma - r, \mu) = (-\mu)^r \sum_{s=0}^{r} \frac{(-1)^s}{s!\,(r-s)!}\,\psi(-\sigma - s, -\sigma - s, \mu)$$

$$= (-1)^r\,\mu^{r-\sigma}\,e^{\mu} \sum_{0}^{r} \frac{1}{s!\,(r-s)!\,(-\mu)^s}\,[\sigma + s, \mu]!\,. \tag{13}$$

Hence our uniform expansion can be written

$$\int_{u=0} e^{-F}(u - u_0)^{\sigma}\,G\,du = \frac{e^{-F_0+\mu}}{F_1^{\sigma+1}} \sum_{r=0}^{\infty} U_r\,u_0^{\,r} \sum_{s=0}^{r} \frac{1}{s!\,(r-s)!\,(-\mu)^s} \times [\sigma + s, \mu]!\,, \tag{14}$$

where $\mu = -F_1u_0$. Basically, only the single incomplete factorial function $[\sigma, \mu]!$ is required, because the others can be deduced through the recurrence relation

$$[p, \mu]! = p[p - 1, \mu]! + \mu^p\,e^{-\mu}. \tag{15}$$

Special cases

Simple pole. Here $\sigma = -1$ and the basis is the exponential integral

$$[-1, \mu]! = -Ei(-\mu) \tag{16}$$

discussed in Chapters II–IV. Now for large $|\mu|$ the multiplier of U_r in the r-sum must decrease with rising $|\mu|$. Since $u_0{}^r \propto \mu^r$ it follows that all surviving components after summing over s in (14) are of degrees lower than μ^{-r}. This observation leads to a neat rule for evaluating the s-sum: write down the term $s = 0$ and subtract from it those components of its own asymptotic series which are of degree μ^{-r} and higher. Since

$$-\frac{1}{r!}Ei(-\mu) = \frac{e^{-\mu}}{r!\,\mu}\left[1 - \frac{1!}{\mu} + \frac{2!}{\mu^2} - \dots\right], \tag{17}$$

the rule gives the s-sum required here as

$$\frac{e^{-\mu}}{r!\,\mu}\left\{-\mu\, e^{\mu}\, Ei(-\mu) - \left[1 - \frac{1!}{\mu} + \frac{2!}{\mu^2} - \dots + \frac{(r-1)!}{(-\mu)^{r-1}}\right]\right\}. \tag{18}$$

Inverse square root branch point. Here $\sigma = -\frac{1}{2}$ and the basis is the error function

$$[-\tfrac{1}{2}, \mu]! = \sqrt{\pi}\,\{1 - \phi(\sqrt{\mu})\}. \tag{19}$$

4. Uniform Expansion, F Quadratic at a Stationary Point

There are two distinct ways of tackling the problem of the mutual approach of a stationary point and a singularity, leading to two types of uniform expansion. The direct approach treated in the present section is to expand $F(u)$ around its stationary point and regard $(u - u_0)^\sigma$ as an additional factor in the integrand. In the alternative approach of Section 5 increased emphasis will be placed on the fast-varying nature of the factor $(u - u_0)^\sigma$ by fixing the expansion point as the stationary point of the modified exponent $\bar{F} = F - \sigma\ln(u - u_0)$; in general the resultant uniform expansion is more powerful, but in practice this is often offset by the increased complexity of the equation locating the stationary point.

We suppose the integral to have been reduced to a standard form in which the stationary point of $F(u)$ is at $u = 0$. Retaining in its original exponential form only the quadratic term in $F(u)$ and expanding the rest in rising powers of u:

$$\int_{\text{s.p.}} e^{-F}(u - u_0)^\sigma\, G\, du = \sum_{r=0}^{\infty} (\text{coefficient of } u^r \text{ in } G\, e^{-F + \frac{1}{2}F_2 u^2})$$

$$\times\left(\int_{\text{s.p.}} e^{-\frac{1}{2}F_2 u^2}\,(u - u_0)^\sigma\, u^r\, du\right). \tag{20}$$

To evaluate the integral on the right, the factor u^r is expanded by the binomial theorem in powers of $v = (u - u_0)\sqrt{F_2}$, giving

$$F_2^{-\frac{1}{2}(\sigma+1+r)} e^{-\frac{1}{2}F_2u_0^2} \sum_{s=0}^{r} \binom{r}{s} (u_0\sqrt{F_2})^{r-s} \int_{\text{s.p.}} e^{-vu_0\sqrt{F_2}-\frac{1}{2}v^2} v^{\sigma+s}\, dv.$$

The correct choice of phase for v when the latter is negative is determined by the location of the stationary point $v = -u_0\sqrt{F_2}$. Our convention throughout (e.g. Chapter VI, Section 3) has been to arrange that the path through the stationary point at $u = 0$ shall vary from the line $u = -\infty$ to ∞ when F_2 is positive (Quadratic Contour $\rightarrow$), to the line $u = -i\infty$ to $i\infty$ when F_2 is negative (Quadratic Contour $\uparrow$), and we saw that passage from one extreme to the other could be effected through the equivalence $\sqrt{F_2} \equiv -i\sqrt{(-F_2)}$. In our formal theory, therefore, F_2 is being regarded as having a *negative* imaginary part. To adhere to this convention, the stationary point $v = -u_0\sqrt{F_2}$ must be regarded as possessing a positive imaginary part if $u_0 > 0$, but a negative imaginary part if $u_0 < 0$. Comparison with the integral representations

$$D_p(\mu) = (2\pi)^{-\frac{1}{2}}\, i^{\mp p}\, e^{\frac{1}{4}\mu^2} \int_{-\infty}^{\infty} e^{\pm i\mu v - \frac{1}{2}v^2} v^p\, dv\,, \qquad -1 \equiv e^{\pm i\pi}, \tag{21}$$

then gives the required uniform expansion as

$$\int_{\text{s.p.}} e^{-F}(u-u_0)^\sigma G\, du = \left(\frac{2\pi}{F_2}\right)^{\frac{1}{2}} e^{-F_0+\frac{1}{4}\mu^2} \left(-\frac{u_0}{\mu}\right)^{\sigma} \sum_{r=0}^{\infty} \mathscr{U}_r u_0^r$$

$$\times \sum_{s=0}^{r} \frac{1}{s!\,(r-s)!\,(-\mu)^s} D_{\sigma+s}(\mu)\,, \tag{22}$$

where the sign allotted in $\mu = \pm u_0\sqrt{(-F_2)} = \pm iu_0\sqrt{F_2}$ is to be correlated with $u_0 \gtrless 0$. (This is equivalent to choosing the sign such that $|\mathrm{ph}\,\mu| < \frac{1}{2}\pi$ always). The coefficients

$$\mathscr{U}_r = r! \times \text{coefficient of } u^r \text{ in } G(u)\, e^{-F(u)+F_0+\frac{1}{2}F_2u^2} \tag{23}$$

are exactly as listed in Chapter X (7), (8).

Only two successive orders of the parabolic cylinder functions are essentially involved in (22), because the others can be deduced through the recurrence relation

$$D_p(\mu) = \mu D_{p-1}(\mu) - (p-1)D_{p-2}(\mu)\,. \tag{24}$$

Special cases

No singularity. When $\sigma = 0$ the addition formula for parabolic cylinder functions reduces the s-summation to

$$\frac{e^{-\frac{1}{4}\mu^2}}{r!\,(-\mu)^r} D_r(0) = \frac{\cos \frac{1}{2}\pi r}{(-\mu\sqrt{2})^r\,(\frac{1}{2}r)!}\, e^{-\frac{1}{4}\mu^2}. \tag{25}$$

Our uniform expansion thereupon becomes

$$\int_{\text{s.p.}} e^{-F}\, G\, du = \left(\frac{2\pi}{F_2}\right)^{\frac{1}{2}} e^{-F_0} \sum_{r=0,2,4,\ldots} \frac{\mathscr{U}_r}{(2F_2)^{\frac{1}{2}r}\,(\frac{1}{2}r)!}. \tag{26}$$

This sum is equivalent to the more familiar ΣQ_{2r} in Chapter VI (2), but with contributions now less sophisticatedly ordered—simply according to inverse powers of F_2.

Simple pole.† When $\sigma = -1$ the bases can conveniently be taken as

$$D_{-1}(\mu) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\, e^{\frac{1}{4}\mu^2}\,\{1 - \phi(2^{-\frac{1}{2}}\mu)\}, \qquad D_0(\mu) = e^{-\frac{1}{4}\mu^2}. \tag{27}$$

The former appears once only in the s-summation, at $s = 0$, and a rule analogous to that of Section 3 appertains: to evaluate the s-sum, write down this term $s = 0$ and subtract from it those components of its own asymptotic expansion for $|\text{ph}\,\mu| < \frac{1}{2}\pi$ which are of degree μ^{-r} and higher. Since

$$\frac{(\frac{1}{2}\pi)^{\frac{1}{2}}\, e^{\frac{1}{4}\mu^2}}{r!}\,\{1 - \phi(2^{-\frac{1}{2}}\mu)\} = \frac{e^{-\frac{1}{4}\mu^2}}{r!\,\mu}\left[1 - \frac{1}{\mu^2} + \frac{1.3}{\mu^4} - \ldots\right], \tag{28}$$

the rule gives the s-sum required here as

$$\frac{e^{-\frac{1}{4}\mu^2}}{r!\,\mu}\left\{(\tfrac{1}{2}\pi)^{\frac{1}{2}}\mu\, e^{\frac{1}{2}\mu^2}[1 - \phi(2^{-\frac{1}{2}}\mu)] - \left[1 - \frac{1}{\mu^2} + \frac{1.3}{\mu^4} - \ldots \right.\right.$$

$$\left.\left.\ldots \frac{1.3.5\ldots(2R-1)}{(-\mu^2)^R}\right]\right\}, \tag{29}$$

where the letter R denotes $\frac{1}{2}r - 1$ if r is even, and $\frac{1}{2}r - \frac{1}{2}$ if r is odd. When μ becomes imaginary, $2^{-\frac{1}{2}}\mu = iy$ say, $\phi(iy)$ can more conveniently be replaced by $2\pi^{-\frac{1}{2}}\, i\, \text{erfi}\, y$, where $\text{erfi}\, y = \int_0^y e^{v^2}\, dv$ is Dawson's integral, tabulated for

† For earlier work on this special case, see Ott (1943) and van der Waerden (1950).

real y in Dawson (1897), Stäblein and Schläfer (1943), Terrill and Sweeny (1944), Rosser (1948) and Lohmander and Rittsten (1958).

Inverse square root branch point. When $\sigma = -\frac{1}{2}$ the bases may be taken as

$$D_{-\frac{1}{2}}(\mu) = (\mu/2\pi)^{\frac{1}{2}} K_{\frac{1}{4}}(\tfrac{1}{4}\mu^2), \qquad D_{\frac{1}{2}}(\mu) = (\mu^3/8\pi)^{\frac{1}{2}} \{K_{\frac{1}{4}}(\tfrac{1}{4}\mu^2) + K_{\frac{3}{4}}(\tfrac{1}{4}\mu^2)\}. \tag{30}$$

Values can be obtained from the tabulations of $I_{\pm\frac{1}{4}}$ and $I_{\pm\frac{3}{4}}$ in "Tables of Bessel functions of fractional order" (1949).

5. Uniform Expansion, $\bar{F}$ Quadratic at a Stationary Point

In this more sophisticated approach greater emphasis is placed on the fast-varying nature of the factor $(u - u_0)^\sigma$ by including it in the fast-varying portion of the integrand, at least to the limited extent of taking as expansion point the stationary point of

$$\bar{F} = F - \sigma \ln(u - u_0). \tag{31}$$

As we have seen in Section 1, when u_0 is small the normal asymptotic expansion expressed in derivatives of $\bar{F}$ is plagued by excessively large contributions to those higher derivatives from the logarithmic portion of $\bar{F}$. This difficulty can be circumvented by expanding the logarithmic portion *only as far as is necessary to maintain the location of the stationary point*, i.e. only up to its quadratic term, and retaining the remainder as additional factors in the integrand. The expansion is thereby changed into one in the derivatives F_3, F_4, etc.

We suppose the integral to have been reduced to a standard form $\int_{\text{s.p.}} e^{-F}\, G\, du$ in which the stationary point of $\bar{F}(u)$ is set at $u = 0$. The condition $\bar{F}_1 = 0$ entails $F_1 = -\sigma/u_0$, so

$$F = F_0 - u\sigma/u_0 + \sum_{j=2}^{\infty} F_j\, u^j/j!\,. \tag{32}$$

Retaining in their original exponential forms the linear and quadratic terms in F, and expanding the rest in rising powers of u,

$$\int_{\text{s.p.}} e^{-F}\, G\, du = \sum_{r=0}^{\infty} (\text{coefficient of } u^r \text{ in } G\, e^{-F+F_1 u+\frac{1}{2}F_2 u^2})$$

$$\times\left(\int_{\text{s.p.}} e^{u\sigma/u_0 - \frac{1}{2}F_2u^2}\,(u-u_0)^\sigma\,u^r\,du\right). \tag{33}$$

To evaluate the integral on the right, the factor u^r is expanded by the binomial theorem in powers of $v = (u - u_0)\sqrt{F_2}$, giving

$$F_2^{-\frac{1}{2}(\sigma+1+r)}\,e^{-\frac{1}{2}F_2u_0^2+\sigma}\sum_0^r\binom{r}{s}(u_0\sqrt{F_2})^{r-s}\int_{\text{s.p.}} e^{-v(u_0\sqrt{F_2}-\sigma/u_0\sqrt{F_2})-\frac{1}{2}v^2}\,v^{\sigma+s}\,dv\,.$$

By supposition, the stationary point of $\bar{F}(u)$ is at $u = 0$, i.e. at $v = -u_0\sqrt{F_2}$. Provided $\Re(\sigma) \leqslant 0$, the path through this stationary point is similar to that considered in the previous section, and the final result is

$$\int_{\text{s.p.}} e^{-F}\,G\,du = \left(\frac{2\pi}{F_2}\right)^{\frac{1}{2}} e^{-F_0+\frac{1}{4}\mu^2-\frac{1}{4}\bar{\mu}^2+\sigma}\left(-\frac{u_0}{\mu}\right)^{\sigma}\sum_{r=0}^{\infty}\mathscr{U}_r\,u_0^{\,r}$$

$$\times\sum_{s=0}^{r}\frac{1}{s!\,(r-s)!\,(-\mu)^s}\,D_{\sigma+s}(\bar{\mu}),\qquad \Re(\sigma)\leqslant 0\,, \tag{34}$$

where $\bar{\mu} = \mu + \sigma/\mu$. The symbols

$$\mu = \pm u_0\sqrt{(-F_2)}\,,\qquad u_0 \gtrless 0\,,$$

and $\mathscr{U}_r$ have the same notational significance as before, apart of course from the changed *values* throughout due to the stationary point labelled "$u = 0$" now being defined by $\bar{F}' = 0$ in place of $F' = 0$ as in Section 4.

The path must be altered when $\Re(\sigma) > 0$, because then $\bar{F} \to +\infty$ as $u \to u_0$, indicating the appearance of some second stationary point between u_0 and one of the extrema of the earlier path (e.g. $u = \pm\infty$ in Quadratic Contour →). Now we have already seen in Chapter VI, Section 2, that before evaluation an integral must be dissected into a set of simpler integrals, each involving just one critical point. To isolate the contribution from the one stationary point under consideration, that at $u = 0$, the path must terminate at $u = u_0$ when $\Re(\sigma) > 0$. Thus if $\Re(u_0) > 0$ the standard path is to run from $u = -\infty$ through $u = 0$ and on to $u = u_0$; if $\Re(u_0) < 0$ it is to run from $u = u_0$ through $u = 0$ and on to $u = +\infty$. Comparison with the integral representation

$$p!\,D_{-1-p}(v) = e^{-\frac{1}{4}v^2}\int_0^{\infty} e^{-vv-\frac{1}{2}v^2}\,v^p\,dv \tag{35}$$

then shows the required uniform expansion to be

$$\int_{\text{s.p.}} e^{-F} G\, du = F_2^{-\frac{1}{2}} e^{-F_0 - \frac{1}{4}\nu^2 + \frac{1}{4}\bar{\nu}^2 + \sigma} \left(-\frac{u_0}{\nu}\right)^{\sigma} \sum_{r=0}^{\infty} \mathscr{U}_r u_0^r$$

$$\times \sum_{s=0}^{r} \frac{(\sigma + s)!}{s!\,(r - s)!\,(-\nu)^s} D_{-1-\sigma-s}(-\bar{\nu}), \qquad \Re(\sigma) > 0, \tag{36}$$

where $\bar{\nu} = \nu - \sigma/\nu$ and

$$\nu = \pm\, u_0 \sqrt{F_2}, \qquad \Re(u_0) \gtrless 0.$$

It is interesting to compare this with the uniform expansion appropriate to $\Re(\sigma) \leqslant 0$. Drawing upon known relations between parabolic cylinder functions, and noting $\mu = i\nu$ and $\bar{\mu} = i\bar{\nu}$, (34) can be reconstructed to look like (36) but with $D_{-1-\sigma-s}(-\bar{\nu})$ replaced by

$$D_{-1-\sigma-s}(-\bar{\nu}) + e^{-i\pi(\sigma+s)} D_{-1-\sigma-s}(\bar{\nu}).$$

Special cases

$F_{j\geqslant 2} \to 0$. This is the limit, treated directly in Section 1, in which the predominant contributions to the derivatives $\bar{F}_{j\geqslant 2}$ are those from the singular factor, so $\bar{F}_{j\geqslant 2} \to \sigma(j-1)!/u_0^{\,j}$.

Considering first the case $\Re(\sigma) > 0$, $-\bar{\nu} \simeq +\sigma/\nu$ is large and positive. Each parabolic cylinder function in (36) can accordingly be approximated by the leading term of its asymptotic expansion for positive variable, $(-\bar{\nu})^{-1-\sigma-s} \exp(-\frac{1}{4}\bar{\nu}^2)$. The s-sum then reads

$$e^{-\frac{1}{4}\bar{\nu}^2} \left(\frac{\nu}{\sigma}\right)^{1+\sigma} \sum_{s=0}^{r} \frac{(\sigma + s)!}{s!\,(r - s)!\,(-\sigma)^s},$$

leading to the approximation

$$\int_{\text{s.p.}} e^{-F} G\, du \to \frac{e^{-F_0}}{\sqrt{F_2}} \frac{e^{\sigma}}{\sigma^{\sigma+\frac{1}{2}}} \sum_{r=0}^{\infty} \mathscr{U}_r \left(\frac{\sigma}{F_2}\right)^{\frac{1}{2}r} \sum_{s=0}^{r} \frac{(\sigma + s)!}{s!\,(r - s)!\,(-\sigma)^s}, \qquad \Re(\sigma) > 0. \tag{37}$$

This agrees with (5^+), since in the same limit $\mathscr{U}_r \to G_r$.

When $\Re(\sigma) < 0$, $\bar{\mu} \simeq \sigma/\mu$ is large and negative. Each parabolic cylinder function in (34) can accordingly be approximated by the leading term of its asymptotic expansion for negative variable,

$$\frac{(2\pi)^{\frac{1}{2}}\, e^{\frac{1}{4}\bar{\mu}^2}}{(-\bar{\mu})^{1+\sigma+s}\,(-1-\sigma-s)!}\,.$$

This leads to the approximation

$$\int_{\text{s.p.}} e^{-F}\, G\, du \to \frac{2\pi\, e^{-F_0}}{\sqrt{F_2}}\, \frac{e^{\sigma}}{(-\sigma)^{\sigma+\frac{1}{2}}} \sum_{r=0}^{\infty} \mathscr{U}_r \left(\frac{\sigma}{F_2}\right)^{\frac{1}{2}r}$$

$$\times \sum_{s=0}^{r} \frac{1}{s!\,(r-s)!\,(-1-\sigma-s)!\,\sigma^s}\,, \qquad \Re(\sigma) < 0\,, \qquad (38)$$

in agreement with (5^-).

Simple pole. When $\sigma = -1$ the only non-trivial basis is the error function; but because powers of the variable $\bar{\mu} = \mu - \mu^{-1}$ produce a variety of positive and negative powers of μ our rule for evaluating s-sums is unhelpful, so each sum has to be separately determined. The first few are

$$r = 0 \qquad (\tfrac{1}{2}\pi)^{\frac{1}{2}}\, e^{\frac{1}{4}\bar{\mu}^2}\{1 - \phi(2^{-\frac{1}{2}}\bar{\mu})\}$$

$$r = 1 \qquad (\tfrac{1}{2}\pi)^{\frac{1}{2}}\, e^{\frac{1}{4}\bar{\mu}^2}\left\{1 - \phi - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-\frac{1}{4}\bar{\mu}^2}}{\mu}\right\}$$

$$r = 2 \qquad \tfrac{1}{2}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\, e^{\frac{1}{4}\bar{\mu}^2}\left\{1 - \phi - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-\frac{1}{4}\bar{\mu}^2}}{\mu}\left(1 + \frac{1}{\mu^2}\right)\right\}$$

$$r = 3 \qquad \tfrac{1}{6}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\, e^{\frac{1}{4}\bar{\mu}^2}\left\{1 - \phi - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-\frac{1}{4}\bar{\mu}^2}}{\mu}\left(1 + \frac{1}{\mu^4}\right)\right\}$$

$$r = 4 \qquad \tfrac{1}{24}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\, e^{\frac{1}{4}\bar{\mu}^2}\left\{1 - \phi - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-\frac{1}{4}\bar{\mu}^2}}{\mu}\left(1 - \frac{2}{\mu^4} + \frac{1}{\mu^6}\right)\right\}. \qquad (39)$$

Inverse square root branch point. When $\sigma = -\frac{1}{2}$ the bases are essentially the modified Bessel functions $K_{\frac{1}{4}}$ and $K_{\frac{3}{4}}$, as specified in (30).

Bose–Einstein condensation. As an elegant demonstration of the power of the uniform expansion (34), we apply it to the cardinal integral occurring in the theory of Bose–Einstein condensation. This is distinguished by an integrand so singular that the steepest descents expansions of Chapter VI and our first uniform expansion (22) all prove inadequate. In such a condensation, considered to be responsible at root for the macroscopic quantum phenomena exhibited by liquid helium below $T = 2{\cdot}186°$K (for review see Dingle 1952), the mean number $\bar{n}_0$ of particles in the lowest state is of the same order of magnitude as the mean number $\bar{n}_e$ in all excited states together. The existence of such a condensation is easily proved once it is assumed that the mean occupation number per state in Bose–Einstein statistics is

$$\bar{n}_i = \frac{1}{\omega_0^{-1}\, e^{\varepsilon_i/kT} - 1}, \tag{40}$$

with ω_0 determined by the conservation condition $N = \Sigma \bar{n}_i$. The difficulty lies in establishing (40), because the most reliable general procedure for deriving statistical distribution functions—the Darwin–Fowler method of selector variables applied to a closed system described by a canonical ensemble, followed by steepest descents evaluation—appears to fail at and below the prospective condensation temperature (G. Schubert 1946, 1947).† It is true that alternative special derivations are available (Dingle 1948, 1952; Fraser 1951), but it is mathematically and physically more instructive to discover why the usual procedure breaks down and how it can be improved so as to cover such regions.

Following therefore the Darwin–Fowler method applied to a canonical ensemble, the partition function for a closed system of conserved elements obeying Bose–Einstein statistics is exactly

$$Z = \frac{1}{2\pi i}\oint \frac{d\omega}{\omega^{N+1}\,\Pi(1 - \omega z_i)} \tag{41}$$

† Several authors (e.g. ter Haar 1952, Landsberg 1954) have argued that the theory of Bose–Einstein condensation is much simpler for an *open* system, because the grand canonical distribution function $\rho(E, N) \propto e^{(\mu N - E)/kT}$ leads directly and exactly to

$$n_i = 1/(e^{(\varepsilon_i - \mu)/kT} - 1).$$

On closer examination, development of a valid theory of the condensation for open systems is seen to be actually harder. The central difficulty is that derivations of the *precisely exponential* form of the population weighting factor $(e^{\mu N/kT})$ hinge on assuming $\rho(E, N)$ to be a strongly-varying function of N. This is unjustified in a Bose–Einstein condensation, for which $\mu = O(N^{-1})$.

The difficulty can be surmounted, but only at the expense of making the derivation for an open system more complex than for a closed system. For instance, the correct $\rho(E, N)$ can first be evaluated; the mathematics turns out to be somewhat similar to (41)–(51). Alternatively, the model can be changed to a more complicated and artificial one in which the "particle reservoir" is never in the condensed state, for instance by imagining the reservoir particles to be kept apart by repulsive forces.

where $z_i = e^{-\varepsilon_i/kT}$ and (for brevity of exposition) the states are assumed to be non-degenerate. The function

$$\bar{F}(\omega) = (N + 1)\ln \omega + \Sigma \ln (1 - \omega z_i) \tag{42}$$

has a stationary point when $\omega = \omega_0$, where

$$0 = \omega_0 \bar{F}_1 = N + 1 - \sum \frac{\omega_0 z_i}{1 - \omega_0 z_i}. \tag{43}$$

Derivatives at this point are

$$\bar{F}_j = -\frac{(j-1)!}{\omega_0{}^j}\left\{(-1)^j (N+1) + \sum \left(\frac{\omega_0 z_i}{1 - \omega_0 z_i}\right)^j\right\}. \tag{44}$$

The second derivative $\bar{F}_2$ is negative, corresponding to a maximum in $\bar{F}$ (minimum in integrand) for a path along the real axis, so the contour through the stationary point is to be taken parallel to the imaginary axis. The steepest descents expansion in Chapter VI (3) leads to the result

$$\begin{aligned} \ln \mathbf{Z} = & -\tfrac{1}{2}\ln 2\pi - (N+1)\ln \omega_0 - \sum_i \ln (1 - \omega_0 z_i) \\ & - \tfrac{1}{2}\ln \sum_i \omega_0{}^{-1} z_i/(1 - \omega_0 z_i)^2 \ldots . \end{aligned} \tag{45}$$

Under normal conditions in which no single energy level contributes disproportionately, the fourth and later terms in (45) can be dropped, since for example the fourth (last quoted) term is then proportional only to the logarithm of the size of the system whereas the third is proportional to the size itself. Accepting such truncation, mean occupation numbers are given by

$$\bar{n}_i = z_i \frac{d \ln \mathbf{Z}}{dz_i} = z_i \left(\frac{\partial \ln \mathbf{Z}}{\partial z_i} + \frac{\partial \ln \mathbf{Z}}{\partial \omega_0}\frac{\partial \omega_0}{\partial z_i}\right) = \frac{1}{(\omega_0 z_i)^{-1} - 1}, \tag{46}$$

the partial derivative $\partial \ln \mathbf{Z}/\partial \omega_0$ vanishing to this approximation through (43).

Assuming for the moment the validity of (46) at least as regards scale, let us examine the magnitudes of all relevant quantities when $\bar{n}_0$, $\bar{n}_e$, and of course $N = \bar{n}_0 + \bar{n}_e$, are comparable. In this regime we must take explicit account of the deviation of $(\omega_0 z_0)^{-1} = 1 + \bar{n}_0{}^{-1}$ from unity in denominators for $i = 0$. It is immediately seen that the dominant $i = 0$ contribution in the fourth term of (45) is now equal and opposite to the corresponding contribution in the third term. Further examination discloses contributions of comparable magnitude in all later terms, jeopardizing all confidence in truncation and therefore in (46).

This breakdown of the steepest descents expansion has its origin in the rapidly increasing magnitude of the derivatives of $\bar{F}$,

$$\bar{F}_{j\geqslant 2} \sim -\frac{(j-1)!}{\omega_0{}^j}\{\bar{n}_0{}^j + O(\bar{n}_e{}^{\frac{2}{3}j})\} \tag{47}$$

in which the first, dominant contributions come from the factor $(1-\omega z_0)^{-1}$ in the integrand of (41). This integrand is so anomalously singular that our earlier uniform expansion (22) completely fails to cope. For in deriving that expansion it was tacitly assumed the usual steepest descents expansion would have been perfectly all right if one singular factor had been absent. But if the offending factor $(1-\omega z_0)^{-1}$ in (41) were excluded when determining the location of the stationary point, this expansion point would have been forced towards the next pole from $(1-\omega z_1)^{-1}$, recreating the problem; and so on if further factors are removed.

Our second uniform expansion (34) is not subject to this kind of failure because the expansion point is not moved when a singular factor is singled out for special treatment. In the present problem the singularity is a simple pole ($\sigma = -1$) at $\omega = z_0{}^{-1}$, a distance $u_0 = z_0{}^{-1} - \omega_0$ from the stationary point. Under condensation conditions $u_0 \sim (z_0\bar{n}_0)^{-1}$ and $F_2 \sim -z_0{}^2\bar{n}_e{}^{\frac{4}{3}}$, so $\bar{\mu} \sim -\bar{n}_0/\bar{n}_e{}^{\frac{2}{3}}$ is huge and negative. This was the limit leading to (38) $\equiv$ (5^-). Specializing to our case,

$$\int_{\text{s.p.}} e^{-F}\,G\,du \to \frac{2\pi i}{e}(-\bar{F}_2)^{-\frac{1}{2}}\,e^{-F_0}, \tag{48}$$

whereby

$$Z = \frac{1}{ez_0\,\omega_0{}^{N+1}}\left(\prod_{i\neq 0}\frac{1}{1-\omega_0 z_i}\right)\left\{1 - O\left(\frac{\bar{n}_e{}^{\frac{4}{3}}}{\bar{n}_0{}^2}\right)\right\}. \tag{49}$$

In principle the parameter ω_0 could be eliminated exactly through (43). Limiting ourselves here to the approximation $(\omega_0 z_0)^{-1} = 1 + \bar{n}_0{}^{-1}$, so

$$\frac{1}{\omega_0{}^{N+1}} = z_0{}^{N+1}\left(1+\frac{1}{\bar{n}_0}\right)^{N+1} \simeq z_0{}^{N+1}\,e^{N/\bar{n}_0}, \tag{50}$$

we have

$$Z = e^{\bar{n}_e/\bar{n}_0}\,z_0{}^N\prod_{i\neq 0}\frac{1}{1-z_i/z_0}. \tag{51}$$

Under condensation conditions the mean occupation numbers are therefore

$$\bar{n}_{i\neq 0} = z_i \frac{d}{dz_i} \ln Z = \frac{1}{z_0/z_i - 1} = \frac{1}{e^{(\varepsilon_i - \varepsilon_0)/kT} - 1}, \tag{52}$$

$$\bar{n}_0 = z_0 \frac{d}{dz_0} \ln Z = N - \sum_{i\neq 0} \frac{1}{z_0/z_i - 1} = N - \bar{n}_e, \tag{53}$$

confirming condensation of the "excess" particles into the ground state. The fluctuations are

$$\overline{(\Delta n_{i\neq 0})^2} = z_i \frac{d}{dz_i} \bar{n}_i = \bar{n}_i(\bar{n}_i + 1), \tag{54}$$

$$\overline{(\Delta n_0)^2} = z_0 \frac{d}{dz_0} \bar{n}_0 = \sum_{i\neq 0} \bar{n}_i(\bar{n}_i + 1). \tag{55}$$

6. The Snare of Attempting Expansion about the Singularity

At first sight simpler and more elegant uniform expansions might be thought to result from developing F and G about the singular point $u = u_0$ rather than about the critical point $u = 0$. For in the first place higher corrections in the former procedure would be represented in the integrand by factors $(u - u_0)^{\sigma + r}$ compared with the latter's more complicated $(u - u_0)^{\sigma} u^r$. Secondly, there would apparently be a major simplification in dealing with poles, because by expanding about $u = u_0$ rather than $u = 0$ we would have

$$\frac{f(u)}{u - u_0} = \frac{f(u_0)}{u - u_0} + f'(u_0) + \frac{u - u_0}{2!} f''(u_0) + \ldots, \tag{56}$$

thereby partitioning off the pole as a single term. However, choice of u_0 as expansion point can lead to serious difficulties. Indeed, unless great care is taken in re-ordering, the resulting expansions are likely to be severely restricted in range, failing to satisfy requirements of uniformity as the gap between singularity and critical point widens.

The type of problem before us stipulates a certain behaviour of the fast-varying portion of the integrand $e^{-F(u)}$ in the immediate vicinity of the critical point; satisfactory ordering of terms in the resultant series is dependent on this knowledge. When singularity and critical point almost coalesce it is reasonably safe to extrapolate the form of e^{-F} near the first from its known form near the second, and also to expect comparative magnitudes of successive terms in the resultant asymptotic expansion not to be seriously affected by the small shift in expansion point; but increasing separation between singularity and critical point can quickly invalidate both suppositions.

As an illustration, consider the case of a simple pole. The r-sequences derived in Sections 3 and 4 constitute valid uniform expansions because the general rth term has degree lower than [separation between pole and critical point]$^{-r}$. This regular sequence of orders is destroyed if the transcendental function is extracted from each term in the sequence; and such extraction is easily shown to be equivalent to expanding about the pole and partitioning off the polar contribution as in (56). Collating the transcendental functions from the various terms, the factor multiplying the transcendental function $(\frac{1}{2}\pi)^{\frac{1}{2}}\, e^{\frac{1}{2}\mu^2}\{1 - \phi(2^{-\frac{1}{2}}\mu)\}$ in the r-sum of Section 4—with $\sigma = -1$ because of the simple pole—is for example

$$\sum_0^\infty \frac{\mathscr{U}_r u_0^r}{r!} = \frac{1}{2\pi i}\oint G\, e^{-F+F_0+\frac{1}{2}F_2 u^2}\frac{du}{u}\sum_0^\infty \left(\frac{u_0}{u}\right)^r$$

$$= \frac{1}{2\pi i}\oint \frac{G\, e^{-F+F_0+\frac{1}{2}F_2u^2}}{u-u_0}\,du = G(u_0)\, e^{-F(u_0)+F_0+\frac{1}{2}F_2u_0^2}, \tag{57}$$

corresponding to $f(u_0)$ in the first term on the right of (56).

Exercises

1. Defining

$$I_\sigma = \int_0^\infty e^{-u^2}(u+x)^\sigma\, du\,,$$

derive through integration by parts the reduction formula

$$I_\sigma = x\, I_{\sigma-1} + \tfrac{1}{2}(\sigma-1)\, I_{\sigma-2} + \tfrac{1}{2}x^{\sigma-1}\,.$$

2. Show that $I_0 = \frac{1}{2}\sqrt{\pi}$ and

$$I_{-1} = \frac{1}{2x}\sum_0^\infty \left\{\frac{(r+\frac{1}{2})!}{x^{2r}} - \frac{(r+1)!}{x^{2r+1}}\right\}.$$

3. For large $|\mu|$, $D_p(\mu) \sim \mu^p e^{-\frac{1}{4}\mu^2}$, so the s-sum in (22) can then be approximated by

$$\mu^\sigma e^{-\frac{1}{4}\mu^2} \sum_0^r (-1)^s/s!\,(r-s)!\,.$$

Show that this vanishes unless $r = 0$, thereby establishing the expected limiting form

$$\int_{\text{s.p.}} e^{-F} (u-u_0)^\sigma G\, du \sim (2\pi/F_2)^{\frac{1}{2}} e^{-F_0} (-u_0)^\sigma G_0, \qquad |u_0| \text{ large.}$$

4. Starting from the representation

$$\bar{\Lambda}_s(-x) = -\frac{x}{s!} P \int_0^\infty \frac{u^s e^{-u}\, du}{u - x}$$

for one of the basic terminants to be introduced in Chapter XXI, take (in the notation of Section 4) $\sigma = -1$, $u_0 = x$ and $\mu = ix\sqrt{s}$, and hence establish the important uniform expansion

$$\bar{\Lambda}_s(-x) = \frac{(2\pi)^{\frac{1}{2}} s^{s-\frac{1}{2}} e^{-s}}{s!} \left\{ E + \frac{s}{12}\left(E - 1 - \frac{1}{x^2 s}\right) \right.$$

$$+ \frac{s(5s+3)}{90}\left(E - 1 - \frac{1}{x^2 s} - \frac{3}{x^4 s^2}\right)$$

$$\left. - \frac{s(119s - 240)}{13{,}440}\left(E - 1 - \frac{1}{x^2 s} - \frac{3}{x^4 s^2} - \frac{15}{x^6 s^3}\right) + O\left(\frac{1}{x^{10} s^2}\right)\right\},$$

where

$$E(x,s) = x(2s)^{\frac{1}{2}} e^{-\frac{1}{2}x^2 s} \operatorname{erfi}[x(\tfrac{1}{2}s)^{\frac{1}{2}}],$$

erfi being Dawson's integral.

5. Show that there is no point in developing a similar type of expansion for the other basic terminant

$$\Lambda_s(x) = x^{s+1} e^x [-s-1, x]!,$$

because the structurally more elementary expansion in Chapter VIII (26) is then uniform for all $\Re(x) > 0$. (And when $\Re(x) < 0$ a complex terminant is more easily expressed through the $\overline{\Lambda}$).

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Chapter XII

Derivation of Asymptotic Power Series from Homogeneous Differential Equations

1. Series Solutions of Differential Equations

In the second half of the nineteenth century, mathematicians directed a great deal of attention towards classifying singularities of differential equations and deriving series solutions. Outstanding among these investigations were those of Frobenius (1874) and Thomé (1873, 1879). The former's systematic method of determining two independent *convergent* series solutions is too well known to need summarizing here. Thomé's main contribution was the demonstration that a differential equation with an irregular singularity of finite rank also possesses formal solutions consisting of an exponential function times what later became known as an "asymptotic power series". Consider, for example, the confluent hypergeometric equation

$$xF'' + (c - x)F' - aF = 0 . \tag{1}$$

For very large $|x|$ the dominant terms are expected to be

$$x(F'' - F') \sim 0 .$$

This limiting equation has two solutions $F^- \sim$ constant and $F^+ \sim e^x$. To obtain the correct leading term in F^- for large $|x|$, we substitute $F^- \sim x^\sigma$ in the full differential equation and equate to zero the coefficient of the highest power, x^σ, giving $\sigma = -a$. Thomé's formal solution is then

$$F^- = x^{-a}\sum_0^\infty A_r x^{-r} , \tag{2}$$

in which a simple recurrence relation for the A_r follows on substituting (2) in the differential equation. On writing $F^+ = e^x f^+$ and applying the same procedure to the differential equation satisfied by f^+, the second formal solution is easily found to be

$$F^+ = e^x x^{a-c} \sum_0^\infty B_r x^{-r}. \tag{3}$$

The weakness in this simple procedure is that it offers no straightforward way of correlating such asymptotic solutions with the convergent series by which, historically, most functions have been defined. Thus in our illustration, knowing two independent formal solutions to the differential equation to be F^- and F^+ is of little avail until we also know how to express the defining absolutely convergent series

$$F(a, c, x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

in terms of F^- and F^+ for all phase regions. There are five ways in which this correlation difficulty can be resolved:

(i) After removing the exponential dependence, each trial solution can be written as a Mellin transform instead of as a power series. This simple but powerful extension of Thomé's method forms the main subject of the present chapter.

(ii) The convergent series solutions can be determined by the procedure introduced by Frobenius, and the asymptotic solutions derived therefrom by the method of Mellin transforms treated in Chapter II. In practice this does not amount to much more than a circuitous variant of (i). Specifically, (i) and (ii) differ in the stage at which exponential factors are extracted—straight from the differential equation in (i), from the Frobenius series solutions in (ii). A detailed discussion of this version will therefore not be undertaken.

(iii) Where possible, the solutions can be framed as Laplace-type integral representations, and asymptotic power series derived therefrom by the methods of Chapter IV. The potential advantage of this approach is that the same representations may well prove suitable starting points for deriving more sophisticated (e.g. large-order) expansions by the methods of Chapters V–XI.

The problem of how to express solutions of differential equations as integral representations suitable for subsequent asymptotic analysis was intensively studied by Horn in a large number of papers spread over the first quarter of this century, of which the most important are listed in the references. Moreover, the derivation of integral representations from certain important differential equations has attracted especial attention. Outstanding among these specialized investigations are those of Hankel (1869) and Schläfli (1871) for the Bessel equation, of Whittaker (1903, 1904) for the parabolic cylinder and confluent hypergeometric equations, and—to quote a case where explicit integral representations have yet to be found—of Dougall (1916) for the Mathieu equation. A systematic account of these well-documented researches is beyond the scope of this book.

(iv) Instead of finding solutions useful only for large $|x|$, asymptotic solutions can be sought which are valid when one of the parameters in the differential equation is large—e.g. whenever either a or c is large in the confluent hypergeometric equation—so permitting the required correlations at small x. This approach is carried through in Chapters XIII, XIV and XVI.

(v) As regards the future, a strong case can be argued for a radical innovation in defining functions: in place of convergent (Frobenius) series, definitions could be based on asymptotic (Thomé) series, supplemented by terminants (Chapters XXI onwards) so as to constitute explicit rather than symbolic expressions. For experience over the last century in the practical applications of mathematical functions in physics, astronomy, chemistry, engineering and "applied mathematics"—hydrodynamics, aerodynamics, etc.—has amply demonstrated two related points:

(a) Asymptotic solutions are generally more widely applicable than convergent solutions, for the reasons propounded in Chapter I, Sections 1 and 4.

(b) Classification of solutions by their exponential-like behaviour over a wide range (e.g. for all large $|x|$) is generally more helpful in applications than classification by behaviour in a narrow region, e.g. that surrounding the origin $x = 0$. Indeed, for some of the more important differential equations it was long ago found desirable to introduce secondary definitions conforming to the first-stated criterion; witness the introductions of a Bessel function K of the third kind to supplement the original J and Y functions, and of the Whittaker functions W_{km} to supplement the original $F(a, c, x)$ $(\equiv M_{km})$ confluent hypergeometric solutions. An incidental advantage of classification by exponential-like behaviour would be the avoidance thereby of breakdowns in defining independent solutions to a differential equation, e.g. $J_p(x)$ and $J_{-p}(x)$ ceasing to be mutually independent solutions to the Bessel equation at integer values of p.

If definitive solutions were thus to be chosen from asymptotic forms, convergent series solutions to the same differential equation would be identified essentially through known analytic behaviour of terminants. We do not adopt this new direction of approach here, despite its evident long-term advantages outlined above, because throughout the book we wish to illustrate each method of deriving asymptotic expansions by application to familiar, conventionally defined, functions.

2. Method of Mellin Transforms

It is instructive to commence with the more familiar related problem of investigating convergent series solutions. For simplicity of exposition, let us suppose the given homogeneous linear differential equation has been reduced to a form in which all coefficients are algebraic in the independent variable (most conveniently polynomials), and thus write it as

$$\sum_{p,q} C_{pq} x^p \left(\frac{d}{dx}\right)^q y = 0 . \tag{4}$$

On substituting the series form $y = \Sigma v_s x^s$, a tractable two-term recurrence relation is obtained for the v_s only if C_{pq} vanishes for all but two values of $p-q$. If this condition is not fulfilled, the approximate behaviour of y must be elucidated in fast-varying regions, and factors describing such regional variations successively removed from y until the new differential equation finally does satisfy the wanted condition.

Once this has been achieved, the more powerful Mellin representation becomes equally feasible. For substitution in the new differential equation of

$$y = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} M(m)\, x^{-m}\, dm \tag{5}$$

then leads to a soluble first-order linear difference equation for $M(m)$. On rescaling m if necessary, this can be reduced to the simple form

$$\frac{M(m+1)}{M(m)} = \mu \frac{(m+a)(m+b)\,..}{(m+A)(m+B)\,..}. \tag{6}$$

The solution of (6) with most poles, i.e. that combining (in some specific manner) all possible solutions to the differential equation, is

$$M(m) \propto (-1)^{\mathcal{N}} \mu^m (m+a-1)!\,(m+b-1)!\,\ldots$$
$$\times(-m-A)!\,(-m-B)!\,\ldots, \tag{7}$$

where $\mathcal{N}$ is the number of factorials containing $(-m)$. Adjustment is then made according to the convergent solution which is to be investigated. For instance, if the convergent series is to be that containing only powers $x^{-a}, x^{1-a}, x^{2-a}, \ldots$, i.e. those generated by the poles of $(m+a-1)!$, the poles of $(m+b-1)!$ and other increasing factorials—corresponding to other convergent solutions—are to be removed by transferring these factorials to the denominator, so selecting the representation

$$M(m) \propto (-1)^{\mathcal{N}'} \mu^m \frac{(m+a-1)!}{(-m-b)! \ldots} (-m-A)!\,(-m-B)! \ldots . \tag{8}$$

In general, each Mellin representation is valid only over a limited phase sector, essentially because the onset of exponential behaviour in y for large values of x at other phases leads to an essential singularity in the function represented. The limitation is easily overcome by extracting the exponential dependence before attempting to find the Mellin representation within each sector. Once the differential equation has been reduced to a form in which C_{pq} vanishes for all but two values of $p-q$, attention is directed to that subset of non-vanishing terms in (4) which contain the highest power of x, i.e. to the equation

$$\sum_q C_{p_{\max}q} \left(\frac{d}{dx}\right)^q y = 0 . \tag{9}$$

Provided there are two or more non-vanishing coefficients $C_{p_{\max}q}$, the substitution $y = e^{\sigma x}$ reduces (9) to a non-trivial algebraic equation soluble for σ. If on the other hand in the original independent variable there is but one $C_{p_{\max}q}$, this variable needs to be changed to that power of the old which produces two $C_{p_{\max}q}$ in the new differential equation. For instance, the Airy equation $d^2y/d\mathfrak{x}^2 - \mathfrak{x}y = 0$ requires a change $\mathfrak{x} = x^{2/3}$, transforming it to

$$\frac{d^2y}{dx^2} + \frac{1}{3x}\frac{dy}{dx} - \tfrac{4}{9}y = 0 .$$

Here $\sigma^2 - \frac{4}{9} = 0$, so the exponential dependences are

$$\exp\left(\pm\tfrac{2}{3}x\right) = \exp\left(\pm\tfrac{2}{3}\mathfrak{x}^{3/2}\right)$$

Once the Mellin representations are known, the asymptotic expansion can be determined exactly as in Chapter II, Section 2.

3. Confluent Hypergeometric Function $F(a, c, x)$

We shall suppose $F(a, c, x)$ has been defined as that solution to the differential equation

$$xF'' + (c-x)F' - aF = 0 \tag{10}$$

which tends to unity as $x \to 0$, whatever the value assigned to the parameter c. [This last stipulation eliminates any possibility of contribution from the second solution $x^{1-c} F(a - c + 1,\ 2 - c,\ x)$.]

As noted in Section 1, the two possible exponential dependences are $F^+ \sim e^x$ and $F^- \sim e^0$, so we need to introduce the two Mellin representations

$$2\pi i\, F = \begin{cases} \displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} M(m)\, x^{-m}\, dm & (11) \\[2ex] \displaystyle e^x \int_{\gamma - i\infty}^{\gamma + i\infty} \overline{M}(m)\, x^{-m}\, dm & (12) \end{cases}$$

which will prove valid over different phase sectors. To satisfy the differential equation (10), we require that for all x

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \{m(m - c + 1)\, M(m)\, x^{-m-1} + (m - a)\, M(m)\, x^{-m}\}\, dm = 0. \quad (13)$$

Similarly, to satisfy the differential equation obeyed by $f = e^{-x}F$, namely

$$xf'' + (c + x)f' + (c - a)f = 0, \quad (14)$$

we have, for all x,

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \{m(m - c + 1)\, \overline{M}(m)\, x^{-m-1} - (m - c + a)\, \overline{M}(m)\, x^{-m}\}\, dm = 0. \quad (15)$$

If M and $\overline{M}$ are free from poles or other singularities in the strip bounded in the complex plane by the vertical lines $\Re(m) = \gamma,\ \gamma - 1$, the value of m can be increased by unity in the second members of (13) and (15), leading to the difference equations

$$\frac{M(m + 1)}{M(m)} = -\frac{m(m - c + 1)}{m - a + 1}, \qquad \frac{\overline{M}(m + 1)}{\overline{M}(m)} = \frac{m(m - c + 1)}{m - c + a + 1}. \quad (16)$$

This requirement of a singularity-free strip can be relaxed to the condition that the dividing line "$\gamma - i\infty$ to $\gamma + i\infty$" shall be drawn, with such indentations as necessary, so as to separate sequences of poles originating in factorials with arguments decreasing with m, from sequences of poles originating in factorials with arguments increasing with m.

At this point we can select the required functional form $F(a, c, x)$ by demanding that its series in rising powers of x shall start at x^0 whatever the value of c. The only allowed factorial in M and $\overline{M}$ which increases with m

is then $(m - 1)!$, specializing the solutions of (16) to

$$M(m) \propto \frac{(-1)^m (m-1)!\,(-m+a-1)!}{(-m+c-1)!},$$

$$\overline{M}(m) \propto \frac{(m-1)!\,(-m+c-a-1)!}{(-m+c-1)!}. \tag{17}$$

The proportionality factors are fixed by the limit $F(a, c, x) \to 1$ as $x \to 0$. M and $\overline{M}$ must therefore possess unit residues at $m = 0$, so

$$2\pi i\, F(x) = \begin{cases} \dfrac{(c-1)!}{(a-1)!} \displaystyle\int \dfrac{(m-1)!\,(-m+a-1)!}{(-m+c-1)!} (-x)^{-m}\, dm, \\ \qquad\qquad |\mathrm{ph}(-x)| < \tfrac{1}{2}\pi, & (18) \\ \dfrac{(c-1)!}{(c-a-1)!} e^x \displaystyle\int \dfrac{(m-1)!\,(-m+c-a-1)!}{(-m+c-1)!} x^{-m}\, dm, \\ \qquad\qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. & (19) \end{cases}$$

Here the phase sectors for validity have been determined by convergence requirements towards the limits of integration, $\gamma \pm i\infty$, as in Chapter II, Section 2.

The asymptotic expansions now follow exactly as in Chapter II, Section 5.

4. Modified Bessel Function

We shall suppose $I_p(x)$ has been defined as that solution to the differential equation

$$x^2 I'' + xI' - (x^2 + p^2)I = 0 \tag{20}$$

which tends to $(\frac{1}{2}x)^p/p!$ as $x \to 0$. (This stipulation eliminates any contribution from the second solution $I_{-p}(x)$.)

For very large $|x|$ the dominant terms in (20) are expected to be

$$x^2(I'' - I) = 0, \tag{21}$$

which has the two solutions $I^+ \sim e^x$ and $I^- \sim e^{-x}$, so we introduce the two Mellin representations

$$2\pi i\, I(x) = e^{\pm x} \int_{\gamma - i\infty}^{\gamma + i\infty} M^{\pm}(m)\, x^{-m}\, dm, \tag{22}$$

which will prove valid over different phase sectors. To satisfy the differential equation

$$\left\{x^2 \frac{d^2}{dx^2} + x(1 \pm 2x)\frac{d}{dx} - (p^2 \mp x)\right\} I e^{\mp x} = 0\,, \tag{23}$$

we must have

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \{(m^2 - p^2)\, M^{\pm}(m)\, x^{-m} \mp (2m - 1)\, M^{\pm}(m)\, x^{-m+1}\}\, dm = 0\,. \tag{24}$$

Subject to the reservations noted in the previous section, the value of m in the second term of (24) can be increased by unity, reducing the problem to solving the difference equations

$$\frac{M^{\pm}(m + 1)}{M^{\pm}(m)} = \pm \frac{(m + p)\,(m - p)}{2(m + \frac{1}{2})}\,. \tag{25}$$

At this juncture we can select the required functional form $I_p(x)$ by demanding that its series in rising powers of x shall start at x^p rather than at x^{-p}. The only allowed factorial in M^+ and M^- which increases with m is therefore $(m + p - 1)!$, specializing the solutions of (25) to

$$M^{\pm}(m) = N(p)\, \frac{(m + p - 1)!\,(-m - \frac{1}{2})!}{(\pm 2)^m\,(-m + p)!}\,, \tag{26}$$

where $N(p)$ is a proportionality factor. To produce from $M^{\pm}$ a residue $(\frac{1}{2})^p/p!$ at $m = -p$ we have to take $N(p) = 1/\sqrt{\pi}\ \ (\pm 1)^p$, so

$$2\pi i\, I(x) = \frac{e^{\pm x}}{\sqrt{\pi}\,(\pm 1)^p} \int \frac{(m + p - 1)!\,(-m - \frac{1}{2})!}{(-m + p)!}\, (\pm 2x)^{-m}\, dm\,,$$

$$|\mathrm{ph}(\pm x)| < \tfrac{1}{2}\pi\,. \tag{27}$$

There is no need to pursue this example independently, because a change to the integration variable $m' = m + p$ and comparison with (18) and (19) leads to the relations

$$p!\, I_p(x) = (\tfrac{1}{2}x)^p\, e^{\pm x}\, F(p + \tfrac{1}{2},\ 2p + 1,\ \mp 2x)\,. \tag{28}$$

EXERCISES

1. The error function $\phi(x)$ can be defined as that solution to

$$\phi'' + x\phi' = 0$$

which behaves as $2x/\sqrt{\pi}$ for sufficiently small x. Find a pair of Mellin representations from this differential equation.

2. The incomplete factorial function $(p, x)!$ can be defined as that solution to

$$\left\{\frac{d^2}{dx^2} + \left(1 - \frac{p}{x}\right)\frac{d}{dx}\right\}(p, x)! = 0$$

which behaves as $x^{p+1}/(p + 1)$ for sufficiently small x. Find a pair of Mellin representations from this differential equation.

3. The Bessel function $J_p(x)$ is defined as that solution to

$$J'' + J'/x + (1 - p^2/x^2)J = 0$$

which behaves as $(\frac{1}{2}x)^p/p!$ for sufficiently small x. Derive the Mellin representation

$$2\pi i\, J_p(x) = (\tfrac{1}{2}x)^p \int \frac{(m - 1)!}{(p - m)!} (\tfrac{1}{4}x^2)^{-m}\, dm\,,$$

and hence the convergent and asymptotic expansions.

4. The parabolic cylinder function $D_p(x)$ is defined as that solution to

$$D'' + (p + \tfrac{1}{2} - \tfrac{1}{4}x^2)D = 0$$

which possesses the asymptotic behaviour $x^p e^{-\frac{1}{4}x^2}$, $x \to \infty$. Derive the Mellin representation

$$2\pi i\; 2^{\frac{1}{2}p}(-p - 1)!\, D_p(x) = e^{-\frac{1}{4}x^2} \int (2m - 1)!\,(-m - \tfrac{1}{2}p - 1)!\,(2x^2)^{-m}\, dm\,,$$

and hence the convergent and asymptotic expansions.

5. The associated Laguerre function $L_t^{\delta}(x)$ is defined as that solution to

$$xL'' + (\delta + 1 - x)L' + tL = 0$$

which possesses the asymptotic behaviour $L_t^{\delta}(x) \sim (-x)^t/t!$, $x \to \infty$. Derive the Mellin representation

$$2\pi i\; t!\,(-t - 1)!\,(-t - 1 - \delta)!\, L_t^{\delta}(x) = (-1)^t \int (m - 1)!\,(m-1-\delta)!$$
$$\times\,(-m - t - 1)!\; x^{-m}\, dm\,,$$

and hence the convergent and asymptotic expansions.

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Chapter XIII

Derivation of Asymptotic Expansions from Homogeneous Differential Equations (Phase-Integral, Liouville-Green or W.K.B. Approach)†

1. Reduction of Differential Equation to Standard Form

In all the later sections of this chapter we shall suppose that the given homogeneous linear differential equation, assumed for expository purposes to be of second order, has been reduced to a standard form

$$\frac{d^2y}{dx^2} = X(x)\ y, \tag{1}$$

where $X(x)$ is made as nearly free from poles and branch points as is conveniently possible, and certainly free from the two following types of variation to which the ensuing theory is not immediately applicable:

$$X(x) \to a + bx^{\frac{1}{2}} \quad \text{as}\ x \to 0, \tag{2a}$$

$$X(x) \to c/x^2 \quad \text{as} \quad x \to 0 \quad \text{or} \quad x \to \infty. \tag{2b}$$

To accomplish such reduction it is usually necessary to change both independent and dependent variables. Starting from a differential equation

$$\frac{d^2u}{dz^2} + f(z)\frac{du}{dz} + g(z)u = 0 \tag{3}$$

the original independent variable z is first written as a function of some new independent variable x, so

$$z = z(x), \qquad \frac{d^2}{dz^2} = \frac{1}{z_x}\frac{d}{dx}\frac{1}{z_x}\frac{d}{dx} = \frac{1}{z_x{}^2}\frac{d^2}{dx^2} - \frac{z_{xx}}{z_x{}^3}\frac{d}{dx},$$

† The titling question is reviewed in Section 12.

where for example z_x signifies dz/dx. The equation (3) is thereby transformed to

$$\frac{d^2u}{dx^2} + \left(fz_x - \frac{z_{xx}}{z_x}\right)\frac{du}{dx} + gz_x{}^2u = 0.$$

Changing now the dependent variable from u to y, where

$$u = y \ \exp\{-\tfrac{1}{2}\textstyle\int(fz_x - z_{xx}/z_x)dx\}, \tag{4}$$

the differential equation assumes the required form (1) with

$$X(x) = (\tfrac{1}{4}f^2 - g)z_x{}^2 + \tfrac{1}{2}f_xz_x + \tfrac{3}{4}z_{xx}{}^2/z_x{}^2 - \tfrac{1}{2}z_{xxx}/z_x. \tag{5}$$

The general formula (5) shows that unwanted variations in g such as (cf. 2(a) and 2(b)) $g(z) \sim a + bz^{\frac{1}{2}}$ and $g(z) \sim c/z^2$ cannot be removed by the tempting trial transformations $z = x^2$ and $z = 1/x$ respectively, since with the first the last two terms in (5) contribute $3/4x^2$ to $X(x)$, while with the second the initial term contributes $-c/x^2$, so both leading to the unacceptable form 2(b).

As remarked by Rudolph Langer back in 1931 the most efficacious transformation for removing such undesirable poles and branch points in this context is $z = e^x$, for which (4) and (5) specialize to

$$y = u \ \exp\tfrac{1}{2}(\textstyle\int fdz - x), \qquad X(x) = (\tfrac{1}{4}f^2 + \tfrac{1}{2}df/dz - g)e^{2x} + \tfrac{1}{4}. \tag{6}$$

As examples, the Bessel equation

$$\frac{d^2J}{dz^2} + \frac{1}{z}\frac{dJ}{dz} + \left(1 - \frac{p^2}{z^2}\right)J = 0$$

is thereby reduced to standard form (1) with

$$X(x) = p^2 - e^{2x}; \tag{7}$$

while the confluent hypergeometric and Whittaker differential equations, which are of course essentially equivalent, namely

$$\frac{d^2F}{dz^2} + \left(\frac{c}{z} - 1\right)\frac{dF}{dz} - \frac{a}{z}F = 0,$$

$$\frac{d^2W}{dz^2} + \left(-\tfrac{1}{4} + \frac{k}{z} - \frac{m^2 - \frac{1}{4}}{z^2}\right)W = 0,$$

lead to

$$X(x) = \tfrac{1}{4}e^{2x} - ke^x + m^2, \qquad \text{where } k = \tfrac{1}{2}c - a, \qquad m = \tfrac{1}{2}c - \tfrac{1}{2}. \quad (8)$$

2. Liouville–Green Approximations

The solutions to $d^2y/dx^2 = Xy$ when X is constant are $y = \exp(\pm X^{\frac{1}{2}}x)$. Physical and mathematical considerations alike indicate that when $X(x)$ is slowly varying or large in magnitude, $y \sim \exp(\pm\int X^{\frac{1}{2}}dx)$ should prove much more accurate solutions than $y \sim \exp(\pm X^{\frac{1}{2}}x)$. From a physical standpoint, on writing $X(x) = -k^2(x)$ the differential equation would appropriately describe the passage of an oscillation of wave number $k(x)$ and amplitude $y(x)$ along a conductor (in the most general sense) which possesses continuously changing characteristics: for instance an alternating voltage along a transmission line which has varying self-inductance and capacity, or an alternating electric field along a wave guide of varying cross-section. Now it is the *phase* of the wave rather than its peak amplitude at a point which primarily governs propagation beyond that point, especially for large k. Phase changes kdx along consecutive elemental sections are additive, so the total phase change up to a point x is the "phase integral" $\int^x kdx$, starting from some chosen origin. We therefore expect the solutions $y(x)$ to be dominated by $\exp(\pm i\int kdx) \equiv \exp(\pm\int X^{\frac{1}{2}}dx)$, the accuracy of these dependences being high when k is either slowly-varying or large in magnitude. The two quantities neglected in this physical approach are (i) variations in amplitude modulus, and (ii) reflected waves.

From a mathematical standpoint, two differentiations of the trial function give

$$\frac{d^2}{dx^2}\exp(\pm\int X^{\frac{1}{2}}dx) = (X \pm \tfrac{1}{2}X^{-\frac{1}{2}}X')\exp(\pm\int X^{\frac{1}{2}}dx),$$

so it displays the correct variation provided $X' \ll 2X^{3/2}$. This non-linear inequality can be broken down into the conditions that X must either be slowly varying, or else of large magnitude, or both, in agreement with the arguments from physics.

To seek reasonably accurate solutions we set $y = U\exp(\pm\int X^{\frac{1}{2}}dx)$, whereupon substitution in the differential equation (1) yields the exact relation

$$\pm\frac{U''}{2X^{\frac{1}{2}}U}+\frac{U'}{U}+\frac{X'}{4X}=0. \tag{9}$$

If the first contribution is neglected, $U = X^{-\frac{1}{4}}$, leading to

$$y_{\pm} \simeq X^{-\frac{1}{4}}\exp\left(\pm\int X^{\frac{1}{2}}dx\right), \tag{10}$$

the "Liouville–Green approximations". Estimation from (9) of the error committed shows that, barring pathological cases in which second or higher derivatives are abnormally large, it suffices to express the fractional error as $O(X'/X^{3/2})$.

Turning for a moment from these approximations to exact features, it is clear from (9) that the two full solutions can be chosen so as to differ only in the sign prefacing the square root $X^{\frac{1}{2}}$. We shall therefore write the correction factors to the two Liouville–Green approximations as $C(\pm X^{\frac{1}{2}})$.

3. Connection Formulae

The Liouville–Green approximations fail completely at a "turning point", defined by $X(x_0) = 0$. They cannot be continued analytically through such a point because the approximations are the leading terms in *asymptotic*, not convergent, expansions; at the turning point there are Stokes discontinuities to be contended with in addition to the easily appraised branch-point behaviour of $X^{-\frac{1}{4}}$ and $X^{\frac{1}{2}}$. Thus while (10) furnishes the leading terms of *some* pair of solutions to the differential equation a sufficient distance away in either direction from a turning point, a selected Liouville–Green approximation (e.g. $X^{-\frac{1}{4}}\exp\left(\int X^{\frac{1}{2}}dx\right)$) does not correctly specify even the leading term of a continuous solution to the differential equation on both sides of a turning point simultaneously.

We assume $X(x)$ to be real on the real axis,† and expansible as a Taylor series about its turning point x_0. In this first volume we will deal exclusively with by far the commonest case, that in which $X(x)$ varies linearly with $x - x_0$ at the turning point, i.e. its first derivative there, X_1, does not vanish. Then on one side the solutions are of exponential type while on the other they are oscillatory. (The alterations required in the

† This is not a material restriction, since we must choose the coordinate x such that $\mathrm{ph}(x-x_0) = 0$ coincides with a ray on which $X(x)$ is real; for otherwise "exponentially decreasing" and "purely exponentially increasing" at $\mathrm{ph}(x - x_0) = 0$ lose their precise meaning.

theory of connection formulae to cope with other circumstances are in themselves not difficult, but consequential changes are cumulative. By the time transitional and uniform expansions are broached the relevant results are so different their parallel development would invite confusion).

Without real loss of generality we can temporarily suppose X_1 to be positive, in which case $X(x)$ is positive and the solutions are of exponential type when $x > x_0$. Let y^- and y^+ denote those *continuous* solutions to the differential equation $d^2y/dx^2 = X(x)y$ which are respectively exponentially decreasing and purely exponentially increasing with increasing $x > x_0$. On the real axis these are $y^\pm = X^{-\frac{1}{4}}\mathscr{y}_\pm$ where

$$\mathscr{y}_- = \exp\left(-\int_{x_0}^{x} X^{\frac{1}{2}}dx\right) C(-X^{\frac{1}{2}}), \qquad \mathscr{y}_+ = \exp\left(\int_{x_0}^{x} X^{\frac{1}{2}}dx\right) C(X^{\frac{1}{2}}). \quad (11)$$

Here we have fixed the ratio of outer constant multipliers by assigning definite limits of integration. Following the lines of argument developed in Chapter I, Section 2, and amplified in exercises at the end of that chapter, the asymptotic expansions of the two continuous solutions $y^\pm$ at other phases can be derived by imagining the turning point x_0 to be bypassed via some roughly semicircular diversion through (say) the upper half of the complex plane.

(i) At $\mathrm{ph}(x - x_0) = 0$, $\int_{x_0}^x X^{\frac{1}{2}}dx$ is real and positive. The asymptotic series $\mathscr{y}_-$ is therefore recessed relative to the associated series $\mathscr{y}_+$ and consequently has no Stokes rays in the neighbourhood, say up to $\mathrm{ph}(x - x_0) = \theta$. Hence extension in range is permitted:

$$y^- = X^{-\frac{1}{4}}\mathscr{y}_-, \qquad 0 \leqslant \mathrm{ph}(x - x_0) < \theta. \quad (12)$$

(ii) At a certain phase $\mathrm{ph}(x - x_0) = \theta$, the phase of $\int_{x_0}^x X^{\frac{1}{2}}dx$ reaches π. The integral is again real, but now negative. The series $\mathscr{y}_-$ is at peak exponential dominance relative to $\mathscr{y}_+$ and is therefore on one of its Stokes rays. Close to the turning point,

$$\mathrm{ph}\int_{x_0}^{x} X^{\frac{1}{2}}dx \simeq \mathrm{ph}\ \tfrac{2}{3}(x - x_0)^{\frac{3}{2}}X_1^{\frac{1}{2}} = \tfrac{3}{2}\ \mathrm{ph}\ (x - x_0),$$

whereby

$$\theta \to \tfrac{2}{3}\pi \qquad \text{as} \qquad x \to x_0. \quad (13)$$

From the arguments outlined in the first chapter, we know that terms are homogeneous in phase in a series on one of its Stokes rays, and concurrently also in the associated series with alternating signs. Since the leading terms in $\mathscr{y}_-$ and $\mathscr{y}_+$ are real on all Stokes rays, both series are then homogeneously real. These realities whenever $\int_{x_0}^x X^{\frac{1}{2}}dx$ is real can naturally be verified from determining equations, such as (31) and (46) to follow. All we need in the present context is the weaker result of phase equality.

In the complete asymptotic expansion for y^-, the factor multiplying y_- is continued across its Stokes ray at phase angle θ, but there will be generated a formal discontinuity proportional to y_+ and $\frac{1}{2}\pi$ out of phase with y_-. Hence

$$y^- = X^{-\frac{1}{4}}(y_- + \alpha\, e^{\frac{1}{2}i\pi} y_+), \qquad \theta < \operatorname{ph}(x - x_0) < \phi, \tag{14}$$

where $\phi \doteqdot \frac{4}{3}\pi$ represents the next Stokes ray (one of y_+). At this stage of the argument all that is asserted about α is its reality.

(iii) By its genesis y^- is real when x is real, and thus in particular at $\operatorname{ph}(x - x_0) = \pi$. Since at this phase $\operatorname{ph} X^{\frac{1}{2}} = \operatorname{ph}(x - x_0)^{\frac{1}{2}} = \frac{1}{2}\pi$, y_- and y_+ are precisely complex conjugates here since they differ only in the sign prefacing the root $X^{\frac{1}{2}}$.

Moreover, at this phase $\operatorname{ph}\int_{x_0}^{x} X^{\frac{1}{2}}\,dx = \frac{3}{2}\pi$ and $X^{-\frac{1}{4}} = (-X)^{-\frac{1}{4}} e^{-\frac{1}{4}i\pi}$. The reality condition for

$$y^- = (-X)^{-\frac{1}{4}}(e^{-\frac{1}{4}i\pi} y_- + \alpha\, e^{\frac{1}{4}i\pi} y_+), \qquad \operatorname{ph}(x - x_0) = \pi, \tag{15}$$

therefore dictates the value $\alpha = 1$, whence the connection formula is

y_-	$\leftarrow$	y^-	$\rightarrow$	$y_- + i y_+$. (16)
Exponential side		Continuous solution		Oscillatory side

(iv) At $\operatorname{ph}(x - x_0) = 0$, y_+ is at peak exponential dominance relative to y_-, so has a Stokes ray there. In the asymptotic expansion for y^+, the factor multiplying y_+ is continued as we move off this ray, but there will be a formal discontinuity proportional to y_- and $\frac{1}{2}\pi$ out of phase with y_+, i.e.

$$y^+ = X^{-\frac{1}{4}}(y_+ + \tfrac{1}{2}\beta e^{\frac{1}{2}i\pi} y_-), \qquad 0 < \operatorname{ph}(x - x_0) < \theta. \tag{17}$$

(By careful comparison between the series y_+ and y_- at phases 0 and θ it is possible to identify β with α. But this step is tricky, and as we shall soon see redundant).

(v) The discontinuity caused by crossing the Stokes ray of y_- at phase angle θ has already been evaluated as $e^{\frac{1}{2}i\pi} y_+$. Hence

$$y^+ = X^{-\frac{1}{4}}\{y_+ + \tfrac{1}{2}\beta e^{\frac{1}{2}i\pi}(y_- + e^{\frac{1}{2}i\pi} y_+)\} = X^{-\frac{1}{4}}\{(1 - \tfrac{1}{2}\beta) y_+ + \tfrac{1}{2}\beta e^{\frac{1}{2}i\pi} y_-\},$$
$$\theta < \operatorname{ph}(x - x_0) < \phi. \tag{18}$$

(vi) By its genesis y^+ is real when x is real. At this phase,

$$y^+ = (-X)^{-\frac{1}{4}}\{(1 - \tfrac{1}{2}\beta)e^{-\frac{1}{4}i\pi}\mathscr{y}_+ + \tfrac{1}{2}\beta e^{\frac{1}{4}i\pi}\mathscr{y}_-\}, \quad \mathrm{ph}(x - x_0) = \pi. \tag{19}$$

Since at this phase $\mathscr{y}_+$ and $\mathscr{y}_-$ are complex conjugates, the reality condition demands $1 - \frac{1}{2}\beta = \frac{1}{2}\beta$, i.e. $\beta = 1$, whence the connection formula on the real axis is

$$\underset{\text{Exponential side}}{\mathscr{y}_+} \quad \leftarrow \quad \underset{\text{Continuous solution}}{y^+} \quad \rightarrow \quad \underset{\text{Oscillatory side}}{\tfrac{1}{2}(\mathscr{y}_+ + i\mathscr{y}_-)}\ . \tag{20}$$

We will apply these exact connection formulae (16) and (20) in due course (Section 6). In the meantime it is instructive to note their effect on the Liouville–Green approximations. Inserting the appropriate modulus signs to remove our initial temporary restriction to positive X_1,

$$X^{-\frac{1}{4}}\exp\left(-\left|\textstyle\int_{x_0}^{x} X^{\frac{1}{2}}dx\right|\right) \leftrightarrow 2(-X)^{-\frac{1}{4}}\sin\left\{\left|\textstyle\int_{x_0}^{x}(-X)^{\frac{1}{2}}dx\right| + \tfrac{1}{4}\pi\right\}, \tag{21}$$

$$X^{-\frac{1}{4}}\exp\left(+\left|\textstyle\int_{x_0}^{x} X^{\frac{1}{2}}dx\right|\right) \leftrightarrow (-X)^{-\frac{1}{4}}\cos\left\{\left|\textstyle\int_{x_0}^{x}(-X)^{\frac{1}{2}}dx\right| + \tfrac{1}{4}\pi\right\}. \tag{22}$$

These connections, incomplete in the sense of linking only leading terms in solutions, can alternatively be derived through replacing the original differential equation by its approximation close to the turning point, namely

$$d^2y/dx^2 \simeq (x - x_0)X_1 y. \tag{23}$$

For brevity of exposition we will again suppose X_1 to be positive. The substitution $z = (x - x_0)X_1{}^{\frac{1}{3}}$ reduces (23) to the Airy equation $d^2y/dz^2 = zy$, for which the asymptotic behaviour of two independent solutions has already been thoroughly investigated in Chapter I, question 2 and 3, and Chapter II, questions 12–15. The leading terms are

$$\tfrac{1}{2}\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}e^{-\frac{2}{3}z^{3/2}} \leftarrow Ai(z) \rightarrow \pi^{-\frac{1}{2}}(-z)^{-\frac{1}{4}}\sin\{\tfrac{2}{3}(-z)^{\frac{3}{2}} + \tfrac{1}{4}\pi\} \tag{24}$$

$$\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}e^{+\frac{2}{3}z^{3/2}} \leftarrow Bi(z) \rightarrow \pi^{-\frac{1}{2}}(-z)^{-\frac{1}{4}}\cos\{\tfrac{2}{3}(-z)^{\frac{3}{2}} + \tfrac{1}{4}\pi\}. \tag{25}$$

To the same accuracy as (23),

$$\int_{x_0}^{x} X^{\frac{1}{2}}dx \simeq \tfrac{2}{3}(x - x_0)^{\frac{3}{2}}X_1{}^{\frac{1}{2}} = \tfrac{2}{3}z^{\frac{3}{2}}. \tag{26}$$

Re-conversion of (24) and (25) to Liouville–Green approximations thus reproduces (21) and (22). However, it must be firmly emphasized that it is logically impossible from such limiting comparisons with Airy functions to construct valid derivations of exact and complete connections between solutions. Specifically, the wholesale neglect of higher derivatives in (23) and (26) allows the hypothetical addition of dimensionless quantities like $X_2/X_1^{\frac{4}{3}}$ and X_1X_3/X_2^2 to the phase constant $\frac{1}{4}\pi$; though their smallness would naturally be assured by the same qualitative requirements as those for the Liouville–Green approximations to be reasonably accurate, namely X is slowly varying or else of large magnitude. Furthermore, comparison with Airy functions can never precisely determine outer multiplying constants. For it is only close to the turning point, when $z \ll Z$ say, that Airy functions of z constitute adequate approximations, so actual solutions will contain asymptotic expansions of the more complicated variety

$$\frac{1}{(-\frac{1}{6})!\,(-\frac{5}{6})!}\sum_{r=0}^{\infty}\frac{(r-\frac{1}{6})!\,(r-\frac{5}{6})!}{r!\,(\pm\frac{4}{3}z^{\frac{3}{2}})^r}\sum_{s=0}^{r}a_{rs}\left\{\pm\tfrac{4}{3}\left(\frac{z}{Z}\right)^{3/2}\right\}^s$$

where $a_{r0} = 1$ for all r. On passing to the limits $z \to \pm\infty$ to seek outer multipliers of solutions away from the turning point, the result would be

$$\frac{1}{(-\frac{1}{6})!\,(-\frac{5}{6})!}\sum_{r=0}^{\infty}\frac{(r-\frac{1}{6})!\,(r-\frac{5}{6})!}{r!}\frac{a_{rr}}{Z^{\frac{3}{2}r}} = 1 + O(Z^{-\frac{3}{2}}),$$

and not just unity as from the simple Airy expansions.

4. Controversies about the Connection Formulae between Liouville–Green Approximations

These connection formulae between leading terms have for long been the subject of uncertainty and dispute, largely to be ascribed to disconcerting limitations in published derivations. For instance, though all ostensibly following the method of comparison with Airy functions—or equivalently Bessel functions of order one-third—Gans (1915), Jeffreys (1924, 1953, 1956) and Kramers (1926, 1929) did not always arrive at identical results. Moreover, Zwaan's (1929) potentially more powerful idea of determining Stokes multipliers through reality requirements on the real axis to either side of the turning point—which we have just exploited to the full in our derivation of exact connection formulae—until recently (e.g. Olver 1965) remained largely undeveloped, its earlier application having been only approximate and restricted to the deduction of (21).

The chief controversy has been whether the connection formulae are truly bi-directional as unequivocally asserted by the double-headed arrows in

(21) and (22). Above all, it has repeatedly been queried whether the left side of (22) necessarily implies the right. Analysis of the literature discloses three main sources of this doubt:

(i) Misunderstandings as to the meaning of "connection formulae" have inadvertently been fostered through variations and looseness in phraseology adopted by expositors.

The connection formulae we are concerned with are closely analogous to the more familiar "continuation formulae" of mathematical analysis, of which we have seen examples in Chapter I (12) and Chapter II (42), (53), (60). There are two minor differences, both stemming from their linkage of *asymptotic* expansions rather than of closed forms or absolutely convergent series: firstly, because the asymptotic expansions become unsuitable close to the turning point, this vicinity is bypassed and the symbol $\leftrightarrow$ introduced in lieu of an equality; secondly, because of Stokes discontinuities, phases on each side of a connection have to be specified, whereas continuation formulae link variables with a stated phase difference whatever their separate phases may be.

Specifically, the strict meaning of (22) is that the component solution which is exponentially increasing away from the turning point—and free from any exponentially decreasing portion—goes over on the other side of the turning point into the oscillatory solution whose leading term has the magnitude and phase stated in (22); and vice-versa, but this has not been seriously questioned. Now (21) and (22) indicate that exponentially increasing and decreasing solutions, vastly disparate in magnitude far to that side of the turning point, nevertheless go over on the other side into oscillatory solutions, this oscillatory pair remaining of mutually comparable magnitude even at great distances from the turning point. Consequently it is not permissible to replace the premise *component solution which is an exponentially increasing function and free from any exponentially decreasing portion* by a weaker version such as *solution which is closely approximated by an exponentially increasing function*. If in some application an exponentially increasing solution is suspected to be contaminated by an exponentially decreasing component of unknown magnitude, no quantitative prediction can be made of either the magnitude or phase of its oscillatory continuation on the other side of the turning point. This observation is salutary in reminding us that there may not always be sufficient data available to allow (22) to be invoked; it throws no light on the *validity* of (22). For these reasons, Fröman and Fröman's (1965, 1966) contentions on unidirection are unrelated to mathematical connection formulae as we understand them; their true significance and value lie in pointing the safer directions for connection in applications where information may be incomplete.

(ii) In any deductive theory, a method drawing on a limited number of propositions can turn out to be sufficient for proving a theorem but not its converse. This canny observation has on occasion been overlooked in the present context. Once an approach has been successful in establishing a connection formula operating in one direction, but not necessarily the reverse, the unjustified inference has been drawn that the connection formula is unidirectional. Now, on the whole, mathematical analysis of exponentially increasing transcendental functions is more complicated than that of the corresponding decreasing functions; for instance, their integral representations tend to be highly singular. The effect of this is augmented by the historical circumstance of relatively less attention having been paid to the increasing functions, presumably because of their slighter importance in science—at root a result of the lower probability of their satisfying localization requirements in space or time, such as boundary or stability conditions. Accordingly, it should occasion no surprise when a given technique yields the continuation of the exponentially decreasing Liouville–Green approximation, but not the continuation of the increasing approximation. The point is amply illustrated by Zwaan's approach. Adopting this, the decreasing solution can be successfully extrapolated across the turning point by introducing a single quite well-known characteristic feature of asymptotic expansions, namely persistence of the dominant series. By contrast, extrapolation of the increasing solution requires a considerably more detailed knowledge of the deportment of Stokes discontinuities; indeed, Zwaan's idea of determining Stokes multipliers through reality requirements was for many years believed to be powerless here (e.g. Langer 1934b).

(iii) Additional to sources (i) and (ii) springing from faulty reasoning, there has been a doubt of undeniable substance linked with a theoretical over-permissiveness in Poincaré's prescription of an asymptotic expansion. As noted in Chapter I, Section 5, Poincaré's specification permits, though it does not demand, abandonment in an asymptotic expansion of any portions which are multiplied by an exponentially decreasing function of the variable in the phase region considered. Abandonment of exponentially small portions of a solution can indeed destroy the bi-directional attributes of (21) and (22). If they are discarded in the solutions well away from a turning point, but not in the derivation of connection formulae, the bi-directional property of (21) is unaffected; but (22) is effectively reduced in application to a unidirectional relation valid only from right to left—i.e. the exponentially increasing form could be inferred from the oscillatory, but not the reverse. Moreover, if exponentially small components are also discarded *in the course of* deriving connection formulae, (21) can then only be proved as a unidirectional relation, valid from left to right—i.e. the oscillatory form could be inferred from the exponentially decreasing, but not the reverse. So in the past, with the over-permissive Poincaré prescription as basis, it all came down to the subjective question of an investigator's predilection for discarding exponentially small components; hence the bewildering diversity of views recorded in the literature. (It is instructive and highly entertaining to read the following references in sequence: Langer 1934b, pp. 559–564; Jeffreys 1956; Heading 1962, pp. 10–13, 59, 87–89, 97; Fröman and Fröman 1965, pp. 2, 7, 86–89, 97; Dingle 1965; Crothers 1971).

