

On The Evans Function and Its Applications to Periodic Waves

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Ongoing Project — Last Update: 09 January 2025

Popular Science Description

Mathematical models describing natural phenomena have often given rise to seemingly cyclic, and repetitive patterns, which have manifested themselves in a wide-ranging set of topics. From population dynamics in a biological setting to the evolution of stellar structures and celestial dynamics, we have oftentimes witnessed wave-like behavior, or at least some permanent structure that follows a certain pattern. An immediate question that plays a pivotal role in applied mathematics is that of the *stability* of such behavior. For instance, when it comes to traveling water waves, the term stability refers to the wave maintaining its shape as it moves forward in time, while small disturbances diminish without causing major effects. However, non-linearity among a large number of interesting evolving systems makes it a challenging task to provide an answer right away. It is also what keeps mathematicians in business since the study of stability in a diverse set of applications demands the development of newer theories and feasible techniques. The central topic of this thesis revolves around one such theory, namely the *Evans function*.

One can conveniently think of this function as a stability detector device. Mathematically speaking, it connects two theoretical perspectives from functional analysis and dynamical systems on stability. It is named after the American mathematician John Evans, who originally trained as a medical doctor, became fascinated by mathematics, and eventually left his medical career to pursue a PhD in math. Having introduced the Evans function in the 1970s, his interest in mathematics was influenced by his medical background, particularly the Hodgkin-Huxley model for nerve impulse propagation, formulated about 20 years earlier. While Hodgkin and Huxley had shown that nerve impulses could travel as waves, Evans focused on proving their stability. Though he didn't fully solve this problem, he developed a theory with applications extending beyond neuroscience.

In this dissertation, the aim is to introduce the main ideas surrounding this method, and eventually see one of its many applications in [stability analysis](#)

of solitons. In particular, the generalized Korteweg-De Vries (gKdV) equation. We will be further extending the application of the theory in investigating the transverse instability of gKdV.

Introduction

To be completed.

For now, enjoy this scene of nature's wild beauty! :)



1 Chapter 1: Preliminaries

We begin by recalling some basic concepts in linear functional analysis, which play a central role in the stability theory of dynamical systems. To establish the connection and necessity of these topics to differential equations, we first provide a short introduction to some basic results in the theory of ODEs. Since the periodic Evans function serves as our main analytical tool, this chapter will subsequently focus further on generalizing the notion of periodic matrices to infinite-dimensional spaces through Floquet theory. *The introductory prerequisites are mainly based on Kapitula-Promislow's *Spectral and Dynamical Stability of Nonlinear Waves* and Teschl's *Ordinary Differential Equations and Dynamical Systems**

1.1 Some Known Results in The Theory of ODEs

Notation: Given $\mathbf{y}(x, \boldsymbol{\lambda}) \in \mathbb{C}^n$, $\mathbf{A}(x, \boldsymbol{\lambda}) \in \mathbb{C}^{n \times n}$ and $\boldsymbol{\lambda} \in \mathbb{C}^m$, one can write a system of linear ordinary differential equations using partial derivatives:

$$\partial_x \mathbf{y} = \underbrace{\mathbf{A}(x, \boldsymbol{\lambda}) \mathbf{y}}_{f(x, \mathbf{y})} \text{ with the initial conditions: } \mathbf{y}(x_0, \boldsymbol{\lambda}) = \mathbf{y}_0(\boldsymbol{\lambda}) \quad (1)$$

Existence and Uniqueness of Solutions: The continuity of $f(x, \mathbf{y})$ ensures existence. However, it does not guarantee uniqueness. For that, we require the Lipschitz condition.

Definition 1.1 (Lipschitz Condition). *For a fixed x The function $f(x, \mathbf{y})$ is said to satisfy a Lipschitz condition in $\Omega \subseteq \mathbb{C} \times \mathbb{C}^n$ if there exists a real-valued constant $L \geq 0$ such that:*

$$|f(x, \mathbf{y}_2) - f(x, \mathbf{y}_1)| \leq L|\mathbf{y}_2 - \mathbf{y}_1| \quad \text{when } (x, \mathbf{y}_1), (t, \mathbf{y}_2) \in \Omega.$$

Remark 1.1. *The Lipschitz condition puts a "cap" on how much $f(x, \mathbf{y})$ can "stretch" the points \mathbf{y}_1 and \mathbf{y}_2 from one another.*

Lemma 1. *Assume that $\Omega \subseteq \mathbb{C} \times \mathbb{C}^n$. Assume that Ω is a convex and bounded set, and that f is continuously differentiable in a neighborhood of $\bar{\Omega}$ (where $\bar{\Omega}$ denotes the closure of the set Ω). Then, f satisfies a Lipschitz condition in Ω .*

Theorem 1 (local in \mathbf{y}). *Assume that f is a continuous function in a neighborhood of $(x_0, \mathbf{y}_0) \in \mathbb{C} \times \mathbb{C}^n$ and satisfies a Lipschitz condition there. Then there exists an open interval around x_0 in which*

$$\begin{cases} \mathbf{y}'(x) = f(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0 \end{cases} \quad (2)$$

has a unique solution $\mathbf{y}(x)$.

Theorem 2 (global in \mathbf{y}). Assume that \mathbf{f} is a continuous function and satisfies a Lipschitz condition **in a strip** $\{(x, \mathbf{y}) \in \mathbb{C} \times \mathbb{C}^n; |x - x_0| \leq a\} =: I(a)$. Then the initial value problem

$$\begin{cases} \mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}(x)), \\ \mathbf{y}(x_0) = \mathbf{y}_0 \end{cases} \quad (3)$$

has a unique solution in the entire strip $I(a)$.

1.2 Linear Homogeneous Systems

The following lemma is a direct consequence of theorems 1 and 2

Lemma 2 (Existence and Uniqueness of Solutions). Suppose that for fixed λ the matrix $\mathbf{A}(x, \lambda)$ is **continuous** in x on the (possibly infinite) open interval $I := (a, b)$. Then for each $x_0 \in I$, and $\mathbf{y}_0 \in \mathbb{C}^n$, there exists a unique solution to the system 1. Furthermore, the solution is as smooth in x and λ as are the coefficient matrix $\mathbf{A}(x, \lambda)$ and the initial data \mathbf{y}_0 .

The linearity of the system motivates the investigation of a *linear space* V consisting of solutions \mathbf{y} that satisfy 1. As with any well-defined linear space, one can examine the linear dependence of its elements, construct a basis, and explore the resulting linear algebraic properties.

