SOLVING A NETWORK DESIGN PROBLEM USING BENDERS DECOMPOSITION.

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1. PROBLEM QUESTIONS

1.1. Solve the problem directly, i.e., without using Benders decomposition and report the resulting solution:

a) Objective function value in its two components: i) investment costs and, ii) exploitation costs

Using our dataset, the AMPL implementation and the Cplex (version 12.8.0.0) solver we obtain in 52 iterations that the value of the objective function for the solution that minimize the total cost is ***14.409,13 €***.

These costs can be segmented in Investment costs, resulted from the addition of new lines, and exploitation costs as follows:

Total Cost = Investment Cost + Exploitation Cost
14.409€ = 54€ + 14.355€

b) List of candidate links $a \in \hat{A}$ which will be included in the solution of the problem, i.e., links $a \in \hat{A}$ for which y_a=1

In the final solution of the problem, the active arcs, those arcs that must be added to the network so that both the investment and the exploration costs are minimized, are the following:

Added arcs: **(3,6)**, **(4,5)**, **(6,14)**, **(13,17)** and **(14,13)**
Nod added arcs: **(10,5)**, **(11,22)**, **(15,13)**, **(16,15)** and **(21,18)**

1.2. Solve the problem using Benders decomposition. Once the algorithm has been satisfactorily implemented, report the problem's solution starting from two different initial solutions for the binary decision variables. These two points are: $y_a = 0 \; and \; y_a = 1 \; \forall \; a \in \; \hat{A}$.

In both cases report always a summary of the iterations carried out by the algorithm by means of a table or list in which, for each iteration the following appears reported:

a) Objective function's value of the Master Problem

b) Objective function's value for the Subproblem z_D resulting from decision variables y as well as the investment and exploitation costs.

Finally, report what led the algorithm to stop and the final solution in the form of the subset of links within the candidates in $a \in \; \hat{A}$ that have been included and excluded. Report also the value of the flows on the network links for the final network's configuration.

Case 1: Starting at $y_a = 0 \; \forall \, a \in \hat{A}$

Applying the benders algorithm from this initial point and setting a limit of maximum number of iterations equal to 100 and a tolerance value equal to 1e-10 we obtain, at each iteration, the following results:

Iteration	Lower Bound	Upper Bound	Investment Cost	Explotation Cost	Relative Gap
1	-1,00E+07	16209,77	0	16209,77	1,0016
2	-72137,15	14875,53	55	14820,53	1,2062
3	12338,14	15676,64	22	15654,64	0,2706
4	12338,14	15916,9	22	15894,9	0,2901
5	12340,14	15363,77	24	15339,77	0,2450
6	12346,14	14918,98	30	14888,98	0,2084
7	12348,14	15656,9	32	15624,9	0,2680
8	12348,14	14650,98	32	14618,98	0,1865
9	12350,14	15373,77	34	15339,77	0,2448
10	12356,14	14658,98	40	14618,98	0,1864
11	12368,14	14527,13	52	14475,13	0,1746
12	12527,13	14677,13	52	14625,13	0,1716
13	12529,13	14409,13	54	14355,13	0,1501
14	14389,13	14532,98	34	14498,98	0,0100
15	14399,13	15239,92	44	15195,92	0,0584
16	14409,13	14409,13	54	14355,13	0,0000

Table 1: Results at each Iteration (y=0)

Graphically, it can be shown how both the value of the master (red line) problem and the value of the sub problem (blue line) converge at the optimal solution. Notice, however, as we already knew, that although the algorithm converges to the optimal solution, the value of the master problem is not always better at each iteration.

