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NUMERICAL EVALUATION OF THE FERMI-DIRAC INTEGRALS

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ABSTRACT

The readily available tables of the Fermi-Dirac integrals occurring in the degenerate electron equation of state have only modest precision. For some computational purposes it is desirable to have these functions available to near machine precision. We discuss the numerical techniques required to evaluate these functions, list selected results with ~12 digit precision, and provide listings of FORTRAN subroutines to produce the tables and to provide accurate interpolation in the tables.

Subject headings: equation of state — numerical methods

I. INTRODUCTION

The equation of state for a nonrelativistic degenerate electron gas is (for example, Chandrasekhar 1939)

$$n_e = \frac{4\pi}{h^3}(2m_e k_B T)^{3/2} F_{1/2}(\eta), \tag{1}$$

$$P_e = \frac{8\pi}{3h^3}(2m_e k_B T)^{3/2} k_B T F_{3/2}(\eta), \tag{2}$$

and

$$F_n(\eta) = \int_0^\infty \frac{x^n\,dx}{1+\exp(x-\eta)}, \quad n > -1, \tag{3}$$

where n_e is the electron density, h is Planck's constant, m_e is the mass of an electron, k_B is the Boltzmann constant, T is the temperature, P_e is the electron pressure, η is the degeneracy parameter, and F_n is the Fermi-Dirac integral of order n. In a typical application, one wants to calculate the pressure given n_e and T. Equation (1) is solved for $F_{1/2}(\eta)$, which is used to find η, which is then used to evaluate $F_{3/2}(\eta)$, and finally equation (2) gives P_e. This process is reversed if the problem is to calculate n_e from P_e and T. Fermi-Dirac integrals of integral order appear in the theory of heat conduction by electrons (for example, Clayton 1968), and $F_{-1/2}$ is needed for modeling collective effects in opacity calculations (for example, Boerker 1987). In all cases, tables or functional fits for values of $F_n(\eta)$ are required.

A number of authors have provided approximations to these functions. In one of the classic references, McDougall and Stoner (1939) use elegant mathematical techniques to produce tables of moderate accuracy (five significant figures at small η and a table spacing of 0.1 for $-4 \le \eta \le 20$). Arpigny (1963) provides polynomial fits accurate to a few parts per million. Larson and Demarque (1964) use the approximation

$$\frac{2}{3}F_{3/2}(\eta) = F_{1/2}(\eta)\frac{\left[1+0.1938F_{1/2}(\eta)\right]^{5/3}}{1+0.12398F_{1/2}(\eta)}, \tag{4}$$

which is accurate to within 0.02% for $\eta < 30$. This equation is convenient for stellar evolution calculations using equations (1) and (2) because there is no need to calculate η explicitly. Bludman and Van Riper (1977) give other approximate formulae for a more extensive selection of Fermi-Dirac integrals, including relativistic effects. However, the accuracy of their approximations is sometimes as low as several percent. Several of the standard stellar evolution texts (for example, Chiu 1968; Clayton 1968; Cox and Giuli 1968) provide limited tables.

The principal thesis of this paper is that special functions such as the Fermi-Dirac integrals can be readily available with the same level of accuracy as we expect for the trigonometric and exponential functions. We show that elementary numerical techniques intelligently applied on modern digital computers may be used to evaluate equation (3) to near machine precision. In § II we show some interesting techniques for relatively easily numerically integrating equation (3). Section III describes some numerical tests of the accuracy of these techniques. A simple, efficient, and accurate algorithm and tables are given in § IV that allow numerical evaluation of equation (3) in applications programs. We also provide a listing of the integration program so that the interested reader can readily extend the tables if desired. Section V is a summary.

II. NUMERICAL INTEGRATION TECHNIQUES

In general, numerical quadrature must be used to evaluate equation (3) for conditions of partial degeneracy (η near zero), and attention must be paid to several complications. For $n < 0$, the integrand has a singularity at the origin. For nonintegral n, certain derivatives of the integrand have a singularity at the origin. Finally, the interval of integration is infinite. This section describes the techniques and strategies that avoid numerical difficulties associated with these features and allow one to obtain excellent accuracy with only a small computational effort. The algorithm we adopt is based on a pair of extrapolation procedures and readily produces accurate results without manual input beyond setting up the program and checking the output for consistency of the extrapolations.

The difficulty with a singularity in the integrand is obvious, and it may be avoided by techniques discussed by, for example, Davis and Rabinowitz (1967) and Acton (1970). The only singular case we shall consider is $n=-\frac{1}{2}$, and that singularity is eliminated by a simple change of variables.

The difficulty with singularities in the derivatives is more subtle. Consider the example of Simpson's rule applied to the evaluation of a definite integral. Simpson's rule is derived by fitting a parabola to three equidistant points and integrating the parabola. For m (an odd integer) equally spaced points, $m \geq 5$, the three-point rule may be applied to a sequence of three-point subintervals to obtain

$$\int_a^b f(x)\,dx=\frac{H}{3}\left[f(a)-f(b)+4\sum_{j=1}^{(m-1)/2} f(a+(2j-1)H)+2\sum_{j=1}^{(m-1)/2} f(a+2jH)\right]-\frac{b-a}{180}H^4 f^{(4)}(\xi), \tag{5}$$

where H is the integration step size and ξ is some point in the open interval $(a,b)=(a,a+(m-1)H)$ (Isaacson and Keller 1966). The truncation error term is valid only if the first four derivatives of the integrand are bounded in the interval of integration. For $n=\frac{1}{2}$ and $\frac{3}{2}$, the first and second derivatives, respectively, are singular at the origin, and Simpson's rule is less accurate than shown in equation (5). This is because, for $n=\frac{1}{2}$, the slope of the integrand at the origin is infinite, and the parabola used to fit the integrand at the first three integration nodes must have a finite slope and therefore cannot make a very accurate fit. The $n=\frac{3}{2}$ case is less accurate because the parabolic fit has similar trouble with the infinite second and higher derivatives of the finite integrand.

This difficulty is not peculiar to Simpson's rule. For example, the trapezoidal rule,

$$\int_a^b f(x)\,dx=\frac{H}{2}\left[f(a)+f(b)=2\sum_{j=1}^{m-2} f(a+jH)\right]-\frac{b-a}{12}H^2 f^{(2)}(\xi), \tag{6}$$

requires the first two derivatives to be finite. Gaussian quadrature of order N requires that the first $2N$ derivatives be bounded, which is even more restrictive.

The difficulty with singular integrands and derivatives for half-integer values of $n \geq -\frac{1}{2}$ may be eliminated by the simple change of variables $z^2=x$ applied to equation (3):

$$F_n(\eta)=2\int_0^\infty \frac{z^{2n+1}\,dz}{1+\exp(z^2-\eta)}. \tag{7}$$

Although this transformation can be used for integral values of n, it is not necessary.

Evaluation of equation (3) or equation (7) begins with application of Simpson's rule over the finite interval $[0,t]$ for two values of t. If the values of t are properly chosen, then we can extrapolate to the integral over $[0,\infty]$. Consider the general case of numerically evaluating improper integrals of the form

$$S(\eta)=\int_a^\infty f(\eta,x)\,dx=\lim_{t\to\infty} S_t(\eta), \tag{8}$$

where

$$S_t(\eta)=\int_a^t f(\eta,x)\,dx \tag{9}$$

and where a is a finite constant. The limit in equation (8) is obtained by means of extrapolation procedures. This may be done by means of any of several transforms that were originally devised for doing Laplace transforms numerically.

Define the B and G transforms (Gray and Atchison 1968) as

$$G[S(\eta);t,k]=\frac{S_{t+k}(\eta)-R_t(\eta,k)S_t(\eta)}{1-R_t(\eta,k)}, \quad R_t\neq 1, \tag{10}$$

and

$$B[S(\eta);t,k]=\frac{S_{kt}(\eta)-\rho_t(\eta,k)S_t(\eta)}{1-\rho_t(\eta,k)}, \quad \rho_t\neq 1, \tag{11}$$

where

$$R_t(\eta,k)=\frac{f(\eta,t+k)}{f(\eta,t)}, \quad k>0, \tag{12}$$

and

$$\rho_t(\eta,k)=\frac{kf(\eta,kt)}{f(\eta,t)}, \quad k>1. \tag{13}$$

These transforms have the property that

$$\lim_{t\to\infty} G[S(\eta);t,k]=\lim_{t\to\infty} B[S(\eta);t,k]=S(\eta). \tag{14}$$

Gray and Atchison show that if $\lim_{t\to\infty} R_t(\eta,k)\neq 0$ or 1, G converges to $S(\eta)$ faster than $S_{t+k}(\eta)$. A similar theorem was given for the B transform. It can be shown that, if S_t is evaluated exactly, the G transformation is exact for $f=\exp(-x)$, and the B transformation is exact for $f=x^{-s}$, $s>1$ for finite t and k. Experience has shown that these transforms do help the convergence in evaluating the Fermi-Dirac integrals. Because the Fermi-Dirac integrand decays exponentially, the G transformation is more accurate than the B transformation.

Davis and Rabinowitz (1967) and Acton (1970) discuss several practical strategies for evaluating improper integrals of the first kind. While some of these alternative approaches undoubtedly would be successful (and may be useful to check the present results), most of them would require more user effort than the B and G transforms.

The $S_t(\eta)$ are evaluated by numerical integration and contain truncation and round-off errors. Aitken extrapolation is used to improve the accuracy of the $S_t(\eta)$ and is based on the assumption that, for sufficiently large m, the truncation error vanishes smoothly and monotonically as m increases. If we use μ, 2μ, and 4μ subintervals, where $\mu = m-1$, then we assume

$$I = I_\mu + cH^p ,$$

$$I = I_{2\mu} + c(H/2)^p , \tag{15}$$

and

$$I = I_{4\mu} + c(H/4)^p ,$$

where I is the exact value of the integral, I_μ is the approximate value using μ subintervals, and c and p are free parameters. Equations (15) can be solved for I, c, and p given H and the I_m:

$$I = I_{4\mu} - \frac{(I_{4\mu} - I_{2\mu})^2}{I_{4\mu} - 2I_{2\mu} + I_\mu} , \tag{16}$$

$$p = \log_{10}\left[(I - I_\mu)/(I - I_{2\mu})\right] / \log_{10}(2) , \tag{17}$$

and

$$c = (I - I_\mu)/H^p . \tag{18}$$

For integrands with finite fourth derivatives, $p \approx 4$ for Simpson's rule.

III. NUMERICAL TESTS

Because extrapolations are usually considered risky at best, we present a small selection of test problems with known results to show the properties of both Aitken extrapolation and the B and G transforms. First we show some results using elementary integrands, and then we show results for the $n=0$ Fermi-Dirac integral.

Table 1 shows the results of applying Aitken extrapolation to four different integrals whose integrands consist of elementary functions. In the second column, "Exact" indicates the analytic value, a literal integer denotes the number of subintervals μ in a Simpson's rule integration, and A_μ denotes the value of I computed from equation (16). For the extrapolated values, the value of the exponent p computed from equation (17) is given.

The first two integrands and their derivatives are well behaved, so we get good accuracy even for fairly coarse grids. Note that p is 4, just as we expected.

The third integrand has a singularity at $x=0$, just outside the interval of integration. Not only are the relative errors somewhat worse than in the first two cases, but p approaches 4 only asymptotically as the grid is refined.

TABLE 1

PERFORMANCE OF AITKEN EXTRAPOLATION

Integral	Mesh Size	Value	Relative Error	p
$\int_0^5 \exp(-x)\,dx$	Exact	0.99326205300	0.0	...
	30	0.99326629677	6.3×10^{-6}	...
	60	0.99326231889	2.7×10^{-7}	...
	120	0.99326206963	1.7×10^{-8}	...
	A_{30}	0.99326205297	3.5×10^{-11}	4.00
$\int_0^5 x\exp(-x)\,dx$ ...	Exact	0.95957231801	0.0	...
	30	0.95955947075	1.3×10^{-5}	...
	60	0.95957151175	8.4×10^{-7}	...
	120	0.95957226756	5.3×10^{-8}	...
	A_{30}	0.95957231818	1.9×10^{-10}	3.99
$\int_1^{32} x^{-2}\,dx$	Exact	0.9687500	0.0	...
	30	1.0095655	4.2×10^{-2}	...
	60	0.9741378	5.6×10^{-3}	...
	120	0.9692386	5.0×10^{-4}	...
	240	0.9687851	3.6×10^{-5}	...
	480	0.9687523	2.4×10^{-6}	...
	A_{30}	0.9684523	3.1×10^{-4}	2.85
	A_{120}	0.9687497	2.8×10^{-7}	3.79
$\int_0^1 x^{1.5}\,dx$	Exact	0.4	0.0	...
	8	0.400077249447	1.9×10^{-4}	...
	16	0.400013713469	3.4×10^{-5}	...
	32	0.400002427846	6.1×10^{-6}	...
	64	0.400000429413	1.1×10^{-6}	...
	A_8	0.399999990249	2.4×10^{-8}	2.49
	A_{16}	0.399999999388	1.5×10^{-9}	2.50

The second and higher derivatives of the fourth integrand are singular at $x=0$. The relative errors are better than in the third case, but p is only approximately 2.5 for integrands with a singular second derivative. This is expected on empirical grounds, and furthermore we expect that p is only approximately 1.5 for integrands with a singular first derivative. In all four cases, the extrapolated values are more accurate than $I_{4\mu}$.