Theorem 3. Assume there are k solutions: Let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k$ be solutions of 1, then the following statements are equivalent:

- a) $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k$ are linearly independent in V .
- b) $\mathbf{y}_1(x), \mathbf{y}_2(x), \dots, \mathbf{y}_k(x)$ are linearly independent in $\mathbb{C}^n \forall x \in I$
- c) $\mathbf{y}_1(x_0), \mathbf{y}_2(x_0), \dots, \mathbf{y}_k(x_0)$ are linearly independent in \mathbb{C}^n for some $x_0 \in I$.

At first glance, Theorem 3 may appear to involve a circular chain of assumptions, as at least one condition must hold to establish the subsequent implications. Nevertheless, its utility becomes evident once it is demonstrated that there exist n linearly independent solutions, which collectively form a basis for the solution space.

Theorem 4 (n linearly independent solutions). Let $\mathbf{A}(x)$ be a continuous $n \times n$ matrix on an open interval I . The solutions of $\mathbf{y}' = \mathbf{A}(x)\mathbf{y}$ form a linear space V of dimension n .

Proof. Recall that a set of vectors form a basis for a space if they **span** that space **and** are **linearly independent**.

Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be a basis of \mathbb{C}^n . Let $\mathbf{y}_i, i = 1, 2, 3, \dots, n$ be a set of solutions to

$$\begin{cases} \mathbf{y}' = \mathbf{A}(x)\mathbf{y}, & x_0 \in I \\ \mathbf{y}(x_0) = \mathbf{e}_1 \end{cases}$$

Then by theorem 3, $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are linearly independent in V .

Now assume (for the sake of contradiction) that they do not span the space. That is, assume there exists $\mathbf{y} \in V$ that **cannot** be written as a linear combination of $\mathbf{y}_1, \dots, \mathbf{y}_n$. Formally, we assume that

$$\mathbf{y} \notin \text{span}\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$$

Then $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n, \mathbf{y}$ are linearly **independent**.

However, using theorem 3, this would imply that $\mathbf{y}_1(x_0), \mathbf{y}_2(x_0), \dots, \mathbf{y}_n(x_0), \mathbf{y}(x_0)$ are linearly independent in \mathbb{R}^n , which is impossible! The number of linearly independent elements of a vector space cannot exceed its dimension. In our case, we cannot have $n + 1$ linearly independent vectors in an n -dimensional space \mathbb{R}^n . We have therefore arrived at a contradiction.

Hence $V = \text{span}\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$

Putting these two results together: The vectors $\mathbf{y}_1, \dots, \mathbf{y}_n$ span the solution space V and are linearly independent $\implies \mathbf{y}_1, \dots, \mathbf{y}_n$ form a basis.

$$\therefore \dim(V) = n.$$

□

Recall that one can build a so-called **fundamental matrix** by concatenating the linearly-independent solutions (basis vectors) $\mathbf{y}_1, \dots, \mathbf{y}_n$ in an $n \times n$ matrix,

$$\Psi(x) := (\mathbf{y}_1, \dots, \mathbf{y}_n)(x) \in \mathbb{C}^{n \times n}.$$

The matrix itself is a solution to 1:

$$\begin{aligned} \partial_x \Psi(x) &= (\partial_x \mathbf{y}_1, \dots, \partial_x \mathbf{y}_n)(x) = (\mathbf{A} \mathbf{y}_1 \dots \mathbf{A} \mathbf{y}_n) = \mathbf{A}(\mathbf{y}_1, \dots, \mathbf{y}_n) = \mathbf{A}\Psi, \\ &\text{with initial value condition } \Psi(x_0) = (\mathbf{e}_1 \dots \mathbf{e}_n). \end{aligned}$$

Remark 1.2. If Ψ is a solution to 1, then for every nonsingular constant $n \times n$ matrix \mathbf{B} , the product $\Psi\mathbf{B}$ is also a solution:

$$\partial_x(\Psi\mathbf{B}) = \mathbf{A}(x)(\Psi\mathbf{B}).$$

In order to construct a particular solution aligning with the initial data, we introduce the Principal Fundamental Matrix Solution (Principal FMS).

Definition 1.2 (Principal FMS). Let $\Psi\mathbf{B}$ be a solution to 1, where Ψ denotes the fundamental matrix solution, and \mathbf{B} is a nonsingular constant $n \times n$ matrix. Fix $B := \Psi^{-1}(x_0)$. So that:

$$\Psi(x_0)\mathbf{B} = \Psi(x_0)\Psi^{-1}(x_0) = I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

We define the principal FMS to be $\Phi(x, x_0) := \Psi(x)\Psi^{-1}(x_0)$
so that the solution would be of the form: $\Psi(x) = \Phi(x, x_0)\Psi(x_0)$

In some applied mathematics literature, particularly in control theory, the principal FMS is called "The State Transition Matrix", since it tracks the transition of the system from the initial starting point x_0 . The state transition matrix provides a systematic way to analyze the stability and controllability of linear systems, determine solutions explicitly, and understand the time evolution of dynamic systems in engineering, physics, and applied mathematics.

Remark 1.3 (Principal FMS in scalar systems). *It is well known that the dimension of an n -dimensional linear system can often be reduced to $n = 1$. In such cases, our focus shifts to a single column of the fundamental solution matrix Ψ . Consequently, the solution \mathbf{y} to 1 takes the form:*

$$\mathbf{y}(x) = \Phi(x, x_0)\mathbf{y}_0$$

Theorem 5 (General non-homogeneous case). *Consider the non-homogeneous version of 1:*

$$\partial_x \mathbf{y} = \mathbf{A}(x, \lambda)\mathbf{y} + \mathbf{f}(x, \mathbf{y}(x)) \text{ with initial conditions } \mathbf{y}(x_0, \lambda) = \mathbf{y}_0.$$

The above equation has the unique solution of the form:

$$\mathbf{y}(x) = \Phi(x, x_0)\mathbf{y}_0 + \int_{x_0}^x \Phi(x, \tau)\Phi(\tau, x_0)^{-1}\mathbf{f}(\tau, \mathbf{y}(\tau))d\tau$$

An explicit form of solutions depends heavily on algebraic properties of the matrix \mathbf{A} . We survey the well-known results surrounding this problem.