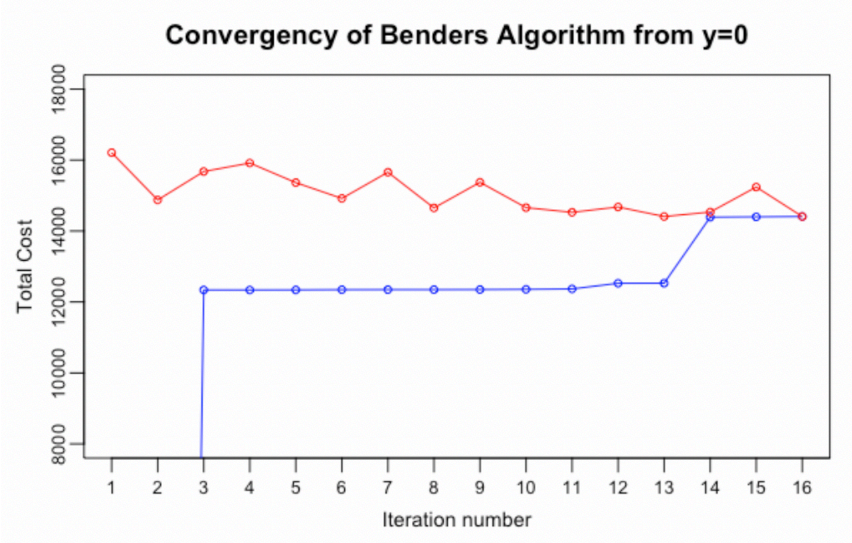

Figure 1: Representation of the convergence between MP and SP (y=1)

Next, we show the value of the binding variables, arcs added to the network, obtained by the master problem at each iteration until the optimal solution is reached.

Iter	(4,5)	(10,5)	(3,6)	(14,13)	(15,13)	(16,15)	(21,18)	(13,17)	(6,14)	(11,22)
1	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	0	0	1	0	0
3	1	0	0	0	0	0	0	0	1	0
4	0	0	0	1	0	0	0	0	1	0
5	1	0	0	1	0	0	0	0	1	0
6	0	0	1	0	0	0	0	0	1	0
7	0	1	0	1	0	0	0	0	1	0
8	1	0	1	0	0	0	0	0	1	0
9	1	1	0	1	0	0	0	0	1	0
10	0	1	1	0	0	0	0	0	1	0
11	1	0	1	0	0	0	0	1	1	0
12	0	0	1	1	0	0	0	1	1	0
13	1	0	1	1	0	0	0	1	1	0
14	1	0	1	1	0	0	0	0	1	0
15	1	0	0	1	0	0	0	1	1	0
16	1	0	1	1	0	0	0	1	1	0

Table 2: Results at each iteration for binding variables (y=0)

Finally, we present the results obtained using the benders algorithm for case 1. We obtain in 16 iterations that, just as in the first question, the value of the objective function for the solution that minimize the total cost is ***14.409,13 €***

Since the number of iterations has been less than 100 (maximum number allowed) the algorithm stopped because the relative gap was less than the tolerance defined.

Again, the segmented costs resulted from the addition of new lines are

Total Cost = Investment Cost + Exploitation Cost

14.409€ = 54€ + 14.355€

Regarding the arcs to be added:

Added arcs: **(3,6)**, **(4,5)**, **(6,14)**, **(13,17)** and **(14,13)**
Nod added arcs: **(10,5)**, **(11,22)**, **(15,13), (16,15)** and **(21,18)**

The optimal flows related to both two origins are:

Arc	Flow 11	Flow 3
{1,2}	0	0
{2,1}	0	0
{1,4}	0	0
{4,1}	0	0
{2,3}	0	0
{3,2}	0	0
{5,4}	0	15
{4,9}	0	15
{9,4}	15	0
{5,10}	0	0
{5,6}	0	0
{6,5}	0	30
{6,3}	0	0
{3,7}	0	15
{7,3}	0	0
{14,6}	0	0
{7,14}	0	0
{14,7}	0	0
{7,19}	0	15
{19,7}	0	0
{8,9}	0	0
{9,8}	0	15
{8,11}	0	0
{11,8}	15	0
{9,12}	0	0
{12,9}	15	0
{9,10}	0	0
{10,9}	0	0
{10,13}	0	0

Arc	Flow 11	Flow 3
{13,10}	0	0
{22,11}	0	0
{11,12}	135	0
{12,11}	0	0
{12,15}	0	0
{15,12}	0	0
{12,13}	105	0
{13,12}	0	15
{13,14}	15	0
{13,15}	0	0
{17,13}	0	0
{14,18}	0	45
{18,14}	0	0
{15,16}	0	0
{16,22}	0	0
{22,16}	0	0
{16,17}	0	0
{17,16}	0	0
{17,18}	30	0
{18,17}	0	0
{17,20}	30	0
{20,17}	0	0
{18,21}	15	30
{20,22}	0	0
{22,20}	0	0
{20,23}	30	15
{23,20}	0	0
{20,21}	0	0
{21,20}	0	15