Table 2 shows the effect of using the B and G transforms on three of the same integrals listed in Table 1. For the value of the integral labeled $I(t)$, t is the upper limit of integration. The relative errors in column (4) are the errors in the Aitken-extrapolated numerical value in column (3) relative to the exact value in column (5). The relative errors in the last column are the errors of the exact value of $I(t)$ compared with the exact value of the improper integral. In all three cases, we used $k=1$ for G and $k=1.1$ for B. The coarse-mesh integrations used $m=31$. For the first integrand, both B and G, which are evaluated from the inexact, numerically evaluated $I(t)$ using $m=31$, are closer to the exact value of 1.0 than the exact values of $I(t)$. This is seen by comparing lines 4 and 5 of column (4) to lines 1–3 of column (6). The B and G transforms are only as accurate as the integrals over the finite intervals that go into them, and in all three cases the transforms converge faster than the integrals over increasingly large intervals. For $f=x^{-2}$ the B transform is more accurate, and for the exponential cases the G transform is more accurate .This is the expected behavior.

TABLE 2

B and G Transform Test Cases

Integrand (1)	Integral (2)	Numerical Value (3)	Relative Error (4)	Exact Value (5)	Relative Error (6)
x^{-2}	$I(32)$	0.9684523167	3.1×10^{-4}	0.96875000000	3.13×10^{-2}
	$I(33)$	0.9693553673	3.5×10^{-4}	0.96969696970	3.03×10^{-2}
	$I(35.2)$	0.9711370042	4.7×10^{-4}	0.97159090909	2.84×10^{-2}
	B	0.9979838784	2.0×10^{-3}	1.0	0.0
	G	0.9835818869	1.6×10^{-2}	1.0	0.0
$\exp(-x)$	$I(5.0)$	0.99326205297	3.0×10^{-11}	0.99326205300	6.74×10^{-3}
	$I(5.5)$	0.99591322850	6.0×10^{-11}	0.99591322856	4.09×10^{-3}
	$I(6.0)$	0.99752124772	1.0×10^{-10}	0.99752124782	2.48×10^{-3}
	B	1.00122793606	1.2×10^{-3}	1.0	0.0
	G	0.99999999985	1.5×10^{-10}	1.0	0.0
$x\exp(-x)$	$I(5.0)$	0.95957231818	1.8×10^{-10}	0.95957231801	4.04×10^{-2}
	$I(5.5)$	0.97343598596	3.2×10^{-10}	0.97343598565	2.66×10^{-2}
	$I(6.0)$	0.98264873529	5.4×10^{-10}	0.98264873476	1.74×10^{-2}
	B	1.01167219454	1.2×10^{-2}	1.0	0.0
	G	1.00088757607	8.9×10^{-4}	1.0	0.0

TABLE 3

Fermi-Dirac Integrals of Order Zero

η	Numerical	Exact
−7.5......	5.529314753618E−04	5.529314753608E−04
−6.0......	2.475685137735E−03	2.475685137730E−03
−4.5......	1.104774484861E−02	1.104774484859E−02
−3.0......	4.858735157382E−02	4.858735157374E−02
−1.5......	2.014132779829E−01	2.014132779828E−01
0.0......	6.931471805600E−01	6.931471805599E−01
1.5......	1.701413277983E+00	1.701413277983E+00
3.0......	3.048587351574E+00	3.048587351574E+00

TABLE 4

Numerical Values of $S_{4.5}(3.5)$ for $n=1/2$

m	Numerical Values	First Differences
31......	4.837066395928551209	...
61......	4.837065652433605814	$+7.435\times10^{-7}$
121......	4.837065652628581104	-1.950×10^{-10}
241......	4.837065652641143496	-1.256×10^{-11}
481......	4.837065652641934662	-7.912×10^{-13}

Table 3 shows some results for the Fermi-Dirac integral of order zero, which is known analytically (Sommerfeld 1928). If we make the change of variables $w=1+\exp(x-\eta)$ and set $n=0$, equation (3) becomes

$$F_0(\eta)=\int_{1+\exp(-\eta)}^{\infty}\frac{dw}{w(w-1)}$$

$$=\ln\left(\frac{w-1}{w}\right)\Bigg|_{1+\exp(-\eta)}^{\infty}=\ln[\exp(\eta)+1]. \quad (19)$$

Even for $m=121$, which is a rather modest number of integration nodes by present standards, the differences between the numerical G transform and the analytic values are in the twelfth digit.

Table 4 illustrates an apparent difficulty with the Aitken extrapolation technique applied to integrands with fourth derivatives that oscillate in the interval of integration, as is the case for $F_{1/2}(\eta)$. The truncation error can change sign as the mesh is refined because the point at which the fourth derivative is evaluated, ξ, is a function of step size. The integral values for $\eta=3.5$, $t=4.5$, and $m=31$, 61, 121, 241, and 481 integration points at first decrease, then increase as the mesh is refined. Aitken extrapolation cannot be performed unless the error is monotonically vanishing. Even then, extrapolation will not be reliable unless the mesh is sufficiently refined that ξ has neared its asymptotic value. The relative error in all cases is small for such a small number of nodes (which is precisely why we employ the $x=z^2$ transformation), so, strictly speaking, the Aitken extrapolation may not be required in all cases except as a diagnostic tool.

Since we are interested in tables accurate to near machine accuracy, some consideration must be given to rounding errors. As the number of subintervals is increased in a numerical integration, the truncation errors decrease. However, the accumulated rounding errors grow approximately as $m^{1/2}$. For a given integral, numerical method, and computer word size, there will be a finite number of subintervals that will produce the minimum total error in the desired result. Therefore, some rounding tests were run to see whether this is an important factor in the present study. Printouts of values of the FORTRAN functions EXP and DEXP for selected arguments showed them to have accuracy near machine precision, as expected. The integration program was run with 200–51,200 subintervals for $F_0(0)$ using both equation (3) and the transformed variable. The results were compared with the exact values from equation (19). In double precision on a CRAY-1 (128 bit words), the relative error was approximately 1×10^{-17} for equation (3), with upper integration limits in the range 15.0–16.5 and for 51,200 subintervals using Simpson's rule, and appears to be still well above the minimum error. The transformed variable of equation (7) gave only slightly better accuracy, which is the expected result for integral values of n. The relative errors for various numbers of subintervals suggest that the minimum error will occur near 6000 subintervals with a 64 bit word and at less than 200 subintervals with a 32 bit word for the present example. This is one reason why scientific programming is usually done in double precision on short–word-length machines. By using double precision to calculate the Fermi-Dirac integrals to single-precision accuracy, we can ignore rounding errors in the final results on machines with 60–64 bits per single-precision word.

The techniques described here may be applied to other integrals as well. For example, the Fermi-Dirac integrals may be generalized to include relativistic effects in the equation of state (for example, Cox and Giuli 1968; Bludman and Van Riper 1977). Let $\beta = kT/m_e c^2$, and

$$F_n(\eta,\beta) = \int_0^\infty \frac{x^n(1+0.5\beta x)^{1/2}}{1+\exp(x-\eta)}\,dx, \qquad n > -1. \quad (20)$$

Then we need $F_n(\eta,\beta)$ for $n=\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ for the relativistic analogs of equations (1) and (2). This generalization is required if the condition $\beta\eta \ll 1$ is violated (Mitalas 1972). The $z^2 = x$ transformation is useful in this case, too.

IV. FERMI-DIRAC INTEGRAL EVALUATION IN APPLICATIONS PROGRAMS

It would be inefficient to use the techniques of the previous section to evaluate a Fermi-Dirac integral every time an applications program needs a value. Therefore the numerical integration program was used to generate accurate tables to be used with a separate function subprogram to evaluate F_n in applications programs. Table 5 gives the values for $-5 \le \eta \le 25$. Outside that range, series expansions provide good accuracy.

Cox and Giuli (1968) show that for $\eta \le 0$ a good approximation is

$$F_n(\eta) = \Gamma(n+1)\exp(\eta)\sum_{r=0}^{\infty}(-1)^r\frac{\exp(r\eta)}{(r+1)^{n+1}}, \quad n > -1, \quad (21)$$

where Γ is the gamma function. Using $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(\frac{1}{2}) = \pi^{1/2}$ (for example, Abramowitz and Stegun 1965), we specialize this series to

$$F_{1/2}(\eta) = \frac{\pi^{1/2}}{2}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\exp(j\eta)}{j^{3/2}} \quad (22)$$

and

$$F_{3/2}(\eta) = \frac{3\pi^{1/2}}{4}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\exp(j\eta)}{j^{5/2}}. \quad (23)$$

Only five terms are used for $\eta \le -5$, which provide 12 digit accuracy for the worst cases, $F_{1/2}(-5)$ and $F_{3/2}(-5)$.

For $\eta > 25$, another asymptotic form is used. For integrals of the form

$$I(\eta) = \int_0^\infty \frac{\phi'(u)\,du}{\exp(u-\eta)+1}$$

we use

$$I(\eta) \approx \phi(\eta) + 2\sum_{j=1}^{\infty} C_{2j}\phi^{(2j)}(\eta), \quad (24)$$

for large η. This series expansion has errors of order $\exp(-\eta)$. The C_{2j} are given by

$$C_{2j} = \sum_{i=1}^{\infty}(-1)^{i+1}i^{-2j} = (1-2^{-2j+1})\zeta(2j), \quad (25)$$

where ζ is the Riemann zeta function. The first five coefficients are $C_2 = \pi^2/12$, $C_4 = 7\pi^4/720$, $C_6 = 31\pi^6/30{,}240$, $C_8 = 127\pi^8/1{,}209{,}600$, and $C_{10} = 511\pi^{10}/47{,}900{,}160$. After evaluating the necessary derivatives, the final expansions are

$$F_n(\eta) = \frac{\eta^{n+1}}{n+1}\left[1+\sum_{r=1}^{\infty}2C_{2r}\left(\prod_{k=n-2r+2}^{n+1}k\right)\eta^{-2r}\right],$$

$$n > 0, \quad \eta \gg 1. \quad (26)$$

This expression specializes to

$$F_{1/2}(\eta) \approx \frac{2}{3}\eta^{3/2} + \frac{\pi^2}{12}\eta^{-1/2} + \frac{7\pi^4}{960}\eta^{-5/2} + \frac{31\pi^6}{4608}\eta^{-9/2} + \frac{1397\pi^8}{81{,}920}\eta^{-13/2} \quad (27)$$

and

$$F_{3/2}(\eta) \approx \frac{2}{5}\eta^{5/2} + \frac{\pi^2}{4}\eta^{1/2} - \frac{7\pi^4}{960}\eta^{-3/2} - \frac{31\pi^6}{10{,}752}\eta^{-7/2} - \frac{381\pi^8}{81{,}920}\eta^{-11/2}. \quad (28)$$

For intermediate values of η, we interpolate in Table 5, which was computed for $n = -\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ and for $-5 \le \eta \le 25$ with a spacing of 0.05 in η. The listing of the integration program FDTAB is given in Table 6. The program uses an adaptive algorithm to choose the upper integration limit t for each value of η. A provisional value of t is incremented by steps of 0.5 in the integration variable until the value of the integrand at t is some fraction, say 10^{-6}, of the maximum value. The comments in the listing in Table 6 provide the documentation necessary to run the integration program. For the parameters given in the listing, the integration program evaluates 501 integrals (including compilation, loading, and input/output) in 451 s on a Cray 1 computer. Users with a