5. Complete Phase-Integral Expansions on the Exponential Side

Let $y^{\pm}$ denote two *continuous* solutions to the given differential equation

$$d^2y/dx^2 = Xy. \tag{1}$$

Along the real axis on the exponential side, we suppose these solutions to be specified by asymptotic expansions $\mathscr{Y}_{\pm}$, thus

$$(y^{\pm})_{\text{exp}} = \mathscr{Y}_{\pm} = X^{-\frac{1}{4}} \exp\left(\pm \left|\int_{x_0}^{x} X^{\frac{1}{2}} dx\right|\right) Y_{\pm} = X^{-\frac{1}{4}} \exp\left(\pm \varepsilon \int_{x_0}^{x} X^{\frac{1}{2}} dx\right) Y_{\pm} \tag{27}$$

where x_0 denotes the turning point, ε = sign of $(x - x_0)$ on the exponential side, and the positive root of X is understood.

Substitution of (27) in (1) leads to the equation

$$\frac{d^2 Y_{\pm}}{dx^2} + \left(\pm 2\varepsilon X^{\frac{1}{2}} - \frac{X'}{2X}\right)\frac{dY_{\pm}}{dx} + \frac{5(X')^2 - 4XX''}{16X^2} Y_{\pm} = 0. \tag{28}$$

As a consequence of our supposition on X being slowly varying, or else of large magnitude well away from the turning point, $\pm 2\varepsilon X^{\frac{1}{2}}$ is the most strongly varying and largest factor in (28). Taking this term to the left of the equation and all other terms to the right, then performing one integration by parts,

$$\mp \varepsilon Y_{\pm} = \frac{1}{2X^{\frac{1}{2}}}\frac{dY_{\pm}}{dx} + \frac{1}{32}\int^{x} \frac{5(X')^2 - 4XX''}{X^{5/2}} Y_{\pm} dx. \tag{29}$$

This equation can be solved by iteration: substitution in the right-hand side of the zero-order contribution to Y, unity say, gives the first-order correction; substitution of this gives the second-order correction, and so on. Denoting the rth order correction by Y_r,

$$Y_{+} = \sum_{0}^{\infty} Y_r, \qquad Y_{-} = \sum_{0}^{\infty} (-1)^r Y_r, \qquad Y_0 = 1, \tag{30}$$

where

$$-\varepsilon Y_{r+1} = \tfrac{1}{2}X^{-\frac{1}{2}}dY_r/dx + \tfrac{1}{32}\int^{x} X^{-\frac{5}{2}}\{5(X')^2 - 4XX''\}Y_r dx. \tag{31}$$

(The ε has been planted in (31) rather than in (30) so as to make all $Y_{r\geqslant 1}$ positive; c.f. (44) and (45) to follow).

Iteration through (31) can be greatly simplified by introducing some new independent variable, q say, which converts the multipliers into as near polynomial form as attainable:

$$AY_{r+1} = P(q)dY_r/dq + \int^{q} Q(q)Y_r dq, \tag{32}$$

where P and Q are preferably polynomials in q. The algebraic forms produced by the integration $\int X^{-\frac{3}{2}}X''dx$ frequently suggest a suitable q, as illustrated by the following simple examples. For Bessel functions $X(x) = p^2 - e^{2x}$ (equation 7), and the integral is proportional to $q = p/(p^2 - e^{2x})^{\frac{1}{2}}$. For parabolic cylinder functions $X(x) = \frac{1}{4}x^2 - p - \frac{1}{2}$, and the integral is proportional to $q = x/(x^2 - 4p - 2)^{\frac{1}{2}}$. The Whittaker function provides a more advanced illustration; here $X(x) = \frac{1}{4}e^{2x} - ke^x + m^2$ (equation 8), and the integral is proportional to

$$\left(\frac{e^x - 2k + 2\kappa}{e^x - 2k - 2\kappa}\right)^{\frac{1}{2}} + \frac{2\kappa(2\kappa - k)}{k(e^{2x} - 4ke^x + 4m^2)^{\frac{1}{2}}}, \qquad \kappa = (k^2 - m^2)^{\frac{1}{2}};$$

the first term by itself proves the best choice for q. As an important consequence of this way of selecting q, the terms of highest degree in the polynomials P, Q and Y_r are essentially the same for the whole class of problems under investigation, because when $X \to X_1(x - x_0)$ near the turning point x_0, $q \to a(x - x_0)^{-\frac{1}{2}}$ and so

$$AY_{r+1} \to q^4 dY_r/dq + \tfrac{5}{4}\int_0^q q^2 Y_r dq. \tag{33}$$

In this limit the solution reducing to unity at $r = 0$ is

$$Y_r \to \left(\frac{3q^3}{A}\right)^r \frac{(r - \frac{1}{6})!\,(r - \frac{5}{6})!}{2\pi\ \ r!}, \tag{34}$$

with $A = 4\varepsilon X_1^{\frac{1}{2}} a^3$.

The lower limit of integration in (31) and (32) may be chosen at will; moreover, it may be shifted in each successive iteration. This corresponds to a freedom of choice in constant outer multiplier within a fractional range of breadth $O(A^{-1})$, the same in general magnitude as the fractional error $O(Z^{-\frac{2}{3}})$ which would result from limiting comparisons with Airy functions as discussed at the end of Section 3. The exact connection formulae are unaffected by this freedom provided the identical set Y_r is inserted in the two expansions $Y_\pm = \Sigma(\pm 1)^r Y_r$, because this already ensures that Y_+ and Y_- can differ only in the sign prefacing $X^{\frac{1}{2}}$ as stipulated in (11).

The freedom in choice of constants is at once a strength and a weakness of the present "linear" approach to phase-integral theory. On the one hand it can advantageously be exploited in moulding the expansion to conform to some known limit, thereby facilitating identification of a solution—frequently a tough problem with differential equations. But, at the same time, expansions which in reality differ only trivially—by having different constant multipliers expanded in inverse powers of the large parameter in the function—may appear to bear little manifest resemblance to each other until their logarithms are extracted.

6. Complete Phase-Integral Expansions on the Oscillatory Side

Along the real axis on this side, (27) can better be written

$$y_+ = (-X)^{-\frac{1}{4}} e^{-i\Psi} Y_+, \qquad y_- = -i(-X)^{-\frac{1}{4}} e^{i\Psi} Y_-, \tag{35}$$

where

$$\Psi = \left| \int_{x_0}^{x} (-X)^{\frac{1}{2}} dx \right| + \tfrac{1}{4}\pi. \tag{36}$$

According to our exact connection formulae (20) and (16), on the oscillatory side

$$(y^+)_{\text{osc}} = \tfrac{1}{2}(\mathscr{y}_+ + i\mathscr{y}_-) = \tfrac{1}{2}(-X)^{-\frac{1}{4}}(e^{-i\Psi}Y_+ + e^{i\Psi}Y_-), \tag{37}$$

$$(y^-)_{\text{osc}} = \mathscr{y}_- + i\mathscr{y}_+ = i(-X)^{-\frac{1}{4}}(e^{-i\Psi}Y_+ - e^{i\Psi}Y_-). \tag{38}$$

These solutions are real because Y_+ and Y_- are complex conjugates here, this interrelation following from (30) taken together with the fact that when X is negative the Y_r are real for even r and imaginary for odd r. To express $y^{\pm}$ in terms of real quantities, we define

$$\mathscr{Y}_r = i^r Y_r, \tag{39}$$

$$\begin{aligned} \mathscr{Y}_{\text{even}} &= \sum_0^\infty (-1)^r \mathscr{Y}_{2r} = \Re(Y_+) = \Re(Y_-); \\ \mathscr{Y}_{\text{odd}} &= \sum_0^\infty (-1)^r \mathscr{Y}_{2r+1} = -\Im(Y_+) = \Im(Y_-). \end{aligned} \tag{40}$$

Then

$$\begin{aligned} (y^+)_{\text{osc}} &= (-X)^{-\frac{1}{4}}(\mathscr{Y}_{\text{even}} \cos\Psi + \mathscr{Y}_{\text{odd}} \sin\Psi), \\ (y^-)_{\text{osc}} &= 2(-X)^{-\frac{1}{4}}(\mathscr{Y}_{\text{even}} \sin\Psi - \mathscr{Y}_{\text{odd}} \cos\Psi). \end{aligned} \tag{41}$$

7. Late Terms in Phase-Integral Expansions

Though transformation to polynomial form greatly eases the iterative process, the sheer length of expressions renders it inconvenient to cope with direct applications of the recurrence relation (32) beyond about $r = 6$. Fortunately a complete asymptotic expansion can be derived for late terms.

Let us examine what each application of (32) does to some typical ingredient of Y_r, $q^{\alpha r+\beta}$ say. The differential coefficient dY_r/dq produces a multiplier $(\alpha r + \beta)$ in that contribution, while the integration produces a multiplier like $1/(\alpha r + \gamma)$; thus in the one iteration the importance of the differential component compared with the integral component has been augmented by a factor $O(r^2)$. Consequently when r is large (32) can be approximated by

$$AY_{r+1} \to P(q)\; dY_r/dq, \tag{42}$$

which has as solution

$$Y_r \to \xi\;\; (r+\eta)! \Big/ \left(-A\int_\zeta^q P^{-1}dq\right)^{r+\eta+1}. \tag{43}$$

Since (42) holds only for $r \gg 1$, the original initial condition $Y_0 = 1$ cannot be utilized directly to fix the constants ξ, η and ζ. Instead they can be inferred from (34), describing the behaviour of Y_r when $q \gg 1$. The power of $3q^3/A = (-A\int_\infty{}^q q^{-4}dq)^{-1}$ prescribed by (34) immediately dictates the values $\zeta = \infty$ (assuming q to be defined positively) and $\eta = -1$, while the limit

$$(r-\tfrac{1}{6})!\,(r-\tfrac{5}{6})!/r! \to (r-1)!\,\{1 + O(r^{-1})\}, \qquad r \gg 1,$$

confirms the value of η and identifies $\xi = 1/2\pi$. The explicit limiting form is therefore

$$Y_r \to (r-1)!/2\pi\mathscr{F}_0{}^r, \qquad r \gg 1, \tag{44}$$

where

$$\mathscr{F}_0 = A\int_q^\infty P^{-1}dq = 2\varepsilon\int_{x_0}^x X^{\frac{1}{2}}dx. \tag{45}†$$

Thus twice the positive Liouville–Green exponent plays the role of the "singulant" introduced in Chapter VII when finding late terms $Q_{2r\gg 1}$ in asymptotic expansions derived from integral representations in which the fast-varying factor in the integrand exhibits a maximum part way through the range of integration. From the arguments adduced in Chapter VII, Section 6, one would expect it to be possible to expand a late term $Y_{r\gg 1}$ as an asymptotic expansion in which each term is the product of a factorial, a power of $\mathscr{F}_0$, and a coefficient closely related to an early Y contribution. By trial substitution in the once-differentiated version of (32),

† $\mathscr{F}_0 = 2\left|\int_{x_0}^x X^{1/2}\,dx\right|$ when x, x_0 and X are real.

the formally exact expansion is found to be

$$Y_r = \frac{1}{2\pi \mathscr{F}_0^{\,r}} \sum_{s=0} (r-s-1)!\; Y_s\,(-\mathscr{F}_0)^s. \tag{46}$$

Towards the end of Section 5 we noted how the lower limit of integration in (32) may be chosen at will and moreover shifted in each successive iteration, this freedom corresponding to the possibility of multiplying the function by an expansion (independent of q) in inverse powers of the large parameter in the function. Because of this, the argument so far leading to (46) is incomplete, since it has not ruled out the contingency of the correct sequence of integration limits determining the Y_s differing from the chosen set defining the Y_r. [This is a particular case of our cautionary observation towards the end of Section 3 on the logical impossibility of deriving complete connections via limiting comparisons, in this instance comparison with (34)]. The actual coincidence between the two sets Y_s and Y_r in (46) can most readily be demonstrated from our interpretative theory of asymptotic expansions (Chapters XXI onwards), which expresses the increment incurred in crossing a Stokes ray quantitatively as a function of late terms. According to this, the second contribution to (20) for example, which represents an increment incurred on crossing the Stokes ray near phase $\frac{2}{3}\pi$, is expressed as a function of late terms, and thereby through (46) as a series in the Y_s; whereas the first contribution to (20) is simply a series in the Y_r. The two contributions prove to be complex conjugates at phase π, as required, only if the Y_s and Y_r sets are identical. In other words, the reality condition in (19) and (20) at phase π is synonomous with coincidence between the Y_s and Y_r sets (Chapter XXIV, Section 1).

Our discovery of the relation (46) greatly enhances the power of the phase–integral approach, especially since it expresses the general late term as an expansion in quantities already known, namely early terms $Y_0, Y_1, Y_2, \ldots$ along with $|\int_{x_0}^{x} X^{\frac{1}{2}}\,dx|$. Our interpretative theory (Chapters XXI onwards) converts these late terms into "terminants", factors which in principle precisely terminate the expansions ΣY_r and $\Sigma(-1)^r Y_r$ at any selected value of r. As far as straightforward *numerical* analysis is concerned, the solutions so obtained are still limited in precision, because the terminants are themselves expressed through (46) as asymptotic expansions. In a second stage, these terminant expansions can themselves be closed with new terminants; and so on stage after stage. But, excepting the neighbourhood of the turning point—where alternative expansions are in any case more suitable (Chapter XV)—even the first stage results prove to be around $10^4 - 10^6$ times more accurate than those hitherto available for the phase-integral method (Chapter XXIV, Section 3).

8. Variants†

First variant.

When X is a complicated function, it may prove more convenient to start from Liouville-Green approximations derived from the regnant part of it rather than the whole. Writing

$$X(x) = \chi(x) + \Delta(x), \qquad (y^{\pm})_{\text{exp}} = \chi^{-\frac{1}{4}} \exp\left(\pm\varepsilon \textstyle\int_{x_0}^{x} \chi^{\frac{1}{2}}\, dx\right)\ Y_{\pm}, \tag{47}$$

where $\chi(x_0) = 0$, the recurrence relation replacing (31) is readily found to be

$$-\varepsilon Y_{r+1} = \tfrac{1}{2}\chi^{-\frac{1}{2}} dY_r/dx + \tfrac{1}{32}\int^{x} \chi^{-5/2}\{5(\chi')^2 - 4\chi\chi'' - 16\chi^2\Delta\} Y_r dx. \tag{48}$$

The whole foregoing theory, including connection formulae and late terms, applies unchanged apart from the obvious replacement of X by χ and redefinition of Q to incorporate the Δ term.

Second variant

Instead of replacing $X^{\frac{1}{2}} = \chi^{\frac{1}{2}}(1 + \Delta/\chi)^{\frac{1}{2}}$ by $\chi^{\frac{1}{2}}$ throughout the Liouville-Green approximations, the compromise of planting $\chi^{\frac{1}{2}}(1 + \Delta/2\chi)$ in the exponentials may prove fruitful, though the resultant theory is more intricate. Writing

$$(y^{\pm})_{\text{exp}} = \chi^{-\frac{1}{4}} \exp\left(\pm\varepsilon \textstyle\int_{x_0}^{x} \chi^{\frac{1}{2}}[1 + \Delta/2\chi] dx\right)\ \sum_{0}^{\infty} (\pm 1)^r Y_r^{\pm} \tag{49}$$

where again $\chi(x_0) = 0$, the recurrence relations replacing (31) are

$$-\varepsilon Y_{r+1}^{\pm} = \tfrac{1}{2}\chi^{-\frac{1}{2}} dY_r^{\pm}/dx \pm \tfrac{1}{2}\varepsilon\chi^{-1}\Delta\ \ Y_r^{\pm}$$

$$+\tfrac{1}{32}\int^{x} \chi^{-\frac{5}{2}}\{5(\chi')^2 - 4\chi\chi'' + 4\chi\,\Delta^2 \pm 8\varepsilon\chi^{\frac{1}{2}}\chi'\Delta \mp 8\varepsilon\chi^{\frac{3}{2}}\Delta'\} Y_r^{\pm} dx. \tag{50}$$

Since the two solutions are still distinguished only by the sign prefacing the square root $\chi^{\frac{1}{2}}$ in the exponents and in the associated sets Y_r, the arguments leading to our exact connection formulae still appertain.

As in Section 5, some new independent variable q is next introduced

† A radical variant is examined separately in Chapter XVI.

to convert the multipliers into as near polynomial form as attainable:

$$A Y_{r+1}^{\pm} = P dY_r^{\pm}/dq \mp R Y_r^{\pm} + \int^q Q^{\pm} Y_r^{\pm}\, dq. \tag{51}$$

The algebraic forms produced by the integration $\int \chi^{-\frac{3}{2}}(\chi'' - \Delta^2)dx$ suggest possibilities. When at the turning point $q \to a(x - x_0)^{-\frac{1}{2}}$, $R(q) \propto q^2$ gives contributions of lower degree than those from the leading terms of $P(q)$ and $Q^{\pm}(q)$, and (33) and (34) thereby still correctly predict terms of highest degree.

Turning to the problem of evaluating late terms from (51), in one iteration the relative importance of the differential component $P dY_r^{\pm}/dq$ over the integral component is augmented by a factor $O(r^2)$, but only by a factor $O(r)$ over the component $RY_r^{\pm}$. Thus it is salutary to absorb the latter term into the leading approximation for $r \gg 1$, by noting that

$$AY_{r+1}^{\pm} \to P dY_r^{\pm}/dq \mp R Y_r^{\pm} = \exp(\pm \int RP^{-1} dq)\, P\frac{d}{dq}\left(Y_r^{\pm} \exp(\mp \int RP^{-1} dq)\right).$$

Introducing accordingly the new dependent variables

$$\bar{Y}_r^{\pm} = Y_r^{\pm} \exp(\pm \textstyle\int_q^{\infty} RP^{-1} dq), \tag{52†}$$

(42) and (43) still apply but with $\bar{Y}$'s replacing Y's. Near the turning point $\int_q^{\infty} RP^{-1}\, dq \propto q^{-1}$, so the exponentials do not affect identification of the terms of highest degree. From the arguments of Section 7 there follows

$$\bar{Y}_r^{\pm} \to \frac{(r-1)!}{2\pi \mathscr{F}_0^{\,r}}, \qquad r \gg 1,$$

where the singulant is

$$\mathscr{F}_0 = A\int_q^{\infty} P^{-1} dq = 2\varepsilon \int_{x_0}^{x} \chi^{\frac{1}{2}} dx. \tag{53‡}$$

† $\displaystyle\int_q^{\infty} RP^{-1}\, dq = \varepsilon \int_{x_0}^{x} \chi^{-1/2}\, \Delta\, dx.$

‡ $\mathscr{F}_0 = 2\left|\displaystyle\int_{x_0}^{x} \chi^{1/2}\, dx\right|$ when x, x_0 and χ are real.

By trial substitution in the once-differentiated version of (51), the formally exact expansion is found to be

$$\overline{Y}_r^{\pm} = \frac{1}{2\pi\mathscr{F}_0{}^r} \sum_{s=0} (r-s-1)!\, Y_s^{\mp}(-\mathscr{F}_0)^s \tag{54}$$

where

$$AY_{s+1}^{\pm} = PdY_s^{\pm}/dq + \int^q (Q^{\mp}Y_s^{\pm} \mp RdY_s^{\pm}/dq)\ dq.$$

By virtue of the correlations $Q^{\pm} = Q^{\mp} \pm dR/dq$ these recurrence relations coincide with those of (51), so furnishing the most striking demonstration up to this point of the reciprocity between late terms in one solution to a second-order homogeneous differential equation, and early terms in the second solution. An argument similar to that indicated in Section 7 further permits us to infer, via reference to the connection formulae, congruity of integration constants and thereby coincidence between the sets $Y_s^{\pm}$ and $\overline{Y}_r^{\pm}$ (Chapter XXIV, question 1).

9. Bessel Functions J_p and Y_p

$J_p(z)$ and $Y_p(z)$ are solutions to the Bessel equation

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} = \left(\frac{p^2}{z^2} - 1\right)y. \tag{55}$$

The substitution $z = e^x$ reduces this to our chosen standard form

$$d^2y/dx^2 = Xy, \qquad \text{with} \qquad X = p^2 - e^{2x}. \tag{56}$$

The turning point is at $z = p$; and X is positive and the solutions therefore exponentially varying when $z < p$, i.e. $\varepsilon = -1$.

$p > z > 0$

A more suitable independent variable is suggested by the integral $\int X^{-\frac{3}{2}}X''dx = 4/(p^2 - z^2)^{\frac{1}{2}}$. We set

$$q = p/(p^2 - z^2)^{\frac{1}{2}}, \tag{57}$$

because over most of the range under investigation this variable will be of order unity. In terms of q, the solutions which respectively decrease or increase exponentially away from the turning point at $q = \infty$ are

$$J_p = (q/2\pi p)^{\frac{1}{2}} e^{-p\Xi} Y_-, \qquad Y_p = -(2q/\pi p)^{\frac{1}{2}} e^{p\Xi} Y_+, \tag{58}$$

where

$$\Xi = \tanh^{-1} q^{-1} - q^{-1}, \tag{59}$$

$$Y_- = \sum_0^\infty (-1)^r Y_r, \qquad Y_+ = \sum_0^\infty Y_r, \tag{60}$$

and the Y_r are determined from the recurrence relation

$$2pY_{r+1} = q^2(q^2 - 1)dY_r/dq + \tfrac{1}{4}\int_0^q (5q^2 - 1)Y_r dq \tag{61}$$

starting from $Y_0 = 1$. The lower limit of integration in (61) has been selected so as to produce the simplest possible expansions. With this selection made, the multiplying constants in (58) have then been adjusted to conform to standard definitions of these Bessel functions (see below).

Explicitly, the recurrence relation (61) yields the first few terms as follows:

$$Y_0 = 1, \qquad Y_1 = \frac{q}{24p}(5q^2 - 3), \qquad Y_2 = \frac{q^2}{1{,}152p^2}(385q^4 - 462q^2 + 81),$$

$$Y_3 = \frac{q^3}{414{,}720p^3}(425{,}425q^6 - 765{,}765q^4 + 369{,}603q^2 - 30{,}375),$$

$$\begin{aligned} Y_4 = \frac{q^4}{39{,}813{,}120p^4}&(185{,}910{,}725q^8 - 446{,}185{,}740q^6 + 349{,}922{,}430q^4 \\ &- 94{,}121{,}676q^2 + 4{,}465{,}125). \end{aligned} \tag{62}$$

These are in complete accord with the evaluations from integral representations, Chapter VIII, Section 5.

Confirmation of the multiplying constants in (58) is most readily sought by letting $z \to \infty$ on the oscillatory side (see over). It is, however, instructive to verify them also through limiting behaviour as $z \to 0$ on the present exponential side. In the limit $q \to 1$ (62) reduces to

$$Y_{\pm} = 1 \cdot \pm \frac{1}{12p} + \frac{1}{288p^2} \mp \frac{139}{51{,}840p^3} - \frac{571}{2{,}488{,}320p^4} \cdots \quad (63)$$

These are the well known Stirling–Laplace series (e.g. Chapter VIII, Section 2) for

$$Y_+ = \frac{p!\,e^p}{(2\pi)^{\frac{1}{2}}p^{p+\frac{1}{2}}}, \qquad Y_- = \frac{(2\pi)^{\frac{1}{2}}p^{p-\frac{1}{2}}e^{-p}}{(p-1)!}.$$

Since as $q \to 1$

$$e^{-p\tanh^{-1}q^{-1}} = \left(\frac{q-1}{q+1}\right)^{\frac{1}{2}p} \to \left(\frac{z}{2p}\right)^p,$$

the complete phase-integral expansions of (58) reduce to the limits

$$J_p(z) \to (\tfrac{1}{2}z)^p/p!, \qquad Y_p(z) \to -(p-1)!/\pi(\tfrac{1}{2}z)^p.$$

These limiting dependences as $z \to 0$ conform to standard definitions.

Late terms By (45) and (61) the singulant is

$$\mathscr{F}_0 = 2p\int_q^\infty \frac{dq}{q^2(q^2-1)} = 2p\Xi, \quad (64)$$

twice the positive Liouville–Green exponent, as expected. Hence (46) expresses the late terms as

$$Y_{r \gg 1} = \frac{(r-1)!}{2\pi(2p\Xi)^r}\left\{1 - \frac{\Xi q(5q^2-3)}{12(r-1)} + \frac{\Xi^2 q^2(385q^4 - 462q^2 + 81)}{288(r-1)(r-2)} \cdots\right\}. \quad (65)$$

This result is again in complete accord with the evaluation from integral representations, Chapter VIII, Section 5, taking into account of course the minor differences in notation.

$z > p > 0$

On this oscillatory side of the turning point, q and Y_{2r+1} are imaginary. To express the expansions in terms of real quantities, we replace q by $i\mathscr{q}$, i.e.

$$\mathscr{q} = p/(z^2 - p^2)^{\frac{1}{2}}, \tag{66}$$

and rewrite the contributions to the series as $\mathscr{Y}_r = i^r Y_r$, thus

$$\mathscr{Y}_0 = 1, \qquad \mathscr{Y}_1 = \frac{\mathscr{q}}{24p}(5\mathscr{q}^2 + 3), \qquad \mathscr{Y}_2 = \frac{\mathscr{q}^2}{1{,}152p^2}(385\mathscr{q}^4 + 462\mathscr{q}^2 + 81),$$

$$\mathscr{Y}_3 = \frac{\mathscr{q}^3}{414{,}720p^3}(425{,}425\mathscr{q}^6 + 765{,}765\mathscr{q}^4 + 369{,}603\mathscr{q}^2 + 30{,}375),$$

$$\mathscr{Y}_4 = \frac{\mathscr{q}^4}{39{,}813{,}120p^4}(185{,}910{,}725\mathscr{q}^8 + 446{,}185{,}740\mathscr{q}^6 + 349{,}922{,}430\mathscr{q}^4 + 94{,}121{,}676\mathscr{q}^2 + 4{,}465{,}125). \tag{67}$$

Then by our exact connection formulae (41),

$$\begin{aligned} J_p &= (2\mathscr{q}/\pi p)^{\frac{1}{2}}[\mathscr{Y}_{\text{even}} \sin(p\Upsilon + \tfrac{1}{4}\pi) - \mathscr{Y}_{\text{odd}} \cos(p\Upsilon + \tfrac{1}{4}\pi)], \\ Y_p &= -(2\mathscr{q}/\pi p)^{\frac{1}{2}}[\mathscr{Y}_{\text{even}} \cos(p\Upsilon + \tfrac{1}{4}\pi) + \mathscr{Y}_{\text{odd}} \sin(p\Upsilon + \tfrac{1}{4}\pi)], \end{aligned} \tag{68}$$

where

$$\Upsilon = \mathscr{q}^{-1} - \tan^{-1}\mathscr{q}^{-1} = -i\Xi, \tag{69}$$

$$\mathscr{Y}_{\text{even}} = \sum_0^\infty (-1)^r \mathscr{Y}_{2r}, \qquad \mathscr{Y}_{\text{odd}} = \sum_0^\infty (-1)^r \mathscr{Y}_{2r+1}. \tag{70}$$

There is again complete agreement with the evaluations from integral representations, Chapter VIII, Section 5.

The multiplying constants in (68)—and therefore those in (58)—are easily confirmed by letting $z \to \infty$. When $\mathscr{q} \doteqdot p/z$ is small,

$$\Upsilon = \mathscr{q}^{-1} - \tan^{-1}\mathscr{q}^{-1} \doteqdot \mathscr{q}^{-1} - \tfrac{1}{2}\pi \doteqdot z/p - \tfrac{1}{2}\pi,$$

so

$$J_p \to (2/\pi z)^{\frac{1}{2}} \sin(z - \tfrac{1}{2}\pi p + \tfrac{1}{4}\pi), \qquad Y_p \to -(2/\pi z)^{\frac{1}{2}} \cos(z - \tfrac{1}{2}\pi p + \tfrac{1}{4}\pi).$$

These asymptotes have already been elicited by less sophisticated asymptotic methods, e.g. Chapter XII, question 3.

Late terms. The singulant and the resultant late terms are still given by (64) and (65) respectively, but the following notational changes are needed to express everything in terms of real quantities:

$$\mathscr{F}_0 = 2ip\Upsilon, \tag{71}$$

$$\mathscr{Y}_{r \gg 1} = \frac{(r-1)!}{2\pi(2p\Upsilon)^r}\left\{1 - \frac{\Upsilon q(5q^2+3)}{12(r-1)} + \frac{\Upsilon^2 q^2(385q^4+462q^2+81)}{288(r-1)(r-2)} \cdots\right\}. \tag{72}$$

10. Whittaker Functions for large k

The functions $W_{k,m}(z)$ and $W_{-k,m}(-z)$ are independent solutions to the Whittaker equation

$$\frac{d^2W}{dz^2} = \left(\tfrac{1}{4} - \frac{k}{z} + \frac{m^2 - \frac{1}{4}}{z^2}\right)W. \tag{73}$$

Direct application of the foregoing methods would lead to the appearance of logarithmic as well as algebraic quantities in the higher terms, these originating from the singularity in (73) at $z = 0$. As noted in Section 1, this difficulty can be overcome by changing the independent variable to x, where $z = e^x$. With $W = e^{\frac{1}{2}x}y$,

$$d^2y/dx^2 = Xy, \qquad X = \tfrac{1}{4}e^{2x} - ke^x + m^2. \tag{74}$$

When $k \gg m$ it is convenient to adopt the first variant of Section 8 and divide X into its dominant part $\chi = \frac{1}{4}z(z-4k)$ and the remainder $\Lambda = m^2$. The relevant turning point of the major part is at $z = 4k$; and χ is positive, and the solutions therefore exponentially varying, when $z > 4k > 0$.

$> 4k > 0$.

More suitable independent variables are suggested by the integral

$$\int \chi^{-\frac{3}{2}}\chi'' dx = -\frac{2}{k}\left(\frac{z}{z-4k}\right)^{\frac{1}{2}} - \frac{4}{[z(z-4k)]^{\frac{1}{2}}},$$

from which we select

$$q = \left(\frac{z}{z-4k}\right)^{\frac{1}{2}}. \tag{75}$$

In terms of q, the solutions which respectively decrease or increase exponentially away from the turning point at $q = \infty$ are

$$W_{km}(z) = A_{km}q^{\frac{1}{2}}e^{-2k\Xi}Y_-, \qquad \mathscr{W}_{km}(z) = \mathscr{A}_{km}q^{\frac{1}{2}}e^{+2k\Xi}Y_+, \tag{76}$$

where

$$\Xi = q/(q^2-1) - \coth^{-1}q, \tag{77}$$

$$Y_- = \sum_0^\infty (-1)^r Y_r, \qquad Y_+ = \sum_0^\infty Y_r, \tag{78}$$

and the Y_r, commencing at $Y_0 = 1$, are determined from the recurrence relation

$$8kY_{r+1} = (q^2-1)^2 dY_r/dq + \tfrac{1}{4}\int^q (5q^2 - 2 - \eta/q^2)Y_r dq, \tag{79}$$

with $\eta = 16m^2 - 1$.

In (76) we have introduced as the exponentially increasing solution

$$\mathscr{W}_{km}(z) = (2\cos\pi k)^{-1}\{W_{-k,m}(ze^{i\pi}) + W_{-k,m}(ze^{-i\pi})\} \tag{80}$$

so as to deal throughout with a function which is real on the real axis. The two sides of (80) relate to segregated ranges (currently $z > 4k > 0$ on the left and $-z > -4k > 0$ on the right), so because of Stokes discontinuities we cannot draw directly upon this definition to deduce the asymptotic expansion of $\mathscr{W}$ from that of W in the same range.

Identification of solutions is most easily carried through by passing to the limit $z \to \infty$, i.e. $q = 1$. For then

$$q^{\frac{1}{2}}e^{-2k\Xi} \to k^{-k}e^k z^k e^{-\frac{1}{2}z}, \qquad q^{\frac{1}{2}}e^{2k\Xi} \to k^k e^{-k} z^{-k} e^{\frac{1}{2}z},$$

while by definition†

$$W_{km}(z) \to z^k e^{-\frac{1}{2}z}, \qquad \mathscr{W}_{km}(z) \to z^{-k}e^{\frac{1}{2}z}.$$

Hence

$$A_{km} = k^k e^{-k}/(Y_-)_{q=1}, \qquad \mathscr{A}_{km} = k^{-k}e^k/(Y_+)_{q=1}. \tag{81}$$

† $W_{km}(z) = (k+m-\frac{1}{2})!(k-m-\frac{1}{2})!\, z^k e^{-(1/2)z}$

$$\times \sum_{r=0}^{\infty} 1/r!(k+m-\tfrac{1}{2}-r)!\,(k-m-\tfrac{1}{2}-r)!(-z)^r.$$

This asymptotic power series is not restricted to the range $z > 4k > 0$, so can be drawn upon to determine $\mathscr{W}$ from W.

The conceptually-simplest identifiable pair of solutions results from the choice unity for the lower limit of integration in (79). Then $(Y_+)_{q=1} = (Y_-)_{q=1} = Y_0 = 1$, and the first few terms are as follows:

$$Y_0 = 1, \qquad Y_1 = \frac{1}{96k}\{5q^3 - 6q - (3\eta - 1) + 3\eta/q\},$$

$$Y_2 = \frac{1}{18{,}432k^2}\{385q^6 - 924q^4 - 10(3\eta - 1)q^3 - 6(7\eta - 114)q^2 + 12(3\eta - 1)q + (9\eta^2 + 102\eta - 143) - 6\eta(3\eta - 1)/q + 9\eta(\eta - 8)/q^2\}, \tag{82}$$

in agreement with Chapter VIII, equation (103).

However, a neater pair of expansions can be constructed by allowing the lower limit to shift in each iteration in such a way that all Y_{2r} are even in q and all Y_{2r+1} odd. The expansions for $Y_\mp$ then start with

$$1 \mp \frac{1}{96k}(5q^3 - 6q + 3\eta/q)\ldots,$$

so

$$(Y_\mp)_{q=1} = 1 \mp \frac{3\eta - 1}{96k}\ldots.$$

Solutions will be identifiable if subsequent integration constants are chosen such that for $q = 1$ the $Y_\mp$ reduce to some known series commencing as above. The appearance of the combination $(k + m - \frac{1}{2})!\,(k - m - \frac{1}{2})!$ in some integral representations and expansions for $W_{k,m}(z)$ led to the discovery that

$$(Y_-)_{q=1} = (Y_+)_{q=1}^{-1} = (2\pi)^{\frac{1}{2}}k^k e^{-k}/[(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}} \tag{83}$$

fulfil both commencement and symmetry conditions. Expansions of (83) are most easily effected via Barnes' (1899) generalization of Stirling's series, videlicet

$$(k + \varepsilon)! = (2\pi)^{\frac{1}{2}}k^{k+\varepsilon+\frac{1}{2}}e^{-k}\exp\left(\sum_{r=1}^{\infty}(-1)^{r-1}\frac{B_{r+1}(\varepsilon+1)}{r(r + 1)k^r}\right), \tag{84}$$

where the B's are Bernoulli polynomials. To the first five orders,

$$(Y_{\mp})_{q=1} = 1 \mp \frac{3\eta - 1}{96k} + \frac{9\eta^2 - 6\eta + 1}{18{,}432k^2}$$

$$\mp \frac{135\eta^3 + 8{,}505\eta^2 - 51{,}795\eta + 4{,}027}{26{,}542{,}080k^3}$$

$$+ \frac{405\eta^4 + 103{,}140\eta^3 - 656{,}370\eta^2 + 255{,}684\eta - 16{,}123}{10{,}192{,}158{,}720k^4} \cdots . \tag{85}$$

Choosing successive integration constants to conform to (85) leads to the following results:

$$A_{km} = \mathscr{A}_{km}^{-1} = (2\pi)^{-\frac{1}{2}}[(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}}, \tag{86}$$

$$Y_0 = 1, \qquad Y_1 = \frac{1}{96k}(5q^3 - 6q + 3\eta/q),$$

$$Y_2 = \frac{1}{18{,}432k^2}$$

$$\times \{385q^6 - 924q^4 - 6(7\eta - 114)q^2 + 36(3\eta - 4) + 9\eta(\eta - 8)/q^2\},$$

$$Y_3 = \frac{1}{26{,}542{,}080k^3}\{425{,}425q^9 - 1{,}531{,}530q^7 - 117(175\eta - 17{,}436)q^5$$

$$+ 1{,}620(33\eta - 734)q^3 + 675(\eta^2 - 68\eta + 384)q + 4{,}050(3\eta - 16)/q$$

$$+ 135\eta(\eta - 8)(\eta - 24)/q^3\}. \tag{87}$$

Late terms. By (45) and (79) the singulant is

$$\mathscr{F}_0 = 8k\int_q^{\infty} (q^2 - 1)^{-2}dq = 4k\Xi, \tag{88}$$

twice the positive Liouville–Green exponent. Late terms can then be written down from (46). Those in the neater expansion (87) are

$$Y_{r\geqslant 1} = \frac{(r-1)!}{2\pi(4k\Xi)^r}\left[1 - \frac{\Xi(5q^3 - 6q + 3\eta/q)}{24(r-1)}\right.$$

$$\left. + \frac{\Xi^2\{385q^6 - 924q^4 - 6(7\eta - 114)q^2 + 36(3\eta - 4) + 9\eta(\eta - 8)/q^2\}}{1{,}152(r-1)(r-2)} \cdots\right]. \tag{89}$$

$4k > z > 0$

On this oscillatory side of the turning point, q and Y_{2r+1} are imaginary. To express the asymptotic expansions in terms of real quantities, we replace q by $i\mathscr{q}$, i.e.

$$\mathscr{q} = \left(\frac{z}{4k - z}\right)^{\frac{1}{2}}, \tag{90}$$

and rewrite the contributions to the series as $\mathscr{Y}_r = i^r Y_r$. Those corresponding to (87) are

$$\mathscr{Y}_0 = 1, \qquad \mathscr{Y}_1 = \frac{1}{96k}(5\mathscr{q}^3 + 6\mathscr{q} + 3\eta/\mathscr{q}),$$

$$\mathscr{Y}_2 = \frac{1}{18{,}432k^2}\{385\mathscr{q}^6 + 924\mathscr{q}^4 - 6(7\eta - 114)\mathscr{q}^2 - 36(3\eta - 4)$$

$$+ 9\eta(\eta - 8)/\mathscr{q}^2\}, \tag{91}$$

etc. Then by the connection formulae (41),

$$W_{km}(z) = [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}}(2\mathscr{q}/\pi)^{\frac{1}{2}}$$
$$\times [\mathscr{Y}_{\text{even}} \sin(2k\Upsilon + \tfrac{1}{4}\pi) - \mathscr{Y}_{\text{odd}} \cos(2k\Upsilon + \tfrac{1}{4}\pi)], \tag{92}$$

$$\mathscr{W}_{km}(z) = [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{-\frac{1}{2}}(2\pi\mathscr{q})^{\frac{1}{2}}$$
$$\times [\mathscr{Y}_{\text{even}} \cos(2k\Upsilon + \tfrac{1}{4}\pi) + \mathscr{Y}_{\text{odd}} \sin(2k\Upsilon + \tfrac{1}{4}\pi)], \tag{93}$$

where

$$\Upsilon = \cot^{-1}\mathscr{q} - \mathscr{q}/(\mathscr{q}^2 + 1) = -i\Xi, \tag{94}$$

$$\mathscr{Y}_{\text{even}} = \sum_0^\infty (-1)^r \mathscr{Y}_{2r}, \qquad \mathscr{Y}_{\text{odd}} = \sum_0^\infty (-1)^r \mathscr{Y}_{2r+1}. \tag{95}$$

Late terms. The singulant and the resultant late terms are still given by (88) and (89) respectively, but the following notational changes are needed to express everything in terms of real quantities:

$$\mathscr{F}_0 = 4ik\Upsilon, \tag{96}$$

$$\mathscr{Y}_{r\gg 1} = \frac{(r-1)!}{2\pi(4k\Upsilon)^r}\left[1 - \frac{\Upsilon(5\varphi^3 + 6\varphi + 3\eta/\varphi)}{24(r-1)}\right.$$

$$\left. + \frac{\Upsilon^2\{385\varphi^6 + 924\varphi^4 - 6(7\eta - 114)\varphi^2 - 36(3\eta - 4) + 9\eta(\eta - 8)/\varphi^2\}}{1{,}152(r-1)(r-2)}\cdots\right]. \tag{97}$$

11. Whittaker Functions for large $\kappa = (k^2 - m^2)^{\frac{1}{2}}$

When m is not much smaller than k, the full expression for X, i.e.

$$X = \tfrac{1}{4}e^{2x} - ke^x + m^2 = \tfrac{1}{4}\{z - 2(k - \kappa)\}\{z - 2(k + \kappa)\}, \tag{98}$$

must be retained in the Liouville–Green factors. The sign allotted to the square root $\kappa = (k^2 - m^2)^{\frac{1}{2}}$ will be that of k.

There are two turning points, at $z = 2(k + \kappa)$ and $z = 2(k - \kappa)$. X is positive, and the solutions therefore exponentially varying, when $z > 2(k + \kappa) > 0$ or $z < 2(k - \kappa)$. The second of these regions will not be treated here, because expansions can be deduced from those in the first region in several ways, providing admirably instructive exercises for the reader, e.g. questions 6, 7 at the end of this chapter.

$z > 2(k + \kappa) > 0$

A more suitable independent variable is suggested by the integral

$$\int X^{-\frac{1}{2}}X''dx = -\frac{2k}{\kappa^2}\left(\frac{z - 2k + 2\kappa}{z - 2k - 2\kappa}\right)^{\frac{1}{2}}$$

$$- \frac{4(2\kappa - k)}{\kappa}\frac{1}{[(z - 2k + 2\kappa)(z - 2k - 2\kappa)]^{\frac{1}{2}}},$$

from which we select

$$q = \left(\frac{z - 2k + 2\kappa}{z - 2k - 2\kappa}\right)^{\frac{1}{2}}. \tag{99}$$

In terms of q and the abbreviations

$$A = 1 + k/\kappa, \qquad a = 1 - k/\kappa, \qquad B = 1 + 3k/\kappa, \qquad b = 1 - 3k/\kappa, \tag{100}$$

the solutions which respectively decrease or increase exponentially away from the turning point at $q = \infty$ are

$$W_{km}(z) = C_{km}\left(\frac{Aq^2 + a}{2q}\right)^{\frac{1}{2}} e^{-2\kappa\Xi} Y_-,$$

$$\mathscr{W}_{km}(z) = \mathscr{C}_{km}\left(\frac{Aq^2 + a}{2q}\right)^{\frac{1}{2}} e^{+2\kappa\Xi} Y_+, \tag{101}$$

where

$$\Xi = \frac{q}{q^2 - 1} - \left(\frac{k}{\kappa}\right)\coth^{-1} q + \left(\frac{m}{\kappa}\right)\tanh^{-1}\frac{1}{q}\left(\frac{k - \kappa}{k + \kappa}\right)^{\frac{1}{2}}, \tag{102}$$

(with positive signs allotted to k/κ and m/κ),

$$Y_- = \sum_0^\infty (-1)^r Y_r, \qquad Y_+ = \sum_0^\infty Y_r, \tag{103}$$

and the Y_r, commencing at $Y_0 = 1$, are determined from the recurrence relation

$$16\kappa Y_{r+1} = (Aq^4 - Bq^2 - b + a/q^2)dY_r/dq$$
$$+ \tfrac{1}{2}\int^q (5Aq^2 - B - b/q^2 + 5a/q^4)Y_r dq. \tag{104}$$

Identification of solutions is most easily carried through by passing to the limit $z \to \infty$, i.e. $q \to 1$. Then, for example,

$$[(Aq^2 + a)/2q]^{\frac{1}{2}} e^{-2\kappa\Xi} \to (k + m)^{-\frac{1}{2}(k+m)}(k - m)^{-\frac{1}{2}(k-m)} e^k z^k e^{-\frac{1}{2}z},$$

whereas by definition $W_{km}(z) \to z^k e^{-\frac{1}{2}z}$. Hence

$$C_{km} = (k + m)^{\frac{1}{2}(k+m)}(k - m)^{\frac{1}{2}(k-m)} e^{-k}/(Y_-)_{q=1}, \tag{105}$$

and similarly

$$\mathscr{C}_{km} = (k + m)^{-\frac{1}{2}(k+m)}(k - m)^{-\frac{1}{2}(k-m)} e^k/(Y_+)_{q=1}. \tag{106}$$

The conceptually-simplest identifiable pair of solutions results from the choice unity for the lower limit of integration in (104). Then $(Y_-)_{q=1} = Y_0 = 1$, and

$$Y_0 = 1, \qquad Y_1 = \frac{1}{192\kappa}(5Aq^3 - 3Bq + 8k + 3b/q - 5a/q^3) \tag{107}$$

etc, in agreement with Chapter VIII, equation (131).

A neater pair of expansions can be obtained by allowing the lower limit to shift in each iteration in such a way that all Y_{2r} are even in q and all Y_{2r+1} odd. The expansions for $Y_{\mp}$ then start with

$$1 \mp \frac{1}{192\kappa}(5Aq^3 - 3Bq + 3b/q - 5a/q^3)\ \ldots\ldots,$$

so

$$(Y_{\mp})_{q=1} = 1 \pm k/24\kappa^2\ \ldots\ldots$$

As anticipated from analogy with the parallel results in the preceding section, these prove to be essentially the beginnings of the expansions for $[(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!]^{\mp\frac{1}{2}}$ in powers of κ^{-2}. We therefore determine the integration constants successively to achieve agreement at $q = 1$ with the series

$$(Y_-)_{q=1} = (Y_+)_{q=1}^{-1} = (2\pi)^{\frac{1}{2}}(k+m)^{\frac{1}{2}(k+m)}(k-m)^{\frac{1}{2}(k-m)}$$

$$\times\, e^{-k}/[(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!]^{\frac{1}{2}} = 1 + \frac{k}{24\kappa^2} + \frac{k^2}{1{,}152\kappa^4}$$

$$-\frac{k(4{,}027k^2 - 3{,}024\kappa^2)}{414{,}720\kappa^6} - \frac{k^2(16{,}123k^2 - 12{,}096\kappa^2)}{39{,}813{,}120\kappa^8}\ \ldots\ldots \tag{108}$$

With shifting integration constants chosen to give expansions consistent with (108),

$$C_{km} = \mathscr{C}_{km}{}^{-1} = [(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!/2\pi]^{\frac{1}{2}}, \tag{109}$$

$$Y_0 = 1, \qquad Y_1 = \frac{1}{192\kappa}(5Aq^3 - 3Bq + 3b/q - 5a/q^3),$$

$$Y_2 = \frac{1}{73{,}728\kappa^2}\{385A^2q^6 - 462ABq^4 + 3(27B^2 - 134Ab)q^2$$

$$+ 4(199 - 451k^2/\kappa^2) + 3(27b^2 - 134aB)/q^2 - 462ab/q^4 + 385a^2/q^6\}. \tag{110}$$

Late terms. The singulant is twice the positive Liouville–Green exponent, i.e.

$$\mathscr{F}_0 = 4\kappa\Xi. \tag{111}$$

Late terms can then be written down from (46). For example those in the expansion (110) are

$$Y_{r \gg 1} = \frac{(r-1)!}{2\pi(4\kappa\Xi)^r}\left[1 - \frac{\Xi(5Aq^3 - 3Bq + 3b/q - 5a/q^3)}{48(r-1)} + \frac{\Xi^2\{385A^2q^6 \ldots + 385a^2/q^6\}}{4{,}608(r-1)(r-2)} \ldots\right]. \tag{112}$$

$2(k + \kappa) > z > 2(k - \kappa)$

On this oscillatory side of the turning point, q and Y_{2r+1} are imaginary. To express the asymptotic expansions in terms of real quantities, we replace q by $i\mathscr{q}$, i.e.

$$\mathscr{q} = \left(\frac{2\kappa - 2k + z}{2\kappa + 2k - z}\right)^{\frac{1}{2}}, \tag{113}$$

and rewrite the contributions to the series as $\mathscr{Y}_r = i^r Y_r$. Those corresponding to (110) are

$$\mathscr{Y}_0 = 1, \qquad \mathscr{Y}_1 = \frac{1}{192\kappa}(5A\mathscr{q}^3 + 3B\mathscr{q} + 3b/\mathscr{q} + 5a/\mathscr{q}^3),$$

$$\mathscr{Y}_2 = \frac{1}{73{,}728\kappa^2}\{385A^2\mathscr{q}^6 + 462AB\mathscr{q}^4 + 3(27B^2 - 134Ab)\mathscr{q}^2 - 4(199 - 451k^2/\kappa^2) + 3(27b^2 - 134aB)/\mathscr{q}^2 + 462ab/\mathscr{q}^4 + 385a^2/\mathscr{q}^6\}. \tag{114}$$

Then by the connection formulae (41),

$$W_{km}(z) = [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}}\left(\frac{A\mathscr{q}^2 - a}{\pi\mathscr{q}}\right)^{\frac{1}{2}} \times [\mathscr{Y}_{\text{even}}\sin(2\kappa\Upsilon + \tfrac{1}{4}\pi) - \mathscr{Y}_{\text{odd}}\cos(2\kappa\Upsilon + \tfrac{1}{4}\pi)], \tag{115}$$

$$\mathscr{W}_{km}(z) = [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{-\frac{1}{2}} \left(\frac{\pi(A\mathscr{q}^2 - a)}{\mathscr{q}}\right)^{\frac{1}{2}}$$

$$\times\ [\mathscr{Y}_{\text{even}} \cos(2\kappa\Upsilon + \tfrac{1}{4}\pi) + \mathscr{Y}_{\text{odd}} \sin(2\kappa\Upsilon + \tfrac{1}{4}\pi)], \tag{116}$$

where

$$\Upsilon = \left(\frac{k}{\kappa}\right)\cot^{-1}\mathscr{q} - \frac{\mathscr{q}}{\mathscr{q}^2 + 1} - \left(\frac{m}{\kappa}\right)\tan^{-1}\frac{1}{\mathscr{q}}\left(\frac{k - \kappa}{k + \kappa}\right)^{\frac{1}{2}} = -i\Xi, \tag{117}$$

$$\mathscr{Y}_{\text{even}} = \sum_0^\infty (-1)^r \mathscr{Y}_{2r}, \qquad \mathscr{Y}_{\text{odd}} = \sum_0^\infty (-1)^r \mathscr{Y}_{2r+1}. \tag{118}$$

Late terms. The singulant and the resultant late terms are still given by (111) and (112) respectively, but the following notational changes are needed to express everything in terms of real quantities:

$$\mathscr{F}_0 = 4i\kappa\Upsilon, \tag{119}$$

$$\mathscr{Y}_{r \gg 1} = \frac{(r-1)!}{2\pi(4\kappa\Upsilon)^r}\left[1 - \frac{\Upsilon(5A\mathscr{q}^3 + 3B\mathscr{q} + 3b/\mathscr{q} + 5a/\mathscr{q}^3)}{48(r-1)}\right.$$

$$\left. + \frac{\Upsilon^2\{385A^2\mathscr{q}^6 \ldots\ldots + 385a^2/\mathscr{q}^6\}}{1{,}152(r-1)(r-2)} \ldots\ldots\right]. \tag{120}$$

12. History and Semantics

Historical Summary

As far as is known, the idea on which the phase-integral method is based first appeared in F. Carlini's (1817) investigation of Bessel functions of large order. In 1837 Joseph Liouville and George Green independently published more general formulations, the former in his "Second mémoire sur le développement des fonctions ou parties de fonctions en séries dont diverse termes sont assujetis à satisfaire à une même équation différentielle du second ordre contenant un paramètre variable," the latter in his paper "On the motion of waves in a variable canal of small depth and width". Lord Rayleigh (1912) and Richard Gans (1915) developed the theory in its application to electromagnetic (light) propagation, Gans' paper being of outstanding historical interest on account of its inclusion of connection formulae between leading terms in exponential and oscillatory regions. Harold Jeffreys (1924) rediscovered the connection formulae and applied them to Mathieu functions; his work was generalized by Samuel Goldstein (1928) to cover zeros of higher order at the turning point. In the meantime the whole method was being redeveloped as a tool in the new subject of quantum mechanics by Gregor Wentzel, Hendrik Kramers and Leon Brillouin (1926), all three authors apparently being unfamiliar with the earlier research; in particular, Kramers (1926, 1929) derived the connection formulae—their third known discovery—in much the same way as Jeffreys. A potentially more powerful method of deriving these was suggested soon afterwards by A. Zwaan (1929). Later broadly parallel approaches to the general problem of solving homogeneous linear differential equations may be summarized as follows:

(a) In Langer's method (1931–1949, Dingle 1956, Olver 1956, 1958) the exponential-like Liouville–Green approximations (10) are replaced as starting points by the more sophisticated Airy-like forms

$$y_- \doteqdot \left(\frac{dz}{dx}\right)^{-\frac{1}{2}} Ai\{z(x)\}, \qquad y_+ \doteqdot \left(\frac{dz}{dx}\right)^{-\frac{1}{2}} Bi\{z(x)\},$$

where (c.f. equation 26)

$$z(x) = \left\{\tfrac{3}{2}\int_{x_0}^{x} X^{\frac{1}{2}} dx\right\}^{\frac{2}{3}}.$$

(b) In Kemble's method (1935), extended by Fröman and Fröman (1965), the original differential equation of the second order (1) is replaced by a pair of simultaneous differential equations of the first order for the multipliers of the two Liouville–Green approximations.

(c) In our general interpretative theory of asymptotic expansions (Chapters XXI onwards) the connection formulae come out naturally and exactly, independently of whether connection across the turning point is made directly along the real axis or via some roughly semi-circular diversion through the complex plane (Chapter XXIV, Sections 1 and 2).

Investigations on conditions, regions of validity, and upper bounds have been undertaken by Blumenthal (1912), Seifert (1943), Olver (1961, 1963, 1965) and Fröman and Fröman (1965).

Of the theory outlined in this Chapter, that part dealing with asymptotic expansions valid in regions of exponential variation is largely based on work the author did in collaboration with Siebe Jorna (1964).

The investigations detailed here on late terms (cf. Jorna's 1967 approach), exact connection formulae and resultant complete asymptotic expansions valid in regions of oscillatory variation, are new. These researches brought to light a number of inexactitudes in published work—mainly algebraic slips and imprecise identifications.

Problem of title

It is no exaggeration to assert that apart from digital computation the method under discussion is the only one available for solving, over an extensive region, second-order differential equations when their coefficients are not extremely simple functions of the independent variable. Apart from its intrinsic mathematical importance, the method is therefore especially valuable in those many areas of science where barrier transmission problems are encountered—from quantum mechanics through ionospheric theory, semiconductors to tides and earthquakes.

The very breadth of its applicability has led to the adoption of diverse titles for essentially the same body of theory. Historically-minded mathematicians sometimes call it the "Liouville–Green method"; most physicists cling to the name "W.K.B. method" commemorating the first authors to introduce it as a technique in quantum mechanics; and those primarily interested in wave propagation—especially radio waves through the ionosphere—prefer the description "phase-integral method" (Section 2).

While "W.K.B." is undeniably parochial, within quantum mechanical bounds it is rational in honouring just those three physicists who originally introduced the method in that ascendant field. Addit on to W.K.B. of an arbitrarily selected extra initial upsets even this limited rationality.

The title "asymptotic approximation method" suggested some years ago is still less felicitous. First, because when divorced from its immediate context the appellation becomes confusingly inexplicit; for instance, the procedure for determining Thomé or Stokes type expansions has equal warrant to this title. More serious is the inclusion of the word "approximation", with its imputation of inexactitude. The development of asymptotic theory has been impeded before through prejudgments on alleged inherent inexactitude, which sooner or later become more and more deeply embodied.

Naming wide-ranging methods after principal originators is rarely compatible with acceptable assignation of credit, since no account can be taken of partial anticipations or of subsequent major developments. Thus "Liouville–Green method" is the best choice of its type, but underweighs the earlier work of Carlini, and, more seriously, the later vitally important researches on connection formulae. For these reasons the author has come to the view that "phase-integral method" is the most appropriate general title, supplemented by the designations "Liouville–Green factors" or "Liouville–Green approximations" for the leading terms—the former when viewed as the multipliers of complete expansions, the latter when viewed as approximations to the full solutions.

Exercises

1. From the recurrence relation (32) between Y_{r+1} and Y_r show that when $q \gg 1$

$$Y_r \simeq (r - \tfrac{1}{6})!\,(r - \tfrac{5}{6})!/2\pi r!\mathscr{F}_0{}^r$$

for all r, where $\mathscr{F}_0 \simeq A/3q^3$ is the singulant. By substituting on each side, verify (for these leading contributions) the formula (46) for late terms.
[The following series, usable only if r is large, can be derived from the theory of hypergeometric functions:

$$\frac{(r-a-1)!\,(r-b-1)!}{r!} = \frac{1}{a!\,b!}\sum_{s=0}(-1)^s(r-s-a-b-2)! \times \frac{(s+a)!\,(s+b)!}{s!}\Bigg].$$

2. Specializing results derived in the text, write down asymptotic expansions for the parabolic cylinder function

$$D_p(x) = 2^{\frac{1}{2}p+\frac{1}{4}}x^{-\frac{1}{2}}W_{\frac{1}{2}p+\frac{1}{4},\,-\frac{1}{4}}(\tfrac{1}{2}x^2)$$

in the two ranges $x > (4p+2)^{\frac{1}{2}}$ and $0 < x < (4p+2)^{\frac{1}{2}}$.

3. Starting from the parabolic cylinder, simple harmonic oscillator, Weber or Hermite differential equation

$$d^2y/dx^2 = (\tfrac{1}{4}x^2 - p - \tfrac{1}{2})y,$$

extract the Liouville–Green factors and hence show how asymptotic solutions depend upon the recurrence relation

$$-Y_{r+1} = (x^2 - 4p - 2)^{-\frac{1}{4}}\frac{dY_r}{dx} + \int^x \left\{\frac{5}{16}x^2(x^2 - 4p - 2)^{-\frac{5}{2}} - \tfrac{1}{8}(x^2 - 4p - 2)^{-\frac{3}{2}}\right\} Y_r\,dx.$$

By introducing the parameter $q = x(x^2 - 4p - 2)^{-\frac{1}{2}}$ reduce this to the polynomial form

$$(4p + 2)Y_{r+1} = (q^2 - 1)^2 dY_r/dq + \tfrac{1}{4}\int^q (5q^2 - 2)Y_r dq.$$

Demonstrate how choice of a fixed zero integration constant leads directly to the favoured expansions of Section 10 for the special case $k = \frac{1}{2}p + \frac{1}{4}$, $\eta = 0$. By examining the limit $x \to \infty$, i.e. $q \to 1$, identify the asymptotic expansion for the exponentially decreasing parabolic cylinder function $D_p(x)$ over the range $x > (4p + 2)^{\frac{1}{2}}$.

4. Apply the appropriate connection formula to the results of question 3, and deduce the asymptotic expansion for $D_p(x)$ over the range $x < (4p + 2)^{\frac{1}{2}}$.

5. In the recurrence relation of question 3, choose instead integration constants shifting in each successive iteration such that $Y_0 = 1$, $Y_{r \neq 0} = 0$ when $q = 0$. By examining behaviour in the limit $x = 0$, i.e. $q = 0$, identify the asymptotic expansion for $D_p(x)$ over the range $x < (4p + 2)^{\frac{1}{2}}$, given the limiting values

$$D_p(o) = 2^{\frac{1}{2}p}\sqrt{\pi}/(-\tfrac{1}{2}p - \tfrac{1}{2})!, \qquad D'_p(o) = -2^{\frac{1}{2}p+\frac{1}{2}}\sqrt{\pi}/(-\tfrac{1}{2}p - 1)!.$$

6. Find asymptotic expansions for $W_{km}(z)$ and $\mathscr{W}_{km}(z)$ in the range $z < 2(k - \kappa)$ by reversing the sign of κ in those obtained in the text for $z > 2(k + \kappa) > 0$, and re-identifying through examining dependences at the limits $z \to \infty e^{i\theta}$, $\theta \to \pm \pi$.

7. Apply connection formulae across the turning point at $z = 2(k - \kappa)$, and find asymptotic expansions for $W_{km}(z)$ and $\mathscr{W}_{km}(z)$ in the range $z < 2(k - \kappa)$ from those obtained in the text for $2(k + \kappa) > z > 2(k - \kappa)$.

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Chapter XIV

Non-linear Approach to Phase-Integral Theory

1. The Non-linear Equation and its Series Solution

In the preceding chapter the equation

$$d^2y/dx^2 = Xy \tag{1}$$

was shown to have exponentially increasing and decreasing solutions on the real axis, taking the form

$$y_\pm = X^{-\frac{1}{4}} \exp\left(\pm\left|\int_{x_0}^{x} X^{\frac{1}{2}}\,dx\right|\right) \quad Y_\pm \quad = X^{-\frac{1}{4}} \exp\left(\pm\,\varepsilon\int_{x_0}^{x} X^{\frac{1}{2}}\,dx\right) \quad Y_\pm, \tag{2}$$

where x_0 denotes the turning point, $\varepsilon =$ sign of $x - x_0$ on the exponential side, and the positive root of X is understood. We then studied the second-order *linear* differential equations obeyed by $Y_\pm$.

The alternative approach is to place the dependent variable in the phase-integral itself by writing

$$y_\pm = \exp\left(\pm\left|\int k_\pm\,dx\right|\right) = \exp\left(\pm\,\varepsilon\int k_\pm\,dx\right). \tag{3}$$

Apart from an imaginary factor, k is the physically significant wave number. Substitution in (1) leads to the deceptively simple-looking *non-linear* equation

$$k_\pm{}^2 \pm \varepsilon k_\pm{}' = X. \tag{4}$$

When the wave number is either slowly varying or else large in magnitude, its derivative in (4) is relatively small, so $k \sim X^{\frac{1}{2}}$ (positive root), whence $k' \sim \frac{1}{2}X^{-\frac{1}{2}}X'$. Substituting this rough value for k' in (4),

$$k_\pm \simeq X^{\frac{1}{2}} \mp \tfrac{1}{4}\varepsilon X'/X, \qquad \text{i.e.} \qquad y_\pm \simeq X^{-\frac{1}{4}} \exp\left(\pm\left|\int X^{\frac{1}{2}}\,dx\right|\right), \tag{5}$$

the Liouville–Green approximations. Higher approximations are more easily generated from a recurrence relation. Let

$$k_+ = \sum_{s=-1}^{\infty} k_s, \qquad k_- = \sum_{s=-1}^{\infty} (-1)^{s+1} k_s, \tag{6}$$

where $k_s = O(X^{-\frac{1}{2}s})$. Substituting in (4),

$$2\sum_{t<s}\sum k_t k_s + \sum k_s^2 + \varepsilon \sum k_s' = X. \tag{7}$$

The equation linking contributions of order $O(X^{-\frac{1}{2}s})$, $s \neq -2$, is therefore

$$2 \sum_{-1 \leqslant t < \frac{1}{2}s} k_t k_{s-t} + (k_{\frac{1}{2}s}^2) + \varepsilon k_s' = 0, \tag{8}$$

the term in brackets being present only when s is even. Starting from $k_{-1} = X^{\frac{1}{2}}$, we deduce the recurrence relation

$$-X^{\frac{1}{2}} k_{s+1} = \tfrac{1}{2}\varepsilon k_s' + k_0 k_s + k_1 k_{s-1} + k_2 k_{s-2} + \dots + (\tfrac{1}{2}k_{\frac{1}{2}s}^2). \tag{9}$$

Thus

$$k_{-1} = X^{\frac{1}{2}}, \qquad k_0 = -\varepsilon \frac{X'}{4X}, \qquad k_1 = -\frac{1}{32X^{5/2}} \{5(X')^2 - 4XX''\},$$

$$k_2 = -\frac{\varepsilon}{64X^4} \{15(X')^3 - 18XX'X'' + 4X^2X'''\},$$

$$k_3 = -\frac{1}{2048X^{11/2}} \{1105(X')^4 - 1768X(X')^2X'' + 448X^2X'X'''$$
$$+ 304X^2(X'')^2 - 64X^3X^{iv}\}. \tag{10}$$

According to (3) it is their integrals we actually require, so we define

$$K_s = \varepsilon \int^x k_s \, dx. \tag{11}$$

Evaluation of K_0 is immediate. Integration of other k's is greatly simplified by introducing some new independent variable, q say, which converts the $k_r\, dx/dq$ into polynomials in q. Exactly as with the linear theory, the terms resulting from the integration $\int X^{-3/2}X''\, dx$ frequently suggest a suitable q.

In phase-integral theory it is frequently advantageous to switch from the non-linear K-representation to the linear Y-representation developed in the preceding chapter, or vice-versa. Hence, rather than set up an independent theory, at each stage we shall relate new quantities to the old. For a start, by equating equal orders in the identities

$$\sum_0^\infty Y_r = \exp\left(\sum_1^\infty K_s\right), \qquad \sum_1^\infty K_s = \ln\left(\sum_0^\infty Y_r\right), \tag{12}$$

we find the respective conversions

$$\begin{aligned}
&Y_1 = K_1, \qquad Y_2 = K_2 + \tfrac{1}{2}K_1{}^2, \qquad Y_3 = K_3 + K_1K_2 + \tfrac{1}{6}K_1{}^3,\\
&Y_4 = K_4 + K_1K_3 + \tfrac{1}{2}K_2{}^2 + \tfrac{1}{2}K_1{}^2K_2 + \tfrac{1}{24}K_1{}^4,\\
&Y_5 = K_5 + K_1K_4 + K_2K_3 + \tfrac{1}{2}K_1{}^2K_3 + \tfrac{1}{2}K_1K_2{}^2 + \tfrac{1}{6}K_1{}^3K_2 + \tfrac{1}{120}K_1{}^5,
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&K_1 = Y_1, \qquad K_2 = Y_2 - \tfrac{1}{2}Y_1{}^2, \qquad K_3 = Y_3 - Y_1Y_2 + \tfrac{1}{3}Y_1{}^3,\\
&K_4 = Y_4 - Y_1Y_3 - \tfrac{1}{2}Y_2{}^2 + Y_1{}^2Y_2 - \tfrac{1}{4}Y_1{}^4,\\
&K_5 = Y_5 - Y_1Y_4 - Y_2Y_3 + Y_1{}^2Y_3 + Y_1Y_2{}^2 - Y_1{}^3Y_2 + \tfrac{1}{5}Y_1{}^5.
\end{aligned} \tag{14}$$

When X is negative the k_s and K_s are real for even s and imaginary for odd s. It is expedient to define

$$\mathscr{k}_s = i^s k_s, \qquad \mathscr{K}_s = i^s K_s, \tag{15}$$

whereby

$$\begin{aligned}
&\mathscr{k}_{-1} = (-X)^{\frac{1}{2}}, \qquad \mathscr{k}_0 = -\varepsilon\frac{X'}{4X}, \qquad \mathscr{k}_1 = -\frac{1}{32(-X)^{5/2}}\{5(X')^2 - 4XX''\},\\
&\mathscr{k}_2 = -\frac{\varepsilon}{64X^4}\{15(X')^3 - 18XX'X'' + 4X^2X'''\},\\
&\mathscr{k}_3 = -\frac{1}{2048(-X)^{11/2}}\{1105(X')^4 - 1768X(X')^2X'' + 448X^2X'X'''\\
&\qquad\qquad + 304X^2(X'')^2 - 64X^3X^{iv}\}.
\end{aligned} \tag{16}$$

Correspondingly the identities (12) now read

$$\sum_0^\infty i^{-r}\mathscr{Y}_r = \exp\left(\sum_1^\infty i^{-s}\mathscr{K}_s\right), \qquad \sum_1^\infty i^{-s}\mathscr{K}_s = \ln\left(\sum_0^\infty i^{-r}\mathscr{Y}_r\right), \tag{17}$$

from which it is easily proved that the conversions (13) and (14) apply equally to the new quantities. Moreover, separating the real and imaginary parts of the first identity in (17),

$$\mathscr{Y}_{\text{even}} = \sum_0^\infty (-1)^r \mathscr{Y}_{2r} = \exp\left(\sum_1^\infty (-1)^s \mathscr{K}_{2s}\right) \cos\left(\sum_0^\infty (-1)^s \mathscr{K}_{2s+1}\right), \quad (18)$$

$$\mathscr{Y}_{\text{odd}} = \sum_0^\infty (-1)^r \mathscr{Y}_{2r+1} = \exp\left(\sum_1^\infty (-1)^s \mathscr{K}_{2s}\right) \sin\left(\sum_0^\infty (-1)^s \mathscr{K}_{2s+1}\right). \quad (19)$$

2. Complete Phase-Integral Expansions on Exponential and Oscillatory Sides

Let $y^\pm$ denote the two *continuous* solutions to the given differential equation (1) which, apart from being expressed in the K-representation rather than the Y-representation, coincide with (27) and (41) of the preceding chapter. These are

$$(y^\pm)_{\text{exp}} = X^{-\frac{1}{4}} \exp\left(\pm\left|\int_{x_0}^{x} X^{\frac{1}{2}}\,dx\right|\right) \exp\left(\sum_1^\infty (\pm 1)^s K_s\right), \quad (20)$$

$$(y^+)_{\text{osc}} = (-X)^{-\frac{1}{4}} \exp\left(\sum_1^\infty (-1)^s \mathscr{K}_{2s}\right) \cos\left(\left|\int_{x_0}^{x} (-X)^{\frac{1}{2}}\,dx\right| + \tfrac{1}{4}\pi - \sum_0^\infty (-1)^s \mathscr{K}_{2s+1}\right), \quad (21)$$

$$(y^-)_{\text{osc}} = 2(-X)^{-\frac{1}{4}} \exp\left(\sum_1^\infty (-1)^s \mathscr{K}_{2s}\right) \sin\left(\left|\int_{x_0}^{x} (-X)^{\frac{1}{2}}\,dx\right| + \tfrac{1}{4}\pi - \sum_0^\infty (-1)^s \mathscr{K}_{2s+1}\right). \quad (22)$$

The importance of these new forms lies in their greater suitability for determining zeros and eigenvalues, subjects studied in depth in the second volume.

3. Late Terms in the Phase-Integral Expansions

Expression for late terms, $k_{s \gg 1}$ and $K_{s \gg 1}$, can be obtained by successive approximation from the non-linear recurrence relation (9) by a process indicated in question 1. It is, however, simpler to fall back on the known complete expansion for $Y_{r\gg 1}$. To this end, we equate equal high orders in the second identity of (12) to obtain the asymptotic expansion

$$\begin{aligned} K_s = {} & Y_s - Y_1 Y_{s-1} + (Y_1^2 - Y_2)Y_{s-2} - (Y_1^3 - 2Y_1Y_2 + Y_3)Y_{s-3} \\ & + \{Y_1^4 - 3Y_1^2Y_2 + (Y_2^2 + 2Y_1Y_3) - Y_4\}Y_{s-4} \\ & - \{Y_1^5 - 4Y_1^3Y_2 + 3Y_1(Y_2^2 + Y_1Y_3) - 2(Y_2Y_3 + Y_1Y_4) + Y_5\}Y_{s-5} \\ & + \{Y_1^6 - 5Y_1^4Y_2 + 2Y_1^2(3Y_2^2 + 2Y_1Y_3) - (Y_2^3 + 6Y_1Y_2Y_3 \\ & + 3Y_1^2Y_4) + (Y_3^2 + 2Y_2Y_4 + 2Y_1Y_5) - Y_6\}Y_{s-6} - \dots \end{aligned} \tag{23}$$

An intriguing symbolic representation of this is

$$K_s = \{1 + Y_1E^{-1} + Y_2E^{-2} + Y_3E^{-3} + \dots\}^{-1}\, Y_s, \tag{23a}$$

where the operator E^{-1} acts only on the final Y_s, $E^{-1}\, Y_s = Y_{s-1}$. According to (46) of the preceding chapter,

$$Y_s = \frac{(s-1)!}{2\pi\,\mathscr{F}_0^{\,s}}\left\{1 - \frac{Y_1\mathscr{F}_0}{s-1} + \frac{Y_2\mathscr{F}_0^{\,2}}{(s-1)(s-2)} - \dots\right\}, \tag{24}$$

where

$$\mathscr{F}_0 = 2\left|\int_{x_0}^{x} X^{\frac{1}{2}}\, dx\right|. \tag{25}$$

Substituting in (23) and assembling contributions of like order,

$$\begin{aligned} K_s = {} & \frac{(s-1)!}{2\pi\mathscr{F}_0^{\,s}}\left\{1 - \frac{2Y_1\mathscr{F}_0}{s-1} + \frac{2Y_1^2\mathscr{F}_0^{\,2}}{(s-1)(s-2)} - \frac{2(Y_1^3 - Y_1Y_2 + Y_3)\mathscr{F}_0^{\,3}}{(s-1)(s-2)(s-3)}\right. \\ & + \frac{2Y_1(Y_1^3 - 2Y_1Y_2 + 2Y_3)\mathscr{F}_0^{\,4}}{(s-1)(s-2)(s-3)(s-4)} \\ & \left. - \frac{2[Y_1^5 - (3Y_1^2 - Y_2)(Y_1Y_2 - Y_3) - Y_1Y_4 + Y_5]\,\mathscr{F}_0^{\,5}}{(s-1)(s-2)(s-3)(s-4)(s-5)} + \dots\right\} \end{aligned} \tag{26}$$

$$\begin{aligned} = {} & \frac{(s-1)!}{2\pi\,\mathscr{F}_0^{\,s}}\left\{1 - \frac{2K_1\mathscr{F}_0}{s-1} + \frac{2K_1^2\mathscr{F}_0^{\,2}}{(s-1)(s-2)} - \frac{2(2K_1^3 + 3K_3)\,\mathscr{F}_0^{\,3}}{3(s-1)(s-2)(s-3)}\right. \\ & + \frac{2K_1(K_1^3 + 6K_3)\,\mathscr{F}_0^{\,4}}{3(s-1)(s-2)(s-3)(s-4)} \\ & \left. - \frac{2(2K_1^5 + 30K_1^2K_3 + 15K_5)\,\mathscr{F}_0^{\,5}}{15(s-1)(s-2)(s-3)(s-4)(s-5)} + \dots\right\}. \end{aligned} \tag{27}$$

On the oscillatory side it is expedient to write

$$\mathscr{F}_0 = i\mathbf{F}_0 \qquad \text{where} \qquad \mathbf{F}_0 = 2\left|\int_{x_0}^{x} (-X)^{\frac{1}{2}}\,dx\right|. \tag{28}$$

In changing to (15), (27) transforms to

$$\mathscr{K}_s = \frac{(s-1)!}{2\pi\,\mathbf{F}_0^{\,s}}\left\{1 - \frac{2\mathscr{K}_1\mathbf{F}_0}{s-1} + \frac{2\mathscr{K}_1^{\,2}\,\mathbf{F}_0^{\,2}}{(s-1)(s-2)} - \frac{2(2\mathscr{K}_1^{\,3} + 3\mathscr{K}_3)\,\mathbf{F}_0^{\,3}}{3(s-1)(s-2)(s-3)} + \dots\right\}. \tag{29}$$

4. Attempts to Find Approximate Solutions in Simple Closed Forms

Writing $y = \exp(\int k dx)$, the solutions to the original differential equation $d^2y/dx^2 = Xy$ are equivalent to those of

$$k^2 + k' = X. \tag{30}$$

Encouraged by the apparent simplicity of this formulation, it is tempting to seek approximate solutions to (30) in terms of elementary functions. The search can be simplified by drawing on known properties of the exact results.

Well away from the turning point, the solutions are $k \simeq X^{\frac{1}{2}}$ (positive or negative root), so we put $k = X^{\frac{1}{2}}\kappa$ and get

$$\kappa^2 + \tfrac{1}{2}X^{-\frac{3}{2}}X'\kappa + X^{-\frac{1}{2}}\kappa' = 1. \tag{31}$$

Except near the turning point, $\kappa' \simeq 0$; this allows replacement of $X^{-\frac{1}{2}}\kappa'$ by some approximation valid near the turning point. It is convenient to introduce the new independent variable

$$u = 4X^{\frac{3}{2}}/X'. \tag{32}$$

We already know that only the first derivative is important near the turning point, so it is sufficient to write $du \simeq 6X^{\frac{1}{2}}\,dx$, reducing (31) to the neat form

$$\kappa^2 + \frac{2}{u}\kappa + 6\frac{d\kappa}{du} = 1. \tag{33}$$

If the derivative of κ is omitted in (33)—or for that matter in the exact equation (31)—the solutions are

$$\kappa = u^{-1}\{(1 + u^2)^{\frac{1}{2}} - 1\}. \tag{34}$$

The first two terms in the expansion for large u are then correctly given, namely $\kappa = 1 - u^{-1} \ldots$, i.e. $k = X^{\frac{1}{2}} - X'/4X \ldots$, the two solutions arising from the alternative signs of $X^{\frac{1}{2}}$. Bailey (1954) spotted that with *one* of these roots—that appropriate to positive u and $X^{\frac{1}{2}}$, corresponding to the solution exponentially increasing away from the turning point—(34) displays no singularities in the *opposite* limit, $\kappa \to \frac{1}{2}u$. Thus (34) describes this exponentially increasing solution y_+ reasonably quantitatively for $u \gg 1$, and qualitatively for small u to the extent of tending to a finite constant at the turning point. Bailey was thereby led to believe that phase-integral solutions could be approximated by

$$y_+ = X^{-\frac{1}{4}} \exp\left(\int_{x_0}^{x} X^{\frac{1}{2}}\{1 + (X')^2/16X^3\}^{\frac{1}{2}}\, dx\right), \quad y_- = y_+ \textstyle\int y_+^{-2}\, dx. \tag{35}$$

If this pair (35) proved acceptably accurate, they would have the advantage over the Liouville–Green approximations of good behaviour as the turning point is approached for *all* power relations $X \propto (x - x_0)^n$, $n \geqslant -2$.

To assess the accuracy of (35) in the usual case where X goes to zero linearly at the turning point, we treat the case of perfect linearity throughout, $X = X_1(x - x_0)$. Then

$$\begin{aligned}\ln(y_+ X_1^{\frac{1}{6}}) &= \int_{x_0}^{x} X^{\frac{1}{2}}\{1 + (X')^2/16X^3\}^{\frac{1}{2}}\, dx - \tfrac{1}{4}\ln z \\ &= \tfrac{1}{6}(1 + 16z^3)^{\frac{1}{2}} + \tfrac{1}{12}\ln\left\{\frac{(1 + 16z^3)^{\frac{1}{2}} - 1}{(1 + 16z^3)^{\frac{1}{2}} + 1}\right\} - \tfrac{1}{4}\ln z - \tfrac{1}{6}(1 + \ln 2)\end{aligned} \tag{36}$$

where $z = X_1^{\frac{1}{3}}(x - x_0)$ as in Chapter XIII (26). When $z \gg 1$ we have correctly

$$y_+ X_1^{\frac{1}{6}} \to (2e)^{-\frac{1}{6}}\, z^{-\frac{1}{4}}\, e^{\frac{2}{3}z^{3/2}}. \tag{37}$$

But as $z \to 0$ the limit approached is unity instead of the correct value

$$(2e)^{-\frac{1}{6}} \sqrt{\pi}\, Bi(0) = (6e)^{-\frac{1}{6}} (\sqrt{\pi})/(-\tfrac{1}{3})! = 0{\cdot}82195. \tag{38}$$

Even this rough correspondence to the actual value must be reckoned largely fortuitous, as the slope at the turning point is nowhere near the correct form:

$$y_+ X_1^{\frac{1}{6}} \to 1 + \tfrac{2}{3}z^3, \quad \text{while} \quad (2e)^{-\frac{1}{6}} \sqrt{\pi}\, Bi(z) \to {\cdot}82195(1 + {\cdot}72901z). \tag{39}$$

As expected from this wrong shape, the continuation to negative z is unacceptable qualitatively as well as quantitatively, becoming complex instead of remaining real like *Bi*. (There are in fact simple empirical expressions which are better on the exponential side than (35), e.g.

$$y_{\pm} = (X + a_{\pm})^{-\frac{1}{4}} \exp\left(\pm \int_{x_0}^{x} X^{\frac{1}{2}}\, dx\right)$$

where the parameters $a_{\pm}$ are adjusted to give correct values at the turning point.)