1.2.1 Constant Matrices and Invariant Subspaces

Block Diagonlizability and Consequences (KP)

The most familiar case is when the matrix \mathbf{A} is x -independent:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \text{ subjected to the initial data } \mathbf{y}(x_0) = \mathbf{y}_0$$

This would have a solution of the form:

$$\mathbf{y} = \mathbf{y}_0 \exp(\mathbf{A}x) = \mathbf{y}_0 \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} x^k$$

Understanding how to interpret the above expression becomes the central focus when dealing with constant matrices. Ideally, our goal is to eliminate the matrix structure, which leads us to our first encounter with **eigenvalues**, and prompts the formulation of the following insightful ansatz.

Let the solution have the form $\mathbf{y} = \mathbf{v}e^{\lambda x}$. Then plugging this into the differential equation 1.2.1, we'd have a classic eigenvalue problem:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \iff \lambda\mathbf{v}e^{\lambda x} = \mathbf{A}\mathbf{v}e^{\lambda x} \iff \mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Assuming that the matrix \mathbf{A} is diagonalizable, we will have n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, resulting in n distinct eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$.

$$\mathbf{y}_i = \mathbf{v}_i e^{\lambda_i x} \text{ for } i = 1, 2, \dots, n$$

Any other solution to 1.2.1 can be written as a linear combination of these:

$$\mathbf{y}(\mathbf{x}) = c_1 \mathbf{v}_1 e^{\lambda_1 x} + c_2 \mathbf{v}_2 e^{\lambda_2 x} + \dots + c_n \mathbf{v}_n e^{\lambda_n x}$$

As is often the case, there may be an insufficient number of eigenvectors to fully describe the system. This occurs when the algebraic multiplicity $m_a(\lambda) > 1$, leading to a loss of distinction among eigenvectors. Our aim is now to derive a description of the structure of $e^{\mathbf{A}x}$, which does not depend on the diagonalizability condition.

Suppose that a matrix \mathbf{B} has the following structure:

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_1 & 0 & \dots & 0 \\ 0 & \mathbf{B}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{B}_n \end{pmatrix}$$

where \mathbf{B}_k are quadratic matrices of order n_k , $k = 1, 2, \dots, m$. Such that: $n_1 + n_2 + \dots + n_m = n$. The rest of the elements in \mathbf{B} are 0.

Although we mainly work with the exponential function, the following results hold for any entire function f . Passing \mathbf{B} into f , we have:

$$f(\mathbf{B}) = \begin{pmatrix} f(\mathbf{B}_1) & 0 & \dots & 0 \\ 0 & f(\mathbf{B}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(\mathbf{B}_n) \end{pmatrix}$$

So, this structure of \mathbf{B} is good when computing matrix exponentials, assuming that $f(\mathbf{B}_n)$ are easy to compute.

If there is an invertible matrix \mathbf{T} such that $\mathbf{A} = \mathbf{T}\mathbf{B}\mathbf{T}^{-1}$, then

$$f(\mathbf{A}) = \mathbf{T}f(\mathbf{B})\mathbf{T}^{-1}.$$

so the task is then to compute $f(\mathbf{B}_n)$.

If \mathbf{A} is diagonalizable, then \mathbf{B} can be chosen to be diagonal. The fact that \mathbf{A} is diagonalizable implies that there is a basis of eigenvalues. Furthermore, $\mathbf{A} = \mathbf{T}\mathbf{B}\mathbf{T}^{-1}$ can be interpreted as a change of basis to \mathbb{C}^n . As a result of this change, In the new basis we get the matrix \mathbf{B} . In general, \mathbf{B} won't be diagonal but we still can make a change of basis so that \mathbf{B}_n is still simple; and $f(\mathbf{B}_n)$ can be therefore computed. In order to carry out these steps rigorously, we will need some concepts from linear algebra.

Definition 1.3 (Direct Sum). *Let \mathcal{V} be an n -dimensional linear space over \mathbb{C} , and $\mathcal{V}_1, \dots, \mathcal{V}_n$ be subspaces in \mathcal{V} . ($\mathcal{V}_n \subset \mathcal{V}$, \mathcal{V}_n linear space) then \mathcal{V} is called the **Direct Sum** of $\mathcal{V}_1, \dots, \mathcal{V}_m$ if all $\mathbf{v} \in \mathcal{V}$ can be uniquely written as:*

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_m$$

where $\mathbf{v}_k \in \mathcal{V}_k$, $k = 1, \dots, m$. To describe this situation, we use the following notation:

$$\mathcal{V} = \mathcal{V}_1 \bigoplus \mathcal{V}_2 \bigoplus \dots \bigoplus \mathcal{V}_m$$

Definition 1.4 (Invariant Subspace). *A subspace \mathcal{V}_n is invariant under \mathbf{A} if $u \in \mathcal{V}_n \implies Au \in \mathcal{V}_n$.*

Definition 1.5 (Kernel). *The nullspace (kernel) of \mathbf{A} is defined by*

$$\ker(\mathbf{A}) = \mathcal{N}(\mathbf{A}) = \{v \in \mathcal{V} : \mathbf{A}v = 0\}$$

. Note that $\mathcal{N}(\mathbf{A}v)$ is invariant under \mathbf{A} .

In the case where $\dim \mathcal{V} = 2$, let $\mathbf{A} : \mathcal{V} \rightarrow \mathcal{V}$ with eigenvalues 1 and 2. Then: $p_a(\lambda) = (\lambda - 1)(\lambda - 2)$, So \mathbf{A} can be diagonalizable. Let

$$\begin{aligned} \mathcal{V}_1 &= \{v : Av = v\} = \mathcal{N}(A - I) \\ \mathcal{V}_2 &= \{v : Av = 2v\} = \mathcal{N}(A - 2I) \end{aligned}$$

Since the eigenvectors form a basis of \mathcal{V} :

$$\mathcal{V} = \mathcal{V}_1 \bigoplus \mathcal{V}_2 = \mathcal{N}(A - I) \bigoplus \mathcal{N}(A - 2I)$$

We now generalize this idea by formulating the following theorem.

Theorem 6. *Let \mathbf{A} be a linear mapping on a linear space \mathcal{V} , and assume that $p(\lambda)$ is a polynomial such that $p(\mathbf{A}) = 0$. If $p(\lambda) = p_1(\lambda)p_2(\lambda)$, where the polynomials $p_1(\lambda)$ and $p_2(\lambda)$ are relatively prime, then:*

$$\mathcal{V} = \mathcal{N}(p_1(\mathbf{A})) \bigoplus \mathcal{N}(p_2(\mathbf{A}))$$

Furthermore, the spaces $\mathcal{N}(p_1(\mathbf{A}))$ and $\mathcal{N}(p_2(\mathbf{A}))$ are invariant under \mathbf{A} .