TABLE 5

FERMI-DIRAC INTEGRAL TABLES

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
-5.00	1.18861109542E-02	5.95717690518E-03	8.94638226041E-03	2.23792483585E-02
-4.95	1.24924922767E-02	6.26184689623E-03	9.40450090972E-03	2.35259405430E-02
-4.90	1.31296464168E-02	6.58205793906E-03	9.88604778973E-03	2.47313496821E-02
-4.85	1.37991180988E-02	6.91859850888E-03	1.03922178058E-02	2.59984783703E-02
-4.80	1.45025278229E-02	7.27229663899E-03	1.09242664705E-02	2.73304823222E-02
-4.75	1.52415753973E-02	7.64402185906E-03	1.14835129426E-02	2.87306781371E-02
-4.70	1.60180436166E-02	8.03468722346E-03	1.20713432149E-02	3.02025514521E-02
-4.65	1.68338020896E-02	8.44525143346E-03	1.26892134574E-02	3.17497655039E-02
-4.60	1.76908112195E-02	8.87672105705E-03	1.33386535235E-02	3.33761701183E-02
-4.55	1.85911263407E-02	9.33015285013E-03	1.40212706264E-02	3.50858111486E-02
-4.50	1.95369020164E-02	9.80665618324E-03	1.47387531936E-02	3.68829403843E-02
-4.45	2.05303964975E-02	1.03073955776E-02	1.54928749072E-02	3.87720259538E-02
-4.40	2.15739763489E-02	1.08335933548E-02	1.62854989393E-02	4.07577632439E-02
-4.35	2.26701212427E-02	1.13865324043E-02	1.71185823894E-02	4.28450863608E-02
-4.30	2.38214289216E-02	1.19675590726E-02	1.79941809350E-02	4.50391801599E-02
-4.25	2.50306203337E-02	1.25780861791E-02	1.89144537032E-02	4.73454928698E-02
-4.20	2.63005449394E-02	1.32195961620E-02	1.98816683731E-02	4.97697493402E-02
-4.15	2.76341861903E-02	1.38936443593E-02	2.08982065193E-02	5.23179649420E-02
-4.10	2.90346671793E-02	1.46018624289E-02	2.19665692072E-02	5.49964601518E-02
-4.05	3.05052564607E-02	1.53459619120E-02	2.30893828493E-02	5.78118758510E-02
-4.00	3.20493740379E-02	1.61277379440E-02	2.42694053349E-02	6.07711893745E-02
-3.95	3.36705975136E-02	1.69490731175E-02	2.55095324433E-02	6.38817313432E-02
-3.90	3.53726683994E-02	1.78119415008E-02	2.68128045530E-02	6.71512033165E-02
-3.85	3.71594985765E-02	1.87184128158E-02	2.81824136583E-02	7.05876963023E-02
-3.80	3.90351769016E-02	1.96706567798E-02	2.96217107046E-02	7.41997101640E-02
-3.75	4.10039759458E-02	2.06709476139E-02	3.11342132561E-02	7.79961739653E-02
-3.70	4.30703588577E-02	2.17216687211E-02	3.27236135085E-02	8.19864672947E-02
-3.65	4.52389863341E-02	2.28253175380E-02	3.43937866578E-02	8.61804426136E-02
-3.60	4.75147236853E-02	2.39845105613E-02	3.61487996408E-02	9.05884486733E-02
-3.55	4.99026479737E-02	2.52019885526E-02	3.79929202582E-02	9.52213550497E-02
-3.50	5.24080552062E-02	2.64806219217E-02	3.99306266954E-02	1.00090577841E-01
-3.45	5.50364675560E-02	2.78234162907E-02	4.19666174540E-02	1.05208106583E-01
-3.40	5.77936405857E-02	2.92335182383E-02	4.41058217061E-02	1.10586532432E-01
-3.35	6.06855704414E-02	3.07142212247E-02	4.63534100882E-02	1.16239077664E-01
-3.30	6.37185009828E-02	3.22689716953E-02	4.87148059442E-02	1.22179626564E-01
-3.25	6.68989308109E-02	3.39013753619E-02	5.11956970343E-02	1.28422757734E-01
-3.20	7.02336201505E-02	3.56152036573E-02	5.38020477207E-02	1.34983777905E-01
-3.15	7.37295975395E-02	3.74144003602E-02	5.65401116438E-02	1.41878757296E-01
-3.10	7.73941662735E-02	3.93030883841E-02	5.94164449026E-02	1.49124566599E-01
-3.05	8.12349105476E-02	4.12855767237E-02	6.24379197484E-02	1.56738915629E-01
-3.00	8.52597012333E-02	4.33663675504E-02	6.56117388064E-02	1.64740393732E-01
-2.95	8.94767012221E-02	4.55501634471E-02	6.89454498346E-02	1.73148511996E-01
-2.90	9.38943702615E-02	4.78418747700E-02	7.24469610297E-02	1.81983747342E-01
-2.85	9.85214692039E-02	5.02466271247E-02	7.61245568902E-02	1.91267588569E-01
-2.80	1.03367063582E-01	5.27697689400E-02	7.99869146429E-02	2.01022584408E-01
-2.75	1.08440526419E-01	5.54168791216E-02	8.40431212418E-02	2.11272393671E-01
-2.70	1.13751540173E-01	5.81937747657E-02	8.83026909427E-02	2.22041837562E-01
-2.65	1.19310097715E-01	6.11065189089E-02	9.27755834578E-02	2.33356954213E-01
-2.60	1.25126502226E-01	6.41614282903E-02	9.74722226921E-02	2.45245055535E-01
-2.55	1.31211365897E-01	6.73650810947E-02	1.02403516063E-01	2.57734786443E-01
-2.50	1.37575607310E-01	7.07243246489E-02	1.07580874394E-01	2.70856186530E-01
-2.45	1.44230447380E-01	7.42462830332E-02	1.13016232394E-01	2.84640754271E-01
-2.40	1.51187403702E-01	7.79383645729E-02	1.18722069685E-01	2.99121513812E-01
-2.35	1.58458283206E-01	8.18082691670E-02	1.24711432407E-01	3.14333084434E-01
-2.30	1.66055172940E-01	8.58639954103E-02	1.30997955347E-01	3.30311752741E-01
-2.25	1.73990428870E-01	9.01138474612E-02	1.37595884604E-01	3.47095547653E-01

TABLE 5—*Continued*

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
-2.20	1.82276662540E-01	9.45664416016E-02	1.44520100753E-01	3.64724318265E-01
-2.15	1.90926725458E-01	9.92307124368E-02	1.51786142491E-01	3.83239814627E-01
-2.10	1.99953691061E-01	1.04115918674E-01	1.59410230729E-01	4.02685771510E-01
-2.05	2.09370834141E-01	1.09231648419E-01	1.67409293101E-01	4.23107995215E-01
-2.00	2.19191607586E-01	1.14587823925E-01	1.75800988854E-01	4.44554453459E-01
-1.95	2.29429616326E-01	1.20194705725E-01	1.84603734063E-01	4.67075368403E-01
-1.90	2.40098588381E-01	1.26062896077E-01	1.93836727139E-01	4.90723312847E-01
-1.85	2.51212342904E-01	1.32203341649E-01	2.03519974566E-01	5.15553309627E-01
-1.80	2.62784755160E-01	1.38627335365E-01	2.13674316801E-01	5.41622934244E-01
-1.75	2.74829718372E-01	1.45346517338E-01	2.24321454288E-01	5.68992420732E-01
-1.70	2.87361102401E-01	1.52372874809E-01	2.35483973501E-01	5.97724770788E-01
-1.65	3.00392709268E-01	1.59718741014E-01	2.47185372941E-01	6.27885866148E-01
-1.60	3.13938225530E-01	1.67396792897E-01	2.59450089011E-01	6.59544584211E-01
-1.55	3.28011171570E-01	1.75420047599E-01	2.72303521667E-01	6.92772916885E-01
-1.50	3.42624847909E-01	1.83801857638E-01	2.85772059760E-01	7.27646092613E-01
-1.45	3.57792278659E-01	1.92555904708E-01	2.99883105970E-01	7.64242701554E-01
-1.40	3.73526152308E-01	2.01696192039E-01	3.14665101212E-01	8.02644823840E-01
-1.35	3.89838760057E-01	2.11237035236E-01	3.30147548421E-01	8.42938160849E-01
-1.30	4.06741931990E-01	2.21193051556E-01	3.46361035584E-01	8.85212169405E-01
-1.25	4.24246971379E-01	2.31579147562E-01	3.63337257917E-01	9.29560198809E-01
-1.20	4.42364587521E-01	2.42410505115E-01	3.81109039041E-01	9.76079630572E-01
-1.15	4.61104827504E-01	2.53702565673E-01	3.99710351057E-01	1.02487202074E+00
-1.10	4.80477007392E-01	2.65471012878E-01	4.19176333372E-01	1.07604324462E+00
-1.05	5.00489643323E-01	2.77731753418E-01	4.39543310154E-01	1.12970364385E+00
-1.00	5.21150383108E-01	2.90500896170E-01	4.60848806290E-01	1.18596817544E+00
-0.95	5.42465938899E-01	3.03794729652E-01	4.83131561718E-01	1.24495656285E+00
-0.90	5.64442021581E-01	3.17629697801E-01	5.06431543996E-01	1.30679344860E+00
-0.85	5.87083277545E-01	3.32022374154E-01	5.30789959008E-01	1.37160854848E+00
-0.80	6.10393228507E-01	3.46989434465E-01	5.56249259657E-01	1.43953680685E+00
-0.75	6.34374215084E-01	3.62547627885E-01	5.82853152469E-01	1.51071855299E+00
-0.70	6.59027344806E-01	3.78713746768E-01	6.10646601967E-01	1.58529965808E+00
-0.65	6.84352445244E-01	3.95504595249E-01	6.39675832741E-01	1.66343169262E+00
-0.60	7.10348022927E-01	4.12936956717E-01	6.69988329124E-01	1.74527208392E+00
-0.55	7.37011228668E-01	4.31027560347E-01	7.01632832377E-01	1.83098427344E+00
-0.50	7.64337829874E-01	4.49793046837E-01	7.34659335350E-01	1.92073787361E+00
-0.45	7.92322190394E-01	4.69249933561E-01	7.69119074540E-01	2.01470882374E+00
-0.40	8.20957258332E-01	4.89414579313E-01	8.05064519522E-01	2.11307954495E+00
-0.35	8.50234562238E-01	5.10303148838E-01	8.42549359723E-01	2.21603909342E+00
-0.30	8.80144215947E-01	5.31931577396E-01	8.81628488544E-01	2.32378331193E+00
-0.25	9.10674932276E-01	5.54315535540E-01	9.22357984821E-01	2.43651497925E+00
-0.20	9.41814045677E-01	5.77470394359E-01	9.64795091670E-01	2.55444395699E+00
-0.15	9.73547543822E-01	6.01411191393E-01	1.00899819274E+00	2.67778733371E+00
-0.10	1.00586010801E+00	6.26152597450E-01	1.05502678594E+00	2.80676956584E+00
-0.05	1.03873516213E+00	6.51708884543E-01	1.10294145474E+00	2.94162261521E+00
0.00	1.07215492994E+00	6.78093895153E-01	1.15280383709E+00	3.08258608284E+00
0.05	1.10610050002E+00	7.05321013021E-01	1.20467659210E+00	3.22990733864E+00
0.10	1.14055189805E+00	7.33403135660E-01	1.25862336459E+00	3.38384164694E+00
0.15	1.17548816569E+00	7.62352648754E-01	1.31470874765E+00	3.54465228730E+00
0.20	1.21088744528E+00	7.92181402603E-01	1.37299824338E+00	3.71261067068E+00
0.25	1.24672706971E+00	8.22900690757E-01	1.43355822188E+00	3.88799645045E+00
0.30	1.28298365645E+00	8.54521230946E-01	1.49645587879E+00	4.07109762833E+00
0.35	1.31963320502E+00	8.87053148405E-01	1.56175919146E+00	4.26221065478E+00
0.40	1.35665119684E+00	9.20505961673E-01	1.62953687396E+00	4.46164052398E+00
0.45	1.39401269663E+00	9.54888570901E-01	1.69985833113E+00	4.66970086299E+00
0.50	1.43169245434E+00	9.90209248713E-01	1.77279361184E+00	4.88671401524E+00
0.55	1.46966500683E+00	1.02647563361E+00	1.84841336165E+00	5.11301111807E+00
0.60	1.50790477830E+00	1.06369472588E+00	1.92678877510E+00	5.34893217428E+00
0.65	1.54638617868E+00	1.10187288607E+00	2.00799154776E+00	5.59482611785E+00
0.70	1.58508369924E+00	1.14101583577E+00	2.09209382828E+00	5.85105087349E+00
0.75	1.62397200461E+00	1.18112866084E+00	2.17916817054E+00	6.11797341033E+00

TABLE 5—*Continued*

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
0.80	1.66302602066E+00	1.22221581683E+00	2.26928748620E+00	6.39596978958E+00
0.85	1.70222101760E+00	1.26428113659E+00	2.36252499767E+00	6.68542520633E+00
0.90	1.74153268790E+00	1.30732783985E+00	2.45895419177E+00	6.98673402547E+00
0.95	1.78093721851E+00	1.35135854467E+00	2.55864877417E+00	7.30029981188E+00
1.00	1.82041135715E+00	1.39637528067E+00	2.66168262473E+00	7.62653535501E+00
1.05	1.85993247247E+00	1.44237950381E+00	2.76812975392E+00	7.96586268788E+00
1.10	1.89947860780E+00	1.48937211259E+00	2.87806426039E+00	8.31871310081E+00
1.15	1.93902852858E+00	1.53735346553E+00	2.99156028984E+00	8.68552714976E+00
1.20	1.97856176335E+00	1.58632339975E+00	3.10869199517E+00	9.06675465981E+00
1.25	2.01805863840E+00	1.63628125047E+00	3.22953349812E+00	9.46285472355E+00
1.30	2.05750030634E+00	1.68722587137E+00	3.35415885238E+00	9.87429569495E+00
1.35	2.09686876851E+00	1.73915565550E+00	3.48264200827E+00	1.03015551786E+01
1.40	2.13614689178E+00	1.79206855673E+00	3.61505677898E+00	1.07451200145E+01
1.45	2.17531841978E+00	1.84596211155E+00	3.75147680845E+00	1.12054862594E+01
1.50	2.21436797893E+00	1.90083346106E+00	3.89197554089E+00	1.16831591629E+01
1.55	2.25328107974E+00	1.95667937310E+00	4.03662619197E+00	1.21786531416E+01
1.60	2.29204411351E+00	2.01349626425E+00	4.18550172161E+00	1.26924917479E+01
1.65	2.33064434496E+00	2.07128022185E+00	4.33867480852E+00	1.32252076373E+01
1.70	2.36906990111E+00	2.13002702564E+00	4.49621782628E+00	1.37773425310E+01
1.75	2.40730975680E+00	2.18973216917E+00	4.65820282108E+00	1.43494471767E+01
1.80	2.44535371721E+00	2.25039088079E+00	4.82470149106E+00	1.49420813065E+01
1.85	2.48319239770E+00	2.31199814414E+00	4.99578516718E+00	1.55558135919E+01
1.90	2.52081720150E+00	2.37454871816E+00	5.17152479562E+00	1.61912215969E+01
1.95	2.55822029528E+00	2.43803715643E+00	5.35199092167E+00	1.68488917288E+01
2.00	2.59539458329E+00	2.50245782601E+00	5.53725367501E+00	1.75294191874E+01
2.05	2.63233368010E+00	2.56780492548E+00	5.72738275644E+00	1.82334079119E+01
2.10	2.66903188231E+00	2.63407250238E+00	5.92244742588E+00	1.89614705266E+01
2.15	2.70548413958E+00	2.70125446988E+00	6.12251649166E+00	1.97142282855E+01
2.20	2.74168602507E+00	2.76934462276E+00	6.32765830108E+00	2.04923110145E+01
2.25	2.77763370568E+00	2.83833665260E+00	6.53794073207E+00	2.12963570540E+01
2.30	2.81332391214E+00	2.90822416228E+00	6.75343118603E+00	2.21270131992E+01
2.35	2.84875390926E+00	2.97900067966E+00	6.97419658166E+00	2.29849346404E+01
2.40	2.88392146642E+00	3.05065967061E+00	7.20030334986E+00	2.38707849025E+01
2.45	2.91882482845E+00	3.12319455123E+00	7.43181742952E+00	2.47852357832E+01
2.50	2.95346268706E+00	3.19659869939E+00	7.66880426425E+00	2.57289672920E+01
2.55	2.98783415294E+00	3.27086546553E+00	7.91132879989E+00	2.67026675875E+01
2.60	3.02193872852E+00	3.34598818285E+00	8.15945548287E+00	2.77070329154E+01
2.65	3.05577628161E+00	3.42196017668E+00	8.41324825923E+00	2.87427675461E+01
2.70	3.08934701986E+00	3.49877477338E+00	8.67277057443E+00	2.98105837117E+01
2.75	3.12265146609E+00	3.57642530849E+00	8.93808537369E+00	3.09112015439E+01
2.80	3.15569043473E+00	3.65490513430E+00	9.20925510296E+00	3.20453490113E+01
2.85	3.18846500903E+00	3.73420762687E+00	9.48634171048E+00	3.32137618570E+01
2.90	3.22097651943E+00	3.81432619249E+00	9.76940664878E+00	3.44171835367E+01
2.95	3.25322652282E+00	3.89525427356E+00	1.00585108772E+01	3.56563651569E+01
3.00	3.28521678289E+00	3.97698535405E+00	1.03537148648E+01	3.69320654133E+01
3.05	3.31694925140E+00	4.05951296440E+00	1.06550785935E+01	3.82450505297E+01
3.10	3.34842605051E+00	4.14283068600E+00	1.09626615622E+01	3.95960941973E+01
3.15	3.37964945599E+00	4.22693215527E+00	1.12765227901E+01	4.09859775150E+01
3.20	3.41062188147E+00	4.31181106727E+00	1.15967208215E+01	4.24154889290E+01
3.25	3.44134586357E+00	4.39746117895E+00	1.19233137301E+01	4.38854241742E+01
3.30	3.47182404788E+00	4.48387631210E+00	1.22563591238E+01	4.53965862157E+01
3.35	3.50205917594E+00	4.57105035585E+00	1.25959141496E+01	4.69497851905E+01
3.40	3.53205407292E+00	4.65897726895E+00	1.29420354988E+01	4.85458383505E+01
3.45	3.56181163619E+00	4.74765108170E+00	1.32947794124E+01	5.01855700060E+01
3.50	3.59133482465E+00	4.83706589763E+00	1.36542016861E+01	5.18698114692E+01
3.55	3.62062664876E+00	4.92721589489E+00	1.40203576765E+01	5.35994009997E+01
3.60	3.64969016135E+00	5.01809532746E+00	1.43933023062E+01	5.53751837489E+01
3.65	3.67852844903E+00	5.10969852607E+00	1.47730900697E+01	5.71980117074E+01
3.70	3.70714462429E+00	5.20201989897E+00	1.51597750394E+01	5.90687436505E+01
3.75	3.73554181820E+00	5.29505393244E+00	1.55534108710E+01	6.09882450870E+01

