

# On The Evans Function and Its Applications to Periodic Waves

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Ongoing Project — Last Update: 27 April 2025

## Popular Science Description

Mathematical models describing natural phenomena have often given rise to seemingly cyclic, and repetitive patterns, which have manifested themselves in a wide-ranging set of topics. From population dynamics in a biological setting to the evolution of stellar structures and celestial dynamics, we have often-times witnessed wave-like behavior, or at least some permanent structure that follows a certain pattern. An immediate question that plays a pivotal role in applied mathematics is that of the *stability* of such behavior. For instance, when it comes to traveling water waves, the term stability refers to the wave maintaining its shape as it moves forward in time, while small disturbances diminish without causing major effects. However, non-linearity among a large number of interesting evolving systems makes it a challenging task to provide an answer right away. It is also what keeps mathematicians in business since the study of stability in a diverse set of applications demands the development of newer theories and feasible techniques. The central topic of this thesis revolves around one such theory, namely the *Evans function*.

One can conveniently think of this function as a stability detector device. Mathematically speaking, it connects two theoretical perspectives from functional analysis and dynamical systems on stability. It is named after the American mathematician John Evans, who originally trained as a medical doctor, became fascinated by mathematics, and eventually left his medical career to pursue a PhD in math. Having introduced the Evans function in the 1970s, his interest in mathematics was influenced by his medical background, particularly the Hodgkin-Huxley model for nerve impulse propagation, formulated about 20 years earlier. While Hodgkin and Huxley had shown that nerve impulses could travel as waves, Evans focused on proving their stability. Though he didn't fully solve this problem, he developed a theory with applications extending beyond neuroscience.

In this dissertation, the aim is to introduce the main ideas surrounding this

method, and eventually see one of its many applications in stability analysis of solitons. In particular, the generalized Korteweg-De Vries (gKdV) equation. We will be further extending the application of the theory in investigating the transverse instability of gKdV.

## Introduction

To be completed.

For now, enjoy this scene of nature's wild beauty! :)





# Chapter 1

## Preliminaries

We begin by recalling some basic concepts in linear functional analysis, which play a central role in the stability theory of dynamical systems. To establish the connection and necessity of these topics to differential equations, we first provide a short introduction to some basic results in the theory of ODEs. Since the periodic Evans function serves as our main analytical tool, this chapter will subsequently focus further on generalizing the notion of periodic matrices to infinite-dimensional spaces through Floquet theory. The introductory prerequisites are mainly based on the following references:

- T. Kapitula and K. Promislow, *Spectral and Dynamical Stability of Non-linear Waves*, Springer, 2013.
- G. Teschl, *Ordinary Differential Equations and Dynamical Systems*, American Mathematical Society, 2012.

### Some Known Results in The Theory of ODEs

We begin by recalling some results for systems of linear ODEs:

$$\frac{d}{dx} \mathbf{y} = \underbrace{\mathbf{A}(x)\mathbf{y}}_{\mathbf{f}(x,\mathbf{y})} \text{ with the initial conditions: } \mathbf{y}(x_0) = \mathbf{y}_0 \quad (1.1)$$

**Existence and Uniqueness of Solutions:** The continuity of  $\mathbf{f}(x, \mathbf{y})$  ensures existence. However, it does not guarantee uniqueness. For that, we require the Lipschitz condition.

**Definition 1.0.1** (Lipschitz Condition). *For a fixed  $x$  The function  $\mathbf{f}(x, \mathbf{y})$  is said to satisfy a Lipschitz condition in  $\Omega \subseteq \mathbb{C} \times \mathbb{C}^n$  if there exists a real-valued constant  $L \geq 0$  such that:*

$$|\mathbf{f}(x, \mathbf{y}_2) - \mathbf{f}(x, \mathbf{y}_1)| \leq L|\mathbf{y}_2 - \mathbf{y}_1| \quad \text{when } (x, \mathbf{y}_1), (x, \mathbf{y}_2) \in \Omega.$$

**Remark 1.0.1.** *The Lipschitz condition puts a "cap" on how much  $\mathbf{f}(x, \mathbf{y})$  can "stretch" the points  $\mathbf{y}_1$  and  $\mathbf{y}_2$  from one another.*

**Theorem 1.** Assume that  $\mathbf{f}$  is a continuous function and satisfies a Lipschitz condition in a strip  $\{(x, \mathbf{y}) \in \mathbb{C} \times \mathbb{C}^n; |x - x_0| \leq a\} =: I(a)$ . Then the initial value problem

$$\begin{cases} \mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0 \end{cases} \quad (1.2)$$

has a unique solution in the entire strip  $I(a)$ .

### Linear Homogeneous Systems

The following lemma is a direct consequence of theorem 1

**Lemma 1** (Existence and Uniqueness of Solutions). Suppose that the matrix  $\mathbf{A}(x)$  is **continuous** in  $x$  on the (possibly infinite) open interval  $I := (a, b)$ . Then for each  $x_0 \in I$ , and  $\mathbf{y}_0 \in \mathbb{C}^n$ , there exists a unique solution to the system 1.1. Furthermore, the solution is as smooth in  $x$  as are the coefficient matrix  $\mathbf{A}(x)$  and the initial data  $\mathbf{y}_0$ .

The linearity of the system motivates the investigation of a *linear space*  $V$  consisting of **functions**  $\mathbf{y}$  that satisfy 1.1. As with any well-defined linear space, one can examine the linear dependence of its elements, construct a basis, and explore the resulting linear algebraic properties.

**Theorem 2** (Linearly Independent Solutions). Let  $\mathbf{A}(x)$  be a continuous  $n \times n$  matrix on an open interval  $I$ . The solutions of  $\mathbf{y}' = \mathbf{A}(x)\mathbf{y}$  form a linear space  $V$  of dimension  $n$ .

The above theorem follows naturally from the principle of superposition. Recall how one often makes use of matrix functions  $x \mapsto \Psi(x)$ , where each column of the  $n \times n$  matrix  $\Psi(x)$  is a solution to 1.1. When choosing the column vectors, it is most practical to consider  $n$ -linearly independent solutions  $\mathbf{y}_1, \dots, \mathbf{y}_n$  that span the solution space. In such case, we call  $\Psi(x)$  the *fundamental matrix solution (FMS)*. In addition, requiring that the FMS yields the identity matrix  $\mathbf{I}$  at the starting point  $x_0$  gives us the *Principal Fundamental Solution Matrix (Principal FMS)* at  $x_0$ .

$$\Psi(x) := (\mathbf{y}_1, \dots, \mathbf{y}_n)(x) \in \mathbb{C}^{n \times n}, \text{ subject to initial condition } \Psi(x_0) = \mathbf{I}.$$

**Notation:** we define the principal FMS to be  $\Phi(x, x_0) := \Psi(x)\Psi^{-1}(x_0)$

so that the solution would be of the form:  $\mathbf{y}(x) = \Phi(x, x_0)\mathbf{y}_0$

In some applied mathematics literature, particularly in control theory, the principal FMS is called "The State Transition Matrix", since it tracks the transition of the system from the initial starting point  $x_0$ . The state transition matrix provides a systematic way to analyze the stability and controllability of linear systems, determine solutions explicitly, and understand the time evolution of dynamic systems in engineering, physics, and applied mathematics.

**Remark 1.0.2.** It is well known that the dimension of an  $n$ -dimensional linear system can often be reduced to  $n = 1$ . In such cases, our focus shifts to a single column of the fundamental solution matrix  $\Phi$ . Consequently, the solution  $\mathbf{y}$  to 1.1 takes the form:

$$\mathbf{y}(x) = \Phi(x, x_0)\mathbf{y}_0$$

**Theorem 3** (General non-homogeneous case). Consider the non-homogeneous version of 1.1:

$$\frac{d}{dx}\mathbf{y} = \mathbf{A}(x)\mathbf{y} + \mathbf{f}(x) \text{ with the initial conditions: } \mathbf{y}(x_0) = \mathbf{y}_0$$

The above equation has the unique solution of the form:

$$\mathbf{y}(x) = \Phi(x, x_0)\mathbf{y}_0 + \int_{x_0}^x \Phi(x, x_0)\Phi(\tau, x_0)^{-1}\mathbf{f}(\tau, \mathbf{y}(\tau))d\tau$$

### 25.03 Needs Fixing

An explicit form of solutions depends heavily on algebraic properties of the matrix  $\mathbf{A}$ . We recall a well-known result surrounding this problem.

**Theorem 4.** Let  $A$  be a linear mapping on a vector space  $\mathcal{V}$  over  $\mathbb{C}$ . Then there are subspaces  $\mathcal{V}_-, \mathcal{V}_0, \mathcal{V}_+$  which are invariant under  $A$  and such that:

- a)  $\mathcal{V} = \mathcal{V}_- \oplus \mathcal{V}_0 \oplus \mathcal{V}_+$ .
- b) all eigenvalues of  $A: \mathcal{V}_- \rightarrow \mathcal{V}_-$  have a negative real part.
- c) all eigenvalues of  $A: \mathcal{V}_0 \rightarrow \mathcal{V}_0$  have the real part equal to zero.
- d) all eigenvalues of  $A: \mathcal{V}_+ \rightarrow \mathcal{V}_+$  have a positive real part.

This classification becomes of great use when studying spectral stability of dynamical systems.

## 1.1 Elements of Functional Analysis

### Banach and Sobolev Spaces

Recall that every complete normed vector space constitutes a Banach space. Throughout this dissertation, we will extensively use and apply a number of already established results and theorems on Banach spaces. In particular, the focus would initially be on the basic Sobolev spaces, which combine properties of the  $L^p$  norm of a function and its (weak) derivatives up to a given order. One could formulate such notions in the following sense:

**Definition 1.1.1** (Weak Derivative). *Let  $u \in L^1_{loc}(\Omega)$ , where  $\Omega \subseteq \mathbb{R}^n$  is an open set. A function  $v \in L^1_{loc}(\Omega)$  is called the weak derivative of  $u$  with respect to  $x_i$  (denoted  $\frac{\partial u}{\partial x_i} = v$ ) if*

$$\int_{\Omega} u(x) \frac{\partial \varphi(x)}{\partial x_i} dx = - \int_{\Omega} v(x) \varphi(x) dx$$

for all test functions  $\varphi \in C_c^\infty(\Omega)$ , where  $C_c^\infty(\Omega)$  is the space of infinitely differentiable functions with compact support in  $\Omega$ .