Theorem 7 (Cayley Hamilton).

According to Cayley – Hamilton theorem, we have that:

$$p_A(A) = 0; p_A(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_m)^{n_m}$$

where λ_j are eigenvalues with multiplicities n_j .

Furthermore, theorem 9 states that

$$F^n = \mathcal{V}_1 \bigoplus \mathcal{V}_2 \bigoplus \dots \bigoplus \mathcal{V}_m$$

where $\mathcal{V}_i = \mathcal{V}_n(p_i(A)) = \mathcal{N}((A - \lambda_i I)^{n_i})$.

Denote by A_k the restriction of the mapping A to \mathcal{V}_k , let I_k denote the identity mapping on \mathcal{V}_k , and put $N_k = A_k - \lambda_k I_k$. Then N_k is nilpotent, because $N_k^{n_k} = 0$, and $A_k = \lambda_k I_k + N_k$. Choose a basis in each subspace \mathcal{V}_k , and let I_k, N_k also denote the matrix of corresponding mapping in this basis. In the hereby chosen basis in \mathbb{C}^n , the matrix of the mapping A has the form:

$$T^{-1}AT = \begin{pmatrix} \lambda_1 I_1 + N_1 & 0 & \dots & 0 \\ 0 & \lambda_2 I_2 + N_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m I_m + N_m \end{pmatrix}$$

$A = TBT^{-1}$ so:

$$e^A = \begin{pmatrix} e^{x\lambda_1} e^{xN_1} & 0 & \dots & 0 \\ 0 & e^{x\lambda_2} e^{xN_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{x\lambda_m} e^{xN_m} \end{pmatrix}$$

Corollary 1. Let A be a linear mapping on a vector space \mathcal{V} over \mathbb{C} . Then there are subspaces $\mathcal{V}_-, \mathcal{V}_0, \mathcal{V}_+$ which are invariant under A and such that:

- a) $\mathcal{V} = \mathcal{V}_- \bigoplus \mathcal{V}_0 \bigoplus \mathcal{V}_+$.
- b) all eigenvalues of $A: \mathcal{V}_- \rightarrow \mathcal{V}_-$ have a negative real part.
- c) all eigenvalues of $A: \mathcal{V}_0 \rightarrow \mathcal{V}_0$ have the real part equal to zero.
- d) all eigenvalues of $A: \mathcal{V}_+ \rightarrow \mathcal{V}_+$ have a positive real part.

This classification becomes of great use when studying *spectral stability* of dynamical systems.

1.3 Elements of Functional Analysis

1.3.1 Banach and Sobolev Spaces

Recall that every complete normed vector space constitutes a Banach space. Throughout this dissertation, we will extensively use and apply a number of already established results and theorems on Banach spaces. In particular, the focus would initially be on the basic Sobolev spaces, which combine properties of the L^p norm of a function and its (weak) derivatives up to a given order. One could formulate such notions in the following sense:

Definition 1.6 (Weak Derivative). *Let $u \in L^1_{loc}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open set. A function $v \in L^1_{loc}(\Omega)$ is called the weak derivative of u with respect to x_i (denoted $\frac{\partial u}{\partial x_i} = v$) if*

$$\int_{\Omega} u(x) \frac{\partial \varphi(x)}{\partial x_i} dx = - \int_{\Omega} v(x) \varphi(x) dx$$

for all test functions $\varphi \in C_c^\infty(\Omega)$, where $C_c^\infty(\Omega)$ is the space of infinitely differentiable functions in Ω .

Definition 1.7 (The L^p Space). *Let $u : \mathbb{R} \rightarrow \mathbb{C}$ be Lebesgue measurable function. For $p \geq 1$, the L^p space over \mathbb{R} is defined as:*

$$L^p(\mathbb{R}) = \left\{ u : \mathbb{R} \rightarrow \mathbb{C} \mid \int_{\mathbb{R}} |u(x)|^p dx < \infty \right\}$$

Recall that $L^2(\mathbb{R})$ is equipped with the following inner product:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} dx.$$

With its associated norm being:

$$\|u\|_{L^p} = \|u\|_p = \left(\int_{\mathbb{R}} |u(x)|^p dx \right)^{1/p}.$$

Definition 1.8 (The $W^{k,p}$ Norm). *For k -times weakly differentiable functions $u : \mathbb{R} \rightarrow \mathbb{C}$ and $p \geq 1$, the $W^{k,p}$ norm is defined as:*

$$\|u\|_{W^{k,p}(\mathbb{R})} = \left(\sum_{j=0}^k \left\| \frac{\partial^j}{\partial x^j} u \right\|_{L^p(\mathbb{R})}^p \right)^{1/p}$$

Remark 1.4. Unlike the L^p norm, which only accounts for the size of a function, the $W^{k,p}$ norm provides us with information on derivatives.

The above norm is associated with the Sobolev space $W^{k,p}(\mathbb{R})$, consisting of functions $u : \mathbb{R} \rightarrow \mathbb{C}$ with a finite $W^{k,p}$ norm,

$$W^{k,p}(\mathbb{R}) := \{u : \|u\|_{W^{k,p}} < \infty\}.$$

Since the ultimate goal is to apply this theory to a practical setting, we fix $p = 2$, and $k = 0$, reducing $W^{k,p}(\mathbb{R})$ to the Hilbert space of square-integrable functions $L^2(\mathbb{R})$, and thereby avoiding unnecessary abstraction:

$$H^k := W^{k,2} \text{ and } H^0(\mathbb{R}) = L^2(\mathbb{R}).$$

1.3.2 Bounded and Closed Operators

Definition 1.9 (Dense Subset). *Let X be a Banach space. A subset $Y \subset X$ is said to be dense in X if for every $x \in X$, and for every $\epsilon > 0$, there exists $y \in Y$, such that $\|x - y\| < \epsilon$*

In more general terms, denseness in a Banach space means that one can approximate any element of the Banach space by elements of a dense subset of that space.