TABLE 5—*Continued*

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
3.80	3.76372317365E+00	5.38879519122E+00	1.59540508098E+01	6.29573882066E+01
3.85	3.79169183920E+00	5.48323831871E+00	1.63617476963E+01	6.49770518295E+01
3.90	3.81945096336E+00	5.57837803701E+00	1.67765539723E+01	6.70481213558E+01
3.95	3.84700368942E+00	5.67420914690E+00	1.71985216866E+01	6.91714887164E+01
4.00	3.87435315072E+00	5.77072652761E+00	1.76277025010E+01	7.13480523238E+01
4.05	3.90150246632E+00	5.86792513656E+00	1.80641476964E+01	7.35787170247E+01
4.10	3.92845473710E+00	5.96580000889E+00	1.85079081781E+01	7.58643940521E+01
4.15	3.95521304220E+00	6.06434625704E+00	1.89590344820E+01	7.82060009790E+01
4.20	3.98178043587E+00	6.16355907008E+00	1.94175767807E+01	8.06044616728E+01
4.25	4.00815994457E+00	6.26343371313E+00	1.98835848882E+01	8.30607062500E+01
4.30	4.03435456446E+00	6.36396552658E+00	2.03571082668E+01	8.55756710317E+01
4.35	4.06036725908E+00	6.46514992531E+00	2.08381960318E+01	8.81502985000E+01
4.40	4.08620095734E+00	6.56698239790E+00	2.13268969574E+01	9.07855372551E+01
4.45	4.11185855182E+00	6.66945850566E+00	2.18232594823E+01	9.34823419731E+01
4.50	4.13734289716E+00	6.77257388176E+00	2.23273317149E+01	9.62416733640E+01
4.55	4.16265680875E+00	6.87632423025E+00	2.28391614388E+01	9.90644981311E+01
4.60	4.18780306162E+00	6.98070532505E+00	2.33587961181E+01	1.01951788931E+02
4.65	4.21278438947E+00	7.08571300897E+00	2.38862829022E+01	1.04904524332E+02
4.70	4.23760348385E+00	7.19134319259E+00	2.44216686318E+01	1.07923688779E+02
4.75	4.26226299359E+00	7.29759185330E+00	2.49649998430E+01	1.11010272552E+02
4.80	4.28676552422E+00	7.40445503412E+00	2.55163227728E+01	1.14165271729E+02
4.85	4.31111363765E+00	7.51192884271E+00	2.60756833638E+01	1.17389688148E+02
4.90	4.33530985191E+00	7.62000945019E+00	2.66431272691E+01	1.20684529375E+02
4.95	4.35935664096E+00	7.72869309011E+00	2.72186998571E+01	1.24050808661E+02
5.00	4.38325643471E+00	7.83797605729E+00	2.78024462158E+01	1.27489544913E+02
5.05	4.40701161898E+00	7.94785470678E+00	2.83944111577E+01	1.31001762656E+02
5.10	4.43062453568E+00	8.05832545269E+00	2.89946392241E+01	1.34588492000E+02
5.15	4.45409748294E+00	8.16938476716E+00	2.96031746897E+01	1.38250768604E+02
5.20	4.47743271542E+00	8.28102917923E+00	3.02200615666E+01	1.41989633648E+02
5.25	4.50063244459E+00	8.39325527375E+00	3.08453436086E+01	1.45806133795E+02
5.30	4.52369883912E+00	8.50605969035E+00	3.14790643157E+01	1.49701321162E+02
5.35	4.54663402525E+00	8.61943912231E+00	3.21212669375E+01	1.53676253287E+02
5.40	4.56944008730E+00	8.73339031559E+00	3.27719944780E+01	1.57731993101E+02
5.45	4.59211906813E+00	8.84791006768E+00	3.34312896988E+01	1.61869608894E+02
5.50	4.61467296967E+00	8.96299522668E+00	3.40991951233E+01	1.66090174285E+02
5.55	4.63710375345E+00	9.07864269019E+00	3.47757530404E+01	1.70394768196E+02
5.60	4.65941334124E+00	9.19484940439E+00	3.54610055080E+01	1.74784474818E+02
5.65	4.68160361558E+00	9.31161236299E+00	3.61549943571E+01	1.79260383589E+02
5.70	4.70367642043E+00	9.42892860627E+00	3.68577611946E+01	1.83823589157E+02
5.75	4.72563356182E+00	9.54679522016E+00	3.75693474073E+01	1.88475191362E+02
5.80	4.74747680843E+00	9.66520933525E+00	3.82897941651E+01	1.93216295199E+02
5.85	4.76920789230E+00	9.78416812590E+00	3.90191424244E+01	1.98048010799E+02
5.90	4.79082850949E+00	9.90366880932E+00	3.97574329313E+01	2.02971453399E+02
5.95	4.81234032071E+00	1.00237086447E+01	4.05047062246E+01	2.07987743315E+02
6.00	4.83374495199E+00	1.01442849322E+01	4.12610026392E+01	2.13098005919E+02
6.05	4.85504399542E+00	1.02653950124E+01	4.20263623092E+01	2.18303371611E+02
6.10	4.87623900972E+00	1.03870362651E+01	4.28008251704E+01	2.23604975796E+02
6.15	4.89733152099E+00	1.05092061087E+01	4.35844309637E+01	2.29003958859E+02
6.20	4.91832302334E+00	1.06319019994E+01	4.43772192378E+01	2.34501466141E+02
6.25	4.93921497955E+00	1.07551214303E+01	4.51792293521E+01	2.40098647912E+02
6.30	4.96000882177E+00	1.08788619308E+01	4.59905004791E+01	2.45796659354E+02
6.35	4.98070595211E+00	1.10031210655E+01	4.68110716075E+01	2.51596660532E+02
6.40	5.00130774337E+00	1.11278964339E+01	4.76409815447E+01	2.57499816374E+02
6.45	5.02181553959E+00	1.12531856693E+01	4.84802689192E+01	2.63507296645E+02
6.50	5.04223065676E+00	1.13789864385E+01	4.93289721834E+01	2.69620275932E+02
6.55	5.06255438339E+00	1.15052964406E+01	5.01871296158E+01	2.75839933615E+02
6.60	5.08278798117E+00	1.16321134066E+01	5.10547793236E+01	2.82167453848E+02
6.65	5.10293268554E+00	1.17594350989E+01	5.19319592449E+01	2.88604025538E+02
6.70	5.12298970633E+00	1.18872593102E+01	5.28187071514E+01	2.95150842325E+02
6.75	5.14296022829E+00	1.20155838634E+01	5.37150606500E+01	3.01809102559E+02

TABLE 5—*Continued*

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
6.80	5.16284541172E+00	1.21444066105E+01	5.46210571857E+01	3.08580009281E+02
6.85	5.18264639301E+00	1.22737254323E+01	5.55367340434E+01	3.15464770203E+02
6.90	5.20236428521E+00	1.24035382377E+01	5.64621283501E+01	3.22464597686E+02
6.95	5.22200017858E+00	1.25338429632E+01	5.73972770771E+01	3.29580708725E+02
7.00	5.24155514109E+00	1.26646375723E+01	5.83422170417E+01	3.36814324924E+02
7.05	5.26103021901E+00	1.27959200546E+01	5.92969849097E+01	3.44166672481E+02
7.10	5.28042643736E+00	1.29276884260E+01	6.02616171971E+01	3.51638982169E+02
7.15	5.29974480044E+00	1.30599407277E+01	6.12361502718E+01	3.59232489313E+02
7.20	5.31898629235E+00	1.31926750255E+01	6.22206203561E+01	3.66948433779E+02
7.25	5.33815187743E+00	1.33258894098E+01	6.32150635278E+01	3.74788059948E+02
7.30	5.35724250075E+00	1.34595819947E+01	6.42195157226E+01	3.82752616704E+02
7.35	5.37625908856E+00	1.35937509178E+01	6.52340127355E+01	3.90843357413E+02
7.40	5.39520254878E+00	1.37283943397E+01	6.62585902227E+01	3.99061539908E+02
7.45	5.41407377138E+00	1.38635104433E+01	6.72932837034E+01	4.07408426469E+02
7.50	5.43287362886E+00	1.39990974336E+01	6.83381285614E+01	4.15885283808E+02
7.55	5.45160297666E+00	1.41351535372E+01	6.93931600463E+01	4.24493383052E+02
7.60	5.47026265352E+00	1.42716770018E+01	7.04584132759E+01	4.33233999726E+02
7.65	5.48885348198E+00	1.44086660961E+01	7.15339232373E+01	4.42108413735E+02
7.70	5.50737626867E+00	1.45461191089E+01	7.26197247883E+01	4.51117909351E+02
7.75	5.52583180475E+00	1.46840343491E+01	7.37158526593E+01	4.60263775195E+02
7.80	5.54422086626E+00	1.48224101452E+01	7.48223414546E+01	4.69547304221E+02
7.85	5.56254421451E+00	1.49612448448E+01	7.59392256537E+01	4.78969793701E+02
7.90	5.58080259638E+00	1.51005368145E+01	7.70665396130E+01	4.88532545208E+02
7.95	5.59899674472E+00	1.52402844394E+01	7.82043175672E+01	4.98236864604E+02
8.00	5.61712737865E+00	1.53804861225E+01	7.93525936304E+01	5.08084062022E+02
8.05	5.63519520391E+00	1.55211402849E+01	8.05114017976E+01	5.18075451850E+02
8.10	5.65320091316E+00	1.56622453651E+01	8.16807759460E+01	5.28212352720E+02
8.15	5.67114518632E+00	1.58037998186E+01	8.28607498366E+01	5.38496087492E+02
8.20	5.68902869082E+00	1.59458021180E+01	8.40513571150E+01	5.48927983237E+02
8.25	5.70685208196E+00	1.60882507522E+01	8.52526313127E+01	5.59509371226E+02
8.30	5.72461600316E+00	1.62311442265E+01	8.64646058488E+01	5.70241586913E+02
8.35	5.74232108624E+00	1.63744810620E+01	8.76873140307E+01	5.81125969924E+02
8.40	5.75996795170E+00	1.65182597957E+01	8.89207890555E+01	5.92163864039E+02
8.45	5.77755720900E+00	1.66624789796E+01	9.01650640112E+01	6.03356617185E+02
8.50	5.79508945679E+00	1.68071371810E+01	9.14201718778E+01	6.14705581414E+02
8.55	5.81256528319E+00	1.69522329822E+01	9.26861455284E+01	6.26212112895E+02
8.60	5.82998526600E+00	1.70977649798E+01	9.39630177301E+01	6.37877571902E+02
8.65	5.84734997300E+00	1.72437317849E+01	9.52508211455E+01	6.49703322795E+02
8.70	5.86465996211E+00	1.73901320225E+01	9.65495883336E+01	6.61690734012E+02
8.75	5.88191578167E+00	1.75369643316E+01	9.78593517507E+01	6.73841178055E+02
8.80	5.89911797065E+00	1.76842273647E+01	9.91801437514E+01	6.86156031475E+02
8.85	5.91626705884E+00	1.78319197876E+01	1.00511996590E+02	6.98636674861E+02
8.90	5.93336356708E+00	1.79800402795E+01	1.01854942421E+02	7.11284492830E+02
8.95	5.95040800749E+00	1.81285875321E+01	1.03209013301E+02	7.24100874010E+02
9.00	5.96740088359E+00	1.82775602501E+01	1.04574241188E+02	7.37087211031E+02
9.05	5.98434269060E+00	1.84269571507E+01	1.05950657943E+02	7.50244900510E+02
9.10	6.00123391553E+00	1.85767769631E+01	1.07338295333E+02	7.63575343043E+02
9.15	6.01807503741E+00	1.87270184289E+01	1.08737185029E+02	7.77079943190E+02
9.20	6.03486652750E+00	1.88776803014E+01	1.10147358608E+02	7.90760109462E+02
9.25	6.05160884938E+00	1.90287613456E+01	1.11568847553E+02	8.04617254316E+02
9.30	6.06830245921E+00	1.91802603380E+01	1.13001683258E+02	8.18652794133E+02
9.35	6.08494780582E+00	1.93321760664E+01	1.14445897023E+02	8.32868149216E+02
9.40	6.10154533091E+00	1.94845073297E+01	1.15901520056E+02	8.47264743775E+02
9.45	6.11809546921E+00	1.96372529380E+01	1.17368583481E+02	8.61844005913E+02
9.50	6.13459864863E+00	1.97904117119E+01	1.18847118326E+02	8.76607367621E+02
9.55	6.15105529036E+00	1.99439824826E+01	1.20337155537E+02	8.91556264760E+02
9.60	6.16746580911E+00	2.00979640920E+01	1.21838725970E+02	9.06692137056E+02
9.65	6.18383061314E+00	2.02523553921E+01	1.23351860393E+02	9.22016428086E+02
9.70	6.20015010450E+00	2.04071552451E+01	1.24876589492E+02	9.37530585266E+02
9.75	6.21642467908E+00	2.05623625230E+01	1.26412943866E+02	9.53236059845E+02
9.80	6.23265472680E+00	2.07179761080E+01	1.27960954028E+02	9.69134306890E+02