**Definition 1.1.2** (The  $L^p$  Space). *Let  $u : \mathbb{R} \rightarrow \mathbb{C}$  be Lebesgue measurable function. For  $p \geq 1$ , the  $L^p$  space over  $\mathbb{R}$  is defined as:*

$$L^p(\mathbb{R}) = \left\{ u : \mathbb{R} \rightarrow \mathbb{C} \mid \int_{\mathbb{R}} |u(x)|^p dx < \infty \right\}$$

With its associated norm being:

$$\|u\|_{L^p} = \|u\|_p = \left( \int_{\mathbb{R}} |u(x)|^p dx \right)^{1/p}.$$

**Remark 1.1.1** (Special case:  $p = 2$ ). *Recall that  $L^2(\mathbb{R})$  is equipped with the following inner product:*

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} dx.$$

**Definition 1.1.3** (The  $W^{k,p}$  Norm). *For  $k$ -times weakly differentiable functions  $u : \mathbb{R} \rightarrow \mathbb{C}$  and  $p \geq 1$ , the  $W^{k,p}$  norm is defined as:*

$$\|u\|_{W^{k,p}(\mathbb{R})} = \left( \sum_{j=0}^k \left\| \frac{\partial^j}{\partial x^j} u \right\|_{L^p(\mathbb{R})}^p \right)^{1/p}$$

**Remark 1.1.2.** *Unlike the  $L^p$  norm, which only accounts for the size of a function, the  $W^{k,p}$  norm provides us with information on derivatives.*

The above norm is associated with the Sobolev space  $W^{k,p}(\mathbb{R})$ , consisting of functions  $u : \mathbb{R} \rightarrow \mathbb{C}$  with a finite  $W^{k,p}$  norm,

$$W^{k,p}(\mathbb{R}) := \{u : \|u\|_{W^{k,p}} < \infty\}.$$

Since the ultimate goal is to apply this theory to a practical setting, we fix  $p = 2$ , and  $k = 0$ , reducing  $W^{k,p}(\mathbb{R})$  to the Hilbert space of square-integrable functions  $L^2(\mathbb{R})$ , and thereby avoiding unnecessary abstraction:

$$H^k := W^{k,2} \text{ and } H^0(\mathbb{R}) = L^2(\mathbb{R}).$$

### Bounded and Closed Operators

**Definition 1.1.4** (Dense Subset). *Let  $X$  be a Banach space. A subset  $Y \subset X$  is said to be dense in  $X$  if for every  $x \in X$ , and for every  $\epsilon > 0$ , there exists  $y \in Y$ , such that  $\|x - y\| < \epsilon$*

*In more general terms, denseness in a Banach space means that one can approximate any element of the Banach space by elements of a dense subset of that space.*

**Remark 1.1.3.** *It is useful to think about a differential operator  $\mathcal{L}$  as an unbounded linear operator on a Banach space  $X$ , which is only defined on a dense subspace  $D(\mathcal{L})$  of  $X$ . We call such an operator **densely defined**. Throughout the text, we assume that  $\mathcal{L}$  is indeed densely defined.*

**Definition 1.1.5** (Closed Operator). *We say that  $\mathcal{L}$  is **closed** if a sequence  $\{u_j\} \subset D(\mathcal{L})$  converges (in norm of  $X$ ) to some  $u$  and if the sequence  $\{\mathcal{L}u_j\}$  converges to some  $v$ , then it follows that  $u \in D(\mathcal{L})$  and  $\mathcal{L}(u) = v$ .*

**Definition 1.1.6** (Bounded Operator). *The operator  $\mathcal{L} : Y \mapsto X$  is bounded from  $Y$  to  $X$  if the norm  $\|\mathcal{L}u\|_X$  is bounded over the unit sphere:*

$$\sup\{\|\mathcal{L}u\|_X : u \in Y, \|u\|_Y = 1\} < \infty$$

**Remark 1.1.4.** *Since operators act on elements and transform them, boundedness of  $\mathcal{L}$  refers to a situation where the set of the transformed functions  $\mathcal{L}v$  (the image) is bounded. In other words, there is a limit to the "strength" of  $\mathcal{L}$  when it comes to transforming the functions. This interpretation only applies over bounded sets, allowing us to formulate the above definition by stating that a bounded operator maps bounded sets to bounded sets.*

**Definition 1.1.7.** *We denote the **space of bounded linear operators** from  $Y$  into  $X$  by  $\mathcal{B}(Y, X)$ . As for notation, if  $Y = X$ , we simply write  $\mathcal{B}(X)$ . Also note that  $\mathcal{B}(Y, X)$  together with the following norm constitutes a Banach space over the unit sphere  $\|u\|_Y = 1$*

$$\|\mathcal{L}\|_{\mathcal{B}(X, Y)} := \sup_{\|u\|_Y=1} \|\mathcal{L}u\|_X$$

**Definition 1.1.8** (Compactness of  $\mathcal{L}$ ). *If for each bounded sequence  $\{u_j\} \subset Y$  the sequence  $\{\mathcal{L}u_j\} \subset X$  has a convergent subsequence, then the operator  $\mathcal{L}$  is said to be compact.*

*A compact operator is bounded. Furthermore, the sum of two compact operators is compact, and the composition of a compact operator and a bounded operator is compact.*

### Stability Through Spectral Analysis

Let  $\mathcal{L}: D(\mathcal{L}) \subset X \rightarrow X$  be a closed and densely defined linear operator on a Banach space  $X$ .

Recall that the spectrum of  $\mathcal{L}$  is a generalization of eigenvalues for operators on infinite-dimensional spaces. Formally, we have that:

**Definition 1.1.9** (Spectrum). *The spectrum of a linear operator  $\mathcal{L}$  is defined as:*

$$\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid (\mathcal{L} - \lambda\mathcal{I}) \text{ does not have a bounded inverse}\}.$$

An introductory understanding of how spectrum can help with stability analysis lies in the question of "where" in the spectrum the value  $\lambda$  lies. We have the following "regions":

**Stable Spectrum:** The Stable Spectrum consists of those values in the spectrum whose real parts are strictly negative. These correspond to modes that decay exponentially over time. In mathematical terms:

$$\sigma_{\text{stable}}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \Re(\lambda) < 0\}$$

**Center Spectrum:** The Center Spectrum consists of values in the spectrum lying on the imaginary axis, corresponding to neutral or oscillatory dynamics. This set can be written as:

$$\sigma_{\text{center}}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \Re(\lambda) = 0\}$$

**Unstable Spectrum:** The Unstable Spectrum consists of values in the spectrum whose real parts are strictly positive, corresponding to modes that grow exponentially over time. It is expressed as:

$$\sigma_{\text{unstable}}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \Re(\lambda) > 0\}$$

**Definition 1.1.10** (Resolvent Set). *The resolvent set of  $\mathcal{L}$ , denoted by  $\rho(\mathcal{L})$ , is the set of all complex numbers  $\lambda \in \mathbb{C}$  for which the operator  $(\mathcal{L} - \lambda\mathcal{I})$  is invertible and the inverse  $(\mathcal{L} - \lambda\mathcal{I})^{-1}$  is a bounded operator on  $X$ :*

$$\rho(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid (\mathcal{L} - \lambda\mathcal{I}) \text{ is invertible (bijective)} \text{ and } (\mathcal{L} - \lambda\mathcal{I})^{-1} \in \mathcal{B}(X)\}.$$

**Definition 1.1.11** (Resolvent of  $\mathcal{L}$ ). *The inverse operator  $\|(\mathcal{L} - \lambda\mathcal{I})^{-1}\|$  is called the resolvent of  $\mathcal{L}$ .*

**Definition 1.1.12** (Adjoint Operator). *Given a linear operator  $\mathcal{L} : X \rightarrow Y$  between two Hilbert spaces  $X$  and  $Y$ , the adjoint operator  $\mathcal{L}^* : Y \rightarrow X$  is defined by the property:*

$$\langle \mathcal{L}x, y \rangle_Y = \langle x, \mathcal{L}^*y \rangle_X$$

for all  $x \in X$  and  $y \in Y$ . Here,  $\langle \cdot, \cdot \rangle_X$  and  $\langle \cdot, \cdot \rangle_Y$  are the inner products in the spaces  $X$  and  $Y$ , respectively.

**Definition 1.1.13** (Fredholm Operator). *A densely defined, closed operator  $\mathcal{L}: D(\mathcal{L}) \subset X \rightarrow X$  between Banach spaces is called a **Fredholm operator** if:*

1. *The kernel of  $\mathcal{L}$ ,  $\ker(\mathcal{L})$ , is finite-dimensional.*
2. *The range  $\mathcal{R}(\mathcal{L})$  is closed and has a finite codimension in  $X$ .*

*The above criteria can be encoded in the Fredholm index, defined as:*

$$\text{ind}(\mathcal{L}) = \dim[\ker(\mathcal{L})] - \text{codim}[\mathcal{R}(\mathcal{L})].$$

*This leads to the statement: An operator is Fredholm if and only if it has a finite Fredholm index.*

At the beginning of this section, we portrayed the elements of the spectrum  $\sigma(\mathcal{L})$  as complex numbers  $\lambda$  for which  $(\mathcal{L} - \lambda I)$  is **not** invertible. In a finite-dimensional setting, this simply implies the existence of a non-trivial kernel. However, once we extend this concept to infinite-dimensional spaces, there are multiple situations where invertibility fails due to different reasons, and hence the elements  $\lambda$  of the spectrum would naturally partition the spectrum into subsets, each having implications for the stability analysis. At this stage of the discussion, we dichotomize the spectra into two categories:

1. The Point Spectrum  $\sigma_p$
2. The Essential Spectrum  $\sigma_{\text{ess}}$

**Definition 1.1.14** (The Point Spectrum). *The Point Spectrum consists of those values  $\lambda$  in the spectrum of  $\mathcal{L}$  for which  $\mathcal{L} - \lambda I$  is not injective. The point spectrum is the set of eigenvalues of the operator, and it can be written as:*

$$\sigma_p(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \mathcal{L} - \lambda I \text{ is not injective and } \lambda \text{ is an eigenvalue}\}$$

*Indeed, this is the most familiar type of spectrum, corresponding to eigenvalues in finite-dimensional spaces.*

**Definition 1.1.15** (Essential Spectrum). *The essential spectrum consists of those  $\lambda \in \mathbb{C}$  such that  $\mathcal{L} - \lambda I$  is either **not** Fredholm, or is Fredholm with a non-zero Fredholm index.*

## Floquet Theory

**Motivating Question:** Given a linear system of ordinary differential equations with periodic variable coefficients  $A(t)$ , will it follow from periodicity of  $A$  that the solution  $\mathbf{x}(t)$  is also periodic? A Naive guess would simply suggest "yes". This, however, is generally not the case. We therefore seek methods to understand and analyze the solutions to periodic linear systems of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \quad \mathbf{A}(t+T) = \mathbf{A}(t) \tag{1.3}$$

### Scalar Systems:

In the case of constant matrices, it is well-known that the solutions are conveniently of the form  $\mathbf{x}(t) = \mathbf{x}_0 \exp(t\mathbf{A})$ . In fact, if we consider all solutions, namely the Fundamental Matrix Solution (FMS)  $\Phi(t)$ , we can express  $\Phi$  as an exponential of the matrix  $\mathbf{A}$ .

When it comes to variable matrices  $\mathbf{A}(t)$ , while existence and uniqueness theorems ensure the existence of the FMS, it soon becomes obvious that  $\Phi$  is generally *not* the exponential of any matrix. In the case of **periodic** matrices, we have an additional structure imposed on our solutions, making it possible to come up with an investigation of asymptotic behavior of solutions. This is the core idea of **Floquet Theory**, which exploits the mathematical properties of periodic matrices, and seeks to decompose  $\Phi$  to a periodic part, and an exponential of a **constant** matrix.