This failure of the proposed pair (35) is to be traced to the rising importance of the derivative in (33) as the turning point is approached. If $\kappa \to \frac{1}{2}u$, as would be indicated by solving the rest of the equation, this neglected third contribution would be three times the retained second contribution! The actual situation is that for small u these are the two equally important terms in (33), so the first approximation is $\kappa_0 \propto u^{-\frac{1}{3}}$, *not* $\propto u$ as earlier inferred. To cancel $\kappa_0{}^2 \propto u^{-\frac{2}{3}}$ requires a correction $\kappa_1 \propto u^{\frac{1}{3}}$, and so on. The initial proportionality constant, that for κ_0, is then fixed by the unit term on the right-hand side of (33). It is now clear that *every* one of these terms in the non-linear equation (33) plays a vital rôle in establishing the shape of its solutions. Prospects of worthwhile representation in closed form by elementary functions therefore seem bleak: and attempts to improve matters by matching two series, one valid for $u > 1$ and the other for $u < 1$, would merely amount to cumbersome approximation to standard results.

Exercises

1. Starting from the solution $k_s \propto (s-1)!/\{2\varepsilon\int_{x_0}^{x} X^{\frac{1}{2}}\, dx\}^s$ to $-X^{\frac{1}{2}}k_{s+1} = \frac{1}{2}\varepsilon k_s{}'$, evolve by an iterative process a formula for determining late terms $k_{s \gg 1}$ and $K_{s \gg 1}$ from the full recurrence relation

$$-X^{\frac{1}{2}}k_{s+1} = \tfrac{1}{2}\varepsilon k_s{}' + k_0 k_s + k_1 k_{s-1} + \dots + (\tfrac{1}{2}k_{\frac{1}{2}s}{}^2).$$

2. From the Y_r found in Chapter XIII, questions 2 and 3, for the parabolic cylinder function, deduce the following values concerned in the non-linear representation:

$$K_1 = -\frac{5q^3 - 6q}{48(p + \frac{1}{2})}, \qquad K_2 = \frac{5q^6 - 12q^4 + 9q^2 - 2}{64(p + \frac{1}{2})^2},$$

$$K_3 = -\frac{1}{829{,}440(p + \frac{1}{2})^3}(99{,}450q^9 - 358{,}020q^7 + 477{,}738q^5 - 279{,}720q^3 + 61{,}565q).$$

3. Compare the results of question 2 with the sequence constructed by Darwin (1949). Show that while in his formulation the contributions corresponding to K_{even} prove somewhat simpler, they do not form a regular sequence when taken together with those corresponding to K_{odd}, which would greatly complicate evaluation of terminants.

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Chapter XV

Derivation of Uniform Asymptotic Expansions from Homogeneous Differential Equations

1. The Status of Uniform Expansions in the Asymptotics of Differential Equations

The asymptotic expansions derived in the two preceding chapters contain expansions in rising powers of q^3/A, and thus fail as they stand if the parameter q begins to approach $A^{\frac{1}{3}}$ in magnitude, which happens near the turning point. By calculating late terms and thence the terminant (Chapters XXI onwards), the range for a given minimum accuracy can be greatly extended, but the expansions cease to be propitious when q^3/A exceeds unity. As we remarked in Chapter X, Section 1, if—for a function satisfying a second-order differential equation—the value of the function and its derivative at some point could be ascertained, the gap could more than adequately be filled by a Taylor expansion about this point. But a specific solution to a homogeneous second-order differential equation presupposes selection of two initially-arbitrary multiplying constants, so there is no way of finding values at such a point without extending some explicit form of the chosen solution over the full range right up to this point. Because of this, uniform expansions play a more vital rôle here than in the evaluation of integral representations, from which values at isolated points can of course always be ascertained. Nonetheless, for the reasons adduced in Chapter X, Section 1, it is still usually best to regard the uniform expansions primarily as tools for determining values of the function and its derivative at the turning point, thereafter to be discarded in favour of the simpler Taylor expansions about this point.

2. Solution through Mellin Transforms

Let two continuous solutions to the given differential equation

$$d^2y/dx^2 = Xy \tag{1}$$

be denoted by†

$$y^{\pm} = X^{-\frac{1}{4}} \exp\left(\pm \varepsilon \int_{x_0}^{x} X^{\frac{1}{2}}\, dx\right) Y^{\pm} \tag{2}$$

where x_0 corresponds to the turning point, ε is the sign of $(x - x_0)$ on the exponential side, and the positive root of X is understood. Then, as in Chapter XIII, Section 5,

$$\mp \varepsilon Y^{\pm} = \frac{1}{2X^{\frac{1}{2}}} \frac{dY^{\pm}}{dx} + \frac{1}{32} \int^{x} \frac{5(X')^2 - 4XX''}{X^{5/2}}\, Y^{\pm}\, dx. \tag{3}$$

Introducing the new independent variable q chosen there, (3) reduces to

$$\pm AY^{\pm} = P(q)\, dY^{\pm}/dq + \int^{q} Q(q)\, Y^{\pm}\, dq. \tag{4}$$

For the present we shall restrict attention to the more important solution containing Y^{-}, the one falling away exponentially from the turning point.

As a consequence of the way it has been selected, q is large close to the turning point and then $P \doteqdot q^4$ and $Q \doteqdot \frac{5}{4}q^2$, so the leading approximations to Y^{-} are given by the solutions to

$$-AY_0^{-} = q^4 \frac{dY_0^{-}}{dq} + \frac{5}{4} \int_0^{q} q^2\, Y_0^{-}\, dq. \tag{5}$$

For brevity, we introduce from the start the parameter $z = A/6q^3$ which proves most convenient in the ensuing argument, and thus examine this equation in the form

$$Y_0^{-} = \frac{1}{2} \frac{dY_0^{-}}{dz} + \frac{5}{72} \int_{\infty}^{z} Y_0^{-} \frac{dz}{z^2}. \tag{6}$$

Writing Y^{-} as a Mellin–Barnes type of integral,

$$Y^{-}(z) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} M(m)\, (2z)^m\, dm, \tag{7}$$

† Our notation here is deliberately slightly different from that of Chapter XIII, Section 5. The uniform expansions $Y^{\pm}$ will extend across the turning point, whereas the expansions $Y_{\pm}$ exhibited Stokes discontinuities and so could not be extended.

individual terms in (6) may be expressed as follows:

$$\frac{1}{2}\frac{dY_0^-}{dz} = \frac{1}{2\pi i}\int_{\gamma-1-i\infty}^{\gamma-1+i\infty} (m+1)\,M_0(m+1)\,(2z)^m\,dm,$$

$$\frac{5}{72}\int_\infty^z Y_0^-\frac{dz}{z^2} = \frac{5}{72\pi i}\int_{\gamma-1-i\infty}^{\gamma-1+i\infty} m^{-1}\,M_0(m+1)\,(2z)^m\,dm.$$

Provided no poles are enclosed in the strip $\gamma - 1 < \Re(m) < \gamma$, (6) therefore implies

$$(m+\tfrac{1}{6})(m+\tfrac{5}{6})\,M_0(m+1) = mM_0(m), \tag{8}$$

which is satisfied by

$$M_0(m) = (m-1)!\,(-m-\tfrac{1}{6})!\,(-m-\tfrac{5}{6})!/(-\tfrac{1}{6})!\,(-\tfrac{5}{6})!. \tag{9}$$

Here the proportionality constant has been chosen such that the pole at $m = 0$ produces unit contribution; i.e. through (7) $Y_0^- \to 1$ for sufficiently large z. As expected from the correspondence between phase-integral and stationary-point expansions, $M_0(m)$ is the Mellin transform of the leading basic function $T_0^e(z)$ introduced in Chapter X, Section 5 when finding a uniform expansion for $\int_{\text{s.p.}} e^{-F}\,G\,du$, F quadratic to cubic in u.

Higher-order contributions to Y^- are evaluated through substituting (7) in the unabridged equation

$$Y^- = \tfrac{1}{2}\left(\frac{P}{q^4}\right)\frac{dY^-}{dz} + \frac{5}{72}\int_\infty^z \left(\frac{4Q}{5q^2}\right) Y^-\frac{dz}{z^2}, \tag{10}$$

with the factors expressed in rising powers of $(z/A)^{\frac{1}{3}}$, and solving by iteration to find $M(m)$ as a sequence in powers of $A^{-\frac{1}{3}}$. For problems in which the lower limit of integration in the recurrence relation for the Y_r was not fixed for all iterations, this sequence will include arbitrary constants: once the uniform expansion is otherwise complete, these can be chosen to conform to the phase-integral expansions of Chapters XIII and XIV.

Each contribution to the sequence is then to be expressed in the simplest possible way as $P_n(m')\,M_r(m')$, where $P_n(m)$ is a polynomial in m of degree n say, and M_r is the Mellin transform of a T_r. From Chapter X (62) and (63),

$$M_r^e(m) = \left(1 - \frac{6m}{r-1}\right) M_{r-2}^e(m), \qquad r \text{ even}, \tag{11}$$

$$M_r^o(m) = \left(1 - \frac{6m}{r+2}\right) M_{r-2}^o(m), \qquad r \text{ odd}, \tag{12}$$

$$M_1^o(m) = M_2^e(m) = (1-6m)\,M_0^e(m), \tag{13}$$

so such an expression can be broken down into M^e of even order only, or M^o of odd order together with $M_0{}^e$; or of course a mixture. We standardize on the first, since in our experience it leads to the most easily handled expressions for both theoretical and practical purposes. With this choice, we start from $P_n(m)\,M_r{}^e(m)$. The term m^n in $P_n(m)$ is then cancelled by subtracting the appropriate multiple of $M^e_{r+2n}/M_r{}^e$, this being a polynomial of degree n. The remainder is a polynomial $P_{n-1}(m)$, of which the term m^{n-1} is next cancelled by subtracting the appropriate multiple of $M^e_{r+2n-2}/M_r{}^e$, leaving $P_{n-2}(m)$; and so on until $P_n(m)\,M_r{}^e(m)$ is dissected into M^e_{r+2n}, $M^e_{r+2n-2}, \ldots M_r{}^e$. The uniform expansion follows immediately on translating into the T^e.

The second solution y^+ to the differential equation (1), that containing Y^+, increases exponentially away from the turning point. This solution is therefore initially defined where it is at peak exponential dominance relative to the first solution y^-, i.e. on a Stokes ray of its constituent uniform asymptotic expansion Y^+. Consequently, by Chapter X, Section 5, in transmuting Y^- into Y^+ we have to replace the T^e by the barred functions $\bar{T}^e$ in addition to reversing the sign prefacing z.

The earlier phase-integral expansions can be retrieved on inserting asymptotic series for the T's (or $\bar{T}$'s). On the other hand, introduction of their convergent series leads to "transitional expansions" effective in the immediate vicinity of the turning point; but, as already remar~~ked, in pract~~ice it is best to pass at once to values of the function and its derivative right at the turning point—signified by $z = 0$—and construct therefrom the convergent Taylor series about this point.

3. Bessel Functions $J_p(x)$, $Y_p(x)$

Referring to Chapter XIII, Section 9, two solutions to Bessel's differential equation can be expressed as

$$J_p(x) = (q/2\pi p)^{\frac{1}{2}}\, e^{-p\Xi}\, Y^-, \qquad Y_p(x) = -(2q/\pi p)^{\frac{1}{2}}\, e^{p\Xi}\, Y^+, \tag{14}$$

where

$$q = p/(p^2 - x^2)^{\frac{1}{2}}, \qquad \Xi = \tanh^{-1}q^{-1} - q^{-1}, \tag{15}$$

and

$$\pm 2pY^\pm = q^2(q^2-1)\, dY^\pm/dq + \tfrac{1}{4}\int_0^q (5q^2 - 1)\, Y^\pm\, dq. \tag{16}$$

Detailed calculations are necessary for only one solution, so we continue awhile with $Y^- \to J$ alone.

On introducing the parameter $z = p/3q^3$, (16) gives

$$Y^- = \frac{1}{2}\left\{1 - \left(\frac{3z}{p}\right)^{\frac{2}{3}}\right\}\frac{dY^-}{dz} + \frac{5}{72}\int_\infty^z Y^-\left\{1 - \frac{1}{5}\left(\frac{3z}{p}\right)^{\frac{2}{3}}\right\}\frac{dz}{z^2}. \qquad (17)$$

Expressing Y^- as the Mellin–Barnes type of integral (7) and substituting, (17) leads to the difference equation

$$(m + \tfrac{1}{6})(m + \tfrac{5}{6})\,M(m + 1) - mM(m) = (3/2p)^{\frac{2}{3}}\,(m + \tfrac{1}{6})^2\,M(m + \tfrac{1}{3}). \qquad (18)$$

This can be solved iteratively on writing

$$M(m) = \mathscr{M}(m)\,M_0(m), \qquad (19)$$

where $M_0(m)$, the solution to (18) with the right-hand side neglected, is defined by (9). Then in terms of the summation operator S defined in (for example) Nörlund (1923) and Milne–Thomson (1933),

$$\mathscr{M}(m) = -\left(\frac{3}{2p}\right)^{\frac{2}{3}} S\,\frac{(m + \frac{1}{6})\,(m - \frac{2}{3})!\,(-m - \frac{1}{2})!}{m!\,(-m - \frac{5}{6})!}\,\mathscr{M}(m + \tfrac{1}{3}), \qquad (20)$$

and substitution of one order on the right evolves the next higher order on the left.

Starting with the zero order $\mathscr{M}_0 = 1$, the first order is

$$\mathscr{M}_1(m) = -\left(\frac{3}{2p}\right)^{\frac{2}{3}} S\,\frac{(m + \frac{1}{6})\,(m - \frac{2}{3})!\,(-m - \frac{1}{2})!}{m!\,(-m - \frac{5}{6})!}. \qquad (21)$$

Summations of this variety can be accomplished by noting the difference

$$\Delta\frac{(m + \alpha)!\,(-m + a - 1)!}{(m + \beta - 1)!\,(-m + b)!} = \{m(\beta + b - \alpha - a - 2)$$

$$+ b(\alpha + 1) - \beta(a + 1)\}\,\frac{(m + \alpha)!\,(-m + a)!}{(m + \beta)!\,(-m + b)!}. \qquad (22)$$

In the present instance,

$$\mathscr{M}_1(m) = \frac{3}{5}\left(\frac{3}{2p}\right)^{\frac{2}{3}}\frac{(m - \frac{2}{3})!\,(-m + \frac{1}{2})!}{(m - 1)!\,(-m - \frac{5}{6})!}. \qquad (23)$$

Substituting $\mathscr{M}_1(m + \frac{1}{3})$ on the right of (20), it is similarly found that

$$\mathscr{M}_2(m) = -\frac{9}{50}\left(\frac{3}{2p}\right)^{\frac{4}{3}}\left(m - \frac{47}{42}\right)\frac{(m - \frac{1}{3})!\,(-m + \frac{1}{2})!}{(m - 1)!\,(-m - \frac{7}{6})!}. \qquad (24)$$

The next summation encountered is quite different:

$$\mathscr{M}_3(m) = \frac{9}{50}\left(\frac{3}{2p}\right)^2 S\left(m^2 - \frac{1}{36}\right)\left(m + \frac{1}{2}\right)\left(m - \frac{11}{14}\right). \tag{25}$$

Here one can either resort to the theory of Bernoulli polynomials (e.g. Nörlund 1923), according to which $S(m^{r-1}) = B_r(m)/r$, or simply difference a general fifth-degree polynomial and equate coefficients. Either way,

$$\mathscr{M}_3(m) = \frac{9}{250}\left(\frac{3}{2p}\right)^2 m\left(m^2 - \frac{1}{4}\right)\left(m^2 - \frac{20}{7}m + \frac{1459}{756}\right). \tag{26}$$

Continuing the iteration through one further order and reverting to M via (19), the result is

$$\begin{aligned}
&(-\tfrac{1}{6})!\,(-\tfrac{5}{6})!\,M(m) = (m-1)!\,(-m-\tfrac{1}{6})!\,(-m-\tfrac{5}{6})! \\
&\quad + \frac{3}{5}\left(\frac{3}{2p}\right)^{\frac{2}{3}}(m-\tfrac{2}{3})!\,(-m-\tfrac{1}{6})!\,(-m+\tfrac{1}{2})! \\
&\quad + \frac{9}{50}\left(\frac{3}{2p}\right)^{\frac{4}{3}}(m+\tfrac{1}{6})\left(m-\frac{47}{42}\right)(m-\tfrac{1}{3})!\,(-m-\tfrac{5}{6})!\,(-m+\tfrac{1}{2})! \\
&\quad + \frac{9}{250}\left(\frac{3}{2p}\right)^{2}(m^2-\tfrac{1}{4})\left(m^2-\frac{20}{7}m+\frac{1459}{756}\right)m!\,(-m-\tfrac{1}{6})!\,(-m-\tfrac{5}{6})! \\
&\quad + \frac{27}{5000}\left(\frac{3}{2p}\right)^{8/3}(m+\tfrac{5}{6})\left(m^3-\frac{73}{14}m^2+\frac{45{,}613}{5{,}292}m-\frac{1{,}567{,}331}{349{,}272}\right) \\
&\quad \times (m+\tfrac{1}{3})!\,(-m-\tfrac{1}{6})!\,(-m+\tfrac{1}{2})! \qquad + \cdots.
\end{aligned} \tag{27}$$

Each contribution has next to be expressed in the simplest way as a polynomial times an M_r^e. Writing

$$M(m) = \sum_{s=0}^{\infty} (3/2p)^{\frac{2}{3}s}\,\mathbf{M}_s(m + \tfrac{1}{3}s), \tag{28}$$

we have

$$\left.\begin{aligned}
\mathbf{M}_0(m) &= (m-1)!\,(-m-\tfrac{1}{6})!\,(-m-\tfrac{5}{6})!/(-\tfrac{1}{6})!\,(-\tfrac{5}{6})! = M_0{}^e(m)\\
\mathbf{M}_1(m) &= \tfrac{3}{5}(m-1)!\,(-m+\tfrac{5}{6})!\,(-m+\tfrac{1}{6})!/(-\tfrac{1}{6})!\,(-\tfrac{5}{6})!\\
&= -\tfrac{1}{10}(m-\tfrac{5}{6})\,M_2{}^e(m)\\
\mathbf{M}_2(m) &= \frac{3}{200}\,(m-\tfrac{7}{6})\left(m-\frac{25}{14}\right)M_4{}^e(m)\\
\mathbf{M}_3(m) &= -\frac{1}{400}\left(m-\frac{3}{2}\right)\left(m^2-\frac{34}{7}m+\frac{625}{108}\right)M_6{}^e(m)\\
\mathbf{M}_4(m) &= \frac{7}{16{,}000}\left(m-\frac{11}{6}\right)\left(m^3-\frac{129}{14}m^2+\frac{147{,}421}{5{,}292}m\right.\\
&\qquad\left.-\frac{21{,}875}{792}\right)M_8{}^e(m).
\end{aligned}\right\} \tag{29}$$

Noting that $M_4{}^e = -2(m-\tfrac{1}{2})\,M_2{}^e$, subtraction of $M_4{}^e/20$ from $\mathbf{M}_1$ cancels the term of highest degree, leaving $M_2{}^e/30$; hence

$$\mathbf{M}_1 = \tfrac{1}{60}\,(2M_2{}^e + 3M_4{}^e). \tag{30}$$

Moreover, since

$$M_8{}^e = \tfrac{36}{35}\,(m-\tfrac{7}{6})\,(m-\tfrac{5}{6})\,M_4{}^e,$$

subtraction of $7M_8{}^e/480$ from $\mathbf{M}_2$ leaves

$$-\tfrac{1}{70}\,(m-\tfrac{7}{6})\,M_4{}^e;$$

and because

$$M_6{}^e = -\tfrac{6}{5}(m-\tfrac{5}{6})\,M_4{}^e,$$

subtraction of $M_6{}^e/84$ then leaves $M_4{}^e/210$, whence

$$\mathbf{M}_2 = \tfrac{1}{3360}\,(16M_4{}^e + 40M_6{}^e + 49M_8{}^e). \tag{31}$$

Likewise,

$$\mathbf{M}_3 = \frac{1}{259{,}200}(320M_6{}^e + 1120M_8{}^e + 2106M_{10}{}^e + 2079M_{12}{}^e), \tag{32}$$

$$\begin{aligned}\mathbf{M}_4 = \frac{1}{750{,}968{,}064{,}000}&(349{,}476{,}480M_8{}^e + 1{,}572{,}644{,}160M_{10}{}^e \\ &+ 3{,}854{,}423{,}056M_{12}{}^e + 5{,}966{,}024{,}064M_{14}{}^e \\ &+ 4{,}894{,}004{,}115\, M_{16}{}^e).\end{aligned} \tag{33}$$

As a consequence of the representation (cf. (7))

$$T_r{}^e(z) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} M_r{}^e(m)\,(2z)^m\,dm, \tag{34}$$

the displaced transforms $M_r{}^e(m+\sigma)$ translate to $(2z)^{-\sigma}T_r{}^e(z)$. Hence by (28) Y^- in (14) is given by

$$\begin{aligned}Y^- = T_0{}^e &+ \frac{1}{40\,(3p^2z)^{1/3}}(2T_2{}^e + 3T_4{}^e) + \frac{3}{4{,}480\,(3p^2z)^{2/3}}(16T_4{}^e + 40T_6{}^e + 49T_8{}^e) \\ &+ \frac{1}{76{,}800\,(3p^2z)}(320T_6{}^e + 1120T_8{}^e + 2106T_{10}{}^e + 2079T_{12}{}^e) \\ &+ \frac{3}{445{,}018{,}112{,}000(3p^2z)^{4/3}}(349{,}476{,}480T_8{}^e + 1{,}572{,}644{,}160T_{10}{}^e \\ &+ 3{,}854{,}423{,}056T_{12}{}^e + 5{,}966{,}024{,}064T_{14}{}^e + 4{,}894{,}004{,}115T_{16}{}^e)\ \ldots,\end{aligned} \tag{35}†$$

the same result as found from the integral representation for $J_p(x)$ (Chapter X, question 5). The second solution is obtained through reversing the sign prefacing z and changing to the barred basic functions, thus

$$Y^+ = \overline{T}_0{}^e(-z) - \frac{1}{40\,(3p^2z)^{1/3}}\{2\overline{T}_2{}^e(-z) + 3\overline{T}_4{}^e(-z)\} + \ldots \tag{36}$$

where $z = p/3q^3$ as in (35), the same result as found from the integral representation for $Y_p(x)$ (Chapter X, question 6).

† $(3p^2z)^{-1/3} = q/p$.

Inserting asymptotic series for the T's and $\bar{T}$'s, we recover the phase-integral expansions of Chapter XIII, Section 9. On inserting instead the convergent series, interesting "transitional expansions" are obtained which are effective close to the turning point; but, as recommended in Section 1, we shall pass straight to values at the turning point $z = 0$, where only the T's of lower order in each group contribute. On doing this before and after differentiation, the results are

$$J_p(p) = (24\pi^3 p^2 z)^{-\frac{1}{6}} e^{-z} Y^- \Big|_{z=0}$$

$$= \frac{(-\frac{2}{3})!\, 6^{\frac{1}{3}}}{2\pi\sqrt{3}\, p^{\frac{1}{3}}} \left(1 - \frac{3\beta}{35 p^{\frac{4}{3}}} - \frac{1}{225 p^2} \cdots\right), \tag{37}$$

$$J_p{}'(p) = -\left(\frac{3z^2}{8\pi^3 p^4}\right)^{\frac{1}{6}} \frac{d}{dz}(z^{-\frac{1}{6}} e^{-z} Y^-)\Big|_{z=0}$$

$$= \frac{(-\frac{2}{3})!\, 6^{\frac{1}{3}}}{2\pi\sqrt{3}\, p^{\frac{1}{3}}} \left(\frac{6\beta}{p^{\frac{1}{3}}} - \frac{1}{5p} + \frac{23\beta}{525 p^{\frac{7}{3}}} \cdots\right), \tag{38}$$

$$Y_p(p) = -\left(\frac{3\pi^3 p^2 z}{8}\right)^{-\frac{1}{6}} e^z Y^+ \Big|_{z=0}$$

$$= -\frac{(-\frac{2}{3})!\, 6^{\frac{1}{3}}}{2\pi\, p^{\frac{1}{3}}} \left(1 + \frac{3\beta}{35 p^{\frac{4}{3}}} - \frac{1}{225 p^2} \cdots\right), \tag{39}$$

$$Y_p{}'(p) = \left(\frac{24 z^2}{\pi^3 p^4}\right)^{\frac{1}{6}} \frac{d}{dz}(z^{-\frac{1}{6}} e^z Y^+)\Big|_{z=0}$$

$$= \frac{(-\frac{2}{3})!\, 6^{\frac{1}{3}}}{2\pi\, p^{\frac{1}{3}}} \left(\frac{6\beta}{p^{\frac{1}{3}}} + \frac{1}{5p} + \frac{23\beta}{525 p^{\frac{7}{3}}} \cdots\right), \tag{40}$$

where $\beta = (-\frac{1}{3})!/6^{\frac{2}{3}}(-\frac{2}{3})! = 0{\cdot}15308275$, all in agreement with results from integral representations (Chapter VIII (60), (65), (74) and (75)). These asymptotic expansions for values at the turning point act as centrepieces in the otherwise absolutely convergent series for $J_p(x)$ and $Y_p(x)$ already derived and examined in Chapter VIII, Sections 5–6. The accuracy could be increased either by laboriously evaluating more terms in (35), or better by deriving a general formula for late terms. This preferential extension would entail asymptotic solution of a difference equation [(18)], a matter we shall defer to the second volume.

4. Whittaker Functions for Large k

Referring to Chapter XIII, Section 10, two independent solutions to the Whittaker equation are

$$
\begin{aligned}
W_{k\mu}(x) &= \left(\frac{(k+\mu-\frac{1}{2})!\,(k-\mu-\frac{1}{2})!}{2\pi}\right)^{\frac{1}{2}} q^{\frac{1}{2}} e^{-2k\Xi}\, Y^{-}, \\
\mathscr{W}_{k\mu}(x) &= \left(\frac{2\pi}{(k+\mu-\frac{1}{2})!\,(k-\mu-\frac{1}{2})!}\right)^{\frac{1}{2}} q^{\frac{1}{2}} e^{+2k\Xi}\, Y^{+},
\end{aligned}
\tag{41}
$$

where

$$
q = \left(\frac{x}{x-4k}\right)^{\frac{1}{2}}, \qquad \Xi = \frac{q}{q^2-1} - \coth^{-1} q, \tag{42}
$$

and, with $\eta = 16\mu^2 - 1$,

$$
\pm 8k\, Y^{\pm} = (q^2-1)^2\, dY^{\pm}/dq + \tfrac{1}{4}\int^{q} (5q^2 - 2 - \eta/q^2)\, Y^{\pm}\, dq. \tag{43}
$$

Setting $z = 4k/3q^3$,

$$
Y^{-} = \frac{1}{2}\left\{1 - \left(\frac{3z}{4k}\right)^{\frac{2}{3}}\right\}^2 \frac{dY^{-}}{dz} + \frac{5}{72}\int^{z} Y^{-}\left\{1 - \frac{2}{5}\left(\frac{3z}{4k}\right)^{\frac{2}{3}} - \frac{\eta}{5}\left(\frac{3z}{4k}\right)^{\frac{4}{3}}\right\}\frac{dz}{z^2}. \tag{44}
$$

On substituting the Mellin–Barnes type of integral (7), this leads to the difference equation

$$
\begin{aligned}
(m+\tfrac{1}{6})(m+\tfrac{5}{6})\, M(m+1) - mM(m) &= 2\left(\frac{3}{8k}\right)^{\frac{2}{3}} (m+\tfrac{1}{6})^2\, M(m+\tfrac{1}{3}) \\
&\quad - \left(\frac{3}{8k}\right)^{\frac{4}{3}} (m^2 - \tfrac{1}{3}m - \tfrac{1}{36}\eta)\, M(m-\tfrac{1}{3}).
\end{aligned}
\tag{45}
$$

The solution to (45) with the right-hand side neglected is $M_0(m)$ as defined by (9). Extracting this, the iteration formula for its factor is

$$
\begin{aligned}
\mathscr{M}(m) &= -2\left(\frac{3}{8k}\right)^{\frac{2}{3}} S\,\frac{(m+\frac{1}{6})(m-\frac{2}{3})!\,(-m-\frac{1}{2})!}{m!\,(-m-\frac{5}{6})!}\,\mathscr{M}(m+\tfrac{1}{3}) \\
&\quad + \left(\frac{3}{8k}\right)^{\frac{4}{3}} S\,(m-\tfrac{1}{6})(m^2 - \tfrac{1}{3}m - \tfrac{1}{36}\eta)\,\frac{(m-\frac{4}{3})!\,(-m-\frac{1}{2})!}{m!\,(-m-\frac{1}{6})!} \\
&\quad \times \mathscr{M}(m-\tfrac{1}{3}).
\end{aligned}
\tag{46}
$$

Substitution on the right of one order in the first summation, together with an order lower in the second summation, generates the next higher order on the left. The summations are performed in just the same way as those of the preceding section. Reinstating the factor M_0, the result can be arranged in the following form best suited to subsequent calculation:

$$
\begin{aligned}
(-\tfrac{1}{6})!\,(-\tfrac{5}{6})!\,M(m) = \Bigg[1 &+ \frac{36}{125}\left(\frac{3}{8k}\right)^2 \left\{m^5 - \frac{65}{14}m^4 + \frac{2935}{378}m^3\right. \\
&- \left(\frac{2915}{504} - \frac{25}{72}\eta\right)m^2 + \left.\left(\frac{16823}{9072} - \frac{25}{54}\eta\right)m + c\right\}\Bigg]\,(m-1)! \\
&\times (-m-\tfrac{1}{6})!\,(-m-\tfrac{5}{6})! + \frac{6}{5}\left(\frac{3}{8k}\right)^{\frac{2}{3}}(m-\tfrac{2}{3})!\,(-m-\tfrac{1}{6})!\,(-m+\tfrac{1}{2})! \\
&- \frac{18}{25}\left(\frac{3}{8k}\right)^{\frac{4}{3}}\left\{\left(m-\frac{1}{2}\right)\left(m^2 - \frac{65}{42}m + \frac{103}{252}\right) + \frac{25}{108}\eta\right\} \\
&\times (m-\tfrac{1}{3})!\,(-m-\tfrac{5}{6})!\,(-m-\tfrac{1}{2})! \\
&- \tfrac{1}{12}\eta\left(\frac{3}{8k}\right)^{\frac{4}{3}}(m-\tfrac{4}{3})!\,(-m+\tfrac{1}{6})!\,(-m-\tfrac{1}{2})!\ \ldots. \qquad (47)
\end{aligned}
$$

The constant c in (47) may be selected at will, since it only affects the multiplier of the solution to the differential equation. To make it correspond to the favoured expansion (86)–(87) of Chapter XIII, Section 10, we need

$$1 + \frac{36}{125}\left(\frac{3}{8k}\right)^2 c \ldots \equiv \quad \text{constant terms in} \quad Y_0 - Y_1 + Y_2 - \ldots,$$

making $c = 125(3\eta - 4)/2592$.

Writing

$$
\begin{aligned}
M(m) = M_0{}^e(m) &+ \frac{6}{5}\left(\frac{3}{8k}\right)^{\frac{2}{3}}\mathbf{M}_1(m+\tfrac{1}{3}) - \left(\frac{3}{8k}\right)^{\frac{4}{3}}\{\tfrac{18}{25}\mathbf{M}_2(m+\tfrac{2}{3}) \\
&+ \tfrac{1}{12}\eta\, M_0{}^e(m-\tfrac{1}{3})\} + \frac{36}{125}\left(\frac{3}{8k}\right)^2\mathbf{M}_3(m)\ \ldots, \qquad (48)
\end{aligned}
$$

we have

$$\mathbf{M}_1(m) = -\tfrac{1}{6}(m - \tfrac{5}{6})\,M_2{}^e = \tfrac{1}{36}\,(2M_2{}^e + 3M_4{}^e),$$

$$\mathbf{M}_2(m) = \{\tfrac{1}{6}\,(m - \tfrac{7}{6})\,(m^2 - \tfrac{121}{42}m + \tfrac{475}{252}) + \tfrac{25}{648}\eta\}\,M_2{}^e$$

$$= -\tfrac{5}{9072}\,\{70(2-\eta)\,M_2{}^e + 183M_4{}^e + 195M_6{}^e + 147M_8{}^e\},$$

$$\mathbf{M}_3(m) = -\frac{5}{163{,}296}\{56(7-45\eta)\,M_0{}^e + 28(7-45\eta)\,M_2{}^e$$

$$+ 27(9-35\eta)\,M_4{}^e + 240M_6{}^e + 1260M_8{}^e + 3969M_{10}{}^e\}. \tag{49}$$

The displaced transforms $M_r{}^e(m+\sigma)$ translate to $(2z)^{-\sigma}\,T_r{}^e(z)$. Hence, by (48), Y^- in (41) is given by

$$Y^- = T_0{}^e + \frac{1}{40(6k^2z)^{\frac{1}{3}}}\,(2T_2{}^e + 3T_4{}^e) + \frac{1}{4480(6k^2z)^{\frac{2}{3}}}\,\{-420\eta z\,T_0{}^e$$

$$+ 70(2-\eta)\,T_2{}^e + 183\,T_4{}^e + 195T_6{}^e + 147T_8{}^e\} - \frac{z}{134{,}400(6k^2z)}$$

$$\times\,\{56(7-45\eta)\,T_0{}^e + 28(7-45\eta)\,T_2{}^e + 27(9-35\eta)\,T_4{}^e + 240T_6{}^e$$

$$+ 1260T_8{}^e + 3969T_{10}{}^e\}\ \ldots, \tag{50}†$$

the same result as found from the integral representation for $W_{k\mu}$ (Chapter X, question 7). The second solution requires reversal of the sign prefacing z and change to the barred functions, thus

$$Y^+ = \bar{T}_0{}^e(-z) - \frac{1}{40(6k^2z)^{\frac{1}{3}}}\,\{2\bar{T}_2{}^e(-z) + 3\bar{T}_4{}^e(-z)\}\ \ldots \tag{51}$$

where $z = 4k/3q^3$ as in (50).

Passing straightaway to values at the turning point $z = 0$, the resultant asymptotic expansions are

$$W_{k\mu}(4k) = \left\{\frac{(k+\mu-\frac{1}{2})!\,(k-\mu-\frac{1}{2})!}{\pi}\right\}^{\frac{1}{2}}\left(\frac{k}{6z}\right)^{\frac{1}{6}} e^{-z}\,Y^-\Bigg|_{z=0}$$

$$= \frac{(-\frac{2}{3})!}{\pi}\,[(k+\mu-\tfrac{1}{2})!\,(k-\mu-\tfrac{1}{2})!]^{\frac{1}{2}}\left(\frac{k}{12}\right)^{\frac{1}{6}}$$

$$\times\left\{1 + \frac{3\beta(9-35\eta)}{560\,.\,4^{\frac{1}{3}}\,.\,k^{\frac{4}{3}}} - \frac{7-45\eta}{14{,}400k^2}\ldots\right\}, \tag{52}$$

† $(6k^2z)^{-1/3} = q/2k$.

$$W_{k\mu}'(4k) = \left\{\frac{(k+\mu-\frac{1}{2})!\,(k-\mu-\frac{1}{2})!}{\pi}\right\}^{\frac{1}{2}} \left(\frac{3z^2}{2048k}\right)^{\frac{1}{6}} \frac{d}{dz}(z^{-\frac{1}{6}} e^{-z} Y^-)\bigg|_{z=0}$$

$$= -\frac{(-\frac{1}{3})!}{\pi}[(k+\mu-\tfrac{1}{2})!\;(k-\mu-\tfrac{1}{2})!]^{\frac{1}{2}}\left(\frac{3}{1024k}\right)^{\frac{1}{6}}$$

$$\times\left\{1-\frac{4^{\frac{1}{3}}}{60\beta k^{\frac{2}{3}}}-\frac{1}{3150k^2}\cdots\right\}, \tag{53}$$

both in agreement with results from integral representations (Chapter VIII (119), (124)), and

$$\mathscr{W}_{k\mu}(4k) = \left\{\frac{\pi}{(k+\mu-\frac{1}{2})!\,(k-\mu-\frac{1}{2})!}\right\}^{\frac{1}{2}}\left(\frac{32k}{3z}\right)^{\frac{1}{6}} e^z Y^+\bigg|_{z=0}$$

$$= (-\tfrac{2}{3})!\,[(k+\mu-\tfrac{1}{2})!\,(k-\mu-\tfrac{1}{2})!]^{-\frac{1}{2}}\left(\frac{9k}{4}\right)^{\frac{1}{6}}$$

$$\times\left\{1-\frac{3\beta(9-35\eta)}{560\,.\,4^{\frac{1}{3}}\,.\,k^{\frac{4}{3}}}-\frac{7-45\eta}{14{,}400k^2}\cdots\right\}, \tag{54}$$

$$\mathscr{W}_{k\mu}'(4k) = \left\{\frac{\pi}{(k+\mu-\frac{1}{2})!\,(k-\mu-\frac{1}{2})!}\right\}^{\frac{1}{2}}\left(\frac{3z^2}{32k}\right)^{\frac{1}{6}}\frac{d}{dz}(z^{-\frac{1}{6}} e^z Y^+)\bigg|_{z=0}$$

$$= (-\tfrac{1}{3})!\,[(k+\mu-\tfrac{1}{2})!\,(k-\mu-\tfrac{1}{2})!]^{-\frac{1}{2}}\left(\frac{81}{1024k}\right)^{\frac{1}{6}}$$

$$\times\left\{1+\frac{4^{\frac{1}{3}}}{60\beta k^{\frac{2}{3}}}-\frac{1}{3150k^2}\cdots\right\}. \tag{55}$$

Having succeeded in finding these values, we can now proceed exactly as in Chapter VIII, Section 8 to find the otherwise absolutely convergent Taylor expansions for $W_{k\mu}(x)$ and $\mathscr{W}_{k\mu}(x)$ about the turning point.

5. Whittaker Functions for large $\kappa = (k^2 - \mu^2)^{\frac{1}{2}}$

As in Chapter XIII, Section 11, we introduce the abbreviations

$$A = 1 + k/\kappa, \qquad a = 1 - k/\kappa, \qquad B = 1 + 3k/\kappa, \qquad b = 1 - 3k/\kappa. \tag{56}$$

Throughout, the sign of the square root $\kappa = (k^2 - \mu^2)^{\frac{1}{2}}$ will be taken to be that of k. In terms of these, two solutions to the Whittaker equations are

$$W_{k\mu}(x) = \tfrac{1}{2}\left\{\frac{(k+\mu-\frac{1}{2})!\,(k-\mu-\frac{1}{2})!}{\pi}\right\}^{\frac{1}{2}}\left(\frac{Aq^2+a}{q}\right)^{\frac{1}{2}} e^{-2\kappa\Xi}\,Y^-,$$

$$\mathscr{W}_{k\mu}(x) = \left\{\frac{\pi}{(k+\mu-\frac{1}{2})!\,(k-\mu-\frac{1}{2})!}\right\}^{\frac{1}{2}}\left(\frac{Aq^2+a}{q}\right)^{\frac{1}{2}} e^{+2\kappa\Xi}\,Y^+, \tag{57}$$

where

$$q = \left(\frac{x-2k+2\kappa}{x-2k-2\kappa}\right)^{\frac{1}{2}},$$

$$\Xi = \frac{q}{q^2-1} - \left(\frac{k}{\kappa}\right)\coth^{-1}q + \left(\frac{\mu}{\kappa}\right)\tanh^{-1}\frac{1}{q}\left(\frac{k-\kappa}{k+\kappa}\right)^{\frac{1}{2}} \tag{58}$$

(with positive signs allocated to k/κ and μ/κ), and

$$\pm\, 16\kappa\, Y^{\pm} = (Aq^4 - Bq^2 - b + a/q^2)\,dY^{\pm}/dq + \tfrac{1}{4}\int^{q}(5Aq^2 - B - b/q^2 + 5a/q^4)Y^{\pm}\,dq. \tag{59}$$

On setting $z = 8\kappa/3Aq^3$ and substituting the Mellin–Barnes type of integral (7), this leads to the difference equation

$$(m+\tfrac{1}{6})(m+\tfrac{5}{6})M(m+1) - mM(m) = \frac{B}{A}\left(\frac{3A}{16\kappa}\right)^{\frac{2}{3}}(m+\tfrac{1}{6})^2 M(m+\tfrac{1}{3}) + \frac{b}{a}\left(\frac{3A}{16\kappa}\right)^{\frac{4}{3}}(m-\tfrac{1}{6})^2 M(m-\tfrac{1}{3}) - \frac{a}{A}\left(\frac{3A}{16\kappa}\right)^{2}(m-\tfrac{1}{6})(m-\tfrac{5}{6}) \times M(m-1). \tag{60}$$

Extracting $M_0(m)$ as defined by (9), the iteration formula for its factor is

$$\begin{aligned}\mathscr{M}(m) = &-\frac{B}{A}\left(\frac{3A}{16\kappa}\right)^{\frac{2}{3}} S\,\frac{(m+\frac{1}{6})(m-\frac{2}{3})!\,(-m-\frac{1}{2})!}{m!\,(-m-\frac{5}{6})!}\,\mathscr{M}(m+\tfrac{1}{3})\\ &-\frac{b}{a}\left(\frac{3A}{16\kappa}\right)^{\frac{4}{3}} S\,\frac{(m-\frac{1}{6})^3(m-\frac{4}{3})!\,(-m-\frac{1}{2})!}{m!\,(-m-\frac{1}{6})!}\,\mathscr{M}(m-\tfrac{1}{3})\\ &-\frac{a}{A}\left(\frac{3A}{16\kappa}\right)^{2} S\,\frac{(m-\frac{1}{6})^2(m-\frac{5}{6})^2}{m(m-1)}\,\mathscr{M}(m-1).\end{aligned} \tag{61}$$

Substitution on the right of one order in the first summation, together with an order lower in the second and two orders in the third, generates the next higher order on the left. The only new type of summation encountered is illustrated by the initial contribution from the third summation:

$$S\frac{(m-\frac{1}{6})^2\,(m-\frac{5}{6})^2}{m(m-1)} = S\,(m^2 - m - \tfrac{5}{18}) + \frac{25}{1296}\,S\frac{1}{m(m-1)}$$

$$= (\tfrac{1}{3}m^3 - m^2 + \tfrac{7}{18}\,m) - \frac{25}{1296}\frac{1}{m-1}. \tag{62}$$

Reinstating the factor M_0, the result can be arranged in the following form best suited to subsequent calculation:

$$\cancel{(-\tfrac{1}{6})!}\,(-\tfrac{5}{6})!\,M(m) = \Bigg[1 + \left(\frac{3A}{16\kappa}\right)^2 \left\{\frac{9}{250}\left(\frac{B}{A}\right)^3 \left(m^5 - \frac{20}{7}m^4 + \frac{635}{378}m^3\right.\right.$$

$$\left. + \frac{5}{7}m^2 - \frac{1459}{3024}m\right) + \frac{9}{35}\frac{bB}{A^2}\left(m^4 - \frac{92}{27}m^3 + \frac{23}{6}m^2 - \frac{127}{81}m\right)$$

$$\left.\left. - \frac{a}{3A}(m^3 - 3m^2 + \tfrac{13}{4}\,m) + c\right\}\right]\,(m-1)!\,(-m-\tfrac{1}{6})!\,(-m-\tfrac{5}{6})!$$

$$+ \frac{3}{5}\frac{B}{A}\left(\frac{3A}{16\kappa}\right)^{\frac{4}{3}} (m-\tfrac{2}{3})!\,(-m-\tfrac{1}{6})!\,(-m+\tfrac{1}{2})!$$

$$- \left(\frac{3A}{16\kappa}\right)^{\frac{4}{3}} \left\{\frac{9}{50}\left(\frac{B}{A}\right)^2 (m-\tfrac{1}{2})\,(m-\tfrac{47}{42})\,(m+\tfrac{1}{6})\right.$$

$$\left. + \frac{6b}{7A}(m^2 - \tfrac{4}{3}m + \tfrac{19}{36})\right\}\,(m-\tfrac{1}{3})!\,(-m-\tfrac{5}{6})!\,(-m-\tfrac{1}{2})!$$

$$- \left(\frac{3A}{16\kappa}\right)^{\frac{4}{3}} \frac{3b}{7A}(m^2 - \tfrac{4}{3}m + \tfrac{19}{36})\,(m-\tfrac{4}{3})!\,(-m+\tfrac{1}{6})!\,(-m-\tfrac{1}{2})!$$

$$+ \left(\frac{3A}{16\kappa}\right)^2 \frac{5a}{36A}(m-2)!\,(-m+\tfrac{5}{6})!\,(-m+\tfrac{1}{6})!\,\ldots. \tag{63}$$

In making the arbitrary constant c in (63) correspond to the favoured expansion (109)–(110) of Chapter XIII, Section 11, we must remember to include the additional contribution to the pole at $m = 0$ which comes from the last term,† so

$$1 + \left(\frac{3A}{16\kappa}\right)^2 \left(c + \frac{5a}{36A} \times \frac{1}{-1} \times \frac{5}{6} \times \frac{1}{6}\right) \ldots \equiv \text{constant terms in } Y_0 - Y_1 + Y_2 - \ldots,$$

making

$$c = \frac{25a}{1296A} + \frac{199 - 451\,k^2/\kappa^2}{648A^2}.$$

Writing

$$\begin{aligned} M(m) = M_0{}^e(m) &+ \frac{3}{5}\frac{B}{A}\left(\frac{3A}{16\kappa}\right)^{\frac{2}{3}} \mathbf{M}_1(m + \tfrac{1}{3}) \\ &- \left(\frac{3A}{16\kappa}\right)^{\frac{4}{3}} \left\{\frac{9}{50}\left(\frac{B}{A}\right)^2 \mathbf{M}_2{}^{(1)}(m + \tfrac{2}{3}) + \frac{6b}{7A}\mathbf{M}_2{}^{(2)}(m + \tfrac{2}{3}) + \right. \\ &\left. + \frac{3b}{7A}\mathbf{M}_2{}^{(3)}(m - \tfrac{1}{3})\right\} + \left(\frac{3A}{16\kappa}\right)^2 \left\{\frac{9}{250}\left(\frac{B}{A}\right)^3 \mathbf{M}_3{}^{(1)}(m) \right. \\ &\left. + \frac{9bB}{35A^2}\mathbf{M}_3{}^{(2)}(m) - \frac{a}{3A}\mathbf{M}_3{}^{(3)}(m) + cM_0{}^e(m) + \frac{5a}{36A}M_0{}^e(m-1)\right\}, \end{aligned} \tag{64}$$

we have

$$\begin{aligned} \mathbf{M}_1(m) &= \tfrac{1}{36}(2M_2{}^e + 3M_4{}^e), \\ \mathbf{M}_2{}^{(1)}(m) &= -\tfrac{1}{12}(m - \tfrac{7}{6})(m - \tfrac{25}{14})M_4{}^e \\ &= -\tfrac{5}{3024}(16M_4{}^e + 40M_6{}^e + 49M_8{}^e), \\ \mathbf{M}_2{}^{(2)}(m) &= \tfrac{1}{6}(m^2 - \tfrac{8}{3}m + \tfrac{67}{36})M_2{}^e = \tfrac{1}{216}(28M_2{}^e + 24M_4{}^e + 15M_6{}^e), \\ \mathbf{M}_2{}^{(3)}(m) &= (m^2 - \tfrac{2}{3}m + \tfrac{7}{36})M_0{}^e = \tfrac{1}{36}(4M_0{}^e + 3M_4{}^e), \\ \mathbf{M}_3{}^{(1)}(m) &= -\frac{5}{163{,}296}(1792M_0{}^e + 896M_2{}^e - 432M_4{}^e - 2760M_6{}^e \\ &\qquad - 3465M_8{}^e + 3969M_{10}{}^e), \\ \mathbf{M}_3{}^{(2)}(m) &= -\frac{1}{11{,}664}(1981M_0{}^e - 112M_2{}^e - 324M_4{}^e - 600M_6{}^e - 945M_8{}^e), \\ \mathbf{M}_3{}^{(3)}(m) &= \tfrac{1}{216}(100M_0{}^e - 58M_2{}^e - 27M_4{}^e - 15M_6{}^e). \end{aligned} \tag{65}$$

† Omission of this subtle contribution invalidates the terms of highest order in Jorna's calculation (1964).

With the displaced transforms $M_r^e(m + \sigma)$ translating to $(2z)^{-\sigma}\, T_r^e(z)$,

$$
\begin{aligned}
Y^- = {} & T_0^e + \frac{1}{60}\left(\frac{9}{512A\kappa^2 z}\right)^{\frac{1}{3}} B(2T_2^e + 3T_4^e) + \frac{1}{10{,}080}\left(\frac{9}{512A\kappa^2 z}\right)^{\frac{2}{3}} \\
& \times \{3B^2(16T_4^e + 40T_6^e + 49T_8^e) - 40Ab(24zT_0^e + 28T_2^e \\
& + [24 + 18z]T_4^e + 15T_6^e)\} - \frac{z}{453{,}600}\left(\frac{9}{512A\kappa^2 z}\right) \{B^3(1792T_0^e \\
& + 896T_2^e - 432T_4^e - 2760T_6^e - 3465T_8^e + 3969T_{10}^e) \\
& + 20AbB(1981T_0^e - 112T_2^e - 324T_4^e - 600T_6^e - 945T_8^e) \\
& + 700A^2a([175 - 360z]T_0^e - 116T_2^e - 54T_4^e - 30T_6^e) \\
& - 1400A(199 - 451k^2/\kappa^2)\, T_0^e\} \ldots,
\end{aligned} \tag{66†}
$$

the same result as found from the integral representation for $W_{k\mu}(x)$ (Chapter X, question 8). To deduce the second solution Y^+, we have simply to reverse the sign prefacing z, where still $z = 8\kappa/3Aq^3$, and change to the $\overline{T}$'s.

In particular, at the turning point $z = 0$ the asymptotic expansions for the two Whittaker functions are as follows:

$$
\begin{aligned}
W_{k\mu}(2k + 2\kappa) &= \left\{\frac{(k + \mu - \frac{1}{2})!\,(k - \mu - \frac{1}{2})!}{\pi}\right\}^{\frac{1}{2}} \left(\frac{A^2\kappa}{24z}\right)^{\frac{1}{6}} e^{-z}\, Y^- \Big|_{z=0} \\
&= \frac{(-\frac{2}{3})!}{\pi} [(k + \mu - \tfrac{1}{2})!\,(k - \mu - \tfrac{1}{2})!]^{\frac{1}{2}} \left(\frac{A^2\kappa}{48}\right)^{\frac{1}{6}} \\
&\quad \times \left[1 - \frac{3\beta(B^2 + 15Ab)}{560A^{\frac{2}{3}}\kappa^{\frac{4}{3}}} - \frac{1}{921{,}600A\kappa^2}\{64B^3\right. \\
&\quad \left. + 1415AbB + 4375A^2a - 50A(199 - 451k^2/\kappa^2)\} \ldots\right] \\
&= \frac{(-\frac{2}{3})!}{\pi} [(k + \mu - \tfrac{1}{2})!\,(k - \mu - \tfrac{1}{2})!]^{\frac{1}{2}} \left\{\frac{(k + \kappa)^2}{48\kappa}\right\}^{\frac{1}{6}} \\
&\quad \times \left[1 + \frac{3\beta(9k^2 + 6k\kappa - 4\kappa^2)}{140\kappa^{8/3}\,(k + \kappa)^{\frac{2}{3}}}\right. \\
&\quad \left. - \frac{14k^3 + 14k^2\kappa - 7k\kappa^2 - 8\kappa^3}{1800\kappa^4(k + \kappa)} \ldots\right],
\end{aligned} \tag{67}
$$

† $(9/512A\kappa^2 z)^{1/3}$ $3q/16\kappa$.

$$W_{k\mu}'(2k + 2\kappa) = \left\{\frac{(k + \mu - \frac{1}{2})!\,(k - \mu - \frac{1}{2})!}{\pi}\right\}^{\frac{1}{2}} \left(\frac{3z^2}{512\,\kappa A^2}\right)^{\frac{1}{6}}$$

$$\times \frac{d}{dz}\left\{1 + \frac{a}{8}\left(\frac{9z^2}{A\kappa^2}\right)^{\frac{1}{3}} \cdots\right\} z^{-\frac{1}{6}} e^{-z}\, Y^{-}\bigg|_{z=0}$$

$$= -\frac{(-\frac{1}{3})!}{\pi}\,[(k + \mu - \tfrac{1}{2})!\,(k - \mu - \tfrac{1}{2})!]^{\frac{1}{2}} \left(\frac{3}{256\,\kappa A^2}\right)^{\frac{1}{6}}$$

$$\times \left[1 - \frac{B + 5a}{120\beta A^{\frac{1}{3}}\kappa^{\frac{4}{3}}} + \frac{1}{6{,}451{,}200 A\kappa^2}\{736B^3 - 7825AbB\right.$$

$$- 45a(1925A^2 - 32B^2 - 480Ab)$$

$$\left. + 350A(199 - 451k^2/\kappa^2)\} \cdots\right]$$

$$= -\frac{(-\frac{1}{3})!}{\pi}\,[(k + \mu - \tfrac{1}{2})!\,(k - \mu - \tfrac{1}{2})!]^{\frac{1}{2}} \left(\frac{3\kappa}{256(k + \kappa)^2}\right)^{\frac{1}{6}}$$

$$\times \left[1 + \frac{k - 3\kappa}{60\beta\kappa^{\frac{4}{3}}(k + \kappa)^{\frac{1}{3}}}\right.$$

$$\left. + \frac{277k^3 + 7k^2\kappa - 296k\kappa^2 - 4\kappa^3}{25{,}200\kappa^4(k + \kappa)} \cdots\right], \qquad (68)$$

both in agreement with results from integral representations (Chapter VIII (144), (148)), and

$$\mathscr{W}_{k\mu}(2k + 2\kappa) = \left\{\frac{\pi}{(k + \mu - \frac{1}{2})!\,(k - \mu - \frac{1}{2})!}\right\}^{\frac{1}{2}} \left(\frac{9A^2\kappa}{8z}\right)^{\frac{1}{6}} e^{z}\, Y^{+}\bigg|_{z=0}$$

$$= (-\tfrac{2}{3})!\,[(k + \mu - \tfrac{1}{2})!\,(k - \mu - \tfrac{1}{2})!]^{-\frac{1}{2}} \left(\frac{9(k + \kappa)^2}{16\kappa}\right)^{\frac{1}{6}}$$

$$\times \left[1 - \frac{3\beta(9k^2 + 6k\kappa - 4\kappa^2)}{140\kappa^{8/3}(k + \kappa)^{\frac{4}{3}}}\right.$$

$$\left. - \frac{14k^3 + 14k^2\kappa - 7k\kappa^2 - 8\kappa^3}{1800\kappa^4(k + \kappa)} \cdots\right], \qquad (69)$$

$$\mathscr{W}_{k\mu}'(2k+2\kappa) = \left\{\frac{\pi}{(k+\mu-\frac{1}{2})!\,(k-\mu-\frac{1}{2})!}\right\}^{\frac{1}{2}} \left(\frac{81z^2}{512\kappa A^2}\right)^{\frac{1}{6}}$$

$$\times \frac{d}{dz}\left\{1 + \frac{a}{8}\left(\frac{9z^2}{A\kappa^2}\right)^{\frac{1}{3}} \cdots\right\} z^{-\frac{1}{6}} e^z Y^+ \bigg|_{z=0}$$

$$= (-\tfrac{1}{3})!\,[(k+\mu-\tfrac{1}{2})!\,(k-\mu-\tfrac{1}{2})!]^{-\frac{1}{2}} \left(\frac{81\kappa}{256(k+\kappa)^2}\right)^{\frac{1}{6}}$$

$$\times \left[1 - \frac{k-3\kappa}{60\beta\kappa^{\frac{4}{3}}(k+\kappa)^{\frac{1}{3}}}\right.$$

$$\left. + \frac{277k^3 + 7k^2\kappa - 296k\kappa^2 - 4\kappa^3}{25{,}200\kappa^4(k+\kappa)} \cdots\right]. \tag{70}$$

Exercises

1. Find uniform expansions for the Poiseuille functions

$$Pe(r,\sigma) = -2e^{\frac{1}{4}\pi\sigma}\,\mathfrak{I}[(\tfrac{1}{2}i\sigma - \tfrac{1}{2})!\,r(2i\sigma)^{\frac{1}{2}}]^{-1}\,W_{\frac{1}{2}i\sigma,0}(2i\sigma r^2),$$

$$Qe(r,\sigma) = -\pi e^{-\frac{1}{4}\pi\sigma}[1 + e^{-\pi\sigma}]^{-1}\,\mathfrak{R}[(\tfrac{1}{2}i\sigma - \tfrac{1}{2})!\,r(2i\sigma)^{\frac{1}{2}}]^{-1}\,W_{\frac{1}{2}i\sigma,0}(2i\sigma r^2).$$

These are solutions to the differential equation

$$d^2y/dr^2 + r^{-1}\,dy/dr - 4\sigma^2(1-r^2)\,y = 0$$

arising in the theory of steady hydrodynamic flow (Lauwerier 1953).

2. From the results in Section 4, deduce the uniform expansion of the parabolic cylinder function

$$D_p(x) = 2^{\frac{1}{2}p+\frac{1}{4}}\,x^{-\frac{1}{2}}\,W_{\frac{1}{2}p+\frac{1}{4},-\frac{1}{4}}(\tfrac{1}{2}x^2).$$

3. Treating A, B, a and b in Section 5 as independently adjustable parameters, show how the choice $A = B = 1$, $a = b = 0$ and $\kappa = -\frac{1}{8}p$ leads to uniform expansions for the modified Bessel functions K_p and $\mathscr{K}_p$.

References

Most of the papers by Erdélyi, Jorna, Langer and Olver cited at the end of Chapter XIII include discussions on uniform expansion. Pertinent additional references are as follows:

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Chapter XVI

Derivation of Asymptotic Expansions from Homogeneous Differential Equations in which the Second Derivative is Relatively Insignificant

1. Recurrence Relations for the Two Solutions

The theory of the two preceding chapters evolved from an initial step wherein the first derivative was removed in order to reduce second-order equations to a standard form $d^2y/dx^2 = X(x)y$. Such a starting point is patently inapposite for one exceptional class of problem, namely where the first derivative is abnormally significant compared to the second derivative—for example, through multiplication by a large parameter as in the important case of a confluent hypergeometric function $F(a, c, x)$ for large $|c|$.

We shall write the differential equation in the form

$$d^2y/dx^2 + \{f(x) + \delta(x)\}dy/dx + g(x)y = 0, \tag{1}$$

where over the proposed range f is large in the sense $f > |y''/y'|$. To avoid inessential sub-division into various cases, we shall further suppose the direction of x to have been chosen such that this inequality breaks down to the *right* of the proposed range, i.e. f is negative beyond the right edge.

One solution to (1) is roughly $\exp(-\int f^{-1}g dx)$, so we set

$$y_- = \exp(-\textstyle\int f^{-1}g\, dx)\ Y_-. \tag{2}$$

The resultant exact equation for Y_- is

$$\frac{d^2 Y_-}{dx^2} + \left(f + \delta - \frac{2g}{f}\right)\frac{dY_-}{dx} + \frac{g^2 + f'g - fg' - fg\delta}{f^2}\, Y_- = 0. \tag{3}$$

As a consequence of our supposition, f in the second term is the largest single coefficient in (3). Taking the corresponding term to the left of the equation and all other terms to the right, then integrating by parts,

$$-Y_- = \frac{1}{f}\frac{dY_-}{dx} + \frac{f' - 2g + f\delta}{f^2} Y_-$$
$$+ \int^x \frac{g^2 - 3f'g + 2(f')^2 + f(g' - g\delta - f'' + f'\delta) - f^2\delta'}{f^3} Y_- \, dx. \qquad (4)$$

This can be solved by iteration, since substitution in the right-hand side of the zero-order contribution to Y_-, $Y_0^- = 1$ say, generates the first-order correction $Y_1^- = 0(f^{-1})$ and so on. Later theory is rendered neater on eliminating g in favour of the new composite function

$$G = g - \tfrac{1}{2}f' - \tfrac{1}{2}f\delta. \qquad (5)$$

In terms of G,

$$y_- = f^{-\frac{1}{2}} \exp\left[-\int(f^{-1}G + \tfrac{1}{2}\delta)dx\right] \sum_{r=0}^{\infty} (-1)^r Y_r^-, \qquad (6)$$

$$Y_{r+1}^- = \frac{1}{f}\frac{dY_r^-}{dx} - \frac{2G}{f^2} Y_r^-$$
$$+ \int^x \frac{(G - \frac{1}{2}f')(G - \frac{3}{2}f') + f(G' - \frac{1}{2}f'') - \frac{1}{4}f^2(\delta^2 + 2\delta')}{f^3} Y_r^- \, dx. \qquad (7)$$

In this first solution to the original differential equation (1), fy' almost balances gy. Anticipating that in the second solution fy' will instead almost balance the other contribution y'', we expect $y \sim \exp(-\int f\,dx)$ for large f. Setting therefore

$$y_+ \propto w \exp(-\textstyle\int f\,dx),$$

the exact equation for w is

$$d^2w/dx^2 + (-f + \delta)dw/dx + (G - \tfrac{1}{2}f' - \tfrac{1}{2}f\delta)w = 0. \qquad (8)$$

But in the same notation (1) reads

$$d^2y/dx^2 + (f + \delta)dy/dx + (G + \tfrac{1}{2}f' + \tfrac{1}{2}f\delta)y = 0, \qquad (9)$$

so results for this second solution can be written down from those for the first by holding G invariant and formally changing the sign of either f, or both δ and dx. Explicitly,

$$y_+ = f^{-\frac{1}{2}} \exp\left[-\int (f - f^{-1}G + \tfrac{1}{2}\delta)\, dx\right] \sum_{r=0}^{\infty} Y_r^+ \tag{10}$$

where

$$Y_{r+1}^+ = \frac{1}{f}\frac{dY_r^+}{dx} + \frac{2G}{f^2} Y_r^+$$

$$+ \int^x \frac{(G + \frac{1}{2}f')(G + \frac{3}{2}f') - f(G' + \frac{1}{2}f'') - \frac{1}{4}f^2(\delta^2 + 2\delta')}{f^3} Y_r^+\, dx. \tag{11}$$

Since the sequences defined by (6) and (10) are asymptotic rather than convergent, they each represent a continuous solution only within a range bounded by the nearest Stokes rays, or zeros of f. Consequently, a separate identification is obligatory in each range. This needs to be effected individually for each differential equation investigated, since evidence points to there being no connection formulae for these asymptotic series solutions $y_\pm$ which are tractable, universal and absolutely precise—as are those applying to phase-integral pairs. Briefly, this major difference arises in the following way. In generalizing continuation theory from Airy to phase-integral expansions, the simple linear equations uniquely determining the two Stokes multipliers are unaltered, essentially because the two asymptotic series are still precisely complex conjugates on the real axis to one side of the turning point (Chapter XIII, Section 3). On the other hand, continuation theory for the present pair $y_\pm$ requires instead generalization of a method applicable in the simplest instance to parabolic cylinder functions (Chapter I, Section 2). Now the functional equation $\alpha(p)\,\alpha(-p-1) = -2\sin \pi p$ determining the two Stokes multipliers in the asymptotic expansions for D_p and $\mathscr{D}_p$ is uniquely soluble because it can involve only the one given parameter p. But on generalization the functional equation becomes intractable, because in principle it could now involve all derivatives of f, G(or g) and δ at the extinction point defined below. The intricate analysis involved in this question of linkage is postponed until Chapter XXV, Section 1.

Iteration through (7) and (11) can be greatly simplified by introducing some new independent variable, q say, which converts the multipliers into as near polynomial form as attainable:

$$AY_{r+1}^\pm = P(q)dY_r^\pm/dq \pm R(q)Y_r^\pm + \int^q Q^\pm(q)Y_r^\pm\, dq. \tag{12}$$

The algebraic forms produced by the integration $\int f^{-2}(G' \pm \frac{1}{2}f'')dx$ suggest possibilities. (The parameter q can very frequently be chosen to be the same in the two solutions. For brevity of exposition we shall assume this equality.) As a consequence of the method of selecting q, the terms of highest degree in the polynomials $P(q)$, $R(q)$, $Q^{\pm}(q)$ and $Y_r^{\pm}(q)$ are the same—apart from multiplying constants—for an extensive class of problems, namely all those in which f goes to zero linearly at an "extinction point", x_0 say; thus $f \to (-f_1)(x_0 - x)$. For then $q \to a(x_0 - x)^{-1}$ and

$$A \to -a^2 f_1, \qquad P \to q^3, \qquad R \to (2\rho + 1)q^2, \qquad Q^+ \to \rho(\rho - 1)q,$$
$$Q^- \to (\rho + 1)(\rho + 2)q \tag{13}$$

with $\rho = -f_1^{-1}G_0 - \frac{1}{2} = -f_1^{-1}g_0 - 1$; G_0, f_1 and g_0 being the respective values of G, f' and g at the extinction point. In this limit the solutions of (12) are

$$Y_r^+ \to \left(\frac{q^2}{2A}\right)^r \frac{(2r + \rho)!}{\rho!\,r!} = \left(\frac{2q^2}{A}\right)^r \frac{(r + \frac{1}{2}\rho)!\,(r + \frac{1}{2}\rho - \frac{1}{2})!}{(\frac{1}{2}\rho)!\,(\frac{1}{2}\rho - \frac{1}{2})!\,r!}, \qquad q \gg 1, \tag{14}$$

$$Y_r^- \to \left(\frac{q^2}{2A}\right)^r \frac{(2r - \rho - 1)!}{(-\rho - 1)!\,r!}, \qquad q \gg 1. \tag{15}$$

2. Late Terms in the Asymptotic Expansions

The recurrence relations (12) are the same in form as those discussed earlier in the context of variants to the phase-integral procedure (Chapter XIII, Section 8). This is hardly surprising, because the problem in hand may be regarded as a rather extreme variant. Proceeding more or less analogously, we introduce the new quantities

$$\overline{Y}_r^{\pm} = Y_r^{\pm} \exp\left(\pm \int RP^{-1}\,dq\right) \tag{16}$$

so as to absorb the middle term on the right side of (12) into the leading approximation for $r \gg 1$. Throughout, the integration constant in the factor $\exp(\int RP^{-1}dq) = \exp(2\int f^{-1}Gdx)$ will be fixed such that in the limit $q \gg 1$ this factor becomes $q^{2\rho+1}$.

Continuing for brevity only with the plus solution, the limiting form is found to be

$$\overline{Y}_r^+ \to \xi\ (r + \eta)! \Big/ \left(-A \int_{\zeta}^{q} P^{-1}\,dq\right)^{r+\eta+1}, \qquad r \gg 1. \tag{17}$$

The constants ξ, η and ζ have now to be identified by comparison with (14) concerned with the limit $q \gg 1$. The power of $2q^2/A = (-A\int_\infty^q q^{-3}dq)^{-1}$ prescribed by (14) dictates the values $\zeta = \infty$ on the positive q side of the extinction point, and $\eta = \rho - \frac{1}{2}$, while the limit

$$(r + \tfrac{1}{2}\rho)!\,(r + \tfrac{1}{2}\rho - \tfrac{1}{2})!/r! \to (r + \rho - \tfrac{1}{2})!\,\{1 + O(r^{-1})\} \tag{18}$$

confirms the value of η and identifies $\xi = (\frac{1}{2}A)^{\rho+\frac{1}{2}}/(\frac{1}{2}\rho)!\,(\frac{1}{2}\rho - \frac{1}{2})!$.

Extending the argument of Chapter XIII, Section 8, the formally exact expansion for $\overline{Y}_r^{\,+}$ is found to be

$$\overline{Y}_r^{\,+} = \frac{A^{\rho+\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}\rho!\,\mathscr{F}_0^{\,r+\rho+\frac{1}{2}}} \sum_{s=0} (r - s + \rho - \tfrac{1}{2})!\,\mathscr{Y}_s^{\,-}(-\mathscr{F}_0)^s, \tag{19}$$

where the $\mathscr{Y}_s^{\,-}$ satisfy a recurrence relation to be examined shortly, and

$$\mathscr{F}_0 = A\int_q^\infty P^{-1}dq = \int_x^{x_0} f\,dx \tag{20}$$

plays the role of the singulant introduced in Chapter VII when finding late terms $L_{r\gg 1}^{(\sigma)}$ in asymptotic expansions derived from integral representations in which the fast-varying factor in the integrand behaves linearly at a limit of integration. The $\mathscr{Y}_s^{\,-}$ in (19) are required to satisfy the recurrence relation

$$A\mathscr{Y}_{s+1}^{-} = P d\mathscr{Y}_s^{\,-}/dq + \int^q (Q^+\mathscr{Y}_s^{\,-} - R d\mathscr{Y}_s^{\,-}/dq)\,dq. \tag{21}$$

Integrating by parts and introducing the correlation

$$Q^- = Q^+ + dR/dq, \tag{22}$$

this is seen to agree with the recurrence relation (12) for $Y_r^{\,-}$, *except perhaps in respect of lower limits of integration.*

Similarly, the formally exact expansion for $\overline{Y}_r^{\,-}$ is

$$\overline{Y}_r^{\,-} = \frac{A^{-\rho-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}(-\rho - 1)!\,\mathscr{F}_0^{\,r-\rho-\frac{1}{2}}} \sum_{s=0} (r - s - \rho - \tfrac{3}{2})!\,\mathscr{Y}_s^{\,+}(-\mathscr{F}_0)^s, \tag{23}$$

with the same reservation, in determining the $\mathscr{Y}_s^{\,+}$, on allowed differences in integration constants compared with the recurrence relation for $Y_r^{\,+}$.

It will be recalled that congruity of integration constants between the recurrence relations determining early and late terms in phase-integral expansions is intimately correlated with ruling connection formulae (Chapter XIII, Section 7). For the $y_\pm$ of the present chapter no such universal connection formulae apply, and consequently there is no standard pattern of connection between the sets of integration constants, or therefore between the sets Y_s and $\mathscr{Y}_s$.

An instructive comparison illustrating this absence of a fixed relation is between

(i) the coefficients when P, R, Q^+ and Q^- represent the functions listed in (13), in which case combining (19) with the expansion

$$\frac{(r+\frac{1}{2}\rho)!\,(r+\frac{1}{2}\rho-\frac{1}{2})!}{r!\,(r+\rho-\frac{1}{2})!} = 1 - \frac{\rho(\rho-1)}{4(r+\rho-\frac{1}{2})} + \frac{\rho(\rho-1)(\rho-2)(\rho-3)}{32(r+\rho-\frac{1}{2})(r+\rho-\frac{3}{2})} - \dots \tag{24}$$

leads us to infer coincidence between the sets Y_s^- and $\mathscr{Y}_s^-$, and similarly on interchanging ρ and $-1-\rho$ between Y_s^+ and $\mathscr{Y}_s^+$; and

(ii) the coefficients for confluent hypergeometric expansions, which we shall find in Section 4 to differ and indeed to be interconnected in a somewhat abstruse fashion.

The integration constants in these late-term formulae have therefore to be ascertained apiece for each problem. Three alternative modes of derivation are available:

(a) With the exercise of perspicacity and ingenuity, the correct set $\mathscr{Y}$ can sometimes be divined by manipulating the late-term formulae to accord with symmetries, limits or other behaviour observed to be common to all terms in the original asymptotic expansion. (This approach is well illustrated in a somewhat similar context in Chapter XIX, Sections 3 and 5).

(b) In an obvious extension of the method adopted earlier in this section, the constants in $\mathscr{Y}_1$, $\mathscr{Y}_2$, ... can be successively identified by calculating further terms of decreasing degree in the Y_r's, not just the term of highest degree as hitherto, (14) and (15).

(c) The interconnection between the sets $\mathscr{Y}_s$ and Y_s can be deduced from continuation formulae for asymptotic solutions to the particular differential equation under examination. Such continuation formulae may be independently established, e.g. from the simpler asymptotic *power* series solutions.

This potent approach is illustrated in Section 4.

3. Asymptotic Expansions for Confluent Hypergeometric Functions when $|c|$ is Large

We shall suppose $F(a, c, x)$ to have been defined as that solution to the differential equation

$$\frac{d^2F}{dx^2} + \left(\frac{c}{x} - 1\right)\frac{dF}{dx} - \frac{a}{x}F = 0 \tag{25}$$

which tends to unity as $x \to 0$ irrespective of the value assigned to the parameter c. [This last stipulation eliminates any possibility of contribution from the second solution $x^{1-c}F(a - c + 1,\ 2 - c,\ x)$]. The coefficient $f(x)$ of our standard form (1) to be regarded as 'large' must include $(c/x - 1)$ but need not coincide with it, and it is in fact profitable to choose this cardinal quantity f so as to remain invariant under major transformations underlying the theory of the confluent hypergeometric function, namely $(a, c, x) \to (c - a,\ c, -x)$ in the Kummer relation, $(a, c, x) \to (a - c + 1,\ 2 - c,\ x)$ in converting to the second solution, and their amalgamation $(a, c, x) \to (1 - a,\ 2 - c, -x)$. The choice

$$f = (c - 2a)/x - 1 \tag{26}$$

fulfils these invariance desiderata. With the consequential allocations $\delta = 2a/x$, $g = -a/x$ and $G = -(2a - 1)(c - 2a)/2x^2$, the recurrence relation (7) becomes

$$Y_{r+1}^- = \frac{x}{C - x}\frac{dY_r^-}{dx} + \frac{(2a - 1)C}{(C - x)^2}Y_r^-$$
$$+ (a - 1)\int^x \frac{2(a - 1)C - ax}{(C - x)^3}Y_r^-\,dx, \tag{27}$$

where for brevity $C = c - 2a$. A more suitable independent variable is suggested by the integration

$$\int \frac{G' \pm \frac{1}{2}f''}{f^2}\,dx \propto \frac{C}{C - x} + \ln\left|\frac{x}{C - x}\right|, \tag{28}$$

from which we select

$$q = C/(C - x). \tag{29}\dagger$$

† In the alternative Whittaker notation $q = k/(k - \frac{1}{2}x)$.

This reduces the recurrence relation to the neater "polynomial form"

$$CY_{r+1}^{-} = q^2(q-1)dY_r^{-}/dq + (2a+1)q^2Y_r^{-} + (a-1)\int_0^q \{(a-2)q+a\}\, Y_r^{-}\, dq. \tag{30}$$

Here the lower limit of integration has been set so as to produce the simplest-looking solution—not the most readily identifiable, which would ensue from the alternative choice $q = 1$ $(x = 0)$. The first few terms of $\mathscr{Y}_-$ are as follows:

$$\mathscr{Y}_- = \left(\frac{q}{C}\right)^a \left[1 - \frac{aq}{2C}\{(a+1)q + 2(a-1)\} + \frac{a(a+1)q^2}{24C^2} \times \left\{3(a+2)(a+3)q^2 + 4(a+2)(3a-5)q + 12(a-1)(a-2)\right\} - \ldots\right]. \tag{31}$$

With the same definition (29) of q,

$$CY_{r+1}^{+} = q^2(q-1)dY_r^{+}/dq - (2a-1)q^2Y_r^{+} + a\int_0^q \{(a+1)q+a-1\}\, Y_r^{+}\, dq. \tag{32}$$

This differs from the recurrence relation for the first solution only by the exchange $a \to 1-a$. Now, of the additional exponential factors in $\mathscr{Y}_+$ relative to $\mathscr{Y}_-$, the integration constant in $\exp(2\int f^{-1}G\,dx) = \exp(\int RP^{-1}\,dq)$ has already been fixed (Section 2). To keep the number of interdependent quantities down to the minimum we choose $\exp(-\int f\,dx) = \exp(A\int_q^\infty P^{-1}\,dq)$ $= \exp(\mathscr{F}_0)$ where $\mathscr{F}_0$ is the singulant. On these understandings the first few terms of $\mathscr{Y}_+$ are as follows:

$$\mathscr{Y}_+ = C^C e^{-C} x^{1-c} e^x \left(\frac{q}{C}\right)^{1-a} \left[1 + \frac{(a-1)q}{2C}\{(a-2)q+2a\} + \frac{(a-1)(a-2)q^2}{24C^2}\{3(a-3)(a-4)q^2 + 4(a-3)(3a+2)q + 12a(a+1)\}\ldots\right]. \tag{33}$$

Identifications.

$x < C$. Of our two solutions, $\mathscr{Y}_+$ cannot be involved in $F(a, c, x)$ in this region, because its outer factors evidently correspond to the unwanted "second solution"

$$x^{1-c}F(a-c+1,\ 2-c,\ x) = x^{1-c}\,e^x\,F(1-a,\ 2-c,\ -x).$$

Concentrating therefore on $\mathscr{Y}_-$, at the identification point $x = 0$, i.e. $q = 1$, our series becomes

$$\frac{1}{C^a}\left[1 - \frac{a(3a-1)}{2C} + \frac{a(a+1)(27a^2-17a+2)}{24C^2}\ \ldots\right] = \frac{(C+a-1)!}{(C+2a-1)!}, \tag{34}$$

so, since $F(a, c, 0) = 1$ by definition of this confluent hypergeometric function,

$$F(a, c, x) = \frac{(C+2a-1)!}{(C+a-1)!}\,\mathscr{Y}_-, \qquad |\mathrm{ph}\,\mathscr{F}_0| < \pi. \tag{35}$$

In passing we may also note that in this same region $x < C$,

$$\mathscr{Y}_+ = (-1)^{1-a}\frac{(-C-a)!}{(-C-2a+1)!}\,C^C\,e^{-C}\,x^{1-c}\,e^x\,F(1-a,\ 2-c,\ -x),$$
$$\mathrm{ph}\,\mathscr{F}_0 = 0. \tag{36}$$

$x > C$. Throughout our investigations in earlier chapters on confluent hypergeometric functions, we have written the solution to the differential equation (25) which does not vary exponentially when $x \to \infty$ as $x^{-a}\psi(a, c, x)$, where $\psi \to 1$, and the solution which increases exponentially along the positive real axis as $x^{a-c}\,e^x\,\psi(1-a,\ 2-c,\ -x)$. Since this limit corresponds to $-q \to 0$, where the series (31) and (33) terminate at their initial entries, identification is immediate; thus

$$x^{-a}\psi(a, c, x) = (-1)^a\mathscr{Y}_-$$
$$= \left(-\frac{q}{C}\right)^a\left[1 - \frac{aq}{2C}\{(a+1)q + 2(a-1)\} + \ldots\right], \tag{37}$$

$$x^{a-c}\,e^x\,\psi(1-a,\ 2-c,\ -x) = (-1)^{1-a}C^{-C}\,e^C\,\mathscr{Y}_+$$
$$= x^{1-c}e^x\left(-\frac{q}{C}\right)^{1-a}\left[1 + \frac{(a-1)q}{2C}\{(a-2)q + 2a\} + \ldots\right]. \tag{38}$$

4. Late Terms in Confluent Hypergeometric Expansions

We shall confine our attention to late terms $Y_{r\gg 1}^{-}$ in $y_{-} \propto \Sigma(-1)^r Y_r^{-}$. Corresponding results for $Y_{r\gg 1}^{+}$ in $y_{+} \propto \Sigma Y_r^{+}$ follow on replacing a by $(1-a)$.