TABLE 5—*Continued*

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
9.85	6.24884063170E+00	2.08739948915E+01	1.29520650412E+02	9.85226785278E+02
9.90	6.26498277209E+00	2.10304177749E+01	1.31092063365E+02	1.00151495768E+03
9.95	6.28108152064E+00	2.11872436685E+01	1.32675223153E+02	1.01800029057E+03
10.00	6.29713724453E+00	2.13444714924E+01	1.34270159963E+02	1.03468425418E+03
10.05	6.31315030554E+00	2.15021001752E+01	1.35876903899E+02	1.05156832253E+03
10.10	6.32912106017E+00	2.16601286551E+01	1.37495484984E+02	1.06865397337E+03
10.15	6.34504985976E+00	2.18185558786E+01	1.39125933166E+02	1.08594268823E+03
10.20	6.36093705056E+00	2.19773808013E+01	1.40768278309E+02	1.10343595237E+03
10.25	6.37678297389E+00	2.21366023873E+01	1.42422550202E+02	1.12113525476E+03
10.30	6.39258796618E+00	2.22962196089E+01	1.44088778558E+02	1.13904208811E+03
10.35	6.40835235911E+00	2.24562314472E+01	1.45766993009E+02	1.15715794882E+03
10.40	6.42407647970E+00	2.26166368913E+01	1.47457223115E+02	1.17548433701E+03
10.45	6.43976065038E+00	2.27774349383E+01	1.49159498358E+02	1.19402275648E+03
10.50	6.45540518909E+00	2.29386245935E+01	1.50873848146E+02	1.21277471471E+03
10.55	6.47101040941E+00	2.31002048701E+01	1.52600301812E+02	1.23174172284E+03
10.60	6.48657662058E+00	2.32621747889E+01	1.54338888617E+02	1.25092529571E+03
10.65	6.50210412763E+00	2.34245333786E+01	1.56089637747E+02	1.27032695176E+03
10.70	6.51759323142E+00	2.35872796753E+01	1.57852578317E+02	1.28994821312E+03
10.75	6.53304422880E+00	2.37504127226E+01	1.59627739367E+02	1.30979060553E+03
10.80	6.54845741259E+00	2.39139315716E+01	1.61415149870E+02	1.32985565836E+03
10.85	6.56383307172E+00	2.40778352805E+01	1.63214838725E+02	1.35014490459E+03
10.90	6.57917149130E+00	2.42421229149E+01	1.65026834760E+02	1.37065988084E+03
10.95	6.59447295266E+00	2.44067935471E+01	1.66851166737E+02	1.39140212728E+03
11.00	6.60973773345E+00	2.45718462568E+01	1.68687863344E+02	1.41237318771E+03
11.05	6.62496610770E+00	2.47372801304E+01	1.70536953204E+02	1.43357460950E+03
11.10	6.64015834589E+00	2.49030942611E+01	1.72398464870E+02	1.45500794359E+03
11.15	6.65531471501E+00	2.50692877488E+01	1.74272426828E+02	1.47667474448E+03
11.20	6.67043547864E+00	2.52358597001E+01	1.76158867494E+02	1.49857657024E+03
11.25	6.68552089699E+00	2.54028092282E+01	1.78057815222E+02	1.52071498248E+03
11.30	6.70057122697E+00	2.55701354526E+01	1.79969298296E+02	1.54309154635E+03
11.35	6.71558672227E+00	2.57378374992E+01	1.81893344935E+02	1.56570783054E+03
11.40	6.73056763338E+00	2.59059145005E+01	1.83829983295E+02	1.58856540724E+03
11.45	6.74551420770E+00	2.60743655948E+01	1.85779241463E+02	1.61166585219E+03
11.50	6.76042668954E+00	2.62431899268E+01	1.87741147465E+02	1.63501074460E+03
11.55	6.77530532020E+00	2.64123866472E+01	1.89715729262E+02	1.65860166721E+03
11.60	6.79015033804E+00	2.65819549127E+01	1.91703014751E+02	1.68244020624E+03
11.65	6.80496197849E+00	2.67518938859E+01	1.93703031766E+02	1.70652795138E+03
11.70	6.81974047417E+00	2.69222027354E+01	1.95715808080E+02	1.73086649582E+03
11.75	6.83448605484E+00	2.70928806353E+01	1.97741371403E+02	1.75545743619E+03
11.80	6.84919894755E+00	2.72639267657E+01	1.99779749381E+02	1.78030237261E+03
11.85	6.86387937663E+00	2.74353403122E+01	2.01830969603E+02	1.80540290863E+03
11.90	6.87852756374E+00	2.76071204659E+01	2.03895059593E+02	1.83076065125E+03
11.95	6.89314372793E+00	2.77792664235E+01	2.05972046818E+02	1.85637721092E+03
12.00	6.90772808569E+00	2.79517773872E+01	2.08061958682E+02	1.88225420148E+03
12.05	6.92228085096E+00	2.81246525645E+01	2.10164822531E+02	1.90839324025E+03
12.10	6.93680223523E+00	2.82978911683E+01	2.12280665652E+02	1.93479594792E+03
12.15	6.95129244750E+00	2.84714924165E+01	2.14409515273E+02	1.96146394860E+03
12.20	6.96575169442E+00	2.86454555326E+01	2.16551398562E+02	1.98839886981E+03
12.25	6.98018018024E+00	2.88197797449E+01	2.18706342630E+02	2.01560234244E+03
12.30	6.99457810691E+00	2.89944642870E+01	2.20874374532E+02	2.04307600079E+03
12.35	7.00894567407E+00	2.91695083973E+01	2.23055521262E+02	2.07082148253E+03
12.40	7.02328307914E+00	2.93449113193E+01	2.25249809761E+02	2.09884042868E+03
12.45	7.03759051732E+00	2.95206723015E+01	2.27457266912E+02	2.12713448366E+03
12.50	7.05186818161E+00	2.96967905971E+01	2.29677919540E+02	2.15570529522E+03
12.55	7.06611626291E+00	2.98732654640E+01	2.31911794415E+02	2.18455451447E+03
12.60	7.08033495000E+00	3.00500961653E+01	2.34158918255E+02	2.21368379587E+03
12.65	7.09452442956E+00	3.02272819682E+01	2.36419317718E+02	2.24309479719E+03
12.70	7.10868488627E+00	3.04048221449E+01	2.38693019410E+02	2.27278917955E+03
12.75	7.12281650278E+00	3.05827159721E+01	2.40980049881E+02	2.30276860741E+03
12.80	7.13691945977E+00	3.07609627312E+01	2.43280435629E+02	2.33303474850E+03
12.85	7.15099393598E+00	3.09395617078E+01	2.45594203096E+02	2.36358927388E+03

TABLE 5—*Continued*

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
12.90	7.16504010822E+00	3.11185121921E+01	2.47921378673E+02	2.39443385794E+03
12.95	7.17905815143E+00	3.12978134788E+01	2.50261988695E+02	2.42557017832E+03
13.00	7.19304823870E+00	3.14774648667E+01	2.52616059447E+02	2.45699991598E+03
13.05	7.20701054127E+00	3.16574656592E+01	2.54983617160E+02	2.48872475514E+03
13.10	7.22094522859E+00	3.18378151637E+01	2.57364688014E+02	2.52074638332E+03
13.15	7.23485246835E+00	3.20185126919E+01	2.59759298135E+02	2.55306649128E+03
13.20	7.24873242647E+00	3.21995575597E+01	2.62167473601E+02	2.58568677307E+03
13.25	7.26258526717E+00	3.23809490872E+01	2.64589240435E+02	2.61860892599E+03
13.30	7.27641115298E+00	3.25626865985E+01	2.67024624613E+02	2.65183465057E+03
13.35	7.29021024473E+00	3.27447694216E+01	2.69473652058E+02	2.68536565061E+03
13.40	7.30398270165E+00	3.29271968887E+01	2.71936348643E+02	2.71920363313E+03
13.45	7.31772868131E+00	3.31099683360E+01	2.74412740191E+02	2.75335030840E+03
13.50	7.33144833972E+00	3.32930831035E+01	2.76902852476E+02	2.78780738988E+03
13.55	7.34514183128E+00	3.34765405350E+01	2.79406711223E+02	2.82257659428E+03
13.60	7.35880930888E+00	3.36603399783E+01	2.81924342107E+02	2.85765964152E+03
13.65	7.37245092384E+00	3.38444807849E+01	2.84455770754E+02	2.89305825472E+03
13.70	7.38606682602E+00	3.40289623102E+01	2.87001022742E+02	2.92877416018E+03
13.75	7.39965716376E+00	3.42137839132E+01	2.89560123602E+02	2.96480908744E+03
13.80	7.41322208396E+00	3.43989449565E+01	2.92133098815E+02	3.00116476918E+03
13.85	7.42676173205E+00	3.45844448068E+01	2.94719973816E+02	3.03784294130E+03
13.90	7.44027625208E+00	3.47702828337E+01	2.97320773991E+02	3.07484534285E+03
13.95	7.45376578667E+00	3.49564584111E+01	2.99935524680E+02	3.11217371607E+03
14.00	7.46723047706E+00	3.51429709160E+01	3.02564251176E+02	3.14982980635E+03
14.05	7.48067046313E+00	3.53298197291E+01	3.05206978725E+02	3.18781536224E+03
14.10	7.49408588342E+00	3.55170042345E+01	3.07863732527E+02	3.22613213546E+03
14.15	7.50747687514E+00	3.57045238197E+01	3.10534537737E+02	3.26478188085E+03
14.20	7.52084357421E+00	3.58923778758E+01	3.13219419462E+02	3.30376635641E+03
14.25	7.53418611524E+00	3.60805657971E+01	3.15918402765E+02	3.34308732328E+03
14.30	7.54750463156E+00	3.62690869814E+01	3.18631512663E+02	3.38274654571E+03
14.35	7.56079925529E+00	3.64579408296E+01	3.21358774129E+02	3.42274579109E+03
14.40	7.57407011726E+00	3.66471267461E+01	3.24100212089E+02	3.46308682993E+03
14.45	7.58731734712E+00	3.68366441385E+01	3.26855851428E+02	3.50377143584E+03
14.50	7.60054107329E+00	3.70264924176E+01	3.29625716982E+02	3.54480138555E+03
14.55	7.61374142302E+00	3.72166709974E+01	3.32409833548E+02	3.58617845888E+03
14.60	7.62691852237E+00	3.74071792950E+01	3.35208225875E+02	3.62790443876E+03
14.65	7.64007249626E+00	3.75980167308E+01	3.38020918670E+02	3.66998111120E+03
14.70	7.65320346846E+00	3.77891827281E+01	3.40847936599E+02	3.71241026531E+03
14.75	7.66631156161E+00	3.79806767136E+01	3.43689304279E+02	3.75519369326E+03
14.80	7.67939689725E+00	3.81724981166E+01	3.46545046291E+02	3.79833319030E+03
14.85	7.69245959580E+00	3.83646463698E+01	3.49415187168E+02	3.84183055478E+03
14.90	7.70549977663E+00	3.85571209087E+01	3.52299751404E+02	3.88568758807E+03
14.95	7.71851755801E+00	3.87499211719E+01	3.55198763448E+02	3.92990609462E+03
15.00	7.73151305717E+00	3.89430466009E+01	3.58112247708E+02	3.97448788194E+03
15.05	7.74448639030E+00	3.91364966401E+01	3.61040228553E+02	4.01943476057E+03
15.10	7.75743767255E+00	3.93302707367E+01	3.63982730306E+02	4.06474854412E+03
15.15	7.77036701805E+00	3.95243683409E+01	3.66939777251E+02	4.11043104920E+03
15.20	7.78327453996E+00	3.97187889057E+01	3.69911393631E+02	4.15648409549E+03
15.25	7.79616035041E+00	3.99135318870E+01	3.72897603647E+02	4.20290950568E+03
15.30	7.80902456058E+00	4.01085967433E+01	3.75898431461E+02	4.24970910548E+03
15.35	7.82186728066E+00	4.03039829359E+01	3.78913901192E+02	4.29688472362E+03
15.40	7.83468861991E+00	4.04996899291E+01	3.81944036921E+02	4.34443819186E+03
15.45	7.84748868663E+00	4.06957171896E+01	3.84988862688E+02	4.39237134494E+03
15.50	7.86026758821E+00	4.08920641871E+01	3.88048402493E+02	4.44068602062E+03
15.55	7.87302543109E+00	4.10887303936E+01	3.91122680297E+02	4.48938405965E+03
15.60	7.88576232084E+00	4.12857152840E+01	3.94211720019E+02	4.53846730577E+03
15.65	7.89847836210E+00	4.14830183359E+01	3.97315545543E+02	4.58793760572E+03
15.70	7.91117365865E+00	4.16806390292E+01	4.00434180711E+02	4.63779680922E+03
15.75	7.92384831337E+00	4.18785768468E+01	4.03567649326E+02	4.68804676896E+03
15.80	7.93650242830E+00	4.20768312738E+01	4.06715975153E+02	4.73868934061E+03
15.85	7.94913610461E+00	4.22754017979E+01	4.09879181919E+02	4.78972638279E+03
15.90	7.96174944264E+00	4.24742879095E+01	4.13057293313E+02	4.84115975711E+03