Without loss of generality, let us assume that the period is of length  $\pi$ . We begin by considering a scalar problem

$$\dot{x}(t) = a(t)x(t), \quad a(t + \pi) = a(t) \quad (1.4)$$

We have previously seen that a general solution  $\Phi(t)$  can be written in the form:

$$\Phi(t) = \exp\left(\int_0^t a(s) ds\right)$$

Define an average and a (net) deviation from the average

$$\bar{a} := \frac{1}{\pi} \int_0^\pi a(s) ds, \quad p(t) = \int_0^t (a(s) - \bar{a}) ds. \quad (1.5)$$

Since  $a(t)$  is periodic, the net deviation  $p(t)$  from its mean is also periodic, since the behavior repeats itself in every cycle. Consider the exponential  $P(t) = e^{p(t)}$ . Together combined, we see that

$$\Phi(t) = P(t)e^{\bar{a}t}$$

### Floquet Decomposition

A central result of Floquet theory is that the FMS  $\Phi(x)$  can be factored into the product of a periodic matrix and an exponential term, provided that  $A$  is a periodic matrix:

$$\Phi(x) = P(x)e^{Bx}, \quad P(x+T) = P(x),$$

where  $P(x) \in \mathbb{C}^{n \times n}$  is periodic with period  $T$ , and  $B \in \mathbb{C}^{n \times n}$  is a constant matrix, known as the *Floquet matrix*.

The eigenvalues  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  of  $B$  are called the *Floquet exponents* and are related to the Floquet multipliers via

$$\rho_k = e^{\lambda_k T}, \quad k = 1, 2, \dots, n.$$

The decomposition implies that the solutions of the system can be written in the form

$$\mathbf{y}(x) = P(x)e^{Bx}\mathbf{c},$$

where  $\mathbf{c} \in \mathbb{C}^n$  is a constant vector. This representation separates the periodic oscillations from the exponential growth or decay, and essentially provides a canonical form for each fundamental matrix solution. The culmination of these efforts result in Floquet's theorem, which shows that there is a periodic, time-dependent change of coordinates that transforms the original periodic problem into a homogeneous linear system with constant coefficients.

### Stability Analysis via Floquet Multipliers

The Floquet multipliers  $\rho_k$  determine the stability of the system:

If all  $|\rho_k| < 1$ , the corresponding solutions decay exponentially, and the system is stable.

If any  $|\rho_k| > 1$ , the corresponding solutions grow exponentially, and the system is unstable.

If  $|\rho_k| = 1$  for some  $k$ , the system is marginally stable, and the long-term behavior depends on higher-order terms.

The connection between the Floquet multipliers and stability makes Floquet theory a powerful tool for analyzing periodic systems, particularly in the context of stability of traveling waves and other periodic structures.



## Chapter 2

# The Evans Function

### Introduction

Since the late 19th century, numerous approaches have been developed for the stability analysis of nonlinear waves. The modern approach over the last few decades has been mainly based on a natural unification of ideas from functional analysis and dynamical systems. This fruitful effort has resulted in the development of the Evans Function.

Introduced by J. Evans in the 1970s, the Evans function has since become a central concept in the stability analysis of nonlinear waves. In theory, we are discussing a complex-valued analytic function whose zeros correspond to eigenvalues of a linearized operator. It generalizes the Wronskian determinant and is particularly well-suited for boundary-value problems (BVPs) and eigenvalue problems on unbounded domains. The key advantage of the Evans function is its ability to encode spectral information, such as eigenvalue multiplicity and stability, in an analytically tractable form.

After a short introduction to the Evans function for the Sturm-Liouville problem, we will shift our focus on the **periodic** Evans function, which provides us with a rich framework for studying the stability of periodic traveling wave solutions in parabolic systems. Its primary purpose is to identify and characterize the spectrum of the linearized operator about such waves, which is composed entirely of continuous spectrum. That is, the set of all complex numbers  $\lambda \in \mathbb{C}$  for which the operator  $(\mathcal{L} - \lambda I)$  is injective and has dense range, but is not surjective.

This chapter presents the theoretical framework, mathematical definitions, and properties of the Periodic Evans Function based on *Kapitula and Promislow's "spectral and dynamical stability of nonlinear waves"* [reference here] and Gardner's foundational work. [reference here]

### Sturm-Liouville Operators

The most familiar setting to discuss stability analysis of boundary value problems is perhaps the *Sturm-Liouville* operators on a bounded domain with separated boundary conditions. This is a specific class of differential operators which takes the form:

$$\mathcal{L}(p) := \partial_x^2 p + a_1(x) \partial_x p + a_0(x) p,$$

with spatial coefficients  $a_0, a_1 \in \mathcal{C}[-1, 1]$  and real-valued.

The associated BVP is of the form:

$$\mathcal{L}p = \lambda p \text{ for } x \in [-1, 1]$$

with separated boundary conditions,

$$(b_1^-, b_2^-) \begin{pmatrix} p \\ \partial_x p \end{pmatrix}(-1) = 0. \quad (2.1)$$

$$(b_1^+, b_2^+) \begin{pmatrix} p \\ \partial_x p \end{pmatrix}(+1) = 0. \quad (2.2)$$

We introduce boundary vectors

$$\mathbf{b}^{a\pm} := (b_1^\pm, b_2^\pm) \text{ with the norm } \|\mathbf{b}^{a\pm}\| = 1$$

**Remark 2.0.1.** *The above restriction on the norm is a convention to ensure that the boundary conditions depend on the direction of the newly introduced  $\mathbf{b}^{a\pm}$  and not its magnitude.*

### The Construction of The Evans Function

We begin by rewriting the eigenvalue problem 2 as a dynamical system:

$$\text{Let } \mathbf{Y} := \begin{bmatrix} p(x) \\ p'(x) \end{bmatrix} \text{ so that we have } \mathbf{Y}' = \mathbf{A}(x, \lambda) \mathbf{Y}$$

$$\text{where } \mathbf{A}(x, \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - a_0(x) & -a_1(x) \end{pmatrix}$$

Instead of solving explicitly for  $\mathbf{Y}(x)$ , we intend to understand the geometric structure of the solution space.

**Definition 2.0.1** (Boundary Subspaces). *Let  $\mathbf{b}^\pm := (-b_2^\pm, b_1^\pm)^\top$ . The boundary space is defined as:*

$$\mathbb{B}_\pm = \text{span } \{\mathbf{b}^\pm\} = \{\text{solutions } \mathbf{Y}(x, \lambda) \text{ that satisfy the boundary conditions at } x = \pm 1\}$$

From the theory of ordinary differential equations, we have that there exists  $n$ -linearly independent solutions  $\mathbf{Y}_1 \dots \mathbf{Y}_n$  that satisfy the dynamical system. Using the notation discussed in chapter 1 for the principal FMS, we introduce the *Jost* solutions:

$$J^\pm(x, \lambda) = \Psi(x, \lambda)\Psi^{-1}(\pm 1, \lambda)\mathbf{b}^\pm$$

$$J^\pm(\pm, \lambda) = \mathbf{b}^\pm \text{ when the domain of } x \text{ is bounded.}$$

We can therefore write every other solution as a linear combination of  $J^+$  and  $J^-$ .

**Remark 2.0.2** (Detecting Eigenvalues via The Evans Function). *If  $\lambda = \lambda_0$  is an eigenvalue, then the linear independency breaks, and hence the spanned system collapses. That is, at an eigenvalue, the Jost solutions coincide up to a scalar multiple.*

**Definition 2.0.2** (The Evans Matrix). *We consider an  $n \times n$  complex-valued matrix  $\mathbf{E}(\lambda)$  of the spectral parameter  $\lambda$  that has columns  $\mathbf{J}^\pm(0, \lambda)$ :*

$$\mathbf{E}(\lambda) = \begin{bmatrix} | & | \\ \mathbf{J}^- & \mathbf{J}^+ \\ | & | \end{bmatrix}(0, \lambda)$$

We are now finally well-equipped to introduce the Evans function.

**Definition 2.0.3** (The Evans Function). *Taking the determinant of the Evans matrix, we will have the Evans function, which serves as the Wronskian of the system in finite dimensions.*

$$E(\lambda) = \det \mathbf{E}(\lambda)$$

**Example 1** (A simplified Sturm-Liouville problem). *Consider the Sturm-Liouville operator defined on the interval  $[-1, 1]$ :*

$$\mathcal{L}(p) = \partial_x^2 p \text{ with the eigenvalue problem: } \partial_x^2 p = \lambda p, \quad x \in [-1, 1]$$

We impose Dirichlet boundary conditions:

$$p(-1) = 0, \quad p(1) = 0$$

These correspond to boundary vectors:

$$\mathbf{b}^- = (b_1^-, b_2^-) = (1, 0), \quad \mathbf{b}^+ = (b_1^+, b_2^+) = (1, 0)$$

We compute the Evans Function analytically and visualize its magnitude to identify eigenvalues.

The eigenvalue problem will first be rewritten as a first-order system:

$$\mathbf{Y} := \begin{bmatrix} p \\ p' \end{bmatrix} \text{ such that } \mathbf{Y}' = \mathbf{A}(x, \lambda)\mathbf{Y}, \quad \mathbf{A}(x, \lambda) = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}$$

The boundary conditions  $p(\pm 1) = 0$  should also be rewritten in vector form:

$$(b_1^\pm, b_2^\pm) \begin{bmatrix} p \\ p' \end{bmatrix} (\pm 1) = b_1^\pm p(\pm 1) = 0$$

Since  $b_1^\pm = 1$ , we have  $p(\pm 1) = 0$ . The boundary vectors are:

$$\mathbf{b}^\pm = (-b_2^\pm, b_1^\pm)^\top = (0, 1)^\top$$

Hence, the boundary subspaces are:

$$\mathbb{B}_\pm = \text{span}\{\mathbf{b}^\pm\} = \text{span} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These represent solutions satisfying the boundary conditions at  $x = \pm 1$ .

We now shift our focus to computing the fundamental matrix solution.

