Well-Founded Relations

An operator $\succ$ is called a *partial order* on a set $\mathcal{N}$ iff it satisfies the following two conditions:

Irreflexivity $\forall n \in \mathcal{N} : \neg(n \succ n)$

Transitivity $\forall m, n, p \in \mathcal{N} : (m \succ n) \wedge (n \succ p) \Rightarrow (m \succ p)$

A partial order $\succ$ on $\mathcal{N}$ is called a *total order* iff it also satisfies the condition:

Completeness $\forall m, n \in \mathcal{N} : (m \succ n) \vee (n \succ m) \vee (m = n)$

A partial order $\succ$ on a set $\mathcal{N}$ is said to be *well-founded* iff there is no infinite descending chain of the form:

$$n_1 \succ n_2 \succ n_3 \succ \ldots$$

with all the n_i in $\mathcal{N}$. This condition can be expressed formally in terms of functions$^{\square}$ as

$$\neg \exists f \in [Nat \to \mathcal{N}] : \forall i \in Nat : f[i] \succ f[i+1]$$

Any partial order on a finite set is obviously well-founded. The relation $>$ is a well-founded total order on the set *Nat* of natural numbers. A well-founded partial (or total) order $\succ$ on a set $\mathcal{N}$ is also a well-founded partial (or total) order on any subset of $\mathcal{N}$.

A useful well-founded total order is the relation $\succ_k$ on k-tuples of natural numbers, defined by letting

$$\langle a_1, \ldots, a_k \rangle \succ_k \langle b_1, \ldots, b_k \rangle$$

iff there exists i in $1 \,..\, k$ such that $a_i > b_i$ and $a_j = b_j$ for all j in $1 \,..\, (i-1)$. Since a k-tuple of natural numbers is a function$^{\square}$ from $1 \,..\, k$ to *Nat*, this definition can be written formally as

$$\begin{aligned} a \succ_k b \;\triangleq\; & \wedge\; a \in [1 \,..\, k \to Nat] \\ & \wedge\; b \in [1 \,..\, k \to Nat] \\ & \wedge\; \exists i \in 1 \,..\, k : \wedge\; a[i] > b[i] \\ & \qquad\qquad\qquad\quad \wedge\; \forall j \in 1 \,..\, (i-1) : a[j] > b[j] \end{aligned}$$

(This isn't a TLA$^+$ definition because we can't write $\succ_k$ in TLA$^+$; we would have to define the operator for a particular value of n.)

We can generalize these relations $\succ_k$ to the well-founded total order $\succ$ on the set of all finite sequences of natural numbers by defining $m \succ n$ to be true iff either (i) sequence m is longer than sequence n or (ii) they both have length k and $m \succ_k n$. The TLA$^+$ definition of $\succ$ is easily written using the operators *Seq* and *Len* defined in the standard *Sequences* module$^{\square}$.

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