TABLE 5—*Continued*

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
15.95	7.97434254187E+00	4.26734891014E+01	4.16250332983E+02	4.89299132813E+03
16.00	7.98691550099E+00	4.28730048688E+01	4.19458324542E+02	4.94522296335E+03
16.05	7.99946841785E+00	4.30728347094E+01	4.22681291565E+02	4.99785653324E+03
16.10	8.01200138951E+00	4.32729781234E+01	4.25919257588E+02	5.05089391120E+03
16.15	8.02451451223E+00	4.34734346135E+01	4.29172246110E+02	5.10433697358E+03
16.20	8.03700788148E+00	4.36742036844E+01	4.32440280594E+02	5.15818759965E+03
16.25	8.04948159196E+00	4.38752848437E+01	4.35723384465E+02	5.21244767162E+03
16.30	8.06193573759E+00	4.40766776010E+01	4.39021581111E+02	5.26711907463E+03
16.35	8.07437041156E+00	4.42783814683E+01	4.42334893883E+02	5.32220369673E+03
16.40	8.08678570627E+00	4.44803959601E+01	4.45663346097E+02	5.37770342891E+03
16.45	8.09918171341E+00	4.46827205929E+01	4.49006961031E+02	5.43362016504E+03
16.50	8.11155852392E+00	4.48853548858E+01	4.52365761927E+02	5.48995580191E+03
16.55	8.12391622802E+00	4.50882983599E+01	4.55739771993E+02	5.54671223923E+03
16.60	8.13625491522E+00	4.52915505387E+01	4.59129014399E+02	5.60389137959E+03
16.65	8.14857467432E+00	4.54951109479E+01	4.62533512280E+02	5.66149512848E+03
16.70	8.16087559340E+00	4.56989791154E+01	4.65953288735E+02	5.71952539427E+03
16.75	8.17315775988E+00	4.59031545713E+01	4.69388366829E+02	5.77798408823E+03
16.80	8.18542126047E+00	4.61076368479E+01	4.72838769591E+02	5.83687312451E+03
16.85	8.19766618122E+00	4.63124254795E+01	4.76304520015E+02	5.89619442012E+03
16.90	8.20989260748E+00	4.65175200028E+01	4.79785641060E+02	5.95594989495E+03
16.95	8.22210062398E+00	4.67229199565E+01	4.83282155651E+02	6.01614147178E+03
17.00	8.23429031477E+00	4.69286248813E+01	4.86794086678E+02	6.07677107622E+03
17.05	8.24646176326E+00	4.71346343202E+01	4.90321456996E+02	6.13784063675E+03
17.10	8.25861505220E+00	4.73409478181E+01	4.93864289428E+02	6.19935208472E+03
17.15	8.27075026374E+00	4.75475649222E+01	4.97422606759E+02	6.26130735431E+03
17.20	8.28286747938E+00	4.77544851813E+01	5.00996431745E+02	6.32370838256E+03
17.25	8.29496678001E+00	4.79617081468E+01	5.04585787104E+02	6.38655710935E+03
17.30	8.30704824589E+00	4.81692333717E+01	5.08190695523E+02	6.44985547738E+03
17.35	8.31911195670E+00	4.83770604112E+01	5.11811179655E+02	6.51360543222E+03
17.40	8.33115799150E+00	4.85851888223E+01	5.15447262119E+02	6.57780892223E+03
17.45	8.34318642877E+00	4.87936181641E+01	5.19098965502E+02	6.64246789862E+03
17.50	8.35519734638E+00	4.90023479977E+01	5.22766312356E+02	6.70758431541E+03
17.55	8.36719082164E+00	4.92113778861E+01	5.26449325203E+02	6.77316012946E+03
17.60	8.37916693128E+00	4.94207073941E+01	5.30148026529E+02	6.83919730040E+03
17.65	8.39112575146E+00	4.96303360885E+01	5.33862438792E+02	6.90569779071E+03
17.70	8.40306735778E+00	4.98402635382E+01	5.37592584412E+02	6.97266356566E+03
17.75	8.41499182528E+00	5.00504893136E+01	5.41338485780E+02	7.04009659330E+03
17.80	8.42689922844E+00	5.02610129873E+01	5.45100165256E+02	7.10799884452E+03
17.85	8.43878964121E+00	5.04718341334E+01	5.48877645165E+02	7.17637229297E+03
17.90	8.45066313699E+00	5.06829523283E+01	5.52670947802E+02	7.24521891509E+03
17.95	8.46251978866E+00	5.08943671499E+01	5.56480095430E+02	7.31454069013E+03
18.00	8.47435966855E+00	5.11060781780E+01	5.60305110280E+02	7.38433960009E+03
18.05	8.48618284848E+00	5.13180849942E+01	5.64146014552E+02	7.45461762976E+03
18.10	8.49798939974E+00	5.15303871819E+01	5.68002830414E+02	7.52537676671E+03
18.15	8.50977939314E+00	5.17429843262E+01	5.71875580003E+02	7.59661900127E+03
18.20	8.52155289894E+00	5.19558760141E+01	5.75764285426E+02	7.66834632653E+03
18.25	8.53330998693E+00	5.21690618343E+01	5.79668968758E+02	7.74056073837E+03
18.30	8.54505072637E+00	5.23825413772E+01	5.83589652044E+02	7.81326423539E+03
18.35	8.55677518607E+00	5.25963142350E+01	5.87526357298E+02	7.88645881896E+03
18.40	8.56848343433E+00	5.28103800014E+01	5.91479106502E+02	7.96014649321E+03
18.45	8.58017553895E+00	5.30247382722E+01	5.95447921610E+02	8.03432926500E+03
18.50	8.59185156729E+00	5.32393886444E+01	5.99432824545E+02	8.10900914394E+03
18.55	8.60351158621E+00	5.34543307171E+01	6.03433837200E+02	8.18418814238E+03
18.60	8.61515566211E+00	5.36695640909E+01	6.07450981436E+02	8.25986827539E+03
18.65	8.62678386093E+00	5.38850883679E+01	6.11484279086E+02	8.33605156079E+03
18.70	8.63839624815E+00	5.41009031522E+01	6.15533751953E+02	8.41274001913E+03
18.75	8.64999288879E+00	5.43170080491E+01	6.19599421812E+02	8.48993567366E+03
18.80	8.66157384743E+00	5.45334026659E+01	6.23681310404E+02	8.56764055036E+03
18.85	8.67313918820E+00	5.47500866113E+01	6.27779439445E+02	8.64585667794E+03
18.90	8.68468897477E+00	5.49670594957E+01	6.31893830619E+02	8.72458608781E+03
18.95	8.69622327042E+00	5.51843209310E+01	6.36024505583E+02	8.80383081409E+03

TABLE 5—*Continued*

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
19.00	8.70774213795E+00	5.54018705307E+01	6.40171485963E+02	8.88359289360E+03
19.05	8.71924563975E+00	5.56197079098E+01	6.44334793357E+02	8.96387436587E+03
19.10	8.73073383779E+00	5.58378326851E+01	6.48514449334E+02	9.04467727313E+03
19.15	8.74220679362E+00	5.60562444747E+01	6.52710475435E+02	9.12600366029E+03
19.20	8.75366456837E+00	5.62749428983E+01	6.56922893171E+02	9.20785557497E+03
19.25	8.76510722275E+00	5.64939275771E+01	6.61151724026E+02	9.29023506747E+03
19.30	8.77653481707E+00	5.67131981339E+01	6.65396989455E+02	9.37314419075E+03
19.35	8.78794741125E+00	5.69327541930E+01	6.69658710884E+02	9.45658500050E+03
19.40	8.79934506478E+00	5.71525953800E+01	6.73936909712E+02	9.54055955503E+03
19.45	8.81072783677E+00	5.73727213222E+01	6.78231607310E+02	9.62506991537E+03
19.50	8.82209578594E+00	5.75931316483E+01	6.82542825020E+02	9.71011814520E+03
19.55	8.83344897062E+00	5.78138259884E+01	6.86870584157E+02	9.79570631085E+03
19.60	8.84478744874E+00	5.80348039743E+01	6.91214906009E+02	9.88183648135E+03
19.65	8.85611127787E+00	5.82560652388E+01	6.95575811835E+02	9.96851072836E+03
19.70	8.86742051518E+00	5.84776094166E+01	6.99953322868E+02	1.00557311262E+04
19.75	8.87871521748E+00	5.86994361434E+01	7.04347460312E+02	1.01434997518E+04
19.80	8.88999544120E+00	5.89215450568E+01	7.08758245344E+02	1.02318186849E+04
19.85	8.90126124240E+00	5.91439357953E+01	7.13185699116E+02	1.03206900077E+04
19.90	8.91251267680E+00	5.93666079992E+01	7.17629842750E+02	1.04101158051E+04
19.95	8.92374979973E+00	5.95895613099E+01	7.22090697343E+02	1.05000981647E+04
20.00	8.93497266617E+00	5.98127953704E+01	7.26568283965E+02	1.05906391766E+04
20.05	8.94618133074E+00	6.00363098248E+01	7.31062623659E+02	1.06817409337E+04
20.10	8.95737584773E+00	6.02601043190E+01	7.35573737440E+02	1.07734055315E+04
20.15	8.96855627106E+00	6.04841784998E+01	7.40101646299E+02	1.08656350679E+04
20.20	8.97972265431E+00	6.07085320155E+01	7.44646371198E+02	1.09584316437E+04
20.25	8.99087505072E+00	6.09331645160E+01	7.49207933076E+02	1.10517973622E+04
20.30	9.00201351320E+00	6.11580756520E+01	7.53786352842E+02	1.11457343294E+04
20.35	9.01313809431E+00	6.13832650759E+01	7.58381651381E+02	1.12402446537E+04
20.40	9.02424884627E+00	6.16087324414E+01	7.62993849552E+02	1.13353304464E+04
20.45	9.03534582101E+00	6.18344774034E+01	7.67622968187E+02	1.14309938211E+04
20.50	9.04642907008E+00	6.20604996181E+01	7.72269028093E+02	1.15272368943E+04
20.55	9.05749864475E+00	6.22867987430E+01	7.76932050052E+02	1.16240617849E+04
20.60	9.06855459593E+00	6.25133744368E+01	7.81612054819E+02	1.17214706144E+04
20.65	9.07959697425E+00	6.27402263597E+01	7.86309063123E+02	1.18194655071E+04
20.70	9.09062583000E+00	6.29673541729E+01	7.91023095670E+02	1.19180485896E+04
20.75	9.10164121317E+00	6.31947575389E+01	7.95754173138E+02	1.20172219912E+04
20.80	9.11264317342E+00	6.34224361216E+01	8.00502316181E+02	1.21169878439E+04
20.85	9.12363176013E+00	6.36503895861E+01	8.05267545429E+02	1.22173482822E+04
20.90	9.13460702235E+00	6.38786175986E+01	8.10049881483E+02	1.23183054430E+04
20.95	9.14556900885E+00	6.41071198266E+01	8.14849344924E+02	1.24198614662E+04
21.00	9.15651776809E+00	6.43358959388E+01	8.19665956304E+02	1.25220184938E+04
21.05	9.16745334823E+00	6.45649456052E+01	8.24499736153E+02	1.26247786706E+04
21.10	9.17837579714E+00	6.47942684968E+01	8.29350704975E+02	1.27281441440E+04
21.15	9.18928516239E+00	6.50238642860E+01	8.34218883250E+02	1.28321170639E+04
21.20	9.20018149129E+00	6.52537326463E+01	8.39104291433E+02	1.29366995827E+04
21.25	9.21106483082E+00	6.54838732523E+01	8.44006949953E+02	1.30418938555E+04
21.30	9.22193522771E+00	6.57142857800E+01	8.48926879219E+02	1.31477020399E+04
21.35	9.23279272840E+00	6.59449699063E+01	8.53864099610E+02	1.32541262958E+04
21.40	9.24363737904E+00	6.61759253093E+01	8.58818631487E+02	1.33611687861E+04
21.45	9.25446922550E+00	6.64071516685E+01	8.63790495181E+02	1.34688316758E+04
21.50	9.26528831341E+00	6.66386486643E+01	8.68779711003E+02	1.35771171329E+04
21.55	9.27609468808E+00	6.68704159782E+01	8.73786299238E+02	1.36860273274E+04
21.60	9.28688839458E+00	6.71024532931E+01	8.78810280149E+02	1.37955644324E+04
21.65	9.29766947771E+00	6.73347602928E+01	8.83851673974E+02	1.39057306230E+04
21.70	9.30843798199E+00	6.75673366622E+01	8.88910500928E+02	1.40165280772E+04
21.75	9.31919395169E+00	6.78001820874E+01	8.93986781200E+02	1.41279589755E+04
21.80	9.32993743082E+00	6.80332962557E+01	8.99080534959E+02	1.42400255006E+04
21.85	9.34066846312E+00	6.82666788552E+01	9.04191782349E+02	1.43527298381E+04
21.90	9.35138709208E+00	6.85003295755E+01	9.09320543490E+02	1.44660741759E+04
21.95	9.36209336093E+00	6.87342481068E+01	9.14466838481E+02	1.45800607045E+04
22.00	9.37278731267E+00	6.89684341409E+01	9.19630687394E+02	1.46946916170E+04