For  $\lambda = -\mu^2 < 0$ , the second-order equation  $\partial_x^2 p = -\mu^2 p$  has solutions:

$$p(x) = Ae^{\mu x} + Be^{-\mu x}$$

Thus:

$$\mathbf{Y}(x) = \begin{bmatrix} p \\ p' \end{bmatrix} = \begin{bmatrix} Ae^{\mu x} + Be^{-\mu x} \\ A\mu e^{\mu x} - B\mu e^{-\mu x} \end{bmatrix}$$

The fundamental matrix solution  $\Psi(x, \lambda)$  is:

$$\Psi(x, \lambda) = \begin{bmatrix} e^{\mu x} & e^{-\mu x} \\ \mu e^{\mu x} & -\mu e^{-\mu x} \end{bmatrix}$$

Evaluate at the boundaries:

$$\Psi(\pm 1, \lambda) = \begin{bmatrix} e^{\pm\mu} & e^{\mp\mu} \\ \mu e^{\pm\mu} & -\mu e^{\mp\mu} \end{bmatrix}$$

Compute  $\Psi^{-1}(\pm 1, \lambda)$ :

$$\det \Psi(\pm 1, \lambda) = e^{\pm\mu}(-\mu e^{\mp\mu}) - e^{\mp\mu}(\mu e^{\pm\mu}) = -2\mu$$

$$\Psi^{-1}(\pm 1, \lambda) = \frac{1}{-2\mu} \begin{bmatrix} -\mu e^{\mp\mu} & -e^{\mp\mu} \\ -\mu e^{\pm\mu} & e^{\pm\mu} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{\mp\mu} & \frac{1}{2\mu}e^{\mp\mu} \\ \frac{1}{2}e^{\pm\mu} & -\frac{1}{2\mu}e^{\pm\mu} \end{bmatrix}$$

The Jost solutions are:

$$J^\pm(x, \lambda) = \Psi(x, \lambda)\Psi^{-1}(\pm 1, \lambda)\mathbf{b}^\pm$$

Since  $\mathbf{b}^\pm = (0, 1)^\top$ , take the second column of  $\Psi^{-1}(\pm 1, \lambda)$ :

$$\Psi^{-1}(-1, \lambda)\mathbf{b}^- = \begin{bmatrix} \frac{1}{2\mu}e^{\mu} \\ -\frac{1}{2\mu}e^{-\mu} \end{bmatrix}, \quad \Psi^{-1}(1, \lambda)\mathbf{b}^+ = \begin{bmatrix} \frac{1}{2\mu}e^{-\mu} \\ -\frac{1}{2\mu}e^{\mu} \end{bmatrix}$$

$$J^-(x, \lambda) = \begin{bmatrix} e^{\mu x} & e^{-\mu x} \\ \mu e^{\mu x} & -\mu e^{-\mu x} \end{bmatrix} \begin{bmatrix} \frac{1}{2\mu} e^\mu \\ -\frac{1}{2\mu} e^{-\mu} \end{bmatrix} = \begin{bmatrix} \frac{1}{\mu} \sinh \mu \\ \cosh \mu \end{bmatrix} \text{ at } x = 0$$

$$J^+(x, \lambda) = \begin{bmatrix} e^{\mu x} & e^{-\mu x} \\ \mu e^{\mu x} & -\mu e^{-\mu x} \end{bmatrix} \begin{bmatrix} \frac{1}{2\mu} e^{-\mu} \\ -\frac{1}{2\mu} e^\mu \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mu} \sinh \mu \\ \cosh \mu \end{bmatrix} \text{ at } x = 0$$

The Evans Matrix at  $x = 0$  is:

$$\mathbf{E}(\lambda) = [J^-(0, \lambda) \quad J^+(0, \lambda)] = \begin{bmatrix} \frac{1}{\mu} \sinh \mu & -\frac{1}{\mu} \sinh \mu \\ \cosh \mu & \cosh \mu \end{bmatrix}$$

The Evans Function is the determinant:

$$E(\lambda) = \det \mathbf{E}(\lambda) = \left( \frac{1}{\mu} \sinh \mu \right) \cosh \mu - \left( -\frac{1}{\mu} \sinh \mu \right) \cosh \mu = \frac{2}{\mu} \sinh \mu \cosh \mu = \frac{1}{\mu} \sinh 2\mu$$

Since  $\mu = \sqrt{-\lambda}$ , we have:

$$E(\lambda) = \frac{1}{\sqrt{-\lambda}} \sinh(2\sqrt{-\lambda})$$

For real  $\lambda > 0$ , let  $\sqrt{-\lambda} = i\sqrt{\lambda}$ , so:

$$E(\lambda) = \frac{1}{i\sqrt{\lambda}} \sinh(2i\sqrt{\lambda}) = \frac{1}{\sqrt{\lambda}} \sin(2\sqrt{\lambda})$$

The magnitude is:

$$|E(\lambda)| = \left| \frac{1}{\sqrt{\lambda}} \sin(2\sqrt{\lambda}) \right|$$

The zeros of  $E(\lambda)$  occur when:

$$\sinh(2\sqrt{-\lambda}) = 0 \implies 2\sqrt{-\lambda} = in\pi \implies \lambda = \frac{n^2\pi^2}{4}, \quad n = 1, 2, 3, \dots$$

These are the eigenvalues of  $\mathcal{L}$  under Dirichlet boundary conditions, corresponding to:

$$\lambda_1 = \frac{\pi^2}{4} \approx 2.467, \quad \lambda_2 = \pi^2 \approx 9.869, \quad \lambda_3 = \frac{9\pi^2}{4} \approx 22.206$$

The magnitude  $|E(\lambda)|$  is plotted for  $\lambda \in (0, 40]$  to observe its zeros, which indicate the eigenvalues. The plot, shown in Figure 2.1, is generated using numerical computation of  $|E(\lambda)| = \left| \frac{1}{\sqrt{\lambda}} \sin(2\sqrt{\lambda}) \right|$ .

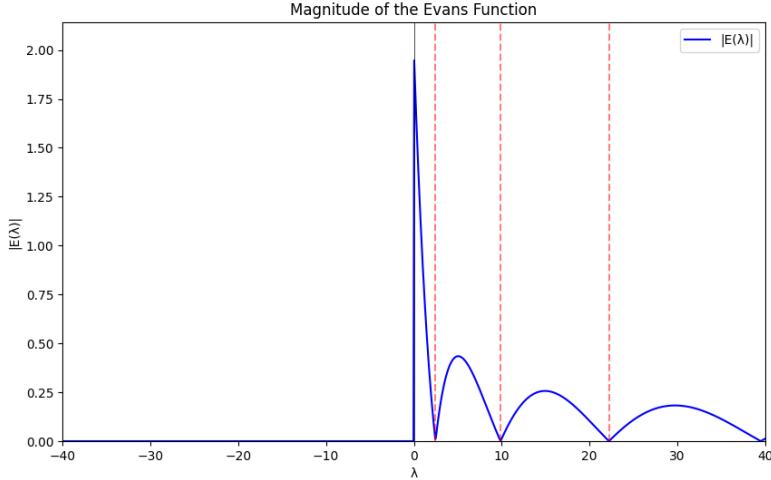


Figure 2.1: Magnitude of the Evans Function  $|E(\lambda)|$  for  $\lambda \in (0, 40]$ . The plot shows zeros at  $\lambda = \frac{n^2\pi^2}{4}$  (marked by red dashed lines), corresponding to the eigenvalues of the Sturm-Liouville operator with Dirichlet boundary conditions.

Note that  $\sin(2\sqrt{\lambda})$  causes oscillations, modulated by the  $\frac{1}{\sqrt{\lambda}}$  factor, which affects the amplitude.

### The Periodic Evans Function

We will now be primarily concerned with the stability analysis of periodic systems. A class of solutions to higher order dispersive equations (such as the Korteweg-de Vries equations) comprise of periodic travelling waves. The motivating idea is to see how one can extend the notion of the roots of the Evans function developed in the previous section, and show that they coincide with the associated eigenvalues of the linearized operator  $\mathcal{L}$ . In fact, one can consider the prototypical differential operator (not necessarily of order 2) that is obtained as a result of linearization about a heteroclinic (or homoclinic) equilibria:

$$\mathcal{L}p = \frac{d^n}{dx^n}p + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}p + \dots + a_1(x)p' + a_0(x)p, \text{ for } n \geq 1 \text{ and } x \in \mathbb{R}. \quad (2.3)$$

with smooth, spatially periodic coefficients  $a_j(x+T) = a_j(x)$  for  $j = 0, \dots, n-1$ .

What makes the periodic case different than BVPs with separated boundary conditions is the lack of point spectrum associated with the operator

$\mathcal{L} : H^n(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ . As we did previously with the separated boundary conditions, we rewrite the equation as a dynamical system, reducing it to the form:

$$\mathbf{Y}' = \mathbf{A}(x, \lambda)\mathbf{Y}$$

where  $\mathbf{A}(x + T, \lambda) = \mathbf{A}(x, \lambda)$  is now a periodic matrix of coefficients.

It is now convenient to apply the Floquet's theorem, which ensures the existence of the matrix  $\mathbf{B}(\lambda)$ , and we know that every fundamental matrix solution  $\Phi(x, \lambda)$  can be decomposed in the following sense:

$$\Phi(x, \lambda) = \mathbf{P}(x, \lambda)e^{\mathbf{B}(\lambda)x} \text{ with } \mathbf{P}(x + T, \lambda) = \mathbf{P}(x, \lambda) \text{ being periodic.}$$

While we have developed our machinery based on the point spectrum and eigenvalues, it is of course challenging to work directly with the essential spectrum. Since we are equipped with Floquet theory, one can in fact decompose the essential spectrum into a union of point eigenvalues via a Bloch-wave decomposition that reduces the system to a subproblem on a bounded domain.

$$\text{Let } p = e^{i\mu x}q \text{ for } \mu \in [-1, 1]$$

. The eigenvalue problem can be rewritten as:

$$\mathcal{L}_\mu q := \left( \frac{d}{dx} + i\mu \right)^n q + a_{n-1}(x) \left( \frac{d}{dx} + i\mu \right)^{n-1} q + \dots + a_1(x) \left( \frac{d}{dx} + i\mu \right) q + a_0(x) q \quad (2.4)$$

it can be shown that

$$\lambda \in \sigma_{ess}(\mathcal{L}) \iff \lambda \in \sigma_{pt}(\mathcal{L}_\mu)$$

As discussed in the previous section, the Evans functions (both classical and periodic) are naturally defined for first-order systems. It therefore becomes necessary to reduce the  $n$ -th order eigenvalue problem to a system of first-order ODEs.

$$\mathbf{Y} := (p, p', \dots, p^{n-1})^\top$$

The eigenvalue problem can therefore be reduced to:  $\mathbf{Y}'(x, \lambda) = \mathbf{A}(x, \lambda)\mathbf{Y}$ .

$$\text{where } \mathbf{A}(x, \lambda) = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ \lambda - a_0(x) & -a_1(x) & \dots & \dots - a_{n-2}(x) & -a_{n-1}(x) \end{bmatrix}$$

The dynamical systems perspective would also provide us with the suitable environment to allow analysis using invariant subspaces. For the classical Evans function, boundary conditions are by default assumed to be separated. This is in contrast with the periodic setting, where the reduced system satisfies:

$$\mathbf{Y}(T, \lambda) = e^{i\mu T} \mathbf{Y}(0, \lambda)$$

This phase factor introduces a non-separated boundary condition, which prevents a direct application of the classical Evans function framework.