Definition 1.10 (Closed Operator). *Let $(X, \|\cdot\|_X)$ be a Banach space. The linear operator $\mathcal{L} : D(\mathcal{L}) \rightarrow X$ is densely defined if it is a linear operator $D(\mathcal{L}) \rightarrow X$ where $D(\mathcal{L})$ is a dense subspace of X . We say that \mathcal{L} is **closed** if a sequence $\{u_j\} \subset D(\mathcal{L})$ converges (in norm of X) to some u and if the sequence $\{\mathcal{L}u_j\}$ converges to some v , then it follows that $u \in D(\mathcal{L})$ and $\mathcal{L}(u) = v$.*

Definition 1.11 (Bounded Operator). *The operator $\mathcal{L} : Y \mapsto X$ is bounded from Y to X if the norm $\|\mathcal{L}u\|_X$ is bounded over the unit sphere:*

$$\sup\{\|\mathcal{L}u\|_X : u \in Y, \|u\|_Y = 1\} < \infty$$

Remark 1.5. *Since operators act on elements and transform them, boundedness of \mathcal{L} refers to a situation where the set of the transformed functions $\mathcal{L}v$ (the image) is bounded. In other words, there is a limit to the "strength" of \mathcal{L} when it comes to transforming the functions. This interpretation only applies over bounded sets, allowing us to formulate the above definition by stating that a bounded operator maps bounded sets to bounded sets.*

We denote the **space of bounded linear operators** from Y into X by $\mathcal{B}(Y, X)$. As for notation, if $Y = X$, we simply write $\mathcal{B}(X)$. Also note that $\mathcal{B}(Y, X)$ together with the following norm constitutes a Banach space *over the unit sphere* $\|u\|_Y = 1$

$$\|\mathcal{L}\|_{\mathcal{B}(X,Y)} := \sup\|\mathcal{L}u\|_X$$

Definition 1.12 (Compactness of \mathcal{L}). *If for each bounded sequence $\{u_j\} \subset Y$ the sequence $\{\mathcal{L}u_j\} \subset X$ has a convergent subsequence, then the operator \mathcal{L} is said to be compact.*

A compact operator is bounded. Furthermore, the sum of two compact operators is compact, and the composition of a compact operator and a bounded operator is compact.

1.3.3 Stability Through Spectral Analysis

Let $\mathcal{L} : X \mapsto X$ be a bounded, closed, densely defined linear operator on a Banach space X .

Recall that the spectrum of \mathcal{L} is a generalization of eigenvalues for operators on infinite-dimensional spaces. Formally, we have that:

Definition 1.13 (Spectrum). *The spectrum of a linear operator \mathcal{L} is defined as:*

$$\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid (\mathcal{L} - \lambda\mathcal{I}) \text{ does not have a bounded inverse}\}.$$

An introductory understanding of how spectrum can help with stability analysis lies in the question of "where" in the spectrum the value λ lies. We have the following "regions":

Stable Spectrum: The Stable Spectrum consists of those values in the spectrum whose real parts are strictly negative. These correspond to modes that decay exponentially over time. In mathematical terms:

$$\sigma_{\text{stable}}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \Re(\lambda) < 0\}$$

Center Spectrum: The Center Spectrum consists of values in the spectrum lying on the imaginary axis, corresponding to neutral or oscillatory dynamics. This set can be written as:

$$\sigma_{\text{center}}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \Re(\lambda) = 0\}$$

Unstable Spectrum: The Unstable Spectrum consists of values in the spectrum whose real parts are strictly positive, corresponding to modes that grow exponentially over time. It is expressed as:

$$\sigma_{\text{unstable}}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \Re(\lambda) > 0\}$$

Definition 1.14 (Resolvent Set). *The resolvent set of \mathcal{L} , denoted by $\rho(\mathcal{L})$, is the set of all complex numbers $\lambda \in \mathbb{C}$ for which the operator $(\mathcal{L} - \lambda\mathcal{I})$ is invertible (i.e., it has a bounded inverse) and the inverse $(\mathcal{L} - \lambda\mathcal{I})^{-1}$ is a bounded operator on X :*

$$\rho(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid (\mathcal{L} - \lambda\mathcal{I}) \text{ is invertible (bijective)} \text{ and } (\mathcal{L} - \lambda\mathcal{I})^{-1} \in \mathcal{B}(X)\}.$$

Definition 1.15 (Resolvent of \mathcal{L}). *The inverse operator $\|(\mathcal{L} - \lambda\mathcal{I})^{-1}\|$ is called the resolvent of \mathcal{L} .*

Definition 1.16 (Adjoint Operator). *Given a linear operator $\mathcal{L} : X \rightarrow Y$ between two Hilbert spaces X and Y , the adjoint operator $\mathcal{L}^* : Y \rightarrow X$ is defined by the property:*

$$\langle \mathcal{L}x, y \rangle_Y = \langle x, \mathcal{L}^*y \rangle_X$$

for all $x \in X$ and $y \in Y$. Here, $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ are the inner products in the spaces X and Y , respectively.

Definition 1.17 (Fredholm Operator). *A bounded linear operator $\mathcal{L} : Y \rightarrow X$ between Banach spaces is called a **Fredholm operator** if:*

- a) *The kernel of \mathcal{L} , $\ker(\mathcal{L})$, is finite-dimensional.*
- b) *The range $\mathcal{R}(\mathcal{L}(X))$ is closed and has a finite codimension in X .*

The above criteria can be encoded in the Fredholm index, defined as:

$$ind(\mathcal{L}) = \dim[\ker(\mathcal{L})] - codim[\mathcal{R}(\mathcal{L})].$$

This leads to the statement: An operator is Fredholm if and only if it has a finite Fredholm index.

At the beginning of this section, we portrayed the elements of the spectrum $\sigma(\mathcal{L})$ as complex numbers λ for which $(\mathcal{L} - \lambda I)$ is **not** invertible. In a finite-dimensional setting, this simply implies the existence of a non-trivial kernel. However, once we extend this concept to infinite-dimensional spaces, there are multiple situations where invertibility fails due to different reasons, and hence the elements λ of the spectrum would naturally partition the spectrum into subsets, each having implications for the stability analysis. At this stage of the discussion, we dichotomize the spectra into two categories:

- a) The Point Spectrum σ_p
- b) The Essential Spectrum σ_{ess}

Definition 1.18 (The Point Spectrum). *The Point Spectrum consists of those values λ in the spectrum of \mathcal{L} for which $\mathcal{L} - \lambda I$ is not injective, The point spectrum is the set of eigenvalues of the operator, and it can be written as:*

$$\sigma_p(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \mathcal{L} - \lambda I \text{ is not injective and } \lambda \text{ is an eigenvalue}\}$$

Indeed, this is the most familiar type of spectrum, corresponding to eigenvalues in finite-dimensional spaces.