The leading term is very easily found. Specializing general results obtained in Section 2, (16) reduces to $\bar{Y}_r^{-} = Y_r^{-}(q-1)^{2a-1}$ while according to (20) the required singulant is

$$\mathscr{F}_0 = C\int_q^\infty \frac{dq}{q^2(q-1)} = C\Xi, \qquad \Xi = \ln\left|\frac{q}{q-1}\right| - \frac{1}{q}. \tag{39}$$

Without need of further information, the leading term from (23) is seen to be

$$Y_r^{-} \sim \frac{(r+a-\frac{3}{2})!}{(2\pi)^{\frac{1}{2}}(a-1)!\,(q-1)^{2a-1}C^r\Xi^{r+a-\frac{1}{2}}}. \tag{40}$$

To extend (40), we have to unravel the interconnection between the sets $\mathscr{Y}_s^{+}$ and Y_s^{+}. Now several times over in our investigations of asymptotic power series for confluent hypergeometric functions (Chapter I, questions 4 and 5; Chapter II, Section 5; Chapter III, Section 4; Chapter IV, Section 4; Chapter XII, Section 3), we have shown that the Stokes formal discontinuity in the asymptotic power series $(-x)^{-a}\psi(a,c,x)$ amounts to $[(c-a-1)!/(a-1)!]x^{a-c}e^x\psi(1-a,\ 2-c,\ -x)$. Translated into our current terminology via (37) and (38), the Stokes discontinuity in y_{-} is thereby $[(-1)^{1-a}/(a-1)!](C+a-1)!C^{-C}e^C y_{+}$. As will be confirmed by our general interpretative theory of Chapters XXI onwards, such a discontinuity is generated by summing late terms in y_{-}, so this sum must incorporate the multiplier $(C+a-1)!\,C^{-C}e^C$. Since we know $\Sigma\mathscr{Y}_s^{+}$ and $\Sigma Y_s^{+} \propto y_{+}$ can differ only by a factor of the form $1 + O(C^{-1})$, their quotient must be precisely

$$\Gamma_+(C) = \frac{\Sigma\mathscr{Y}_s^{+}}{\Sigma Y_s^{+}} = \frac{(C+a-1)!}{(2\pi)^{\frac{1}{2}}C^{C+a-\frac{1}{2}}\,e^{-C}}$$

$$= \exp\left\{\frac{B_2(a)}{1.2.C} - \frac{B_3(a)}{2.3.C^2} + \frac{B_4(a)}{3.4.C^3} - \cdots\right\}$$

$$= 1 + \frac{a^2-a+\frac{1}{6}}{2C} + \frac{3a^4-10a^3+10a^2-3a+1/12}{24C^2}\cdots. \tag{41}$$

Thus to procure the $\mathscr{Y}_s^+$ coefficients, we have only to multiply the series ΣY_s^+ in (33) by $\Gamma_+(C)$, then re-separate the orders C^{-s}. Compare, for example,

$$Y_1^+ = \frac{(a-1)q\{(a-2)q+2a\}}{2C},$$

$$\mathscr{Y}_1^+ = \frac{(a-1)(a-2)q^2+2a(a-1)q+a^2-a+\frac{1}{6}}{2C}. \tag{42}$$

In this way (40) is easily extended to

$$\begin{aligned} Y_{r\geqslant 1}^- &= \frac{(r+a-\frac{3}{2})!}{(2\pi)^{\frac{1}{2}}(a-1)!\,(q-1)^{2a-1}C^r\Xi^{r+a-\frac{1}{2}}} \\ &\times\left[1-\frac{\Xi\{(a-1)(a-2)q^2+2a(a-1)q+a^2-a+\frac{1}{6}\}}{2(r+a-\frac{3}{2})}\right. \\ &+\frac{\Xi^2}{24(r+a-\frac{3}{2})(r+a-\frac{5}{2})}\{3(a-1)(a-2)(a-3)(a-4)q^4 \\ &+4(a-1)(a-2)(a-3)(3a+2)q^3+(a-1)(a-2)(18a^2 \\ &+6a+1)q^2+2a(a-1)(6a^2-6a+1)q \\ &\left.+3a^4-10a^3+10a^2-3a+\tfrac{1}{12}\}\ldots\right], \end{aligned} \tag{43}$$

in agreement with the corresponding result in Chapter VIII (83) derived from the integral representation.

Exercises

Specialize all results to apply to the incomplete factorial functions

$$(p,x)! = \frac{x^{p+1}e^{-x}}{p+1}F(1,p+2,x)$$

and

$$[p,x]! = x^p e^{-x}\psi(1,p+2,x).$$

Notes and References

For the special case of the confluent hypergeometric function, expansions similar to those of this chapter and the next have been derived in the following references:

Jorna, S. (1965). *Proc. Roy. Soc.* **A 284**, 531–539.
Kazarinoff, N. D. (1957). *J. Math. Mech.* **6**, 341–360.

Because of the specific nature of these treatments, it was puzzling to see what general relation the assumptions bore to those entailed in the phase-integral method.

The wider investigations detailed in this chapter, extended in the next to uniform expansions, are new; so also is the evaluation of late terms. The approach is now recognized as the next stage beyond the second variant to the phase-integral method (Chapter XIII, Section 8); the theory remains much the same, but the different limiting behaviour of the coefficients P, Q and R in the recurrence relation radically alters the expression for late terms, with ensuing consequences.

The main application is to the calculation of eigenvalues and eigenfunctions. As we shall see in the second volume, the phase-integral methods of Chapters XIII and XIV are suited to one class of eigenvalue problem, the present approach to another; as expected, the second variant to the phase-integral method links the two.

Chapter XVII

Derivation of Uniform Asymptotic Expansions from Homogeneous Differential Equations in which the Second Derivative is Significant only over a Short Range

1. Solution Through Mellin Transforms

The asymptotic expansions derived in the preceding chapter contain expansions in rising powers of q^2/A, and thus fail as they stand if the parameter q begins to approach $A^{\frac{1}{2}}$ in magnitude, which happens near the extinction point. Uniform expansions holding for large $|A|$ will now be developed. However, for the reasons detailed in Chapter X, Section 1, and Chapter XV, Section 1, once values of the function and its derivative at the extinction point have been deduced from such a uniform expansion, we favour replacement by the simpler Taylor expansion about this point, for all but certain specialized applications.

Under the conditions laid down in the last chapter, let two continuous solutions to the differential equation

$$d^2y/dx^2 + (f + \delta)dy/dx + (G + \tfrac{1}{2}f' + \tfrac{1}{2}f\delta)y = 0 \tag{1}$$

be denoted by†

$$\begin{aligned} y^- &= f^{-\frac{1}{2}} \exp\left[-\textstyle\int(f^{-1}G + \tfrac{1}{2}\delta)dx\right] Y^-, \\ y^+ &= f^{-\frac{1}{2}} \exp\left[-\textstyle\int(f - f^{-1}G + \tfrac{1}{2}\delta)dx\right] Y^+. \end{aligned} \tag{2}$$

† Comparing with the notation of the last chapter, the new uniform expansions $Y^\pm$ will extend across the extinction point, whereas the previous expansions $Y_\pm$ exhibited Stokes discontinuities and so could not be extended.

Integration constants in the exponents of (2) are inessential to our argument. In practice it is convenient to choose them so as to make the proportionalities of (12) and (13) into equalities, as we did in Chapter XVI, Section 3.

Then, as in Chapter XVI, Section 1,

$$\pm Y^{\pm} = \frac{1}{f}\frac{dY^{\pm}}{dx} \pm \frac{2G}{f^2} Y^{\pm}$$
$$+ \int^x \frac{(G \pm \frac{1}{2}f')(G \pm \frac{3}{2}f'') \mp f(G' \pm \frac{1}{2}f'') - \frac{1}{4}f^2(\delta^2 + 2\delta')}{f^3} Y^{\pm} dx. \quad (3)$$

Introducing the new independent variable q chosen there, (3) reduces to

$$\pm AY^{\pm} = P(q)dY^{\pm}/dq \pm R(q)Y^{\pm} + \int^q Q^{\pm}(q)Y^{\pm}dq. \quad (4)$$

The argument will be continued in detail only for the solution Y^-, since that for Y^+ runs parallel until the last stage.

Due to the way it was selected, q is large close to the extinction point, and $P \simeq q^3$, $R \simeq (2\rho + 1)q^2$ and $Q^- \simeq (\rho + 1)(\rho + 2)q$ where $\rho = -f_1^{-1}G_0 - \frac{1}{2}$. The leading approximations to Y^- are then given by the solutions to

$$-AY_0^- = q^3 dY_0^-/dq - (2\rho + 1)q^2 Y_0^- + (\rho + 1)(\rho + 2)\int_0^q qY_0^- dq. \quad (5)$$

For brevity, we introduce from the start the parameter $z = A^{\frac{1}{2}}/q$ which proves most convenient in the ensuing argument, and thus examine this equation in the form

$$Y_0^- = \frac{1}{z}\frac{dY_0^-}{dz} + \frac{2\rho + 1}{z^2} Y_0^- + (\rho + 1)(\rho + 2)\int_\infty^z Y_0^- \frac{dz}{z^3}. \quad (6)$$

Writing Y^- as a Mellin–Barnes type of integral

$$Y^- = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} M(m)(2z^2)^m dm, \quad (7)$$

(6) implies, as in the argument of Chapter XV, Section 2,

$$(2m + \rho + 1)(2m + \rho + 2)M_0(m + 1) = mM_0(m), \quad (8)$$

which is satisfied by

$$M_0(m) = (m - 1)!\,(-2m - \rho - 1)!/(-\rho - 1)!\,. \quad (9)$$

Here the proportionality constant has been chosen such that the pole at $m = 0$ produces unit contribution; i.e. through (7) $Y_0^- \to 1$ for sufficiently large z. $M_0(m)$ is the Mellin transform of the leading basic function $\mathscr{T}_{-\rho-1}(z)$ introduced in Chapter X, Section 3 when finding a uniform expansion for $\int_{\text{limit}} e^{-F} G u^{-\rho-1} du$, F linear to quadratic in u.

Higher order contributions to Y^- are evaluated through substituting (7) in the unabridged equation

$$-AY^- = \left(\frac{P}{q^3}\right) q^3 \frac{dY^-}{dq} - \left(\frac{R}{(2\rho + 1)q^2}\right)(2\rho + 1)q^2 Y^-$$

$$+ (\rho + 1)(\rho + 2)\int_0^q \left(\frac{Q^-}{(\rho + 1)(\rho + 2)q}\right) qY^- dq, \tag{10}$$

with the factors expressed in rising powers of $zA^{-\frac{1}{4}}$, and solving by iteration to find $M(m)$ as a sequence in powers of $A^{-\frac{1}{4}}$. Each contribution to the sequence is then to be expressed in the simplest way as $P_n(m')M_r(m')$, where $P_n(m)$ is a polynomial in m of degree n say, and M_r is the Mellin transform of a $\mathscr{T}_{r-\rho-1}$. From X (23),

$$M_r(m) = \left(1 - \frac{2m}{r - \rho - 1}\right) M_{r-1}(m), \tag{11}$$

so we can cancel the term m^n in $P_n(m)$ by subtracting the appropriate multiple of M_{r+n}/M_r. This leaves a new polynomial $P_{n-1}(m)$, in which the term m^{n-1} can be cancelled by subtracting a multiple of M_{r+n-1}/M_r, leaving $P_{n-2}(m)$; and so on until $P_n(m)M_r(m)$ is dissected into $M_{r+n}, M_{r+n-1}, \ldots M_r$. The uniform expansion follows immediately on translating into the $\mathscr{T}_{r-\rho-1}$.

Relative to y^-, the second solution y^+ contains two extra fast-varying factors:

$$\exp\left(-\int f\,dx\right) \propto \exp\left(A\int_q^\infty P^{-1} dq\right) = \exp(\mathscr{F}_0), \tag{12}$$

$$\exp\left(2\int f^{-1} G\,dx\right) \propto \exp\left(\int RP^{-1} dq\right) \underset{q \gg 1}{\to} \propto q^{2\rho+1}. \tag{13}$$

This solution is therefore initially defined where it is at peak exponential dominance relative to the first solution y^-, i.e. on a Stokes ray of its constituent uniform asymptotic expansion Y^+. Consequently, by Chapter X, Sections 2 and 3, in transmuting Y^- into Y^+ we have to replace the $\mathscr{T}$ by the barred functions $\bar{\mathscr{T}}$ in addition to changing z to iz.

The expansions of the preceding chapter can be retrieved on inserting asymptotic series for the $\mathscr{T}$'s (or $\bar{\mathscr{T}}$'s). On the other hand, introduction of their convergent series leads to "transitional expansions" effective in the immediate vicinity of the extinction point; though in practice it is best to pass at once to values of the function and its derivative right at the extinction point—signified by $z = 0$—and construct therefrom the simpler conditionally-convergent Taylor series about this point.

2. Confluent Hypergeometric Function $F(a, c, x)$ for large c

Referring to Chapter XVI, Section 3, this is expressed by

$$F(a, c, x) = \frac{(C + 2a - 1)!}{(C + a - 1)!}\left(\frac{q}{C}\right)^a Y^-, \tag{14}$$

where $C = c - 2a,\ q = C/(C - x)$, and

$$-CY^- = q^2(q - 1)dY^-/dq + (2a - 1)q^2 Y^-$$

$$+ (a - 1)\int_0^q \{(a - 2)q + a\}Y^- dq. \tag{15}$$

On introducing the parameter $z = C^{\frac{1}{2}}/q = (C - x)/\sqrt{C}$, (15) gives

$$Y^- = \left(1 - \frac{z}{\sqrt{C}}\right)\frac{dY^-}{zdz} - \frac{2a - 1}{z^2}Y^-$$

$$+ (a - 1)\int_\infty^z \{a - 2 + \frac{az}{\sqrt{C}}\}\, Y^- \frac{dz}{z^3}. \tag{16}$$

Expressing Y^- as the Mellin–Barnes type of integral (7) and substituting, (16) leads to the difference equation

$$(2m - a + 1)(2m - a + 2)M(m + 1) - mM(m)$$

$$= (2C)^{-\frac{1}{2}}(2m + a)(2m - a + 1)M(m + \tfrac{1}{2}). \tag{17}$$

This can be solved iteratively on writing

$$M(m) = \mathscr{M}(m)M_0(m), \tag{18}$$

where $M_0(m)$, the solution to (17) with the right-hand side neglected, is defined by (9) with $\rho = -a$. Then

$$\mathscr{M}(m) = -\left(\frac{2}{C}\right)^{\frac{1}{2}} S\, \frac{(m+\frac{1}{2}a)(m-\frac{1}{2})!}{m!}\, \mathscr{M}(m+\tfrac{1}{2}), \tag{19}$$

and substitution of one order on the right evolves the next higher order on the left. The summations encountered are simpler than those discussed in Chapter XV, Section 3, so skipping details we record the result obtained on reverting to M:

$$\begin{aligned}(a-1)!M(m) = (-2m+a-1)!\Bigg[&(m-1)! - \frac{1}{3}\left(\frac{2}{C}\right)^{\frac{1}{2}}(2m+3a-2)\\ &\times (m-\tfrac{1}{2})! + \frac{1}{18}\left(\frac{2}{C}\right)\{4m^2+3(4a-3)m+9a^2-15a+5\}m!\\ &-\frac{1}{1{,}620}\left(\frac{2}{C}\right)^{\frac{3}{2}}\{80m^3+60(6a-5)m^2+2(270a^2-495a+179)m\\ &+270a^3-810a^2+675a-138\}(m+\tfrac{1}{2})!\dots\Bigg].\end{aligned} \tag{20}$$

Each contribution has next to be expressed in the simplest way as a polynomial times an M_r. Writing

$$M(m) = \sum_{s=0}^{\infty} (2/C)^{\frac{1}{2}s}\mathbf{M}_s(m+\tfrac{1}{2}s), \tag{21}$$

we have

$$\mathbf{M}_0(m) = M_0(m), \qquad \mathbf{M}_1(m) = -\tfrac{1}{3}a(2m+3a-3)M_1(m)$$

$$= -\tfrac{1}{3}a\,\{2(2a-1)M_1 - (a+1)M_2\},$$

$$\mathbf{M}_2(m) = \frac{a(a+1)}{18}\{4m^2 + (12a-17)m + 9(a-1)(a-2)\}M_2$$

$$= \frac{a(a+1)}{18}\left\{\left(16a^2 - \frac{39}{2}a + 5\right)M_2 - (a+2)(8a-\tfrac{7}{2})M_3\right.$$

$$\left. + (a+2)(a+3)M_4\right\},$$

$$\mathbf{M}_3(m) = -\frac{a(a+1)(a+2)}{810}\{40m^3 + 30(6a-11)m^2 + (270a^2 - 1{,}035a$$

$$+ 899)m + 135(a^3 - 6a^2 + 11a - 6)\}M_3$$

$$= -\frac{a(a+1)(a+2)}{1{,}620}\{2(320a^3 - 690a^2 + 427a - 69)M_3$$

$$- 3(a+3)(160a^2 - 175a + 38)M_4 + 15(a+3)(a+4)(8a-3)M_5$$

$$- 10(a+3)(a+4)(a+5)M_6\}. \tag{22}$$

The displaced transforms $M_r(m+\sigma)$ translate to $(2z^2)^{-\sigma}\mathscr{T}_{r+a-1}(z)$. Hence, by (21), Y^- in (14) is given by

$$Y^- = \mathscr{T}_{a-1} - \frac{a}{3(z\sqrt{C})}\{2(2a-1)\mathscr{T}_a - (a+1)\mathscr{T}_{a+1}\}$$

$$+ \frac{a(a+1)}{36(z\sqrt{C})^2}\{(32a^2 - 39a + 10)\mathscr{T}_{a+1} - (a+2)(16a-7)\mathscr{T}_{a+2}$$

$$+ 2(a+2)(a+3)\mathscr{T}_{a+3}\} - \frac{a(a+1)(a+2)}{1{,}620(z\sqrt{C})^3}\{2(320a^3 - 690a^2$$

$$+ 427a - 69)\mathscr{T}_{a+2} - 3(a+3)(160a^2 - 175a + 38)\mathscr{T}_{a+3}$$

$$+ 15(a+3)(a+4)(8a-3)\mathscr{T}_{a+4} - 10(a+3)(a+4)(a+5)\mathscr{T}_{a+5}\}\ldots, \tag{23}$$

in agreement with the result found from the integral representation for $F(a, c, x)$, Chapter X, question 3.

Passing straightaway to limiting values at the extinction point $z = 0$, the resultant asymptotic expansions are

$$F(a, c, C) = \frac{(C + 2a - 1)!}{(C + a - 1)!\, C^{\frac{1}{2}a}} z^{-a} Y^{-}\Big|_{z=0}$$

$$= \frac{(\frac{1}{2}a - 1)!\,(C + 2a - 1)!}{2\,(a - 1)!\,(C + a - 1)!}\left(\frac{2}{C}\right)^{\frac{1}{2}a}\left[1 - \frac{2}{3}\frac{(2a - 1)!\,(\frac{1}{2}a - \frac{1}{2})!}{(\frac{1}{2}a - 1)!}\left(\frac{2}{C}\right)^{\frac{1}{2}}\right.$$

$$+ \frac{a(32a^2 - 39a + 10)}{72}\left(\frac{2}{C}\right)$$

$$\left. - \frac{320a^3 - 690a^2 + 427a - 69}{810}\frac{(\frac{1}{2}a + \frac{1}{2})!}{(\frac{1}{2}a - 1)!}\left(\frac{2}{C}\right)^{\frac{3}{2}} \cdots\right], \tag{24}$$

$$F'(a, c, C) = -\frac{(C + 2a - 1)!}{(C + a - 1)!\, C^{\frac{1}{2}a + \frac{1}{2}}}\frac{d}{dz} z^{-a} Y^{-}\Big|_{z=0}$$

$$= \frac{(\frac{1}{2}a - \frac{1}{2})!\,(C + 2a - 1)!}{2\,(a - 1)!\,(C + a - 1)!}\left(\frac{2}{C}\right)^{\frac{1}{2}a + \frac{1}{2}}\left[1 - \frac{(4a - 1)(\frac{1}{2}a)!}{3\,(\frac{1}{2}a - \frac{1}{2})!}\left(\frac{2}{C}\right)^{\frac{1}{2}}\right.$$

$$+ \frac{(a + 1)(32a^2 - 23a + 3)}{72}\left(\frac{2}{C}\right)$$

$$\left. - \frac{640a^3 - 900a^2 + 329a - 24}{1{,}620}\frac{(\frac{1}{2}a + 1)!}{(\frac{1}{2}a - \frac{1}{2})!}\left(\frac{2}{C}\right)^{\frac{3}{2}} + \cdots\right], \tag{25}$$

in agreement with results from integral representations in Chapter VIII (93) and (94).

Having ascertained these, we can now proceed exactly as in Chapter VIII, Section 7, to find the otherwise conditionally-convergent Taylor expansion for $F(a, c, x)$ about the extinction point.

3. Incomplete Factorial Function

The results of the preceding section greatly simplify in this important special case. Starting from the relation

$$(p, x)! = \frac{x^{p+1} e^{-x}}{p + 1} F(1, p + 2, x), \tag{26}$$

(14) + (23) reduce to

$$(p, x)! = \frac{x^{p+1}e^{-x}}{p - x}\left[\mathcal{T}_0 - \frac{2}{3z\sqrt{p}}(\mathcal{T}_1 - \mathcal{T}_2) + \frac{1}{6(z\sqrt{p})^2}(\mathcal{T}_2 - 9\mathcal{T}_3 + 8\mathcal{T}_4)\right.$$

$$\left. + \frac{2}{45(z\sqrt{p})^3}(2\mathcal{T}_3 + 23\mathcal{T}_4 - 125\mathcal{T}_5 + 100\mathcal{T}_6)\ldots\right], \tag{27}$$

where $z = (p - x)/\sqrt{p}$ (cf. Chapter X, question 2). At the extinction point $x = p$,

$$(p, p)! = (\tfrac{1}{2}\pi)^{\frac{1}{2}}p^{p+\frac{1}{2}}e^{-p}\left\{1 - \frac{2\sqrt{2}}{3\sqrt{\pi}\,p^{\frac{1}{2}}} + \frac{1}{12p} + \frac{4\sqrt{2}}{135\sqrt{\pi}\,p^{\frac{3}{2}}}\cdots\right\}, \tag{28}$$

in agreement with VIII (33).

The derivative of the incomplete factorial function $(p, x)! = \int_0^x t^p e^{-t}dt$ is of course $x^p e^{-x}$, so the corresponding value at the extinction point is $p^p e^{-p}$.

Exercises

1. Show that the uniform expansion for $\psi(a, c, x)$ is

$$\left(\frac{x}{x - C}\right)^a\left[\mathcal{T}_{a-1}(-z) - \frac{a}{3z\sqrt{C}}\{2(2a-1)\mathcal{T}_a(-z) - (a+1)\mathcal{T}_{a+1}(-z)\}\ldots\right]$$

where $C = c - 2a$ and $z = (C - x)/\sqrt{C}$ as in the text.

2. Deduce the value at the extinction point $x = C$:

$$\psi(a, c, C) = \frac{(\frac{1}{2}a - 1)!\,(2C)^{\frac{1}{2}a}}{2\,(a - 1)!}\left[1 + \frac{2}{3}\,\frac{(2a - 1)(\frac{1}{2}a - \frac{1}{2})!}{(\frac{1}{2}a - 1)!}\left(\frac{2}{C}\right)^{\frac{1}{2}}\right.$$

$$+ \frac{a(32a^2 - 39a + 10)}{72}\left(\frac{2}{C}\right)$$

$$\left. + \frac{320a^3 - 690a^2 + 427a - 69}{810}\,\frac{(\frac{1}{2}a + \frac{1}{2})!}{(\frac{1}{2}a - 1)!}\left(\frac{2}{C}\right)^{\frac{3}{2}} + \ldots\right].$$

3. Specialize the results of the preceding questions to the incomplete factorial function

$$[p, x]! = x^p e^{-x}\psi(1, p + 2, x).$$

Chapter XVIII

Derivation of Asymptotic Power Series from Inhomogeneous Differential Equations

1. Solution by Variation of Parameters

Let y_1 and y_2 be two independent solutions to the homogeneous equation

$$d^2y/dx^2 + f(x)\,dy/dx + g(x)\,y = 0. \tag{1}$$

The Wronskian W of these solutions is easily determined (apart from a constant factor) from the relation

$$W = y_2\,dy_1/dx - y_1\,dy_2/dx = \exp(-\textstyle\int f(x)\,dx). \tag{2}$$

Then by the method of variation of parameters the general solution to the inhomogeneous equation

$$d^2y/dx^2 + f(x)\,dy/dx + g(x)\,y = \zeta(x) \tag{3}$$

is

$$y = y_1 \int y_2\, W^{-1}\,\zeta\,dx - y_2 \int y_1\, W^{-1}\,\zeta\,dx. \tag{4}$$

It is clear from (4) that if the solutions to the homogeneous equation are known only as series, convergent or asymptotic, the required solution to the inhomogeneous equation is by this approach inconveniently expressed in terms of *products* of series. For this reason, an alternative approach needs to be developed towards series solutions of inhomogeneous differential equations.

2. Particular Integrals as Power Series and Mellin Representations: Example of the Modified Struve Function

It is usually a simple matter to find both convergent and asymptotic power series satisfying an inhomogeneous equation, though the universal relationship between the two is often difficult to establish. Consider, for example, the modified Struve equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (p^2 + x^2)y = \frac{4}{\sqrt{\pi}\,(p-\frac{1}{2})!}(\tfrac{1}{2}x)^{p+1}. \tag{5}$$

When $|x|$ is small, the dominant terms on the left-hand side are $\propto x^0 y$, so the series for the particular integral begins at x^{p+1} and proceeds in steps of x^2. Substitution of such a form leads easily to the absolutely convergent power series

$$\mathbf{L}_p(x) = (\tfrac{1}{2}x)^{p+1} \sum_0^{\infty} (\tfrac{1}{4}x^2)^s/(s+\tfrac{1}{2})!\,(s+p+\tfrac{1}{2})!, \tag{6}$$

defining the "modified Struve function". On the other hand, when $|x|$ is large $-x^2 y$ dominates on the left side of (5), and the series begins at x^{p-1} and proceeds in steps of x^{-2}. Substitution of this form leads to the asymptotic power series

$$\phi_p(x) = -\frac{(\frac{1}{2}x)^{p-1}}{\pi} \sum_0^{\infty} \frac{(r-\frac{1}{2})!}{(-r+p-\frac{1}{2})!\,(-\frac{1}{4}x^2)^r}. \tag{7}$$

This asymptotic series (7) is at all phases *a* particular integral in the sense of satisfying the differential equation, but its interpretation changes across its Stokes rays at $\operatorname{ph} x = 0, \pm\pi$.

Instead of recommencing from (5), by the method of Chapter II, Section 2 we could have converted the convergent series (6) into the Mellin representation

$$\mathbf{L}_p(x) = (\tfrac{1}{2}x)^{p+1} \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \frac{(m-1)!\,(-m)!}{(-m+\frac{1}{2})!\,(-m+p+\frac{1}{2})!}(-\tfrac{1}{4}x^2)^{-m}\,dm, \tag{8}$$

and argued that although convergence difficulties prevent continuous distortion from this path around the poles of $(m-1)!$ to a new path around the poles of $(-m)!$ at $m = 1, 2, 3, \ldots$, the integral for this new path, i.e.

$$\phi_p(x) = (\tfrac{1}{2}x)^{p+1} \frac{1}{2\pi i}\int^{(1^-, 2^-, \ldots)} \frac{(m-1)!\,(-m)!}{(-m+\frac{1}{2})!\,(-m+p+\frac{1}{2})!}(-\tfrac{1}{4}x^2)^{-m}\,dm, \tag{9}$$

is nonetheless certainly *a* particular integral of the inhomogeneous equation; it will therefore differ from $\mathbf{L}_p$ only by multiples of the two solutions $I_{\pm p}$ to the homogeneous equation, these multiples varying according to the phase sector.

3. Connecting the Series by Mellin Transforms

When $|x| \gg 1$ the contribution to the convergent series (6) from large late terms is of type $e^x + e^{-x}$. Proceeding as in Chapter II we find the alternative series

$$\mathbf{L}_p(x) = (\tfrac{1}{2}x)^{p+1} e^{\pm x} \sum_0^\infty a_t^{\pm} x^t,$$

where

$$a_t^{\pm} = -\frac{4^{p+1}}{\sqrt{\pi}} \frac{(\mp 2)^t}{(t+1)!} \left\{ \frac{(t+p+\frac{1}{2})!}{(t+2p+1)!} - V(t) \right\} \tag{10}$$

with

$$V(t) = \frac{1}{2^{p-\frac{1}{2}}(p-\frac{1}{2})!} \int_0^1 v^{p-\frac{1}{2}}(1-\tfrac{1}{2}v)^{t+p+\frac{1}{2}}\, dv.$$

These are equivalent to the Mellin representations

$$\mathbf{L}_p(x) = \begin{cases} \dfrac{(2x)^{p+1} e^x}{\sqrt{\pi}} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \left\{ \frac{(-m+p+\frac{1}{2})!}{(-m+2p+1)!} - V(-m) \right\} \\ \qquad \times (m-2)!\,(2x)^{-m}\, dm, \qquad |\mathrm{ph}\, x| < \frac{1}{2}\pi, & (11) \\ \dfrac{(2x)^{p+1} e^{-x}}{\sqrt{\pi}} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \left\{ \frac{(-m+p+\frac{1}{2})!}{(-m+2p+1)!} - V(-m) \right\} \\ \qquad \times (m-2)!\,(-2x)^{-m}\, dm, \qquad |\mathrm{ph}(-x)| < \frac{1}{2}\pi. & (12) \end{cases}$$

Noting that the integral V has no divergences and therefore no poles, we move the path of integration to the right past the poles of $(-m+p+\frac{1}{2})!$ and find

$$\mathbf{L}_p(x) = \begin{cases} \dfrac{e^x}{(2\pi x)^{\frac{1}{2}} (p-\frac{1}{2})!\,(-p-\frac{1}{2})!} \displaystyle\sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(2x)^r} \\ \qquad + O(x^{p-1}) + O(x^{-\frac{1}{2}} e^{-x}), \qquad |\mathrm{ph}\, x| < \frac{1}{2}\pi & (13) \\ \dfrac{e^{-x} \sin \pi p}{(2\pi x)^{\frac{1}{2}} (p-\frac{1}{2})!\,(-p-\frac{1}{2})!} \displaystyle\sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(-2x)^r} \\ \qquad + O(x^{p-1}) + O(x^{-\frac{1}{2}} e^{x}), \qquad x \text{ real and negative.} & (14) \end{cases}$$

The contribution $O(x^{p-1})$ is already known as $\phi_p(x)$. It has a Stokes ray at $\mathrm{ph}\, x = 0$, so will generate a discontinuity in form with the phase $i x^{p-1}$ when real x is changed from positive to negative. This must be exactly cancelled by the simultaneous discontinuity in form generated by the series $O(x^{-\frac{1}{2}} e^x)$, because it is clear from the definition (6) that for real x $\mathbf{L}_p(x)$ contains no portion with the phase $i\,x^{p-1}$. The series $O(x^{-\frac{1}{2}}\,e^{-x})$ included in the complete asymptotic expansion must therefore be present equally (i.e. with algebraically the same multiplier) for real positive x as for real negative x—the first contribution to (14). It combines with the series $O(x^{-\frac{1}{2}}\,e^{x})$ to give the known asymptotic expansion for $I_{-p}(x)$, x real and positive [e.g. IV (23)]. Hence

$$\mathbf{L}_p(x) = \phi_p(x) + I_{-p}(x), \qquad \operatorname{ph} x = 0. \tag{15}$$

This argument could be developed to yield asymptotic expansions valid for the various other phase sectors; but we will instead derive these in the course of illustrating an alternative approach.

4. Connecting the Series by Integral Representation

In

$$S(x) = \sum_0^\infty \frac{(\frac{1}{4}x^2)^s}{(s+\frac{1}{2})!\,(s+p+\frac{1}{2})!} = \frac{2}{\sqrt{\pi}} \sum_0^\infty \frac{x^{2s}}{(2s+1)!}\,\frac{s!}{(s+p+\frac{1}{2})!} \tag{16}$$

we proceed as in Chapter III and choose as familiar function

$$S^0(x) = \sum_0^\infty \frac{x^{2s}}{(2s+1)!} = \frac{\sinh x}{x}.$$

We then need the sub-representation

$$\frac{s!}{(s+p+\frac{1}{2})!} = \frac{1}{(p-\frac{1}{2})!}\int_0^1 u^s\,(1-u)^{p-\frac{1}{2}}\,du.$$

With these choices,

$$\mathbf{L}_p(x) = (\tfrac{1}{2}x)^{p+1}\,S(x) = \frac{2(\frac{1}{2}x)^p}{\sqrt{\pi}\,(p-\frac{1}{2})!}\int_0^1 (1-u^2)^{p-\frac{1}{2}}\sinh ux\,du,$$

$$\Re(p) > -\tfrac{1}{2}. \tag{17}$$

Combination with the representation (cf. III (30))

$$I_p(x) = \frac{2(\frac{1}{2}x)^p}{\sqrt{\pi}\,(p-\frac{1}{2})!}\int_0^1 (1-u^2)^{p-\frac{1}{2}}\cosh ux\,du, \qquad \Re(p) > -\tfrac{1}{2}, \tag{18}$$

provides the more convenient variants

$$\mathbf{L}_p(x) = \pm\left\{I_p(x) - \frac{2(\frac{1}{2}x)^p}{\sqrt{\pi}\,(p-\frac{1}{2})!}\int_0^1 (1-u^2)^{p-\frac{1}{2}}\,e^{\mp ux}\,du\right\}, \qquad \Re(p) > -\tfrac{1}{2}.$$

(19)

Direct derivation of an asymptotic expansion from any of these forms, though straightforward, perforce leads to a cumbersome result because there is no neat answer to the finite-range integral $\int_0^x v^{2n} e^{\mp v}\,dv$. To make the integration range infinite we proceed as in Chapter IV, Section 4 when confronted by a similar problem with the confluent hypergeometric function. Dissecting $\int_0^x \{1 + v^2/(-x^2)\}^{p-\frac{1}{2}} e^{-v}\,dv$ into $\int_0^\infty - \int_x^\infty$, and noting in the second integral that by our phase convention making fractional powers definite:

$$\left\{1 + \frac{v^2}{(-x^2)}\right\}^{p-\frac{1}{2}} = \begin{pmatrix} e^{i\pi(p-\frac{1}{2})} \\ e^{-i\pi(p-\frac{1}{2})} \\ \sin \pi p \end{pmatrix} \left(\frac{v^2}{x^2} - 1\right)^{p-\frac{1}{2}}, \qquad \left.\begin{matrix} 0 < \mathrm{ph}\, x < \frac{1}{2}\pi \\ 0 > \mathrm{ph}\, x > -\frac{1}{2}\pi \\ \mathrm{ph}\, x = 0 \end{matrix}\right\},$$

we have

$$\mathbf{L}_p(x) = I_p(x) + \frac{(\frac{1}{2}x)^{p-1}}{\sqrt{\pi}\,(p-\frac{1}{2})!}\left[-\int_0^\infty \left(1 - \frac{v^2}{x^2}\right)^{p-\frac{1}{2}} e^{-v}\,dv \right.$$
$$\left. + \begin{pmatrix} e^{i\pi(p-\frac{1}{2})} \\ e^{-i\pi(p-\frac{1}{2})} \\ \sin \pi p \end{pmatrix} \times \int_x^\infty \left(\frac{v^2}{x^2} - 1\right)^{p-\frac{1}{2}} e^{-v}\,dv \right]. \tag{20}$$

The middle contribution is the integral version of $\phi_p(x)$, while apart from the set of phase factors the last contribution is

$$\frac{2}{\pi} K_p(x) = \frac{1}{\sin \pi p} \{I_{-p}(x) - I_p(x)\} \tag{21}$$

by III (33). Hence

$$\mathbf{L}_p(x) = \phi_p(x) + \begin{pmatrix} 1 - i\cot \pi p \\ 1 + i \cot \pi p \\ 1 \end{pmatrix} I_{-p}(x) + \begin{pmatrix} i\cot \pi p \\ -i\cot \pi p \\ 0 \end{pmatrix} I_p(x) \tag{22}$$

$$= \phi_p(x) + (2\pi x)^{-\frac{1}{2}} \left[e^x \psi_p(-x) + \begin{pmatrix} \sin \pi p - i \cos \pi p \\ \sin \pi p + i\cos \pi p \\ \sin \pi p \end{pmatrix} e^{-x} \psi_p(x) \right],$$

$$\left.\begin{matrix} 0 < \mathrm{ph}\, x < \frac{1}{2}\pi \\ 0 > \mathrm{ph}\, x > -\frac{1}{2}\pi \\ \mathrm{ph}\, x = 0 \end{matrix}\right\}, \tag{23}$$

where

$$\psi_p(x) = \frac{1}{(p-\frac{1}{2})!\,(-p-\frac{1}{2})!}\sum_0^\infty \frac{(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(-2x)^r} \tag{24}$$

as in Chapter IV, equation (24). Note the mutual cancellation of the expected Stokes discontinuities in ϕ_p and $(2\pi x)^{-\frac{1}{2}}\, e^x\, \psi_p(-x)$, a cancellation argued on other grounds in Section 3.

Exercises

1. Show that the simplest asymptotic power series for a particular integral to the Anger inhomogeneous differential equation

$$x^2\mathbf{J}'' + x\mathbf{J}' + (x^2 - p^2)\mathbf{J} = \pi^{-1}(x-p)\sin \pi p$$

is $\mathbf{\tilde{J}}_p(x) = \pi^{-1}\sin \pi p\,\{\mathscr{C}_p(x) - \mathscr{S}_p(x)\}$ where

$$\mathscr{C}_p(x) = \frac{1}{x(\frac{1}{2}p-\frac{1}{2})!\,(-\frac{1}{2}p-\frac{1}{2})!}\sum_0^\infty \frac{(r+\frac{1}{2}p-\frac{1}{2})!\,(r-\frac{1}{2}p-\frac{1}{2})!}{(-\frac{1}{4}x^2)^r},$$

$$\mathscr{S}_p(x) = \frac{p}{x^2(\frac{1}{2}p)!\,(-\frac{1}{2}p)!}\sum_0^\infty \frac{(r+\frac{1}{2}p)!\,(r-\frac{1}{2}p)!}{(-\frac{1}{4}x^2)^r}, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi,$$

as in Chapter IV, questions 4 and 5.

2. Show that the simplest asymptotic power series for a particular integral to the Weber inhomogeneous differential equation

$$x^2\mathbf{E}'' + x\mathbf{E}' + (x^2 - p^2)\mathbf{E} = -\frac{x+p}{\pi} - \frac{(x-p)\cos \pi p}{\pi}$$

is

$$\mathbf{\tilde{E}}_p(x) = -\frac{1+\cos \pi p}{\pi}\,\mathscr{C}_p(x) - \frac{1-\cos \pi p}{\pi}\,\mathscr{S}_p(x).$$

3. Show that the simplest convergent series for a particular integral to

$$x^2C'' + xC' + (x^2 - p^2)C = -2\,\pi^{-1}p\sin \tfrac{1}{2}\pi p$$

is

$$C(x) = \sum_0^\infty (-\tfrac{1}{4}x^2)^s/(s+\tfrac{1}{2}p)!\,(s-\tfrac{1}{2}p)!.$$

Choose

$$C^0(x) = \sum_0^\infty (-x^2)^s/(2s)! = \cos x,$$

$$(2s)!/(s + \tfrac{1}{2}p)!\,(s - \tfrac{1}{2}p)! = 2\,\pi^{-1} \int_0^{\frac{1}{2}\pi} (2\cos u)^{2s} \cos pu\,du,$$

and deduce the integral

$$C(x) = 2\,\pi^{-1} \int_0^{\frac{1}{2}\pi} \cos(x\cos u)\cos pu\,du.$$

(This is a combination of Anger and Weber functions,

$$C(x) = \mathbf{J}_p(x)\cos\tfrac{1}{2}\pi p + \mathbf{E}_p(x)\sin\tfrac{1}{2}\pi p.)$$

4. Show that the simplest convergent series for a particular integral to

$$x^2S'' + xS' + (x^2 - p^2)S = 2\,\pi^{-1}x\cos\tfrac{1}{2}\pi p$$

is

$$S(x) = \sum_0^\infty (-1)^s\,(\tfrac{1}{2}x)^{2s+1}/(s + \tfrac{1}{2} + \tfrac{1}{2}p)!\,(s + \tfrac{1}{2} - \tfrac{1}{2}p)!.$$

Deduce the integral

$$S(x) = 2\,\pi^{-1} \int_0^{\frac{1}{2}\pi} \sin(x\cos u)\cos pu\,du.$$

(This is another combination of Anger and Weber functions,

$$S(x) = \mathbf{J}_p(x)\sin\tfrac{1}{2}\pi p - \mathbf{E}_p(x)\cos\tfrac{1}{2}\pi p.)$$

5. From the results of questions (3) and (4) deduce the Anger integral

$$\mathbf{J}_p(x) = \pi^{-1} \int_0^\pi \cos(p\theta - x\sin\theta)\,d\theta$$

and the Weber integral

$$\mathbf{E}_p(x) = \pi^{-1} \int_0^\pi \sin(p\theta - x\sin\theta)\,d\theta.$$

6. By combining the results of question 5 with Schläfli's representations of J_p and Y_p (Chapter III, questions 11 and 13), show that

$$\mathbf{J}_p(x) = J_p(x) + \pi^{-1} \sin \pi p \int_0^\infty e^{-p\omega - x\sinh\omega}\, d\omega, \qquad \Re(x) > 0,$$

and

$$\mathbf{E}_p(x) = -Y_p(x) - \pi^{-1} \int_0^\infty (e^{p\omega} + e^{-p\omega} \cos \pi p)\, e^{-x\sinh\omega}\, d\omega, \qquad \Re(x) > 0.$$

7. Proceeding from the results of question 6 as in Chapter IV, questions 4–6, derive the asymptotic expansions

$$\mathbf{J}_p(x) = J_p(x) + \tilde{\mathbf{J}}_p(x), \qquad \mathbf{E}_p(x) = -Y_p(x) + \tilde{\mathbf{E}}_p(x),$$

where $\tilde{\mathbf{J}}_p$ and $\tilde{\mathbf{E}}_p$ are the asymptotic power expansions derived in questions 1 and 2 respectively.

8. Show that the simplest asymptotic power series for a particular integral to the Lommel inhomogeneous differential equation

$$x^2 S'' + xS' - (p^2 - x^2)S = x^{q+1}$$

is

$$S_{qp}(x) = \frac{x^{q-1}}{\{\tfrac{1}{2}(p - q - 1)\}!\,\{-\tfrac{1}{2}(p + q + 1)\}!}$$
$$\times \sum_0^\infty \frac{\{r + \tfrac{1}{2}(p - q - 1)\}!\,(\{r - \tfrac{1}{2}(p + q + 1)\}!}{(-\tfrac{1}{4}x^2)^r}, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi,$$

while the simplest convergent series for a (different) particular integral is

$$s_{qp}(x) = \tfrac{1}{4}x^{q+1}(\tfrac{1}{2}q - \tfrac{1}{2} + \tfrac{1}{2}p)!\,(\tfrac{1}{2}q - \tfrac{1}{2} - \tfrac{1}{2}p)!$$
$$\times \sum_0^\infty \frac{(-\tfrac{1}{4}x^2)^s}{(s + \tfrac{1}{2}q + \tfrac{1}{2} + \tfrac{1}{2}p)!\,(s + \tfrac{1}{2}q + \tfrac{1}{2} - \tfrac{1}{2}p)!}.$$

9. In

$$S(x) = \sum_0^\infty (-\tfrac{1}{4}x^2)^s/(s + \tfrac{1}{2}q + \tfrac{1}{2} + \tfrac{1}{2}p)!\,(s + \tfrac{1}{2}q + \tfrac{1}{2} - \tfrac{1}{2}p)!,$$

choose

$$S^0(x) = \sum_0^\infty (-\tfrac{1}{4}x^2)^s/s!\,(s + \nu)! = (\tfrac{1}{2}x)^{-\nu} J_\nu(x).$$

Hence establish the following representations of the Lommel function:

$$s_{qp}(x) = 2^q(\tfrac{1}{2}x)^{\frac{1}{2}q+\frac{1}{2}\mp\frac{1}{2}p}(\tfrac{1}{2}q - \tfrac{1}{2} \pm \tfrac{1}{2}p)!\int_0^{\frac{1}{2}\pi} \cos^{q\mp p}\theta \sin^{-\frac{1}{2}q+\frac{1}{2}\mp\frac{1}{2}p}\theta$$

$$\times J_{\frac{1}{2}q+\frac{1}{2}\pm\frac{1}{2}p}(x\sin\theta)\,d\theta.$$

10. In the above $S(x)$ choose

$$S^0(x) = \sum_0^\infty (\tfrac{1}{4}x^2)^s = (1 - \tfrac{1}{4}x^2)^{-1}, \qquad |x| < 2,$$

and

$$\frac{1}{(s+\alpha)!\,(s+\beta)!} = -\frac{1}{2\pi i\cos\pi\beta}\int_\infty^{(0+)} \frac{J_{\alpha-\beta}(-2u)\,du}{(-u^2)^s\,(-u)^{\alpha+\beta+1}}.$$

Hence establish the following representations of the Lommel function:

$$s_{qp}(x) = \frac{(2x)^{q+1}(\tfrac{1}{2}q - \tfrac{1}{2} \pm \tfrac{1}{2}p)!\,(\tfrac{1}{2}q - \tfrac{1}{2} \mp \tfrac{1}{2}p)!}{8\pi i\sin\tfrac{1}{2}\pi(q\mp p)}$$

$$\times \int_\infty^{(0+)} \frac{J_{\pm p}(-v)\,dv}{(-v)^q\,(v^2 - x^2)}, \qquad |x/v| < 1,$$

i.e. with the poles at $\pm x$ enclosed within the contour.

11. By combining the two representations derived in the last question show that

$$s_{qp}(x) = \frac{(2x)^{q+1}}{8i\sin\pi p}\left\{\frac{(\tfrac{1}{2}q - \tfrac{1}{2} + \tfrac{1}{2}p)!}{(-\tfrac{1}{2}q - \tfrac{1}{2} + \tfrac{1}{2}p)!}\int_\infty^{(0+)} \frac{J_{-p}(-v)\,dv}{(-v)^q\,(v^2 - x^2)}\right.$$

$$\left. - \frac{(\tfrac{1}{2}q - \tfrac{1}{2} - \tfrac{1}{2}p)!}{(-\tfrac{1}{2}q - \tfrac{1}{2} - \tfrac{1}{2}p)!}\int_\infty^{(0+)} \frac{J_p(-v)\,dv}{(-v)^q\,(v^2 - x^2)}\right\}, \qquad |x/v| < 1.$$

12. Extracting the contributions from the poles at $\pm x$, and thereafter modifying the path of integration to run from 0 to $+i\infty$ and 0 to $-i\infty$, prove that the result of the last question can be transformed to

$$s_{qp}(x) = S_{qp}(x) - \frac{2^{q-1}\pi}{\sin\pi p}\left\{\frac{(\tfrac{1}{2}q - \tfrac{1}{2} + \tfrac{1}{2}p)!}{(-\tfrac{1}{2}q - \tfrac{1}{2} + \tfrac{1}{2}p)!}J_{-p}(x)\right.$$

$$\left. - \frac{(\tfrac{1}{2}q - \tfrac{1}{2} - \tfrac{1}{2}p)!}{(-\tfrac{1}{2}q - \tfrac{1}{2} - \tfrac{1}{2}p)!}J_p(x)\right\}$$

where

$$S_{qp}(x) = \frac{2^{q+1}}{(-\frac{1}{2}q - \frac{1}{2} + \frac{1}{2}p)!\,(-\frac{1}{2}q - \frac{1}{2} - \frac{1}{2}p)!}$$

$$\times \int_0^\infty \frac{K_p(wx)}{w^q(w^2+1)}\,dw, \qquad |p| + q < 1,$$

the simplest and most convenient integral representation of the Lommel function (Dingle 1959).

13. Combining the results of question 12 with those of Chapter IV, question 8, write down the asymptotic expansion for $s_{qp}(x)$.

14. Comparing the asymptotic power series of S_{qp} and W_{km}, derive the integral representation

$$S_{qp}(x) = \tfrac{1}{4}x^{q+1}\int_0^\infty W_{\frac{1}{2}q,\frac{1}{2}p}(u)\,e^{\frac{1}{2}u - \frac{1}{4}x^2/u}\,u^{-\frac{1}{2}q-2}\,du.$$

15. Convert the representation of Chapter IX, question 2 to

$$S_{qp}(x) = \frac{2^{q-1}\,(\frac{1}{2}x)^p}{\{\frac{1}{2}(p-q-1)\}!}\int_0^\infty [\tfrac{1}{2}(p+q-1),\ u]!\,e^{u-\frac{1}{4}x^2/u}\,u^{-p-1}\,du$$

$$= \frac{x^{q-1}}{\{\frac{1}{2}(p-q-1)\}!\,\{-\frac{1}{2}(p+q+1)\}!}$$

$$\times \int_0^\infty\int_0^\infty \frac{e^{-v-w}\,v^{\frac{1}{2}(p-q-1)}\,w^{-\frac{1}{2}(p+q+1)}\,dv\,dw}{1+4vw/x^2}.$$

16. Establish the representation

$$S_{qp}(x) = \frac{x^q\{\frac{1}{2}(p+q-1)\}!}{\{\frac{1}{2}(p-q-1)\}!\,(q-1)!}\int_1^\infty \{A_{p+q}(ux) + A_{-p-q}(ux)\}$$

$$\times \frac{(u^2-1)^{q-1}}{u^{p+q-1}}\,du,$$

where $A_p(\mu) = -(\pi/\sin \pi p)\,\mathbf{J}_{-p}(\mu)$ is essentially the Anger function of negative order.

17. Specializing question 8, show that the simplest convergent series for a particular integral to the Struve inhomogeneous differential equation

$$x^2\mathbf{H}'' + x\mathbf{H}' - (p^2 - x^2)\mathbf{H} = \frac{4}{\sqrt{\pi}\,(p - \frac{1}{2})!}(\tfrac{1}{2}x)^{p+1}$$

is

$$\mathbf{H}_p(x) = (\tfrac{1}{2}x)^{p+1}\sum_0^\infty \frac{(-\frac{1}{4}x^2)^s}{(s + \frac{1}{2})!\,(s + p + \frac{1}{2})!} = \frac{2^{1-p}}{\sqrt{\pi}\,(p - \frac{1}{2})!}s_{pp}(x),$$

while the simplest asymptotic power series for a (different) particular integral is

$$\frac{(\frac{1}{2}x)^{p-1}\cos\pi p}{\pi^2}\sum_0^\infty \frac{(r - p - \frac{1}{2})!\,(r - \frac{1}{2})!}{(-\frac{1}{4}x^2)^r} = \frac{2^{1-p}}{\sqrt{\pi}\,(p - \frac{1}{2})!}S_{pp}(x).$$

18. Specializing question 12, establish the relation

$$\mathbf{H}_p(x) = Y_p(x) + \frac{2^{1-p}}{\sqrt{\pi}\,(p - \frac{1}{2})!}S_{pp}(x).$$

19. Show that the simplest asymptotic power series for a particular integral to the modified Lommel inhomogeneous differential equation

$$x^2\tilde{S}'' + x\tilde{S}' - (p^2 + x^2)\tilde{S} = -x^{q+1}$$

is

$$\tilde{S}_{qp}(x) = \frac{x^{q-1}}{\{\frac{1}{2}(p - q - 1)\}!\,\{-\frac{1}{2}(p + q + 1)\}!}$$

$$\times\sum_0^\infty \frac{\{r + \frac{1}{2}(p - q - 1)\}!\,\{r - \frac{1}{2}(p + q + 1)\}!}{(\frac{1}{4}x^2)^r}.$$

20. By reference to the absolutely convergent series, or otherwise, show that the modified Struve function discussed in the text is expressed by

$$\mathbf{L}_p(x) = -i^{1-p}\mathbf{H}_p(ix) = I_{-p}(x) - \frac{2^{1-p}}{\sqrt{\pi}\,(p - \frac{1}{2})!}\tilde{S}_{pp}(x).$$

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Chapter XIX

Derivation of Asymptotic Expansions from Inhomogeneous Differential Equations

1. Asymptotic Expansion for a "Particular Integral"

In Chapters XIII and XIV we investigated in some detail the asymptotic expansions for the solutions to the homogeneous differential equation

$$d^2y/dx^2 - Xy = 0 \tag{1}$$

where $X(x)$ is either slowly varying or else of large magnitude. The asymptotic expansion for a 'particular integral' of the inhomogeneous equation

$$d^2y/dx^2 - Xy = Z, \tag{2}$$

where $Z(x)$ is like $X(x)$ slowly varying or else of large magnitude, is somewhat easier to find.

When the quotient Z/X is slowly varying, the procedure is simple and straightforward. For then the term d^2y/dx^2 in (2) is certainly the least important, enabling y to be determined through iteration of (2) displayed in the form

$$y = -\frac{Z}{X} + \frac{1}{X}\frac{d^2y}{dx^2}.$$

Ordering contributions by the power of X^{-1} they contain, the required expansion is therefore

$$y = \sum_0^\infty y_r, \quad \text{where} \quad y_{r+1} = \frac{1}{X}\frac{d^2}{dx^2}y_r \quad \text{commencing at} \quad y_0 = -\frac{Z}{X}. \tag{3}$$

When the quotient Z/X is fast varying, the operator d^2/dx^2 will no longer be equivalent to some small algebraic quantity of order $O(x^{-2})$. For instance, if the quotient contains e^x, $d^2/dx^2 \sim 1$; while if it contains a high power x^p, $d^2/dx^2 \sim p(p-1)/x^2$. Such predictably large contributions, $d^2/dx^2 \sim \Delta(x)$, must be subtracted out before expansion. To this end we write in (2)

$$X(x) = \chi(x) + \Delta(x) \tag{4}$$

and iterate the form

$$y = -\frac{Z}{\chi} + \frac{1}{\chi}\left(\frac{d^2}{dx^2} - \Delta\right)y.$$

The required extension of (3) to cope with fast variation is therefore

$$y = \sum_0^\infty y_r, \quad \text{where} \quad y_{r+1} = \frac{1}{\chi}\left(\frac{d^2}{dx^2} - \Delta\right)y_r \text{ commencing at } y_0 = -\frac{Z}{\chi}. \tag{5}$$

The sequences (3) and (5) are asymptotic rather than convergent. Consequently each such sequence represents a single continuous particular integral only within a range bounded by the nearest Stokes rays or turning points. To represent the analytic continuation of this same particular integral outwith this range in general requires the addition of expansions satisfying the homogeneous differential equation; their multiplying factors within a specified phase sector are most easily found by comparison with the simpler asymptotic *power* series of the preceding chapter, for as we shall see in the next section there can be no exact *general* connection formulae.

2. Late Terms

Early terms of the expansion for a 'particular integral' y tend to bear little resemblance to early terms of the asymptotic series solutions to the homogeneous equation, $Y_\pm$ of Chapters XIII and XIV. Indeed, as far as exact early terms in y are concerned, introduction of the variable q which proved such a boon in simplifying $Y_\pm$ here actually complicates the calculation. On the other hand, on deploying the arguments in Chapter VII, Section 6 concerning inter-relations between late and early terms, one would anticipate the following correlations:

Late terms in $y \rightarrow$ terminant of that expansion $\rightarrow$ discontinuity on crossing Stokes ray $\rightarrow$ early terms in Y_+ and Y_-,

because the only new expansions which can materialize on crossing a Stokes ray of a 'particular integral' expansion are the two solutions to the homogeneous equation (1). Consequently, late terms $y_{r \gg 1}$ are perforce similar in structure to $Y_{r\gg 1}$. Expressed differently, it should always prove practicable to expand $y_{r\gg 1}$ as an asymptotic expansion in which each term is the product of a factorial, a power of a singulant closely related to the one dictating $Y_{r\gg 1}$, and a coefficient closely related to a Y contribution.

To find this asymptotic representation of a late term explicitly, we first recast (5) in a form in which the basic iteration operator dominant for large r is a *first* derivative as was the case for $Y_{r\gg 1}$ in Chapter XIII (42). This is accomplished through the substitution

$$y_r = \chi^{-\frac{1}{4}}\, \mathfrak{Y}_{2r}, \tag{6}$$

for then

$$\mathfrak{Y}_{r+2} = \left\{\left(\frac{1}{\chi^{\frac{1}{2}}}\frac{d}{dx}\right)^2 + \frac{5(\chi')^2 - 4\chi\chi'' - 16\chi^2\Delta}{16\chi^3}\right\}\mathfrak{Y}_r, \tag{7}$$

starting at $\mathfrak{Y}_0 = -Z/\chi^{\frac{3}{4}}$. Introducing a new independent variable as in Chapter XIII, Section 5, this equation can be reduced to the more convenient 'polynomial' form

$$A^2\mathfrak{Y}_{r+2} = 4P(q)\left\{\frac{d}{dq}P(q)\frac{d}{dq} + Q(q)\right\}\mathfrak{Y}_r \tag{8}$$

where q, A and P have exactly the same meaning as in Chapter XIII, Section 5, while Q has the same minor modification to include Δ as required in Chapter XIII (48). The terms of highest degree in the polynomials $P(q)$, $Q(q)$ and $\mathfrak{Y}_r(q)$ are essentially the same for the whole class of problems in which $\chi \to \chi_1(x - x_0)$ and $Z \to Z_0$ at the turning point x_0, since in this neighbourhood

$$A^2\mathfrak{Y}_{r+2} \to 4q^4\left\{\frac{d}{dq}q^4\frac{d}{dq} + \frac{5}{4}q^2\right\}\mathfrak{Y}_r. \tag{9}$$

The solution reducing to the correct expression at $r = 0$, namely

$$\mathfrak{Y}_0 = -Z/\chi^{\frac{3}{4}} \to -2Z_0(\varepsilon q^3/\chi_1 A)^{\frac{1}{4}} = -2Z_0(q^3/|\chi_1|A)^{\frac{1}{4}},$$

is

$$\mathfrak{Y}_r \to -\frac{Z_0}{2\pi|\chi_1|^{\frac{1}{4}}}\left\{\frac{12q^3}{A}\right\}^{r+\frac{1}{4}}(\tfrac{1}{2}r - \tfrac{1}{3})!\,(\tfrac{1}{2}r - \tfrac{2}{3})!\,. \tag{10}$$

Let us next examine what the recurrence relation (8) does to some typical term in $\mathfrak{Y}_r$, $q^{\alpha r+\beta}$ say. The differential operator produces some multiplier like $(\alpha r + \beta)(\alpha r + \beta')$, so in the one iteration the relative importance of this component has been increased by a factor $O(r^2)$. Thus when r is large (8) can be approximated by

$$A^2\mathfrak{Y}_{r+2} \to (2P(q)d/dq)^2\mathfrak{Y}_r,$$

or more simply

$$A\mathfrak{Y}_{r+1} \to 2P(q)\frac{d}{dq}\mathfrak{Y}_r. \tag{11}$$

A sufficiently general solution to this difference-differential equation is

$$\mathfrak{Y}_r \to \xi\,(r+\eta)!\Big/\left(-\tfrac{1}{2}A\int_\zeta^q P^{-1}dq\right)^{r+\eta+1}. \tag{12}$$

Since this holds only for $r \gg 1$, the constants ξ, η and ζ cannot be ascertained directly from the qualifying condition at $r = 0$, namely $\mathfrak{Y}_0 = -Z/\chi^{\frac{1}{4}}$. They can, however, be inferred from (10), which shows that when $q \gg 1$ in addition to $r \gg 1$,

$$\mathfrak{Y}_r \to -\frac{Z_0}{(\pi|\chi_1|)^{\frac{1}{2}}}\left(\frac{6q^3}{A}\right)^{r+\frac{1}{2}}(r-\tfrac{1}{2})!\,. \tag{13}$$

Comparing (12) and (13), the power of $6q^3/A = (-\frac{1}{2}A\int_\infty^q q^{-4}dq)^{-1}$ dictates the values $\zeta = \infty$ (assuming q to be defined positively) and $\eta = -\frac{1}{2}$; the factorial confirms this value of η; and the multiplier identifies $\xi = -Z_0(\varepsilon/\pi\chi_1)^{\frac{1}{2}}$. The explicit limiting form is therefore

$$\mathfrak{Y}_r \to -\frac{Z_0}{(\pi|\chi_1|)^{\frac{1}{2}}}\frac{(r-\frac{1}{2})!}{\mathscr{F}_0^{\,r+\frac{1}{2}}}, \qquad r \gg 1, \tag{14}$$

where

$$\mathscr{F}_0 = \tfrac{1}{2}A\int_q^\infty P^{-1}dq = \varepsilon\int_{x_0}^x \chi^{\frac{1}{2}}dx. \tag{15†}$$

Thus the positive Liouville–Green exponent in the phase-integral ("W.K.B.")

† $\mathscr{F}_0 \quad \left|\int_{x_0}^x \chi^{1/2}\,dx\right|$ when all these quantities are real.

solutions to the homogeneous equation $d^2y/dx^2 - \chi y = 0$ plays the role of the "singulant" introduced in Chapter VII when finding late terms $L_{r \gg 1}$ in asymptotic expansions derived from integral representations in which the fast-varying factor in the integral decreases steadily away from a limit of integration, the variation being linear at this limit.

As already argued, the full expansion—of which (14) constitutes the leading term—will contain coefficients Y_s intimately correlated with those in Chapter XIII (48). By trial substitution in the complete equation (8), the formally exact expansion for a late term in the particular integral (5) is found to be

$$y_r = -Z_0\left(\frac{\varepsilon}{\pi\chi_1\mathscr{F}_0}\right)^{\frac{1}{2}} \frac{1}{\chi^{\frac{1}{2}}\mathscr{F}_0{}^{2r}} \sum_{s=0} (2r - s - \tfrac{1}{2})!\, Y_s(-\mathscr{F}_0)^s, \tag{16}$$

where

$$AY_{s+1} = PdY_s/dq + \int^q QY_s dq, \qquad Y_0 = 1. \tag{17}$$

This is seen to correspond to the earlier recurrence relation in Chapter XIII (48), but since the integration constants in the latter could have been chosen arbitrarily, corresponding to the free selection of constant multiplier for a solution to a homogeneous differential equation, the integration constants in (17) have to be determined independently. For the same reason there can be no connection formulae, involving simultaneously our asymptotic solutions to the inhomogeneous and corresponding homogeneous differential equations, which are tractable, universal and absolutely precise.

In practice the correct set $Y.$ can conveniently be picked out through (i) symmetry and limit requirements common to all terms in the asymptotic expansion $y = \Sigma y_r$ (illustrated in Sections 3 and 5), or (ii) calculating further terms of decreasing degree in the $\mathfrak{Y}_r$'s rather than just the term of highest degree (10), or (iii) continuation formulae independently established, e.g. from the asymptotic *power* series solutions. (The last-mentioned approach was illustrated in a somewhat similar context in Chapter XVI, Section 4).

3. Anger Function

As a simple example requiring no partition of X, we derive an asymptotic expansion for the function

$$A_p(z) = \int_0^\infty e^{p\omega - z\sinh\omega} d\omega, \qquad \Re(z) > 0, \tag{18}$$

valid when p is large and $z > p$. By virtue of the relation

$$A_p(z) = \frac{\pi}{\sin \pi p}\left(J_{-p}(z) - \mathbf{J}_{-p}(z)\right), \tag{19}$$

this is essentially the Anger function of negative order.

From the angle of the present chapter, $A_p(z)$ is that continuous particular integral of the inhomogeneous differential equation

$$z^2 d^2y/dz^2 + z dy/dz - (p^2 - z^2)y = p + z \tag{20}$$

which for large enough z behaves as

$$\int_0^\infty e^{-z\sinh\omega}d\omega \doteq \int_0^\infty e^{-z\omega}d\omega = z^{-1}.$$

The substitution $z = e^x$ reduces (20) to our standard form

$$d^2y/dx^2 - Xy = Z, \qquad \text{with} \qquad X = p^2 - e^{2x}, \qquad Z = p + e^x. \tag{21}$$

Viewed as a function of z, $Z/X = (p - z)^{-1}$ is not fast varying, so no partition of X is required here. Starting at

$$y_0(x) = -\frac{Z}{X} = \frac{1}{e^x - p}$$

the recurrence relation $y_{r+1} = X^{-1}d^2y_r/dx^2$ establishes the asymptotic expansion

$$y = \sum_0^\infty y_r(x) = \frac{1}{z - p}\left\{1 - \frac{z}{(z - p)^3} + \frac{z(p + 9z)}{(z - p)^6} - \frac{z(p^2 + 54pz + 225z^2)}{(z - p)^9} \cdots\right\}. \tag{22}$$

This behaves as z^{-1} for large enough z, so correctly represents $A_p(z)$ until the next Stokes ray or turning point, here $z = p$; thus (22) is the asymptotic expansion for $A_p(z)$ so long as $z > p$. The supplementary series needed when $z < p$ can be ascertained in a variety of ways; e.g. by comparison with the corresponding asymptotic *power* series (cf. Chapter XVIII, question 7), from our general interpretative theory (Chapter XXVI, Section 4), or from steepest descents applied to the defining integral (18). It is

$$\left(\frac{2\pi q}{p}\right)^{\frac{1}{2}} e^{p\Xi}\left\{1+\frac{q}{24p}(5q^2-3)+\frac{q^2}{1{,}152\,p^2}(385q^4-462q^2+81)+\ldots\right\} \quad (23)$$

where

$$q = p/(p^2-z^2)^{\frac{1}{2}}, \qquad \Xi = \tanh^{-1}q^{-1}-q^{-1}. \quad (24)$$

This is the expansion for $-\pi Y_p(z)$ and needs no further discussion (Chapters VIII, Section 6, and XIII, Section 9).

Late terms.

Returning to the asymptotic expansion (22), the singulant dominating late terms is equal to the positive Liouville–Green exponent in the associated function $Y_p(z)$, so $\mathscr{F}_0 = p\Xi$. Moreover, at the turning point $z = p$ we have $Z_0 = 2p$ and $X_1 = -2p^2$. In Chapter XIII, Section 9 we have already shown that $\varepsilon = -1$ and derived a set of coefficients Y_0 to Y_4. Assuming for the moment the integration constants in these to be appropriate to the present context, (16) gives the late terms as

$$y_r = -\frac{1}{p}\left(\frac{2q}{\pi\Xi}\right)^{\frac{1}{2}}\frac{(2r-\frac{1}{2})!}{(p\Xi)^{2r}}\left\{1-\frac{\Xi q(5q^2-3)}{24(2r-\frac{1}{2})}\right.$$

$$\left.+\frac{\Xi^2 q^2(385q^4-462q^2+81)}{1{,}152(2r-\frac{1}{2})(2r-\frac{3}{2})}\ldots\right\}. \quad (25)$$

Making the appropriate changes of notation for the range $z > p$,

$$y_r = \frac{1}{p}\left(\frac{2\varphi}{\pi\Upsilon}\right)^{\frac{1}{2}}\frac{(-1)^r(2r-\frac{1}{2})!}{(p\Upsilon)^{2r}}\left\{1-\frac{\Upsilon\varphi(5\varphi^2+3)}{24(2r-\frac{1}{2})}\right.$$

$$\left.+\frac{\Upsilon^2\varphi^2(385\varphi^4+462\varphi^2+81)}{1{,}152(2r-\frac{1}{2})(2r-\frac{3}{2})}\ldots\right\} \quad (26)$$

where as in Chapter XIII, Section 9,

$$\varphi = p/(z^2-p^2)^{\frac{1}{2}}, \qquad \Upsilon = \varphi^{-1}-\tan^{-1}\varphi^{-1}. \quad (27)$$

This choice of integration constants is readily verified on considering the limit $z \gg p$. For it is easily seen from the argument leading to (22) that in this limit

$$y_r \to (-1)^r\{1.3.5\ldots(2r-1)\}^2/z^{2r+1} = (-1)^r 2^{2r}\{(r-\tfrac{1}{2})!\}^2/\pi z^{2r+1}.$$

In the same limit $\varphi \to 0$, $\Upsilon\varphi \to 1$ and (26) reduces to

$$y_r \to \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{(-1)^r(2r-\frac{1}{2})!}{z^{2r+1}} \left(1 - \frac{1}{16r} + \frac{1}{512r^2} \cdots\right).$$

Insertion in each alternative of the Stirling-Laplace type asymptotic expansion for a factorial of a half-integer, namely

$$(R-\tfrac{1}{2})! = (2\pi)^{\frac{1}{2}} R^R e^{-R}(1 - 1/24R + 1/1{,}152R^2\ldots),$$

confirms their identity.

The asymptotic expansion (22) and formula for late terms (26) agree with those obtained in Chapter VIII, question 11 directly from the integral representation (18).

4. Modified Struve Function

As an example necessitating partition of X, we derive an asymptotic expansion satisfying the modified Struve equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - (p^2+z^2)y = \frac{4}{\sqrt{\pi}\,(p-\frac{1}{2})!}(\tfrac{1}{2}z)^{p+1}. \tag{28}$$

The substitution $z = e^x$ reduces this to the standard form

$$d^2y/dx^2 - Xy = Z, \qquad \text{with} \qquad X = p^2 + e^{2x},$$

$$Z = \{2^{1-p}/\sqrt{\pi}\,(p-\tfrac{1}{2})!\}\, e^{(p+1)x}. \tag{29}$$

Depending on the relative magnitudes of p^2 and e^{2x}, the quotient Z/X varies in behaviour from $e^{(p-1)x}$ to $e^{(p+1)x}$. The equivalent value of the operator d/dx then varies from $p-1$ to $p+1$, i.e. centering on p, so as regards magnitudes $d^2/dx^2 \sim p^2$. The best partitioning of X is therefore to extract $\Delta(x) = p^2$, leaving $\chi = e^{2x}$. Starting at

$$y_0(x) = -\frac{Z}{\chi} = -\frac{2^{1-p}}{\sqrt{\pi}\,(p-\frac{1}{2})!}\,e^{(p-1)x},$$

the recurrence relation $y_{r+1} = e^{-2x}(d^2/dx^2 - p^2)y_r$ establishes the asymptotic expansion

$$\phi_p(z) = \sum_0^\infty y_r = -\frac{1}{\sqrt{\pi}\,(p-\frac{1}{2})!}\,(\tfrac{1}{2}z)^{p-1}\left\{1 + \frac{1-2p}{z^2} + \frac{(1-2p)3(3-2p)}{z^4}\right.$$
$$\left. + \frac{(1-2p)3(3-2p)5(5-2p)}{z^6} + \ldots\right\}. \tag{30}$$

This can be ordered a trifle more succinctly to read

$$\phi_p(z) = -\frac{1}{\sqrt{\pi}\,(p-\frac{1}{2})!}\,(\tfrac{1}{2}z)^{p-1}\left\{1 - \frac{2p}{z^2} + \frac{12p^2+z^2}{z^4}\right.$$
$$\left. - \frac{24p(5p^2+z^2)}{z^6} + \ldots\right\}.$$

The analogous expansion for the unmodified Struve function was derived in Chapter VIII, question 13 directly from its integral representation. It is this version which is customarily quoted (e.g. Watson 1944, Section 10.43). But the original form (30) is superior on two counts: first, through displaying its general term exactly,

$$\phi_p(z) = -\frac{(\frac{1}{2}z)^{p-1}}{\pi}\sum_0^\infty \frac{(r-\frac{1}{2})!}{(-r+p-\frac{1}{2})!\,(-\frac{1}{4}z^2)^r}, \tag{31}$$

and second, because it is identical with the asymptotic *power* series in Chapter XVIII (7) already fully discussed – a coincidence between 'large variable' and 'large order' expansions almost unique among the better known mathematical functions.

Incidentally, our formula for late terms (16) + (17) is inapplicable in this degenerate instance because $\chi = e^{2x}$ possesses no zero—corresponding to a turning point—in the finite x-plane, so neither χ_1 nor Z_0 can be assigned meaningful values. This observation correlates with our preceding remarks:

by a fluke the 'large order' expansion for Struve functions reverts to the lower class of expansions called power series and treated earlier in this book.

5. Lommel Function of Two Variables

The asymptotic power series satisfying the equation

$$z^2 d^2y/dz^2 + z dy/dz - (p^2 - z^2)y = z^{q+1} \tag{32}$$

is easily found to be

$$S_{qp}(z) = z^{q-1}\left[1 - \frac{(q-1)^2 - p^2}{z^2} + \frac{\{(q-1)^2 - p^2\}\{(q-3)^2 - p^2\}}{z^4} - \dots\right], \quad |\mathrm{ph}\, z| < \tfrac{1}{2}\pi. \tag{33}$$

This is no use when p or q is large. S_{qp} has been the subject of numerous earlier exercises: Chapter IV, questions 8, 11; Chapter IX, question 2; Chapter XVIII, questions 8–16. Our task now is to find, from the inhomogeneous differential equation (32), the best asymptotic expansion when p and q are large.

The substitution $z = e^x$ reduces (32) to the standard form

$$d^2y/dx^2 - Xy = Z, \quad \text{with} \quad X = p^2 - e^{2x}, \quad Z = e^{(q+1)x}. \tag{34}$$

Depending on the relative magnitudes of p^2 and e^{2x}, the quotient Z/X varies in behaviour from $e^{(q-1)x}$ to $e^{(q+1)x}$. The equivalent value of the operator d/dx then varies from $q - 1$ to $q + 1$, i.e. centering on q, so as regards magnitudes $d^2/dx^2 \sim q^2$. The best partition of X into Δ and χ is therefore

$$\Delta = q^2, \qquad \chi = \rho^2 - e^{2x}, \tag{35}$$

where $\rho^2 = p^2 - q^2$. Starting at

$$y_0(x) = -\frac{Z}{\chi} = \frac{e^{(q+1)x}}{e^{2x} - \rho^2},$$

the recurrence relation

$$y_{r+1} = \frac{1}{e^{2x} - \rho^2}\left(q^2 - \frac{d^2}{dx^2}\right)y_r$$

yields the asymptotic expansion

$$\begin{aligned} y = \frac{z^{q+1}}{z^2 - \rho^2}\Bigg[1 &+ \frac{(2q-1)z^4 - 6z^2\rho^2 - (2q+1)\rho^4}{(z^2-\rho^2)^3} \\ &+ (z^2-\rho^2)^{-6}\{3(2q-1)(2q-3)z^8 + 4(2q^2 - 39q + 49)z^6\rho^2 \\ &- 2(24q^2 - 34q - 175)\, z^4\rho^4 + 12(2q^2 + 9q + 7)z^2\rho^6 + (2q+1)^2\rho^8\} \\ &+ (z^2-\rho^2)^{-9}\{15(2q-1)(2q-3)(2q-5)z^{12} + 2(104q^3 - 2{,}092q^2 \\ &+ 6{,}646q - 5{,}333)z^{10}\rho^2 - 9(120q^3 - 396q^2 - 2{,}470q + 5{,}975)z^8\rho^4 \\ &+ 4(200q^3 + 1{,}596q^2 - 4{,}286q - 15{,}267)z^6\rho^6 + (200q^3 - 4{,}036q^2 \\ &- 17{,}346q - 16{,}815)z^4\rho^8 - 6(40q^3 + 196q^2 + 286q + 135)z^2\rho^{10} \\ &- (2q+1)^3\rho^{12}\} + \ldots\Bigg]. \end{aligned} \tag{36}$$

When $|z| > |\rho|$ this may be identified with the large-z expansion (33), and thereby with $S_{qp}(z)$. Because of the breakdown in the expansion at $z = \rho$, the identity cannot be continued through to $|z| < |\rho|$; in fact, to represent $S_{qp}(z)$ in this range there has to be added a supplementary expansion for a solution to the homogeneous Bessel equation of order p.

Late terms

The effect of the chosen partition (35) is to relate the theory to a re-ordered homogeneous equation for order $\rho = (p^2 - q^2)^{\frac{1}{2}}$. Referring to Chapter XIII Section 9, the "turning point" is now at $z = \rho$; and, since χ is positive and the solutions therefore exponential when $z < \rho$, i.e. to the left of this turning point, $\varepsilon = -1$. Pursuing the analogy, the appropriate new independent variable is

$$\mu = \rho/(\rho^2 - z^2)^{\frac{1}{2}} = \rho/(\rho^2 - e^{2x})^{\frac{1}{2}}, \tag{37}$$

and this reduces the recurrence relation

$$Y_{s+1} = \tfrac{1}{2}\chi^{-\frac{1}{4}} dY_s/dx + \tfrac{1}{32}\int^{x} \chi^{-5/2}\{5(\chi')^2 - 4\chi\chi'' - 16\chi^2\Delta\}Y_s dx$$

to

$$2\rho Y_{s+1} = \mu^2(\mu^2 - 1)dY_s/d\mu + \tfrac{1}{4}\int^{\mu}\left(5\mu^2 - 1 - \frac{4q^2}{\mu^2 - 1}\right)Y_s d\mu. \tag{38}$$

The present quest provides an excellent example of the prediction of appropriate integration constants via symmetry and limit mandates. Thus:

(a) In each numerator of (36), successive terms proceed in powers of $\rho^2/z^2 = \mu^2/(\mu^2 - 1)$; the numerators are homogeneously even in μ. On the assumption of a fixed lower limit of integration in (38), only $\mu = 0$ as applied to the $5\mu^2 - 1$ section of the integrand would guarantee this symmetry. The scant chance of evasion through a varying lower limit, which would necessarily be zero when s is even, is ruled out by reference to Section 3, where it was confirmed that the fixed integration constant $\mu = 0$ *is* the correct choice for the Anger function, and therefore for Lommel functions in which $q = -1$ or 0.

(b) In each numerator of (36), all terms involving q cancel when $z = \rho$; i.e. all terms become independent of q when $\mu \to \infty$. This demands the limit $\mu = \infty$ as applied to the $-q^2/(\mu^2 - 1)$ section of the integral.

With these two integration limits,

$$Y_0 = 1, \qquad Y_1 = \frac{1}{24\rho}\{\mu(5\mu^2 - 3) + 12q^2 \tanh^{-1}\mu^{-1}\}. \tag{39}$$

The calculation of the remaining quantities needed in (16) is easy. The singulant is

$$\mathscr{F}_0 = \rho\int_{\mu}^{\infty}\frac{d\mu}{\mu^2(\mu^2 - 1)} = \rho U \qquad \text{where} \qquad U = \tanh^{-1}\mu^{-1} - \mu^{-1}. \tag{40}$$

Moreover, at the "turning point" $z = \rho$ we have $Z_0 = \rho^{q+1}$, $\chi_1 = -2\rho^2$. Equation (16) then gives the late terms as

$$y_r = -\rho^{q-1}\left(\frac{\mu}{2\pi U}\right)^{\frac{1}{2}} \frac{(2r-\frac{1}{2})!}{(\rho U)^{2r}}\left[1 - \frac{U}{24(2r-\frac{1}{2})} \times \{\mu(5\mu^2 - 3) + 12q^2 \tanh^{-1}\mu^{-1}\} + \dots\right]. \tag{41}$$

Making the appropriate changes of notation for the range $z > \rho$,

$$y_r = \rho^{q-1}\left(\frac{\nu}{2\pi V}\right)^{\frac{1}{2}} \frac{(-1)^r(2r-\frac{1}{2})!}{(\rho V)^{2r}}\left[1 - \frac{V}{24(2r-\frac{1}{2})} \times \{\nu(5\nu^2 + 3) + 12q^2 \tan^{-1}\nu^{-1}\} + \dots\right], \tag{42}$$

where

$$\nu = \rho/(z^2 - \rho^2)^{\frac{1}{2}}, \qquad V = \nu^{-1} - \tan^{-1}\nu^{-1}. \tag{43}$$

Exercises

1. Obtain the first three terms of an asymptotic solution, valid for large x or p, to the equation

$$d^2y/dx^2 + (p + \tfrac{1}{2} - \tfrac{1}{4}x^2)y = 1$$

through iterating the recurrence relation

$$y_{r+1} = \frac{2}{x^2 - 4p}\left(2\frac{d^2}{dx^2} + 1\right)y_r, \qquad y_0 = -\frac{4}{x^2 - 4p}.$$

2. Referring as necessary to Chapter XIII, Section 10 and questions 2–4, derive a formula for late terms in the preceding expansion.