To handle the periodicity, we decompose solutions into Bloch waves, setting:

$$\mathbf{Y}(x) = e^{i\mu x} \tilde{\mathbf{Y}}(x)$$

Plugging this into the reduced system, we get:

$$\partial_x \tilde{\mathbf{Y}} = (\mathbf{A}(x, \lambda) - i\mu \mathbf{I}_n) \tilde{\mathbf{Y}},$$

satisfying the periodic conditions:  $\tilde{\mathbf{Y}}(0, \lambda) = \tilde{\mathbf{Y}}(T, \lambda)$

We can now write our original  $\mathbf{Y}$  as:

$$\partial_x \mathbf{Y} = \mathbf{A}(x, \mu, \lambda) \mathbf{Y}, \text{ such that } \mathbf{Y}(T, \lambda, \mu) = \mathbf{Y}(0, \lambda, \mu). \quad (2.5)$$

We have thereby successfully transformed the periodic eigenvalue problem into a Bloch-wave problem. Despite the Bloch reduction, the periodicity still introduces difficulties for defining an Evans function because of the lack of separation in the boundary conditions. To address this, we embed the system into a larger space by defining an extended vector:

$$\mathbf{W} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} \text{ such that } \mathbf{Z}' = \mathbf{0} \quad (2.6)$$

where  $\mathbf{Z}$  is an auxiliary variable that allows us to enforce conditions on  $\mathbf{Y}$  at different points. The system is then rewritten as:

$$\partial_x \mathbf{W} = \begin{bmatrix} \mathbf{A}(x, \lambda, \mu) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{W}. \quad (2.7)$$

As a prerequisite to what follows, we first consider the following theorem.

**Theorem 5.** *Let  $\mathbf{W}$  be as in 2.6, and consider the boundary value problem*

$$\mathbf{Y}' = \mathbf{A}(x) \mathbf{Y} \text{ such that } \mathbf{Y}(0) = \mathbf{Y}(T) \quad (2.8)$$

*then  $\mathbf{W}$  solves 2.8 with boundary conditions  $\mathbf{Y}(0) = \mathbf{Z}(0)$  and  $\mathbf{Y}(T) = \mathbf{Z}(T)$  if and only if  $\mathbf{Y}$  solves 2.8 with boundary conditions  $\mathbf{Y}(0) = \mathbf{Y}(T)$*

*Proof.* (See Erik's Handwritten Note) □

This larger system allows us to convert the problem into one with separated boundary conditions in the following sense:

Let  $\mathbf{e}_i = (e_1 \dots e_n)$  be a canonical basis for  $\mathbb{C}^n$ . we set  $\mathbf{b}_l^\pm = (e_l, e_l)^\top$  for  $l = 1, \dots, n$ . and then let us introduce the boundary subspaces for the larger system:

$$\mathbf{W}(0) \in \mathbb{B}_0 := \text{span}\{\mathbf{b}_1^-, \dots, \mathbf{b}_n^-\} \text{ and } \mathbf{W}(T) \in \mathbb{B}_T := \text{span}\{\mathbf{b}_1^+, \dots, \mathbf{b}_n^+\}$$

We now have a dynamical system with separated boundary conditions! This would allow us to apply the already established theory for classical Evans function using Jost solutions.

It is easy to see that the fundamental solution matrix  $\Phi(x, \lambda, \mu)$  for 2.5 satisfies and induces a FMS for 2.7.

$$\Psi(x, \lambda, \mu) = \begin{pmatrix} \Phi(x, \lambda, \mu) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

This observation is necessary since the construction of Jost eigenfunctions  $\mathbf{J}^\pm$  depends heavily on our understanding of the fundamental solution of the system.

$$\text{On one side we have: } \mathbf{J}_j^-(x, \lambda, \mu) = \Psi(x, \lambda, \mu) \Psi^{-1}(0, \lambda, \mu) \mathbf{b}_j^- = \begin{pmatrix} \Phi(x, \lambda, \mu) \Phi^{-1}(0, \lambda, \mu) \mathbf{e}_j \\ \mathbf{e}_j \end{pmatrix}$$

$$\text{And on the other side: } \mathbf{J}_j^+(x, \lambda, \mu) = \Psi(x, \lambda, \mu) \Psi^{-1}(T, \lambda, \mu) \mathbf{b}_j^+ = \begin{pmatrix} \Phi(x, \lambda, \mu) \Phi^{-1}(T, \lambda, \mu) \mathbf{e}_j \\ \mathbf{e}_j \end{pmatrix}$$

**Definition 2.0.4** (The Periodic Evans Function).

$$E(\lambda, \mu) = \det \begin{bmatrix} \mathbf{J}_1^- & \mathbf{J}_2^- & \cdots & \mathbf{J}_n^- & \mathbf{J}_1^+ & \cdots & \mathbf{J}_n^+ \\ | & | & \cdots & | & | & \cdots & | \\ \cdots & \cdots & & \cdots & \cdots & & \cdots \end{bmatrix} \left( \frac{T}{2}, \lambda, \mu \right) \quad (2.9)$$

**Lemma 2** (KP Lemma 8.4.1). *The Evans function as defined in 2.9 is an entire function of  $\lambda$  for fixed  $\mu$  and of  $\mu$  for fixed  $\lambda$ . It has at most a countable number of zeros, with a zero at  $(\lambda, \mu)$  if and only if  $\lambda$  is an eigenvalue of  $\mathcal{L}_\mu$  defined in 2.4. Moreover, the multiplicity of the Evans function is equal to the algebraic multiplicity of  $\mathcal{L}_\mu$  at  $\lambda$ .*

*Proof.* The periodic Evans function is a determinant function, and from complex analysis we know that in order for  $E(\lambda, \mu)$  to be analytic, one must ensure that the matrix entries (i.e., the Jost solutions) are holomorphic in both variables  $\lambda$  and  $\mu$ . As shown previously, the Jost solutions  $\mathbf{J}_j^\pm$  are in essence a product of the fundamental matrix solutions and their inverses. Since the fundamental matrix solution  $\Psi(x, \lambda, \mu)$  satisfies

$$\frac{d}{dx} \Psi(x, \lambda, \mu) = \mathbf{A}(x, \lambda, \mu) \Psi(x, \lambda, \mu) \text{ subject to } \Psi(0, \lambda, \mu) = \mathbf{I}_n$$

one must therefore focus on the matrix  $\mathbf{A}(x, \lambda, \mu)$ . If  $\mathbf{A}$  depends analytically on  $(\lambda, \mu)$ , then we are done.  $\square$

**Lemma 3** (KP Lemma 8.4.2: Alternative Definition). *Let  $\Phi(x, \lambda, \mu)$  be a fundamental matrix solution to 2.5. The Evans function defined by*

$$E(\lambda, \mu) = \det (\Phi(T, \lambda, \mu) \Phi^{-1}(0, \lambda, \mu) - \mathbf{I}_n) \quad (2.10)$$

*shares the same properties stated in the previous lemma for the definition 2.9*

**Remark 2.0.3.** *The first definition 2.9 is constructed by a determinant that vanishes once periodic eigenfunctions exist. Similarly, the definition 2.10 approaches the problem behind a lens of Floquet theory. The alternative method focuses on the eigenvalues of the monodromy matrix  $\Phi(\mathbf{T})\Phi^{-1}(0)$  which describes how the solutions evolve over one period. The determinant defined in the alternative approach checks if the monodromy matrix has 1 as eigenvalue, implying that a solution repeats exactly after period  $T$ , leading to a periodic eigenfunction.*

**Remark 2.0.4** (Analyticity of the alternative case). *Recall that the monodromy matrix itself is a product of fundamental solutions, which are analytic, hence the alternative definition also produces an entire function.*

**Example 2** (A simple periodic case). *Again, Consider the Sturm-Liouville operator on  $\mathbb{R}$ , associated with an eigenvalue problem:*

$$\mathcal{L}p = \partial_x^2 p + \cos(x)p = \lambda p, \quad x \in \mathbb{R}$$

The coefficient  $\cos(x)$  is periodic with period  $T = 2\pi$ . The operator  $\mathcal{L} : H^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  has an essential spectrum, which we analyze using the Bloch-wave decomposition. For a Bloch parameter  $\mu \in [-1, 1]$ , let  $p(x) = e^{i\mu x}q(x)$ , transforming the problem to:

$$\mathcal{L}_\mu q = \left( \frac{d}{dx} + i\mu \right)^2 q + \cos(x)q = \lambda q$$

Expanding:

$$\frac{d^2q}{dx^2} + 2i\mu \frac{dq}{dx} + (\cos(x) - \mu^2)q = \lambda q$$

The function  $q(x)$  satisfies periodic boundary conditions:

$$q(x + 2\pi) = q(x)$$

We aim to compute the Periodic Evans Function  $E(\lambda, \mu)$  using the alternative definition:

$$E(\lambda, \mu) = \det(\Phi(2\pi, \lambda, \mu) - \mathbf{I}_2)$$

where  $\Phi(x, \lambda, \mu)$  is the fundamental matrix solution. Rewrite the eigenvalue problem as a first-order system. Let:

$$\mathbf{Y} = \begin{bmatrix} q \\ q' \end{bmatrix}$$

The system is:

$$\mathbf{Y}' = \mathbf{A}(x, \lambda, \mu)\mathbf{Y}, \quad \mathbf{A}(x, \lambda, \mu) = \begin{bmatrix} 0 & 1 \\ \lambda + \mu^2 - \cos(x) & -2i\mu \end{bmatrix}$$

with periodic boundary conditions:

$$\mathbf{Y}(2\pi, \lambda, \mu) = \mathbf{Y}(0, \lambda, \mu)$$

The matrix  $\mathbf{A}(x, \lambda, \mu)$  is periodic with period  $2\pi$ . By Floquet theory, the fundamental matrix solution  $\Phi(x, \lambda, \mu)$  satisfies:

$$\Phi(x, \lambda, \mu) = \mathbf{P}(x, \lambda, \mu) e^{\mathbf{B}(\lambda, \mu)x}, \quad \mathbf{P}(x + 2\pi, \lambda, \mu) = \mathbf{P}(x, \lambda, \mu)$$

where  $\Phi(0, \lambda, \mu) = \mathbf{I}_2$ . The monodromy matrix is:

$$\Phi(2\pi, \lambda, \mu) = \Phi(2\pi, \lambda, \mu) \Phi^{-1}(0, \lambda, \mu) = \Phi(2\pi, \lambda, \mu)$$

The Periodic Evans Function is:

$$E(\lambda, \mu) = \det(\Phi(2\pi, \lambda, \mu) - \mathbf{I}_2)$$

A zero of  $E(\lambda, \mu)$  indicates that  $\lambda$  is an eigenvalue of  $\mathcal{L}_\mu$ , corresponding to a periodic eigenfunction.

To compute  $E(\lambda, \mu)$ , solve the ODE:

$$\frac{d}{dx} \Phi = \mathbf{A}(x, \lambda, \mu) \Phi, \quad \Phi(0) = \mathbf{I}_2$$

over  $[0, 2\pi]$  to obtain  $\Phi(2\pi, \lambda, \mu)$ . Then:

$$E(\lambda, \mu) = \det(\Phi(2\pi, \lambda, \mu) - \mathbf{I}_2)$$

We evaluate  $|E(\lambda, \mu)|$  for fixed  $\mu = 0.5$  over  $\lambda \in [-10, 10]$ .