Definition 1.19 (Essential Spectrum). *The essential spectrum consists of those $\lambda \in \mathbb{C}$ such that $\mathcal{L} - \lambda I$ is **not** a Fredholm operator.*

1.4 Floquet Theory

Motivating Question: Given a linear system of ordinary differential equations with periodic variable coefficients $A(t)$, will it follow from periodicity of A that the solution $\mathbf{x}(t)$ is also periodic? A Naive guess would simply suggest "yes". This, however, is generally not the case. We therefore seek methods to understand and analyze the solutions to periodic linear systems of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \quad \mathbf{A}(t+T) = \mathbf{A}(t) \tag{4}$$

1.5 Scalar Systems:

In the case of constant matrices, it is well-known that the solutions are conveniently of the form $\mathbf{x}(t) = \mathbf{x}_0 \exp(t\mathbf{A})$. In fact, if we consider all solutions, namely the Fundamental Matrix Solution (FMS) $\Phi(t)$, we can express Φ as an exponential of the matrix \mathbf{A} .

When it comes to variable matrices $\mathbf{A}(t)$, while existence and uniqueness theorems ensure the existence of the FMS, it soon becomes obvious that Φ is generally *not* the exponential of any matrix. In the case of **periodic** matrices, we have an additional structure imposed on our solutions, making it possible to come up with an investigation of asymptotic behavior of solutions. This is the core idea of **Floquet Theory**, which exploits the mathematical properties of periodic matrices, and seeks to decompose Φ to a periodic part, and an exponential of a **constant** matrix.

Without loss of generality, let us assume that the period is of length π . We begin by considering a scalar problem

$$\dot{x}(t) = a(t)x(t), \quad a(t + \pi) = a(t) \quad (5)$$

We have previously seen that a general solution $\Phi(t)$ can be written in the form:

$$\Phi(t) = \exp\left(\int_0^t a(s) ds\right)$$

Define an average and a (net) deviation from the average

$$\bar{a} := \frac{1}{\pi} \int_0^\pi a(s) ds, \quad p(t) = \int_0^t (a(s) - \bar{a}) ds. \quad (6)$$

Since $a(t)$ is periodic, the net deviation $p(t)$ from its mean is also periodic, since the behavior repeats itself in every cycle. Consider the exponential $P(t) = e^{p(t)}$. Together combined, we see that

$$\Phi(t) = P(t)e^{\bar{a}t}$$

1.6 Floquet Decomposition

A central result of Floquet theory is that the FMS $\Phi(x)$ can be factored into the product of a periodic matrix and an exponential term:

$$\Phi(x) = P(x)e^{Bx}, \quad P(x + T) = P(x),$$

where $P(x) \in \mathbb{C}^{n \times n}$ is periodic with period T , and $B \in \mathbb{C}^{n \times n}$ is a constant matrix, known as the *Floquet matrix*.

The eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of B are called the *Floquet exponents* and are related to the Floquet multipliers via

$$\rho_k = e^{\lambda_k T}, \quad k = 1, 2, \dots, n.$$

The decomposition implies that the solutions of the system can be written in the form

$$\mathbf{y}(x) = P(x)e^{Bx}\mathbf{c},$$

where $\mathbf{c} \in \mathbb{C}^n$ is a constant vector. This representation separates the periodic oscillations (encoded in $P(x)$) from the exponential growth or decay (encoded in e^{Bx}).

1.7 Stability Analysis via Floquet Multipliers

The Floquet multipliers ρ_k determine the stability of the system:

If all $|\rho_k| < 1$, the solutions decay exponentially, and the system is stable.

If any $|\rho_k| > 1$, the solutions grow exponentially, and the system is unstable.

If $|\rho_k| = 1$ for some k , the system is marginally stable, and the long-term behavior depends on higher-order terms.

The connection between the Floquet multipliers and stability makes Floquet theory a powerful tool for analyzing periodic systems, particularly in the context of stability of traveling waves and other periodic structures.

2 Chapter 2: The Periodic Evans Function

2.1 Introduction

Introduced by J. Evans in the 1970s, the Evans function has since become a central concept in the stability analysis of nonlinear waves. In theory, we are discussing a complex-valued analytic function whose zeros correspond to eigenvalues of a linearized operator. It generalizes the Wronskian determinant and is particularly well-suited for boundary-value problems (BVPs) and eigenvalue problems on unbounded domains. The key advantage of the Evans function is its ability to encode spectral information, such as eigenvalue multiplicity and stability, in an analytically tractable form.

After a short introduction to the Evans function for the Sturm-Liouville problem, we will shift our focus on the **periodic** Evans function, which provides us with a rich framework for studying the stability of periodic traveling wave solutions in parabolic systems. Its primary purpose is to identify and characterize the spectrum of the linearized operator about such waves, which is composed entirely of continuous spectrum. That is, the set of all complex numbers $\lambda \in \mathbb{C}$ for which the operator $(\mathcal{L} - \lambda I)$ is injective and has dense range, but is not surjective.

This chapter presents the theoretical framework, mathematical definitions, and properties of the Periodic Evans Function based on *Kapitula and Promislow's "spectral and dynamical stability of nonlinear waves"* [reference here] and Gardner's foundational work. [reference here]

2.2 Sturm-Liouville Operators

The most familiar setting to discuss stability analysis of boundary value problems is perhaps the *Sturm-Liouville* operators on a bounded domain with separated boundary conditions. This is a specific class of differential operators which takes the form:

$$\mathcal{L}(p) := \partial_x^2 p + a_1(x) \partial_x p + a_0(x) p,$$

with spatial coefficients $a_0, a_1 \in \mathcal{C}[-1, 1]$ and real-valued.