3. Explain why the best asymptotic solution to the equation

$$\frac{d^2W}{dz^2} - \left(\frac{1}{4} - \frac{k}{z} + \frac{m^2 - \frac{1}{4}}{z^2}\right)W = z^{q-\frac{1}{2}}$$

for large z, k, m or q comes from iterating

$$y_{r+1} = \frac{1}{\frac{1}{4}e^{2x} - ke^{x} + \rho^{2}} \left(\frac{d^{2}}{dx^{2}} - q^{2}\right) y_{r},$$

$$y_{0} = -\frac{e^{(q+1)x}}{\frac{1}{4}e^{2x} - ke^{x} + \rho^{2}}, \qquad \rho^{2} = m^{2} - q^{2}.$$

Find y_1 and y_2.

4. Referring as necessary to Chapter XIII, Section 11, derive a formula for late terms in question 3.

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Chapter XX

Derivation of Uniform Asymptotic Expansions from Inhomogeneous Differential Equations

1. Behaviour of Solution Close to a Turning Point

Adopting similar assumptions and notations to those in the preceding chapter, $X \to X_1(x - x_0)$ and $Z \to Z_0$ at a true turning point x_0, so in this vicinity the inhomogeneous equation in Chapter XIX (2) can be approximated by

$$d^2y/dx^2 + X_1(x_0 - x)y = Z_0. \tag{1}$$

In terms of the integrals Υ_r introduced in Chapter X, Section 4, the solution is

$$y^0 = \frac{Z_0}{X_1^{\frac{2}{3}}\mu}\Upsilon_0(\mu), \qquad \mu = X_1^{\frac{1}{3}}(x_0 - x); \tag{2}$$

and it would not be difficult to improve the range of accuracy by introducing higher powers of $x_0 - x$ into (1), resulting in higher Υ's supplementing y^0. But only by rarest coincidence could this constitute a *uniform* expansion, because far away from the turning point $y^0 \doteq -Z_0/X_1(x - x_0)$ instead of the correct $-Z/X$. For instance, in the case of the Anger function (Chapter XIX, Section 3) the distant value would be $y^0 \doteq 1/p(x - \ln p)$, which does not even involve the same function of x as the correct answer $1/(e^x - p)$.

Thus while (2) describes behaviour close to a turning point, it is unacceptable as the leading contribution to a uniform expansion.

2. Solution Through Comparison with the Ordinary Asymptotic Expansion

To ensure viability far from the turning point, let us assume the *form* (2) but now determine the two parameters—(i) outer multiplier, (ii) argument of Υ_0—by fitting to the first two contributions $y_0 + y_1$ in the asymptotic expansion Σy_r, specified in Chapter XIX (5). Since by X (35)

$$\Upsilon_0(u) = 1 - 2u^{-3} + \dots,$$

the right fit is

$$y^{(0)} = y_0\Upsilon_0(u), \qquad u = (-2y_0/y_1)^{\frac{1}{3}}. \tag{3}$$

Near the turning point, where $x - x_0$ is very small,

$$y_0 \to \frac{-Z_0}{X_1(x-x_0)}, \qquad y_1 \to \frac{1}{X_1(x-x_0)}\frac{d^2}{dx^2}y_0 = \frac{-2Z_0}{X_1{}^2(x-x_0)^4},$$

and (3) then reduces to (2). The trial leading term (3) is thereby proven uniform through the turning point and out to the far distance, indeed until a second turning point or a singularity is approached.

To find the next contribution we look at the leading term (for large u) in $y_0 + y_1 + y_2 - y^{(0)}$. Owing to the contrived cancellation of $y_0 \propto u^{-1}$ and $y_1 \propto u^{-4}$, this will be of the form $f(x)\,u^{-n}$ where $n \geqslant 5$. (The usual value is $n = 6$). The associated contribution $y^{(1)}$ to the uniform expansion must coincide with this as $u \to \infty$, and must also remain finite everywhere including the turning point $u \to 0$. Since by X (35) and (36) all $\Upsilon(u) \to 1$ as $u \to \infty$, and $\Upsilon_{n-1}(u) \propto u^n$ as $u \to 0$, these two injunctions are met by setting

$$y^{(1)} = f(x)u^{-n}\,\Upsilon_{n-1}(u). \tag{4}$$

The next contribution $y^{(2)}$ is similarly determined from $y_0 + y_1 + y_2 + y_3 - y^{(0)} - y^{(1)}$, and so on until the uniform expansion has been constructed to desired accuracy.

When a function has initially been defined at ph $u = \pm\frac{1}{3}\pi$, which are Stokes rays of the asymptotic series for Υ, these must be replaced by $\overline{\Upsilon}$.

3. Observations on the Choice of Variable

The above procedure, while perfectly systematic and highly successful in practice, admittedly carries an aura of empiricism. The author has made repeated attempts to develop methods independent of the asymptotic expansion in Chapter XIX (5), but these have all proved complicated and unsatisfactory.

The correlation in this problem between empiricism and simplicity is intrinsic, not accidental. It arises because the best variable u to serve as the *argument* of the Υ's is not a convenient variable in which to express the *multipliers* of the Υ's; the rôle of empiricism is to superimpose simplicity requirements on strictly mathematical requirements. To delve still deeper, the reason why no single variable serves satisfactorily is that a uniform expansion is dependent on early and late terms of the ordinary asymptotic expansion simultaneously, and, as we have noted in Section 2 of the preceding chapter, early and late terms in the solution to an inhomogeneous differential equation cannot conveniently be expressed with the same variable.

For illustration, we summarize the consequences of restriction to the one determining variable q, taken together with the other notation chosen in

Chapter XIX, Section 2 for dealing with late terms. Introducing also the variant $z = A/6q^3$ which proved so helpful in the related analysis of Chapter XV, Section 2, the uniform expansion is to be found by solving the formidable equation

$$\left(\frac{P}{q^4}\right)^2 \frac{d^2\mathfrak{Y}}{dz^2} + \frac{1}{2}\frac{d(P/q^4)^2}{dz}\frac{d\mathfrak{Y}}{dz} - \left\{1 - \frac{5}{36z^2}\left(\frac{4PQ}{5q^6}\right)\right\}\mathfrak{Y} = \left(\frac{A}{6}\right)^{\frac{1}{3}} \frac{Zz^{-\frac{1}{3}}}{\chi^{\frac{1}{2}}q^{\frac{1}{2}}}.$$

The leading approximation close to the turning point is

$$\mathfrak{Y}^0 = -Z_0|\tfrac{3}{2}\chi_1|^{-\frac{1}{2}} z^{-\frac{1}{3}}\,\Upsilon_0\{(-9z^2/4)^{\frac{1}{3}}\}.$$

In principle, the next approximation would be calculated by expanding $Z/\chi^{\frac{1}{2}}q^{\frac{1}{2}}$, $4PQ/5q^6$, $d(P/q^4)^2/dz$ and $(P/q^4)^2$ to first order in $(z/A)^{\frac{1}{3}}$ and replacing $\mathfrak{Y}$ by $\mathfrak{Y}^{(0)}$ in the multipliers of the latter three. These contributions do not seem to be expressible as Υ_r's except by an infinite sum. Moreover, it is evident from the wrong variation of $\mathfrak{Y}^{(0)}$ far from the turning point that a true uniform expansion cannot be obtained directly this way. To put this right would necessitate empirical modification of $\mathfrak{Y}^{(0)}$ and thereby abandonment of q as the sole determining variable.

4. Anger Function

According to XIX (22), when $z > p$ this function has the ordinary asymptotic expansion

$$A_p(z) = \sum_0^\infty y_r; \qquad y_0 = (z-p)^{-1}, \qquad y_1 = -z(z-p)^{-4},$$
$$y_2 = z(p+9z)(z-p)^{-7}, \ldots. \tag{5}$$

The leading contribution to the uniform expansion is, by (3),

$$y^{(0)} = (z-p)^{-1}\,\Upsilon_0(u), \qquad u = (2/z)^{\frac{1}{3}}(z-p). \tag{6}$$

For large u, $\Upsilon_0(u) = 1 - 2u^{-3} + 40u^{-6} - \ldots$, so

$$y_0 + y_1 + y_2 - y^{(0)} = -z(z-p)^{-6} + \ldots,$$

and we have to set

$$y^{(1)} = -z(z-p)^{-6}\,\Upsilon_5(u). \tag{7}$$

Proceeding in this way,

$$A_p(z) = \frac{1}{z-p}\left\{\Upsilon_0 - \frac{z}{(z-p)^5}\Upsilon_5 - \frac{z}{(z-p)^7}\Upsilon_7 - \frac{z}{(z-p)^9}\Upsilon_9 \right.$$
$$\left. + \frac{126z^2}{(z-p)^{10}}\Upsilon_{10} - \dots\right\}, \tag{8}$$

in agreement with the result found in Chapter X, question 4, from the integral representation.

5. Lommel Function of Two Variables

The ordinary asymptotic expansion of $S_{qp}(z)$ when $z > \rho$ is given by XIX (36). Derivation therefrom of the uniform expansion differs from that for the Anger function only by magnified algebra. The final answer is:

$$S_{qp}(z) = \frac{z^{q+1}}{z^2-\rho^2}\left\{\Upsilon_0(u) - \Upsilon_5(u)\Big[(2q-1)(14q-1)z^6\right.$$
$$+ 5(4q^2-20q-15)z^4\rho^2 - 3(4q^2+56q+15)z^2\rho^4$$
$$- 9(2q+1)^2\rho^6\Big]\Big/(z^2-\rho^2)^5$$
$$+ \Upsilon_7(u)\Big[(2q-1)(508q^2-16q+1)z^8$$
$$+ 28(48q^3-124q^2-188q-61)z^6\rho^2$$
$$+ 2(296q^3-4556q^2-7150q-1155)z^4\rho^4$$
$$- 36(32q^3+332q^2+216q+35)z^2\rho^6$$
$$\left. - 225(2q+1)^3\rho^8\Big]\Big/(z^2-\rho^2)^7 \dots\right\} \tag{9}$$

where $\rho^2 = p^2 - q^2$ and

$$u = (z^2-\rho^2)/\{(q+\tfrac{1}{2})\rho^4 + 3\rho^2z^2 - (q-\tfrac{1}{2})z^4\}^{\frac{1}{2}}. \tag{10}$$

EXERCISES

1. Show that at the turning point $z = p$, (8) leads to

$$A_p(p) = \frac{\alpha}{p^{\frac{1}{3}}}\left(1 - \frac{1}{10\alpha p^{\frac{2}{3}}} + \frac{3\beta}{35p^{\frac{4}{3}}} \cdots\right), \qquad A_p'(p) = -\frac{\alpha}{p^{\frac{2}{3}}}\left(6\beta + \frac{1}{5p^{\frac{2}{3}}} \cdots\right),$$

in agreement with Chapter VIII, question 12.

2. Show that at the "turning point" $z = \rho$, (9) leads to

$$S_{qp}(\rho) = \tfrac{1}{2}\alpha\rho^{q-\frac{1}{3}}\left(1 + \frac{q + \frac{2}{5}}{\alpha\rho^{\frac{2}{3}}} \cdots\right),$$

$$S_{qp}'(\rho) = -\alpha\rho^{q-\frac{2}{3}}\left(3\beta + \frac{1}{10\rho^{\frac{2}{3}}} \cdots\right).$$

3. Drawing upon the ordinary asymptotic expansion derived in Chapter XIX, question 1, find the first two terms of the uniform solution to

$$d^2y/dx^2 + (p + \tfrac{1}{2} - \tfrac{1}{4}x^2)y = 1.$$

4. Similarly starting from Chapter XIX, question 3, find the first two terms of the uniform solution to

$$\frac{d^2W}{dz^2} - \left(\tfrac{1}{4} - \frac{k}{z} + \frac{m^2 - \frac{1}{4}}{z^2}\right)W = z^{q-\frac{1}{2}}.$$

Chapter XXI

Theory of Terminants

1. Remainders in Alternating and Single-sign Asymptotic Series

Defining

$$S_n = \sum_0^n A_r \tag{1}$$

to be a "partial sum" of an asymptotic series, our objective in this chapter is to interpret S_∞. We start with semi-quantitative considerations which depend on the signs but not the forms of the terms. These arguments are most easily expressed by setting

$$S_\infty = S_n + R_n \tag{2}$$

where R_n is the "remainder" corresponding to the partial sum S_n. With this notation,

$$R_{n-1} - R_n = A_n. \tag{3}$$

In an *alternating series*—one in which successive (late) terms A_r alternate in sign—a partial sum S_n overestimates the function represented when the last term included, A_n, is positive, but underestimates it when A_n is negative. Hence, by (2), R_n has the opposite sign to A_n, but R_{n-1} has the same sign. Then (3) can be expressed as

$$|R_{n-1}| + |R_n| = |A_n|, \tag{4}$$

from which there follow the inequalities

$$|R_n| < |A_n|, \qquad |R_n| < |A_{n+1}|: \tag{5}$$

put into words, the remainder is numerically less than the last term included in the partial sum, and numerically less than the first excluded term. Viewed as a function of n, $|R_n|$ must follow the variation of $|A_n|$ quite closely to satisfy

(5) for all n, i.e. $d|R_n|/dn \sim d|A_n|/dn$. In particular, close to the numerically least term $n = \eta$ specified by $d|A_n|/dn|_\eta \sim 0$, $|R_n|$ should be almost independent of n, causing (4) to reduce to

$$|R_\eta| \simeq \tfrac{1}{2}|A_\eta|,$$

or, recalling the relative signs,

$$R_\eta \simeq -\tfrac{1}{2}A_\eta. \tag{6}$$

Hence the approximate interpretation of S_∞ is

$$S_\infty \simeq A_0 + A_1 + A_2 + \ldots. + A_{\eta-1} + \tfrac{1}{2}A_\eta \tag{7}$$

where A_η is the term of least magnitude. This important general result was first adequately argued by Stieltjes in his classic paper of 1886.

In a *single-sign series*, early partial sums S_n underestimate the function represented, and late partial sums overestimate it, so R_n changes from positive to negative values when a certain term is reached. This reversal in sign of the remainder occurs at or near the least term $n = \eta$. For in this vicinity the term magnitudes can be approximated by

$$A_n \sim a(n-\eta)^2 + b \tag{8}$$

where a and b are certainly positive. From (3),

$$R_n \sim -\int A_{n+\frac{1}{2}}\,dn \sim -\tfrac{1}{3}a(n + \tfrac{1}{2} - \eta)^3 - b(n + \tfrac{1}{2} - \eta) + c. \tag{9}$$

Neglecting c, about which we know next to nothing, (9) points to R_η as the first negative remainder. Returning to (3) we have therefore

$$R_{\eta-1} + |R_\eta| = A_\eta,$$

leading to $-R_\eta < A_\eta$ and the interpretation

$$S_\infty \simeq A_0 + A_1 + A_2 + \ldots. + A_{\eta-1} + TA_\eta \tag{10}$$

where $0 < T < 1$ on the simplifying assumptions made. The coefficient T may lie somewhat outwith these bounds if c is significant, or if two successive A_n are nearly equal so η cannot be approximated by an integer. Hence arguments independent of the detailed form of late terms cannot reliably interpret S_∞ to an accuracy better than about the least term A_η.

The foregoing considerations imply a major difference between alternating and single-sign asymptotic series in respect of ease in interpretation, a difference originating from the slow variation of $|R_n|$ near the least term in the former compared with the fast variation throughout in the latter. Thus while transformations known to accelerate convergence of slowly-convergent series can often be usefully applied to alternating asymptotic series when the terms are expressed numerically, they are generally useless and at best untrustworthy for single-sign asymptotic series. For recent work on such transformations, see Goodwin and Staton (1948), MacFarlane (1949), Cherry (1950), van der Corput (1951), Rosser (1951), and Shanks (1955). Convergence proofs of even the most sophisticated iterative methods of bettering convergence, such as continued fractions (Stieltjes 1894, 1895, Wall 1948) and Padé sequences (review, Baker 1965), fail when terms in an asymptotic series are all of the same sign and phase.

2. Airey's Method of Terminating Alternating Asymptotic Series

Airey's (1937) notably successful compromise between simplicity, accuracy and generality will be illustrated by application to the asymptotic power series in Chapter I (6) for the error function $\phi(x)$ when $|\text{ph}\, x| < \frac{1}{4}\pi$. The sum of the nth and later terms can be rearranged as follows:

$$\begin{aligned}\sum_n^\infty \frac{(r-\frac{1}{2})!}{(-x^2)^r} &= \frac{(n-\frac{1}{2})!}{(-x^2)^n}\left[1 - \frac{n+\frac{1}{2}}{x^2} + \frac{(n+\frac{1}{2})(n+\frac{3}{2})}{x^4} - \dots\right] \\ &= \frac{(n-\frac{1}{2})!}{(-x^2)^n}\left[\left\{1 - \frac{n}{x^2} + \frac{n^2}{x^4} - \,..\right\}\right. \\ &\quad - \frac{1}{2x^2}\left\{1 - \frac{(1+3)n}{x^2} + \frac{(1+3+5)n^2}{x^4} - \,..\right\} \\ &\quad \left. + \frac{1}{4x^4}\left\{1{\cdot}3 - \frac{(1{\cdot}3 + 1{\cdot}5 + 3{\cdot}5)n}{x^2} + \,..\right\} - \dots\right]. \qquad (11)\end{aligned}$$

The first series in (11) is an alternating geometrical progression in $v = n/x^2$, and subsequent series are increasingly complicated derivatives of this. Since

$$1 + 3 + 5 + \dots + (2m+1) = (m+1)^2,$$

the second is

$$\sum_0^\infty (m+1)^2(-v)^m = \frac{d}{dv}\, v \frac{d}{dv}\, v \sum_0^\infty (-v)^m = \frac{1-v}{(1+v)^3}.$$

In the third,

$$1{\cdot}3 + 1{\cdot}5 + \ldots + 3{\cdot}5 + 3{\cdot}7 + \ldots + \ldots (2m+1).(2m+3)$$
$$= \tfrac{1}{2}[\{1 + 3 + \ldots + (2m+3)\}^2 - \{1^2 + 3^2 + \ldots + (2m+3)^2\}]$$
$$= \tfrac{1}{6}(m+1)(m+2)(3m^2 + 11m + 9),$$

leading to the required sum $(3 - 8v + v^2)/(1 + v)^5$. Defining the "terminant" $T_n(x)$ as the function which when multiplied by the nth term in an asymptotic series would correctly terminate that series, we have for this example

$$T_n(x) = \frac{x^2}{x^2+n}\left\{1 - \frac{x^2 - n}{2(x^2+n)^2} + \frac{3x^4 - 8nx^2 + n^2}{4(x^2+n)^4} \cdots\right\}. \tag{12}$$

When x is real, $\phi(x)$ is represented by an alternating asymptotic power series in which the numerically least term is at the nearest integer n to x^2; note then from (12) the slow variation of the terminant and the closeness to $\frac{1}{2}$. When x is imaginary, the asymptotic power series becomes single-sign and the numerically least term is at the nearest integer n to $-x^2$; (12) then completely fails.

3. Termination Methods Requiring Supplementary Information

Airey's interpretation applies to an alternating asymptotic power series in which the general term is known; it does not relate to the specific method by which that series was derived. Naturally, similar results can be found by methods which *do* depend on the derivation process. We indicate the main methods below, because they are all of at least historical importance through their possibility of modification for single-sign series, unlike Airey's:

(i), (ii) Derivation from an integral representation of its terminated asymptotic series either through repeated integration by parts (e.g. Stieltjes 1886) or through introduction of a terminated expansion (Dingle 1957, Jeffreys 1958), followed by special expansion of the remainder integral.

(iii) Conversion of the function into a Mellin representation, evaluation of the residue at each of its first n poles, and coalescence of the remainder as a calculable integral (Dingle 1957).

(iv) Derivation from a homogeneous differential equation of the first part of an asymptotic series, and of a terminating series to this found from the inhomogeneous differential equation satisfied by the difference between the function and the first part of the asymptotic series (Miller 1952, Slater 1960). For a single-sign asymptotic series, further information has to be introduced, e.g. from a difference equation obeyed by the function.

Viewed from the purely theoretical angle, these approaches are disappointingly uninformative; being based on terminated asymptotic series plus remainders, they shed no direct light on problems of divergence and interpretation in asymptotics; and since remainders for alternating and single-sign series have to be evaluated differently (e.g. (iv)), nothing accrues to our understanding of phase dependence and Stokes discontinuities. On the practical side, ease of application and accuracy of results vary greatly from function to function. Illustrating this by method (ii), it is usually difficult to reduce to tractable form the double integral produced by the single-integral form of remainder in a terminated Taylor series (Whittaker and Watson, 1927, pp. 95–96); and almost impossible to do this for the triple integral produced by the double-integral form of remainder in a Bürmann series (loc. cit., pp. 130–131)—in effect ruling out application to stationary-point expansions. The root of the trouble lies in the altogether higher order of complexity of a terminated series with remainder, compared with the "pure" unterminated asymptotic series. In this connection, van der Corput's remarks (1954, pp. 3–4) on *direct* calculations of even the upper bounds to remainders are notably perspicacious: "It is often very difficult to find such an upper bound, or the upper bound obtained may be so weak that it is quite useless .... In complicated problems it is advisable to restrict oneself first to pure asymptotic expansions without trying to find a numerical upper bound for the absolute value of the error term .... Moreover, there reigns in pure asymptotics a—partially still unrevealed—harmony which enables us under general conditions to write down almost immediately the required asymptotic expansions."

4. Termination by Borel Summation

Following successful termination of the asymptotic power series for Fermi–Dirac and Bose–Einstein integrals by methods (ii) and (iii) above (Dingle 1957), the author embarked on a systematic search for a simpler and more general procedure. This search was guided by features noted as common to the pursuit of (ii) and (iii), and by the form of the common answer. By far the best method found was by a form of Borel summation restricted to late terms of "pure" unterminated asymptotic series. Applications to numerous alternating and single-sign power series were published in a series of six papers (Dingle 1958, 1959).

Starting from the equality

$$r! = \int_0^\infty e^{-\varepsilon}\, \varepsilon^r\, d\varepsilon,$$

Borel (1899) observed that the sum of a divergent series could be defined as

$$\sum_0^\infty \frac{a_r}{x^r} = \int_0^\infty e^{-\varepsilon}\, d\varepsilon \sum_0^\infty \frac{a_r}{r!}\left(\frac{\varepsilon}{x}\right)^r, \tag{13}$$

provided the summation on the right converges for *some* range of ε/x and is then extended by analytic continuation, and provided the integral converges. Now as we have seen (e.g. Chapter I, Section 4), in asymptotic series $|a_r|$ behaves essentially like $(r+\alpha)!$ for all but small r—or at least, the series can be written such that this is so. We therefore need to exclude early terms and modify Borel's expression to

$$\sum_n^\infty \frac{a_r}{x^r} = \int_0^\infty \varepsilon^\alpha e^{-\varepsilon}\, d\varepsilon \sum_n^\infty \frac{a_r}{(r+\alpha)!}\left(\frac{\varepsilon}{x}\right)^r. \tag{14}$$

There remains the problem of interpreting the summation under the integral sign for *all* real and positive ε. Our procedure will be to reduce this summation to a distribution of geometrical progressions. The sum to infinity of a convergent geometrical progression is the simple function

$$\frac{\text{initial term}}{1 - \text{ratio between terms}},$$

and, by analytic continuation, this function will be extended as our interpretation for all real and positive ε. Then (14) defines a unique termination to an asymptotic series.

We require a minimum of two basic terminants, one to apply over a phase range between two Stokes rays, the other on a Stokes ray—i.e. where all late terms of an asymptotic series are of the same sign and phase. These will be defined as follows:

$$\sum_n^\infty \frac{(r+\alpha)!}{(-x)^r} = \frac{(n+\alpha)!}{(-x)^n}\Lambda_{n+\alpha}(x), \qquad |\mathrm{ph}\, x| < \pi, \tag{15}$$

$$\sum_n^\infty \frac{(r+\alpha)!}{x^r} = \frac{(n+\alpha)!}{x^n}\bar{\Lambda}_{n+\alpha}(-x), \qquad x \text{ real and positive.} \tag{16}$$

Because of the frequent absence of alternate terms in series of this type, it is convenient to introduce from the start two supplementary basic terminants:

$$\sum_{n,n+2,n+4,\ldots}^\infty (-1)^{\frac{1}{2}r}\frac{(r+\alpha)!}{x^r} = (-1)^{\frac{1}{2}n}\frac{(n+\alpha)!}{x^n}\Pi_{n+\alpha}(x),$$

$$|\mathrm{ph}\, x| < \tfrac{1}{2}\pi, \qquad n \text{ even}, \tag{17}$$

with the obvious adjustment when n is odd, and

$$\sum_{n,n+2,n+4,\ldots}^{\infty} \frac{(r+\alpha)!}{x^r} = \frac{(n+\alpha)!}{x^n} \overline{\Pi}_{n+\alpha}(ix), \qquad x \text{ real and positive.} \tag{18}$$

The consistency relations for these are seen to be

$$\Pi_s(x) = \tfrac{1}{2}\{\Lambda_s(-ix) + \Lambda_s(ix)\}, \qquad |\text{ph}\, x| < \tfrac{1}{2}\pi, \tag{19}$$

$$\overline{\Pi}_s(ix) = \tfrac{1}{2}\{\overline{\Lambda}_s(-x) + \Lambda_s(x)\}, \qquad |\text{ph}\, x| < \pi. \tag{20}$$

Carrying through the interpretative procedure of replacing the factorial by its integral representation, interchanging the order of summation and integration, and summing as for a convergent geometrical progression,

$$\sum_n^{\infty} \frac{(r+\alpha)!}{(-x)^r} = \int_0^{\infty} \varepsilon^{\alpha} e^{-\varepsilon}\, d\varepsilon \sum_n^{\infty} \left(\frac{\varepsilon}{-x}\right)^r = \int_0^{\infty} \varepsilon^{\alpha} e^{-\varepsilon}\, d\varepsilon \frac{[\varepsilon/(-x)]^n}{1+\varepsilon/x}$$
$$= \frac{1}{(-x)^n} \int_0^{\infty} \frac{\varepsilon^{n+\alpha} e^{-\varepsilon}\, d\varepsilon}{1+\varepsilon/x}. \tag{21}$$

Thus in the notation (15),

$$\Lambda_s(x) = \frac{1}{s!} \int_0^{\infty} \frac{\varepsilon^s e^{-\varepsilon}\, d\varepsilon}{1+\varepsilon/x}. \tag{22}$$

The function is extended to $\Re(s) \leqslant -1$ via its absolutely convergent expansion ((49) below), or by changing to the Hankel contour (e.g. Whittaker and Watson 1927, p. 244).

Moreover, once we demand extension to asymptotics of separate accountability of real and imaginary quantities as in the rest of mathematical analysis, the reality of all terms in (16) dictates the modification

$$\overline{\Lambda}_s(-x) = \frac{1}{s!} P \int_0^{\infty} \frac{\varepsilon^s e^{-\varepsilon}\, d\varepsilon}{1-\varepsilon/x}, \qquad x \text{ real and positive.} \tag{23}$$

Here P denotes the principal value as usual. The function is extended into the complex plane via its absolutely convergent expansion ((50) below), or alternatively through the form

$$\overline{\Lambda}_s(-x) = \frac{x^{s+1}}{s!} P \int_0^{\infty} \frac{\sigma^s e^{-\sigma x}\, d\sigma}{1-\sigma}, \qquad \Re(x) > 0. \tag{24}$$

Either repeating the interpretative procedure or relying on (19) and (20), the integral representations for the supplementary basic terminants are found to be

$$\Pi_s(x) = \frac{1}{s!}\int_0^\infty \frac{\varepsilon^s e^{-\varepsilon}\, d\varepsilon}{1 + \varepsilon^2/x^2}, \tag{25}$$

$$\bar{\Pi}_s(ix) = \frac{1}{s!} P \int_0^\infty \frac{\varepsilon^s e^{-\varepsilon}\, d\varepsilon}{1 - \varepsilon^2/x^2}, \qquad x \text{ real and positive.} \tag{26}$$

To make the interpretation general, we need to show how the summation $\sum_n^\infty \{a_r/(r+\alpha)!\}(\varepsilon/x)^r$ in (14) can be reduced to a distribution of geometrical progressions. As we shall see in the first section of the next chapter, deviations from the precise proportionality $|a_r| \propto (r+\alpha)!$ can be dealt with in at least five different ways. In every one the result is expressed as some distribution over a basic terminant, so no fresh problem of interpretation emerges.

To conclude this section with a historical note, before pressing far in applying these results the author searched the early literature deriving from Borel's work so as to check whether these interpretations were altogether new, and discovered a "near-miss" by Watson (1911). He had applied Borel's *original* form (13) to the terms—*including early terms*—of an asymptotic series away from any of its Stokes rays. His approach looked promising only for a relatively small class of alternating series deriving from exponential integrals, and it does not seem to have attracted more than theoretical interest—unlike Airey's later method, which was extensively called upon in early table-making and is still used (e.g. Murnaghan and Wrench 1963, Murnaghan 1965). Our various extensions when applying the idea of Borel summation to asymptotics in 1957–8 were, individually, small and straightforward; partly by design and partly by good fortune, they resulted in an interpretative procedure of remarkable theoretical and practical power (e.g. Wrench 1970, 1971).

5. NATURE OF THE INTERPRETATION

Soon after this interpretative procedure had been developed, Dr J. C. P. Miller (private communication) posed a most interesting question: does an asymptotic expansion have an intrinsic meaning, or has a meaning been imposed on it? We must first dispose of the semantic side issue as to what "meaning" is to imply in this context. Clearly, as a consequence of its ultimate divergence a component asymptotic series has no immediately appreciated meaning; but neither has any code—unless correctly decoded it is gibberish or wrong. "Meaning" here is to be understood as latent meaning, turning to immediately appreciated meaning on decoding.

Two arguments then show that asymptotic *power* expansions, at least, indeed do have an intrinsic rather than an imposed meaning. The first is the more general argument, but does not go quite as far as one could wish. Confining attention to expansions composed of one or more power series, a chosen function at a chosen phase has a *unique* asymptotic expansion,† whether derived from an integral representation for the function, a differential equation it satisfies, or however else. The asymptotic expansion must therefore carry *a* meaning which is altogether independent of the process of derivation. The one reservation in this argument rests on the indefinite article "a"; it would be logically possible, though unlikely in the extreme, for every process of derivation to destroy the same element of information, leading to a meaning which is apparently unique, but is nevertheless not coincident with the meaning of the function expanded. (No such case has come to light).

In the second argument, we suppose the function has been expressed as a Laplace-type integral $L(z) = \int_0^\infty F(\varepsilon)\, e^{-\varepsilon z}\, d\varepsilon$, a form embracing a very wide class of solutions to differential, difference and integral equations. (Moreover, essentially the same argument can be applied to representations containing additional exponentials, e.g. Fermi–Dirac and Bose–Einstein integrals). By the Darboux theorem (Chapter I, Section 1, and Chapter VII, Section 2), the late coefficients in the expansion of $F(\varepsilon)$ in rising powers of ε are determined by the singularity of $F(\varepsilon)$ closest to $\varepsilon = 0$. For the present the singularity will be supposed to lie at $-\varepsilon_0$ where $\varepsilon_0 > 0$. In estimating late terms in the asymptotic expansion of such an integral, it therefore suffices to adopt an approximation of the type $F(\varepsilon) = c(1 + \varepsilon/\varepsilon_0)^{-\alpha-1}$, where α is zero or a positive integer if the singularity is a pole, and fractional (positive or negative) if it is a branch point. Now

$$\int_0^\infty \varepsilon^\alpha (1 + \varepsilon/x)^{-1}\, e^{-\varepsilon}\, d\varepsilon = x^{\alpha+1}\, e^x \int_x^\infty d\zeta \int_0^\infty \sigma^\alpha\, e^{-(1+\sigma)\zeta}\, d\sigma$$

$$= \alpha!\, x^{\alpha+1}\, e^x \int_x^\infty \zeta^{-\alpha-1}\, e^{-\zeta}\, d\zeta = \alpha!\, x^{\alpha+1} \int_0^\infty (x + v)^{-\alpha-1}\, e^{-v}\, dv$$

$$= \alpha!\, x\, \varepsilon_0{}^{-1} \int_0^\infty (1 + \varepsilon/\varepsilon_0)^{-\alpha-1}\, e^{-\varepsilon x/\varepsilon_0}\, d\varepsilon, \tag{27}$$

† This follows from individual equality of coefficients of like powers on comparing two power series for the same function over a range in magnitude of the variable at a fixed phase. For a simple demonstration applied to a single asymptotic power series according to Poincaré's prescription, see Whittaker and Watson 1927, p. 154.

so, in this approximation appropriate to late terms,

$$L(z) \simeq cz^{-1}\, \Lambda_\alpha(\varepsilon_0 z) \tag{28}$$

by (22). Hence for all such Laplace-type integrals, the problems to be overcome in deriving and interpreting late terms can be reduced to the same problems for the much simpler basic terminant. Now examination of the *derivation* of the asymptotic power series—

$$\int_0^\infty \frac{\varepsilon^\alpha e^{-\varepsilon}\, d\varepsilon}{1+\varepsilon/x} = \int_0^\infty \sum_0^\infty \left(-\frac{\varepsilon}{x}\right)^r \varepsilon^\alpha e^{-\varepsilon}\, d\varepsilon = \sum_0^\infty \frac{1}{(-x)^r}\int_0^\infty \varepsilon^{r+\alpha} e^{-\varepsilon}\, d\varepsilon$$

$$= \sum_0^\infty \frac{(r+\alpha)!}{(-x)^r} \tag{29}$$

—reveals three steps, in turn

Expansion of $(1+\varepsilon/x)^{-1}$ as $\sum(-\varepsilon/x)^r$, continued formally into the region $\varepsilon/x > 1$.

Interchange in order of integration and summation.

Replacement of $\int_0^\infty \varepsilon^{r+\alpha} e^{-\varepsilon}\, d\varepsilon$ by $(r+\alpha)!$.

In the *interpretation* (21) of late terms, exactly the same sequence of steps has been retraced backwards.

When $F(\varepsilon)$ has a singularity on the positive real axis, the Laplace-type integral needs additional specification. In particular, a function $L(z)$ which is to be real for positive real z is specified by $L(z) = P\int_0^\infty F(\varepsilon)\, e^{-\varepsilon z}\, d\varepsilon$. Supposing the singularity of $F(\varepsilon)$ to lie at $\varepsilon_0 > 0$, the analogue to (28) is then seen from (23) to be

$$L(z) \simeq cz^{-1}\, \overline{\Lambda}_\alpha(-\varepsilon_0 z), \qquad \mathrm{ph}\, z = 0. \tag{30}$$

The situation therefore corresponds to specification of a function on a Stokes ray of its asymptotic expansion. On this ray, the derivation and interpretation

$$P\int_0^\infty \frac{\varepsilon^\alpha e^{-\varepsilon}\, d\varepsilon}{1-\varepsilon/x} \rightleftarrows \sum_0^\infty \frac{(r+\alpha)!}{x^r}, \qquad \mathrm{ph}\, x = 0, \tag{31}$$

again involve three steps each, differing only in direction of execution.

Thus, both on and off Stokes rays, during derivation and interpretation the information content is the same at every stage. In particular, the "pure" unterminated asymptotic expansion possesses the full meaning of a standardly-coded representation of the function expanded.

All correct mathematical processes are in essence tautological, and the foregoing demonstration that interpretation reverses the path of one method of derivation in no way implies aimless redundancy in some of the processes of asymptotics! The paramount achievement has been to show *how* a vast range of functions can be transformed to a common pattern.

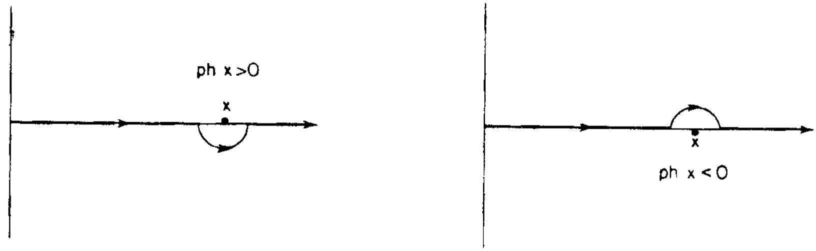
$$\text{Function} = \text{first } n \text{ terms of asymptotic series} + n\text{'th term} \times \text{terminant}, \quad (32)$$

in which the terminant is expressed (adequately at least for large n) by "basic terminants"—quantities with known theoretical properties and already tabulated for real variable (Dingle 1958). As we have seen (e.g. Chapter I, Section 1), it is the Darboux theorem which lies at the heart of reasons *why* this common pattern exists. No rigorous theorems are yet available on either the general existence or applicability of (32). Until they are, full rigour is attainable only by confirmation in specific applications. Fortunately this is simple for functions directly defined by integrals or equations (but extremely difficult for zeros, eigenvalues and other quantities defined through auxiliary conditions). Given an integral, we may determine its complete asymptotic expansion and then confirm rigorously that this formally exactly obeys—over a given phase range—those relations satisfied by the original integral which do not involve any numerical value of the variable; we can next interpret late terms of the expansion and then rigorously confirm the correctness of the result by making n in (32) zero, thereby leaving an integral representation. Given a differential, difference or integral equation, we may determine its complete asymptotic expansion, and rigorously confirm by substitution that it formally exactly satisfies the equation; we can next interpret late terms of the expansion and then rigorously confirm the correctness of the result by substitution in the original equation and reference to boundary conditions.

Within the restricted realm of purely numerical analysis, interpretation can be reduced to the simple assumption of smooth interpolation between terminants of asymptotic series with similar late terms. For as far as numerical work is concerned, we simply *label* basic terminants by the parameters of the general late terms of selected functions (in practice the exponential integrals) and tabulate them from the known function values; then interpolate to evaluate terminants, and hence function values, for more complicated functions.

6. Stokes Discontinuities and Continuation Rules for Asymptotic Expansions

Owing to the presence in (22) of the pole at $\varepsilon = -x$, there is a discontinuity in $\Lambda_s(-x)$ when $\text{ph}\, x = 0$; for $\text{ph}\, x$ slightly positive the path of integration in the complex ε-plane has to circumnavigate the pole by an anticlockwise semicircular diversion below the real axis, whereas for ph x slightly negative the semicircular diversion must be clockwise above the real axis (see diagrams).



The contributions to $\Lambda_s(-x)$ from these semicircular diversions alone are respectively $(i\pi)(-x^{s+1}e^{-x}/s!)$ and $(-i\pi)(-x^{s+1}e^{-x}/s!)$. By definition of an integral's principal value—stopping infinitesimally short of the pole and restarting the same distance beyond—$\overline{\Lambda}_s(-x)$ in (23) excludes such a contribution. Hence

$$\overline{\Lambda}_s(-x) = \Lambda_s(-x) \pm i\pi x^{s+1} e^{-x}/s! \qquad \left.\begin{matrix} 0 < \mathrm{ph}\, x < 2\pi \\ 0 > \mathrm{ph}\, x > -2\pi \end{matrix}\right\} . \tag{33}$$

Also, by (20),

$$\overline{\Pi}_s(ix) = \Pi_s(ix) \pm \tfrac{1}{2}i\pi x^{s+1} e^{-x}/s! \qquad \left.\begin{matrix} 0 < \mathrm{ph}\, x < \pi \\ 0 > \mathrm{ph}\, x > -\pi \end{matrix}\right\} . \tag{34}$$

These two relations provide the key to quantitative prediction of Stokes discontinuities in form of an asymptotic expansion. When all late terms of a component series are of the same sign and phase, i.e. on one of its Stokes rays, the series is to be completed by a distribution over basic terminants such as $\overline{\Lambda}_s(-x)$; on moving off the ray, nominally the same component series is to be completed by this distribution over $\Lambda_s(-x)$; the balance, comprising this distribution over $\pm i\pi x^{s+1} e^{-x}/s!$, is the Stokes discontinuity in form, the upper or lower signs pertaining when leaving the ray towards higher or lower phases respectively. Thus a function can be evaluated from its complete asymptotic expansion in two ways: either we can select the asymptotic expansion appropriate to a phase at which a component series is on a Stokes ray, and interpret with $\overline{\Lambda}$ (or $\overline{\Pi}$); or we can select the expansion appropriate to a phase sector between Stokes rays and interpret with Λ (or Π). The first way is usually the simplest, because asymptotic expansions have fewest components on Stokes rays. This is well illustrated by our standard example in Chapter I of the error function. According to I (10) the complete expansion on a Stokes ray is

$$\phi(y\,e^{\frac{1}{2}i\pi}) = \frac{2i}{\sqrt{\pi}}\int_0^y e^{v^2}\,dv = \frac{i\,e^{y^2}}{y\pi}\sum_0^\infty \frac{(r-\frac{1}{2})!}{y^{2r}}, \qquad \operatorname{ph} y = 0. \tag{35}$$

By (16), the interpretation is

$$\phi(y\,e^{\frac{1}{2}i\pi}) = \frac{i\,e^{y^2}}{y\pi}\left\{\sum_0^{n-1}\frac{(r-\frac{1}{2})!}{y^{2r}} + \frac{(n-\frac{1}{2})!}{y^{2n}}\,\overline{\Lambda}_{n-\frac{1}{2}}(-y^2)\right\}, \tag{36}$$

and specification of the phase (or phase sector) is no longer necessary. Introduction of (33) produces the alternative forms

$$\phi(y\,e^{\frac{1}{2}i\pi}) = \frac{i\,e^{y^2}}{y\pi}\left\{\sum_0^{n-1}\frac{(r-\frac{1}{2})!}{y^{2r}} + \frac{(n-\frac{1}{2})!}{y^{2n}}\,\Lambda_{n-\frac{1}{2}}(-y^2)\right\}$$

$$\mp 1, \qquad \left.\begin{array}{l} 0 < \operatorname{ph} y < \pi \\ 0 > \operatorname{ph} y > -\pi \end{array}\right\}, \tag{37}$$

which are more conveniently written

$$\phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}\,du = -\frac{e^{-z^2}}{z\pi}\left\{\sum_0^{n-1}\frac{(r-\frac{1}{2})!}{(-z^2)^r} + \frac{(n-\frac{1}{2})!}{(-z^2)^n}\,\Lambda_{n-\frac{1}{2}}(z^2)\right\}$$

$$\mp 1, \qquad \left.\begin{array}{ll} \frac{1}{2}\pi < \operatorname{ph} z < \frac{3}{2}\pi & (38) \\ \frac{1}{2}\pi > \operatorname{ph} z > -\frac{1}{2}\pi & (39) \end{array}\right\}.$$

Of these, (39) is the interpretation of I (6), and (38) that of I (13).

Rules

As we have seen in Section 4, the terminant of some specific asymptotic series at a specific phase can be expressed as a distribution over basic terminants of a single kind: Λ or $\overline{\Lambda}$ or Π or $\overline{\Pi}$. The rules A to E of Chapter I, Section 2, for locating Stokes rays and continuing asymptotic expansions across them, follow from the salient behavioural pattern of each kind of basic terminant as the phase changes. As such, they summarize the main pragmatical features of relations already established in this and the two preceding sections, and require only brief derivation and review.

The difference between (16) appropriate to a Stokes ray, and (15) appropriate to a whole phase sector between Stokes rays—and the similar difference between (18) and (17)—leads to the first rule:

A *The Stokes rays for an asymptotic series are determined by those phases for which successive late terms are homogeneous in phase and all of the same sign.*

All the other main rules come from combining (15) and (16) with (33) to get the following relation between interpretations of an asymptotic power series on and immediately to one side of a Stokes ray:

$$\sum_{n}^{\infty} \frac{(r+\alpha)!}{x^r}\bigg|_{\text{ph}\,x=0} = \sum_{n}^{\infty} \frac{(r+\alpha)!}{x^r}\bigg|_{\text{ph}\,x\to\pm 0} \pm i\pi\, x^{\alpha+1}\, e^{-x}; \tag{40}$$

and from the similar combination of (17) and (18) with (34). From the right-hand side of (40), the quotient asymptotic series/associated function is dominated by the exponential e^x; regarded as a function of phase, the magnitude $|e^x| = e^{|x|\cos \text{ph}\, x}$ is at a peak when $\text{ph}\, x = 0$, the Stokes ray, so furnishing the second rule:

B *The Stokes rays for an asymptotic series are determined by those phases for which the series (including its multiplier) attains peak exponential dominance over its associated function.*

As seen from (40), the changes in interpretation of an asymptotic series as its Stokes ray is traversed do not involve any change in the multiplier of the series; hence

C *The factor multiplying an asymptotic series is analytically continued across its own Stokes ray.*

Again from (40), the discontinuity in form on crossing the Stokes ray, say from $\text{ph}\, x \to +0$ to $\text{ph}\, x \to -0$, is the pure imaginary $-2i\pi\, x^{\alpha+1}\, e^{-x}$ proportional to the associated function, while at the Stokes ray $\text{ph}\, x = 0$ every term in the series $\sum(r+\alpha)!/x^r$ is real; hence

D *On crossing its Stokes ray, an asymptotic series generates a discontinuity in form which is, on the ray, $\frac{1}{2}\pi$ out of phase with the series and proportional to its associated function.*

The $\pm$ signs in (40) lead immediately to the final main rule:

E *Half the discontinuity in form occurs on reaching the Stokes ray, and half on leaving it the other side.*

In rules B and D the Stokes discontinuity is being expressed as an associated *function*, not *series*; the distribution over $\pm i\pi\, x^{\alpha+1}\, e^{-x}$ in (40) is thus being treated as an analytic function, e.g. an integral representation, not as an expansion which might itself exhibit Stokes discontinuities (if asymptotic) or become divergent (if only conditionally convergent); this means that *the rules are to be applied to one Stokes discontinuity at a time.* If, as is often convenient in practice, the distribution and thereby the associated function are in some instance actually expressed as asymptotic series, and by mischance one of their Stokes rays were to coincide with a Stokes ray of the basic series (e.g. $\sum(r+\alpha)!/x^r$), the first application of the rules would give the Stokes discontinuity as an asymptotic series, which in turn would have a Stokes discontinuity found by a second application of the rules. In principle, instances could occur where this went on indefinitely, each newly-found contribution to the discontinuity being exponentially smaller than the one found at the previous application of the rules.

7. Properties and Evaluation of Basic Terminants

Expression as incomplete factorial functions

$$\Lambda_{s-1}(x) = x\int_0^\infty (1+u)^{-s} e^{-ux}\,du = x^s e^x\,[-s, x]!, \tag{41}$$

$$\bar{\Lambda}_{s-1}(-x) = x\,P\int_0^\infty (1-u)^{-s} e^{-ux}\,du = P(-x)^s e^{-x}\,[-s, -x]!, \tag{42}$$

where $P(1-u)^{-s} \equiv (u-1)^{-s}\cos\pi s$.

Recurrence and differential relations. Proofs of the following results are straightforward:

$$s\Lambda_s(x) = x\{1 - \Lambda_{s-1}(x)\}, \qquad s\bar{\Lambda}_s(-x) = x\{\bar{\Lambda}_{s-1}(-x) - 1\}, \tag{43}$$

$$s(s-1)\,\Pi_s(x) = x^2\{1 - \Pi_{s-2}(x)\}, \qquad s(s-1)\,\bar{\Pi}_s(ix) = x^2\{\bar{\Pi}_{s-2}(ix) - 1\}. \tag{44}$$

Defining "reduced derivatives" by

$$\Lambda^{(t)}(x) = \frac{1}{t!}\left(\frac{d}{dx}\right)^t \Lambda(x), \qquad \bar{\Lambda}^{(t)}(-x) = \frac{1}{t!}\left(\frac{d}{dx}\right)^t \bar{\Lambda}(-x), \tag{45}$$

and similarly for $\Pi^{(t)}(x)$ and $\bar{\Pi}^{(t)}(ix)$,

$$\left.\begin{aligned} x\Lambda_s^{(1)}(x) &= (s+x+1)\Lambda_s(x) - x,\\ 2x\Lambda_s^{(2)}(x) &= (s+x)\Lambda_s^{(1)}(x) + \Lambda_s(x) - 1,\\ tx\Lambda_s^{(t)}(x) &= (s+x+2-t)\Lambda_s^{(t-1)}(x) + \Lambda_s^{(t-2)}(x), \qquad t > 2. \end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned} x\overline{\Lambda}_s^{(1)}(-x) &= (s-x+1)\overline{\Lambda}_s(-x)+x, \\ 2x\overline{\Lambda}_s^{(2)}(-x) &= (s-x)\,\overline{\Lambda}_s^{(1)}(-x)-\overline{\Lambda}_s(-x)+1, \\ tx\,\overline{\Lambda}_s^{(t)}(-x) &= (s-x+2-t)\,\overline{\Lambda}_s^{(t-1)}(-x)-\overline{\Lambda}_s^{(t-2)}(-x), \qquad t>2\cdot \end{aligned}\right\} \quad (47)$$

$$\left.\begin{aligned} tx\,\Pi_s^{(t)}(x) &= (s+2-t)\,\Pi_s^{(t-1)}(x)-(s+1)\,\Pi_{s+1}^{(t-1)}(x) \\ tx\,\overline{\Pi}_s^{(t)}(ix) &= (s+2-t)\,\overline{\Pi}_s^{(t-1)}(ix)-(s+1)\,\overline{\Pi}_{s+1}^{(t-1)}(ix) \end{aligned}\right\} t \geqslant 1. \quad (48)$$

Absolutely convergent expansions. The following expansions are equivalent to the integral representations (22)–(26):

$$\Lambda_s(x) = \begin{cases} \dfrac{x}{s}\left\{1-\dfrac{x}{s-1}+\dfrac{x^2}{(s-1)(s-2)}-\ldots\right\}-\dfrac{\pi x^{s+1}e^x}{s!\sin\pi s}, \\ \qquad s = \text{non-integer}, \\ \dfrac{x}{s}\left\{1-\dfrac{x}{s-1}+\ldots+\dfrac{(-x)^{s-1}}{(s-1)!}\right\}-\dfrac{(-x)^{s+1}}{s!}\sum\limits_{t=0}^{\infty}\dfrac{x^t(\Psi(t)-\ln x)}{t!}, \\ \qquad s = \text{integer}, \end{cases} \quad (49)$$

where $\Psi(t) = d\ln(t!)/dt$.

$$\overline{\Lambda}_s(-x) = \begin{cases} -\dfrac{x}{s}\left\{1+\dfrac{x}{s-1}+\dfrac{x^2}{(s-1)(s-2)}+\ldots\right\}+\dfrac{\pi x^{s+1}e^{-x}}{s!\tan\pi s}, \\ \qquad s = \text{non-integer}, \\ -\dfrac{x}{s}\left\{1+\dfrac{x}{s-1}+\ldots+\dfrac{x^{s-1}}{(s-1)!}\right\} \\ \qquad -\dfrac{x^{s+1}}{s!}\sum\limits_{t=0}^{\infty}\dfrac{(-x)^t(\Psi(t)-\ln x)}{t!}, \qquad s = \text{integer}. \end{cases} \quad (50)$$

$$\Pi_s(x) = \begin{cases} \dfrac{x^2}{s(s-1)}\left\{1 - \dfrac{x^2}{(s-2)(s-3)} + \dfrac{x^4}{(s-2)(s-3)(s-4)(s-5)} \right. \\ \qquad \left. - \dots\right\} + \dfrac{\pi x^{s+1}\sin(x+\frac{1}{2}\pi s)}{s!\sin \pi s}, \qquad s = \text{non-integer}, \\ \dfrac{x^2}{s(s-1)}\left\{1 - \dfrac{x^2}{(s-2)(s-3)} + \dots + \dfrac{(-x^2)^{\frac{1}{2}s-1}}{(s-2)!}\right\} \\ \qquad + \dfrac{(-1)^{\frac{1}{2}s}x^{s+1}}{s!}\left\{\frac{1}{2}\pi\cos x - \displaystyle\sum_{1,3,..}^{\infty} \dfrac{(-1)^{\frac{1}{2}(t-1)}x^t(\Psi(t)-\ln x)}{t!}\right\}, \\ \qquad s = \text{even integer}, \\ \dfrac{x^2}{s(s-1)}\left\{1 - \dfrac{x^2}{(s-2)(s-3)} + \dots + \dfrac{(-x^2)^{\frac{1}{2}(s-3)}}{(s-2)!}\right\} \\ \qquad + \dfrac{(-1)^{\frac{1}{2}(s-1)}x^{s+1}}{s!}\left\{\frac{1}{2}\pi\sin x + \displaystyle\sum_{0,2,..}^{\infty} \dfrac{(-1)^{\frac{1}{2}t}x^t(\Psi(t)-\ln x)}{t!}\right\}, \\ \qquad s = \text{odd integer}. \end{cases} \tag{51}$$

$$\Pi_s(ix) = \begin{cases} -\dfrac{x^2}{s(s-1)}\left\{1 + \dfrac{x^2}{(s-2)(s-3)} + \right. \\ \qquad \left. + \dfrac{x^4}{(s-2)(s-3)(s-4)(s-5)} + \dots\right\} \\ \qquad - \dfrac{\pi x^{s+1}}{2(s!)\sin \pi s}(e^x - e^{-x}\cos \pi s), \qquad s = \text{non-integer}, \\ -\dfrac{x^2}{s(s-1)}\left\{1 + \dfrac{x^2}{(s-2)(s-3)} + \dots + \dfrac{(x^2)^{\frac{1}{2}s-1}}{(s-2)!}\right\} \\ \qquad + \dfrac{x^{s+1}}{s!}\displaystyle\sum_{1,3,..}^{\infty} \dfrac{x^t(\Psi(t)-\ln x)}{t!}, \qquad s = \text{even integer}, \\ -\dfrac{x^2}{s(s-1)}\left\{1 + \dfrac{x^2}{(s-2)(s-3)} + \dots + \dfrac{(x^2)^{\frac{1}{2}(s-3)}}{(s-2)!}\right\} \\ \qquad - \dfrac{x^{s+1}}{s!}\displaystyle\sum_{0,2,..}^{\infty} \dfrac{x^t(\Psi(t)-\ln x)}{t!}, \qquad s = \text{odd integer}. \end{cases} \tag{52}$$

Continuation formulae. From the preceding expansions:

$$\left.\begin{aligned}\Lambda_s(x\,e^{\pm 2i\pi}) &= \Lambda_s(x) \mp 2i\pi\, e^{\pm i\pi s}\, x^{s+1}\, e^{x}/s\,!,\\ \overline{\Lambda}_s(-x\,e^{\pm 2i\pi}) &= \overline{\Lambda}_s(-x) \pm i\pi(e^{\pm 2i\pi s}+1)\, x^{s+1}\, e^{-x}/s\,!,\end{aligned}\right\}\tag{53}$$

$$\left.\begin{aligned}\Pi_s(x\,e^{\pm i\pi}) &= \Pi_s(x) - \pi\, e^{\pm\frac{1}{2}i\pi s}\, x^{s+1}\, e^{\mp ix}/s\,!,\\ \overline{\Pi}_s(ix\,e^{\pm i\pi}) &= \overline{\Pi}_s(ix) \mp \tfrac{1}{2}i\pi\, x^{s+1}(e^{\pm i\pi s}\, e^{x} - e^{-x})/s\,!.\end{aligned}\right\}\tag{54}$$

Asymptotic expansions. When $|x| \gg 1$ and $|x| \gg s$,

$$\Lambda_s(x) = 1 - \frac{s+1}{x} + \frac{(s+1)(s+2)}{x^2} - \ldots.\tag{55}$$

When either $|x| \gg 1$ or $|s| \gg 1$, or both,

$$\begin{aligned}\Lambda_s(x) = \frac{x}{s+x}\biggl\{1 &- \frac{x}{(s+x)^2} - \frac{x(s-2x)}{(s+x)^4} - \frac{x(s^2-8sx+6x^2)}{(s+x)^6}\\ &- \frac{x(s^3-22s^2x+58sx^2-24x^3)}{(s+x)^8}\ldots\biggr\}.\end{aligned}\tag{56}$$

Analogous expressions for $\Pi(x)$ are

$$\Pi_s(x) = \begin{cases} 1 - \dfrac{(s+1)(s+2)}{x^2} + \dfrac{(s+1)(s+2)(s+3)(s+4)}{x^4} - \ldots & (57)\\[2ex] \dfrac{x^2}{s^2+x^2}\biggl\{1 + \dfrac{s(s^2-3x^2)}{(s^2+x^2)^2} + \dfrac{s^6-20s^4x^2+25s^2x^4-2x^6}{(s^2+x^2)^4} & \\[2ex] \qquad + \dfrac{s(s^8-83s^6x^2+441s^4x^4-385s^2x^6+50x^8)}{(s^2+x^2)^6} + \ldots\biggr\}. & (58)\end{cases}$$

Corresponding expansions for $\overline{\Lambda}(-x)$ may be obtained by reversing the sign of x in (55) and (56), and for $\overline{\Pi}(ix)$ by reversing the sign of x^2 in (57) and (58), but these are of limited value because of failure near $x = s$.

Values for nearly equal order and argument. From Chapter VIII, questions 2 and 3:

$$\Lambda_s(s) = \frac{1}{2} - \frac{1}{8s} + \frac{1}{32s^2} + \frac{1}{128s^3} - \frac{13}{512s^4} + \frac{47}{2{,}048s^5} + \frac{73}{8{,}192s^6} - \frac{2{,}447}{32{,}768s^7} \cdots \tag{59}$$

$$\bar{\Lambda}_s(-s) = -\frac{1}{3} + \frac{4}{135s} + \frac{8}{2{,}835s^2} - \frac{16}{8{,}505s^3} - \frac{8{,}992}{12{,}629{,}925s^4} + \frac{334{,}144}{492{,}567{,}075s^5} + \frac{698{,}752}{1{,}477{,}701{,}225s^6} \cdots. \tag{60}$$

The reduced derivatives required in a Taylor expansion about $x = s$ are given in terms of these values by (46) and (47).

A uniform expansion for $\bar{\Lambda}_s(-x)$ is derived in Chapter XI, question 4.

Values of $\bar{\Pi}_s(ix)$ are most easily calculated as the means of $\bar{\Lambda}_s(-x)$ and $\Lambda_s(x)$ (equation (20)).

Finally, from (58),

$$\left.\begin{aligned} \Pi_s(s) &= \frac{1}{2} - \frac{1}{4s} + \frac{1}{8s^2} + \frac{3}{16s^3} \cdots \\ \Pi_s^{(1)}(s) &= \frac{1}{2s} - \frac{1}{2s^2} + \frac{9}{8s^3} \cdots \\ \Pi_s^{(2)}(s) &= -\frac{1}{4s^2} + \frac{1}{2s^3} \cdots. \end{aligned}\right\} \tag{61}$$

Tabulation. The following tables are reprinted by kind permission of the Council of the Royal Society from the author's paper in Proc. Roy. Soc. **A 244**, 1958, pp. 456–475.

TABLE OF $\Lambda_s(x) = (s!)^{-1}\int_0^\infty \varepsilon^s(1 + \varepsilon/x)^{-1} e^{-\varepsilon} d\varepsilon$

x \ s	−0·5	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0	5·5	6·0
0·1	0·40557	0·20146	0·11889	0·07985	0·05874	0·04601	0·03765	0·03180	0·02750	0·02420	0·02161	0·01952	0·01779	0·01634
0·2	0·51031	0·29867	0·19588	0·14027	0·10722	0·08597	0·07142	0·06094	0·05306	0·04695	0·04209	0·03812	0·03483	0·03206
0·3	0·57474	0·36676	0·25516	0·18997	0·14897	0·12150	0·10212	0·08785	0·07696	0·06841	0·06154	0·05590	0·05119	0·04721
0·4	0·62059	0·41913	0·30353	0·23235	0·18573	0·15353	0·13028	0·11286	0·09940	0·08871	0·08005	0·07290	0·06691	0·06181
0·5	0·65568	0·46146	0·34432	0·26927	0·21856	0·18268	0·15629	0·13622	0·12053	0·10797	0·09772	0·08920	0·08203	0·07590
0·6	0·68376	0·49676	0·37949	0·30194	0·24820	0·20942	0·18043	0·15812	0·14050	0·12628	0·11460	0·10485	0·09659	0·08952
0·7	0·70692	0·52687	0·41031	0·33119	0·27519	0·23408	0·20295	0·17871	0·15941	0·14373	0·13076	0·11988	0·11063	0·10268
0·8	0·72647	0·55300	0·43765	0·35760	0·29992	0·25696	0·22402	0·19814	0·17737	0·16037	0·14625	0·13434	0·12418	0·11542
0·9	0·74326	0·57595	0·46214	0·38164	0·32272	0·27826	0·24382	0·21652	0·19445	0·17628	0·16111	0·14827	0·13727	0·12776
1·0	0·75787	0·59635	0·48426	0·40365	0·34383	0·29817	0·26247	0·23394	0·21072	0·19151	0·17539	0·16170	0·14993	0·13972
1·2	0·78218	0·63112	0·52277	0·44265	0·38178	0·33441	0·29674	0·26624	0·24112	0·22013	0·20237	0·18717	0·17403	0·16257
1·4	0·80167	0·65981	0·55531	0·47627	0·41504	0·36661	0·32757	0·29558	0·26897	0·24655	0·22743	0·21097	0·19665	0·18411
1·6	0·81773	0·68398	0·58327	0·50563	0·44452	0·39550	0·35551	0·32240	0·29462	0·27104	0·25080	0·23327	0·21795	0·20446
1·8	0·83122	0·70469	0·60761	0·53157	0·47087	0·42159	0·38098	0·34705	0·31836	0·29383	0·27266	0·25422	0·23804	0·22373
2·0	0·84274	0·72266	0·62905	0·55469	0·49461	0·44531	0·40432	0·36979	0·34039	0·31510	0·29316	0·27396	0·25703	0·24200
2·5	0·86539	0·75881	0·67304	0·60296	0·54494	0·49630	0·45506	0·41975	0·38924	0·36265	0·33931	0·31867	0·30031	0·28388
3·0	0·88213	0·78625	0·70721	0·64125	0·58558	0·53813	0·49730	0·46187	0·43089	0·40360	0·37941	0·35784	0·33850	0·32108
3·5	0·89505	0·80787	0·73463	0·67246	0·61921	0·57319	0·53311	0·49794	0·46689	0·43930	0·41464	0·39249	0·37250	0·35438
4·0	0·90536	0·82538	0·75717	0·69847	0·64755	0·60306	0·56391	0·52925	0·49838	0·47075	0·44588	0·42340	0·40300	0·38440
4·5	0·91377	0·83989	0·77606	0·72051	0·67181	0·62886	0·59073	0·55671	0·52620	0·49870	0·47380	0·45117	0·43053	0·41162
5·0	0·92078	0·85211	0·79215	0·73944	0·69284	0·65138	0·61432	0·58102	0·55097	0·52372	0·49893	0·47628	0·45552	0·43643
5·5	0·92672	0·86256	0·80602	0·75591	0·71125	0·67125	0·63525	0·60271	0·57318	0·54627	0·52167	0·49910	0·47833	0·45916

6·0	0·93182	0·87161	0·81812	0·77036	0·72752	0·68890	0·65395	0·62219	0·59322	0·56671	0·54237	0·51995	0·49924	0·48005
6·5	0·93625	0·87951	0·82876	0·78317	0·74201	0·70471	0·67078	0·63979	0·61141	0·58533	0·56129	0·53907	0·51847	0·49936
7·0	0·94013	0·88649	0·83821	0·79458	0·75500	0·71895	0·68600	0·65579	0·62800	0·60237	0·57867	0·55668	0·53624	0·51724
7·5	0·94356	0·89269	0·84666	0·80484	0·76672	0·73185	0·69984	0·67039	0·64319	0·61803	0·59468	0·57295	0·55271	0·53380
8·0	0·94661	0·89824	0·85425	0·81410	0·77735	0·74358	0·71249	0·68377	0·65717	0·63247	0·60948	0·58805	0·56803	0·54926
8·5	0·94935	0·90324	0·86111	0·82251	0·78703	0·75432	0·72409	0·69608	0·67006	0·64583	0·62323	0·60209	0·58228	0·56372
9·0	0·95181	0·90776	0·86735	0·83018	0·79589	0·76418	0·73478	0·70745	0·68200	0·65824	0·63601	0·61516	0·59562	0·57722
9·5	0·95405	0·91187	0·87305	0·83721	0·80404	0·77327	0·74465	0·71798	0·69308	0·66979	0·64793	0·62740	0·60811	0·58993
10	0·95609	0·91563	0·87827	0·84367	0·81155	0·78167	0·75381	0·72778	0·70341	0·68056	0·65909	0·63888	0·61984	0·60186
11	0·95966	0·92226	0·88751	0·85514	0·82494	0·79670	0·77025	0·74542	0·72208	0·70010	0·67937	0·65979	0·64127	0·62372
12	0·96269	0·92791	0·89543	0·86504	0·83654	0·80978	0·78460	0·76088	0·73850	0·71735	0·69733	0·67836	0·66037	0·64327
13	0·96530	0·93280	0·90231	0·87365	0·84668	0·82126	0·79725	0·77455	0·75307	0·73270	0·71336	0·69499	0·67751	0·66086
14	0·96756	0·93706	0·90833	0·88123	0·85563	0·83142	0·80848	0·78673	0·76608	0·74644	0·72776	0·70996	0·69298	0·67677
15	0·96955	0·94080	0·91364	0·88794	0·86358	0·84047	0·81852	0·79764	0·77777	0·75883	0·74076	0·72351	0·70701	0·69123
16	0·97130	0·94413	0·91837	0·89392	0·87070	0·84860	0·82755	0·80749	0·78834	0·77005	0·75257	0·73583	0·71980	0·70444
17	0·97286	0·94710	0·92261	0·89930	0·87710	0·85593	0·83572	0·81641	0·79794	0·78027	0·76333	0·74710	0·73155	0·71656
18	0·97427	0·94977	0·92643	0·90416	0·88290	0·86258	0·84314	0·82453	0·80670	0·78960	0·77319	0·75743	0·74228	0·72771
19	0·97553	0·95218	0·92988	0·90856	0·88817	0·86864	0·84992	0·83196	0·81473	0·79817	0·78226	0·76695	0·75220	0·73801
20	0·97667	0·95437	0·93303	0·91258	0·89298	0·87418	0·85613	0·83879	0·82211	0·80607	0·79062	0·77574	0·76139	0·74755

Notes: (i) $\Lambda_s(0) = 0$ for all s. (ii) For small x, it is more accurate to interpolate $s\Lambda_s$ between orders, rather than Λ_s itself.

TABLE OF $\overline{\Lambda}_s(-x) = (s!)^{-1} P \int_0^\infty \varepsilon^s (1 - \varepsilon/x)^{-1} e^{-\varepsilon} d\varepsilon$

x \ s	−0·5	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0	5·5	6·0
0·1	0·18719	−0·14684	−0·16257	−0·11468	−0·07750	−0·05573	−0·04310	−0·03519	−0·02980	−0·02588	−0·02288	−0·02052	−0·01860	−0·01701
0·2	0·35070	−0·13456	−0·25972	−0·22691	−0·16796	−0·12269	−0·09344	−0·07485	−0·06248	−0·05374	−0·04722	−0·04215	−0·03808	−0·03474
0·3	0·49324	−0·06727	−0·30405	−0·32018	−0·26081	−0·19803	−0·15130	−0·11980	−0·09868	−0·08399	−0·07325	−0·06504	−0·05854	−0·05325
0·4	0·61722	0·02809	−0·30622	−0·38876	−0·34833	−0·27775	−0·21573	−0·17037	−0·13894	−0·11704	−0·10124	−0·08936	−0·08009	−0·07262
0·5	0·72478	0·13775	−0·27522	−0·43113	−0·42507	−0·35778	−0·28501	−0·22630	−0·18357	−0·15329	−0·13151	−0·11533	−0·10286	−0·09294
0·6	0·81782	0·25351	−0·21861	−0·44789	−0·48745	−0·43437	−0·35699	−0·28687	−0·23263	−0·19303	−0·16435	−0·14316	−0·12702	−0·11432
0·7	0·89805	0·37017	−0·14273	−0·44088	−0·53327	−0·50431	−0·42932	−0·35101	−0·28586	−0·23643	−0·20002	−0·17310	−0·15273	−0·13686
0·8	0·96698	0·48434	−0·05283	−0·41253	−0·56151	−0·56501	−0·49968	−0·41734	−0·34278	−0·28347	−0·23872	−0·20535	−0·18018	−0·16071
0·9	1·02595	0·59381	0·04671	−0·36557	−0·57198	−0·61451	−0·56591	−0·48435	−0·40266	−0·33398	−0·28053	−0·24012	−0·20954	−0·18602
1·0	1·07616	0·69717	0·15232	−0·30285	−0·56512	−0·65141	−0·62605	−0·55047	−0·46459	−0·38762	−0·32546	−0·27752	−0·24099	−0·21292
1·2	1·15443	0·88265	0·37063	−0·14082	−0·50350	−0·68449	−0·72168	−0·67380	−0·59029	−0·50214	−0·42408	−0·36051	−0·31071	−0·27210
1·4	1·20890	1·03820	0·58491	0·05347	−0·38742	−0·66257	−0·77695	−0·77587	−0·71078	−0·62155	−0·53224	−0·45403	−0·39003	−0·33927
1·6	1·24521	1·16464	0·78467	0·26343	−0·22968	−0·58926	−0·78700	−0·84760	−0·81691	−0·73904	−0·64601	−0·55649	−0·47884	−0·41507
1·8	1·26776	1·26450	0·96392	0·47609	−0·04329	−0·47151	−0·75117	−0·88291	−0·90060	−0·84731	−0·76024	−0·66503	−0·57608	−0·49951
2·0	1·27998	1·34097	1·11990	0·68193	0·15987	−0·31807	−0·67210	−0·87871	−0·95549	−0·93936	−0·86911	−0·77574	−0·67967	−0·59191
2·5	1·28146	1·45163	1·40730	1·12906	0·67884	0·16133	−0·32116	−0·69889	−0·94369	−1·06181	−1·07983	−1·03090	−0·94538	−0·84621
3·0	1·26127	1·48373	1·56762	1·45119	1·13523	0·67678	0·16228	−0·32322	−0·71805	−0·99241	−1·14536	−1·19545	−1·17020	−1·09772
3·5	1·23361	1·47178	1·63530	1·65124	1·48237	1·13967	0·67532	0·16294	−0·32468	−0·73242	−1·03031	−1·21270	−1·29202	−1·29074
4·0	1·20536	1·43821	1·64289	1·75283	1·71438	1·50566	1·14301	0·67422	0·16344	−0·32578	−0·74361	−1·06062	−1·26808	−1·37375
4·5	1·17950	1·39642	1·61551	1·78389	1·84652	1·76374	1·52373	1·14561	0·67337	0·16382	−0·32663	−0·75257	−1·08543	−1·31442
5·0	1·15705	1·35383	1·57051	1·76916	1·90170	1·92289	1·80339	1·53815	1·14770	0·67269	0·16412	−0·32731	−0·75989	−1·10609
5·5	1·13809	1·31414	1·51897	1·72779	1·90287	2·00142	1·98632	1·83594	1·54993	1·14942	0·67214	0·16436	−0·32786	−0·76601

6·0	1·12228	1·27888	1·46738	1·67330	1·86951	2·01991	2·08683	2·03982	1·86313	1·55973	1·15085	0·67167	0·16456	−0·32833
6·5	1·10917	1·24839	1·41920	1·61454	1·81653	1·99726	2·12298	2·16074	2·08554	1·88620	1·56801	1·15206	0·67128	0·16473
7·0	1·09828	1·22241	1·37597	1·55686	1·75452	1·94900	2·11264	2·21433	2·22529	2·12507	1·90601	1·57510	1·15310	0·67095
7·5	1·08921	1·20042	1·33809	1·50316	1·69045	1·88685	2·07135	2·21714	2·29575	2·28213	2·15958	1·92320	1·58124	1·15400
8·0	1·08159	1·18185	1·30536	1·45478	1·62860	1·81914	2·01153	2·18436	2·31206	2·36872	2·33255	2·18996	1·93825	1·58662
8·5	1·07513	1·16613	1·27729	1·41208	1·57131	1·75132	1·94246	2·12874	2·28883	2·39858	2·43447	2·37758	2·21690	1·95157
9·0	1·06963	1·15276	1·25327	1·37483	1·51963	1·68675	1·87066	2·06024	2·23884	2·38555	2·47768	2·49399	2·41802	2·24098
9·5	1·06488	1·14132	1·23270	1·34257	1·47379	1·62722	1·80038	1·98620	2·17247	2·34223	2·47521	2·55024	2·54810	2·45454
10	1·06075	1·13147	1·21503	1·31470	1·43355	1·57351	1·73420	1·91170	2·09771	2·27925	2·43934	2·55851	2·61699	2·59752
11	1·05393	1·11543	1·18653	1·26974	1·36786	1·48357	1·61859	1·77310	1·94415	2·12602	2·30792	2·47723	2·61583	2·70825
12	1·04853	1·10297	1·16475	1·23569	1·31802	1·41417	1·52648	1·65667	1·80508	1·97001	2·14688	2·32802	2·50228	2·65604
13	1·04414	1·09303	1·14766	1·20936	1·27975	1·36081	1·45470	1·56353	1·68890	1·83146	1·99015	2·16179	2·34036	2·51722
14	1·04050	1·08489	1·13391	1·18848	1·24981	1·31937	1·39892	1·49039	1·59567	1·71636	1·85319	2·00581	2·17175	2·34689
15	1·03742	1·07810	1·12259	1·17155	1·22585	1·28660	1·35513	1·43300	1·52197	1·62374	1·73990	1·87123	2·01790	2·17806
16	1·03478	1·07234	1·11309	1·15752	1·20628	1·26016	1·32017	1·38750	1·46361	1·55002	1·64841	1·76005	1·88627	2·02680
17	1·03250	1·06739	1·10500	1·14569	1·18996	1·23839	1·29172	1·35087	1·41693	1·49119	1·57509	1·67004	1·77754	1·89846
18	1·03050	1·06309	1·09801	1·13557	1·17613	1·22014	1·26815	1·32083	1·37905	1·44376	1·51618	1·59753	1·68932	1·79260
19	1·02873	1·05931	1·09191	1·12680	1·16425	1·20460	1·24827	1·29578	1·34776	1·40496	1·46832	1·53884	1·61783	1·70633
20	1·02716	1·05596	1·08654	1·11912	1·15391	1·19118	1·23126	1·27454	1·32150	1·37270	1·42890	1·49082	1·55965	1·63605

Notes: (i) $\overline{\Lambda}_s(0) = 0$ for all s. (ii) For small x, it is more accurate to interpolate $s\overline{\Lambda}_s$ between orders, rather than $\overline{\Lambda}_s$ itself.

TABLE OF $\Pi_s(x) = (s!)^{-1} \int_0^\infty \varepsilon^s [1 + (\varepsilon/x)^2]^{-1} e^{-\varepsilon} d\varepsilon$

x \ s	−0·5	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0	5·5	6·0
0·1	0·36810	0·12910	0·04689	0·01866	0·00843	0·00435	0·00254	0·00164	0·00113	0·00083	0·00063	0·00050	0·00040	0·00033
0·2	0·49106	0·22737	0·10597	0·05175	0·02714	0·01545	0·00954	0·00632	0·00445	0·00328	0·00252	0·00199	0·00161	0·00133
0·3	0·57171	0·30709	0·16380	0·08965	0·05139	0·03118	0·02007	0·01366	0·00976	0·00727	0·00560	0·00444	0·00360	0·00298
0·4	0·63087	0·37365	0·21805	0·12899	0·07875	0·05011	0·03336	0·02323	0·01685	0·01267	0·00982	0·00781	0·00636	0·00527
0·5	0·67676	0·43026	0·26823	0·16817	0·10775	0·07122	0·04878	0·03466	0·02549	0·01935	0·01510	0·01207	0·00984	0·00817
0·6	0·71362	0·47907	0·31440	0·20637	0·13746	0·09377	0·06582	0·04762	0·03549	0·02719	0·02135	0·01714	0·01403	0·01167
0·7	0·74395	0·52160	0·35680	0·24314	0·16729	0·11721	0·08404	0·06181	0·04663	0·03605	0·02850	0·02299	0·01887	0·01574
0·8	0·76936	0·55896	0·39573	0·27828	0·19681	0·14113	0·10313	0·07698	0·05875	0·04581	0·03644	0·02954	0·02434	0·02036
0·9	0·79096	0·59202	0·43150	0·31170	0·22576	0·16523	0·12280	0·09292	0·07167	0·05635	0·04511	0·03674	0·03038	0·02548
1·0	0·80953	0·62145	0·46439	0·34338	0·25397	0·18928	0·14283	0·10944	0·08526	0·06756	0·05442	0·04453	0·03696	0·03108
1·2	0·83974	0·67148	0·52268	0·40168	0·30771	0·23654	0·18329	0·14360	0·11393	0·09162	0·07467	0·06166	0·05155	0·04360
1·4	0·86317	0·71225	0·57245	0·45367	0·35758	0·28199	0·22347	0·17847	0·14390	0·11727	0·09664	0·08051	0·06780	0·05767
1·6	0·88178	0·74597	0·61522	0·50000	0·40352	0·32516	0·26268	0·21333	0·17451	0·14396	0·11984	0·10069	0·08538	0·07305
1·8	0·89684	0·77418	0·65219	0·54130	0·44567	0·36583	0·30051	0·24770	0·20526	0·17123	0·14390	0·12187	0·10404	0·08951
2·0	0·90920	0·79804	0·68435	0·57818	0·48425	0·40392	0·33670	0·28121	0·23577	0·19869	0·16846	0·14376	0·12351	0·10684
2·5	0·93197	0·84576	0·74838	0·65442	0·56688	0·48826	0·41936	0·35998	0·30937	0·26653	0·23041	0·20001	0·17440	0·15281
3·0	0·94724	0·87587	0·79547	0·71300	0·63309	0·55857	0·49087	0·43051	0·37739	0·33107	0·29093	0·25627	0·22640	0·20068
3·5	0·95797	0·89925	0·83097	0·75869	0·68647	0·61706	0·55217	0·49268	0·43894	0·39091	0·34832	0·31073	0·27770	0·24871
4·0	0·96579	0·91677	0·85831	0·79485	0·72985	0·66584	0·60455	0·54707	0·49399	0·44555	0·40173	0·36235	0·32712	0·29571
4·5	0·97165	0·93020	0·87975	0·82386	0·76541	0·70669	0·64933	0·59449	0·54290	0·49497	0·45087	0·41058	0·37399	0·34090
5·0	0·97616	0·94071	0·89685	0·84740	0·79483	0·74108	0·68770	0·63581	0·58621	0·53942	0·49572	0·45524	0·41796	0·38381
5·5	0·97969	0·94908	0·91066	0·86674	0·81935	0·77020	0·72068	0·67186	0·62454	0·57929	0·53648	0·49632	0·45890	0·42421

6·0	0·98250	0·95583	0·92197	0·88276	0·83996	0·79500	0·74913	0·70336	0·65845	0·61500	0·57341	0·53395	0·49679	0·46200
6·5	0·98478	0·96136	0·93132	0·89619	0·85741	0·81624	0·77378	0·73096	0·68848	0·64697	0·60683	0·56834	0·53178	0·49718
7·0	0·98665	0·96594	0·93914	0·90753	0·87229	0·83454	0·79524	0·75522	0·71515	0·67561	0·63702	0·59972	0·56394	0·52984
7·5	0·98820	0·96976	0·94573	0·91716	0·88507	0·85039	0·81398	0·77659	0·73885	0·70130	0·66435	0·62833	0·59352	0·56007
8·0	0·98950	0·97299	0·95134	0·92542	0·89610	0·86418	0·83043	0·79550	0·75998	0·72436	0·68906	0·65441	0·62066	0·58803
8·5	0·99060	0·97574	0·95615	0·93255	0·90568	0·87624	0·84491	0·81227	0·77885	0·74511	0·71145	0·67819	0·64558	0·61385
9·0	0·99154	0·97810	0·96029	0·93873	0·91404	0·88684	0·85772	0·82719	0·79575	0·76381	0·73176	0·69989	0·66847	0·63771
9·5	0·99234	0·98014	0·96389	0·94412	0·92138	0·89620	0·86908	0·84051	0·81092	0·78070	0·75018	0·71971	0·68948	0·65973
10	0·99304	0·98191	0·96703	0·94885	0·92785	0·90448	0·87920	0·85243	0·82458	0·79598	0·76698	0·73782	0·70877	0·68007
11	0·99418	0·98482	0·97222	0·95672	0·93867	0·91844	0·89636	0·87279	0·84805	0·82242	0·79619	0·76961	0·74289	0·71624
12	0·99507	0·98709	0·97630	0·96294	0·94730	0·92964	0·91025	0·88939	0·86734	0·84433	0·82061	0·79639	0·77187	0·74720
13	0·99577	0·98889	0·97955	0·96794	0·95426	0·93875	0·92161	0·90307	0·88334	0·86264	0·84116	0·81907	0·79658	0·77381
14	0·99633	0·99034	0·98219	0·97201	0·95997	0·94624	0·93101	0·91445	0·89673	0·87805	0·85857	0·83840	0·81779	0·79674
15	0·99678	0·99153	0·98435	0·97536	0·96469	0·95247	0·93886	0·92400	0·90804	0·89112	0·87339	0·85497	0·83601	0·81659
16	0·99716	0·99252	0·98615	0·97815	0·96864	0·95771	0·94548	0·93209	0·91765	0·90228	0·88610	0·86925	0·85179	0·83388
17	0·99748	0·99334	0·98766	0·98051	0·97197	0·96214	0·95111	0·93899	0·92588	0·91188	0·89708	0·88161	0·86556	0·84893
18	0·99774	0·99404	0·98894	0·98250	0·97480	0·96592	0·95593	0·94491	0·93296	0·92016	0·90660	0·89239	0·87753	0·86224
19	0·99797	0·99463	0·99003	0·98421	0·97724	0·96917	0·96008	0·95004	0·93911	0·92738	0·91492	0·90177	0·88807	0·87391
20	0·99816	0·99514	0·99097	0·98568	0·97934	0·97199	0·96369	0·95450	0·94448	0·93368	0·92220	0·91008	0·89736	0·88427

Notes: (i) $\Pi_s(0) = 0$ for all s. (ii) For small x, it is more accurate to interpolate $s(s-1)\Pi_s$ between orders, rather than Π_s itself.