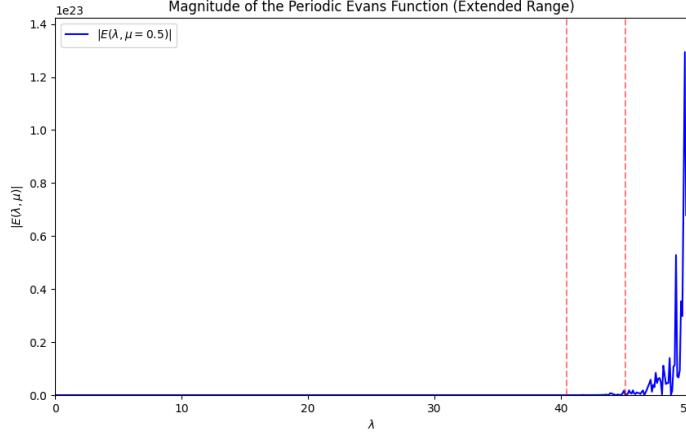


Figure 2.2: Magnitude of the Periodic Evans Function  $|E(\lambda, \mu = 0.5)|$  for  $\lambda \in [-5, 5]$ . Zeros indicate eigenvalues of  $\mathcal{L}_\mu$  for  $\mu = 0.5$ , part of the essential spectrum of the periodic Sturm-Liouville operator.

Red dashed lines mark the zeros of  $|E(\lambda, \mu = 0.5)|$ , where the function dips below a small threshold, indicating eigenvalues of  $\mathcal{L}_\mu$ . These correspond to the essential spectrum bands of  $\mathcal{L}$

### Computations with Periodic Evans Function

A topological approach is often considered in cases where directly locating eigenvalues proves to be difficult. In principle, we will be using a winding number as a "device" to locate eigenvalues of the operator and analyze spectral properties.

**Theorem 6** (KP Theorem 8.4.3). *Consider the operator  $\mathcal{L}$  as defined in 2.3. Let  $C \subset \mathbb{C}$  be a positively oriented simple closed curve. The curve is assumed not to intersect  $\sigma(\mathcal{L})$ . The winding number*

$$W(\mu) = \frac{1}{2\pi i} \oint_C \frac{\partial_\lambda E(\lambda, \mu)}{E(\lambda, \mu)} d\lambda$$

*is constant for  $\mu \in (-1, 1]$ . Moreover, if  $W(0) = 1$ , then the spectra inside of  $C$  forms a smooth, closed curve.*

**Remark 2.0.5** (Intuition and Further Remarks on the Winding Number). *We are essentially investigating how the eigenvalues of the operator  $\mathcal{L}$  change with respect to the parameter  $\mu$ . While observing the problem from a dynamical systems perspective, we develop a topological tool using a contour integral. Since zeros of the periodic Evans function  $E(\lambda, \mu)$  correspond to eigenvalues of the linearized operator, we consider the encirclement of zero in the complex plane to compute how many eigenvalues are enclosed by the curve  $C$ . The winding number turns out to be constant for specific values of  $\mu$ . That is, if  $\mu \in (-1, 1]$ , then the constant winding number suggests that the enclosed eigenvalues are entirely within  $C$  and follow a continuous, smooth path as  $\mu$  varies. This is particularly useful, since it means that the information is predictable, and hence no eigenvalues suddenly appear or disappear in this range.*

**Example 3.** *To demonstrate the above remarks in a simplified numerical setting, we begin with an easy choice for our model. Let*

$$E(\lambda, \mu) = \det \begin{bmatrix} \lambda & \mu + i \\ \mu + i & \lambda \end{bmatrix} = \lambda^2 - (\mu + i)^2$$

*. Having zeros at  $\lambda = \pm(\mu + i)$ , we allow  $\mu$  to vary from  $-1$  to  $1$ , thereby observe the following values for the eigenvalues in the complex plane:*

$$\mu = -1 \implies \lambda = -1 + i \text{ and } 1 - i$$

$$\mu = 0 \implies \lambda = i \text{ and } -i$$

$$\mu = 1 \implies \lambda = 1 + i \text{ and } -1 - i$$

*We expect that the generated graph would form a closed curve, following the predictions of the theorem when  $W(0) = 1$*

*A numerical computation of the winding number takes the form:*

```

1 # Define the periodic Evans function E(lambda, mu)
2 def E(lam, mu):
3     # Eigenvalues form a circle centered at i
4     return lam**2 - (mu + 1j)**2
5
6 # Derivative of E with respect to lambda
7 def dE_dlam(lam, mu):
8     return 2 * lam
9
10 # Function to compute the integrand for the winding number
11 def winding_integrand(lam, mu):
12     return dE_dlam(lam, mu) / E(lam, mu)

```

Listing 2.1: Step 1: Defining the Evans Function

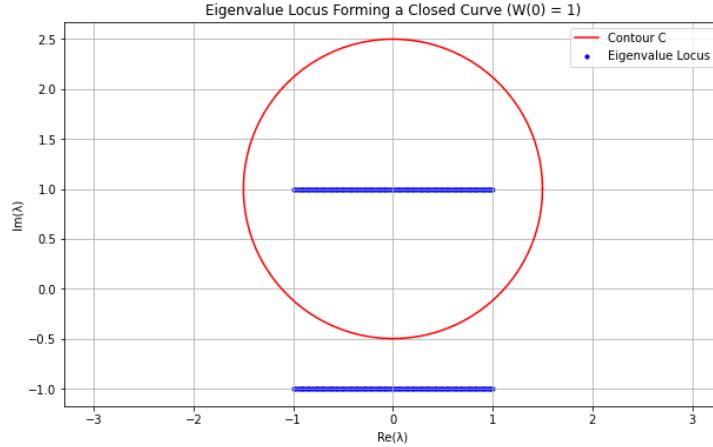


Figure 2.3: The circle encloses the eigenvalue locus: By plotting both roots, we show the full locus of the eigenvalues. Having  $\partial_\lambda E = 2\lambda$ , we parametrize the contour  $C$  as a circle centered at  $(0, 1)$  with radius 1.5. This ensures the contour encloses the eigenvalue circle without intersecting it.  $\lambda = re^{it} = 1.5e^{it}$  for  $t \in [0, 2\pi]$ .

```

1 # Compute the winding number W(mu) over a contour C
2 def compute_winding_number(mu, contour_center=(0, 1),
3     radius=1.5):
4     def real_integrand(t):
5         lam = contour_center[0] + radius * np.cos(t) + 1
6             j * (contour_center[1] + radius * np.sin(t))
7         integrand = winding_integrand(lam, mu)
8         return np.real(integrand * (-radius * np.sin(t)
9             + 1j * radius * np.cos(t)))

```

```

7      def imag_integrand(t):
8          lam = contour_center[0] + radius * np.cos(t) + 1
9              j * (contour_center[1] + radius * np.sin(t))
10             integrand = winding_integrand(lam, mu)
11             return np.imag(integrand * (-radius * np.sin(t)
12                             + 1j * radius * np.cos(t)))
13
14     # Integrate over t from 0 to 2pi
15     real_part, _ = quad(real_integrand, 0, 2 * np.pi)
16     imag_part, _ = quad(imag_integrand, 0, 2 * np.pi)
17     W = (real_part + 1j * imag_part) / (2 * np.pi * 1j)
18     return np.round(np.real(W))  # Should be integer (
19         winding number)

```

Listing 2.2: Step 2: Computing Winding Number

**Remark 2.0.6** (Implications of  $W(0) = 1$ ). *Spectral curves appear once the initial winding number  $W(0) = 1$ , which implies that the distribution of the eigenvalues will not manifest itself as chaotic, but it will rather evolve predictably with  $\mu$  leading to a closed spectral curve.*

A useful extension to the already-established results is presented by the following theorem, which claims that once we expand the periodicity of the problem, the behavior in the structure of eigenvalues still remains predictable.

**Theorem 7** (KP Corollary 8.4.4). *Consider the original eigenvalue problem 2.3 together with periodic boundary conditions on  $[0, \pi]$ . As before, let the curve  $C \subset \mathbb{C}$  be positively oriented and simple closed that does not intersect the spectrum  $\sigma(\mathcal{L})$ . If  $W(0) = m$  for  $\pi$ -periodic Bloch-wave problem, then for each  $\frac{-1}{k} < \mu < \frac{1}{k}$  the corresponding  $k\pi$ -periodic Bloch-wave problem satisfies  $W(\mu) = km$ . Insert Graph Here*

**Remark 2.0.7.** *The above theorem essentially extends the results of the preceding discussion on the winding number to the Bloch-wave spectral problem. The interesting result is the fact that the winding number scales itself accordingly as the periodicity of the problem evolves to being  $k\pi$ -periodic from its original  $\pi$ -periodic form. Having  $W(0) = m$  implies that the Evans function  $E(\lambda, 0)$  encircles the origin exactly  $m$  times along the contour  $C$ . As an intuitive suggestion, we are essentially stating that the number of eigenvalues (including algebraic multiplicities) enclosed by the contour is indeed proportional to  $m$ .*

**Remark 2.0.8** (Increase of the periodic domain). *Transitioning to a  $k\pi$  periodic problem in Bloch-wave form increases the periodic domain length by a factor of  $k$ . Particularly of interest is the fact that the eigenvalues of  $\mathcal{L}$  are thereby rescaled accordingly. The claim that  $W(\mu) = kW(0)$  implies that the spectral structure of the  $k\pi$  periodic problem is akin to having exactly  $k$  copies of the original spectrum, each repeating the same topological winding.*

**Remark 2.0.9.** *Since the winding number tracks the number of eigenvalues enclosed by the contour  $C$ , we arrive at the following conclusion:*

*Increasing periodicity  $\implies$  increase in eigenvalues*

In principle, we have developed a scaling law for spectral topology in Bloch-wave problems.



## Chapter 3

# The Generalized Korteweg-de Vries Equation

### 3.1 Historical Background

What began as a curious observation in 1844 by the Scottish engineer John Scott Russel led to the derivation of a significant partial differential equation applicable to a wide range of physical systems. Russel's empirical relation between the speed of a *solitary* wave, gravitational acceleration, depth, and the maximum amplitude of the wave above the surface was motivated by observing a traveling boat, and the *bow wave* of water preserving its shape and speed for a long time after being detached from the boat's bow. This single humped wave of water was later called a *solitary* wave (or simply a *soliton*).

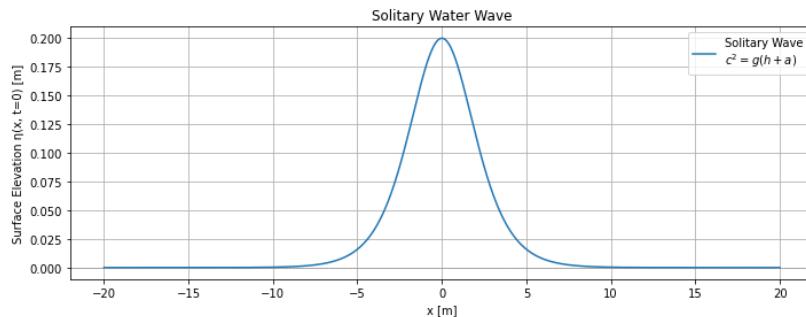


Figure 3.1: Solitary water wave at a fixed time. The wave has a permanent shape described by  $\eta(x, t) = a \operatorname{sech}^2(\kappa(x - ct))$ , where  $a$  is the amplitude and  $c$  is the wave speed. The speed is determined by John Scott Russell's empirical relation  $c^2 = g(h + a)$ , relating the speed to the undisturbed depth  $h$  and the amplitude  $a$ . The graph visualizes the localized, non-dispersive nature of solitary waves, which maintain shape and travel at constant speed.