The associated BVP is of the form:

$$\mathcal{L}p = \lambda p \text{ for } x \in [-1, 1]$$

with separated boundary conditions,

$$(b_1^-, b_2^-) \begin{pmatrix} p \\ \partial_x p \end{pmatrix} (-1) = 0. \quad (7)$$

$$(b_1^+, b_2^+) \begin{pmatrix} p \\ \partial_x p \end{pmatrix} (+1) = 0. \quad (8)$$

We introduce boundary vectors

$$\mathbf{b}^{a\pm} := (b_1^\pm, b_2^\pm) \text{ where they have norm } \|\mathbf{b}^{a\pm}\| = 1$$

Remark 2.1. *The above restriction on the norm is a convention to ensure that the boundary conditions depend on the direction of the newly introduced $\mathbf{b}^{a\pm}$ and not its magnitude.*

2.3 The Construction of The Evans Function

We begin by rewriting the eigenvalue problem 2.2 as a dynamical system:

$$\text{Let } \mathbf{Y} := \begin{bmatrix} p(x) \\ p'(x) \end{bmatrix} \text{ so that we have } \mathbf{Y}' = \mathbf{A}(x, \lambda)\mathbf{Y}$$

$$\text{where } \mathbf{A}(x, \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - a_0(x) & -a_1(x) \end{pmatrix}$$

Instead of solving explicitly for $\mathbf{Y}(x)$, we intend to understand the geometric structure of the solution space.

Definition 2.1 (Boundary Subspaces). *Let $\mathbf{b}^\pm := (-b_2^\pm, b_1^\pm)^\top$, The boundary space*

$$\mathbb{B}_\pm = \text{span } \mathbf{b}^\pm$$

Definition 2.2.

2.4 Periodic Waves and Linearization

Consider a parabolic system of the form

$$u_t = u_{xx} + f(u, u_x),$$

where $u \in \mathbb{R}^n$. Suppose $U(x)$ is a T -periodic traveling wave solution. By the travelling wave ansatz (That is, substituting $u(x, t) = U(x - ct) + \phi(x - ct, t)$ and linearizing about $U(x)$), we obtain the eigenvalue problem

$$L\phi = \lambda\phi,$$

where L is the linearized operator given by

$$L = -\partial_x^2 - c\partial_x + A(x),$$

and $A(x) = \partial_u f(U(x), U'(x))$. The coefficients of L are T -periodic due to the periodicity of $U(x)$.

The eigenvalue problem can be recast as a first-order system:

$$y' = A(x, \lambda)y,$$

where $y = \begin{pmatrix} \phi \\ \phi' \end{pmatrix}$, and

$$A(x, \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - a(x) & -b(x) \end{pmatrix},$$

with $a(x) = \partial_u f(U(x), U'(x))$ and $b(x) = \partial_{u_x} f(U(x), U'(x))$.

2.5 Floquet Theory and the Monodromy Matrix

As discussed earlier in section 1.4, Floquet theory provides the appropriate framework to study the spectrum for systems with periodic coefficients. The fundamental matrix solution $\mathbf{Y}(x, \lambda)$ of the system

$$\mathbf{y}' = \mathbf{A}(x, \lambda)\mathbf{y},$$

is determined by the initial condition $\mathbf{Y}(0, \lambda) = \mathbb{I}$, where \mathbb{I} is the identity matrix.

The monodromy matrix $\mathbf{Y}(T, \lambda)$ captures the behavior of solutions after one period. Its eigenvalues, known as Floquet multipliers, determine the nature of solutions:

$$\mathbf{Y}(T, \lambda)v = \mu v,$$

where μ are the Floquet multipliers.

The spectrum of L on the space of bounded, uniformly continuous functions consists entirely of continuous spectrum. A point $\lambda \in \sigma(L)$ is in the spectrum if and only if $\mathbf{Y}(T, \lambda)$ has an eigenvalue μ on the unit circle $|\mu| = 1$.

Definition 2.3 (Periodic Evans Function). *The Periodic Evans Function $D(\lambda, y)$ is an analytic function constructed to encode the spectral properties of L . For $y \in S^1$ (the unit circle), a point λ is a y -eigenvalue if $\mathbf{Y}(T, \lambda)$ has an eigenvalue y , or equivalently if*

$$\det(\mathbf{Y}(T, \lambda) - yI) = 0.$$

2.6 Construction of the Evans Function

Let $z_i(x, \lambda)$, $1 \leq i \leq n$, and $\tilde{z}_i(x, \lambda)$, $1 \leq i \leq n$, be two sets of solutions to the first-order system satisfying the following conditions:

$$\begin{aligned} z_i(0, \lambda) &= e_i, & 1 \leq i \leq n, \\ \tilde{z}_i(T, \lambda) &= ye_i, & 1 \leq i \leq n, \end{aligned}$$

where e_i are the standard basis vectors in \mathbb{R}^n .

The Evans Function is then defined as

$$D(\lambda, y) = \det(z_1(T, \lambda) \quad \cdots \quad z_n(T, \lambda) \quad \tilde{z}_1(T, \lambda) \quad \cdots \quad \tilde{z}_n(T, \lambda)).$$

2.7 Properties of the Evans Function

- a) **Analyticity:** The function $D(\lambda, y)$ is analytic in both λ (the spectral parameter) and y (the Floquet multiplier). This analyticity arises from its definition as a determinant involving the fundamental matrix solution $Y(T, \lambda)$. Specifically, $D(\lambda, y) = \det(Y(T, \lambda) - yI)$ inherits analyticity from $Y(T, \lambda)$, which is analytic due to the smooth dependence of the underlying system on λ .
- b) **Roots and Multiplicities:** The roots of $D(\lambda, \gamma)$ correspond to the γ -eigenvalues of L . The multiplicity of a root λ_0 is equal to the algebraic γ -multiplicity of λ_0 as an eigenvalue of L . Formally, this means that if λ_0 is a root of $D(\lambda, y)$, the order of the root matches the dimension of the generalized eigenspace of $Y(T, \lambda) - \gamma I$ at λ_0 .
- c) **Independence of Initial Phase:** The value of $D(\lambda, \gamma)$ is independent of the initial phase $x = 0$. This property reflects the periodic nature of the system: any phase shift in the coordinate x simply permutes the basis of solutions without affecting the determinant $D(\lambda, \gamma)$.

The following needs revisions and change:

2.8 Topological Index and the Bundle Structure

The spectral analysis of L benefits greatly from topological methods, particularly the use of vector bundles and associated invariants. This approach provides a global perspective on the distribution of γ -eigenvalues in the complex plane.