TABLE 5—*Continued*

η	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$	$F_{5/2}(\eta)$
22.05	9.38346899000E+00	6.92028873702E+01	9.24812110282E+02	1.48099691086E+04
22.10	9.39413843544E+00	6.94376074884E+01	9.30011127172E+02	1.49258953776E+04
22.15	9.40479569119E+00	6.96725941904E+01	9.35227758070E+02	1.50424726244E+04
22.20	9.41544079927E+00	6.99078471718E+01	9.40462022957E+02	1.51597030519E+04
22.25	9.42607380141E+00	7.01433661294E+01	9.45713941795E+02	1.52775888657E+04
22.30	9.43669473913E+00	7.03791507613E+01	9.50983534519E+02	1.53961322737E+04
22.35	9.44730365369E+00	7.06152007662E+01	9.56270821043E+02	1.55153354866E+04
22.40	9.45790058614E+00	7.08515158441E+01	9.61575821260E+02	1.56352007171E+04
22.45	9.46848557727E+00	7.10880956960E+01	9.66898555039E+02	1.57557301808E+04
22.50	9.47905866765E+00	7.13249400238E+01	9.72239042227E+02	1.58769260956E+04
22.55	9.48961989760E+00	7.15620485306E+01	9.77597302647E+02	1.59987906819E+04
22.60	9.50016930725E+00	7.17994209202E+01	9.82973356103E+02	1.61213261626E+04
22.65	9.51070693646E+00	7.20370568978E+01	9.88367222375E+02	1.62445347631E+04
22.70	9.52123282490E+00	7.22749561692E+01	9.93778921220E+02	1.63684187112E+04
22.75	9.53174701199E+00	7.25131184415E+01	9.99208472375E+02	1.64929802373E+04
22.80	9.54224953694E+00	7.27515434226E+01	1.00465589555E+03	1.66182215740E+04
22.85	9.55274043874E+00	7.29902308215E+01	1.01012121045E+03	1.67441449566E+04
22.90	9.56321975616E+00	7.32291803480E+01	1.01560443673E+03	1.68707526229E+04
22.95	9.57368752775E+00	7.34683917131E+01	1.02110559405E+03	1.69980468129E+04
23.00	9.58414379186E+00	7.37078646285E+01	1.02662470203E+03	1.71260297694E+04
23.05	9.59458858659E+00	7.39475988071E+01	1.03216178027E+03	1.72547037372E+04
23.10	9.60502194988E+00	7.41875939626E+01	1.03771684837E+03	1.73840709640E+04
23.15	9.61544391943E+00	7.44278498096E+01	1.04328992589E+03	1.75141336997E+04
23.20	9.62585453272E+00	7.46683660639E+01	1.04888103235E+03	1.76448941967E+04
23.25	9.63625382705E+00	7.49091424419E+01	1.05449018730E+03	1.77763547098E+04
23.30	9.64664183950E+00	7.51501786612E+01	1.06011741022E+03	1.79085174963E+04
23.35	9.65701860696E+00	7.53914744402E+01	1.06576272059E+03	1.80413848160E+04
23.40	9.66738416609E+00	7.56330294982E+01	1.07142613786E+03	1.81749589309E+04
23.45	9.67773855339E+00	7.58748435554E+01	1.07710768149E+03	1.83092421057E+04
23.50	9.68808180512E+00	7.61169163331E+01	1.08280737086E+03	1.84442366074E+04
23.55	9.69841395737E+00	7.63592475532E+01	1.08852522540E+03	1.85799447053E+04
23.60	9.70873504603E+00	7.66018369387E+01	1.09426126445E+03	1.87163686714E+04
23.65	9.71904510679E+00	7.68446842136E+01	1.10001550738E+03	1.88535107799E+04
23.70	9.72934417515E+00	7.70877891025E+01	1.10578797353E+03	1.89913733076E+04
23.75	9.73963228641E+00	7.73311513310E+01	1.11157868218E+03	1.91299585334E+04
23.80	9.74990947570E+00	7.75747706258E+01	1.11738765265E+03	1.92692687390E+04
23.85	9.76017577794E+00	7.78186467141E+01	1.12321490420E+03	1.94093062083E+04
23.90	9.77043122787E+00	7.80627793242E+01	1.12906045607E+03	1.95500732276E+04
23.95	9.78067586006E+00	7.83071681853E+01	1.13492432750E+03	1.96915720856E+04
24.00	9.79090970888E+00	7.85518130274E+01	1.14080653770E+03	1.98338050736E+04
24.05	9.80113280851E+00	7.87967135812E+01	1.14670710585E+03	1.99767744850E+04
24.10	9.81134519297E+00	7.90418695785E+01	1.15262605112E+03	2.01204826158E+04
24.15	9.82154689607E+00	7.92872807518E+01	1.15856339266E+03	2.02649317643E+04
24.20	9.83173795149E+00	7.95329468346E+01	1.16451914961E+03	2.04101242312E+04
24.25	9.84191839267E+00	7.97788675609E+01	1.17049334106E+03	2.05560623198E+04
24.30	9.85208825293E+00	8.00250426660E+01	1.17648598610E+03	2.07027483354E+04
24.35	9.86224756537E+00	8.02714718857E+01	1.18249710381E+03	2.08501845860E+04
24.40	9.87239636296E+00	8.05181549567E+01	1.18852671323E+03	2.09983733818E+04
24.45	9.88253467846E+00	8.07650916165E+01	1.19457483339E+03	2.11473170356E+04
24.50	9.89266254449E+00	8.10122816035E+01	1.20064148331E+03	2.12970178622E+04
24.55	9.90277999346E+00	8.12597246569E+01	1.20672668196E+03	2.14474781793E+04
24.60	9.91288705766E+00	8.15074205167E+01	1.21283044832E+03	2.15987003064E+04
24.65	9.92298376918E+00	8.17553689235E+01	1.21895280135E+03	2.17506865658E+04
24.70	9.93307015995E+00	8.20035696191E+01	1.22509375997E+03	2.19034392820E+04
24.75	9.94314626174E+00	8.22520223458E+01	1.23125334309E+03	2.20569607818E+04
24.80	9.95321210616E+00	8.25007268467E+01	1.23743156962E+03	2.22112533945E+04
24.85	9.96326772464E+00	8.27496828659E+01	1.24362845841E+03	2.23663194518E+04
24.90	9.97331314848E+00	8.29988901480E+01	1.24984402833E+03	2.25221612875E+04
24.95	9.98334840878E+00	8.32483484386E+01	1.25607829821E+03	2.26787812380E+04
25.00	9.99337353652E+00	8.34980574840E+01	1.26233128686E+03	2.28361816420E+04

short-word machine (that is, 32 bits per word) should convert the small single-precision section of FDTAB to double precision.

Polynomial interpolation in a table with uniform table spacing s involves fitting a polynomial to a small set of tabulated values. We have two conflicting requirements in the present application where high accuracy is sought. On one hand, we already have tables of appreciable size, and we do not want to make them any larger than necessary. On the other hand, our tables with $s = 0.05$ are fairly coarsely spaced since the Fermi-Dirac integrals change significantly from node to node. The solution is to use highly accurate interpolation polynomials.

The first thing we tried was Lagrangian interpolation (for example, Isaacson and Keller 1966), which forces a polynomial of degree j to pass through $j+1$ tabulated function values. Ordinary linear and quadratic interpolation on two and three points, respectively, are familiar examples of this approach. Quadratic interpolation yields only 4 digit accuracy near $\eta = -5$, improving to 8 digits near $\eta = 20$.

TABLE 6

Listing of the Integration Program FDTAB

```
      PROGRAM FDTAB (TTY,TAPE59=TTY,TAPE6,TAPE12)
C +++
C +++ COMPUTE ACCURATE TABLES OF FERMI-DIRAC INTEGRALS
C +++ USING SIMPSON'S RULE WITH EXTRAPOLATION TECHNIQUES
C +++
C +++     TAPE59 = TERMINAL
C +++     TAPE12 = PRINTED OUTPUT FILE WITH DIAGNOSTICS
C +++     TAPE6  = FINAL TABLE
C +++
      DOUBLE PRECISION A, B, BAD, BGT, BINC, BK, BKT, BRATIO, BSAVE
      DOUBLE PRECISION BXFRM, CEXTR, DE, DENOM,, ETA, ETAO, EXTRAP
      DOUBLE PRECISION F, FINT, FMAX, FSUM, GK, GKT, GXFRM, H, PEXTR
      DOUBLE PRECISION RGB, ROB, ROG, VALUE, X, XMAX

      COMMON /X1/ X(2001), F(2001), VALUE(20)
C +++
C +++ INITIAL DATA
C +++

C +++ (A, B) ARE THE STARTING VALUES OF THE INTEGRATION LIMITS (O, T).
      DATA A/0.D+00/, B/1.D+00/

C +++ BINC IS THE INCREMENT IN B USED BY THE ADAPTIVE UPPER INTEGRATION
C +++ LIMIT ROUTINE, BRATIO IS THE MAXIMUM VALUE OF F(B)/MAX(F),
C +++ WHERE THE FUNCTION F IS THE INTEGRAND.
      DATA BINC/0.5D+00/, BRATIO/1.D-06/

C +++ INCREMENTING FACTORS FOR THE B AND G TRANSFORMS
      DATA GK / 1.D+00 / , BK / 1.1D+00 /

C +++ MPTS IS THE NUMBER OF MESH POINTS TO WHICH SIMPSON'S RULE
C +++ IS APPLIED FOR THE COARSEST MESH, MUST BE AN ODD INTEGER
      DATA MPTS /201/

C +++ ETAO IS THE FIRST VALUE OF DEGENERACY PARAMETER, DE IS THE
C +++ INCREMENT IN ETA, AND NE IS THE NUMBER OF ETA VALUES
      DATA ETAO/-5.D+00/, DE/0.05D+00/, NE/601/

C +++ NFAIL IS A DIAGNOSTIC THAT COUNTS EXTRAPOLATION FAILURES
      DATA NFAIL / 0 /

C +++ ITRAP = 1 FOR TRAPEZOIDAL RULE, OTHERWISE SIMPSON'S RULE INTEGRATION
      DATA ITRAP/0/

C +++
C +++ LINK OPENS OUTPUT FILES (SYSTEM-DEPENDENT).
C +++ THESE SYSTEM ROUTINES ARE NOT NEEDED ON MOST SYSTEMS.
C +++
      CALL LINK("UNIT59=TTY//")
      CALL LINK("UNIT12=(TAPE12,CREATE,TEXT)//")
      CALL LINK("UNIT6=(TAPE6,CREATE,TEXT)//")
C +++
C +++ MAKE SURE MPTS IS ODD FOR SIMPSON'S RULE
C +++
      IF (MOD(MPTS,2) .NE. 1 .AND. ITRAP .EQ. 0) THEN
         WRITE (59, 10) MPTS
         WRITE (12, 10) MPTS
         CALL EXIT
      ENDIF
```

Hermite interpolation (for example, Isaacson and Keller 1966) is highly accurate and fits both function values and first derivatives at the table's nodes. The Fermi-Dirac integrals obey the relation (McDougall and Stoner 1939)

$$\frac{dF_n}{d\eta} = nF_{n-1}(\eta), \qquad n > 0, \tag{29}$$

so we need the table for $F_{-1/2}$ for a routine that evaluates $F_{1/2}$ and $F_{3/2}$. Cubic Hermite interpolation, which fits the function values and derivatives at two nodes, provides 7 digit accuracy near $\eta = -5$ and 12 digit near $\eta = 20$. We finally adopted fifth-order Hermite interpolation, which fits function values and derivatives at three nodes. The accuracy is at least 10 digits near $\eta = -5$ and at least 12 near $\eta = 25$.

Table 7 gives the listing of the interpolation program. Subroutine FDSET is called only once before using the function subprogram FD to calculate function values. Internal comments provide the necessary documentation. The least accurate part of this algorithm is the series expansion for large η, providing a relative error of 1.3×10^{-10} at $\eta = 25.33$.