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Half a decade later, in 1895 mathematicians Diederik Korteweg and Gustav de Vries analytically derived and rediscovered a PDE that was formerly introduced by Joseph Valentin Boussinesq. The now called "Korteweg-de Vries" equation models the disturbance of the surface of the shallow water in the presence of solitary water waves.

$$u_t + \underbrace{uu_x}_{\text{nonlinearity}} + \underbrace{u_{xxx}}_{\text{linear dispersion}} = 0. \quad (3.1)$$

The Nonlinearity  $uu_x$  and linear dispersion  $u_{xxx}$  tend to compete in their physical effects on the wave. More precisely, the nonlinear term  $uu_x$  shifts the wave based on its own amplitude  $u(x)$ , causing the larger parts of the wave to move faster than the smaller parts. As a result, the wave steepens, pushing the wave profile into a narrower, taller, and more localized shape. This particular process is portrayed in Burgers' equation, where one lacks the dispersion term and observes discontinuous shocks over time:

$$u_t + uu_x = 0.$$

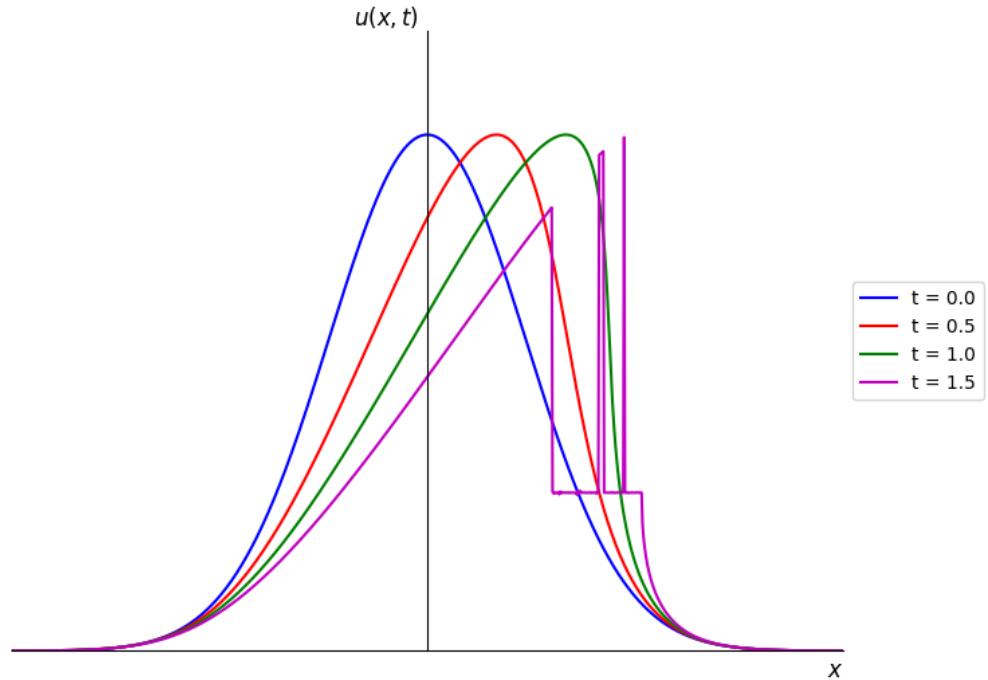


Figure 3.2: The wave front steepens over time due to the effects of nonlinearity, eventually leading to discontinuity.

The dispersion term  $u_{xxx}$  tends to neutralize the extreme effects of the increased slope, as it spreads out the wave profile over time. The KdV equation provides an elegant balance between steepening and spreading the wave, leading to a travelling wave of permanent shape, what can loosely be called a *soliton*. That is, a localized, highly stable wave that retains its **identity** (shape and speed).

Although this problem was mainly motivated by *water* waves, the range of its applications reaches far beyond hydrodynamics, and it has since adjusted and evolved into many forms. In this thesis, we will be mainly discussing the generalized Korteweg-de Vries equation (gKdV), and investigate its spectral stability using the Evans function. The study builds on the foundational work of Bronski and Johnson [1], adapting their methodology to our specific case.

## 3.2 Mathematical & Physical Settings

The main focus will be on the generalized Korteweg-de Vries equation, which serves as a modified version of the classical 3.1:

$$u_t + u_{xxx} + (f(u))_x = 0. \quad (3.2)$$

Motivated by the developed theory on the periodic Evans function, we will be studying periodic traveling wave solutions to 3.2. The general assumption is that the nonlinearity  $f(u)$  is a smooth function. Using the travelling wave ansatz, we are looking for traveling wave solutions of the form  $u(x, t) = u(\xi)$  with  $\xi = x - ct$  and  $c$  being the wave speed. Being in the moving frame  $\xi$  provides us with a wave profile  $u(\xi)$  modelled by an ODE.

$$\begin{aligned} \xi &= x - ct \implies u_t = -cu_\xi, \text{ and similarly } u_{xxx} = u_{\xi\xi\xi} \\ cu_\xi &= u_{\xi\xi\xi} + (f(u))_\xi \\ \int cu_\xi d\xi &= \int u_{\xi\xi\xi} d\xi + \int (f(u))_\xi d\xi \\ \implies cu &= u_{\xi\xi} + f(u) + a \end{aligned}$$

Rearranging terms, we will have the transformed equation:

$$u_{\xi\xi} = -f(u) + cu - a. \quad (3.3)$$

Recall that our goal is to find a periodic solution  $u(\xi)$  and determine whether small perturbations grow (resulting in instability) or decay (leading to stability) using the Evans function.

### Computations: Energy Method

To overcome the challenges of computing and working with a second-order ODE, specially with a general nonlinearity  $f(u)$  we can leverage the energy balance from classical mechanics. From Newton's law we have:

$$m \frac{d^2x}{dt^2} = -\frac{dV}{dx}.$$

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and the energy equations:

$$\frac{1}{2}mv^2 + V(x) = E \text{ where } V(x) \text{ denotes the potential energy and } v = \frac{dx}{dt}.$$

Combined together, we'll have:

$$v = \sqrt{\frac{2}{m}(E - V(x))}$$

We can mimic the same process in the context of gKdV:

$$u_{\xi\xi} = -f(u) + cu - a \text{ is similar to } \frac{d^2u}{d\xi^2} = -\frac{dV}{du}$$

One could treat  $u$  is position  $x$ ,  $\xi$  as time  $t$  and define  $V(u)$  in the following sense to match the force term:

$$V(u) := F(u) - \frac{c}{2}u^2 + au$$

A common trick to isolate  $u_\xi$  and find a periodic travelling wave solution is multiplying both sides by  $u_\xi$  and integrating:

$$\begin{aligned} u_{\xi\xi} &= -f(u) + cu - a \\ u_\xi u_{\xi\xi} &= u_\xi (-f(u) + cu - a) \\ \int \frac{d}{d\xi} \left( \frac{u_\xi^2}{2} \right) d\xi &= \int u_\xi (-f(u) + cu - a) d\xi \\ \frac{u_\xi^2}{2} &= -F(u) + \frac{c}{2}u^2 - au + E \\ \iff u_\xi &= \pm \sqrt{2 \left( -F(u) + \frac{c}{2}u^2 - au + E \right)} \end{aligned}$$

Observe that the  $\pm$  sign behind the solution reflects that the velocity is dependent on whether the height  $u$  is increasing or decreasing along the wave. The periodicity is observed when  $u$  oscillates between roots  $u_-$  and  $u_+$  where one has  $E = V(u)$  at roots and  $V(u) < E$  between them.

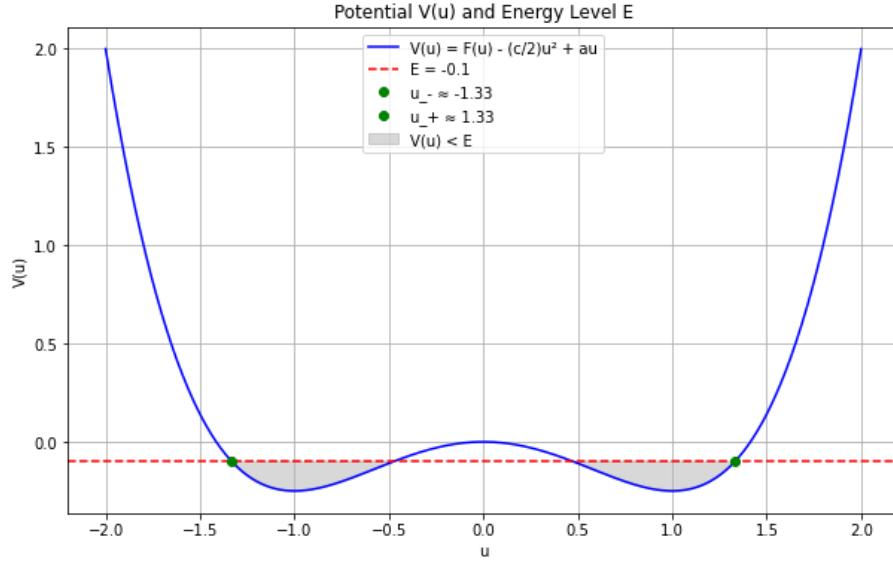


Figure 3.3: Potential  $V(u) = \frac{u^4}{4} - \frac{u^2}{2}$  for the gKdV equation with the choices  $f(u) = u^3$ ,  $c = 1$ , and  $a = 0$ . The energy level  $E = -0.1$  intersects  $V(u)$  at roots  $u_-$  and  $u_+$  with the shaded region ( $V(u) < E$ ).