Arc	Flow 11	Flow 3
{21,24}	0	0
{24,21}	0	0
{24,19}	15	0
{19,24}	0	0
{22,23}	0	0
{23,22}	0	0
{23,24}	15	0
{24,23}	0	0
{4,5}	15	0
{10,5}	0	0
{3,6}	0	135
{14,13}	0	45
{15,13}	0	0
{16,15}	0	0
{21,18}	0	0
{13,17}	75	15
{6,14}	0	105
{11,22}	0	0

Table 3: Optimal flows for origins 3 and 11 (y=0)

Case 2: Starting at $y_a = 1 \; \forall \, a \in \hat{A}$

Applying the benders algorithm from this initial point and setting a limit of maximum number of iterations equal to 100 and a tolerance value equal to 1e-10 we obtain, at each iteration, the following results:

Iteration	Lower Bound	Upper Bound	Investment Cost	Explotation Cost	Relative Gap
1	-1,00E+07	14491,13	54	14355,13	1,0014
2	14355,13	16209,77	0	16209,77	0,1292
3	14357,13	15656,64	2	15654,64	0,0905
4	14357,13	15896,9	2	15894,9	0,1072
5	14359,13	15343,77	4	15339,77	0,0686
6	14365,13	15274,64	10	15264,64	0,0633
7	14367,13	15006,64	12	14994,64	0,0445
8	14375,13	16085,92	20	16065,92	0,1190
9	14377,13	15532,79	22	15510,79	0,0804
10	14377,13	15773,05	22	15751,05	0,0971
11	14379,13	15219,92	24	15195,92	0,0585
12	14385,13	15120,53	30	15090,53	0,0511
13	14385,13	14918,98	30	14888,98	0,0371
14	14387,13	14650,98	32	14618,98	0,0183
15	14387,13	14852,53	32	14820,53	0,0323
16	14387,13	14800,98	32	14768,98	0,0288
17	14389,13	14532,98	34	14498,98	0,0100
18	14405,13	14795,13	50	14745,13	0,0271
19	14407,13	14677,13	52	14625,13	0,0187
20	14407,13	14527,13	52	14475,13	0,0083
21	14409,13	14409,13	54	14355,13	0,0000

Table 4: Results at each Iteration (y=1)

Graphically, it can be shown how both the value of the master (red line) problem and the value of the sub problem (blue line) converge at the optimal solution. Notice, however, that in this case the approximation to the optimal solution by both the master problem and the sub problem are different than those of the previous case because the sequence of solutions for the binding variables is different and depends on the initial seed or starting point. Actually, in this case the algorithm needed more iterations to reach the optimal solution of the problem.

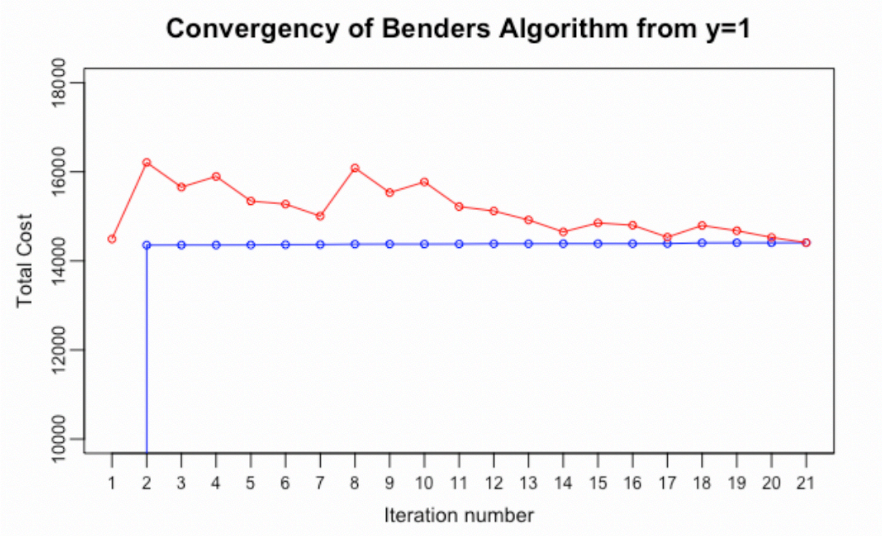

Figure 2: Representation of the convergence between MP and SP (y=1)

Next, we show the value of the binding variables, arcs added to the network, obtained by the master problem at each iteration until the optimal solution is reached.