TABLE OF $\overline{\Pi}_s(ix) = (s!)^{-1} P \int_0^{\infty} \varepsilon^s [1 - (\varepsilon/x)^2]^{-1} e^{-\varepsilon} d\varepsilon$

x \ s	−0·5	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0	5·5	6·0
0·1	0·29637	0·02731	–0·02184	–0·01741	–0·00938	–0·00486	–0·00272	–0·00170	–0·00115	–0·00084	–0·00064	–0·00050	–0·00040	–0·00033
0·2	0·43050	0·08205	–0·03192	–0·04332	–0·03037	–0·01836	–0·01101	–0·00695	–0·00471	–0·00339	–0·00257	–0·00201	–0·00162	–0·00134
0·3	0·53399	0·14975	–0·02445	–0·06510	–0·05592	–0·03826	–0·02459	–0·01598	–0·01086	–0·00779	–0·00585	–0·00457	–0·00368	–0·00302
0·4	0·61891	0·22361	–0·00135	–0·07821	–0·08130	–0·06211	–0·04272	–0·02875	–0·01977	–0·01416	–0·01059	–0·00823	–0·00659	–0·00541
0·5	0·69023	0·29960	0·03455	–0·08093	–0·10326	–0·08755	–0·06436	–0·04504	–0·03152	–0·02266	–0·01689	–0·01306	–0·01042	–0·00852
0·6	0·75079	0·37514	0·08044	–0·07297	–0·11962	–0·11247	–0·08828	–0·06438	–0·04606	–0·03337	–0·02487	–0·01916	–0·01521	–0·01240
0·7	0·80249	0·44852	0·13379	–0·05485	–0·12904	–0·13511	–0·11318	–0·08615	–0·06323	–0·04635	–0·03463	–0·02661	–0·02105	–0·01709
0·8	0·84673	0·51867	0·19241	–0·02746	–0·13079	–0·15403	–0·13783	–0·10960	–0·08271	–0·06155	–0·04624	–0·03551	0·02800	–0·02265
0·9	0·88460	0·58488	0·25442	0·00803	–0·12463	–0·16812	–0·16104	–0·13391	–0·10411	–0·07885	–0·05971	–0·04592	–0·03613	–0·02913
1·0	0·91702	0·64676	0·31829	0·05041	–0·11065	–0·17662	–0·18179	–0·15826	–0·12693	–0·09805	–0·07503	–0·05791	–0·04553	–0·03660
1·2	0·96830	0·75689	0·44670	0·15092	–0·06086	–0·17504	–0·21247	–0·20378	–0·17459	–0·14100	–0·11085	–0·08667	–0·06834	–0·05477
1·4	1·00529	0·84900	0·57011	0·26487	0·01381	–0·14798	–0·22469	–0·24014	–0·22091	–0·18750	–0·15241	–0·12153	–0·09669	–0·07758
1·6	1·03147	0·92431	0·68397	0·38453	0·10742	–0·09688	–0·21574	–0·26260	–0·26114	–0·23400	–0·19761	–0·16161	–0·13045	–0·10530
1·8	1·04949	0·98459	0·78577	0·50383	0·21379	–0·02496	–0·18510	–0·26793	–0·29112	–0·27674	–0·24379	–0·20540	–0·16902	–0·13789
2·0	1·06136	1·03181	0·87447	0·61831	0·32724	0·06362	–0·13389	–0·25446	–0·30755	–0·31213	–0·28797	–0·25089	–0·21132	–0·17495
2·5	1·07343	1·10522	1·04017	0·86601	0·61189	0·32881	0·06695	–0·13957	–0·27722	–0·34958	–0·37026	–0·35612	–0·32253	–0·28116
3·0	1·07170	1·13499	1·13741	1·04622	0·86041	0·60746	0·32979	0·06932	–0·14358	–0·29441	–0·38298	–0·41880	–0·41585	–0·38832
3·5	1·06433	1·13982	1·18496	1·16185	1·05079	0·85643	0·60421	0·33045	0·07110	–0·14656	–0·30783	–0·41010	–0·45976	–0·46818
4·0	1·05536	1·13180	1·20003	1·22565	1·18097	1·05436	0·85346	0·60174	0·33091	0·07248	–0·14887	–0·31861	–0·43254	–0·49468
4·5	1·04663	1·11815	1·19578	1·25220	1·25916	1·19630	1·05723	0·85116	0·59978	0·33126	0·07358	0·15070	–0·32745	–0·45140
5·0	1·03892	1·10297	1·18133	1·25430	1·29727	1·28714	1·20886	1·05959	1·84933	0·59820	0·33152	0·07448	–0·15219	–0·33483
5·5	1·03240	1·08835	1·16249	1·24185	1·30706	1·33633	1·31078	1·21932	1·06155	0·84784	0·59690	0·33173	0·07523	–0·15342

6·0	1·02705	1·07524	1·14275	1·22183	1·29851	1·35441	1·37039	1·33101	1·22818	1·06322	0·84661	0·59581	0·33190	0·07586
6·5	1·02271	1·06395	1·12398	1·19885	1·27927	1·35099	1·39688	1·40027	1·34848	1·23576	1·06465	0·84556	0·59488	0·33203
7·0	1·01921	1·05445	1·10709	1·17572	1·25476	1·33397	1·39932	1·43506	1·42664	1·36372	1·24234	1·06589	0·84467	0·59408
7·5	1·01638	1·04655	1·09237	1·15400	1·22858	1·30935	1·38559	1·44376	1·46947	1·45008	1·37713	1·24803	1·06697	0·84390
8·0	1·01410	1·04004	1·07980	1·13444	1·20297	1·28136	1·36201	1·43406	1·48461	1·50060	1·47101	1·38900	1·25314	1·06794
8·5	1·01224	1·03468	1·06920	1·11729	1·17917	1·25282	1·33328	1·41241	1·47944	1·52220	1·52885	1·48984	1·39959	1·25764
9·0	1·01072	1·03026	1·06031	1·10251	1·15776	1·22547	1·30272	1·38385	1·46042	1·52190	1·55684	1·55457	1·50682	1·40912
9·5	1·00946	1·02660	1·05287	1·08989	1·13891	1·20024	1·27252	1·35209	1·43278	1·50601	1·56157	1·58882	1·57810	1·52224
10	1·00842	1·02355	1·04665	1·07918	1·12255	1·17759	1·24400	1·31974	1·40056	1·47991	1·54922	1·59871	1·61842	1·59969
11	1·00680	1·01884	1·03702	1·06244	1·09640	1·14014	1·19442	1·25926	1·33311	1·41306	1·49364	1·56851	1·62855	1·66599
12	1·00561	1·01544	1·03009	1·05036	1·07728	1·11197	1·15555	1·20878	1·27179	1·34368	1·42210	1·50319	1·58132	1·64965
13	1·00472	1·01291	1·02498	1·04150	1·06322	1·09103	1·12598	1·16904	1·22098	1·28208	1·35176	1·42839	1·50893	1·58903
14	1·00403	1·01097	1·02112	1·03485	1·05272	1·07539	1·10370	1·13856	1·18087	1·23140	1·29047	1·35788	1·43236	1·51183
15	1·00348	1·00945	1·01811	1·02974	1·04472	1·06353	1·08682	1·11532	1·14987	1·19129	1·24033	1·29737	1·36246	1·43465
16	1·00304	1·00824	1·01573	1·02572	1·03849	1·05438	1·07386	1·09750	1·12598	1·16003	1·20049	1·24794	1·30304	1·36562
17	1·00268	1·00725	1·01380	1·02250	1·03353	1·04716	1·06372	1·08364	1·10744	1·13573	1·16921	1·20857	1·25453	1·30751
18	1·00238	1·00643	1·01222	1·01986	1·02951	1·04136	1·05564	1·07268	1·09287	1·11668	1·14469	1·17748	1·21580	1·26016
19	1·00213	1·00574	1·01090	1·01768	1·02621	1·03662	1·04909	1·06387	1·08124	1·10157	1·12529	1·15289	1·18501	1·22217
20	1·00192	1·00516	1·00978	1·01585	1·02345	1·03268	1·04370	1·05666	1·07181	1·08938	1·10976	1·13328	1·16052	1·19180

Notes: (i) $\Pi_s(0) = 0$ for all s. (ii) For small x, it is more accurate to interpolate $s(s-1)\Pi_s$ between orders, rather than Π_s itself.

EXERCISES

1. Derive the absolutely convergent series (49) and (50) for Λ and $\bar{\Lambda}$ by the method of Mellin transforms.

2. Deduce the series (51) and (52) for Π and $\bar{\Pi}$ from (19) and (20).

3. By comparing the absolutely convergent series for $\bar{\Lambda}(-x)$ and $\Lambda(-x)$, establish (33) expressing the Stokes discontinuities.

4. Similarly, establish (34).

Show that to order s^{-3}:

5.

$$\Lambda_s^{(1)}(s) = \frac{1}{4s} - \frac{1}{16s^2} + \frac{3}{64s^3}, \qquad \Lambda_s^{(2)}(s) = -\frac{1}{8s^2} + \frac{1}{16s^3},$$

$$\Lambda_s^{(3)}(s) = \frac{1}{16s^3}.$$

6.

$$\Lambda_s(s + \tfrac{1}{2}) = \frac{1}{2} - \frac{1}{32s^2} + \frac{3}{128s^3},$$

an expansion used in constructing the tables.

7.

$$\bar{\Lambda}_s^{(1)}(-s) = 1 - \frac{1}{3s} + \frac{4}{135s^2} + \frac{8}{2835s^3},$$

$$\bar{\Lambda}_s^{(2)}(-s) = \frac{2}{3s} - \frac{2}{135s^2} - \frac{4}{2835s^3},$$

$$\bar{\Lambda}_s^{(3)}(-s) = -\frac{1}{3s} - \frac{1}{9s^2} - \frac{2}{405s^3}, \qquad \bar{\Lambda}_s^{(4)}(-s) = \frac{8}{135s^3},$$

$$\bar{\Lambda}_s^{(5)}(-s) = \frac{1}{15s^2} + \frac{1}{45s^3}, \qquad \bar{\Lambda}_s^{(6)}(-s) = -\frac{2}{45s^3},$$

$$\bar{\Lambda}_s^{(7)}(-s) = -\frac{1}{105s^3},$$

higher derivatives vanishing to this order.

8.

$$\overline{\Lambda}_s(-s-\tfrac{1}{2}) = \frac{1}{6} - \frac{13}{1{,}080s} + \frac{193}{90{,}720s^2} + \frac{2{,}383}{1{,}088{,}640s^3},$$

an expansion used in constructing the tables.

9.

$$\Pi_s(s+\tfrac{1}{2}) = \frac{1}{2} - \frac{3}{16s^2} + \frac{7}{8s^3},$$

similarly used.

10.

$$\overline{\Pi}_s(is) = \frac{1}{12} - \frac{103}{2{,}160s} + \frac{3091}{181{,}440s^2} + \frac{6457}{2{,}177{,}280s^3},$$

$$\overline{\Pi}_s^{(1)}(is) = \frac{1}{2} - \frac{1}{24s} - \frac{71}{4{,}320s^2} + \frac{9017}{362{,}880s^3},$$

$$\overline{\Pi}_s^{(2)}(is) = \frac{1}{3s} - \frac{151}{2{,}160s^2} + \frac{2771}{90{,}720s^3},$$

$$\overline{\Pi}_s^{(3)}(is) = -\frac{1}{6s} - \frac{1}{18s^2} + \frac{373}{12{,}960s^3}, \qquad \overline{\Pi}_s^{(4)}(is) = \frac{4}{135s^3},$$

$$\overline{\Pi}_s^{(5)}(is) = \frac{1}{30s^2} + \frac{1}{90s^3}, \qquad \overline{\Pi}_s^{(6)}(is) = -\frac{1}{45s^3}, \qquad \overline{\Pi}_s^{(7)}(is) = -\frac{1}{210s^3},$$

higher derivatives vanishing to this order.

11.

$$\overline{\Pi}_s\{i(s+\tfrac{1}{2})\} = \frac{1}{3} - \frac{13}{2{,}160s} - \frac{1{,}321}{90{,}720s^2} + \frac{13{,}949}{1{,}088{,}640s^3},$$

an expansion used in constructing the tables.

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Chapter XXII

Termination of Asymptotic Power Series

1. Methods for Expressing Terminants as Distributions over Basic Terminants

As we have seen in Chapter XXI (15)–(18), an asymptotic power series $\Sigma a_r/x^r$ is interpreted by a basic terminant if $|a_r| = (r + \alpha)!$. To investigate the general problem, we return to XXI (14) and write

$$a_r = (-1)^r(r + \alpha)!\, m_r \tag{1}$$

where m_r, which goes to unity as r tends to infinity, will be called the "modifying factor". Then

$$\sum_n^\infty a_r/x^r = \int_0^\infty \varepsilon^\alpha e^{-\varepsilon} d\varepsilon \sum_n^\infty m_r \;(-\varepsilon/x)^r. \tag{2}$$

To reduce the sum on the right to a distribution over geometrical progressions as required in our interpretative theory, m_r can either be analysed into a convergent series of rth powers, represented as an integral over the rth power of the integration variable, expanded in an inverse descending factorial sequence $(r + \alpha)^{-1}$, $(r + \alpha)^{-1}(r + \alpha - 1)^{-1}$, $\ldots$, expanded in an ascending factorial sequence $(r - n)$, $(r - n)(r - n + 1)$, $\ldots$, or expanded in a descending factorial sequence

$$(r - n),\;\; (r - n)(r - n - 1), \ldots .$$

We now show how to determine the modification to the basic terminant

$$\frac{1}{(n + \alpha)!} \sum_n^\infty \frac{(r + \alpha)!}{(-x)^{r-n}} = \Lambda_{n+\alpha}(x) \tag{3}$$

caused by the modifying factor, for each of these five types of representation.

(i) *m_r as a convergent series of rth powers.* This is simply illustrated by the example of the Riemann zeta function. Insertion of

$$m_r = \zeta(r) = 1 + 2^{-n}\frac{1}{2^{r-n}} + 3^{-n}\frac{1}{3^{r-n}} + \ldots$$

into the sum on the left of (3) modifies the terminant on the right to

$$T_n(x) = \{\Lambda_{n+\alpha}(x) + 2^{-n}\Lambda_{n+\alpha}(2x) + 3^{-n}\Lambda_{n+\alpha}(3x) + \ldots\}/\zeta(n). \quad (4)$$

Since all basic terminants tend to unity for large argument, a series of this nature converges rapidly even for quite small values of n.

(ii) *m_r as an integral over an rth power.* The most important example is when m_r is a quotient of factorials. Insertion of the representation

$$\frac{(r+\beta)!}{(r+\gamma)!} = \frac{1}{(\gamma-\beta-1)!}\int_1^\infty \frac{(v-1)^{\gamma-\beta-1}}{v^{n+\gamma+1}}\left(\frac{1}{v^{r-n}}\right) dv \quad (5)$$

into the sum in (3) modifies the terminant to

$$T_n(x) = \frac{1}{(\gamma-\beta-1)!}\int_1^\infty \frac{(v-1)^{\gamma-\beta-1}}{v^{n+\gamma+1}}\,\Lambda_{n+\alpha}(vx)\,dv \bigg/ \frac{(n+\beta)!}{(n+\gamma)!}. \quad (6)$$

Because of the factor v^{-n} in the integrand, the main contributions come from values of v close to unity, allowing the integral to be expanded in terms of the basic terminant $\Lambda_{n+\alpha}(x)$ and its derivatives. By the reduction formulae in Chapter XXI (46), these derivatives can be expressed by the same basic terminant, its order $n+\alpha$ and powers of x.

(iii) *m_r as an expansion in inverse descending factorials.* Continuing with essentially the same example as in (ii), an alternative way of introducing a quotient of factorials is based on the expansion

$$\frac{(r+a)!\,(r+b)!}{r!} = (r+\alpha)!\left\{1 - \frac{ab}{1!\,(r+\alpha)} + \frac{a(a-1)b(b-1)}{2!\,(r+\alpha)(r+\alpha-1)} - \ldots\right\} \quad (7)$$

where $\alpha = a + b$. The terminant is then modified to

$$T_n(x) = \left\{\Lambda_{n+\alpha}(x) - \frac{ab}{1!\,(n+\alpha)}\Lambda_{n+\alpha-1}(x) + \frac{a(a-1)b(b-1)}{2!\,(n+\alpha)(n+\alpha-1)}\Lambda_{n+\alpha-2}(x) - \ldots\right\} \bigg/ \frac{(n+a)!\,(n+b)!}{n!\,(n+\alpha)!}. \quad (8)$$

This type of terminant series is of especial theoretical interest because numerators in successive coefficients (here 1, ab, etc.) are the same as those of the asymptotic series for the associated function.

(iv) *m_r as an expansion in ascending factorials.* According to the backward difference expansion (Newton 1687),

$$m_r = m_n + \frac{r-n}{1!}\nabla m_n + \frac{(r-n)(r-n+1)}{2!}\nabla^2 m_n + \dots, \tag{9}$$

where $\nabla m_n = m_n - m_{n-1}$. Now

$$\frac{r-n}{x^{r-n}} = x\left(-\frac{d}{dx}\right)\frac{1}{x^{r-n}},$$

$$\frac{(r-n)(r-n+1)}{x^{r-n}} = x^2\left(-\frac{d}{dx}\right)^2\frac{1}{x^{r-n}}, \qquad \text{etc.,}$$

so insertion of (9) into the sum in (3) modifies the terminant to

$$T_n(x) = \Lambda_{n+\alpha}(x) - x\Lambda^{(1)}_{n+\alpha}(x)\frac{\nabla m_n}{m_n} + x^2\Lambda^{(2)}_{n+\alpha}(x)\frac{\nabla^2 m_n}{m_n} - \dots, \tag{10}$$

involving the reduced derivatives introduced in Chapter XXI (45).

(v) *m_r as an expansion in descending factorials.* According to the forward difference expansion (Gregory 1670),

$$m_r = m_n + \frac{r-n}{1!}\Delta m_n + \frac{(r-n)(r-n-1)}{2!}\Delta^2 m_n + \dots, \tag{11}$$

where $\Delta m_n = m_{n+1} - m_n$. Now

$$\frac{(r-n)(r-n-1)}{x^{r-n}} = x\left(-\frac{d}{dx}\right)^2 x.\,\frac{1}{x^{r-n}},$$

$$\frac{(r-n)(r-n-1)(r-n-2)}{x^{r-n}} = x\left(-\frac{d}{dx}\right)^3 x^2.\,\frac{1}{x^{r-n}}, \text{etc.,}$$

so insertion of (11) into the sum in (3) modifies the terminant to

$$T_n(x) = \Lambda_{n+\alpha}(x) - \Lambda^{[1]}_{n+\alpha}(x)\frac{\Delta m_n}{m_n} + \Lambda^{[2]}_{n+\alpha}(x)\frac{\Delta^2 m_n}{m_n} - \dots, \tag{12}$$

involving the compound derivatives

$$\Lambda^{[t]}(x) = \frac{x}{t!}\left(\frac{d}{dx}\right)^t \{x^{t-1}\Lambda(x)\}. \tag{13}$$

These can easily be written down in terms of the reduced derivatives, because the Leibnitz formula gives the coefficients as the familiar binomials, here of order one lower than the superscript. In extenso,

$$\begin{aligned}
&\Lambda^{[1]} = x\Lambda^{(1)}, \qquad \Lambda^{[2]} = x\Lambda^{(1)} + x^2\Lambda^{(2)},\\
&\Lambda^{[3]} = x\Lambda^{(1)} + 2x^2\Lambda^{(2)} + x^3\Lambda^{(3)},\\
&\Lambda^{[4]} = x\Lambda^{(1)} + 3x^2\Lambda^{(2)} + 3x^3\Lambda^{(3)} + x^4\Lambda^{(4)},
\end{aligned} \tag{14}$$

and so on.

Choice between backward or forward differences depends on available information. If terms later than the least are known to the same accuracy as earlier terms, expansion in forward differences should prove the more accurate because the limit $m_r \to 1$ as $r \to \infty$ implies progressively less influential forward differences.

2. Incomplete Factorial Function

To illustrate most effectively the theory of interpretation developed in the preceding chapter and completed in Section 1 of this, our viewpoint from now on will be that some pure unterminated asymptotic expansion (valid over a stated phase sector) has been placed before us, and we are to examine it numerically and analytically. It is appropriate to start with the asymptotic power series representing the incomplete factorial function, because its interpretation is unusually simple in requiring no distribution over basic terminants.

According to II (29) or IV (14),

$$[p, x]! = \frac{x^p e^{-x}}{(-p-1)!}\sum_0^\infty \frac{(r-p-1)!}{(-x)^r}, \qquad |\mathrm{ph}\, x| < \pi. \tag{15}$$

To assess the practical value of this unterminated series as it stands, let $p = -1$, reducing the function to the well-known exponential integral $-Ei(-x)$. When $x = 4$ (say), the sum is

$$1 - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{3}{32} + \tfrac{3}{32} - \tfrac{15}{128} + \cdots.$$

This would have to be broken off at either the first "least term" $-3/32$ or the second, $+3/32$; comparing the sums, the error from such crude truncation is evidently about 10 per cent—i.e. we have one-figure accuracy.

Interpreting (15) by XXI (15),

$$[p,x]! = \frac{x^p e^{-x}}{(-p-1)!}\left\{\sum_0^{n-1}\frac{(r-p-1)!}{(-x)^r} + \frac{(n-p-1)!}{(-x)^n}\Lambda_{n-p-1}(x)\right\}. \tag{16}$$

Again illustrating numerically for the exponential integral $p = -1$, let us choose $n = 3$ and take $\Lambda_3(4)$ from the five-figure tables reproduced in the preceding chapter. A value for $-Ei(-4)$ is then obtained which is only one unit out in the seventh significant figure.

To examine (16) analytically, we set $n = 0$ so as to dispense with the summation, leaving

$$[p,x]! = x^p e^{-x}\Lambda_{-p-1}(x). \tag{17}$$

Substituting the two alternative integral representations in XXI (22), (41),

$$[p,x]! = \begin{cases} \dfrac{x^p e^{-x}}{(-p-1)!}\displaystyle\int_0^\infty \frac{\varepsilon^{-p-1}e^{-\varepsilon}d\varepsilon}{1+\varepsilon/x}, & \Re(p) < 0, \qquad (18) \\[2ex] x^{p+1}e^{-x}\displaystyle\int_0^\infty (1+u)^p e^{-ux}\,du. & \qquad (19) \end{cases}$$

These correspond to III (11), (6) respectively.

3. Logarithm of Factorial

Replacing in late terms of Chapter II (59) [or IV (30)] the $\zeta(r+1)$ by $\sum_{\nu=1}^{\infty} 1/\nu^{r+1}$, the interpretation is seen to be

$$\ln(x!) = \tfrac{1}{2}\ln 2\pi x + x(\ln x - 1) + \frac{1}{\pi}\left\{\sum_{1,3,5,\ldots}^{n-2}\frac{(-1)^{\frac{1}{2}(r-1)}(r-1)!\zeta(r+1)}{(2\pi x)^r}\right.$$
$$\left. + \frac{(-1)^{\frac{1}{2}(n-1)}(n-1)!}{(2\pi x)^n}\sum_1^\infty \frac{\Pi_{n-1}(2\pi\nu x)}{\nu^{n+1}}\right\} \tag{20}$$

where n is an odd integer. Choosing $n = 5$, a result for $\ln(2/\pi)!$ is obtained which is only two units out in the eighth significant figure, a remarkably high accuracy considering the variable chosen for illustration is less than unity!

An alternative starting point is the asymptotic expansion valid on a Stokes ray, derived in Chapter II, question 16. Interpreting this,

$$\ln(iy)! = \tfrac{1}{2}\ln(\pi y/\sinh \pi y) + iy(\ln y - 1) + \tfrac{1}{4}i\pi$$
$$- \frac{i}{\pi}\left\{\sum_{1,3,5,\ldots}^{n-2} \frac{(r-1)!\,\zeta(r+1)}{(2\pi y)^r} + \frac{(n-1)!}{(2\pi y)^n}\sum_1^\infty \frac{\overline{\Pi}_{n-1}(2\pi viy)}{v^{n+1}}\right\}. \tag{21}$$

Again taking $n = 5$, (21) gives a value for the imaginary part of $\ln(2i/\pi)!$ to the same accuracy as that of $\ln(2/\pi)!$ from (20). Though convenient over different phase ranges, the two forms (20) and (21) are equivalent, as may be verified from the relation in Chapter XXI (34) between $\overline{\Pi}$ and Π, $\operatorname{ph} y < 0$.

For analytical examination, we set $n = 1$ in (20) so as to dispense with the first summation (n must be odd). By XXI (25) the second sum then reduces to

$$\sum_1^\infty \frac{\Pi_0(2\pi vx)}{v^2} = \sum_1^\infty \frac{1}{v^2}\int_0^\infty \frac{e^{-\varepsilon}d\varepsilon}{1+(\varepsilon/2\pi vx)^2} = \int_0^\infty \frac{du}{1+(u/2\pi x)^2}\sum_1^\infty \frac{e^{-vu}}{v}$$
$$= -\int_0^\infty \frac{du}{1+(u/2\pi x)^2}\ln(1-e^{-u}) = 2\pi x\int_0^\infty \frac{\tan^{-1}(u/2\pi x)}{e^u - 1}\,du, \tag{22}$$

and (20) becomes

$$\ln(x!) = \tfrac{1}{2}\ln 2\pi x + x(\ln x - 1) + \frac{1}{\pi}\int_0^\infty \frac{\tan^{-1}(u/2\pi x)}{e^u - 1}\,du, \tag{23}$$

which is Binet's "second representation", derived in Chapter III (40).

4. Fermi–Dirac Integral

In Chapter II (68) [or IV (39)] there are available three different asymptotic expansions—one specific to the Stokes ray $\operatorname{ph} x = 0$, one to the phase sector $0 < \operatorname{ph} x < \pi$, and the other to the sector $0 > \operatorname{ph} x > -\pi$. Choosing the first as starting point, we replace the t_r in late terms by $\sum_{v=1}^\infty (-1)^{v-1}/v^r$, and arrive at the interpretation

$$\mathscr{F}_p(x) = \cos \pi p\ \ \mathscr{F}_p(-x) + \frac{2\sin \pi p}{\pi}\left\{\sum_{0,2,4,\ldots}^{n-2} \frac{(r-p-2)!\,t_r}{x^{r-p-1}}\right.$$
$$\left. + \frac{(n-p-2)!}{x^{n-p-1}}\sum_1^\infty \frac{(-1)^{v-1}}{v^n}\,\overline{\Pi}_{n-p-2}(ivx)\right\}, \tag{24}$$

where n is an even integer. Choosing $n = 6$, this gives a result for $\mathscr{F}_{\frac{1}{2}}(2)$ correct to seven figures.

Referring now to Chapter XXI (34), if in (24) $\overline{\Pi}$ were to be replaced by Π, there would be compensating contributions

$$\pm\, i \sin \pi p \sum_1^\infty \frac{(-1)^{\nu-1}}{\nu^{p+1}} e^{-\nu x}, \qquad \left.\begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \end{matrix}\right\} .$$

Since by II (62) this sum is a representation of $\mathscr{F}_p(-x)$, these alternative forms of (24) may be written

$$\mathscr{F}_p(x) = \begin{pmatrix} e^{i\pi p} \\ e^{-i\pi p} \end{pmatrix} \mathscr{F}_p(-x) + \frac{2\sin \pi p}{\pi} \left\{ \sum_{0,2,4,\ldots}^{n-2} \frac{(r-p-2)!\, t_r}{x^{r-p-1}} + \frac{(n-p-2)!}{x^{n-p-1}} \right.$$

$$\left. \times \sum_1^\infty \frac{(-1)^{\nu-1}}{\nu^n} \Pi_{n-p-2}(i\nu x) \right\}, \qquad \begin{matrix} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \end{matrix} . \tag{25}$$

As expected, these are the interpreted versions of the other two asymptotic expansions quoted in Chapter II (68) and IV (39).

To examine (24) analytically, we set $n = 0$ and introduce from Chapter XXI (20), (41), and (42) the representation

$$\overline{\Pi}_{s-1}(iz) = \tfrac{1}{2} z \;\; P\int_0^\infty \{(1+u)^{-s} + (1-u)^{-s}\} e^{-uz} du. \tag{26}$$

Then

$$\mathscr{F}_p(x) - \cos \pi p\, \mathscr{F}_p(-x)$$

$$= \frac{2x^{p+1}}{(p+1)!} \sum_1^\infty (-1)^{\nu-1} \overline{\Pi}_{-p-2}(i\nu x)$$

$$= \frac{x^{p+2}}{(p+1)!} P\int_0^\infty \{(1+u)^{p+1} + (1-u)^{p+1}\} \left\{ \sum_1^\infty (-1)^{\nu-1} \nu e^{-\nu u x} \right\} du$$

$$= \frac{x^{p+2}}{(p+1)!} P\int_0^\infty \{(1+u)^{p+1} + (1-u)^{p+1}\} \frac{e^{ux}}{(e^{ux}+1)^2} du$$

$$= \frac{x^{p+2}}{(p+1)!} P\int_{-\infty}^\infty \frac{(1+u)^{p+1} e^{ux}\, du}{(e^{ux}+1)^2} . \tag{27}$$

This is the integral representation in Chapter III (43) [and IV (39)] at $\operatorname{ph} x = 0$.

5. Confluent Hypergeometric Functions

Substituting in late terms of Chapter III (23) [or IV (21)] the representation

$$\frac{(r+a-1)!}{r!} = \frac{1}{(-a)!}\int_1^\infty \frac{(v-1)^{-a}}{v^{r+1}}\,dv, \tag{28}$$

the interpretation is seen to be

$$\psi(a,c,x) = \frac{1}{(a-1)!\,(a-c)!}\left[\sum_0^{n-1}\frac{(r+a-1)!\,(r+a-c)!}{r!\,(-x)^r}\right.$$

$$\left.+\frac{(n+a-c)!}{(-x)^n\,(-a)!}\int_1^\infty \frac{(v-1)^{-a}}{v^{n+1}}\,\Lambda_{n+a-c}(vx)\,dv\right]. \tag{29}$$

Setting $n = 0$ for analytical examination, (29) reduces to

$$\psi(a,c,x) = \frac{1}{(a-1)!\,(-a)!}\int_1^\infty \frac{(v-1)^{-a}}{v}\,\Lambda_{a-c}(vx)\,dv. \tag{30}$$

Simplifying this integral by writing $v = \mu + 1$ and introducing the representation in Chapter XXI (22) for Λ,

$$\psi(a,c,x) = \frac{1}{(a-c)!\,(a-1)!\,(-a)!}\int_0^\infty \varepsilon^{a-c}e^{-\varepsilon}\,d\varepsilon\int_0^\infty \frac{\mu^{-a}\,d\mu}{\mu+(1+\varepsilon/x)}$$

$$= \frac{1}{(a-c)!}\int_0^\infty \frac{\varepsilon^{a-c}e^{-\varepsilon}\,d\varepsilon}{(1+\varepsilon/x)^a} = \frac{x^{a-c+1}e^x}{(a-c)!}\int_1^\infty \frac{(v-1)^{a-c}e^{-vx}\,dv}{v^a}. \tag{31}$$

This is the integral representation in Chapter III (23) + (24).

The asymptotic expansion of the more familiar confluent hypergeometric function $F(a,c,x)$ involves also the asymptotic power series $\psi(c-a,\ c,\ -x) = \psi(1-a,\ 2-c,\ -x)$, for which the interpretation *on the Stokes ray* is

$$\psi(c-a,\ c,\ -x) = \frac{1}{(c-a-1)!\,(-a)!}\left[\sum_0^{n-1}\frac{(r+c-a-1)!\,(r-a)!}{r!\,x^r}\right.$$

$$\left.+\frac{(n-a)!}{x^n(a-c)!}\int_1^\infty \frac{(v-1)^{a-c}}{v^{n+1}}\,\overline{\Lambda}_{n-a}(-vx)\,dv\right], \qquad \operatorname{ph} x = 0. \tag{32}$$

Referring now to Chapter XXI (33), if in (32) $\overline{\Lambda}$ were to be replaced by Λ, there would be compensating contributions

$$\pm i\pi \frac{x^{1-a}}{(c-a-1)!\,(-a)!\,(a-c)!}\int_1^\infty \frac{(v-1)^{a-c}\,e^{-vx}\,dv}{v^a}$$

$$= \pm i \sin \pi a \,\frac{(a-1)!}{(c-a-1)!}\,x^{c-2a}\,e^{-x}\,\psi(a,c,x), \qquad \left.\begin{array}{l} 0 < \operatorname{ph} x < \pi \\ 0 > \operatorname{ph} x > -\pi \end{array}\right\}, \tag{33}$$

where we have introduced (31) and the reflection formula for factorials. These are seen to correspond to the changes when interpreting the forms of Chapter III (25) [or IV (19)] appropriate to either side of the Stokes ray.

The terminant T_n in (29), defined as that multiplier of the nth term $(n+a-1)!\,(n+a-c)!/n!(-x)^n$ which would correctly terminate the series, can be evaluated numerically in the following three ways:

(a) Starting from the form (29), the simplest argument is to expand the basic terminant in rising powers of $v-1$ and integrate term by term, leading to

$$T_n(x) = \Lambda_{n+a-c}(x) - \frac{a-1}{n+a-1}\,x\Lambda^{(1)}_{n+a-c}(x)$$

$$+ \frac{(a-1)(a-2)}{(n+a-1)(n+a-2)}\,x^2\Lambda^{(2)}_{n+a-c}(x) - \dots. \tag{34}$$

This result also comes out (logically still more directly) on applying the backward difference formula (10) to the asymptotic power series $\Sigma m_r(r+a-c)!/(-x)^r$ with modifying factor $m_r = (r+a-1)!/r!$. Interchange of $a-1$ and $a-c$ provides the alternative

$$T_n(x) = \Lambda_{n+a-1}(x) - \frac{a-c}{n+a-c}\,x\Lambda^{(1)}_{n+a-1}(x)$$

$$+ \frac{(a-c)(a-c-1)}{(n+a-c)(n+a-c-1)}\,x^2\Lambda^{(2)}_{n+a-1}(x) - \dots. \tag{35}$$

(b) On applying the forward difference expansion (12), the result is

$$T_n(x) = \Lambda_{n+a-c}(x) - \frac{a-1}{n+1}\,\Lambda^{[1]}_{n+a-c}(x)$$

$$+ \frac{(a-1)(a-2)}{(n+1)(n+2)}\,\Lambda^{[2]}_{n+a-c}(x) - \dots. \tag{36}$$

This re-ordering of (34) is superior in its initial convergence, particularly when $n + a$ is fairly small.

(c) Application of (8) leads to

$$T_n(x) = \left\{\Lambda_{n+2a-c-1}(x) - \frac{(a-1)(a-c)}{1!\,(n+2a-c-1)}\Lambda_{n+2a-c-2}(x) + \frac{(a-1)(a-2)(a-c)(a-c-1)}{2!\,(n+2a-c-1)(n+2a-c-2)}\Lambda_{n+2a-c-3}(x) - \dots\right\} \div \frac{(n+a-1)!\,(n+a-c)!}{n!\,(n+2a-c-1)!}. \tag{37}$$

To confirm the remark made in Section 1 (iii) on the intimate connection between this type of terminant series and the associated function expressed as an asymptotic series, let Λ be replaced by $\overline{\Lambda}$, so producing by Chapter XXI (33) compensating contributions proportional to

$$1 + \frac{(a-1)(a-c)}{1!\,x} + \frac{(a-1)(a-2)(a-c)(a-c-1)}{2!\,x^2} + \dots,$$

which is indeed the asymptotic series for the associated function $\psi(c-a,\ c,\ -x)$.

Numerical illustrations will be deferred to the next section, after specialization to modified Bessel functions.

6. Modified Bessel Functions

The results of the previous section are easily specialized to modified Bessel functions through III (33), (31):

$$K_p(x) = (\pi/2x)^{\frac{1}{2}} e^{-x} \psi_p(x), \tag{38}$$

$$I_p(x) = (2\pi x)^{-\frac{1}{2}} \left[e^x \psi_p(-x) - \begin{pmatrix} \sin \pi p - i\cos \pi p \\ \sin \pi p + i \cos \pi p \\ \sin \pi p \end{pmatrix} e^{-x}\psi_p(x) \right], \quad \left.\begin{matrix} 0 < \mathrm{ph}\, x < \pi \\ 0 > \mathrm{ph}\, x > -\pi \\ \mathrm{ph}\, x = 0 \end{matrix}\right\}, \tag{39}$$

where $\psi_p(x) = \psi(p + \frac{1}{2}, 2p + 1, 2x)$. The forms of (38) most suited to numerical evaluation would be

$$K_p(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \frac{1}{(p - \frac{1}{2})!\,(-p - \frac{1}{2})!} \left[\sum_0^{n-1} \frac{(r + p - \frac{1}{2})!\,(r - p - \frac{1}{2})!}{r!\,(-2x)^r} + \frac{(n + p - \frac{1}{2})!\,(n - p - \frac{1}{2})!}{n!\,(-2x)^n} T_n(2x) \right], \tag{40}$$

where

$$T_n(z) = \begin{cases} \Lambda_{n-p-\frac{1}{2}}(z) - \dfrac{p - \frac{1}{2}}{n + p - \frac{1}{2}} z\Lambda^{(1)}_{n-p-\frac{1}{2}}(z) + \dfrac{(p - \frac{1}{2})(p - \frac{3}{2})}{(n + p - \frac{1}{2})(n + p - \frac{3}{2})} z^2\Lambda^{(2)}_{n-p-\frac{1}{2}}(z) - \ldots & (41) \\[2ex] \Lambda_{n-p-\frac{1}{2}}(z) - \dfrac{p - \frac{1}{2}}{n + 1} \Lambda^{[1]}_{n-p-\frac{1}{2}}(z) + \dfrac{(p - \frac{1}{2})(p - \frac{3}{2})}{(n + 1)(n + 2)} \Lambda^{[2]}_{n-p-\frac{1}{2}}(z) - \ldots. & (42) \end{cases}$$

Choosing $n = 5$ and retaining only half the least (fifth) term of (41) because of the alternating signs, this gives $K_1(2) = 0{\cdot}13986588$, which is correct to all eight significant figures.

In the analogous calculation of I_1 (2), the terms in the analogue of (41) with $\bar{\Lambda}$ replacing Λ do not alternate in sign, and this series has perforce to be crudely broken off at about its fifth term. Partly because of this, and partly because successive derivatives of $\bar{\Lambda}$ form a less rapidly decreasing sequence than those of Λ, the accuracy is much lower, the value of I_1 (2) obtained being three units high in the sixth figure. The disparity in accuracy has been abnormally emphasized in this example by the ultra-small value of x chosen for illustration; in practice the *convergent* series would always be used for such small values. The increasing accuracy obtained for larger values of x is well shown by the extensive calculations of Wrench (1970, 1971), who finds for example $K_0(2\pi)$ to almost 14 significant figures and $K_1(10)$ to almost 22; and $I_0(2\pi)$ to 9 significant figures, $I_1(10)$ to 15.

7. Lommel Function

By Chapter XVIII, question 8, the asymptotic power series of this function is

$$S_{qp}(x) = \frac{x^{q-1}}{\{\frac{1}{2}(p-q-1)\}!\,\{-\frac{1}{2}(p+q+1)\}!} \times \sum_{s=0}^{\infty} \frac{\{s+\frac{1}{2}(p-q-1)\}!\,\{s-\frac{1}{2}(p+q+1)\}!}{(-\frac{1}{4}x^2)^s}, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{43}$$

This form is not suitable for interpretation, because of the two factorials in the numerator. Replacing $\{s-\frac{1}{2}(p+q+1)\}!$ by $\sqrt{\pi}\,(2s-p-q)!/2^{2s-p-q}\{s-\frac{1}{2}(p+q)\}!$ and writing $s=\frac{1}{2}r$, (43) is transformed to

$$S_{qp}(x) = \frac{\sqrt{\pi}\, 2^{p+q} x^{q-1}}{\{\frac{1}{2}(p-q-1)\}!\,\{-\frac{1}{2}(p+q+1)\}!} \times \sum_{r=0,2,4,\ldots}^{\infty} (-1)^{\frac{1}{2}r} \frac{(r-p-q)!}{x^r} \frac{\{\frac{1}{2}(r+p-q-1)\}!}{\{\frac{1}{2}(r-p-q)\}!}, \tag{44}$$

which complies with one of our chosen standard forms. Substituting in late terms of (44) the usual integral representation for the quotient of factorials, interpreting, and then reinstating the original more convenient summation variable s, the interpreted version of (43) is found to be

$$\begin{aligned} S_{qp}(x) = {} & \frac{x^{q-1}}{\{\frac{1}{2}(p-q-1)\}!\,\{-\frac{1}{2}(p+q+1)\}!} \\ & \times \left[\sum_{s=0}^{n-1} \frac{\{s+\frac{1}{2}(p-q-1)\}!\,\{s-\frac{1}{2}(p+q+1)\}!}{(-\frac{1}{4}x^2)^s} \right. \\ & + \frac{\{n-\frac{1}{2}(p+q)\}!\,\{n-\frac{1}{2}(p+q+1)\}!}{(-\frac{1}{4}x^2)^n(-p-\frac{1}{2})!} \\ & \left. \times \int_1^{\infty} \frac{(v-1)^{-p-\frac{1}{2}}}{v^{n-\frac{1}{2}p-\frac{1}{2}q+1}} \Pi_{2n-p-q}(x\sqrt{v})\, dv \right]. \end{aligned} \tag{45}$$

Setting $n=0$ for analytical examination, (45) reduces to

$$S_{qp}(x) = \frac{2\, x^{q-1}\{-\frac{1}{2}(p+q)\}!}{\{\frac{1}{2}(p-q-1)\}!\,(-p-\frac{1}{2})!} \int_1^{\infty} \frac{(v^2-1)^{-p-\frac{1}{2}}}{v^{1-p-q}} \Pi_{-p-q}(xv)\, dv. \tag{46}$$

Introducing XXI (25) in the form

$$\Pi_{-p-q}(vx) = \frac{(vx)^{1-p-q}}{(-p-q)!}\int_0^\infty \frac{\varepsilon^{-p-q}\, e^{-\varepsilon vx}\, d\varepsilon}{1+\varepsilon^2}, \tag{47}$$

we have

$$S_{qp}(x) = \frac{2^{p+q+1}\sqrt{\pi}}{x^p\{\frac{1}{2}(p-q-1)\}!\,\{-\frac{1}{2}(p+q+1)\}!\,(-p-\frac{1}{2})!}\int_0^\infty \frac{\varepsilon^{-p-q}\, d\varepsilon}{1+\varepsilon^2}$$

$$\times \int_1^\infty (v^2-1)^{-p-\frac{1}{2}}\, e^{-\varepsilon vx}\, dv$$

$$= \frac{2^{q+1}}{\{\frac{1}{2}(p-q-1)\}!\,\{-\frac{1}{2}(p+q+1)\}!}\int_0^\infty \frac{K_p(\varepsilon x)\, d\varepsilon}{\varepsilon^q(1+\varepsilon^2)} \tag{48}$$

provided $|p| + q < 1$. This is the integral representation in Chapter XVIII, question 12.

Defining

$$^{(t)}\Pi(x) = \frac{1}{t!}\left(\frac{d}{d(x^2)}\right)^t \Pi(x), \tag{49}$$

the basic terminant in (45) can be expanded in rising powers of $v - 1$ according to the Taylor series

$$\Pi(x\sqrt{v}) = \sum_0^\infty (v-1)^t\, x^{2t}\ {}^{(t)}\Pi(x). \tag{50}$$

On replacing each resultant integral by its equivalent quotient of factorials, the terminant is expressed as the series

$$T_n(x) = \Pi_{2n-p-q}(x) - \frac{2p-1}{2n+p-q-1}x^2\ {}^{(1)}\Pi_{2n-p-q}(x)$$

$$+ \frac{(2p-1)(2p-3)}{(2n+p-q-1)(2n+p-q-3)}x^4\ {}^{(2)}\Pi_{2n-p-q}(x) - \dots. \tag{51}$$

The same result follows from the appropriate modification to the backward difference expansion (10), which is

$$T_n(x) = \Pi_{2n-p-q}(x) - x^2\ {}^{(1)}\Pi_{2n-p-q}(x)\frac{\nabla m_{2n}}{m_{2n}} + \dots, \tag{52}$$

where by (44) $m_{2n} = \{n + \frac{1}{2}(p-q-1)\}!/\{n - \frac{1}{2}(p+q)\}!$.

Three variants of (51) are available. Rearrangement of the sum in (44) to

$$2\sum_{0,2,\dots}^{\infty} (-1)^{\frac{1}{2}r}\frac{(r-p-q-1)!}{x^r}\frac{\{\frac{1}{2}(r+p-q-1)\}!}{\{\frac{1}{2}(r-p-q-2)\}!}$$

leads to the alternative form

$$T_n(x) = \Pi_{2n-p-q-1}(x) - \frac{2p+1}{2n+p-q-1}\, x^2\ {}^{(1)}\Pi_{2n-p-q-1}(x)$$
$$+ \frac{(2p+1)(2p-1)}{(2n+p-q-1)(2n+p-q-3)}\, x^4\ {}^{(2)}\Pi_{2n-p-q-1}(x) - \cdots. \tag{53}$$

Moreover, as evident from (43), the sign of p can be reversed in (51) and (53).

Defining [cf. (13)]

$$^{[t]}\Pi(x) = \frac{x^2}{t!}\left(\frac{d}{d(x^2)}\right)^t \{x^{2t-2}\,\Pi(x)\}, \tag{54}$$

so that for instance $^{[2]}\Pi = x^2\,{}^{(1)}\Pi + x^4\,{}^{(2)}\Pi$ [cf. (14)], the companion forward difference expansion to e.g. (51) is found to be

$$T_n(x) = \Pi_{2n-p-q}(x) - \frac{2p-1}{2n-p-q+2}\, {}^{[1]}\Pi_{2n-p-q}(x)$$
$$+ \frac{(2p-1)(2p-3)}{(2n-p-q+2)(2n-p-q+4)}\, {}^{[2]}\Pi_{2n-p-q}(x) - \cdots. \tag{55}$$

Numerical illustrations will be deferred to the next section, after specialization to Struve functions.

8. Struve Functions

The results of the previous section are easily specialized to the Struve function $\mathbf{H}_p(x)$ through XVIII, question 18:

$$\mathbf{H}_p(x) = Y_p(x) + \frac{2^{1-p}}{\sqrt{\pi}\,(p-\frac{1}{2})!}\, S_{pp}(x). \tag{56}$$

A simpler type of integral representation than (48) can be found in this special case. Changing the sign of p in (46), writing $v^2 = 1 + u$ and introducing XXI (25),

$$S_{pp}(x) = \frac{x^{p-1}\cos \pi p}{\pi}\int_0^\infty e^{-\varepsilon} d\varepsilon \int_0^\infty \frac{u^{p-\frac{1}{2}}du}{u + (1+\varepsilon^2/x^2)}$$
$$= \frac{x^{p-1}\cos \pi p}{\pi}\int_0^\infty \left(1+\frac{\varepsilon^2}{x^2}\right)^{p-\frac{1}{2}} e^{-\varepsilon}\, d\varepsilon \int_0^\infty \frac{\sigma^{p-\frac{1}{2}}\,d\sigma}{\sigma+1}$$
$$= x^{p-1}\int_0^\infty (1+\varepsilon^2/x^2)^{p-\frac{1}{2}}\, e^{-\varepsilon}\, d\varepsilon, \tag{57}$$

which is equivalent to Chapter IV, question 7.

The form of (56) most suited to numerical evaluation would be

$$\mathbf{H}_p(x) - Y_p(x) = \frac{(\tfrac{1}{2}x)^{p-1}\cos \pi p}{\pi^2}\left[\sum_0^{n-1} \frac{(r-\tfrac{1}{2})!\,(r-p-\tfrac{1}{2})!}{(-\tfrac{1}{4}x^2)^r} + \frac{(n-\tfrac{1}{2})!\,(n-p-\tfrac{1}{2})!}{(-\tfrac{1}{4}x^2)^n}\,T_n(x)\right], \tag{58}$$

where typical choices of terminant series are

$$T_n(x) = \begin{cases} \Pi_{2n-2p}(x) - \dfrac{2p-1}{2n-1}x^{2\,(1)}\Pi_{2n-2p}(x) \\ \quad + \dfrac{(2p-1)(2p-3)}{(2n-1)(2n-3)}x^{4\,(2)}\Pi_{2n-2p}(x) - \dots & (59) \\ \Pi_{2n-2p-1}(x) - \dfrac{2p+1}{2n-1}x^{2\,(1)}\Pi_{2n-2p-1}(x) \\ \quad + \dfrac{(2p+1)(2p-1)}{(2n-1)(2n-3)}x^{4\,(2)}\Pi_{2n-2p-1}(x) - \dots & (60) \\ \Pi_{2n-2p}(x) - \dfrac{2p-1}{2n-2p+2}\,{}^{[1]}\Pi_{2n-2p}(x) \\ \quad + \dfrac{(2p-1)(2p-3)}{(2n-2p+2)(2n-2p+4)}\,{}^{[2]}\Pi_{2n-2p}(x) - \dots\,. & (61) \end{cases}$$

Choosing $n = 4$ and retaining only half the least (fourth) term of (60) because of the alternating signs, this gives $\mathbf{H}_1(4) - Y_1(4) = 0{\cdot}671803$, two units high in the sixth decimal place.

The analogous form for the modified Struve function discussed in Chapter XVIII is easily found to be

$$\mathbf{L}_p(x) - I_{-p}(x) = -\frac{(\tfrac{1}{2}x)^{p-1}\cos \pi p}{\pi^2}\left[\sum_0^{n-1} \frac{(r-\tfrac{1}{2})!\,(r-p-\tfrac{1}{2})!}{(\tfrac{1}{4}x^2)^r} + \frac{(n-\tfrac{1}{2})!\,(n-p-\tfrac{1}{2})!}{(\tfrac{1}{4}x^2)^n}\,T_n(x)\right], \tag{62}$$

where for example the analogue to (60) is

$$T_n(x) = \overline{\Pi}_{2n-2p-1}(ix) - \frac{2p+1}{2n-1} x^2 \ {}^{(1)}\overline{\Pi}_{2n-2p-1}(ix)$$

$$+ \frac{(2p+1)(2p-1)}{(2n-1)(2n-3)} x^4 \ {}^{(2)}\overline{\Pi}_{2n-2p-1}(ix) - \dots . \tag{63}$$

Again taking $n = 4$ and retaining only half the fourth term because neighbouring terms alternate in sign, this gives $\mathbf{L}_1(4) - I_{-1}(4) = 0{\cdot}590193$, six units high in the sixth decimal place.

EXERCISES

1. The integral $f(x) = \int_0^\infty (u+x)^{-1} e^{-u^2} du$ has been discussed and tabulated for positive real x by Goodwin and Staton (1948). Substituting

$$\frac{1}{u+x} = \frac{u}{u^2 - x^2} - \frac{x}{u^2 - x^2},$$

prove the identity $f(x) = e^{-x^2}\{\sqrt{\pi}\,\mathrm{erfi}\,(x) - \frac{1}{2}Ei(x^2)\}$.

Derive the terminated asymptotic expansion

$$f(x) = \frac{1}{2x}\left\{\sum_0^{n-1} \frac{(r-\frac{1}{2})!}{x^{2r}} + \frac{(n-\frac{1}{2})!}{x^{2n}} \overline{\Lambda}_{n-\frac{1}{2}}(-x^2)\right\}$$

$$- \frac{1}{2x^2}\left\{\sum_0^{n-1} \frac{r!}{x^{2r}} + \frac{n!}{x^{2n}} \overline{\Lambda}_n(-x^2)\right\},$$

and explain why both $\overline{\Lambda}$ could simultaneously be replaced by Λ.

2. The Fresnel (1826) integrals are defined by

$$C(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\sqrt{x}} \cos u^2 \, du, \qquad S(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\sqrt{x}} \sin u^2 \, du.$$

Express them as

$$C(x) = \tfrac{1}{2} + (2\pi x)^{-\frac{1}{2}}\{U \sin x - V \cos x\},$$

$$S(x) = \tfrac{1}{2} - (2\pi x)^{-\frac{1}{2}}\{U \cos x + V \sin x\},$$

and derive the terminated asymptotic series

$$\sqrt{\pi}\, U(x) = \sum_{0,2,..}^{n-2} \frac{(-1)^{\frac{1}{2}r}(r-\frac{1}{2})!}{x^r} + \frac{(-1)^{\frac{1}{2}n}(n-\frac{1}{2})!}{x^n}\Pi_{n-\frac{1}{2}}(x), \qquad n \text{ even},$$

$$\sqrt{\pi}\, V(x) = \sum_{1,3,..}^{n-2} \frac{(-1)^{\frac{1}{2}(r-1)}(r-\frac{1}{2})!}{x^r} + \frac{(-1)^{\frac{1}{2}(n-1)}(n-\frac{1}{2})!}{x^n}\Pi_{n-\frac{1}{2}}(x), \qquad n \text{ odd}.$$

Choosing $n = 4, 5$, confirm that $C(4)$ and $S(4)$ are both given correctly to eight significant figures.

3. Express the integrals

$$Ci_p(x) = \int_1^\infty u^{-p}\cos ux\, du, \qquad Si_p(x) = \int_1^\infty u^{-p}\sin ux\, du,$$

(Dingle 1955, see also Kreyszig 1951, 1953) as

$$Ci_p(x) = -\,U_p \sin x + V_p \cos x, \qquad Si_p(x) = U_p \cos x + V_p \sin x,$$

and derive the terminated asymptotic series

$$x\,(p-1)!\,U_p(x) = \sum_{0,2,...}^{n-2} \frac{(-1)^{\frac{1}{2}r}(r+p-1)!}{x^r}$$

$$+ \frac{(-1)^{\frac{1}{2}n}\,(n+p-1)!}{x^n}\Pi_{n+p-1}(x), \qquad n \text{ even},$$

$$x\,(p-1)!\,V_p(x) = \sum_{1,3,...}^{n-2} \frac{(-1)^{\frac{1}{2}(r-1)}(r+p-1)!}{x^r}$$

$$+ \frac{(-1)^{\frac{1}{2}(n-1)}(n+p-1)!}{x^n}\Pi_{n+p-1}(x), \qquad n \text{ odd}.$$

4. Defining Raabe integrals by

$$R_s(x) = x\int_0^\infty \frac{\sin u\, du}{x^2+u^2}, \qquad R_c(x) = -\int_0^\infty \frac{u\cos u\, du}{x^2+u^2},$$

derive the terminated asymptotic series

$$xR_s(x) = \sum_{0,2,\ldots}^{n-2} \frac{r!}{x^r} + \frac{n!}{x^n}\overline{\Pi}_n(ix), \qquad n \text{ even},$$

$$xR_c(x) = \sum_{1,3,\ldots}^{n-2} \frac{r!}{x^r} + \frac{n!}{x^n}\overline{\Pi}_n(ix), \qquad n \text{ odd}.$$

Choosing $n = 4, 5$, confirm that $R_s(4)$ and $R_c(4)$ are given correctly to seven and six significant figures respectively. Show how only one-figure accuracy could reliably have been ascertained from the unterminated series.

5. Replacing in (21) $\overline{\Pi}$ by Π for $\operatorname{ph} y > 0$, deduce the terminated expansion

$$\ln(-x)! = \tfrac{1}{2}\ln\tfrac{1}{2}\pi x - x(\ln x - 1) - \ln\sin\pi x$$
$$-\frac{1}{\pi}\left\{\sum_{1,3,5,\ldots}^{n-2} \frac{(-1)^{\frac{1}{2}(r-1)}(r-1)!\,\zeta(r+1)}{(2\pi x)^r} + \frac{(-1)^{\frac{1}{2}(n-1)}(n-1)!}{(2\pi x)^n}\sum_{1}^{\infty}\frac{\Pi_{n-1}(2\pi\nu x)}{\nu^{n+1}}\right\}.$$

6. Deduce this expansion for $\ln(-x)!$ from (20) by replacing x therein by $xe^{i\pi}$ and applying the continuation formula in Chapter XXI (54).

7. Differentiating (20), deduce the terminated expansion

$$\Psi(x) = \frac{d\ln(x!)}{dx} = \frac{1}{2x} + \ln x - \frac{1}{\pi x}\left\{\sum_{1,3,\ldots}^{n-2} \frac{(-1)^{\frac{1}{2}(r-1)}r!\,\zeta(r+1)}{(2\pi x)^r}\right.$$
$$\left. + \frac{(-1)^{\frac{1}{2}(n-1)}n!}{(2\pi x)^n}\sum_{1}^{\infty}\frac{\Pi_n(2\pi\nu x)}{\nu^{n+1}}\right\},$$

where n is an odd integer. Choosing $n = 5$, confirm that $\Psi(2/\pi)$ is given correctly to within two units in the eighth significant figure.

8. Differentiating (21), deduce the terminated expansion

$$\Psi(iy) = \tfrac{1}{2}i(\pi\coth\pi y - y^{-1}) + \ln y + \frac{1}{\pi y}\left\{\sum_{1,3,\ldots}^{n-2} \frac{r!\,\zeta(r+1)}{(2\pi y)^r}\right.$$
$$\left. + \frac{n!}{(2\pi y)^n}\sum_{1}^{\infty}\frac{\overline{\Pi}_n(2\pi i\nu y)}{\nu^{n+1}}\right\},$$

where n is an odd integer. Choosing $n = 5$, confirm that $\Psi(2i/\pi)$ is given correctly to within two units in the seventh significant figure.

9. Defining the Bose–Einstein integral by

$$\mathscr{B}_p(x) = \frac{1}{p!}\int_0^\infty \frac{\varepsilon^p d\varepsilon}{e^{\varepsilon - x} - 1}$$

on the understanding the principal value is to be taken when $\Re(x) > 0$, derive the terminated expansion

$$\mathscr{B}_p(x) = \cos \pi p \; \mathscr{B}_p(-x) + \frac{2 \sin \pi p}{\pi}\left\{\sum_{0,2,\ldots}^{n-2} \frac{(r-p-2)!\,\zeta(r)}{x^{r-p-1}} + \frac{(n-p-2)!}{x^{n-p-1}}\sum_1^\infty \frac{\Pi_{n-p-2}(ivx)}{v^n}\right\},$$

where n is an even integer. Choosing $n = 6$, obtain the value $\mathscr{B}_{\frac{1}{2}}(2) = -0{\cdot}751549$.

10. Express the parabolic cylinder function as a special case of the confluent hypergeometric function, $D_p(x) = x^p e^{-\frac{1}{4}x^2}\psi(-\frac{1}{2}p, \frac{1}{2}, \frac{1}{2}x^2)$, and deduce the terminated expansion

$$D_p(x) = \frac{x^p e^{-\frac{1}{4}x^2}}{(-p-1)!}\left\{\sum_0^{n-1} \frac{(2r-p-1)!}{r!\,(-2x^2)^r} + \frac{(2n-p-1)!}{n!\,(-2x^2)^n}\, T_n(\tfrac{1}{2}x^2)\right\},$$

where

$$T_n(z) = \Lambda_{n-\frac{1}{2}p-1}(z) + \frac{p+1}{2n-p-1}\, z\Lambda^{(1)}_{n-\frac{1}{2}p-1}(z)$$

$$+ \frac{(p+1)(p+3)}{(2n-p-1)(2n-p-3)}\, z^2\Lambda^{(2)}_{n-\frac{1}{2}p-1}(z) + \ldots .$$

This series terminates when p is a negative integer, so numerical checks are easiest for the function $Hh_m(x) = e^{-\frac{1}{4}x^2}D_{-m-1}(x)$ tabulated in BA I (1931). Choosing $n = 3$, confirm that $Hh_2(3)$ is given correctly to six units in the seventh significant figure.

11. Prove the relation

$$J_p(x) - iY_p(x) = 2i\pi^{-1}e^{\frac{1}{2}i\pi p}K_p(xe^{\frac{1}{2}i\pi}).$$

Hence express the ordinary Bessel functions as

$$J_p(x) = (2/\pi x)^{\frac{1}{2}} (P \cos \phi - Q \sin \phi),$$
$$Y_p(x) = (2/\pi x)^{\frac{1}{2}} (P \sin \phi + Q \cos \phi),$$

where $\phi_p(x) = x - \frac{1}{2}\pi p - \frac{1}{4}\pi$. Deduce the terminated asymptotic series

$$P_p(x) = \frac{\cos \pi p}{\pi} \left\{ \sum_{0,2,\ldots}^{n-2} \frac{(-1)^{\frac{1}{2}r}(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(2x)^r} + \frac{(-1)^{\frac{1}{2}n}(n+p-\frac{1}{2})!\,(n-p-\frac{1}{2})!}{n!\,(2x)^n} T_n(2x) \right\}, \qquad n \text{ even},$$

$$Q_p(x) = \frac{\cos \pi p}{\pi} \left\{ \sum_{1,3,\ldots}^{n-2} \frac{(-1)^{\frac{1}{2}(r-1)}(r+p-\frac{1}{2})!\,(r-p-\frac{1}{2})!}{r!\,(2x)^r} + \frac{(-1)^{\frac{1}{2}(n-1)}(n+p-\frac{1}{2})!\,(n-p-\frac{1}{2})!}{n!\,(2x)^n} T_n(2x) \right\}, \qquad n \text{ odd},$$

where

$$T_n(z) = \Pi_{n-p-\frac{1}{2}}(z) - \frac{p-\frac{1}{2}}{n+p-\frac{1}{2}} z\Pi^{(1)}_{n-p-\frac{1}{2}}(z) + \frac{(p-\frac{1}{2})(p-\frac{3}{2})}{(n+p-\frac{1}{2})(n+p-\frac{3}{2})} z^2\Pi^{(2)}_{n-p-\frac{1}{2}}(z) - \ldots .$$

Choosing $n = 5, 6$, confirm that $J_1(2)$ and $Y_1(2)$ are given correctly to one and two units respectively in their seventh significant figures.

12. According to Chapter IV, questions 4, 5, and Chapter XVIII, questions 1, 2, 6 and 7, two important integrals in the theory of Anger and Weber functions are

$$\mathscr{C}_p(x) = \int_0^\infty e^{-x \sinh \omega} \cosh p\omega \, d\omega,$$

$$\mathscr{S}_p(x) = \int_0^\infty e^{-x \sinh \omega} \sinh p\omega \, d\omega, \qquad \Re(x) > 0.$$

Establish the relations $\mathscr{C}_p(x) = S_{0,p}(x)$ and $\mathscr{S}_p(x) = pS_{-1,p}(x)$, and write down the terminated asymptotic series.

13. Prove that if Z_p represents any one of the Bessel functions J_p, Y_p, $H_p^{(1)}$ or $H_p^{(2)}$, then

$$z^{-1}\int^{z} z^{q}Z_p(z)\,dz = (p+q-1)Z_p(z)S_{q-1,p-1}(z) - Z_{p-1}(z)S_{q,p}(z).$$

14. Terminate the asymptotic series for the modified Lommel function $\tilde{S}_{qp}(x)$ introduced in Chapter XVIII, question 19, and deduce the relation (Dingle 1959)

$$\tilde{S}_{qp}(x) = i^{1-q}\,S_{qp}(ix) - \frac{i\pi 2^{q}}{\{-\frac{1}{2}(p+q+1)\}!\,\{\frac{1}{2}(p-q-1)\}!}K_p(x).$$

15. Prove that if $\tilde{Z}_p$ represents any one of the modified Bessel functions I_p, K_p or $\mathscr{K}_p$, then

$$z^{-1}\int^{z} z^{q}\tilde{Z}_p(z)\,dz = -(p+q-1)\tilde{Z}_p(z)\tilde{S}_{q-1,p-1}(z) + \tilde{Z}_{p-1}(z)\tilde{S}_{q,p}(z).$$

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Chapter XXIII

Termination of Asymptotic Expansions derived from Integral Representations

1. EVALUATION OF $\int_{\text{limit}} e^{-F}\, G\, du$, $F(u)$ LINEAR AT LIMIT

As shown in Chapter V, Section 2, under the conditions (A) $f = F - F_0 \to F_1 u$ as $u \to 0$, (B) F increases steadily up to $+\infty$ towards the upper limit of integration, and (C) the slowly-varying function $G(u)$ is expansible as a power series in zero and positive integer powers of u,

$$\int_{\text{limit}} e^{-F}\, G\, du = F_1^{-1}\, e^{-F_0} \sum_0^{\infty} L_r \tag{1}$$

where

$$L_r = \frac{F_1\, r!}{2\pi i} \oint \frac{G\, du}{f^{r+1}} . \tag{2}$$

By XXI (15), $\sum_n^\infty r!/f^r = (n!/f^n)\,\Lambda_n(-f)$ except on a Stokes ray, so the interpreted version of (1) is

$$\begin{aligned} \int_{\text{limit}} e^{-F}\, G\, du &= \frac{e^{-F_0}}{F_1} \left\{ \sum_0^{n-1} L_r + T_n L_n \right\} , \\ T_n &= \oint \frac{\Lambda_n(-f)\, G\, du}{f^{n+1}} \bigg/ \oint \frac{G\, du}{f^{n+1}} . \end{aligned} \tag{3}$$

The elegance of this expression for the terminant T_n is matched by its intractability.

Fortunately, an alternative form eminently suited to calculation can be derived from VII (13), according to which the contribution to L_r from a singulant $\mathscr{F}_0$ is

$$L_r = F_1(-2\pi \mathscr{F}_2 \mathscr{F}_0)^{-\frac{1}{2}} \sum_{s=0} (r - s - \tfrac{1}{2})!\, \mathscr{Q}_{2s} / \mathscr{F}_0^{\,r-s} . \tag{4}$$

As explained in detail in Chapter VII, Section 6, the chief singulant is here the change in value of F in going from the limit of integration to the nearest stationary point in the F-plane; and the $\mathscr{Q}$'s are as cited in Chapter V, Section 3, evaluated at this stationary point. By XXI (15),

$$\sum_{r=n}^{\infty} (r-s-\tfrac{1}{2})!/\mathscr{F}_0{}^r = \{(n-s-\tfrac{1}{2})!/\mathscr{F}_0{}^n\}\, \Lambda_{n-s-\frac{1}{2}}(-\mathscr{F}_0) \tag{5}$$

except on a Stokes ray, so

$$T_n = \left\{\mathscr{Q}_0 \Lambda_{n-\frac{1}{2}}(-\mathscr{F}_0) + \frac{\mathscr{Q}_2 \mathscr{F}_0}{n-\frac{1}{2}} \Lambda_{n-\frac{3}{2}}(-\mathscr{F}_0) + \frac{\mathscr{Q}_4 \mathscr{F}_0{}^2}{(n-\frac{1}{2})(n-\frac{3}{2})}\right.$$

$$\left.\times\, \Lambda_{n-5/2}(-\mathscr{F}_0) \ldots\right\} \Big/ \left\{\mathscr{Q}_0 + \frac{\mathscr{Q}_2 \mathscr{F}_0}{n-\frac{1}{2}} + \frac{\mathscr{Q}_4 \mathscr{F}_0{}^2}{(n-\frac{1}{2})(n-\frac{3}{2})} \ldots\right\}. \tag{6}$$

On a Stokes ray—i.e. when all late L_r are the same sign and phase—the Λ in (3) and (6) are to be replaced by $\bar{\Lambda}$.

Extensions of these results to $\int_0 e^{-F} u^\sigma G\, du$ are left as exercises (question 2).

A striking demonstration of the power of our method of interpretation is provided by the continuing applicability of (3)–(6) when the initial condition (B) is transcended, in particular when a stationary point of $F(u)$ comes to lie on or near the path of integration. Since the chief singulant is equal to the change in value of F in going from the limit of integration to the nearest stationary point, $|\mathscr{F}_0|$ decreases as this stationary point draws nearer. The simplest common situation is for it to start negative (stationary point below limit of integration in the u-plane), go to nought (stationary point coincident with limit), then become negative again (stationary point within range of integration): this entails a phase change in $\mathscr{F}_0$ of 2π as it passes through its zero. Now, as we have seen in Chapter XXI (Section 6 and again in (53)), there is a discontinuity in $\Lambda_s(-\mathscr{F}_0)$ when ph $\mathscr{F}_0$ passes through zero amounting to $2\pi i\, \mathscr{F}_0{}^{s+1} e^{-\mathscr{F}_0}/s!$. Combining with (6), the resulting discontinuity in $F_1{}^{-1} e^{-F_0}(T_n L_n)$ appearing in (3) is

$$(2\pi/\mathscr{F}_2)^{\frac{1}{2}}\, e^{-(F_0+\mathscr{F}_0)} \sum_0^{\infty} \mathscr{Q}_{2s}.$$

Recalling the meanings here of F_0 (value of F at limit of integration) and $\mathscr{F}_0$ (change in F in going from limit of integration to stationary point), this is recognised as precisely the asymptotic series in Chapter VI (2) arising from expansion at the stationary point. Hence, after $\mathscr{F}_0$ has touched its zero value, a more convenient form is

$$\int_{\text{limit, (s.p.)}} e^{-F}\,G\,du = \int_{\text{exp at limit}} e^{-F}\,G\,du + \int_{\text{s.p.}} e^{-F}\,G\,du. \tag{7}$$

2. Evaluation of $\int_{\text{limit}} e^{-F}\,G\,du$, $F(u)$ Quadratic at Limit

According to our work in Chapter V, Section 3, when $F - F_0 = f^2 \to \frac{1}{2}F_2u^2$ as $u \to 0$, and F increases steadily up to $+\infty$ towards the upper limit of integration,

$$\int_{\text{limit}} e^{-F}\,G\,du = (\pi/2F_2)^{\frac{1}{2}}\, e^{-F_0}\sum_0^\infty Q_r \tag{8}$$

where

$$Q_r = \left(\frac{F_2}{2\pi}\right)^{\frac{1}{2}} \frac{(\frac{1}{2}r - \frac{1}{2})!}{2\pi i} \oint \frac{G\,du}{f^{r+1}}. \tag{9}$$

Interpreting even and odd orders separately,

$$\int_{\text{limit}} e^{-F}\,G\,du = \left(\frac{\pi}{2F_2}\right)^{\frac{1}{2}} e^{-F_0}\left[\left\{\sum_{0,2,\ldots}^{n-2} Q_r + T_nQ_n\right\} + \left\{\sum_{1,3,\ldots}^{m-2} Q_r + T_mQ_m\right\}\right],$$

$$T_n = \oint \frac{\Lambda_{\frac{1}{2}n-\frac{1}{2}}(-f^2)\,G\,du}{f^{n+1}} \Bigg/ \oint \frac{G\,du}{f^{n+1}} \tag{10}$$

(same formula for m).

The alternative asymptotic form in Chapter VII (12),

$$Q_r = \frac{1}{2\pi}\left(\frac{F_2}{-\mathscr{F}_2}\right)^{\frac{1}{2}} \frac{1}{\mathscr{F}_0^{\frac{1}{2}r}} \sum_{s=0} (\tfrac{1}{2}r - s - 1)!\,\mathscr{Q}_{2s}\mathscr{F}_0^{\,s}, \tag{11}$$

leads to the calculable terminant formula

$$T_n = \left\{\mathscr{Q}_0\Lambda_{\frac{1}{2}n-1}(-\mathscr{F}_0) + \frac{\mathscr{Q}_2\mathscr{F}_0}{\frac{1}{2}n - 1}\Lambda_{\frac{1}{2}n-2}(-\mathscr{F}_0) + \frac{\mathscr{Q}_4\mathscr{F}_0^{\,2}}{(\frac{1}{2}n-1)(\frac{1}{2}n-2)}\right.$$

$$\left.\times\,\Lambda_{\frac{1}{2}n-3}(-\mathscr{F}_0)\ldots\right\}\Bigg/\left\{\mathscr{Q}_0 + \frac{\mathscr{Q}_2\mathscr{F}_0}{\frac{1}{2}n-1} + \frac{\mathscr{Q}_4\mathscr{F}_0^{\,2}}{(\frac{1}{2}n-1)(\frac{1}{2}n-2)}\ldots\right\} \tag{12}$$

(same for m). On a Stokes ray the Λ in (10) and (12) are to be barred.

Extensions to $\int_0 e^{-F}\,u^\sigma\,G\,du$ are left as exercises (question 8).

Though remaining valid, the foregoing interpretations are best rephrased when F ceases to increase steadily as a consequence of an additional stationary point of $F(u)$ coming to lie on or near the path of integration. Since the chief singulant or singulant pair is equal to the least change in value of F in going *either*

(a) from its value at the stationary point—here located at the limit of integration—determining the original asymptotic expansion, to a neighbouring stationary point,

or

(b) from its principal value at the stationary point located at the limit, to a conjugate pair of non-principal values,

$|\mathscr{F}_0|$ decreases in magnitude as this additional stationary point (a.s.p.) draws nearer in the F-plane. The simplest common situation is for it to start negative, drop to nought and then become imaginary, entailing a phase change of $\frac{1}{2}\pi$ as it passes through its zero. The discontinuities of the $\Lambda_s(-\mathscr{F}_0)$ in (12) as ph $\mathscr{F}_0$ passes through zero result in discontinuities in $(\pi/2F_2)^{\frac{1}{2}}\, e^{-F_0}[T_n Q_n$ or $T_m Q_m]$ which are equal and together amount to

$$(2\pi/\mathscr{F}_2)^{\frac{1}{2}}\, e^{-(F_0+\mathscr{F}_0)} \sum_0^\infty \mathscr{Q}_{2s}.$$

Hence, after $\mathscr{F}_0$ has gone through its zero value, a more convenient form is

$$\int_{\text{limit, (a.s.p.)}} e^{-F} G\,du = \int_{\text{exp at limit}} e^{-F} G\,du + \int_{\text{a.s.p.}} e^{-F} G\,du. \tag{13}$$

The quite different outcome for expansion on a Stokes ray is noted in question 9.

3. Evaluation of $\int_{\text{limit}} e^{-F} G\,du$, $F(u)$ Cubic at Limit

As shown in Chapter V, Section 4, when $F - F_0 = f^3 \to \frac{1}{6}F_3 u^3$ as $u \to 0$, and F increases steadily up to $+\infty$ towards the upper limit of integration,

$$\int_{\text{limit}} e^{-F} G\,du = \alpha\, F_3{}^{-\frac{1}{3}}\, e^{-F_0} \sum_0^\infty C_r \tag{14}$$

where

$$C_r = \left(\frac{F_3}{6}\right)^{\frac{1}{3}} \frac{(\frac{1}{3}r - \frac{2}{3})!}{(-\frac{2}{3})!} \frac{1}{2\pi i} \oint \frac{G\,du}{f^{r+1}}. \tag{15}$$

Making separate interpretations for $r = (0, 3, 6, \ldots)$, $(1, 4, 7, \ldots)$ and $(2, 5, 8, \ldots)$,

$$\int_{\text{limit}} e^{-F} G\,du = \alpha F_3^{-\frac{1}{3}} e^{-F_0}\left[\left\{\sum_{0,3,\ldots}^{n-3} C_r + T_n C_n\right\} + \left\{\sum_{1,4,\ldots}^{m-3} C_r + T_m C_m\right\}\right.$$

$$\left. + \left\{\sum_{2,5,\ldots}^{l-3} C_r + T_l C_l\right\}\right], \qquad T_n = \oint \frac{\Lambda_{\frac{1}{3}n-\frac{1}{3}}(-f^3)\,G\,du}{f^{n+1}} \Bigg/ \oint \frac{G\,du}{f^{n+1}}. \tag{16}$$

The alternative asymptotic form in Chapter VII (20), which assumes cubic dependence of $F(u)$ at the stationary point determining late terms, leads to the calculable terminant formula

$$T_n = \{(\tfrac{1}{3}n - 1)!\,\mathscr{C}_0\Lambda_{\frac{1}{3}n-1}(-\mathscr{F}_0) - (\tfrac{1}{3}n - \tfrac{4}{3})!\,\mathscr{C}_1\mathscr{F}_0^{\frac{1}{3}}\Lambda_{\frac{1}{3}n-\frac{4}{3}}(-\mathscr{F}_0)$$

$$+ (\tfrac{1}{3}n - 2)!\,\mathscr{C}_3\mathscr{F}_0\Lambda_{\frac{1}{3}n-2}(-\mathscr{F}_0) - \ldots\}$$

$$\div \{(\tfrac{1}{3}n - 1)!\,\mathscr{C}_0 - (\tfrac{1}{3}n - \tfrac{4}{3})!\,\mathscr{C}_1\mathscr{F}_0^{\frac{1}{3}} + (\tfrac{1}{3}n - 2)!\,\mathscr{C}_3\mathscr{F}_0 - \ldots\}. \tag{17}$$

(Same formula for m and l).

Extensions to $\int_0 e^{-F} u^\sigma G\,du$ are left as exercises (question 14).