TABLE 6—*Continued*

```
          MPTSAV = MPTS
          BSAVE  = B
C +++
C +++ OUTER LOOP IS OVER ALL VALUES OF ETA
C +++
          DO 20 N = 1, NE

             ETA = ETA0 + DE * DFLOAT(N - 1)
             BAD = BSAVE

             WRITE (12, 30) ETA
C +++
C +++ THIS LOOP DOES THE THREE INTEGRATIONS REQUIRED FOR THE
C +++ B AND G TRANSFORMS, AFTER WHICH THE TRANSFORMS ARE COMPUTED.
C +++
             DO 40 NGB = 1, 3

C +++ SET APPROPRIATE UPPER INTEGRATION LIMIT
                B = BAD
                IF (NGB .EQ. 2) B = BAD + GK
                IF (NGB .EQ. 3) B = BAD * BK
C +++
C +++ THIS LOOP INCREMENTS THE NUMBER OF MESH POINTS TO DO THE
C +++ THREE INTEGRATIONS REQUIRED FOR EACH AITKEN EXTRAPOLATION.
C +++
             DO 50 NM = 1, 3

   60        CONTINUE

                M2    = ( MPTS - 3 ) / 2
                M4    = M2 + 1
                H     = (B - A) / DFLOAT(MPTS - 1)

                FMAX = -1.D+100

                DO 70 J = 1, MPTS

                   X(J) = A + H * DFLOAT(J - 1)
                   F(J) = FINT( X(J), ETA )

                      IF (F(J) .GE. FMAX) THEN
                         XMAX = X(J)
                         FMAX = F(J)
                      ENDIF

   70           CONTINUE

          IF (NGB .NE. 1 .OR. NM .NE. 1 .OR. F(MPTS).LT.BRATIO*FMAX
         1 .OR. B .GT. 100.D+00) GO TO 80
          B = B + BINC
          BAD = B
          GO TO 60
   80     CONTINUE

               IF (NM .EQ. 1) WRITE (12, 90) FMAX,XMAX,X(MPTS),F(MPTS)

          IF (ITRAP .EQ. 1) THEN
C +++
C +++ USE TRAPEZOIDAL RULE INTEGRATIONS
C +++
```

TABLE 6—*Continued*

```
          FSUM = 0.D+00

          DO 100 I = 1, MPTS
             FSUM = FSUM + F(I)
  100     CONTINUE
          VALUE(NM) = H * ( FSUM - 0.5D+00 * ( F(1) + F(MPTS) ) )
       ELSE
C +++
C +++ INTEGRATE THE F ARRAY WITH SIMPSON'S RULE
C +++
          FSUM = 0.D+00
          DO 110 II = 1, M2
          I = II
          I = M2 - II + 1
             J = M4 - I
             FSUM = FSUM + F(2*J+1) + F(2*J) * 2.D+00
  110     CONTINUE
          VALUE(NM) = H*( 2.D+00*FSUM + F(1) + F(MPTS)+4.D+00*F(MPTS-1) )
     1                / 3.D+00
       ENDIF
          WRITE (12, 120) VALUE(NM), MPTS
          IF (NM .LT. 3) MPTS = 2 * (MPTS - 1) + 1
   50  CONTINUE
C +++
C +++ AITKEN EXTRAPOLATION
C +++
          DENOM = VALUE(3) + VALUE(1) - 2.D+00 * VALUE(2)

          IF (DENOM .NE. 0.D+00) THEN
             EXTRAP = VALUE(3) - ( VALUE(3) -VALUE(2) )**2 / DENOM
          ELSE
             EXTRAP = VALUE(3)
             NFAIL = NFAIL + 1
             WRITE (12, 130)
          ENDIF

          IF (EXTRAP-VALUE(2) .NE. 0.D+00) THEN
             DENOM = (EXTRAP - VALUE(1)) / (EXTRAP - VALUE(2))
          ELSE
             DENOM = 0.D+00
          ENDIF

          IF (DENOM .GT. 0.D+00) THEN
             PEXTR = DLOG10( DENOM ) / DLOG10(2.D+00)
             H = (B - A) / ( DFLOAT(MPTSAV-1) )
             CEXTR = (EXTRAP - VALUE(1) ) / H**PEXTR
```

TABLE 6—*Continued*

```
         ELSE
            NFAIL = NFAIL + 1
            PEXTR    = -99999999.D+00
            CEXTR    = PEXTR
            WRITE (12, 140) EXTRAP, VALUE(1), VALUE(2)
         ENDIF

         RELERR = DABS( (EXTRAP - VALUE(3)) / EXTRAP )
         IF (RELERR .GT. 0.0) THEN
            DIGITS = -ALOG10( RELERR )
         ELSE
            DIGITS = -30.
         ENDIF

         WRITE (12, 150) EXTRAP, PEXTR, CEXTR, H, DIGITS

      IF (NGB .EQ. 1) THEN
         BGT = EXTRAP
         RGB = F(MPTS)
      ENDIF

      IF (NGB .EQ. 2) THEN
         GKT = EXTRAP
         ROG = F(MPTS)
      ENDIF

      IF (NGB .EQ. 3) THEN
         BKT = EXTRAP
         ROB = F(MPTS)
      ENDIF

      MPTS = MPTSAV

   40 CONTINUE
C +++
C +++ CALCULATE THE B AND G TRANSFORMS
C +++
      ROG    = ROG / RGB
      ROB    = BK * ROB / RGB

C +++ LIMITED ERROR CHECKING

      IF (ROB .LE. 0.D+00 .OR. ROG .LE. 0.D+00) THEN
         WRITE (12, 160) ETA, ROB, ROG
         WRITE (59, 160) ETA, ROB, ROG
         CALL EXIT
      ENDIF

      GXFRM = ( GKT - ROG * BGT ) / ( 1.D+00 - ROG )
      BXFRM = ( BKT - ROB * BGT ) / ( 1.D+00 - ROB )

      WRITE (12, 170) GXFRM, BXFRM

         RELERR = DABS( (GXFRM - GKT) / GXFRM )
         IF (RELERR .GT. 0.0) THEN
            DIGITS = -ALOG10( RELERR )
         ELSE
            DIGITS = -30.
         ENDIF
         SETA    = ETA
```

TABLE 6—*Continued*

```
          SGXFRM = GXFRM
          WRITE ( 6, 180) SETA, SGXFRM
          WRITE (12, 200) SETA, SGXFRM, DIGITS
          WRITE (59, 200) SETA, SGXFRM, DIGITS

   20 CONTINUE

      WRITE (59,190) NFAIL
      WRITE (12,190) NFAIL

      CALL EXIT
C +++
C +++ FORMAT BLOCK
C +++
   10 FORMAT (1X,6HMPTS =,I7,2X,22HMUST BE AN ODD INTEGER)
   30 FORMAT (2X/5X,5HETA =,D12.4)
   90 FORMAT (2X,14HINTEGRAND MAX=,D14.6,6H AT X=,D14.6,
     1    12H ENDPOINT X=,D14.6,3H F=,D14.6)
  120 FORMAT (1X,10HINTEGRAL =,D28.20,1X,10HFOR MPTS =,I6)
  130 FORMAT (1X,35HCANNOT PERFORM AITKEN EXTRAPOLATION)
  140 FORMAT (1X,51HCANNOT COMPUTE AITKEN PARAMETERS P AND C - EXTRAP =,
     1    D21.13,1X,12HVALUE(1&2) =,2D21.13)
  150 FORMAT (1X, 23HEXTRAPOLATED INTEGRAL =, D28.20,
     1    3H P=,D12.4,3H C=,D12.4,3H H=,D13.5,8H DIGITS=,F6.1)
  160 FORMAT (1X, 4HETA=,D28.12, 5H ROB=,D28.12, 5H ROG=,D28.12/
     1    1X,28HCANNOT DO B AND G TRANSFORMS)
  170 FORMAT (1X,13HG TRANSFORM =,D28.20,2X,13HB TRANSFORM =,D28.20)
  180 FORMAT (1X,E14.5,E20.12)
  190 FORMAT (//1X,I5,23H EXTRAPOLATION FAILURES)
  200 FORMAT (1X,E14.5,E20.12,5X,F6.1,1X,13HDIGITS G XFRM )

      END
      FUNCTION FINT(X, ETA)
      DOUBLE PRECISION FINT, X, ETA
C +++
C +++ EVALUATE THE INTEGRAND AT (X,ETA)
C +++

C +++ THIS IS THE F-D INTEGRAL OF ORDER 1/2 (SEE EQ. (7))
      FINT = 2.D+00* (X**2) / ( DEXP( X*X - ETA ) + 1.D+00 )

      RETURN
      END
```

TABLE 7

LISTING OF THE INTERPOLATION PROGRAM FDINT

```
      SUBROUTINE FDSET
C +++
C +++ THIS SUBROUTINE INITIALIZES ARRAYS NEEDED BY FUNCTION FD.
C +++ FDSET IS CALLED ONCE (AND ONLY ONCE) BEFORE CALLING FD.
C +++
      COMMON /FDTAB/ F(601,3), AHI(5,2), ALO(5,2)
      DATA PI /3.14159265358979/
C +++
C +++ SET UP F-D INTEGRAL ASSYMPTOTIC EXPANSION COEFFICIENTS
C +++
      A=SQRT(PI)/2.
      ALO(1,1)=A
      ALO(2,1)=-A/(SQRT(2.))**3
      ALO(3,1)=A/(SQRT(3.))**3
      ALO(4,1)=-A/8.
      ALO(5,1)=A/(SQRT(5.))**3
      A=3.*SQRT(PI)/4.
      ALO(1,2)=A
      ALO(2,2)=-A/(SQRT(2.))**5
      ALO(3,2)=A/(SQRT(3.))**5
      ALO(4,2)=-A/32.
      ALO(5,2)=A/(SQRT(5.))**5
      AHI(1,1)=2./3.
      AHI(2,1)=PI**2/12.
      AHI(3,1)=PI**4*7./960.
      AHI(4,1)=PI**6*31./4608.
      AHI(5,1)=PI**8*1397./81920.
      AHI(1,2)=.4
      AHI(2,2)=PI**2/4.
      AHI(3,2)=-AHI(3,1)
      AHI(4,2)=-31.*PI**6/10752.
      AHI(5,2)=-381.*PI**8/81920.
C +++
C +++ READ IN FERMI-DIRAC INTEGRAL TABLES GIVEN IN TABLE 5
C +++
C +++   READ IN THE N = -1/2 TABLE
      READ (5,20) (F(I,1),I=1,601)
C +++   READ IN THE N = 1/2 TABLE
      READ (5,20) (F(I,2),I=1,601)
C +++   READ IN THE N=3/2 TABLE
      READ (5,20) (F(I,3),I=1,601)
      RETURN
C
   20 FORMAT (15X,E20.12)
      END

      FUNCTION FD (ETA,N)
C +++
C +++ COMPUTES THE FERMI-DIRAC INTEGRAL FD FOR DEGENERACY
C +++ PARAMETER ETA.  N=1, ORDER 1/2; N=2, ORDER 3/2.
C +++
      COMMON /FDTAB/ F(601,3), AHI(5,2), ALO(5,2)
      IF (ETA.LT.-5.) GO TO 10
      IF (ETA.GT.25.) GO TO 30
C +++
C +++ FIFTH ORDER HERMITE INTERPOLATION FOR INTERMEDIATE VALUES OF ETA
C +++
      J=(ETA+5.)*20.+1.00
      IF (J.LT.2) J=2
      IF (J.GT.600) J=600
      Y1 = F(J-1,N+1)
      Y2 = F(J,N+1)
      Y3 = F(J+1,N+1)
```

TABLE 7—*Continued*

```
      S1 = F(J-1,N)  * (FLOAT(N)-0.5)
      S2 = F(J,N)    * (FLOAT(N)-0.5)
      S3 = F(J+1,N)  * (FLOAT(N)-0.5)
      X1 = 0.05*FLOAT(J-2)-5.0
      X2 = X1 +0.05
      X3 = X1 + 0.1
      FD = HERMIT5(ETA,X1,Y1,S1,X2,Y2,S2,X3,Y3,S3)
      RETURN
C +++
C +++ EXPANSION FOR SMALL VALUES OF ETA
C +++
   10 CONTINUE
      FD=0.
      DO 20 KK=1,5
      K=6-KK
      FD=FD+ALO(K,N)*EXP(FLOAT(K)*ETA)
   20 CONTINUE
      RETURN
C +++
C +++ EXPANSION FOR LARGE VALUES OF ETA
C +++
   30 CONTINUE
      U=SQRT(ETA)
      L=5+2*N
      FD=0.
      DO 40 KK=1,5
      K=6-KK
      FD=FD+AHI(K,N)*U**(L-4*K)
   40 CONTINUE
      RETURN
      END

      FUNCTION HERMIT5(X, X1, P1, DP1, X2, P2, DP2, X3, P3, DP3)
C +++
C +++ FIFTH ORDER HERMITE INTERPOLATION
C +++
C +++ X = VALUE OF THE INDEPENDENT VARIABLE WHERE THE FUNCTION
C +++        VALUE HERMIT5 IS DESIRED.
C +++ X1, X2, X3 = VALUES OF THE INDEPENDENT VARIABLE AT THE THREE
C +++        INTERPOLATION NODES; MUST BE UNIFORMLY SPACED.
C +++ P1, P2, P3 = FUNCTION VALUES AT THE INTERPOLATION NODES.
C +++ DP1, DP2, DP3 = FIRST DERIVATIVES OF THE FUNCTION AT THE
C +++        INTERPOLATION NODES.
C +++
      H=X2-X1
      XP = X - X2
      A0 = P2
      A1 = DP2
      A4 = (H * (DP3 - DP1) - 2. * (P1+P3-P2-P2))
      A2 = ((P1 + P3 - P2 - P2) * 2. - A4) / (4. * H * H)
      A4 = A4 * 0.25 / H**4
      A5 = 0.25*(H*(DP3+4.*DP2+DP1)-3.*(P3-P1))
      A3 = (0.5 * (P3 - P1) - H * DP2 - A5) / H**3
      A5 = A5 / H**5
      HERMIT5 = ((((A5*XP + A4)*XP + A3)*XP + A2)*XP + A1)*XP + A0
      RETURN
      END
```

V. SUMMARY

We have shown that it is possible, with relatively elementary numerical techniques, to provide Fermi-Dirac integrals with an accuracy of at least one part in 10^{10}. Tables of values of integrals for $-5 \leq \eta \leq 25$ and $n = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$, and $\frac{5}{2}$ with a spacing of 0.05 in η are provided. Source listings for the program that generated the tables and for the program that evaluates Fermi-Dirac integrals of orders $\frac{1}{2}$ and $\frac{3}{2}$ for arbitrary η are provided. Users with different needs should be able to generate easily tables suited to their requirements.

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