**Remark 3.2.1.** *The choice of constants  $c, a, E$  for the purpose of generating the above graph was made so that  $V(u)$  could have a double well.*

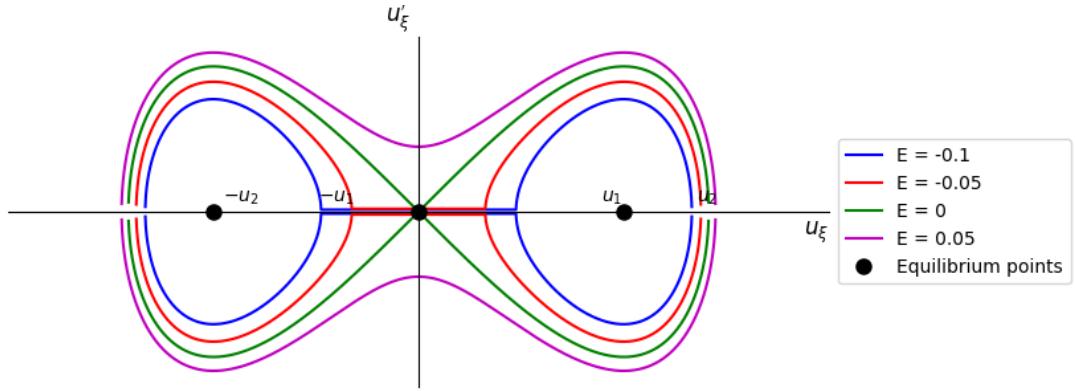


Figure 3.4: Phase portrait of the generalized Korteweg-de Vries equation with potential  $V(u) = \frac{u^4}{4} - \frac{u^2}{2}$ , showing closed trajectories for energy levels  $E = -0.1, -0.05, 0$ , and  $0.05$ . Trajectories represent  $u'_\xi = \pm\sqrt{2(E - V(u))}$

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### Computing the Period $T$ and Reduction to Quadrature

A well-known trick in turning a second-order nonlinear differential equation into an integral equation is - in our case - by treating  $u$  as the independent variable, and  $\xi$  as the dependent. Consider the one-way trip (omitting the negative sign):

$$\begin{aligned}\frac{du}{d\xi} &= \sqrt{2(E - V(u))} \\ d\xi &= \frac{du}{\sqrt{2(E - V(u))}}. \\ \therefore \xi(u) &= \int_{u_-}^{u(\xi)} \frac{du}{\sqrt{2(E - V(u))}}\end{aligned}$$

In order to find  $T$ , one measures how long it takes  $u$  to go from  $u_-$  to  $u_+$  and back to  $u_-$ . The above derived formula can be intuitively interpreted as time itself. Since  $u_\xi = \sqrt{2(E - V(u))}$  is the "speed" of  $u$ , the integral serves as the time  $= \int \frac{\text{distance}}{\text{speed}}$

Hence,  $\xi(u) =: \frac{T}{2}$  computes the time it takes to go from  $u_-$  to  $u_+$ . A full cycle will have the form:

$$T = 2 \int_{u_-}^{u_+} \frac{du}{\sqrt{2(E - V(u))}}$$

### 3.3 Stability Analysis

#### Linearization Around The Wave

We begin by perturbing the periodic solution  $u(\xi)$ . Considering a small disturbance  $u(\xi, t) = u(\xi) + \varepsilon v(\xi, t)$ , we plug the perturbed wave into the original gKdV:

$$(u + \varepsilon v)_t = (u + \varepsilon v)_{\xi\xi\xi} + (f(u + \varepsilon v))_\xi$$

Since  $u_t = 0$  in the frame of reference, the left-hand side reduces to  $\varepsilon v_t$

The  $\xi$ -derivative of the nonlinear term  $f$  can be Taylor expanded in the following sense:

$$(f(u + \varepsilon v))_\xi \approx (f(u) + \varepsilon f'(u)v)_\xi$$

The right-hand side, together with the terms  $u_{\xi\xi\xi}$  and  $\varepsilon v_{\xi\xi\xi}$  will reduce to:

$$u_{\xi\xi\xi} + (f(u))_\xi + \varepsilon (v_{\xi\xi\xi} + (f'(u)v)_\xi)$$

Some Computations Need Revisiting. Complete This Passage

Note that in the moving frame  $\xi = x - ct$ , and assuming a spectral form  $v(\xi, t) = e^{-\mu t}v(\xi)$ , we adjust for the time derivative  $v_t = -\mu v - cv_\xi$ , yielding:

$$-\mu v - cv_\xi = v_{\xi\xi\xi} + (f'(u)v)_\xi \quad (3.4)$$

Rearranging:

$$v_{\xi\xi\xi} + (f'(u)v)_\xi - cv_\xi = -\mu v \quad (3.5)$$

This third-order ODE governs the perturbation  $v(\xi)$ , where  $\mu$  is the spectral parameter whose real part determines stability:  $\text{Re}(\mu) > 0$  indicates instability.

We now rewrite the equations in the form of an eigenvalue problem, using the following operator:

$$\mathcal{L}[u] = -\partial_\xi^2 - f'(u) + c$$

The eigenvalue problem therefore becomes:

$$\begin{aligned} v_{\xi\xi\xi} + (f'(u)v)_\xi - cv_\xi &= -\mu v \\ \partial_\xi \mathcal{L}[u]v &= \mu v \end{aligned}$$

Having derived the periodic traveling wave solution  $u(\xi)$  and its linearized perturbation equation, we now proceed to assess its spectral stability using the Evans function. This section details the steps from formulating the linearized system to analyzing the stability, emphasizing the mathematical rigor and physical intuition behind each step.

### Conversion to a First-Order System

To analyze this using Floquet theory (due to the periodicity of  $u(\xi)$ ), we transform Equation (3.5) into a first-order system. Define:

- $v_1 = v$
- $v_2 = v_\xi$
- $v_3 = v_{\xi\xi}$

Compute derivatives:

$$v_{1\xi} = v_2, \quad (3.6)$$

$$v_{2\xi} = v_3, \quad (3.7)$$

$$v_{3\xi} = v_{\xi\xi\xi} = -\mu v - (f'(u)v)_\xi + cv_\xi \quad (\text{from (3.5)}). \quad (3.8)$$

Expand  $(f'(u)v)_\xi$ :

$$(f'(u)v)_\xi = f''(u)u_\xi v + f'(u)v_\xi \quad (3.9)$$

Substitute:

$$\begin{aligned} v_{\xi\xi\xi} &= -\mu v - [f''(u)u_\xi v + f'(u)v_\xi] + cv_\xi \\ &= -\mu v_1 - f''(u)u_\xi v_1 - f'(u)v_2 + cv_2. \end{aligned} \quad (3.10)$$

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Thus:

$$v_{3\xi} = -\mu v_1 - f''(u)u_\xi v_1 + (c - f'(u))v_2 \quad (3.11)$$

The system becomes:

$$\frac{d}{d\xi} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu - f''(u)u_\xi & c - f'(u) & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (3.12)$$

Define:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \mathbf{H}(\xi; \mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu - f''(u)u_\xi & c - f'(u) & 0 \end{pmatrix} \quad (3.13)$$

So:

$$\mathbf{v}_\xi = \mathbf{H}(\xi; \mu)\mathbf{v} \quad (3.14)$$

**Remark 3.3.1.** Since  $u(\xi)$  is periodic with period  $T$ ,  $\mathbf{H}(\xi; \mu)$  is periodic:  $\mathbf{H}(\xi + T; \mu) = \mathbf{H}(\xi; \mu)$ . This periodicity motivates Floquet theory, where solutions take the form  $\mathbf{v}(\xi) = e^{i\lambda\xi}\mathbf{p}(\xi)$ , with  $\mathbf{p}(\xi + T) = \mathbf{p}(\xi)$ .

**Remark 3.3.2.** Converting to a first-order system allows us to use matrix methods and Floquet theory, essential for periodic coefficients. As seen in previous sections, the Evans function leverages this structure to detect eigenvalues  $\mu$  by examining solution behavior over one period.

The monodromy matrix captures the evolution of solutions over one period  $T$ . Consider the fundamental matrix solution  $\Phi(\xi, \mu)$  to:

$$\Phi_\xi = \mathbf{H}(\xi; \mu)\Phi, \quad \Phi(0, \mu) = \mathbf{I} \quad (3.15)$$

where  $\mathbf{I}$  is the 3x3 identity matrix. The monodromy matrix is:

$$\mathbf{M}(\mu) = \Phi(T, \mu) \quad (3.16)$$

#### Computation

Solving  $\Phi_\xi = \mathbf{H}\Phi$  analytically is challenging due to  $u(\xi)$ 's complexity (e.g., a cnoidal wave). For each  $\mu$ ,  $\mathbf{M}(\mu)$  is a 3x3 matrix whose eigenvalues (Floquet multipliers) determine solution growth.

**Remark 3.3.3.** For  $f(u) = u^3$ ,  $f'(u) = 3u^2$ ,  $f''(u) = 6u$ , so:

$$\mathbf{H}(\xi; \mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu - 6uu_\xi & c - 3u^2 & 0 \end{pmatrix}$$

This depends on the specific  $u(\xi)$ .

Observe that the monodromy matrix links initial conditions at  $\xi = 0$  to  $\xi = T$ , showing periodic behavior. This motivates the Evans function approach.

### 3.4 The Evans Function

The Evans function for periodic problems is defined as:

$$D(\mu, \lambda) = \det(\mathbf{M}(\mu) - \lambda \mathbf{I}) \quad (3.17)$$

Here,  $\lambda$  is a Floquet multiplier. Spectral stability requires  $v(\xi)$  to be bounded, so  $|\lambda| = 1$  (i.e.,  $\lambda = e^{i\kappa}$ ,  $\kappa \in [0, 2\pi]$ ). Thus:

$$D(\mu, e^{i\kappa}) = 0 \quad (3.18)$$

defines the spectrum  $\mu$ .

#### A General Algorithm For Computing

For a given  $\mu$ :

1. We begin by Computing  $\mathbf{M}(\mu)$  as described above.
2. We proceed by forming  $\mathbf{M}(\mu) - e^{i\kappa} \mathbf{I}$ .
3. Calculate  $D(\mu, e^{i\kappa}) = \det(\mathbf{M}(\mu) - e^{i\kappa} \mathbf{I})$  for  $\kappa \in [0, 2\pi]$ .
4. Find  $\mu$  such that  $D(\mu, e^{i\kappa}) = 0$  (e.g., via root-finding or contour integration).

**Example 4.** Suppose  $f(u) = u^3$ ,  $c = 1$ ,  $a = 0$ , and  $u(\xi)$  is a cnoidal wave with period  $T$ .

#### The Spectrum

The spectrum  $\text{spec}(\partial_\xi \mathcal{L}[u])$  comprises  $\mu$  values where  $D(\mu, e^{i\kappa}) = 0$  for some  $\kappa$ .

#### Stability Criteria

- **Stable:** All  $\mu$  have  $\text{Re}(\mu) \leq 0$  (perturbations decay or oscillate).
- **Unstable:** Any  $\mu$  has  $\text{Re}(\mu) > 0$  (perturbations grow exponentially).

Focus on  $\kappa = 0$  ( $\lambda = 1$ ) for long-wavelength perturbations (modulational instability):

$$D(\mu, 1) = 0 \quad (3.19)$$

#### Analysis

1. Plot  $\mu$  roots in the complex plane.
2. Check  $\text{Re}(\mu)$ .
3. Compare with the paper's indices (Section 3):

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- Orientation index: Counts real  $\mu \pmod{2}$ .
- Modulational instability index:  $\Delta > 0$  (stable near origin) or  $\Delta < 0$  (unstable).

**Example 5.** For  $p = 5$  (Section 5 of [1]),  $\Delta < 0$ , and spectra show  $\mu$  with  $\text{Re}(\mu) > 0$ , indicating instability. For  $p = 2$ , test numerically to verify stability trends.

**Remark 3.4.1.** The discriminant  $\Delta$  (Section 3) involves derivatives of conserved quantities (e.g.,  $M = \int u d\xi$ ), linking physical properties to spectral behavior.

### Extensions to The Transverse Instability

## Bibliography

- [1] J. C. Bronski and M. A. Johnson, “The Modulational Instability for a Generalized Korteweg-de Vries Equation”. 2010