Though remaining valid, the foregoing interpretations are best rephrased when F ceases to increase steadily as a consequence of an additional stationary point of $F(u)$ coming to lie on or near the path of integration. The simplest common situation is then for $\mathscr{F}_0$ to start imaginary, drop to nought and then again become imaginary but with reversed sign. Assuming cubic dependence of $\mathscr{F}(u)$ at the additional stationary point (a.s.p.), the resultant discontinuities in $\alpha F_3^{-\frac{1}{3}} e^{-F_0}[T_n C_n$ or $T_m C_m$ or $T_l C_l]$ come out equal, together amounting to

$$i\,3^{\frac{1}{2}}\alpha\mathscr{F}_3^{-\frac{1}{3}} e^{-(F_0+\mathscr{F}_0)}\{\mathscr{C}_0 - \mathscr{C}_1 + \mathscr{C}_3 - \mathscr{C}_4 + \mathscr{C}_6 - \mathscr{C}_7 \ldots\}.$$

Hence, recalling VI (8), after $\mathscr{F}_0$ has gone through its zero value a more convenient form is

$$\int_{\text{limit, (a.s.p.)}} e^{-F} G\,du = \int_{\text{exp at limit}} e^{-F} G\,du + \int_{>} e^{-F} G\,du. \tag{18}$$

4. Evaluation of $\int_{\text{s.p}} e^{-F}\,G\,du$, $F(u)$ Quadratic at Stationary Point

According to our work in Chapter VI, Section 4, when $F - F_0 = f^2 \to \frac{1}{2}F_2u^2$ as $u \to 0$, and F increases steadily up to $+\infty$ towards both limits of integration,

$$\int_{\text{s.p.}} e^{-F}\,G\,du = (2\pi/F_2)^{\frac{1}{2}}\,e^{-F_0}\sum_0^\infty Q_{2r} \tag{19}$$

where

$$Q_{2r} = \left(\frac{F_2}{2\pi}\right)^{\frac{1}{2}}\frac{(r-\frac{1}{2})!}{2\pi i}\oint\frac{G\,du}{f^{2r+1}}. \tag{20}$$

Interpreting by XXI (15),

$$\int_{\text{s.p.}} e^{-F}\,G\,du = \left(\frac{2\pi}{F_2}\right)^{\frac{1}{2}} e^{-F_0}\left\{\sum_0^{n-1} Q_{2r} + T_{2n}Q_{2n}\right\}, \tag{21}$$

where

$$T_{2n} = \oint\frac{\Lambda_{n-\frac{1}{2}}(-f^2)\,G\,du}{f^{2n+1}}\Bigg/\oint\frac{G\,du}{f^{2n+1}}$$

$$= \left\{\mathcal{Q}_0\Lambda_{n-1}(-\mathcal{F}_0) + \frac{\mathcal{Q}_2\mathcal{F}_0}{n-1}\Lambda_{n-2}(-\mathcal{F}_0) + \frac{\mathcal{Q}_4\mathcal{F}_0^{\,2}}{(n-1)(n-2)}\Lambda_{n-3}(-\mathcal{F}_0)\right.$$

$$\left. + \ldots\right\}\Bigg/\left\{\mathcal{Q}_0 + \frac{\mathcal{Q}_2\mathcal{F}_0}{n-1} + \frac{\mathcal{Q}_4\mathcal{F}_0^{\,2}}{(n-1)(n-2)} + \ldots\right\}. \tag{22}$$

These expressions are best rephrased when F ceases to increase steadily up to ∞ towards both limits of integration, in particular when an additional stationary point (a.s.p.) of $F(u)$ comes to lie on or near the path of integration. The discontinuities of the $\Lambda_s(-\mathcal{F}_0)$ in (22) as ph $\mathcal{F}_0$ passes through zero result in a discontinuity in $(2\pi/F_2)^{\frac{1}{2}}\,e^{-F_0}\,T_{2n}\,Q_{2n}$ of

$$(2\pi/\mathcal{F}_2)^{\frac{1}{2}}\,e^{-(F_0+\mathcal{F}_0)}\sum_0^\infty \mathcal{Q}_{2s}.$$

Hence, after $\mathcal{F}_0$ has gone through its zero value, a more convenient form is

$$\int_{\text{s.p.,(a.s.p.)}} e^{-F}\,G\,du = \int_{\text{s.p.}} e^{-F}\,G\,du + \int_{\text{a.s.p.}} e^{-F}\,G\,du, \tag{23}$$

where the contour through a.s.p. ($\rightarrow$ or $\uparrow$) is the same as prescribed through s.p. Note that (23) is not a general proposition on the additivity of contributions from stationary points; it refers solely to additivity, under the stated conditions, of a contribution from a *specific* second stationary point—that determining a singulant dominating late terms in the expansion around the first stationary point, provided this expansion is not on a Stokes ray. More general questions of additivity have already been examined in Chapter VI, Section 2.

On a Stokes ray—i.e. when all late Q_{2r} are of the same sign and phase—the Λ in (22) are to be replaced by $\overline{\Lambda}$. Before invoking (23), which refers to a total phase change of around 2π in the argument of Λ, we have to come off the Stokes ray and express the $\overline{\Lambda}$ in terms of Λ. As noted in Section 2, for quadratic dependence of $F(u)$ at a stationary point the usual phase change in $\mathscr{F}_0$ is $\frac{3}{2}\pi$, i.e. less than 2π, so we come off the Stokes ray on the side $\operatorname{ph}\mathscr{F}_0 < 0$. By XXI (33), $\overline{\Lambda}_s(-\mathscr{F}_0)$ is then to be replaced by $\Lambda_s(-\mathscr{F}_0) - \pi i \mathscr{F}_0^{\,s+1} e^{-\mathscr{F}_0}/s!$, leading to the equivalence

$$\int_{\text{s.p.}} e^{-F}\, G\, du \bigg|_{\operatorname{ph}\mathscr{F}_0 = 0} = \int_{\text{s.p.}} e^{-F}\, G\, du \bigg|_{\operatorname{ph}\mathscr{F}_0 \to -0} - \tfrac{1}{2}\int_{\text{a.s.p.}} e^{-F}\, G\, du. \tag{24}$$

On applying (23) to each of the integrals on the right, the more convenient form of $\int_{\text{s.p.}} e^{-F}\, G\, du$ (*when initially defined on a Stokes ray*) after $\mathscr{F}_0$ has gone through its zero value is seen to be

$$\int_{\text{s.p.},(\text{a.s.p.})} e^{-F}\, G\, du = \tfrac{1}{2}\left\{\int_{\text{s.p.}} e^{-F}\, G\, du + \int_{\text{a.s.p.}} e^{-F}\, G\, du\right\}, \tag{25}$$

where the contour through a.s.p. is the same as prescribed through s.p., i.e. $\rightarrow$ correlating with specification on a Stokes ray (Chapter VI, Section 6).

5. Evaluation of $\int_{\text{s.p.}} e^{-F}\, G\, du$, $F(u)$ Cubic at Stationary Point

As shown in Chapter VI, Sections 5 and 6, even for a single stationary point results for cubic behaviour depend critically on the contour designated. The variants all involve essentially the same set of coefficients and terminants—those already discussed in Section 3 in the context of cubic behaviour at a limit—so for illustrative purposes it will suffice to quote explicit interpretations for just two important typical cases, VI (8) and (12):

$$\int_{>} e^{-F} G\,du = i\,3^{\frac{1}{2}}\alpha\, F_3^{-\frac{1}{3}} e^{-F_0}\left[\left\{\sum_{0,3,..}^{n-3} C_r + T_n C_n\right\} - \left\{\sum_{1,4,\ldots}^{m-3} C_r + T_m C_m\right\}\right], \tag{26}$$

$$\int_{\frac{1}{2}(\backslash\_ - /\overline{\ })} e^{-F} G\,du = \tfrac{3}{2}\alpha\, F_3^{-\frac{1}{3}} e^{-F_0}\left[\left\{\sum_{0,3,\ldots}^{n-3} C_r + T_n C_n\right\} + \left\{\sum_{1,4,\ldots}^{m-3} C_r + T_m C_m\right\}\right]. \tag{27}$$

The influence of an additional stationary point (a.s.p.) also depends critically on the original contour. After $\mathscr{F}_0$ has gone through its zero value, (26) needs no rephrasing because discontinuities in $T_n C_n$ and $T_m C_m$ cancel, but in (27) they add to give the more convenient form

$$\int_{\frac{1}{2}(\backslash\_ - /\overline{\ })} e^{-F} G\,du = \int_{\frac{1}{2}(\backslash\_ - /\overline{\ })} e^{-F} G\,du \Big|_{\text{s.p.}} + \int_{>} e^{-F} G\,du \Big|_{\text{a.s.p.}}. \tag{28}$$

6. Computational Accuracy Obtainable from Terminant Expansions of this Type

The preceding chapter dealt with the case of an asymptotic power series for a function ϕ in which the general late term $A_{n \gg 1}$ could be simply expressed. We now suppose it to be legitimate to regard the problem before us, in which late terms are not simply expressed, as the case of an asymptotic expansion for a function $\Phi(\phi)$ of such a function ϕ. This viewpoint is plainly correct for the factorial function $\Phi = p!$ to be examined in the next section, since $\Phi = e^{\phi}$ where $\phi = \ln p!$ was the asymptotic power series interpreted in Section 3 of the preceding chapter.

If

$$\phi = 1 + A_1 + A_2 + \ldots + A_{m-1} + A_m + \ldots,$$

the late term in $\Phi(\phi)$ of order m will in general involve A_m, $A_{m-1}A_1$, $A_{m-2}A_2$, ... up to $A_{\frac{1}{2}(m+1)}A_{\frac{1}{2}(m-1)}$ if m is odd, or up to $A_{\frac{1}{2}m}{}^2$ if m is even. [For instance, with $\Phi = \phi^2$ the late term in Φ of order m is

$$2\sum_{0}^{\frac{1}{2}(m-1)} A_{m-s}A_s, \qquad m \text{ odd}; \qquad 2\sum_{0}^{\frac{1}{2}m-1} A_{m-s}A_s + A_{\frac{1}{2}m}{}^2, \qquad m \text{ even}].$$

A terminant of given order is formed from late terms of that order and higher. Hence for the same level of sophistication adopted in the numerical analysis, the terminant of order m appropriate to the asymptotic expansion Φ is obtained only to an accuracy comparable with that of the terminant of order $\frac{1}{2}m$ appropriate to the asymptotic power series ϕ. In practice this means poor accuracy if the least term in an asymptotic expansion (as opposed to power series) is earlier than the fourth. The accuracy improves rapidly for $m > 4$; in the terminant expansion (12), for example, the last retainable contributions contain the divisor $(\frac{1}{2}m - 1)!$.

This rapid improvement in accuracy for $m > 4$ raises the question as to whether it might not be preferable to adopt the practice of retaining unmodified all exactly known terms L_r, Q_r or C_r, and calculate the remainder $T_m \times m$'th term entirely from the expansion for late terms. Such a practice is indeed obligatory when too few early terms are known in closed form—e.g. for $W_{km}(4k)$, $W_{km}(2k + 2\kappa)$ and their first derivatives (Sections 11, 12)—but otherwise our experience is that it is more accurate to calculate the remainder as the product of the least or last exactly known term and a terminant expressed as an average in quotient form, as in (6), (12), (17) and (22). The quotient of numerator and denominator expansions, which are so similar they can be truncated in exactly the same manner, largely compensates for errors of estimation in each.

The finite summations and parity sensitivity brought to mind in the earlier part of this discussion could occasion momentary anxiety as to the exactitude of the asymptotic expansions for L_r, Q_r and C_r; should these not strictly be parity-sensitive terminated summations likewise? Such suspicions are speedily dispelled by a variety of arguments. In particular, keeping to the quadratic case:

(a) In the reasoning leading to the expansions in Chapter VII (12) and (13), we showed how exclusion of overlapping terms was safeguarded by omitting $\mathcal{Q}_s$ for *odd* s, not for *large* s; the issue has therefore already been resolved.

(b) After $\mathcal{F}_0$ has gone through its zero value, a contribution proportional to $\Sigma\mathcal{Q}_{2s}$ separates out from the terminant expansion. To coincide with $\int_{\text{a.s.p.}} e^{-F}\, G\, du$ as anticipated, this must be the *infinite* sum $s = 0$ to ∞.

To place this section in perspective, we conclude with the reminder that late terms and terminants are theoretically known exactly as integral representations. We have available two varieties of each—the closed-contour integrals introduced in the text, and the $\int_0^\infty$ representations of questions 3–7, 10–13, 15–19. Terminant expansions of the type treated in the text, which need truncation numerically (though not analytically), are being tolerated only as a compromise between suitability for analytical work and ease of computation. Alternatives are indicated in questions 7, 13 and 19.

7. Factorial Function

In applying the theory of Section 4 to the results of Chapter VIII, Section 2, the sole complication is the division of the Q_{2r} into two regular sequences. Written in decimals, early terms are as follows:

$$Q_2 = +\cdot 08\dot{3}/p, \qquad Q_6 = -\cdot 0^2 2681327160/p^3,$$

$$Q_{10} = +\cdot 0^3 7840392217/p^5, \qquad Q_{14} = -\cdot 0^3 5921664362/p^7,$$

$$Q_{18} = +\cdot 0^3 8394987206/p^9;$$

$$Q_0 = 1, \qquad Q_4 = +\cdot 0^2 347\dot{2}/p^2, \qquad Q_8 = -\cdot 0^3 2294720936/p^4,$$

$$Q_{12} = +\cdot 0^4 6972813758/p^6, \qquad Q_{16} = -\cdot 0^4 5171790908/p^8,$$

$$Q_{20} = +\cdot 0^4 7204895411/p^{10}.$$

Taking into account this division into two sequences, the interpretation of VIII (15) is

$$p! = (2\pi p)^{\frac{1}{2}}\, p^p\, e^{-p}\left[\left\{\sum_{1,3,\ldots}^{n-2} Q_{2r} + T_{2n}{}^{o} Q_{2n}\right\} + \left\{\sum_{0,2,\ldots}^{m-2} Q_{2r} + T_{2m}{}^{e} Q_{2m}\right\}\right], \tag{29}$$

where on combining (22) and VIII (18)

$$\begin{aligned} T_{2n}{}^{o} = {} & \left[\sum_{\nu=1}^{\infty} \frac{1}{\nu^{n+1}} \left\{\Pi_{n-1}(2\pi\nu p) - \frac{\pi^2\nu^2}{72(n-1)(n-2)}\Pi_{n-3}(2\pi\nu p)\right.\right. \\ & \left.\left. - \frac{571\pi^4\nu^4}{155{,}520(n-1)(n-2)(n-3)(n-4)}\Pi_{n-5}(2\pi\nu p)\ldots\right\}\right] \\ & \div \left\{\zeta(n+1) - \frac{\pi^2}{72(n-1)(n-2)}\zeta(n-1)\right. \\ & \left. - \frac{571\pi^4}{155{,}520(n-1)(n-2)(n-3)(n-4)}\zeta(n-3)\ldots\right\}, \end{aligned} \tag{30}$$

$$T_{2m}{}^{e} = \left[\sum_{\nu=1}^{\infty} \frac{1}{\nu^m}\left\{\Pi_{m-2}(2\pi\nu p) + \frac{139\pi^2\nu^2}{1080(m-2)(m-3)}\Pi_{m-4}(2\pi\nu p)\right.\right.$$
$$\left.\left. + \frac{163{,}879\pi^4\nu^4}{1{,}088{,}740(m-2)(m-3)(m-4)(m-5)}\Pi_{m-6}(2\pi\nu p)\ldots\right\}\right]$$
$$\div \left\{\zeta(m) + \frac{139\pi^2}{1080(m-2)(m-3)}\zeta(m-2)\right.$$
$$\left. + \frac{163{,}879\pi^4}{1{,}088{,}740(m-2)(m-3)(m-4)(m-5)}\zeta(m-4)\ldots\right\}. \qquad (31)$$

Choosing $n = 5$ and $m = 6$, the result obtained for $(2/\pi)!$ is ·89816296 as compared with the correct value ·89816233, a creditable accuracy for so small an argument, though as expected from Section 6 inferior to that from Chapter XXII (20) for the logarithm of this factorial.

8. Incomplete Factorial Function

$x < p$

Applying the theory developed in Section 1, the interpretation of VIII (21) + (23) is

$$(p, x)! = \frac{x^{p+1}\,e^{-x}}{p-x}\left\{\sum_{0}^{n-1} L_r + T_n L_n\right\}, \qquad (32)$$

with

$$T_n = \left\{\Lambda_{n-\frac{1}{2}}(p\Xi) - \frac{\Xi}{12(n-\frac{1}{2})}\Lambda_{n-\frac{3}{2}}(p\Xi) + \frac{\Xi^2}{288(n-\frac{1}{2})(n-\frac{3}{2})}\Lambda_{n-5/2}(p\Xi)\right.$$
$$\left. + \frac{139\Xi^3}{51{,}840(n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2})}\Lambda_{n-7/2}(p\Xi)\ldots\right\}$$
$$\div \left\{1 - \frac{\Xi}{12(n-\frac{1}{2})} + \frac{\Xi^2}{288(n-\frac{1}{2})(n-\frac{3}{2})}\right.$$
$$\left. + \frac{139\Xi^3}{51{,}840(n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2})}\ldots\right\} \qquad (33)$$

where

$$\Xi = \ln(p/x) + x/p - 1. \qquad (34)$$

Early L's, from $L_0 = 1$ up to L_7, are quoted in Chapter VIII (21).

Choosing $n = 6$, the result obtained for $(6, 1)!$ is ·05993364 as compared with the correct value ·05993377.

$x > p$

The singulant $\mathscr{F}_0 = -p\Xi$ touches its zero value when $x = p$, so for $x > p$ (7) provides the more convenient form:

$$(p, x)! = \int_{\text{exp at } x} t^p e^{-t}\, dt + \int_{\text{s.p.}} t^p e^{-t}\, dt.$$

Expansion at the stationary point $t = p$ contributes $p!$, and the right-hand side of (32)—expansion at x—now represents $(p, x)! - p! = -[p, x]!$. Hence

$$[p, x]! = \frac{x^{p+1} e^{-x}}{x - p} \left\{ \sum_0^{n-1} L_r + T_n L_n \right\}, \tag{35}$$

where the meanings of L_r and T_n are as before. This result could alternatively have been written down from Chapter VIII (26).

Choosing $n = 3$, the result obtained for $[3, 10]!$ is ·0620165 as compared with the correct value ·0620163.

When p is negative (33) has to be revised, because the previous sole singulant is replaced by a singulant pair. In the notation of Chapter VIII (29) the new terminant is easily found to be

$$T_n = \Im\left\{ e^{i(n+\frac{1}{2})\theta} \Lambda_{n-\frac{1}{2}}(p|\Xi|\, e^{-i\theta}) + \frac{|\Xi|\, e^{i(n-\frac{1}{2})\theta}}{12(n-\frac{1}{2})} \Lambda_{n-\frac{3}{2}}(p|\Xi|\, e^{-i\theta}) + \dots \right\}$$

$$\div \left\{ \sin (n + \tfrac{1}{2})\theta + \frac{|\Xi| \sin (n - \frac{1}{2})\theta}{12(n - \frac{1}{2})} + \dots \right\} \tag{36}$$

Unfortunately, tables of basic terminants have as yet been published only for real argument (Dingle 1958). The essential involvement here of a basic terminant for complex argument albeit in the interpretation of an asymptotic expansion for a real function of a real variable, indicates how widely tables of $\Lambda_s(x)$ for complex x would extend the range of numerical asymptotics. (For complex x, separate calculations of Π, $\bar{\Lambda}$ and $\bar{\Pi}$ are redundant, since these can be deduced from XXI (19), (33) and (34)).

$x \sim p$

The series in Chapter VIII (30) for $(p, x)!$, which is convergent for $|(x - p)/p| < 1$, contains the one-parameter function $(p, p)!$. The asymptotic expansion in VIII (33) for $(p, p)!$ includes that of $\frac{1}{2}(p!)$, so to avoid repeating results obtained in Section 7 we shall extract this contribution and correspondingly omit Q's of even order in the full expansion.

The Q_{2r+1} divide into two regular sequences. Written in decimals, early terms are as follows:

$$Q_1 = -\cdot 53192304056/p^{\frac{1}{2}}, \qquad Q_5 = -\cdot 0^2 2251526089/p^{5/2},$$

$$Q_9 = +\cdot 0^3 56806181/p^{9/2}, \qquad Q_{13} = -\cdot 0^3 37729104/p^{13/2},$$

$$Q_{17} = +\cdot 0^3 48020491/p^{17/2};$$

$$Q_3 = +\cdot 023641024/p^{\frac{3}{2}}, \qquad Q_7 = -\cdot 0^2 15010174/p^{7/2},$$

$$Q_{11} = +\cdot 0^3 54126302/p^{11/2}, \qquad Q_{15} = -\cdot 0^3 47086072/p^{15/2},$$

$$Q_{19} = +\cdot 0^3 74531482/p^{19/2}.$$

Allowing for these separate sequences, the interpretation of VIII (33) is

$$(p,p)! = \tfrac{1}{2}(p!) + (\tfrac{1}{2}\pi p)^{\frac{1}{2}} p^p e^{-p}\left[\left\{\sum_{0,2,..}^{m-2} Q_{2r+1} + T_{2m+1}^{-} Q_{2m+1}\right\}\right.$$
$$\left. + \left\{\sum_{1,3,\ldots}^{n-2} Q_{2r+1} + T_{2n+1}^{+} Q_{2n+1}\right\}\right], \tag{37}$$

where by VIII (34)

$$T_{2m+1}^{\mp} = \left[\sum_{v=1}^{\infty} \frac{1}{v^{m+\frac{3}{2}}}\left\{\Pi_{m-\frac{1}{2}}(2\pi vp) \mp \frac{\pi v}{6(m-\frac{1}{2})}\Pi_{m-\frac{3}{2}}(2\pi vp)\right.\right.$$
$$\left.\left. - \frac{\pi^2 v^2}{72(m-\frac{1}{2})(m-\frac{3}{2})}\Pi_{m-5/2}(2\pi vp)\ldots\right\}\right]$$
$$\div \left\{\zeta(m+\tfrac{3}{2}) \mp \frac{\pi}{6(m-\frac{1}{2})}\zeta(m+\tfrac{1}{2})\right.$$
$$\left. - \frac{\pi^2}{72(m-\frac{1}{2})(m-\frac{3}{2})}\zeta(m-\tfrac{1}{2})\ldots\right\}. \tag{38}$$

The formula for $[p,p]!$ differs from (37) only by having a minus sign after $\frac{1}{2}(p!)$.

9. FERMI–DIRAC INTEGRAL

Application of the theory of Section 4 to the results of Chapter VIII, Section 4 is similar to that for the factorial function already treated in Section 7. The results are:

$$\mathscr{F}_{p-1}(x) = \frac{(1-\tau^2)(p/\tau)^p}{2\,p!}\left\{\frac{\pi}{1-\tau^2(1-2/p)}\right\}^{\frac{1}{2}}\left[\left\{\sum_{1,3,\ldots}^{n-2} Q_{2r} + T_{2n}{}^{o}\, Q_{2n}\right\}\right.$$

$$\left. + \left\{\sum_{0,2,\ldots}^{m-2} Q_{2r} + T_{2m}{}^{e}\, Q_{2m}\right\}\right], \qquad (39)$$

where

$$T_{2n}{}^{o} = \left[\sum_{\nu=1}^{\infty} \frac{1}{\nu^{n+1}}\left\{\Pi_{n-1}(2\pi\nu p) - \frac{(2\pi p)^2\nu^2}{(n-1)(n-2)}\, Q_4 \Pi_{n-3}(2\pi\nu p)\ldots\right\}\right]$$

$$\div \left\{\zeta(n+1) - \frac{(2\pi p)^2}{(n-1)(n-2)}\, Q_4\, \zeta(n-1)\ldots\right\}, \qquad n \text{ odd}, \qquad (40)$$

$$T_{2m}{}^{e} = \left[\sum_{\nu=1}^{\infty} \frac{1}{\nu^{m}}\left\{\Pi_{m-2}(2\pi\nu p) - \frac{(2\pi p)^2\,\nu^2}{(m-2)(m-3)}\, \frac{Q_6}{Q_2}\, \Pi_{m-4}(2\pi\nu p)\ldots\right\}\right]$$

$$\div \left\{\zeta(m) - \frac{(2\pi p)^2}{(m-2)(m-3)}\, \frac{Q_6}{Q_2}\, \zeta(m-2)\ldots\right\}, \qquad m \text{ even}. \qquad (41)$$

These formulae appertain to both ranges $p > x$ and $p < x$, but—for the same labour and level of sophistication adopted in the numerical analysis—prove much more accurate in the former range. To see why, let us return to the initial integral

$$\mathscr{F}_{p-1}(x) = \frac{1}{p!}\int_0^\infty \frac{t^p\, e^{x-t}\, dt}{(e^{x-t}+1)^2},$$

and examine the integrand's behaviour near the stationary point.

$p > x > 0$

In this range the combination $t^p\, e^{-t}$ dominates, because near its maximum at $t = p$

$$(e^{x-t}+1)^{-2} \sim (e^{x-p}+1)^{-2} \sim 1 \qquad \text{when} \qquad p > x.$$

The accuracy from (39)–(41) is then comparable to that from (29)–(31) for the factorial function, and is high because $t^p e^{-t}$ has a plain, almost symmetrical peak.

$0 < p < x$

In this range the dominant combination is $e^{x-t}/(e^{x-t}+1)^2$, which is symmetrical about its maximum at $t = x$. But the factor t^p, fast-rising when p is large, shifts the peak to larger t, and complicates and asymmetrizes its shape. The consequent drop in accuracy of the stationary-point expansion will evidently be most marked when p is slightly smaller than x.

To illustrate this diminished accuracy we take $p = \frac{3}{2}$ and $x = 2$, pertinent to $\mathscr{F}_{\frac{1}{2}}(2)$, and compare results at various stages with those from the analogous calculation of $\frac{3}{2}!$. From Chapter VIII (39) the location of the stationary point is determined from

$$\tau = \frac{\cdot 75}{1 + \tanh^{-1}\tau}.$$

If $\tanh^{-1}\tau$ were replaced by τ the solution would be ·5. Allowing for the underestimate of $\tanh^{-1}\tau$ we start seriously with $\tau \simeq \cdot 49$, giving the next approximation as $\tau \simeq \cdot 75/(1+\tanh^{-1}\cdot 49) = \cdot 488$. Resubstituting a few times for $\tanh^{-1}$, and then refining by Newton's method, the solution is easily found to be $\tau = \cdot 4887747333$. Note that the stationary point is at $t_0 = p/\tau \simeq 3{\cdot}07$, well above 2; this shows how seriously the symmetrical peak of $e^{2-t}/(e^{2-t}+1)^2$ at $t = 2$ has been distorted and displaced by the factor $t^{\frac{3}{2}}$. This complexity in resulting peak shape leads to irregularity in signs and magnitudes of derivatives calculated from VIII (40): compare the regularly alternating signs and smoothly varying magnitudes for the derivatives $F_{j \geq 2} = (-1)^j (j-1)!/p^{j-1}$ of the exponent in the integral for the factorial function. Because of this irregularity, cancellation between contributions of a given order in the asymptotic expansion is less systematic, leaving relatively larger late Q_{2r}'s.

j	F_j for $\mathscr{F}_{\frac{1}{2}}(2)$
2	+ ·53981 67900
3	− ·28979 73956
4	+ ·04755 96802
5	+ ·10644 90408
6	+ ·07705 79930
7	− ·90805 07729
8	+ 2·15174 1139
9	− 1·88039 0227
10	− 1·53230 2285

r	Q_{2r} for $\mathscr{F}_{\frac{1}{2}}(2)$	Q_{2r} for $\frac{3}{2}!$
0	1	1
1	+ ·09082 53514	+ ·0$\dot{5}$
2	− ·01315 00518	+ ·00154 32099
3	− ·00700 37504	− ·00079 44673
4	+ ·00502 14206	− ·00004 53278

Accuracy is adversely affected in two ways:

(i) The least term in the asymptotic expansion is far larger than in the corresponding expansion for $p!$.

(ii) In the asymptotic expansions (40) and (41) for the terminants, the coefficients $(2\pi p)^2 Q_4/Q_0$, $(2\pi p)^2 Q_6/Q_2$, etc. are abnormally large, thwarting accurate evaluations for moderately small n and m.

Illustrating by our example $\mathscr{F}_{\frac{1}{2}}(2)$, $T_6{}^o$ can be reasonably confidently estimated from (40) to two places, ·93. But on attempting to apply (41) to $T_8{}^e$, the correction terms are seen to exceed the first! The best we can do is to retain only the leading term and thus assume $T_8{}^e \sim \Pi_2(3\pi) = \cdot 89$. With these guestimates, $\mathscr{F}_{\frac{1}{2}}(2)$ is given correctly to five significant places.

This problem has been analysed in some detail because the conclusions are almost certainly general. In a stationary-point expansion where the peak is of complicated asymmetrical shape, the asymptotic expansions for the terminants T_{2n} and T_{2m} do not begin to provide precise estimates until $n = 5$ and $m = 6$, thereby demanding exact calculations of Q_{10} and Q_{12}. Extension of the formulae for Q's—quoted up to Q_8 in Chapter V, Section 3—must therefore be assigned high priority for the future, to cope with this obtuse class of problem to the high precision now achieved elsewhere.

10. Bessel Functions $J_p(x)$ and $Y_p(x)$

Applying the theory of Section 4, the interpretation of VIII (48) + (50) is

$$J_p(x) = (q/2\pi p)^{\frac{1}{2}} e^{-p\Xi} \left\{ \sum_0^{n-1} Q_{2r}{}^- + T_{2n} Q_{2n}{}^- \right\}, \tag{42}$$

with

$$\begin{aligned} T_{2n} = &\left\{ \Lambda_{n-1}(2p\Xi) - \frac{\Xi q(5q^2-3)}{12(n-1)} \Lambda_{n-2}(2p\Xi) \right. \\ &\left. + \frac{\Xi^2 q^2(385q^4 - 462q^2 + 81)}{288(n-1)(n-2)} \Lambda_{n-3}(2p\Xi) - \dots \right\} \\ &\div \left\{ 1 - \frac{\Xi q(5q^2-3)}{12(n-1)} + \frac{\Xi^2 q^2(385q^4 - 462q^2 + 81)}{288(n-1)(n-2)} - \dots \right\} \end{aligned} \tag{43}$$

where

$$q = p/(p^2 - x^2)^{\frac{1}{2}}, \qquad \Xi = \tanh^{-1} q^{-1} - q^{-1}. \tag{44}$$

The first five Q_{2r}^- are quoted in Chapter VIII (48). Two additional terms, first published by Bickley (1952), are:

$$Q_{10}^- = -\frac{q^5}{6{,}688{,}604{,}160p^5}\{188{,}699{,}385{,}875q^{10} - 566{,}098{,}157{,}625q^8$$
$$+ 614{,}135{,}872{,}350q^6 - 284{,}499{,}769{,}554q^4 + 49{,}286{,}948{,}607q^2$$
$$- 1{,}519{,}035{,}525\},$$

$$Q_{12}^- = \frac{q^6}{4{,}815{,}794{,}995{,}200p^6}\{1{,}023{,}694{,}168{,}371{,}875q^{12}$$
$$- 3{,}685{,}299{,}006{,}138{,}750q^{10} + 5{,}104{,}696{,}716{,}244{,}125q^8$$
$$- 3{,}369{,}032{,}068{,}261{,}860q^6 + 1{,}050{,}760{,}774{,}457{,}901q^4$$
$$- 127{,}577{,}298{,}354{,}750q^2 + 2{,}757{,}049{,}477{,}875\}.$$

The point on accuracy argued in Section 6 will be illustrated here by calculation of $J_5(3)$, where the least term in the asymptotic expansion is the fourth, Q_6^-, so entailing the over-small value $n = 3$ in (43). Retention of half the terms linear in Ξ, because the series are alternating, leads to the estimate $T_6 = \cdot 533$. This gives $J_5(3) = \cdot 0430280$, four units low in the sixth significant place.

The interpretation of VIII (70) is

$$Y_p(x) = -(2q/\pi p)^{\frac{1}{2}} e^{p\Xi}\left\{\sum_0^{n-1} Q_{2r}^+ + \bar{T}_{2n}Q_{2n}^+\right\}, \tag{45}$$

where the Q^+ differ from the Q^- only by all being prefaced by a positive sign (cf. VIII (70)), and the terminant $\bar{T}$ differs from (43) only by replacement of Λ by $\bar{\Lambda}(-2p\Xi)$. The point about accuracy (Section 6) is shown more strongly here because of the much faster increase in $\bar{\Lambda}_s$ with falling s. The estimate of $\bar{T}_6 = \cdot 63$ is limited to two places, and the resultant accuracy of $Y_5(3) = -1{\cdot}9060$ to five.

$x > p > 0$

By (44), Ξ goes through its zero value when $x = p$. Before it does, i.e. when $x < p$, $J_p(x)$ is expressed through expansion—not on a Stokes ray—at one stationary point, VIII (45), and the singulant is determined by the distance from this stationary point to the other one. Hence after Ξ has gone through its zero value, i.e. when $x > p$, $J_p(x)$ is expressed via (23) as the sum of expansions at both stationary points, in agreement with VIII (51) argued on other grounds.

The interpretation of the resulting expressions in Chapter VIII (53) + (54) is

$$J_p(x) = (2\varphi/\pi p)^{\frac{1}{2}}\left[\left\{\sum_{0,2,\ldots}^{n-2} Q_{2r} + \mathfrak{T}_{2n}Q_{2n}\right\}\sin\left(p\Upsilon + \tfrac{1}{4}\pi\right)\right.$$

$$\left. - \left\{\sum_{1,3,\ldots}^{m-2} Q_{2r} + \mathfrak{T}_{2m}Q_{2m}\right\}\cos\left(p\Upsilon + \tfrac{1}{4}\pi\right)\right], \tag{46}$$

with

$$\mathfrak{T}_{2n} = \left\{\Pi_{n-1}(2p\Upsilon) - \frac{\Upsilon\varphi(5\varphi^2+3)}{12(n-1)}\Pi_{n-2}(2p\Upsilon)\right.$$

$$\left. + \frac{\Upsilon^2\varphi^2(385\varphi^4 + 462\varphi^2 + 81)}{288(n-1)(n-2)}\Pi_{n-3}(2p\Upsilon) - \ldots\right\}$$

$$\div\left\{1 - \frac{\Upsilon\varphi(5\varphi^2+3)}{12(n-1)} + \frac{\Upsilon^2\varphi^2(385\varphi^4 + 462\varphi^2 + 81)}{288(n-1)(n-2)} - \ldots\right\} \tag{47}$$

(same for m), where

$$\varphi = p/(x^2 - p^2)^{\frac{1}{2}}, \qquad \Upsilon = \varphi^{-1} - \tan^{-1}\varphi^{-1}. \tag{48}$$

The first five Q_{2r} are quoted in Chapter VIII (53). Others can be written down from the $Q_{2r}{}^{\pm}$ by substituting $q^2 = -\varphi^2$ and making the sign or phase $(-1)^{\frac{1}{2}r}$ when r is even and $(-1)^{\frac{1}{2}(r-1)}$ when odd.

Choosing $n = 4$ and $m = 5$, the result obtained for $J_6(10)$ is $-{\cdot}01445885$. This is one unit high in its seventh significant figure, a good accuracy considering there would be uncertainty in the third place without the terminant.

When $x < p$, $Y_p(x)$ is expressed through expansion—on a Stokes ray—at one stationary point, VIII (68), and the singulant is determined by the distance from this stationary point to the other one. Hence after Ξ has gone through its zero value, i.e. when $x > p$, $Y_p(x)$ is expressed via (25) as the mean of expansions at both stationary points, in agreement with VIII (71) argued on other grounds.

The interpretation is

$$Y_p(x) = -(2\varphi/\pi p)^{\frac{1}{2}}\left[\left\{\sum_{0,2,\ldots}^{n-2} Q_{2r} + \mathfrak{T}_{2n}Q_{2n}\right\}\cos\left(p\Upsilon + \tfrac{1}{4}\pi\right)\right.$$

$$\left. + \left\{\sum_{1,3,\ldots}^{m-2} Q_{2r} + \mathfrak{T}_{2m}Q_{2m}\right\}\sin\left(p\Upsilon + \tfrac{1}{4}\pi\right)\right], \tag{49}$$

with the same meanings (47) and (48) for the symbols.

The result obtained for $Y_6(10)$ is ·2803528, two units high in the seventh place.

$x \sim p$

The Taylor expansion in Chapter VIII (57) for $J_p(x)$, itself absolutely convergent, contains the one-parameter functions $J_p(p)$ and $J_p{}'(p)$ for which only asymptotic expansions are convenient when p is large.

Applying the theory of Section 5, the interpretation of VIII (60) + (62) is

$$J_p(p) = \cdot 4473073184 p^{-\frac{1}{3}} \left[\left\{ \sum_{0,6,12,\ldots}^{n-6} C_r + T_n C_n \right\} - \left\{ \sum_{4,10,16,\ldots}^{m-6} C_r + T_m C_m \right\} \right] \tag{50}$$

with

$$T_n = \{(\tfrac{1}{3}n - 1)!\, \Pi_{\frac{1}{3}n-1}(2\pi p) - \cdot 1521315(\tfrac{1}{3}n - \tfrac{7}{3})!\, \Pi_{\frac{1}{3}n-7/3}(2\pi p) + \cdot 1754596(\tfrac{1}{3}n - 3)!\, \Pi_{\frac{1}{3}n-3}(2\pi p) \ldots\} \div \{(\tfrac{1}{3}n - 1)! - \cdot 1521315(\tfrac{1}{3}n - \tfrac{7}{3})! + \cdot 1754596(\tfrac{1}{3}n - 3)! \ldots\} \tag{51}$$

(same for m). Early C_r are as follows:

$$C_0 = 1, \qquad C_6 = -1/225p^2 = -\cdot 00\dot{4}/p^2, \qquad C_{12} \doteqdot \cdot 0^3 693735/p^4,$$
$$C_{18} \doteqdot -\cdot 0^3 3538/p^6;$$
$$C_4 = 3\beta/35p^{\frac{4}{3}} = \cdot 013121378/p^{\frac{4}{3}},$$
$$C_{10} = -1213\beta/170{,}625p^{10/3} = -\cdot 0^2 10882894/p^{10/3}, \quad C_{16} \doteqdot \cdot 0^3 4021/p^{16/3}.$$

The interpretation of VIII (65) + (66) is

$$J_p{}'(p) = \cdot 4473073184 p^{-\frac{1}{3}} \left[-\left\{ \sum_{1,7,\ldots}^{n-6} \bar{C}_r + \bar{T}_n \bar{C}_n \right\} + \left\{ \sum_{3,9,\ldots}^{m-6} \bar{C}_r + \bar{T}_m \bar{C}_m \right\} \right] \tag{52}$$

with

$$\bar{T}_n = \{(\tfrac{1}{3}n - \tfrac{4}{3})!\, \Pi_{\frac{1}{3}n-\frac{4}{3}}(2\pi p) + \cdot 7414338(\tfrac{1}{3}n - 2)!\, \Pi_{\frac{1}{3}n-2}(2\pi p) - \cdot 2882551(\tfrac{1}{3}n - \tfrac{10}{3})!\, \Pi_{\frac{1}{3}n-10/3}(2\pi p) \ldots\} \div \{(\tfrac{1}{3}n - \tfrac{4}{3})! + \cdot 7414338(\tfrac{1}{3}n - 2)! - \cdot 2882551(\tfrac{1}{3}n - \tfrac{10}{3})! \ldots\} \tag{53}$$

(same for m). Early $\bar{C}_r$ are:

$$\bar{C}_1 = -6\beta/p^{1/3} = -\cdot 918496472/p^{1/3},$$

$$\bar{C}_7 = -23\beta/525p^{7/3} = -\cdot 0^2 67064822/p^{7/3}, \qquad \bar{C}_{13} \simeq +\cdot 0^3 86091/p^{13/3},$$

$$\bar{C}_{19} \simeq -\cdot 0^3 4078/p^{19/3};$$

$$\bar{C}_3 = -1/5p, \qquad \bar{C}_9 = 947/346{,}500p^3 = \cdot 0^2 273304473/p^3,$$

$$\bar{C}_{15} \simeq -\cdot 0^3 6047/p^5.$$

In the same notation,

$$Y_p(p) = -\cdot 7747590021p^{-1/3}\left[\left\{\sum_{0,6,..}^{n-6} C_r + T_n C_n\right\} + \left\{\sum_{4,10,..}^{m-6} C_r + T_m C_m\right\}\right], \tag{54}$$

$$Y_p'(p) = -\cdot 7747590021p^{-1/3}\left[\left\{\sum_{1,7,..}^{n-6} \bar{C}_r + \bar{T}_n \bar{C}_n\right\} + \left\{\sum_{3,9,..}^{m-6} \bar{C}_r + \bar{T}_m \bar{C}_m\right\}\right]. \tag{55}$$

The power of our interpretative method is impressively demonstrated by evaluating these expressions with the abnormally small value $p = 1$. To do this expeditiously, $\Pi_s(2\pi)$ is first found—by interpolation between tabulated arguments—separately for orders $s = 3\cdot 5\ (\cdot 5)\ 5\cdot 0$, and extended to one or two other s via the recurrence relation in Chapter XXI (44). Interpolation between these orders then gives two of the required awkward quantities, $\Pi_{3\cdot\dot{6}}(2\pi) = \cdot 66162$ and $\Pi_{4\cdot\dot{3}}(2\pi) = \cdot 60617$. The others, $\Pi_{1\cdot\dot{6}}$ and $\Pi_{2\cdot\dot{3}}$, follow via the recurrence relation.

s	$\Pi_s(2\pi)$
3·0	·71944
3·5	·67589
4·0	·63354
4·5	·59276
5·0	·55381
5·5	·51699

Retaining half the third entries in the terminant expansions, the values obtained are

$$T_{18} = \cdot 55251, \qquad T_{16} = \cdot 60461, \qquad \bar{T}_{19} = \cdot 56331, \qquad \bar{T}_{15} = \cdot 6732.$$

From these:

$J_1(1) = \cdot 4400509$, 3 units high in the 7th place.

$Y_1(1) = -\cdot 7812128$, correct to the 7th place.

$J_1'(1) = \cdot 3251467$, 4 units low.

$Y_1'(1) = \cdot 8694698$, 2 units high.

11. Whittaker Function for Large k

Application of the theory of Section 4 is similar to that for Bessel functions (Section 10) so only the most useful results will be recorded below.

$x > 4k > 0$

Over this region only one stationary point contributes. In terms of the parameters

$$q = \{x/(x - 4k)\}^{\frac{1}{2}}, \qquad \Xi = q/(q^2 - 1) - \coth^{-1} q, \qquad \eta = 16m^2 - 1, \tag{56}$$

the interpretation of the simplest expansions obtained in Chapter VIII, (107) + (109), is

$$W_{km}(x) = [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!\,q/2\pi]^{\frac{1}{2}}\, e^{-2k\Xi} \left\{\sum_0^{n-1} Q_{2r}^- + T_{2n} Q_{2n}^-\right\}, \tag{57}$$

with

$$T_{2n} = \{\Lambda_{n-1}(4k\Xi) - a\Lambda_{n-2}(4k\Xi) + b\Lambda_{n-3}(4k\Xi)\ldots\}/(1 - a + b \ldots), \tag{58}$$

where

$$a = \Xi(5q^3 - 6q + 3\eta/q)/24(n - 1), \qquad b = \Xi^2\{385q^6 - 924q^4$$
$$- 6(7\eta - 114)q^2 + 36(3\eta - 4) + 9\eta(\eta - 8)/q^2\}/1152(n - 1)(n - 2).$$

The first three Q_{2r}^- are quoted in Chapter VIII (107). The next three are:

$$Q_6^- = -\frac{1}{26{,}542{,}080k^3}\{425{,}425q^9 - 1{,}531{,}530q^7 - 117(175\eta - 17{,}436)q^5$$
$$+ 1{,}620(33\eta - 734)q^3 + 675(\eta^2 - 68\eta + 384)q + 4{,}050(3\eta - 16)/q$$
$$+ 135\eta(\eta - 8)(\eta - 24)/q^3\},$$

$$Q_8^- = \frac{1}{10{,}192{,}158{,}720k^4}\{185{,}910{,}725q^{12} - 892{,}371{,}480q^{10}$$
$$- 163{,}020(35\eta - 10{,}566)q^8 + 2{,}376(8{,}925\eta - 709{,}208)q^6$$
$$+ 54(1{,}925\eta^2 - 548{,}944\eta + 15{,}843{,}000)q^4 - 68{,}040(9\eta^2 - 308\eta$$
$$+ 2{,}928)q^2 + 540(41\eta^3 + 2{,}630\eta^2 - 25{,}392\eta + 23{,}616)$$
$$+ 113{,}400\eta(\eta - 8)(\eta - 12)/q^2 + 405\eta(\eta - 8)(\eta - 24)(\eta - 48)/q^4\},$$

$$Q_{10}^- = -\frac{1}{34{,}245{,}653{,}299{,}200k^5}\{943{,}496{,}929{,}375q^{15} - 5{,}660{,}981{,}576{,}250q^{13}$$
$$- 311{,}220(62{,}475\eta - 4{,}888{,}136)q^{11} + 42{,}882{,}840(2{,}415\eta - 459{,}574)q^9$$
$$+ 2{,}340(114{,}539\eta^2 - 87{,}412{,}402\eta + 6{,}765{,}933{,}564)q^7$$
$$- 9{,}072(151{,}857\eta^2 - 23{,}090{,}210\eta + 809{,}650{,}230)q^5$$
$$+ 140(1{,}687\eta^3 + 10{,}739{,}598\eta^2 - 597{,}213{,}972\eta + 10{,}381{,}815{,}360)q^3$$
$$+ 200(119{,}637\eta^3 - 11{,}493{,}090\eta^2 + 189{,}853{,}816\eta - 850{,}118{,}976)q$$
$$+ 14{,}175\eta(101\eta^3 + 30{,}680\eta^2 - 512{,}832\eta + 2{,}126{,}592)/q$$
$$+ 1{,}050\eta(5{,}083\eta^3 - 272{,}160\eta^2 + 4{,}463{,}424\eta - 20{,}901{,}888)/q^3$$
$$+ 8{,}505\eta(\eta - 8)(\eta - 24)(\eta - 48)(\eta - 80)/q^5\}.$$

(Q_{10}^- was found by Jorna (1964) using the phase-integral method of Chapter XIII, Section 10. There are slips in the a_{15} and a_{44} components of his equivalents to Q_{10}^- and Q_8^-).

$4k > x > 0$

After Ξ has gone through its zero value, at $x = 4k$ by (56), $W_{km}(x)$ is expressed via (23) as the sum of expansions at both stationary points. In terms of the parameters

$$\varphi = \{x/(4k - x)\}^{\frac{1}{2}}, \qquad \Upsilon = \cot^{-1}\varphi - \varphi/(\varphi^2 + 1), \tag{59}$$

the interpretation of the simplest resultant expressions obtained in Chapter VIII, (112) + (113), is

$$W_{km}(x) = [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!\,2\varphi/\pi]^{\frac{1}{2}}\left[\left\{\sum_{0,2,\ldots}^{n-2} Q_{2r} + \mathfrak{T}_{2n}Q_{2n}\right\}\right.$$
$$\left.\times \sin(2k\Upsilon + \tfrac{1}{4}\pi) - \left\{\sum_{1,3,\ldots}^{l-2} Q_{2r} + \mathfrak{T}_{2l}Q_{2l}\right\}\cos(2k\Upsilon + \tfrac{1}{4}\pi)\right], \tag{60}$$

with

$$\mathfrak{T}_{2n} = \{\Pi_{n-1}(4k\Upsilon) - a\Pi_{n-2}(4k\Upsilon) + b\Pi_{n-3}(4k\Upsilon)\ldots\}/(1 - a + b\ldots) \tag{61}$$

(same for l), where

$$a = \Upsilon(5\varphi^3 + 6\varphi + 3\eta/\varphi)/24(n-1),$$
$$b = \Upsilon^2\{385\varphi^6 + 924\varphi^4 - 6(7\eta - 114)\varphi^2 - 36(3\eta - 4) + 9\eta(\eta - 8)/\varphi^2\}/1152(n-1)(n-2).$$

The first three Q_{2r} are quoted in Chapter VIII (112). Others can be written down from the Q_{2r}^- by substituting $q^2 = -\varphi^2$ and making the sign or phase $(-1)^{\frac{1}{2}r}$ when r is even and $(-1)^{\frac{1}{2}(r-1)}$ when odd.

$x \sim 4k$

The Taylor expansion in Chapter VIII (116) for $W_{km}(x)$, believed to be absolutely convergent, contains the two-parameter functions $W_{km}(4k)$ and $W_{km}'(4k)$ for which only asymptotic expansions are convenient when k is large.

The interpretation of VIII (119) + (121) is

$$W_{km}(4k) = \cdot 563571906k^{\frac{1}{6}}[(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!]^{\frac{1}{2}}\left[\left\{\sum_{0,6,\ldots}^{n-6} \mathbf{C}_r + \mathbf{T}_n\mathbf{C}_n\right\} - \left\{\sum_{4,10,\ldots}^{l-6} \mathbf{C}_r + \mathbf{T}_l\mathbf{C}_l\right\}\right] \tag{62}$$

with

$$\mathbf{T}_n\mathbf{C}_n = (-1)^{\frac{1}{2}n}(4\pi k)^{-\frac{1}{3}n}[\cdot 183776298(\tfrac{1}{3}n - 1)!\,\Pi_{\frac{1}{3}n-1}(4\pi k)$$
$$- \cdot 0^2 27738020(35\eta - 9)(\tfrac{1}{3}n - \tfrac{7}{3})!\,\Pi_{\frac{1}{3}n-7/3}(4\pi k)$$
$$- \cdot 0^2 20153326(45\eta - 7)(\tfrac{1}{3}n - 3)!\,\Pi_{\frac{1}{3}n-3}(4\pi k)\ldots] \tag{63}$$

(same for l). This form of expression has been adopted because the few $\mathbf{C}_r$ as yet exactly known are unlikely to reach as far as the least term. Those known are:

$\mathbf{C}_0 = 1, \qquad \mathbf{C}_4 = 3\beta(35\eta - 9)/560.4^{\frac{1}{3}}k^{\frac{4}{3}} = \cdot 0^3 51662191(35\eta - 9)/k^{\frac{4}{3}},$

$\mathbf{C}_6 = (45\eta - 7)/14{,}400k^2.$

Similarly from VIII (124) + (125),

$$W_{km}'(4k) = \cdot 140892977 k^{\frac{1}{3}}[(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}}$$
$$\times \left[-\left\{ \sum_{1,7,\ldots}^{n-6} \bar{C}_r + \bar{T}_n\bar{C}_n \right\} + \left\{ \sum_{3,9,\ldots}^{l-6} \bar{C}_r + \bar{T}_l\bar{C}_l \right\} \right] \tag{64}$$

with

$$\bar{T}_n\bar{C}_n = (-1)^{\frac{1}{2}(n+1)}(4\pi k)^{-\frac{1}{3}n}[\cdot 49444003(\tfrac{1}{3}n - \tfrac{4}{3})!\,\Pi_{\frac{1}{3}n-\frac{4}{3}}(4\pi k)$$
$$+ \cdot 46188022(\tfrac{1}{3}n - 2)!\,\Pi_{\frac{1}{3}n-2}(4\pi k)$$
$$+ \cdot 024786933(\tfrac{1}{3}n - \tfrac{10}{3})!\,\Pi_{\frac{1}{3}n-10/3}(4\pi k) \ldots] \tag{65}$$

(same for l). Known exact values are:

$$\bar{C}_1 = 12\beta/(4k)^{\frac{1}{3}} = 1{\cdot}1572331/k^{\frac{1}{3}}, \qquad \bar{C}_3 = 1/5k,$$
$$\bar{C}_7 = -2\beta/525.\,4^{\frac{1}{3}}k^{7/3} = -\cdot 0^3 36737558/k^{7/3}.$$

12. Whittaker Function for Large $\kappa = (k^2 - m^2)^{\frac{1}{2}}$

Throughout this section, results will be expressed in terms of the quantities

$$A = 1 + k/\kappa, \qquad a = 1 - k/\kappa, \qquad B = 1 + 3k/\kappa, \qquad b = 1 - 3k/\kappa, \tag{66}$$

though the motivation for these abbreviations came from the phase-integral approach (Chapter XIII) and not expansion from integral representations, ostensibly the subject of the present chapter. In addition to shortening the printing, adoption of these abbreviations from the start extends the generality, since they can then legitimately be regarded as independent parameters.

$x > 2(k + \kappa) > 0$

Over this region only one stationary point contributes. In terms of the parameters

$$q = \left(\frac{x - 2k + 2\kappa}{x - 2k - 2\kappa}\right)^{\frac{1}{2}},$$
$$\Xi = \frac{q}{q^2 - 1} - \left(\frac{k}{\kappa}\right)\coth^{-1} q + \left(\frac{m}{\kappa}\right)\tanh^{-1}\frac{1}{q}\left(\frac{k - \kappa}{k + \kappa}\right)^{\frac{1}{2}} \tag{67}$$

(with positive signs allotted to k/κ and m/κ), the interpretation of the simplest expansions obtained in Chapter VIII, (133) + (136), is

$$W_{km}(x) = [(k+m-\tfrac{1}{2})!\,(k-m-\tfrac{1}{2})!\,(Aq^2+a)/4\pi q]^{\frac{1}{2}}$$
$$\times\, e^{-2\kappa\Xi}\left\{\sum_0^{n-1} Q_{2r}^- + T_{2n}Q_{2n}^-\right\}, \tag{68}$$

with

$$T_{2n} = \{\Lambda_{n-1}(4\kappa\Xi) - \mathscr{a}\Lambda_{n-2}(4\kappa\Xi) + \mathscr{b}\Lambda_{n-3}(4\kappa\Xi)\ldots\}/(1-\mathscr{a}+\mathscr{b}\ldots), \tag{69}$$

where

$$\mathscr{a} = \Xi(5Aq^3 - 3Bq + 3b/q - 5a/q^3)/48(n-1),$$
$$\mathscr{b} = \Xi^2\{385A^2q^6 - 462ABq^4 + 3(27B^2 - 134Ab)q^2 + 4(199 - 451k^2/\kappa^2)$$
$$+ 3(27b^2 - 134aB)/q^2 - 462ab/q^4 + 385a^2/q^6\}/4{,}608(n-1)(n-2).$$

$Q_0^- \equiv Y_0$, $-Q_2^- \equiv Y_1$ and $Q_4^- \equiv Y_2$ are quoted in Chapter XIII (110). The next two are

$$Q_6^-(\equiv -Y_3) = -\frac{1}{212{,}336{,}640\kappa^3}\{425{,}425A^3q^9 - 765{,}765A^2Bq^7$$
$$+ 117A(3{,}159B^2 - 4{,}975Ab)q^5 + 15(32{,}688ABb - 2{,}025B^3$$
$$+ 29{,}645aA^2 + 3{,}980A - 9{,}020Ak^2/\kappa^2)q^3 + 45(3{,}375Ab^2 - 8{,}496AaB$$
$$- 729bB^2 - 796B + 1{,}804Bk^2/\kappa^2)q - 45(3{,}375aB^2 - 8{,}496aAb$$
$$- 729Bb^2 - 796b + 1{,}804bk^2/\kappa^2)/q - 15(32{,}688abB - 2{,}025b^3$$
$$+ 29{,}645Aa^2 + 3{,}980a - 9{,}020ak^2/\kappa^2)/q^3$$
$$- 117a(3{,}159b^2 - 4{,}975aB)/q^5 + 765{,}765a^2b/q^7 - 425{,}425a^3/q^9\},$$

$$Q_8^-(\equiv Y_4) = \frac{1}{163{,}074{,}539{,}520\kappa^4}\{185{,}910{,}725A^4q^{12} - 446{,}185{,}740A^3Bq^{10}$$
$$+ 85{,}510A^2(4{,}293B^2 - 4{,}030Ab)q^8 + 44A(10{,}800{,}000ABb$$
$$- 2{,}139{,}129B^3 + 5{,}829{,}950aA^2 + 208{,}950A - 473{,}550Ak^2/\kappa^2)q^6$$
$$+ 9(16{,}779{,}150A^2b^2 - 37{,}251{,}550A^2aB - 18{,}217{,}476AbB^2$$
$$- 1{,}225{,}840AB + 2{,}778{,}160ABk^2/\kappa^2 + 496{,}125B^4)q^4 + 180(684{,}288aAB^2$$
$$- 1{,}281{,}907abA^2 - 457{,}443ABb^2 - 53{,}332Ab + 120{,}868Abk^2/\kappa^2$$

$$+ 35{,}154bB^3 + 10{,}746B^2 - 24{,}354B^2k^2/\kappa^2)q^2 + 240(1{,}150{,}610$$

$$- 11{,}735{,}533k^2/\kappa^2 + 12{,}442{,}415k^4/\kappa^4) + 180(684{,}288Aab^2$$

$$- 1{,}281{,}907ABa^2 - 457{,}443abB^2 - 53{,}332aB + 120{,}868aBk^2/\kappa^2$$

$$+ 35{,}154Bb^3 + 10{,}746b^2 - 24{,}354b^2k^2/\kappa^2)/q^2 + 9(16{,}779{,}150a^2B^2$$

$$- 37{,}251{,}550a^2Ab - 18{,}217{,}476aBb^2 - 1{,}225{,}840ab + 2{,}778{,}160abk^2/\kappa^2$$

$$+ 496{,}125b^4)/q^4 + 44a(10{,}800{,}000abB - 2{,}139{,}129b^3 + 5{,}829{,}950Aa^2$$

$$+ 208{,}950a - 473{,}550ak^2/\kappa^2)/q^6 + 85{,}510a^2(4{,}293b^2 - 4{,}030aB)/q^8$$

$$- 446{,}185{,}740a^3b/q^{10} + 185{,}910{,}725a^4/q^{12}\}.$$

$2(k + \kappa) > x > 2(k - \kappa)$

After Ξ has gone through its zero value, at $x = 2(k + \kappa)$ by (67), $W_{km}(x)$ is expressed via (23) as the sum of expansions at both stationary points. In terms of the parameters

$$\mathcal{q} = \left(\frac{2\kappa - 2k + x}{2\kappa + 2k - x}\right)^{\frac{1}{2}},$$

$$\Upsilon = \left(\frac{k}{\kappa}\right)\cot^{-1}\mathcal{q} - \frac{\mathcal{q}}{\mathcal{q}^2 + 1} - \left(\frac{m}{\kappa}\right)\tan^{-1}\frac{1}{\mathcal{q}}\left(\frac{k - \kappa}{k + \kappa}\right)^{\frac{1}{2}}, \tag{70}$$

(with positive signs allotted to k/κ and m/κ), the interpretation of the simplest resultant expressions obtained in Chapter VIII, (138) + (129), is

$$W_{km}(x) = [(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!\,(A\mathcal{q}^2 - a)/\pi\mathcal{q}]^{\frac{1}{2}}$$

$$\times \left[\left\{\sum_{0,2,\ldots}^{n-2} Q_{2r} + \mathfrak{T}_{2n}Q_{2n}\right\}\sin(2\kappa\Upsilon + \tfrac{1}{4}\pi)\right.$$

$$\left. - \left\{\sum_{1,3,\ldots}^{l-2} Q_{2r} + \mathfrak{T}_{2l}Q_{2l}\right\}\cos(2\kappa\Upsilon + \tfrac{1}{4}\pi)\right], \tag{71}$$

with

$$\mathfrak{T}_{2n} = \{\Pi_{n-1}(4\kappa\Upsilon) - \mathfrak{a}\Pi_{n-2}(4\kappa\Upsilon) + \mathfrak{b}\Pi_{n-3}(4\kappa\Upsilon)\ldots\}/(1 - \mathfrak{a} + \mathfrak{b}\ldots) \tag{72}$$

(same for l), where

$$\mathfrak{a} = \Upsilon(5A\varphi^3 + 3B\varphi + 3b/\varphi + 5a/\varphi^3)/48(n-1),$$

$$\mathfrak{b} = \Upsilon^2\{385A^2\varphi^6 + 462AB\varphi^4 + 3(27B^2 - 134Ab)\varphi^2 - 4(199 - 451k^2/\kappa^2)$$
$$+ 3(27b^2 - 134aB)/\varphi^2 + 462ab/\varphi^4 + 385a^2/\varphi^6\}/4{,}608(n-1)(n-2).$$

$Q_0 = \mathscr{Y}_0$, $Q_2 = \mathscr{Y}_1$ and $-Q_4 = \mathscr{Y}_2$ are quoted in Chapter XIII (114). Others can be written down from the Q_{2r}^- by substituting $q^2 = -\varphi^2$ and making the sign or phase $(-1)^{\frac{1}{2}r}$ when r is even and $(-1)^{\frac{1}{2}(r-1)}$ when odd.

$x \sim 2(k + \kappa)$

The Taylor expansion in Chapter VIII (141), believed to be absolutely convergent, contains the two-parameter function $W_{km}(2k + 2\kappa)$ and its first derivative, for which only asymptotic expansions are convenient when κ is large.

The interpretation of VIII (144) + (145) is

$$W_{km}(2k + 2\kappa) = .447307318A^{\frac{1}{4}}\kappa^{\frac{1}{2}}[(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}}$$
$$\times\left[\left\{\sum_{0,6,\ldots}^{n-6} \mathrm{C}_r + \mathrm{T}_n\mathrm{C}_n\right\} - \left\{\sum_{4,10,\ldots}^{l-6} \mathrm{C}_r + \mathrm{T}_l\mathrm{C}_l\right\}\right] \tag{73}$$

with

$$\mathrm{T}_n\mathrm{C}_n = (-1)^{\frac{1}{2}n}(4\pi k)^{-\frac{1}{3}n}[\cdot 183776298(\tfrac{1}{3}n - 1)!\,\Pi_{\frac{1}{3}n-1}(4\pi k)$$
$$- \cdot 0^2 34947715(k^2/A\kappa^2)^{\frac{1}{3}}(B^2 + 15Ab)(\tfrac{1}{3}n - \tfrac{7}{3})!\,\Pi_{\frac{1}{3}n-\frac{7}{3}}(4\pi k)$$
$$+ \cdot 0^4 31489572(k^2/A\kappa^2)\{64B^3 + 1415AbB + 4375A^2a$$
$$- 50A(199 - 451k^2/\kappa^2)\}(\tfrac{1}{3}n - 3)!\,\Pi_{\frac{1}{3}n-3}(4\pi k)\ldots] \tag{74}$$

(same for l). The few C_r exactly known are:

$$\mathrm{C}_0 = 1, \qquad \mathrm{C}_4 = -3\beta(B^2 + 15Ab)/560A^{\frac{2}{3}}\kappa^{\frac{4}{3}}$$
$$= -\cdot 0^2 32803446(9k^2 + 6k\kappa - 4\kappa^2)/\kappa^{8/3}(k + \kappa)^{2/3},$$

$$\mathrm{C}_6 = -\{64B^3 + 1415AbB + 4375A^2a$$
$$- 50A(199 - 451k^2/\kappa^2)\}/921{,}600A\kappa^2$$
$$= -(14k^3 + 14k^2\kappa - 7k\kappa^2 - 8\kappa^3)/1800\kappa^4(k + \kappa).$$

Similarly from VIII (148) + (149),

$$W_{km}'(2k + 2\kappa) = \cdot 223653659 A^{-\frac{2}{3}}\kappa^{\frac{1}{6}}[(k + m - \tfrac{1}{2})!\,(k - m - \tfrac{1}{2})!]^{\frac{1}{2}}$$
$$\times \left[- \left\{ \sum_{1,7,\ldots}^{n-6} \overline{\mathbf{C}}_r + \overline{\mathbf{T}}_n\overline{\mathbf{C}}_n \right\} + \left\{ \sum_{3,9,\ldots}^{l-6} \overline{\mathbf{C}}_r + \overline{\mathbf{T}}_l\overline{\mathbf{C}}_l \right\} \right] \qquad (75)$$

with

$$\overline{\mathbf{T}}_n\overline{\mathbf{C}}_n = (-1)^{\frac{1}{2}(n+1)}(4\pi k)^{-\frac{1}{3}n}(Ak/\kappa)^{\frac{1}{3}}[\cdot 39243731(\tfrac{1}{3}n - \tfrac{4}{3})!\,\Pi_{\frac{1}{3}n-\frac{4}{3}}(4\pi k)$$
$$+ \cdot 11547005(k^2/A\kappa^2)^{\frac{1}{3}}(B + 5a)(\tfrac{1}{3}n - 2)!\,\Pi_{\frac{1}{3}n-2}(4\pi k)$$
$$- \cdot 0^5 96061533(k^2/A\kappa^2)\{736B^3 - 7825AbB - 45a(1925A^2 - 32B^2$$
$$- 480Ab) + 350A(199 - 451k^2/\kappa^2)\}(\tfrac{1}{3}n - \tfrac{10}{3})!\,\Pi_{\frac{1}{3}n-10/3}(4\pi k) \ldots] \qquad (76)$$

(same for l). Known exact values are

$$\overline{\mathbf{C}}_1 = 6\beta A^{\frac{1}{3}}/\kappa^{\frac{1}{3}} = \cdot 9184965(k + \kappa)^{\frac{1}{3}}/\kappa^{\frac{2}{3}},$$

$$\overline{\mathbf{C}}_3 = (B + 5a)/20\kappa = (3\kappa - k)/10\kappa^2,$$

$$\overline{\mathbf{C}}_7 = (\beta/1{,}075{,}200A^{\frac{2}{3}}\kappa^{\frac{7}{3}})\{736B^3 - 7825AbB - 45a(1925A^2 - 32B^2$$
$$- 480Ab) + 350A(199 - 451k^2/\kappa^2)\}$$
$$= \cdot 0^4 36448274(k + \kappa)^{-\frac{2}{3}}\kappa^{-14/3}(277k^3 + 7k^2\kappa - 296k\kappa^2 - 4\kappa^3).$$

EXERCISES

1. Starting from (3), show that for linear dependence of $F(u)$ at a limit the discontinuity in $\int_{\text{limit}} e^{-F}\,G\,du$ is

$$\frac{n!\,e^{-F_0}}{2\pi i} \oint \frac{\Delta\,\Lambda_n(-f)\,G\,du}{f^{n+1}} = e^{-F_0}\oint e^{-f}\,G\,du = \int_{s.p.} e^{-F}\,G\,du$$

in agreement with (7).

2. Show from V (14) and VII (28) that for linear dependence of $F(u)$ at a limit,

$$T_n^{(\sigma)} = \oint \frac{\Lambda_{n+\sigma}(-f)\,u^\sigma\,G\,du}{f^{n+\sigma+1}} \bigg/ \oint \frac{u^\sigma\,G\,du}{f^{n+\sigma+1}}$$

$$= \left\{ \bar{\mathscr{Q}}_0\,\Lambda_{n+\sigma-\frac{1}{2}}(-\mathscr{F}_0) + \frac{\bar{\mathscr{Q}}_2\,\mathscr{F}_0}{n + \sigma - \frac{1}{2}}\,\Lambda_{n+\sigma-\frac{3}{2}}(-\mathscr{F}_0) \ldots \right\}$$

$$\div \left\{ \bar{\mathscr{Q}}_0 + \frac{\bar{\mathscr{Q}}_2\,\mathscr{F}_0}{n + \sigma - \frac{1}{2}} \ldots \right\}.$$

3. Representing the factorial in (4) by an integral, derive the expression

$$L_r = \frac{F_1}{(-2\pi \mathscr{F}_2 \mathscr{F}_0)^{\frac{1}{2}} \mathscr{F}_0^r} \int_0^\infty \varepsilon^{r-\frac{1}{2}} e^{-\varepsilon} \, d\varepsilon \sum_0^\infty \left(\frac{\mathscr{F}_0}{\varepsilon}\right)^s \mathscr{Q}_{2s}$$

for the contribution to L_r from a single singulant $\mathscr{F}_0$.

Regarded as a function of the order, A say, $\mathscr{Q}_{2s}(A) \propto A^{-s}$. Denoting by $\mathscr{Q}(A) = \sum_0^\infty \mathscr{Q}_{2s}(A)$ the series obtained by expansion at the stationary point dictating late terms, show that

$$L_r(A) = \frac{F_1}{(2\pi A \mathscr{F}_2)^{\frac{1}{2}} (-A)^r} \int_0^\infty a^{r-\frac{1}{2}} e^{-a\Xi} \, \mathscr{Q}(-a) \, da$$

where $\mathscr{F}_0 = -A\Xi$.

4. Late terms in the expansion of the incomplete factorial function $(p, x)!$ are determined by the expansion at the stationary point, i.e. by the series for $p!$ in Chapter VIII (15). Derive the representation

$$L_r(p) = \frac{p/x - 1}{(2\pi)^{\frac{1}{2}} (-p)^r} \int_0^\infty a^{r-\frac{1}{2}} e^{-a\Xi} \, \mathscr{Q}(-a) \, da, \qquad x < p,$$

where

$$\Xi = \ln (p/x) + x/p - 1, \quad \mathscr{Q}(-a) = 1 - 1/12a + 1/288a^2 + 139/51840a^3 \ldots$$

Integrate term by term to regain VIII (23).

5. Show that $\mathscr{Q}(-a) = 1/\mathscr{Q}(a) = (2\pi a)^{\frac{1}{2}} a^a e^{-a}/a!$ for the factorial function, and deduce the representation

$$L_r(p) = \frac{p/x - 1}{(-p)^r} \int_0^\infty \left(\frac{ax}{p}\right)^a \frac{a^r e^{-ax/p} \, da}{a!}, \qquad x < p,$$

for the rth term in the asymptotic expansion of $(p, x)!$

Setting $r = 0$, obtain the integral

$$\int_0^\infty \frac{a^a e^{-a(x - \ln x)} \, da}{a!} = \frac{x}{1 - x}, \qquad x < 1.$$

6. Summing $L_r(A)$ of question 3 over r from 0 to ∞, establish the "dispersion relation"

$$L(A) = \frac{F_1}{(2\pi A \mathscr{F}_2)^{\frac{1}{2}}} \int_0^\infty \frac{e^{-a\Xi}}{a^{\frac{1}{2}} (1 + a/A)} \mathscr{Q}(-a) \, da$$

between $\int_{\text{limit}}$ and $\int_{\text{s.p.}}$. Illustrate it by deriving

$$(p, xp)! = (xp)^p \, e^{-xp} \int_0^\infty \frac{(ax)^a \, e^{-ax} \, da}{a! \, (1 + a/p)} \, .$$

7. Summing $L_r(A)$ of question 3 over r from n to ∞, derive the terminant

$$T_n(A) = \int_0^\infty \frac{a^{n-\frac{1}{2}} \, e^{-a\Xi}}{1 + a/A} \, \mathscr{Q}(-a) \, da \Bigg/ \int_0^\infty a^{n-\frac{1}{2}} \, e^{-a\Xi} \, \mathscr{Q}(-a) \, da.$$

When n is large, these integrands are strongly peaked at $a = (n - \frac{1}{2})/\Xi$. Expand $\ln \mathscr{Q}$ about this point to obtain the useful approximation

$$T_n(A) \doteq \Lambda_{n-\frac{1}{2}} \, (A\Xi \, [1 - d \ln \mathscr{Q}\{-(n - \tfrac{1}{2})/\Xi\}/\, dn]).$$

Provided n is not too small, the derivative (or equivalent difference) can be estimated from early terms of the usual asymptotic series for $\mathscr{Q}$.

8. Show from V (19) and VII (29) that for quadratic dependence at a limit,

$$T_n^{(\sigma)} = \oint \frac{\Lambda_{\frac{1}{2}n+\frac{1}{2}\sigma-\frac{1}{2}}(-f^2) \, u^\sigma \, G \, du}{f^{n+\sigma+1}} \Bigg/ \oint \frac{u^\sigma \, G \, du}{f^{n+\sigma+1}}$$

$$= \left\{ \overline{\mathscr{Q}}_0 \, \Lambda_{\frac{1}{2}n+\frac{1}{2}\sigma-1}(-\mathscr{F}_0) + \frac{\overline{\mathscr{Q}}_2 \, \mathscr{F}_0}{\frac{1}{2}n + \frac{1}{2}\sigma - 1} \, \Lambda_{\frac{1}{2}n+\frac{1}{2}\sigma-2}(-\mathscr{F}_0) + \ldots \right\}$$

$$\div \left\{ \overline{\mathscr{Q}}_0 + \frac{\overline{\mathscr{Q}}_2 \, \mathscr{F}_0}{\frac{1}{2}n + \frac{1}{2}\sigma - 1} + \ldots \right\}.$$

9. Prove that when the expansion of $\int_{\text{limit}} e^{-F} \, G \, du$, for quadratic dependence of $F(u)$ at the limit, has initially been specified on a Stokes ray,

$$\int_{\text{limit, (a.s.p)}} = \int_{\text{exp. at limit}} - \tfrac{1}{2} \int_{\text{s.p.}} + \tfrac{1}{2} \int_{\text{a.s.p.}}$$

after $\mathscr{F}_0$ has gone through its zero value. [Hint: cf. derivation of (25)].

10. Applying the method of question 3 to (11), derive the expression

$$\mathscr{Q}_r(A) = \frac{1}{2\pi} \left(\frac{F_2}{-\mathscr{F}_2} \right)^{\frac{1}{2}} \frac{1}{(-A)^{\frac{1}{2}r}} \int_0^\infty a^{\frac{1}{2}r-1} \, e^{-a\Xi} \, \mathscr{Q}(-a) \, da$$

for the contribution to Q_r from a single singulant $\mathscr{F}_0 = -A\Xi$.

11. Late terms in the expansion of $J_p(x)$ are determined by the expansion at the other stationary point, i.e. by the series for $Y_p(x)$ in Chapter VIII (70). Derive the representation

$$Q_{2r}(p) = \frac{1}{2\pi(-p)^r}\int_0^\infty \rho^{r-1}\, e^{-2\rho\Xi}\, \mathcal{Q}(-\rho)\, d\rho, \qquad q > 1,$$

for the rth term in the asymptotic expansion of $J_p(x)$, where

$$q = p/(p^2 - x^2)^{\frac{1}{2}}, \qquad \Xi = \tanh^{-1}q^{-1} - q^{-1},$$
$$\mathcal{Q}(-\rho) = 1 - q(5q^2 - 3)/24\rho + \dots .$$

Integrate term by term to regain VIII (50).

Show that $\mathcal{Q}(-\rho) = (2\pi\rho/q)^{\frac{1}{2}}\, e^{\rho\Xi}\, J_\rho(x)$, and deduce the representation

$$Q_{2r}(p) = (2\pi q)^{-\frac{1}{2}}(-p)^{-r}\int_0^\infty \rho^{r-\frac{1}{2}}\, e^{-\rho\Xi}\, J_\rho(x)\, d\rho, \qquad q > 1.$$

Setting $r = 0$, obtain the integral

$$\int_0^\infty \rho^{-\frac{1}{2}} \left\{\frac{p - (p^2 - x^2)^{\frac{1}{2}}}{p + (p^2 - x^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}\rho} e^{\rho(p^2 - x^2)^{\frac{1}{2}}/p}\, J_\rho(x)\, d\rho = \left(\frac{4\pi^2 p^2}{p^2 - x^2}\right)^{\frac{1}{4}} \qquad x < p.$$

12. Summing $Q_{2r}(A)$ of question 10 over r from 0 to ∞, establish the "dispersion relation"

$$Q(A) = \frac{1}{2\pi}\left(\frac{F_2}{-\mathcal{F}_2}\right)^{\frac{1}{2}}\int_0^\infty \frac{e^{-a\Xi}}{a(1 + a/A)}\, \mathcal{Q}(-a)\, da$$

between $\int_{\text{s.p.}}$ and $\int_{\text{a.s.p.}}$. Illustrate it by deriving

$$J_p(x) = \frac{e^{-p\Xi}}{2\pi p^{\frac{1}{2}}}\int_0^\infty \frac{e^{-\rho\Xi}}{\rho^{\frac{1}{2}}(1 + \rho/p)}\, J_\rho(x)\, d\rho,$$

$$J_p(p) = \frac{1}{2\pi p^{\frac{1}{2}}}\int_0^\infty \frac{1}{\rho^{\frac{1}{2}}(1 + \rho/p)}\, J_\rho(p)\, d\rho.$$

13. Summing $Q_r(A)$ of question 10 over alternate r from n to ∞, derive the terminant defined in Section 2:

$$T_n(A) = \int_0^\infty \frac{a^{\frac{1}{2}n-1}\, e^{-a\Xi}}{1 + a/A}\, \mathcal{Q}(-a)\, da \Bigg/ \int_0^\infty a^{\frac{1}{2}n-1}\, e^{-a\Xi}\, \mathcal{Q}(-a)\, da \cdot$$

Expand $\ln \mathscr{Q}$ about the point $a = (\frac{1}{2}n - 1)/\Xi$ to obtain the useful approximation

$$T_n(A) \simeq \Lambda_{\frac{1}{2}n-1}(A\Xi[1 - d\ln \mathscr{Q}\{-(\tfrac{1}{2}n - 1)/\Xi\}/dn]).$$

14. Show from V (31) and VII (30) that for cubic dependence at a limit,

$$\begin{aligned} T_n^{(\sigma)} &= \oint \frac{\Lambda_{\frac{1}{3}n+\frac{1}{3}\sigma-\frac{2}{3}}(-f^3)\, u^\sigma\, G\, du}{f^{n+\sigma+1}} \bigg/ \oint \frac{u^\sigma\, G\, du}{f^{n+\sigma+1}} \\ &= \{(\tfrac{1}{3}n + \tfrac{1}{3}\sigma - 1)!\, \mathscr{C}_0 \Lambda_{\frac{1}{3}n+\frac{1}{3}\sigma-1}(-\mathscr{F}_0) \\ &\quad - (\tfrac{1}{3}n + \tfrac{1}{3}\sigma - \tfrac{4}{3})!\, \mathscr{C}_1\, \mathscr{F}_0^{\frac{1}{3}}\, \Lambda_{\frac{1}{3}n+\frac{1}{3}\sigma-\frac{4}{3}}(-\mathscr{F}_0) + \dots\} \\ &\div \{(\tfrac{1}{3}n + \tfrac{1}{3}\sigma - 1)!\, \mathscr{C}_0 - (\tfrac{1}{3}n + \tfrac{1}{3}\sigma - \tfrac{4}{3})!\, \mathscr{C}_1\, \mathscr{F}_0^{\frac{1}{3}} + \dots\}. \end{aligned}$$

15. Regarded as a function of the order, A say, $\mathscr{C}_s(A) \propto A^{-\frac{1}{3}s}$. Denoting by

$$\mathscr{C}(A) = \mathscr{C}_0(A) - \mathscr{C}_1(A) + \mathscr{C}_3(A) - \mathscr{C}_4(A) + \dots$$

the series obtained with contour > on expansion at the stationary point dictating late terms, derive the expression

$$C_r(A) = \frac{1}{2\sqrt{3}\,\pi} \left(\frac{F_3}{\mathscr{F}_3}\right)^{\frac{1}{3}} \frac{1}{(-A)^{\frac{1}{3}r}} \int_0^\infty a^{\frac{1}{3}r-1}\, e^{-a\Xi}\, \mathscr{C}(-a)\, da$$

for the contribution to C_r from a single singulant $\mathscr{F}_0 = -A\Xi$.

16. Late terms in the expansion of $J_p(p)$ are determined by the singulant pair $\mathscr{F}_0 = \pm 2\pi i p$ and the expansion $\mathscr{C}(-\rho)$, which is here identical with $C(\rho)$, the series for $J_\rho(\rho)$. Show that $C_r = 0$ when r is odd, and

$$C_r(p) = \frac{(-1)^{\frac{1}{2}r}}{\sqrt{3}\,\pi\, p^{\frac{1}{3}r}} \int_0^\infty \rho^{\frac{1}{3}r-1}\, e^{-2\pi\rho}\, C(i\rho)\, d\rho,$$

$$C(i\rho) = 1 - \frac{3\beta}{35\rho^{\frac{4}{3}}} + \frac{1}{225\rho^2} \cdots,$$

when r is even. Integrate term by term to regain VIII (62).

For even r, derive the representation

$$C_r(p) = \frac{2(i/6)^{\frac{1}{3}}}{(-2/3)!} \frac{(-1)^{\frac{1}{2}r}}{p^{\frac{1}{3}r}} \int_0^\infty \rho^{\frac{1}{3}r-\frac{1}{3}}\, e^{-2\pi\rho}\, J_{i\rho}(i\rho)\, d\rho.$$

Setting $r = 0$, obtain the integrals

$$\int_0^\infty \rho^{-\frac{2}{3}} \sin 2\pi\rho \;\; J_\rho(\rho)\, d\rho = 0, \quad \int_0^\infty \rho^{-\frac{2}{3}} \cos 2\pi\rho \;\; J_\rho(\rho)\, d\rho = \tfrac{1}{2}\, 6^{\frac{1}{3}}\, (-\tfrac{2}{3})!$$

17. From the result of question 15, establish the "dispersion relation"

$$C(A) = \frac{1}{2\sqrt{3}\,\pi} \left(\frac{F_3}{\mathscr{F}_3}\right)^{\frac{1}{3}} \int_0^\infty \frac{e^{-a\Xi}}{a\{1 - (a/A)^{\frac{1}{3}} + (a/A)^{\frac{2}{3}}\}}\, \mathscr{C}(-a)\, da$$

between $\int_{\text{s.p.}}$ and $\int_{\text{a.s.p.}}$.