Iteration	(4,5)	(10,5)	(3,6)	(14,13)	(15,13)	(16,15)	(21,18)	(13,17)	(6,14)	(11,22)
1	1	1	1	1	1	1	1	1	1	1
2	0	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0
5	1	0	0	1	0	0	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0
7	1	0	1	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0
9	1	0	0	0	0	0	0	1	0	0
10	0	0	0	1	0	0	0	1	0	0
11	1	0	0	1	0	0	0	1	0	0
12	0	0	1	0	0	0	0	1	0	0
13	0	0	1	0	0	0	0	0	1	0
14	1	0	1	0	0	0	0	0	1	0
15	1	0	1	0	0	0	0	1	0	0
16	0	0	1	1	0	0	0	0	1	0
17	1	0	1	1	0	0	0	0	1	0
18	0	0	1	0	0	0	0	1	1	0
19	0	0	1	1	0	0	0	1	1	0
20	1	0	1	0	0	0	0	1	1	0
21	1	0	1	1	0	0	0	1	1	0

Table 5: Results at each iteration for binding variables (y=1)

Finally, we present the results obtained using the benders algorithm for case 1. We obtain in 21 iterations that, just as in the first question, the value of the objective function for the solution that minimize the total cost is ***14.409,13 €***

Since the number of iterations has been less than 100 (maximum number allowed) the algorithm stopped because the relative gap was less than the tolerance defined.

Again, the segmented costs resulted from the addition of new lines are

Total Cost = Investment Cost + Exploitation Cost

14.409€ = 54€ + 14.355€

Regarding the arcs to be added

Added arcs: **(3,6)**, **(4,5)**, **(6,14)**, **(13,17)** and **(14,13)**
Nod added arcs: **(10,5)**, **(11,22)**, **(15,13), (16,15)** and **(21,18)**

The optimal flows related to both two origins are:

Arc	Flow 11	Flow 3
{1,2}	0	0
{2,1}	0	0
{1,4}	0	0
{4,1}	0	0
{2,3}	0	0
{3,2}	0	0
{5,4}	0	15
{4,9}	0	15
{9,4}	15	0
{5,10}	0	0
{5,6}	0	0
{6,5}	0	30
{6,3}	0	0
{3,7}	0	15
{7,3}	0	0
{14,6}	0	0
{7,14}	0	0
{14,7}	0	0
{7,19}	0	15
{19,7}	0	0
{8,9}	0	0
{9,8}	0	15
{8,11}	0	0
{11,8}	15	0
{9,12}	0	0
{12,9}	15	0
{9,10}	0	0
{10,9}	0	0
{10,13}	0	0

Arc	Flow 11	Flow 3
{13,10}	0	0
{22,11}	0	0
{11,12}	135	0
{12,11}	0	0
{12,15}	0	0
{15,12}	0	0
{12,13}	105	0
{13,12}	0	15
{13,14}	15	0
{13,15}	0	0
{17,13}	0	0
{14,18}	0	45
{18,14}	0	0
{15,16}	0	0
{16,22}	0	0
{22,16}	0	0
{16,17}	0	0
{17,16}	0	0
{17,18}	30	0
{18,17}	0	0
{17,20}	30	0
{20,17}	0	0
{18,21}	15	30
{20,22}	0	0
{22,20}	0	0
{20,23}	30	15
{23,20}	0	0
{20,21}	0	0
{21,20}	0	15

Arc	Flow 11	Flow 3
{21,24}	0	0
{24,21}	0	0
{24,19}	15	0
{19,24}	0	0
{22,23}	0	0
{23,22}	0	0
{23,24}	15	0
{24,23}	0	0
{4,5}	15	0
{10,5}	0	0
{3,6}	0	135
{14,13}	0	45
{15,13}	0	0
{16,15}	0	0
{21,18}	0	0
{13,17}	75	15
{6,14}	0	105
{11,22}	0	0

Table 6: Table 3: Optimal flows for origins 3 and 11 (y=1)