C2. $c \succeq^+ d$ holds, for any elements c and d of $\mathcal{CW}$.

1. Let c be an arbitrary element of $\mathcal{CW}$ and define
   $D \triangleq \{d \in \mathcal{CW} : \neg(c \succeq^+ d)\}$
   It suffices to assume D is nonempty and obtain a contradiction.

   PROOF: Obvious.

2. $d \succ c$ holds, for all $d \in D$.

   PROOF: $\neg(c \succeq^+ d)$ implies $\neg(c \succeq d)$ (because $c \succeq d$ implies $c \succeq^+ d$ by definition of the transitive closure), and $\neg(c \succeq d)$ equals $d \succ c$ by definition of $\succeq$.

3. For all $d \in D$ and all $e \in \mathcal{CW} \setminus D : d \succ e$.

   3.1. It suffices to assume $d \in D$, $e \in \mathcal{CW} \setminus D$, and $\neg(d \succ e)$ and obtain a contradiction

   PROOF: Obvious.

   3.2. $c \succeq^+ e$

   PROOF: By the assumption of 3.1, which implies $e \notin D$, and the definition of D.

   3.3. $e \succeq d$

   PROOF: By the assumption of 3.1, which asserts $\neg(d \succ e)$, and the definition of *succeq*.

   3.4. Q.E.D.

   Steps 3.2 and 3.3 and the definition of the transitive closure imply $c \succeq^+ d$, which by the definition of D contradicts the step 3.1 assumption $d \in D$.

4. D is a dominating set.

   PROOF: We must prove that if $d \in D$, then $d \succ e$ for any candidate e not in D. If e is not in $\mathcal{CW}$, this follows because $\mathcal{CW}$ is a dominating set. If $e \in \mathcal{CW}$, then $e \in \mathcal{CW} \setminus D$ and this follows from step 3.

5. Q.E.D.

   PROOF: D is a subset of $\mathcal{CW}$ by definition. It is a proper subset of $\mathcal{CW}$ because $c \in \mathcal{CW}$ by the step 1 assumption and step 2 implies $c \notin D$. Therefore, step 4 implies that $\mathcal{CW}$ is not the smallest dominating set, which is a contradiction.

CLOSE