18. Derive the relation

$$J_p(p) = \frac{1}{\sqrt{3}\,\pi} \left(\frac{i}{p}\right)^{\frac{1}{3}} \int_0^\infty \frac{1 - (\rho/p)^{\frac{4}{3}}}{\rho^{\frac{2}{3}}(1 + \rho^2/p^2)}\, e^{-2\pi\rho}\, J_{i\rho}(i\rho)\, d\rho.$$

19. Summing $C_r(A)$ of question 15 over every third r from n to ∞, derive the terminant defined in Section 3:

$$T_n(A) = \int_0^\infty \frac{a^{\frac{1}{3}n-1}\, e^{-a\Xi}}{1 + a/A}\, \mathscr{C}(-a)\, da \bigg/ \int_0^\infty a^{\frac{1}{3}n-1}\, e^{-a\Xi}\, \mathscr{C}(-a)\, da.$$

Expand $\ln \mathscr{C}$ about the point $a = (\frac{1}{3}n - 1)/\Xi$ to obtain the useful approximation

$$T_n(A) \simeq \Lambda_{\frac{1}{3}n-1}(A\Xi[1 - d \ln \mathscr{C}\{-(\tfrac{1}{3}n - 1)/\Xi\}/\,dn\,]).$$

20. Show that the interpretation of the expansions in Chapter VIII, question 3, is

$$\overline{\Lambda}_{s-1}(-s) = \left\{ \sum_{0,2\ldots}^{n-2} Q_r + T_n^{\,-}\, Q_n \right\} + \left\{ \sum_{1,3\ldots}^{m-2} Q_r + T_m^{\,+}\, Q_m \right\}$$

with

$$T_n^{\,\mp} = \Bigg[\sum_{\nu=1}^{\infty} \frac{1}{\nu^{n+\frac{3}{2}}} \Bigg\{ \Pi_{n-\frac{1}{2}}(2\pi\nu s) \mp \frac{\pi\nu}{6(n - \frac{1}{2})}\, \Pi_{n-\frac{3}{2}}(2\pi\nu s)$$

$$- \frac{\pi^2\nu^2}{72(n - \frac{1}{2})(n - \frac{3}{2})}\, \Pi_{n-5/2}(2\pi\nu s) \ldots \Bigg\} \Bigg]$$

$$\div \left\{ \zeta(n + \tfrac{3}{2}) \mp \frac{\pi}{6(n - \frac{1}{2})}\, \zeta(n + \tfrac{1}{2}) - \frac{\pi^2}{72(n - \frac{1}{2})(n - \frac{3}{2})}\, \zeta(n - \tfrac{1}{2}) \ldots \right\}.$$

Hence evaluate $\overline{\Lambda}_0\,(-1)$.

21. From Chapter VIII, question 10, deduce

$$K_p(x) = (\pi q/2p)^{\frac{1}{2}}\, e^{-p(q^{-1}-\tanh^{-1}q)} \left\{ \sum_{0}^{n-1} Q_{2r} + T_{2n} Q_{2n} \right\}$$

with

$$T_{2n} = \Re \left\{ e^{in\theta} \Lambda_{n-1}(-2p\Xi\, e^{-i\theta}) + \frac{\Xi q(5q^2-3)}{12(n-1)} e^{i(n-1)\theta} \Lambda_{n-2}(-2p\Xi\, e^{-i\theta}) + \ldots \right\} \div \left\{ \cos n\theta + \frac{\Xi q(5q^2-3)}{12(n-1)} \cos(n-1)\theta + \ldots \right\}.$$

22. From Chapter VIII, question 11, deduce

$$A_p(x) = (x-p)^{-1} \left\{ \sum_{0,2,\ldots}^{n-2} L_r + T_n L_n \right\}$$

with

$$T_n = \left\{ \Pi_{n-\frac{1}{2}}(p\Upsilon) - \frac{\Upsilon\varphi(5\varphi^2+3)}{24(n-\frac{1}{2})} \Pi_{n-\frac{3}{2}}(p\Upsilon) \ldots \right\} \Big/ \left\{ 1 - \frac{\Upsilon\varphi(5\varphi^2+3)}{24(n-\frac{1}{2})} \ldots \right\}.$$

23. From Chapter VIII, question 12, deduce

$$A_p(p) = \alpha p^{-\frac{1}{3}} \left[\left\{ \sum_{0,6,\ldots}^{n-6} C_r + T_n C_n \right\} + \left\{ \sum_{2,8,\ldots}^{l-6} C_r + T_l C_l \right\} + \left\{ \sum_{4,10,\ldots}^{m-6} C_r + T_m C_m \right\} \right]$$

with T's as in (51). [T_n and T_m are needed in $J_p(p)$ and $Y_p(p)$]. Also,

$$A_p'(p) = \alpha p^{-\frac{2}{3}} \left[\left\{ \sum_{1,7,\ldots}^{n-6} \bar{C}_r + \bar{T}_n \bar{C}_n \right\} + \left\{ \sum_{3,9,\ldots}^{m-6} \bar{C}_r + \bar{T}_m \bar{C}_m \right\} + \left\{ \sum_{5,7,\ldots}^{l-6} \bar{C}_r + \bar{T}_l \bar{C}_l \right\} \right]$$

with $\bar{T}$'s as in (53). [$\bar{T}_n$ and $\bar{T}_m$ are needed in $J_p'(p)$ and $Y_p'(p)$].

24. From Chapter VIII (81) + (83), deduce

$$F(a, c, x) = \frac{(c-1)!}{(c-a-1)!}\left(\frac{q}{C}\right)^a \left\{\sum_0^{n-1} L_r^{(a-1)} + T_n L_n^{(a-1)}\right\}$$

with

$$T_n = \{\Lambda_{n+a-\frac{3}{2}}(C\Xi) - \mathfrak{a}\Lambda_{n+a-5/2}(C\Xi) + \mathfrak{b}\Lambda_{n+a-7/2}(C\Xi)\ldots\}/(1 - \mathfrak{a} + \mathfrak{b}\ldots),$$

where

$$\mathfrak{a} = \Xi\{(a-1)(a-2)q^2 + 2a(a-1)q + a^2 - a + \tfrac{1}{6}\}/2(n + a - \tfrac{3}{2}),$$

$$\begin{aligned}\mathfrak{b} = \Xi^2\{&3(a-1)(a-2)(a-3)(a-4)q^4 \\ &+ 4(a-1)(a-2)(a-3)(3a+2)q^3 + (a-1)(a-2)(18a^2 + 6a + 1)q^2 \\ &+ 2a(a-1)(6a^2 - 6a + 1)q \\ &+ 3a^4 - 10a^3 + 10a^2 - 3a + \tfrac{1}{12}\}/24(n + a - \tfrac{3}{2})(n + a - \tfrac{5}{2}).\end{aligned}$$

25. When $x = C$, Ξ in question 24 goes through its zero value and changes phase by 2π. Show that the resulting discontinuity in $[(c-1)!/(c-a-1)!]\ (q/C)^a\ T_n L_n^{(a-1)}$ is

$$\frac{(c-1)!}{(a-1)!} x^{1-c} e^x \left(-\frac{q}{C}\right)^{1-a} \frac{(2\pi)^{\frac{1}{2}} C^{C+a-\frac{1}{2}} e^{-C}}{(C+a-1)!}$$

$$\times \left[1 + \frac{(a-1)(a-2)q^2 + 2a(a-1)q + a^2 - a + \frac{1}{6}}{2C}\ldots\right]$$

$$= \frac{(c-1)!}{(a-1)!} x^{1-c} e^x \left(-\frac{q}{C}\right)^{1-a} \left\{1 + \frac{(a-1)q\{(a-2)q + 2a\}}{2C}\ldots\right\}$$

(cf. Chapter XVI, Section 4). Identify this discontinuity with the first contribution on the right of VIII (84).

26. Specializing results derived in the text to

$$D_p(x) = 2^{\frac{1}{2}p+\frac{1}{4}} x^{-\frac{1}{2}} W_{\frac{1}{2}p+\frac{1}{4}, -\frac{1}{4}}(\tfrac{1}{2}x^2),$$

write down the interpreted asymptotic expansions for $D_p[2(p + \frac{1}{2})^{\frac{1}{2}}]$ and its first derivative.

27. In continuation of Chapter VIII, question 17, derive the interpretation

$$D_p(2\sqrt{p}) = (8/3\pi^3)^{\frac{1}{6}}(-\tfrac{2}{3})!\,p^{\frac{1}{2}p+\frac{1}{6}}\,e^{-\frac{1}{2}p}\Bigg[\Bigg\{\sum_{0,6,..}^{n-6} C_r + T_n{}^e C_n\Bigg\}$$

$$-\Bigg\{\sum_{1,7,\ldots}^{m-6} C_r + T_m{}^o C_m\Bigg\} + \Bigg\{\sum_{3,9,\ldots}^{l-6} C_r + T_l{}^o C_l\Bigg\} - \Bigg\{\sum_{4,10,\ldots}^{k-6} C_r + T_k{}^e C_k\Bigg\}\Bigg]$$

with

$$T_n{}^e C_n = \frac{(-1)^{\frac{1}{2}n}}{\sqrt{3}\,\pi(2\pi p)^{\frac{1}{3}n}}\Bigg\{(\tfrac{1}{3}n-1)!\,\Pi_{\frac{1}{3}n-1}(2\pi p)$$

$$-\frac{67\pi^{\frac{4}{3}}\beta}{560}(\tfrac{1}{3}n-\tfrac{7}{3})!\,\Pi_{\frac{1}{3}n-\frac{7}{3}}(2\pi p)\ldots\Bigg\}$$

(same for k), and

$$T_m{}^o C_m = \frac{(-1)^{\frac{1}{2}(n-1)}\sqrt{3}\,\beta}{\pi^{\frac{2}{3}}(2\pi p)^{\frac{1}{3}n}}\Bigg\{(\tfrac{1}{3}n-\tfrac{4}{3})!\,\Pi_{\frac{1}{3}n-\frac{4}{3}}(2\pi p)$$

$$-\frac{\pi^{\frac{4}{3}}}{360\beta}(\tfrac{1}{3}n-2)!\,\Pi_{\frac{1}{3}n-2}(2\pi p)\ldots\Bigg\}$$

(same for l). The interpretation is more cumbersome than that obtained in question 26 because the C_{odd} are non-zero.

28. Show that in the similar interpretation of the expansion for $p^{-\frac{1}{2}}D_{p+1}(2\sqrt{p})$,

$$T_n{}^e C_n = \frac{(-1)^{\frac{1}{2}n}}{\sqrt{3}\,\pi(2\pi p)^{\frac{1}{3}n}}\Bigg\{(\tfrac{1}{3}n-1)!\,\Pi_{\frac{1}{3}n-1}(2\pi p)$$

$$+\frac{213\pi^{\frac{4}{3}}\beta}{560}(\tfrac{1}{3}n-\tfrac{7}{3})!\,\Pi_{\frac{1}{3}n-\frac{7}{3}}(2\pi p)\ldots\Bigg\},$$

$$T_m{}^o C_m = \frac{(-1)^{\frac{1}{2}(n-1)}3\sqrt{3}\,\beta}{\pi^{\frac{2}{3}}(2\pi p)^{\frac{1}{3}n}}\Bigg\{(\tfrac{1}{3}n-\tfrac{4}{3})!\,\Pi_{\frac{1}{3}n-\frac{4}{3}}(2\pi p)$$

$$+\frac{17\pi^{\frac{4}{3}}}{1080\beta}(\tfrac{1}{3}n-2)!\,\Pi_{\frac{1}{3}n-2}(2\pi p)\ldots\Bigg\}.$$

29. From Chapter VIII, question 23, deduce

$$\int_0^\infty e^{-xu(1+u)^{\frac{1}{2}}}\,du = x^{-1}\left\{\sum_0^{n-1} L_r + T_n L_n\right\}$$

with

$$T_n = \left\{\Lambda_{n-\frac{1}{2}} + \frac{\cdot 1041\dot{6}}{n-\frac{1}{2}}\Lambda_{n-\frac{3}{2}} + \frac{\cdot 00542535}{(n-\frac{1}{2})(n-\frac{3}{2})}\Lambda_{n-5/2} - \ldots\right\}$$

$$\div\left\{1 + \frac{\cdot 1041\dot{6}}{n-\frac{1}{2}} + \frac{\cdot 00542535}{(n-\frac{1}{2})(n-\frac{3}{2})} - \ldots\right\}$$

where $\Lambda = \Lambda(\cdot 47247039x)$. Hence evaluate the integral when $x = 4$.

30. From Chapter VIII, question 25, dealing with the integral used by van der Corput (1954) to illustrate estimation of upper bounds, deduce

$$I(x) = \int_0^\infty e^{-xu(1+u)^{\frac{1}{2}}}\{(1+u)(1+\tfrac{1}{2}u)(1+\tfrac{1}{3}u)\}^{-\frac{1}{2}}du$$

$$= x^{-1}\left\{\sum_0^{n-1} L_r + T_n L_n\right\}$$

with

$$T_n = \left\{\Lambda_{n-\frac{1}{2}} + \frac{\cdot 01208\dot{3}}{n-\frac{1}{2}}\Lambda_{n-\frac{3}{2}} - \frac{\cdot 12377792}{(n-\frac{1}{2})(n-\frac{3}{2})}\Lambda_{n-5/2} + \ldots\right\}$$

$$\div\left\{1 + \frac{\cdot 01208\dot{3}}{n-\frac{1}{2}} - \frac{\cdot 12377792}{(n-\frac{1}{2})(n-\frac{3}{2})}\ldots\right\}$$

where $\Lambda = \Lambda(\cdot 47247039x)$. Hence evaluate $I(4)$.

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Chapter XXIV

Termination of Asymptotic Expansions derived from Homogeneous Differential Equations (Phase-Integral Method)

1. Coincidence between the Y_s and Y_r sets

In Chapter XIII, Section 5, we let $y^{\mp}$ denote two continuous solutions to the given differential equation

$$d^2y/dx^2 = Xy, \tag{1}$$

$X(x)$ being real on the real axis, and showed how these could be specified *along the real axis* on the exponential side of the turning point x_0 by the asymptotic expansions $\mathscr{y}_{\mp}$,

$$(y^{\mp})_{\exp} = \mathscr{y}_{\mp} = X^{-\frac{1}{4}} \exp\left(\mp \varepsilon \int_{x_0}^{x} X^{\frac{1}{2}} dx \right) \ Y_{\mp}, \quad Y_{\mp} = \sum_{0}^{\infty} (\mp 1)^r Y_r, \tag{2}$$

where $\varepsilon =$ sign of $(x - x_0)$ on the exponential side and the positive root of X is understood. The recurrence relation

$$-\varepsilon\, Y_{r+1} = \tfrac{1}{2} X^{-\frac{1}{2}}\, dY_r/dx + \frac{1}{32}\int^{x} X^{-5/2} \{5(X')^2 - 4XX''\}\, Y_r\, dx, \tag{3}$$

starting at $Y_0 = 1$, was then recast in "polynomial form"

$$A\, Y_{r+1} = P(q)\, dY_r/dq + \int^{q} Q(q)\, Y_r\, dq \tag{4}$$

by introducing a simplifying variable q.

Assuming $X(x)$ to vary linearly with $x - x_0$ at the turning point, we derived in Chapter XIII, Section 7, a formally exact expansion for late terms,

$$Y_r = \frac{1}{2\pi\,\mathscr{F}_0{}^r} \sum_{s=0} Y_s\,(-\mathscr{F}_0)^s\,(r-s-1)!\,, \tag{5}$$

where the singulant

$$\mathscr{F}_0 = 2\,\varepsilon \int_{x_0}^{x} X^{\frac{1}{2}}\,dx = A\int_{q}^{\infty} P^{-1}\,dq \tag{6}$$

is real and positive along the real axis on the exponential side of the turning point, and the Y_s satisfy (4). However, at that stage we could not rule out a possibility that the sequence of integration limits in (4) determining the Y_s might differ from the sequence chosen for the early terms Y_r, causing a cumulative divergence between the two sets after the common first term unity. This omission will now be rectified.

Combining (5) with XXI (15), (16) of our interpretative theory, the remainders after n terms in the two asymptotic series $Y_{\mp}$ are seen to be

$$\sum_n^{\infty} (-1)^r Y_r = \frac{1}{2\pi\,(-\mathscr{F}_0)^n} \sum_{s=0} Y_s\,(-\mathscr{F}_0)^s (n-s-1)!\,\Lambda_{n-s-1}(\mathscr{F}_0),$$

$$|\mathrm{ph}\,\mathscr{F}_0| < \pi, \tag{7}$$

$$\sum_n^{\infty} Y_r = \frac{1}{2\pi\,\mathscr{F}_0{}^n} \sum_{s=0} Y_s\,(-\mathscr{F}_0)^s (n-s-1)!\,\overline{\Lambda}_{n-s-1}(-\mathscr{F}_0),$$

$$\mathrm{ph}\,\mathscr{F}_0 = 0. \tag{8}$$

The asymptotic expansions of the two continuous solutions $y^{\mp}$ in other phase sectors can be derived by imagining the turning point x_0 to be circumnavigated via some roughly semicircular diversion through (say) the upper half of the complex plane. Since $\mathscr{F}_0 \propto \int_{x_0}^{x} (x-x_0)^{\frac{1}{2}} dx \propto (x-x_0)^{\frac{3}{2}}$ close to the turning point, ph $\mathscr{F}_0$ on this path starts from zero, along the real axis on the exponential side, and increases up to $\frac{3}{2}\pi$, along the real axis on the oscillatory side. Now as we have seen in XXI (Section 6, and again in (53)), there is a discontinuity in $\Lambda_{n-s-1}(\mathscr{F}_0)$ when ph$\mathscr{F}_0$ passes through π amounting to $2\pi i\,(-\mathscr{F}_0)^{n-s} e^{\mathscr{F}_0}/(n-s-1)!$. From (7), the resultant discontinuity in $\sum_n^{\infty}(-1)^r Y_r$ is $i\,e^{\mathscr{F}_0}\sum_0^{\infty} Y_s$. ($\sum Y_s$ is here terminated by a Λ, since it is not on one of its Stokes rays). Hence the solution

$$(y^-)_{\exp} = y_- = X^{-\frac{1}{4}}\,e^{-\frac{1}{2}\mathscr{F}_0}\,Y_-$$

on the exponential side changes to the solution

$$(y^-)_{\mathrm{osc}} = X^{-\frac{1}{4}} \left\{ e^{-\frac{1}{2}\mathscr{F}_0} \sum_{r=0}^{\infty} (-1)^r Y_r + i\, e^{\frac{1}{2}\mathscr{F}_0} \sum_{s=0}^{\infty} Y_s \right\} \tag{9}$$

on the oscillatory side. In particular, along the real axis on this oscillatory side,

$$(y^-)_{\mathrm{osc}} = |X|^{-\frac{1}{4}} \left\{ e^{-i(\frac{1}{4}\pi - \frac{1}{2}|\mathscr{F}_0|)} \sum_{r=0}^{\infty} i^r |Y_r| + e^{i(\frac{1}{4}\pi - \frac{1}{2}|\mathscr{F}_0|)} \sum_{s=0}^{\infty} (-i)^s |Y_s| \right\}.$$

This is real, as required by continuity of y^- along the real axis, only if its two components are complex conjugates, a condition satisfied only if the Y_s and Y_r sets are identical.

2. Connection Formulae

In the notation of (2), $(y^-)_{\mathrm{exp}} = \mathscr{y}_-$ and $(y^-)_{\mathrm{osc}} = \mathscr{y}_- + i\mathscr{y}_+$ according to (9). The connection formula is therefore

$$\begin{array}{ccccc} \mathscr{y}_- & \leftarrow & y^- & \rightarrow & \mathscr{y}_- + i\mathscr{y}_+ \\ \text{Exponential side} & & \text{Continuous solution} & & \text{Oscillatory side} \end{array} \tag{10}$$

in agreement with XIII (16).

Turning now to the remainder (8) after n terms in the asymptotic series Y_+, we first apply XXI (33) to substitute each $\overline{\Lambda}$ by

$$\overline{\Lambda}_{n-s-1}(-\mathscr{F}_0) = \Lambda_{n-s-1}(-\mathscr{F}_0) + i\pi \mathscr{F}_0^{\;n-s} e^{-\mathscr{F}_0}/(n-s-1)!, \qquad \mathrm{ph}\, \mathscr{F}_0 > 0, \tag{11}$$

thereby bringing the series off its Stokes ray. The basic terminants $\Lambda(-\mathscr{F}_0)$ are continuous from $\mathrm{ph}\,\mathscr{F}_0 \to +0$ up to and including $\frac{3}{2}\pi$ (in fact up to 2π); the initial series Y_+ therefore persists to the oscillatory side. The contribution to $\sum_n^\infty Y_r$ from the second term on the right of (11) is $\frac{1}{2}i\, e^{-\mathscr{F}_0} Y_-$, which as we have seen in Section 1 has to be replaced beyond $\mathrm{ph}\,\mathscr{F}_0 = \pi$ by

$$\tfrac{1}{2}i\, e^{-\mathscr{F}_0}(Y_- + i\, e^{\mathscr{F}_0} Y_+).$$

The asymptotic series Y_+ specified along the real axis on the exponential side has therefore to be replaced by $\frac{1}{2}(Y_+ + i\, e^{-\mathscr{F}_0} Y_-)$ on the oscillatory side, so the connection formula is

$$\begin{array}{ccccc} \mathscr{Y}_+ & \leftarrow & y^+ & \rightarrow & \tfrac{1}{2}(\mathscr{Y}_+ + i\mathscr{Y}_-) \\ \text{Real axis on exponential side} & & \text{Continuous solution} & & \text{Oscillatory side} \end{array} \quad (12)$$

in agreement with XIII (20).

3. Interpretations on the Exponential Side

From (7) the interpretation of the series Y_- entering the exponentially decreasing solution y^- is

$$Y_- = \sum_0^{n-1} (-1)^r Y_r + T_n(-1)^n Y_n, \quad (13)$$

where

$$T_n = \left\{ Y_0 \Lambda_{n-1}(\mathscr{F}_0) - \frac{Y_1 \mathscr{F}_0}{n-1} \Lambda_{n-2}(\mathscr{F}_0) + \frac{Y_2 \mathscr{F}_0^{\,2}}{(n-1)(n-2)} \Lambda_{n-3}(\mathscr{F}_0) - \dots \right\}$$

$$\div \left\{ Y_0 - \frac{Y_1 \mathscr{F}_0}{n-1} + \frac{Y_2 \mathscr{F}_0^{\,2}}{(n-1)(n-2)} - \dots \right\}. \quad (14)$$

The series entering the exponentially increasing solution y^+ is similarly

$$Y_+ = \sum_0^{n-1} Y_r + T_n Y_n, \quad (15)$$

where by (8)

$$T_n = \left\{ Y_0 \bar{\Lambda}_{n-1}(-\mathscr{F}_0) - \frac{Y_1 \mathscr{F}_0}{n-1} \bar{\Lambda}_{n-2}(-\mathscr{F}_0) + \dots \right\} \Big/ \left\{ Y_0 - \frac{Y_1 \mathscr{F}_0}{n-1} + \dots \right\}. \quad (16)$$

These terminant expansions differ only in notation from those derived in Chapter XXIII, Section 4, when interpreting asymptotic series for $\int_{\text{s.p.}} e^{-F} G\,du$, $F(u)$ quadratic at the stationary point. In that context explicit formulae were obtained for $J_p(x)$ and $Y_p(x)$ in the exponential range $p > x > 0$, illustrated numerically by calculation of $J_5(3)$ and $Y_5(3)$; for $W_{km}(x)$ with k large in the range $x > 4k > 0$; and for $W_{km}(x)$ with $\kappa = (k^2 - m^2)^{\frac{1}{2}}$ large in the range $x > 2(k + \kappa) > 0$. Excepting small values of variable and order, where convergent series are more suitable, and the neighbourhood of the turning point, where the uniform and Taylor expansions of Chapter XV excel, the results from these terminant expansions prove to be around 10^4–10^6 times more accurate than those hitherto available for the phase-integral method.

Late terms and terminants can also be expressed as integrals, in close analogy with those of XXIII, questions 10–13. Representing the factorial in (5) by an integral,

$$Y_r = \frac{1}{2\pi \mathscr{F}_0{}^r} \int_0^\infty \sigma^{r-1} e^{-\sigma} d\sigma \sum_{s=0}^{\infty} \left(- \frac{\mathscr{F}_0}{\sigma} \right)^s Y_s .$$

Regarded as a function of the order A, $Y_s \propto A^{-s}$ and $Y_-(A) = \sum_0^\infty (-1)^s \; Y_s(A)$ by (2), so $\sum_0^\infty (- \mathscr{F}_0/\sigma)^s Y_s(A) = Y_-(A\sigma/\mathscr{F}_0)$. Introducing a parametric order $a = A\sigma/\mathscr{F}_0$,

$$Y_r(A) = \frac{1}{2\pi A^r} \int_0^\infty a^{r-1} e^{-a\Xi} \, Y_-(a) \, da, \quad \mathscr{F}_0 = A\Xi. \tag{17}$$

(I.e. $2\pi A^r Y_r$ is the Mellin transform of $e^{-a\Xi} \, Y_-(a)$). The terminants are therefore

$$T_n(A) = \sum_n^\infty (-1)^r \, Y_r(A)/(-1)^n \, Y_n(A)$$

$$= \int_0^\infty \frac{a^{n-1} e^{-a\Xi}}{1 + a/A} \, Y_-(a) \, da \Bigg/ \int_0^\infty a^{n-1} e^{-a\Xi} \, Y_-(a) \, da, \tag{18}$$

$$\bar{T}_n(A) = \sum_n^\infty Y_r(A)/Y_n(A)$$

$$= \mathrm{P} \int_0^\infty \frac{a^{n-1} e^{-a\Xi}}{1 - a/A} \, Y_-(a) \, da \Bigg/ \int_0^\infty a^{n-1} e^{-a\Xi} \, Y_-(a) \, da. \tag{19}$$

To a certain extent these results beg the question because of their dependence on $Y_-(a)$, itself an objective. A useful estimate of the terminant can nevertheless be found by expanding Y_-, or better ln Y_-, around the approximate zenith of the integrands at $a = (n-1)/\Xi$. This gives

$$T_n = [\Lambda_{n-1} - \tfrac{1}{2} l_1^{-2} l_2\{n(n+1)\Lambda_{n+1} - 2l_1 n(n-1)\Lambda_n + l_1^2(n-1)^2\Lambda_{n-1}\}\ldots]$$
$$\div [1 - \tfrac{1}{2} l_1^{-2} l_2\{n(n+1) - 2l_1 n(n-1) + l_1^2(n-1)^2\}\ldots] \quad (20)$$

where $\Lambda = \Lambda(A\Xi l_1)$ and

$$l_1 = 1 - \frac{d}{dn}\ln Y_-\left(\frac{n-1}{\Xi}\right), \qquad l_2 = \frac{d^2}{dn^2}\ln Y_-\left(\frac{n-1}{\Xi}\right).$$

Provided n is not too small, these derivatives can be adequately estimated from the known early terms of the asymptotic expansion. In the analogous result for $\overline{T}$, the Λ are replaced by $\overline{\Lambda}(-A\Xi l_1)$.

4. Interpretations on the Oscillatory Side

To express $(y^{\mp})_{\text{osc}}$ in terms of quantities which are real along this section of the real axis, we reintroduce the definitions of Chapter XIII (39), (40):

$$\mathscr{Y}_r = i^r Y_r, \quad \mathscr{Y}_{\text{even}} = \sum_{0,2,..}^{\infty} (-1)^{\frac{1}{2}r}\mathscr{Y}_r, \quad \mathscr{Y}_{\text{odd}} = \sum_{1,3,..}^{\infty} (-1)^{\frac{1}{2}(r-1)}\mathscr{Y}_r. \quad (21)$$

Then, from the connection formulae (10) and (12) and the expressions (2) for $y_{\mp}$,

$$(y^-)_{\text{osc}} = 2(-X)^{-\frac{1}{4}}(\mathscr{Y}_{\text{even}}\sin\psi - \mathscr{Y}_{\text{odd}}\cos\psi),$$
$$(y^+)_{\text{osc}} = (-X)^{-\frac{1}{4}}(\mathscr{Y}_{\text{even}}\cos\psi + \mathscr{Y}_{\text{odd}}\sin\psi), \quad (22)$$

where

$$\psi = \varepsilon\int_x^{x_0}(-X)^{\frac{1}{2}}\,dx + \tfrac{1}{4}\pi = \tfrac{1}{2}\mathbf{F}_0 + \tfrac{1}{4}\pi. \quad (23)$$

$\mathbf{F}_0 = -i\mathscr{F}_0$ is real and positive along this section of the real axis.

Rewriting (5) in the form

$$\mathscr{Y}_r = \frac{1}{2\pi\,\mathbf{F}_0^{\,r}}\sum_{s=0}\mathscr{Y}_s(-\mathbf{F}_0)^s(r-s-1)!, \quad (24)$$

the interpretations of the asymptotic series (21) are seen to be

$$\mathcal{Y}_{\text{even}} = \sum_{0,2,..}^{n-2} (-1)^{\frac{1}{2}r}\,\mathcal{Y}_r + (-1)^{\frac{1}{2}n}\,\mathfrak{T}_n\,\mathcal{Y}_n,$$

$$\mathcal{Y}_{\text{odd}} = \sum_{1,3,..}^{m-2} (-1)^{\frac{1}{2}(r-1)}\mathcal{Y}_r + (-1)^{\frac{1}{2}(m-1)}\;\mathfrak{T}_m\,\mathcal{Y}_m, \tag{25}$$

with

$$\mathfrak{T}_n = \left\{\mathcal{Y}_0\,\Pi_{n-1}(\mathbf{F}_0) - \frac{\mathcal{Y}_1\mathbf{F}_0}{n-1}\Pi_{n-2}(\mathbf{F}_0) + \frac{\mathcal{Y}_2{\mathbf{F}_0}^2}{(n-1)(n-2)}\,\Pi_{n-3}(\mathbf{F}_0) - \ldots\right\}$$

$$\div \left\{\mathcal{Y}_0 - \frac{\mathcal{Y}_1\mathbf{F}_0}{n-1} + \frac{\mathcal{Y}_2{\mathbf{F}_0}^2}{(n-1)(n-2)} - \ldots\right\} \tag{26}$$

(same for m).

Special cases of this terminant expansion have already been discussed in the preceding chapter. Explicit formulae were obtained for $J_p(x)$ and $Y_p(x)$ in the oscillatory range $x > p > 0$, illustrated numerically by calculation of $J_6(10)$ and $Y_6(10)$; for $W_{km}(x)$ with k large in the range $4k > x > 0$; and for $W_{km}(x)$ with $\kappa = (k^2 - m^2)^{\frac{1}{2}}$ large in the range $2(k + \kappa) > x > 2(k - \kappa)$.

5. Non-linear Phase-Integral Theory

In (2) and (22), solutions to the differential equation (1) were expressed linearly in terms of asymptotic series which have now been interpreted. This is the best form of expression for most purposes, because the linearity of the theory is conducive to maximum progress in its analysis—culminating as we have seen in complete asymptotic expansions and integral representations for the terminants.

For determining zeros and eigenvalues, manipulations are simpler with non-linear representations in which asymptotic series are placed in the arguments of the exponential, sine and cosine, XIV (20)–(22). We have now to supply interpretations of these asymptotic series.

From the formula XIV (27) for late K_s, the summations in XIV (20) become

$$\sum_{1}^{n-1} (-1)^s K_s + T_n(-1)^n K_n, \qquad \sum_{1}^{n-1} K_s + \bar{T}_n K_n, \tag{27}$$

with

$$T_n = \left\{\Lambda_{n-1}(\mathscr{F}_0) - \frac{2K_1\mathscr{F}_0}{n-1}\Lambda_{n-2}(\mathscr{F}_0) + \frac{2K_1{}^2\mathscr{F}_0{}^2}{(n-1)(n-2)}\Lambda_{n-3}(\mathscr{F}_0) - \frac{2(K_1{}^3+3K_3)\mathscr{F}_0{}^3}{3(n-1)(n-2)(n-3)}\Lambda_{n-4}(\mathscr{F}_0) + \dots\right\}$$

$$\div\left\{1 - \frac{2K_1\mathscr{F}_0}{n-1} + \frac{2K_1{}^2\mathscr{F}_0{}^2}{(n-1)(n-2)} - \frac{2(K_1{}^3+3K_3)\mathscr{F}_0{}^3}{3(n-1)(n-2)(n-3)} + \dots\right\} \tag{28}$$

and $\bar{T}_n$ differing only by replacement of all $\Lambda(\mathscr{F}_0)$ by $\bar{\Lambda}(-\mathscr{F}_0)$. From XIV(29) for late $\mathscr{K}_s$, the summations in XIV (21), (22) have interpretations

$$\sum_{2,4,..}^{n-2}(-1)^{\frac{1}{2}s}\mathscr{K}_s + \mathfrak{T}_n(-1)^{\frac{1}{2}n}\mathscr{K}_n, \quad \sum_{1,3,..}^{m-2}(-1)^{\frac{1}{2}(s-1)}\mathscr{K}_s + \mathfrak{T}_m(-1)^{\frac{1}{2}(m-1)}\mathscr{K}_m, \tag{29}$$

with

$$\mathfrak{T}_n = \left\{\Pi_{n-1}(\mathrm{F}_0) - \frac{2\mathscr{K}_1\mathrm{F}_0}{n-1}\Pi_{n-2}(\mathrm{F}_0) + \dots\right\}\Big/\left\{1 - \frac{2\mathscr{K}_1\mathrm{F}_0}{n-1} + \dots\right\} \tag{30}$$

(same for m), We have not succeeded in finding general terms or integral representations.

Exercises

1. In the second variant of the phase-integral method (Chapter XIII, Section 8), the solutions to $d^2y/dx^2 = (\chi + \Delta)y$ along the real axis on the exponential side are expressed as

$$(y^{\pm})_{\exp} = \mathscr{Y}_{\pm} = \chi^{-\frac{1}{4}} e^{\pm\frac{1}{2}(\mathscr{F}_0+\delta)} \sum_0^{\infty} (\pm 1)^r Y_r^{\pm},$$

where

$$\mathscr{F}_0 = 2\varepsilon\int_{x_0}^{x} \chi^{\frac{1}{2}}\,dx, \quad \delta = \varepsilon\int_{x_0}^{x} \chi^{-\frac{1}{2}}\Delta\,dx.$$

Late terms are given by

$$Y_r^{\pm} = \frac{e^{\mp\delta}}{2\pi\mathscr{F}_0{}^r}\sum_{s=0}(r-s-1)!\,Y_s^{\mp}(-\mathscr{F}_0)^s.$$

Show that the discontinuity in $\Sigma(-1)^r Y_r^-$ on going over to the oscillatory side is $ie^{\mathscr{F}_0+\delta}\sum_0^\infty Y_s^+$ (this series being terminated here by a Λ since it is not on one of its Stokes rays), and

$$(y^-)_{\text{osc}} = \chi^{-\frac{1}{4}}\left\{e^{-\frac{1}{2}(\mathscr{F}_0+\delta)}\sum_{r=0}^{\infty}(-1)^r Y_r^- + i\,e^{\frac{1}{2}(\mathscr{F}_0+\delta)}\sum_{s=0}^{\infty} Y_s^+\right\}.$$

According to XIII (50) the recurrence relations for $(-1)^r Y_r^-$ and Y_r^+ differ only in respect of the sign prefacing the root $\chi^{\frac{1}{2}}$, making $\Sigma(-1)^r Y_r^-$ and ΣY_r^+ conjugate complexes along the real axis on the oscillatory side. Show that $(y^-)_{\text{osc}}$ is real there—as required by continuity of y^- along the real axis—only if the $Y_s^\pm$ and $Y_r^\pm$ sets coincide.

2. Verify the connection formula

$\mathscr{y}_-$	$\leftarrow$	y^-	$\rightarrow$	$\mathscr{y}_- + i\mathscr{y}_+$
Exponential side		Continuous solution		Oscillatory side

for this variant.

3. Show that the asymptotic series ΣY_r^+ specified along the real axis on the exponential side has to be replaced by $\frac{1}{2}\{\Sigma Y_r^+ + i\,e^{-\mathscr{F}_0-\delta}\Sigma(-1)^s Y_s^-\}$ on the oscillatory side. Verify the connection formula

$\mathscr{y}_+$	$\leftarrow$	y^+	$\rightarrow$	$\frac{1}{2}(\mathscr{y}_+ + i\mathscr{y}_-)$.
Real axis on exponential side		Continuous solution		Oscillatory side

4. Derive the representation

$$Y_r(k) = \frac{1}{2\pi\,k^r}\int_0^\infty l^{r-1}\,e^{-4l\Xi}\,Y_-(l)\,dl$$

for late terms in the asymptotic series of W_{km} and $\mathscr{W}_{km}$ for large k. Substituting

$$Y_-(l) = \sum_0^\infty(-1)^r\,Y_r = 1 - \frac{1}{96l}(5q^3 - 6q + 3\eta/q) + \dots$$

from XIII (87) and integrating term by term, regain the formula XIII (89) for late terms.

5. Show that the terminant appropriate to the series $Y_- \propto W_{km}$ is

$$T_n(k) = \int_0^\infty \frac{l^{n-1}\, e^{-4l\Xi}}{1 + l/k}\, Y_-(l)\, dl \bigg/ \int_0^\infty l^{n-1}\, e^{-4l\Xi}\, Y_-(l) dl$$

$$= \{\Lambda_{n-1}(4k\Xi) - a\,\Lambda_{n-2}(4k\Xi) \ldots\}/(1 - a \ldots)$$

where $a = \Xi\,(5q^3 - 6q + 3\eta/q)/24(n - 1)$.

6. Show that the terminant appropriate to the series $Y_+ = \sum Y_r \propto \mathscr{W}_{km}$ is

$$\overline{T}_n(k) = P\int_0^\infty \frac{l^{n-1}\, e^{-4l\Xi}}{1 - l/k}\, Y_-(l)\, dl \bigg/ \int_0^\infty l^{n-1}\, e^{-4l\Xi}\, Y_-(l)\, dl$$

$$= \{\overline{\Lambda}_{n-1}(-4k\Xi) - a\,\overline{\Lambda}_{n-2}(-4k\Xi) \ldots\}/(1 - a \ldots).$$

7. Derive the following expansion for the parabolic cylinder function:

$$D_p(x) = (p!/2)^{\frac{1}{2}} \left(\frac{q(q^2 - 1)}{\pi(2p + 1)}\right)^{\frac{1}{4}} \exp\left[-\tfrac{1}{2}\mathscr{F}_0 - K_1 + K_2 - K_3 + \ldots \right.$$
$$\left. + (-1)^n\, T_n K_n\right],$$

where

$$q = \frac{x}{(x^2 - 4p - 2)^{\frac{1}{2}}}, \qquad \mathscr{F}_0 = (2p + 1)\left\{\frac{q}{q^2 - 1} - \coth^{-1} q\right\},$$

the K's are those of Chapter XIV, question 2, and the terminant expansion T_n is given by (28).

Chapter XXV

Termination of Asymptotic Expansions derived from Homogeneous Differential Equations in which the Second Derivative is Relatively Insignificant

1. Connection Formulae

Let $y^{\mp}$ denote two continuous solutions to the differential equation

$$d^2y/dx^2 + (f + \delta)\,dy/dx + (G + \tfrac{1}{2}f' + \tfrac{1}{2}f\delta)y = 0, \tag{1}$$

where f, δ and G are real on the real axis. In Chapter XVI, Section 1, these solutions were specified *along the section of the real axis for which* $f > 0$ by the asymptotic expansions $\mathscr{y}_{\mp}$,

$$(y^-)_{f>0} = \mathscr{y}_- = f^{-\frac{1}{2}} \exp\left[-\textstyle\int (f^{-1}G + \tfrac{1}{2}\delta)\,dx\right]\ Y^-,$$
$$Y^- = \sum_0^{\infty} (-1)^r Y_r^-, \tag{2}$$

$$(y^+)_{f>0} = \mathscr{y}_+ = f^{-\frac{1}{2}} \exp\left[-\textstyle\int (f - f^{-1}G + \tfrac{1}{2}\delta)\,dx\right]\ Y^+,$$
$$Y^+ = \sum_0^{\infty} Y_r^+, \tag{3}$$

assuming the direction of the axis to be such that $f > 0$ when $x < x_0$. The recurrence relations

$$Y_{r+1}^{\mp} = \frac{1}{f}\frac{dY_r^{\mp}}{dx} \mp \frac{2G}{f^2} Y_r^{\mp}$$
$$+ \int^x \frac{(G \mp \frac{1}{2}f')(G \mp \frac{3}{2}f') \pm f(G' \mp \frac{1}{2}f'') - \frac{1}{4}f^2(\delta^2 + 2\delta')}{f^3}\, Y_r^{\mp} dx, \tag{4}$$

starting at $Y_0^{\mp} = 1$, were then recast in 'polynomial form'

$$A Y_{r+1}^{\mp} = P(q)dY_r^{\mp}/dq \mp R(q)Y_r^{\mp} + \int^q Q^{\mp}(q) Y_r^{\mp}\, dq \tag{5}$$

by introducing a simplifying variable q.

Assuming $f(x)$ to vary linearly with $x_0 - x$ at the extinction point x_0, we derived in Chapter XVI, Section 2 formally exact expansions for late terms,

$$Y_r^- = \frac{A^{-\rho-\frac{1}{2}}e^{2\mathscr{G}}}{(2\pi)^{\frac{1}{2}}(-\rho-1)!\,\mathscr{F}_0^{\,r-\rho-\frac{1}{2}}} \sum_{s=0} \mathscr{Y}_s^{\,+}(-\mathscr{F}_0)^s(r-s-\rho-\tfrac{3}{2})!, \quad (6)$$

$$Y_r^+ = \frac{A^{\rho+\frac{1}{2}}e^{-2\mathscr{G}}}{(2\pi)^{\frac{1}{2}}\;\rho!\;\mathscr{F}_0^{\,r+\rho+\frac{1}{2}}} \sum_{s=0} \mathscr{Y}_s^{\,-}(-\mathscr{F}_0)^s(r-s+\rho-\tfrac{1}{2})!\,. \quad (7)$$

Here $\rho = -f_1^{-1}G_0 - \frac{1}{2}$,

$$\mathscr{G} = \int f^{-1}G\,dx = \tfrac{1}{2}\int RP^{-1}\,dq \quad (8)$$

with the integration constant fixed such that in the limit $q \gg 1$ $(x \to x_0)$ $e^{\mathscr{G}} \to q^{\rho+\frac{1}{2}}$, the singulant

$$\mathscr{F}_0 = \int_x^{x_0} f\,dx = A\int_q^{\infty} P^{-1}dq \quad (9)$$

is real and positive along the section of the real axis for which $f > 0$, and the $\mathscr{Y}_s^{\,\mp}$ satisfy (5). However, from the arguments of Chapter XVI, Section 1, indicating absence of connection formulae linking $(y^{\mp})_{f>0}$ and $(y^{\mp})_{f<0}$ which are at once universal and precise, and from specific examples discussed in Chapter XVI, Sections 2 and 4, we showed how the sequences of integration limits in (5) determining the $\mathscr{Y}_s^{\,\mp}$ could differ from those chosen for the early terms $Y_r^{\mp}$, causing a cumulative divergence between the two sets after the common first term unity. We are now in a position to undertake the intricate analysis involved in this question of linkage.

Combining (6) and (7) respectively with XXI (15) and (16) of our interpretative theory, the remainders after n terms in the two asymptotic series $Y^{\mp}$ are seen to be

$$\sum_n^{\infty} (-1)^r Y_r^- = \frac{(-1)^n A^{-\rho-\frac{1}{2}}\,e^{2\mathscr{G}}}{(2\pi)^{\frac{1}{2}}(-\rho-1)!\,\mathscr{F}_0^{\,n-\rho-\frac{1}{2}}} \sum_{s=0} \mathscr{Y}_s^{\,+}(-\mathscr{F}_0)^s(n-s-\rho-\tfrac{3}{2})!$$
$$\times\, \Lambda_{n-s-\rho-\frac{3}{2}}(\mathscr{F}_0), \qquad |\mathrm{ph}\,\mathscr{F}_0| < \pi, \quad (10)$$

$$\sum_n^{\infty} Y_r^+ = \frac{A^{\rho+\frac{1}{2}}e^{-2\mathscr{G}}}{(2\pi)^{\frac{1}{2}}\;\rho!\;\mathscr{F}_0^{\,n+\rho+\frac{1}{2}}} \sum_{s=0} \mathscr{Y}_s^{\,-}(-\mathscr{F}_0)^s(n-s+\rho-\tfrac{1}{2})!$$
$$\times\, \overline{\Lambda}_{n-s+\rho-\frac{1}{2}}(-\mathscr{F}_0), \qquad \mathrm{ph}\,\mathscr{F}_0 = 0. \quad (11)$$

The asymptotic expansions of the two continuous solutions $y^{\mp}$ in other phase sectors can in principle be derived by imagining the extinction point x_0 to be circumnavigated via some roughly semicircular diversion in the complex plane. Since $\mathscr{F}_0 \propto \int_x^{x_0} (x_0 - x)\,dx \propto (x_0 - x)^2$ close to the extinction point, ph $\mathscr{F}_0$ on this path starts from zero, along the real axis on the side $x < x_0$, $f > 0$; and increases up to 2π, along the real axis on the side $x > x_0$, $f < 0$.

The discontinuity in $\Lambda(\mathscr{F}_0)$ as ph $\mathscr{F}_0$ passes through π (Chapter XXI, Section 6 and (53)) leads to a discontinuity in $\sum_n^\infty(-1)^r Y_r^-$ of magnitude

$$D = i\,\frac{(2\pi)^{\frac{1}{2}}\,e^{i\pi(\rho+\frac{1}{2})}A^{-\rho-\frac{1}{2}}\,e^{\mathscr{F}_0+2\mathscr{G}}}{(-\rho-1)!}\sum_0^\infty \mathscr{Y}_s^{+}. \tag{12}$$

Let us define

$$\Gamma_+ = \sum_0^\infty \mathscr{Y}_s^{+} \Big/ \sum_0^\infty Y_s^{+}, \tag{13}$$

where all we can assume at present is that Γ_+ is independent of x and deviates from unity by $O(A^{-1})$. At the phase of this discontinuity (ph $\mathscr{F}_0 = \pi$), ΣY_s^+ is not on one of its Stokes rays, and is therefore terminated with $\Lambda(-\mathscr{F}_0)$ replacing the $\bar{\Lambda}(-\mathscr{F}_0)$ of (11). Explicitly,

$$D = -\frac{(2\pi)^{\frac{1}{2}}e^{i\pi\rho}\,e^{\mathscr{F}_0+2\mathscr{G}}}{(-\rho-1)!\,A^{\rho+\frac{1}{2}}}\,\Gamma_+\sum_0^{n-1} Y_s^{+} + \frac{e^{i\pi\rho}\sin\pi\rho\,e^{\mathscr{F}_0}}{\pi\,\mathscr{F}_0^{\,n+\rho+\frac{1}{2}}}\,\Gamma_+\sum_{s=0}\mathscr{Y}_s^{-}(-\mathscr{F}_0)^s$$
$$\times\,(n-s+\rho-\tfrac{1}{2})!\,\Lambda_{n-s+\rho-\frac{1}{2}}(-\mathscr{F}_0), \qquad \pi < \text{ph}\,\mathscr{F}_0 < 2\pi. \tag{14}$$

On approaching the end of the semicircular diversion, we need expression in terms of the $\bar{\Lambda}(-\mathscr{F}_0)$ because of the discontinuity in $\Lambda(-\mathscr{F}_0)$ at ph $\mathscr{F}_0 = 2\pi$. This is most easily accomplished by modifying XXI (33) to apply to the discontinuity in $\Lambda_s(-x)$ located at ph $x = 2\pi$, instead of that at ph $x = 0$:

$$\bar{\Lambda}_s(-x) = \Lambda_s(-x) \pm i\pi e^{-2i\pi s}x^{s+1}e^{-x}/s! \qquad \left.\begin{matrix} 2\pi < \text{ph}\,x < 4\pi \\ 2\pi > \text{ph}\,x > 0 \end{matrix}\right\}. \tag{15}$$

Substituting in (14) for the relevant phase region ph $\mathscr{F}_0 < 2\pi$,

$$D = -\frac{(2\pi)^{\frac{1}{2}}\,e^{i\pi\rho}\,e^{\mathscr{F}_0+2\mathscr{G}}}{(-\rho-1)!\,A^{\rho+\frac{1}{2}}}\,\Gamma_+\sum_0^{n-1} Y_s^{+} + \frac{e^{i\pi\rho}\sin\pi\rho\,e^{\mathscr{F}_0}}{\pi\,\mathscr{F}_0^{\,n+\rho+\frac{1}{2}}}\,\Gamma_+\sum_{s=0}\mathscr{Y}_s^{-}(-\mathscr{F}_0)^s$$
$$\times\,(n-s+\rho-\tfrac{1}{2})!\,\Lambda_{n-s+\rho-\frac{1}{2}}(-\mathscr{F}_0)$$
$$-\,i\,e^{-i\pi\rho}\sin\pi\rho\,\Gamma_+\Gamma_-\sum_{s=0}^{\infty}(-1)^s Y_s^{-}, \qquad \pi < \text{ph}\,\mathscr{F}_0 < 3\pi, \tag{16}$$

where
$$\Gamma_- = \sum_0^\infty (-1)^s \mathscr{Y}_s^- \Big/ \sum_0^\infty (-1)^s Y_s^-. \tag{17}$$

The second contribution to (16) correctly terminates the first series on its Stokes rays, i.e. on the real axis. Hence *along the section* $(x > x_0)$ *of the real axis for which* $f < 0$,

$$(y^-)_{f<0} = (1 - i\,e^{-i\pi\rho} \sin\pi\rho\; \Gamma_+\Gamma_-)\mathscr{y}_- - \frac{(2\pi)^{\frac{1}{2}}\, e^{i\pi\rho}}{(-\rho-1)!\, A^{\rho+\frac{1}{2}}}\, \Gamma_+ \mathscr{y}_+. \tag{18}$$

Now $\mathscr{y}_+$ includes the multiplier $f^{-\frac{1}{4}} e^{\mathscr{G}} \propto (x_0 - x)^{-\rho-1}$, so its phase factor on this side is $-e^{-i\pi\rho}$; the reality condition applied to the second contribution in (18) then shows that Γ_+ must be real. The other expansion $\mathscr{y}_-$ includes the multiplier $f^{-\frac{1}{4}} e^{-\mathscr{G}} \propto (x_0 - x)^\rho$; its phase factor on this side is $e^{i\pi\rho}$, so the reality condition applied to the first contribution requires $e^{i\pi\rho} - i\sin\pi\rho\;\Gamma_+\Gamma_-$ to be real, i.e. $\Gamma_+\Gamma_- = 1$. This reciprocity simplifies (18) to

$$(y^-)_{f<0} = \cos\pi\rho\,(e^{-i\pi\rho}\mathscr{y}_-) + \frac{(2\pi)^{\frac{1}{2}}}{(-\rho-1)!\, A^{\rho+\frac{1}{2}}}\, \Gamma_+(-\,e^{i\pi\rho}\mathscr{y}_+). \tag{18$'$}$$

Turning to the remainder (11) after n terms in the asymptotic series Y^+, we first apply XXI (33) to bring this series off its Stokes ray:

$$\sum_n^\infty Y_r^+ = \frac{A^{\rho+\frac{1}{2}}\, e^{-2\mathscr{G}}}{(2\pi)^{\frac{1}{2}}\, \rho!\, \mathscr{F}_0^{\,n+\rho+\frac{1}{2}}} \sum_{s=0} \mathscr{Y}_s^-(-\mathscr{F}_0)^s (n - s + \rho - \tfrac{1}{2})!$$

$$\times\, \Lambda_{n-s+\rho-\frac{1}{2}}(-\mathscr{F}_0) + i(\tfrac{1}{2}\pi)^{\frac{1}{2}}\, \frac{A^{\rho+\frac{1}{2}}\, e^{-\mathscr{F}_0 - 2\mathscr{G}}}{\rho!}\, \Gamma_- \sum_0^\infty (-1)^s Y_s^-,$$

$$\pi > \operatorname{ph}\mathscr{F}_0 > 0. \tag{19}$$

The first component suffers no discontinuities until $\operatorname{ph}\mathscr{F}_0 = 2\pi$; on approaching the end of the diversion it is to be re-expressed in terms of the $\overline{\Lambda}(-\mathscr{F}_0)$ via (15). The vicissitudes of the second component were analysed in the preceding paragraph. Assembling the contributions and applying the reality condition to y^+ along the section $(x > x_0)$ of the real axis for which $f < 0$, the relation $\Gamma_+\Gamma_- = 1$ is again found, and y^+ along this section simplifies to

$$(y^+)_{f<0} = -\cos\pi\rho\,(-e^{i\pi\rho}\mathscr{y}_+) - (\tfrac{1}{2}\pi)^{\frac{1}{2}}\, \frac{A^{\rho+\frac{1}{2}}}{\rho!}\, \Gamma_- \sin\pi\rho\,(e^{-i\pi\rho}\mathscr{y}_-). \tag{20}$$

Summarizing our conclusions, analysis of this class of problem establishes that the functions Γ_+ and Γ_- appearing in the connection formulae (18′) and (20) are real, independent of x, deviate from unity by $O(A^{-1})$, and have product unity. Individually, the $\Gamma_\mp$ must be specific to the given differential equation, i.e. dependent on $f(x)$, $\delta(x)$ and $G(x)$. In this sense there are no connection formulae which are tractable, universal and absolutely precise—as are those applying to phase-integral pairs.

2. Terminant Expansions and Integral Representations

From (10), the interpretation of the series Y^- is

$$Y^- = \sum_0^{n-1} (-1)^r Y_r^- + T_n(-1)^n Y_n^- \tag{21}$$

where

$$\begin{aligned} T_n = &\left\{ \mathcal{Y}_0^{\ +} \Lambda_{n-\rho-\frac{1}{2}}(\mathcal{F}_0) - \frac{\mathcal{Y}_1^{\ +}\mathcal{F}_0}{n-\rho-\frac{3}{2}} \Lambda_{n-\rho-\frac{3}{2}}(\mathcal{F}_0) \right. \\ &\left. + \frac{\mathcal{Y}_2^{\ +}\mathcal{F}_0^{\ 2}}{(n-\rho-\frac{3}{2})(n-\rho-\frac{5}{2})} \Lambda_{n-\rho-\frac{5}{2}}(\mathcal{F}_0) - \dots \right\} \\ &\div \left\{ \mathcal{Y}_0^{\ +} - \frac{\mathcal{Y}_1^{\ +}\mathcal{F}_0}{n-\rho-\frac{3}{2}} + \frac{\mathcal{Y}_2^{\ +}\mathcal{F}_0^{\ 2}}{(n-\rho-\frac{3}{2})(n-\rho-\frac{5}{2})} - \dots \right\}. \end{aligned} \tag{22}$$

Similarly,

$$Y^+ = \sum_0^{n-1} Y_r^+ + \overline{T}_n Y_n^+ \tag{23}$$

where

$$\begin{aligned} \overline{T}_n = &\left\{ \mathcal{Y}_0^{\ -} \overline{\Lambda}_{n+\rho-\frac{1}{2}}(-\mathcal{F}_0) - \frac{\mathcal{Y}_1^{\ -}\mathcal{F}_0}{n+\rho-\frac{1}{2}} \overline{\Lambda}_{n+\rho-\frac{3}{2}}(-\mathcal{F}_0) + \dots \right\} \\ &\div \left\{ \mathcal{Y}_0 \quad - \frac{\mathcal{Y}_1^{\ -}\mathcal{F}_0}{n+\rho-\frac{1}{2}} + \dots \right\}. \end{aligned} \tag{24}$$

Integral representations for late terms and terminants, and numerically useful approximations to them, can all be obtained by the methods developed in Chapter XXIV, Section 3. In particular,

$$
\begin{aligned}
Y_r^{+}(A) &= \frac{e^{-2\mathscr{G}}}{(2\pi)^{\frac{1}{2}}\rho!\,A^r}\int_0^\infty a^{r+\rho-\frac{1}{2}}\,e^{-a\mathscr{F}_0/A}\,\Gamma_-(a)Y^-(a)\,da,\\
Y_r^{-}(A) &= \frac{e^{2\mathscr{G}}}{(2\pi)^{\frac{1}{2}}(-\rho-1)!\,A^r}\int_0^\infty a^{r-\rho-\frac{1}{2}}\,e^{-a\mathscr{F}_0/A}\,\Gamma_+(-a)Y^+(-a)\,da.
\end{aligned}
\tag{25}
$$

3. Confluent Hypergeometric Functions for Large $|c|$

$x < C = c - 2a$

On this side the most convenient starting point is the identification in Chapter XVI (35),

$$
F(a, c, x) = \frac{(C + 2a - 1)!}{(C + a - 1)!}\,\mathscr{Y}_-, \qquad |\mathrm{ph}\,\mathscr{F}_0| < \pi. \tag{26}
$$

Using (21) to interpret the series in Chapter XVI (31), we have

$$
\mathscr{Y}_- = \left(\frac{q}{C}\right)^a \left\{\sum_0^{n-1} (-1)^r Y_r^- + T_n(-1)^n Y_n^-\right\}, \tag{27}
$$

where $q = C/(C - x)$ and the first three Y_r^- are

$$
Y_0^- = 1, \qquad Y_1^- = \frac{aq}{2C}\{(a+1)q + 2(a-1)\},
$$

$$
\begin{aligned}
Y_2^- = \frac{a(a+1)q^2}{24C^2}\{3(a+2)(a+3)q^2 &+ 4(a+2)(3a-5)q\\
&+ 12(a-1)(a-2)\}.
\end{aligned}
$$

The terminant expansion involves the singulant

$$
\mathscr{F}_0 = C\Xi, \qquad \Xi = \ln\left|\frac{q}{q-1}\right| - \frac{1}{q}, \tag{28}
$$

the quantity $\rho = -a$, and the coefficients $\mathscr{Y}_s^+$ derived from the Y_s^+ by equating terms of like order in

$$
\sum \mathscr{Y}_s^+ = \Gamma_+(C)\sum Y_s^+, \qquad \Gamma_+(C) = (C + a - 1)!/(2\pi)^{\frac{1}{2}}C^{C+a-\frac{1}{2}}\,e^{-C}.
$$

Inserting this information in (22),

$$T_n = \frac{\{\Lambda_{n+a-3/2}(C\Xi) - \mathfrak{a}\Lambda_{n+a-5/2}(C\Xi) + \mathfrak{b}\Lambda_{n+a-7/2}(C\Xi) - \dots\}}{(1 - \mathfrak{a} + \mathfrak{b} - \dots)} \tag{29}$$

where

$$\mathfrak{a} = \Xi\,\{(a-1)(a-2)q^2 + 2a(a-1)q + a^2 - a + \tfrac{1}{6}\}/2(n + a - \tfrac{3}{2}),$$

$$\begin{aligned}\mathfrak{b} = \Xi^2\,\{&3(a-1)(a-2)(a-3)(a-4)q^4 + 4(a-1)(a-2)(a-3)(3a+2)q^3 \\ &+ (a-1)(a-2)(18a^2 + 6a + 1)q^2 + 2a(a-1)(6a^2 - 6a + 1)q \\ &+ 3a^4 - 10a^3 + 10a^2 - 3a + \tfrac{1}{12}\}/24(n + a - \tfrac{3}{2})(n + a - \tfrac{5}{2}).\end{aligned}$$

The other series solution on this side is

$$-e^{i\pi a}\mathscr{y}_+ = \frac{(-C-a)!}{(-C-2a+1)!}\,C^C\,e^{-C}x^{1-c}\,e^x\,F(1-a,\ 2-c,\ -x),$$

$$\operatorname{ph}\mathscr{F}_0 = 0. \tag{30}$$

$x > C$

Our identifications this side were

$$e^{i\pi a}\mathscr{y}_- = x^{-a}\,\psi(a,c,x), \qquad -e^{-i\pi a}\mathscr{y}_+ = C^Ce^{-C}x^{a-c}e^x\,\psi(1-a,\ 2-c,\ -x). \tag{31}$$

Interpreting the first by (27),

$$x^{-a}\,\psi(a,c,x) = \left(-\frac{q}{C}\right)^a\left\{\sum_0^{n-1}(-1)^rY_r^- + T_n(-1)^nY_n^-\right\}. \tag{32}$$

The relation between the functions $F(a, c, x)$ and $\psi(a, c, x)$ is provided by the connection formula (18′), which asserts

$$\begin{aligned}F(a,c,x) = \frac{(c-1)!}{(c-a-1)!}\cos\pi a\,x^{-a}\,\psi(a,c,x) + \frac{(c-1)!}{(a-1)!}\,x^{a-c}\,e^x \\ \times\,\psi(1-a,2-c,-x), \qquad \operatorname{ph}x = 0,\end{aligned} \tag{33}$$

in agreement with Chapter III (25), [or IV (19).] The second connection formula (20) gives a similar result for $F(1-a,\ 2-c,\ -x)$.

Exercises

1. Find integral representations for late terms $\mathscr{Y}_r^{\mp}$.

2. Find integral representations for terminants appropriate to the asymptotic series Y^+ and Y^-.

3. Expanding part of the integrands obtained in question 2, develop numerically useful approximations to these terminants, analogous to those of Chapter XXIV (20).

4. Specialize all results derived in the text to apply to the incomplete factorial functions

$$(p, x)! = \frac{x^{p+1} e^{-x}}{p+1} F(1, p+2, x), \qquad [p, x]! = x^p e^{-x} \psi(1, p+2, x).$$

Chapter XXVI

Termination of Asymptotic Expansions derived from Inhomogeneous Differential Equations

1. Interpretation on the "Exponential Side"

In Chapter XIX the asymptotic expansion for a particular integral of the inhomogeneous equation

$$d^2y/dx^2 - (\chi + \Delta)y = Z \tag{1}$$

was obtained in the form

$$y = \sum_0^{\infty} y_r \quad \text{where}\ y_{r+1} = \frac{1}{\chi}\left(\frac{d^2}{dx^2} - \Delta\right) y_r \quad \text{commencing at}\quad y_0 = -\frac{Z}{\chi}. \tag{2}$$

Assuming that $\chi \to \chi_1(x - x_0)$ and $Z \to Z_0$ at the turning point x_0, late terms were found to be given by the formula

$$y_r = -Z_0\left(\frac{\varepsilon}{\pi\chi_1}\right)^{\frac{1}{2}} \frac{1}{\chi^{\frac{1}{4}}\mathscr{F}_0^{\,2r+\frac{1}{2}}} \sum_{s=0} Y_s(-\mathscr{F}_0)^s(2r - s - \tfrac{1}{2})!\,, \tag{3}$$

where ε = sign of $(x - x_0)$ on the side where solutions to the homogeneous equation are exponentially varying, and the singulant

$$\mathscr{F}_0 = \varepsilon\int_{x_0}^{x} \chi^{\frac{1}{2}}\,dx \tag{4}$$

is half that pertinent to late terms in the solutions to the homogeneous equation. The Y_s were shown to satisfy the same recurrence relation

$$-\varepsilon Y_{s+1} = \tfrac{1}{2}\chi^{-\frac{1}{2}}\,dY_s/dx + \frac{1}{32}\int^{x} \chi^{-5/2}\{5(\chi')^2 - 4\chi\chi'' - 16\chi^2\Delta\}\,Y_s\,dx,$$
$$Y_0 = 1, \tag{5}$$

as for first variant solutions XIII (47) to the homogeneous equation, with the sequence of integration limits specific to the given inhomogeneous equation, i.e. dependent on $\chi(x)$, $\Delta(x)$ and $Z(x)$. In contrast, for solutions to the homogeneous equation the sequence of limits has only to coincide with that selected for early terms, this selection for early terms being arbitrary —corresponding to the free choice of constant multiplier for a solution to a homogeneous equation. Consequently, all we can assert in general about the Y_s in (3) is their identity with one possible set for the homogeneous equation.

As is clear from the late terms (3), the series $\sum_0^\infty y_r$ is asymptotic rather than convergent, and consequently represents a single continuous particular integral only within a range bounded by the nearest Stokes rays or turning points. Straight interpretation of the series as it stands therefore produces one function when effected on one side of the turning point, and a different function when effected on the other. Extending the notation of Chapter XIII, Sections 5, 6, for solutions to the homogeneous equation, these particular integrals will be labelled y_{exp} and y_{osc}, where the subscripts refer to the behaviour of solutions to the homogeneous equation in the same region, not the behaviour of the particular integrals.

As in Chapters XIII and XXIV, we suppose the coordinate x to have been chosen such that $\chi(x)$ is real along the real axis; for otherwise essential descriptions of solutions to the homogeneous equation as "exponentially decreasing" and "purely exponentially increasing" at $\text{ph}(x - x_0) = 0$ lose their precise meaning. On this understanding, solutions to the homogeneous equation are exponentially varying along the section of the real axis for which χ is positive. The singulant $\mathscr{F}_0$ is then real and positive, and it follows from (3) that late terms in the expansion for the chosen particular integral are all of the same sign and phase. The function y_{exp} is therefore being defined on a Stokes ray of its asymptotic series:

$$y_{\text{exp.}} = \sum_0^{n-1} y_r + \overline{T}_n y_n, \tag{6}$$

where by (3) and XXI (18) the terminant expansion is

$$\overline{T}_n = \left\{ Y_0 \,\overline{\Pi}_{2n-\frac{1}{2}}(i\mathscr{F}_0) - \frac{Y_1 \mathscr{F}_0}{2n - \frac{1}{2}} \overline{\Pi}_{2n-\frac{3}{2}}(i\mathscr{F}_0) + \frac{Y_2 \mathscr{F}_0{}^2}{(2n - \frac{1}{2})(2n - \frac{3}{2})} \right.$$
$$\left. \times\, \overline{\Pi}_{2n-\frac{5}{2}}(i\mathscr{F}_0) - \ldots \right\} \Big/ \left\{ Y_0 - \frac{Y_1 \mathscr{F}_0}{2n - \frac{1}{2}} + \frac{Y_2 \mathscr{F}_0{}^2}{(2n - \frac{1}{2})(2n - \frac{3}{2})} - \ldots \right\}. \tag{7}$$

Late terms and terminants can also be expressed as integrals. Representing the factorial in (3) by an integral and summing as in Chapter XXIV, Section 3,

$$y_r(A) = -Z_0\left(\frac{\varepsilon}{\pi \chi_1}\right)^{\frac{1}{2}} \frac{1}{\chi^{\frac{1}{4}} A^{2r+\frac{1}{2}}} \int_0^\infty a^{2r-\frac{1}{2}} e^{-\frac{1}{2}a\Xi}\, Y_-(a)\, da, \tag{8}$$

where a is a parametric order, A the actual order, and we have set the present singulant—half that for solutions to the homogeneous equation—equal to $\frac{1}{2}A\Xi$ so as to let Ξ continue its earlier meaning as well. Then

$$\bar{T}_n(A) = \sum_n^\infty y_r(A)/y_n(A) = \mathrm{P}\int_0^\infty \frac{a^{2n-\frac{1}{2}}\,e^{-\frac{1}{2}a\Xi}}{1 - a^2/A^2}\,Y_-(a)\,da \Bigg/ \int_0^\infty a^{2n-\frac{1}{2}}\,e^{-\frac{1}{2}a\Xi}\,Y_-(a)da. \tag{9}$$

The terminant of the particular-integral expansion can thus be precisely evaluated once the exponentially decreasing solution to the homogeneous equation is known. Moreover, a useful estimate is simply obtained by expanding $\ln Y_-$ around the approximate zenith of the integrand at $a = (4n-1)/\Xi$ (question 1).

2. Interpretation on the "Oscillatory Side"

Along this section of the real axis χ is negative, $\mathscr{F}_0$ is a positive imaginary and the coefficients Y_s are alternately real and imaginary. As at the equivalent stages in Chapter XIII, Section 6, and Chapter XXIV, Section 4, it is convenient to change to the quantities

$$\mathscr{Y}_s = i^s\,Y_s, \qquad \mathbf{F}_0 = -\,i\,\mathscr{F}_0, \tag{10}$$

these being real and positive along the real axis on the "oscillatory side" of the turning point. In this notation (3) becomes

$$y_r = Z_0\left(\frac{\varepsilon}{\pi\chi_1}\right)^{\frac{1}{2}} \frac{(-1)^r}{(-\chi)^{\frac{1}{4}}\,\mathbf{F}_0^{\,2r+\frac{1}{2}}} \sum_{s=0} \mathscr{Y}_s(-\mathbf{F}_0)^s(2r - s - \tfrac{1}{2})!. \tag{11}$$

When effected this side of the turning point, the interpretation of the asymptotic series $\sum_0^\infty y_r$ is therefore

$$y_{\mathrm{osc}} = \sum_0^{n-1} y_r + T_n y_n, \tag{12}$$

where by (11) and XXI (17) the terminant expansion is

$$T_n = \left\{\mathscr{Y}_0\,\Pi_{2n-\frac{1}{2}}(\mathbf{F}_0) - \frac{\mathscr{Y}_1\,\mathbf{F}_0}{2n-\frac{1}{2}}\,\Pi_{2n-\frac{3}{2}}(\mathbf{F}_0) + \frac{\mathscr{Y}_2\,\mathbf{F}_0^{\,2}}{(2n-\frac{1}{2})(2n-\frac{3}{2})}\right.$$
$$\left.\times\,\Pi_{2n-5/2}(\mathbf{F}_0) - \dots\right\} \Bigg/ \left\{\mathscr{Y}_0 - \frac{\mathscr{Y}_1\,\mathbf{F}_0}{2n-\frac{1}{2}} + \frac{\mathscr{Y}_2\,\mathbf{F}_0^{\,2}}{(2n-\frac{1}{2})(2n-\frac{3}{2})} - \dots\right\}. \tag{13}$$

3. Connection Formula

The connection between these two distinct particular integrals, $y_{\exp}$ of (6) and y_{osc} of (12), is most easily elucidated starting from the remainder term in (6), namely

$$\overline{T}_n y_n = -Z_0\left(\frac{\varepsilon}{\pi\chi_1}\right)^{\frac{1}{2}}\frac{1}{\chi^{\frac{1}{2}}\mathscr{F}_0^{2n+\frac{1}{2}}}\sum_{s=0} Y_s(-\mathscr{F}_0)^s(2n-s-\tfrac{1}{2})!\,\overline{\Pi}_{2n-s-\frac{1}{2}}(i\mathscr{F}_0), \tag{14}$$

because replacement of $\overline{\Pi}(i\mathscr{F}_0)$ by the mean of $\Lambda(\mathscr{F}_0)$ and $\overline{\Lambda}(-\mathscr{F}_0)$, in accordance with XXI (20), then divides the argument into essentially the same steps as studied in Chapter XXIV, Sections 1 and 2. With this reference back in view, we recast (14) in the form

$$\overline{T}_n y_n = -\tfrac{1}{2}Z_0\left(\frac{\varepsilon}{\pi\chi_1}\right)^{\frac{1}{2}}\frac{1}{\chi^{\frac{1}{2}}}(\zeta+\bar{\zeta}),$$

$$\begin{pmatrix}\zeta\\ \bar{\zeta}\end{pmatrix} = \frac{1}{\mathscr{F}_0^{2n+\frac{1}{2}}}\sum_{s=0} Y_s(-\mathscr{F}_0)^s\,(2n-s-\tfrac{1}{2})!\begin{pmatrix}\Lambda_{2n-s-\frac{1}{2}}(\mathscr{F}_0)\\ \overline{\Lambda}_{2n-s-\frac{1}{2}}(-\mathscr{F}_0)\end{pmatrix}, \tag{15}$$

and examine the discontinuities in ζ and $\bar{\zeta}$ which occur when ph $\mathscr{F}_0$ starts from zero, along the real axis on the exponential side, and increases up to $\frac{3}{2}\pi$, along the real axis on the oscillatory side.

As ph $\mathscr{F}_0$ passes through π, there is a discontinuity in $\Lambda_{2n-s-\frac{1}{2}}(\mathscr{F}_0)$ amounting to $2\pi i(-\mathscr{F}_0)^{2n-s+\frac{1}{2}}e^{\mathscr{F}_0}/(2n-s-\frac{1}{2})!$, leading to a discontinuity in ζ of magnitude $2\pi e^{\mathscr{F}_0}Y_+$. Turning to $\bar{\zeta}$, we first apply XXI (33) to substitute each $\overline{\Lambda}$ by

$$\overline{\Lambda}_{2n-s-\frac{1}{2}}(-\mathscr{F}_0) = \Lambda_{2n-s-\frac{1}{2}}(-\mathscr{F}_0) + i\pi\mathscr{F}_0^{2n-s+\frac{1}{2}}e^{-\mathscr{F}_0}/(2n-s-\tfrac{1}{2})!,$$

$$\text{ph}\,\mathscr{F}_0 > 0, \tag{16}$$

thereby bringing the series off its Stokes ray. The basic terminants $\Lambda(-\mathscr{F}_0)$ are continuous from ph $\mathscr{F}_0 \to +0$ up to and including $\frac{3}{2}\pi$ (in fact up to 2π); they join the $\Lambda(\mathscr{F}_0)$ in ζ (remaining after its discontinuity) to produce the basic terminants $\Pi(\mathbf{F}_0)$ involved in $T_n y_n$ of (12), since by (10) and XXI (19),

$$\Lambda(-\mathscr{F}_0) + \Lambda(\mathscr{F}_0) = 2\,\Pi(\mathbf{F}_0), \qquad 2\pi > \text{ph}\,\mathscr{F}_0 > \pi. \tag{17}$$

The contribution to $\bar{\zeta}$ from the second term on the right of (16) is $i\pi e^{-\mathscr{F}_0}Y_-$, which as we have seen in Chapter XXIV, Section 1, has to be replaced beyond ph $\mathscr{F}_0 = \pi$ by

$$i\pi e^{-\mathscr{F}_0}(Y_- + i e^{2\mathscr{F}_0}Y_+).$$

(The discontinuity in Y_- is proportional to exp(singulant in $Y_\pm$), and this singulant is double (4).)

On going across to the oscillatory side, the Stokes discontinuities in $\zeta + \bar{\zeta}$ have totalled

$$\pi(e^{\mathscr{F}_0} Y_+ + i e^{-\mathscr{F}_0} Y_-) = \pi \chi^{\frac{1}{2}}(\mathscr{y}_+ + i\mathscr{y}_-);$$

and on this oscillatory side $\mathscr{y}_+ + i\mathscr{y}_-$ is the expansion of the continuous solution y^+ to the homogeneous equation. Identifying the difference $y_{\mathrm{exp}} - y_{\mathrm{osc}}$ with its sole source, the difference $\overline{T}_n y_n - T_n y_n$ between interpretations of remainders, we find

$$y_{\mathrm{osc}} = y_{\mathrm{exp}} + Z_0(\pi\varepsilon/\chi_1)^{\frac{1}{2}} y^+. \tag{18}$$

In this result, y^+ is that solution to the homogeneous equation which is produced from exactly the same sequence of integration limits in (5) as required in equation (3) for late terms. The outer multiplier in y^+ is determined by this sequence, and is consequently specific to the given inhomogeneous equation. In the absence of such detail individual to each inhomogeneous equation, all we can assert is that the multiplier is a function of the order A and deviates from unity by $O(A^{-1})$. In this sense there is no connection formula between the particular integrals y_{exp} and y_{osc} which is general, tractable and precise, as are the connections between our solutions to the homogeneous equation.

4. Anger Function

The integral

$$A_p(z) = \int_0^\infty e^{p\omega - z\sinh\omega}\,d\omega = \frac{\pi}{\sin \pi p}(J_{-p}(z) - \mathbf{J}_{-p}(z)) \tag{19}$$

provides a particularly neat illustration of the foregoing theory, partly because no partition of X is needed but mainly through coincidence between the pertinent Y_s and those chosen earlier in solutions to the homogeneous Bessel equation.

As shown in Chapter XIX, Section 3, identification of the asymptotic expansion $\Sigma\, y_r$ is most direct when $z > p$, i.e. on the side where solutions to the homogeneous equation are oscillatory. The interpretation is then

$$A_p(z) = y_{\mathrm{osc}} = \sum_0^{n-1} y_r + T_n y_n, \tag{20}$$

where

$$y_0 = 1/(z-p), \qquad y_1 = -z/(z-p)^4, \qquad y_2 = z(p+9z)/(z-p)^7,$$
$$y_3 = -z(p^2 + 54pz + 225z^2)/(z-p)^{10}.$$

From (13),

$$T_n = \left\{\Pi_{2n-1/2}(p\Upsilon) - \frac{\Upsilon q(5q^2+3)}{24(2n-\frac{1}{2})}\Pi_{2n-3/2}(p\Upsilon) + \frac{\Upsilon^2 q^2(385q^4+462q^2+81)}{1152(2n-\frac{1}{2})(2n-\frac{3}{2})}\Pi_{2n-5/2}(p\Upsilon) - \dots\right\}$$

$$\div\left\{1 - \frac{\Upsilon q(5q^2+3)}{24(2n-\frac{1}{2})} + \frac{\Upsilon^2 q^2(385q^4+462q^2+81)}{1152(2n-\frac{1}{2})(2n-\frac{3}{2})} - \dots\right\} \qquad (21)$$

with

$$q = p/(z^2-p^2)^{\frac{1}{2}}, \qquad \Upsilon = q^{-1} - \tan^{-1} q^{-1}.$$

When $z < p$, solutions $y^{\pm}$ to the homogeneous equation are exponentially varying. The alternative particular integral y_{exp} should then be calculated, and the connection formula is needed to deduce A_p. Now $\varepsilon =$ sign of $x - x_0 =$ sign of $z - p$ where $y^{\pm}$ are exponential, so $\varepsilon = -1$; the values of Z and X' at the turning point $z = p$ are $Z_0 = 2p$ and $X_1 = -2p^2$; and $y^+ = -(\frac{1}{2}\pi)^{\frac{1}{2}} Y_p(z)$. Insertion of these in (18) yields the required connection

$$A_p(z) = y_{\text{exp}} - \pi Y_p(z). \qquad (22)$$

The interpretation of the series y_{exp} is left as an exercise (question 2).

Exercises

1. Expanding $\ln Y_-$ around the approximate zenith of the integrand in (9) located at $a = (4n-1)/\Xi$, obtain the useful estimate

$$\bar{T}_n(A) \doteq \bar{\Pi}_{2n-\frac{1}{2}}(\tfrac{1}{2} iA\Xi l), \qquad l = 1 - \tfrac{1}{2}\frac{d}{dn}\ln Y_-\left(\frac{4n-1}{\Xi}\right).$$

2. In the theory of the Anger function, show that the terminant expansion for the series y_{exp} is

$$\bar{T}_n = \left\{\bar{\Pi}_{2n-\frac{1}{2}}(ip\Xi) - \frac{\Xi q(5q^2-3)}{24(2n-\frac{1}{2})}\bar{\Pi}_{2n-\frac{3}{2}}(ip\Xi) + \dots\right\}$$

$$\div\left\{1 - \frac{\Xi q(5q^2-3)}{24(2n-\frac{1}{2})} + \dots\right\}$$

where

$$q = p/(p^2-z^2)^{\frac{1}{2}}, \qquad \Xi = \tanh^{-1} q^{-1} - q^{-1}.$$

3. Derive the terminant expansion

$$T_n = \{\Pi_{2n-\frac{1}{2}}(\rho V) - a\,\Pi_{2n-\frac{3}{2}}(\rho V)\ldots\}/(1 - a\ldots),$$

$$a = \frac{V}{24(2n-\frac{1}{2})}\{v(5v^2+3) + 12q^2\tan^{-1}v^{-1}\},$$

for the series representing the Lommel function $S_{qp}(z)$, the notation being that of Chapter XIX, Section 5.

4. Interpret the alternative particular integral $y_{\text{exp.}}$ to the Lommel inhomogeneous equation, and establish the connection formula

$$S_{qp}(z) = y_{\text{exp}} + (\tfrac{1}{2}\pi\mu)^{\frac{1}{2}}\,\rho^{q-\frac{1}{2}}\,e^{\rho U}$$
$$\times\left[1 + \frac{1}{24\rho}\{\mu(5\mu^2-3) + 12q^2\tanh^{-1}\mu^{-1}\}\ldots\right].$$

(According to general theory the second term should be expressible as $f(p,q)\,Y_p(z)$. We have not succeeded in expressing $f(p,q)$ in closed form).

5. Drawing upon the preliminary calculations of Chapter XIX, questions 1 and 2, define two particular integrals of the equation

$$d^2y/dx^2 + (p + \tfrac{1}{2} - \tfrac{1}{4}x^2)y = 1,$$

and find the connection between them.

6. Similarly, define and connect two particular integrals of the equation

$$\frac{d^2W}{dz^2} - \left(\frac{1}{4} - \frac{k}{z} + \frac{m^2-\frac{1}{4}}{z^2}\right)W = z^{q-\frac{1}{2}}$$

studied in Chapter XIX, questions 3 and 4.

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