2.8.1 The γ -Eigenvalue Index

To count the number of γ -eigenvalues within a closed curve K in the complex plane, we define the γ -eigenvalue index using topological invariants. Consider the γ -eigenvalue bundle $E(K, \gamma)$, constructed by associating to each $\lambda \in K$ the solutions of the system satisfying the boundary condition

$$z(T, \lambda) = \gamma z(0, \lambda).$$

The first Chern number $c_1(E(K, \gamma))$ of this bundle provides a topological count of the γ -eigenvalues inside K , including their algebraic multiplicities:

$$c_1(E(K, \gamma)) = \text{winding number of } D(\lambda, y) \text{ around } K.$$

Here, the winding number is computed as

$$\text{winding number} = \frac{1}{2\pi i} \int_K \frac{D'(\lambda, \gamma)}{D(\lambda, \gamma)} d\lambda,$$

where $D'(\lambda, \gamma)$ denotes the derivative with respect to λ .

2.8.2 Geometric Characterization

The bundle $E(K, \gamma)$ can be viewed as an n -plane bundle over the cylinder $K \times S^1$, where S^1 represents the unit circle of Floquet multipliers γ . The transition functions of the bundle are determined by the Floquet matrix $Y(T, \lambda)$. Specifically, the bundle is defined such that the fibers over λ contain the space of initial conditions leading to solutions satisfying the periodic boundary condition.

The topology of the bundle, encapsulated in its first Chern number, reflects the spectral geometry of L . For instance: - If $c_1(E(K, \gamma)) = 0$, there are no γ -eigenvalues within K . - If $c_1(E(K, \gamma)) > 0$, the number of γ -eigenvalues inside K is equal to $c_1(E(K, \gamma))$, counting multiplicities.

This characterization links the spectral problem to topological invariants, offering a robust framework for stability analysis.

2.9 Applications to Stability Analysis

The Periodic Evans Function plays a pivotal role in determining the stability of periodic traveling waves. Stability is assessed by analyzing the location of the roots of $D(\lambda, \gamma)$ in the complex plane:

- **Spectral Stability:** The wave is spectrally stable if all roots of $D(\lambda, \gamma)$ satisfy $\text{Re}(\lambda) < 0$. In this case, the spectrum lies entirely in the left half-plane, and no perturbations grow exponentially.
- **Spectral Instability:** If any root of $D(\lambda, \gamma)$ lies in the right half-plane $\text{Re}(\lambda) > 0$, the wave is spectrally unstable. Such roots correspond to growing modes, leading to instability.
- **Period Doubling Bifurcations:** When a Floquet multiplier y crosses -1 on the unit circle, it signals a period-doubling bifurcation. The Evans Function can detect this by identifying changes in the γ -eigenvalue index.
- **Hopf Bifurcations:** If a Floquet multiplier crosses the imaginary axis, it indicates the onset of oscillatory instability, which can also be tracked via $D(\lambda, \gamma)$.

In practical applications, numerical computations of $D(\lambda, \gamma)$ provide a direct method for evaluating stability and detecting bifurcations in systems governed by periodic traveling waves.

3 Chapter 3: Applications to the Generalized Korteweg-de Vries Equation

3.1 Historical Background

What began as a curious observation in 1844 by the Scottish engineer John Scott Russel led to the derivation of a significant partial differential equation applicable to a wide range of physical systems. Russel's empirical relation between the speed of a *solitary* wave, gravitational acceleration, depth, and the maximum amplitude of the wave above the surface was motivated by observing a traveling boat, and the *bow wave* of water preserving its shape and speed for a long time after being detached from the boat's bow. This single humped wave of water was later called a *solitary* wave (or simply a *soliton*).

Definition 3.1 (soliton). *A soliton is a localized, highly stable wave that retains its **identity** (shape and speed).*

Half a decade later, in 1895 mathematicians Diederik Korteweg and Gustav de Vries analytically derived and rediscovered a PDE that was formerly introduced by Joseph Valentin Boussinesq. The now called "Korteweg-de Vries" equation models the disturbance of the surface of the shallow water in presence of solitary water waves. Although this problem was mainly motivated by *water* waves, the range of its applications reaches far beyond hydrodynamics. Hence, it has since evolved into many forms. In this thesis, we will be mainly discussing the *generalized* Korteweg-de Vries equation, and investigate its spectral stability using the Evans function.

3.2 The gKdV Equation

Introduction and Preliminaries

The generalized Korteweg–de Vries (gKdV) equation

$$u_t = u_{xxx} + (f(u))_x - cu_x, \quad (9)$$

is a model for weakly nonlinear and dispersive waves. Standing wave solutions of (9) take the form $u(x, t) = u(x)$ in a co-moving reference frame and satisfy the ordinary differential equation

$$u_{xx} + f(u) - cu = a, \quad (10)$$

or equivalently

$$\frac{u_x^2}{2} + F(u) - \frac{cu^2}{2} = au + E, \quad (11)$$

where a and E are constants of integration and $F'(u) = f(u)$.

These solutions are periodic for certain parameter regimes.

Key conserved quantities of the gKdV equation include:

$$\begin{aligned} M &= \int_T u(x) dx \quad (\text{mass}), \\ P &= \int_T u^2(x) dx \quad (\text{momentum}), \\ H &= \int_T \left(\frac{1}{2} u_x^2 + F(u) \right) dx \quad (\text{Hamiltonian}). \end{aligned}$$

These quantities play a critical role in analyzing the stability of periodic solutions.

Spectral Stability and Evans Function

The linearized operator around a standing wave is given by

$$\partial_x L[u]v = -v_t, \tag{12}$$

where $L[u]$ is a second-order operator. Assuming perturbations of the form $v(x, t) = e^{-\mu t} \phi(x)$ leads to the eigenvalue problem

$$\partial_x L[u]\phi = \mu\phi. \tag{13}$$

Spectral stability requires that the spectrum of $\partial_x L[u]$ lies entirely on the imaginary axis.

The Evans function $E(\mu, \lambda)$, defined via the monodromy map of the linearized problem, characterizes the spectrum. For periodic solutions, $E(\mu, e^{i\kappa})$ determines the Floquet exponents.

Instability Indices

The study of spectral stability involves understanding the interplay between the periodic Evans function and the spectrum of the linearized operator. The spectrum near the origin in the spectral plane is particularly significant, as it relates to long-wavelength perturbations of the wave profile. This region is characterized by the null directions of the linearized operator, which are associated with the tangent space to the manifold of periodic solutions.

Two distinct stability indices are introduced to analyze the spectral properties near the origin:

1. The *orientation index* counts modulo 2 the number of real periodic eigenvalues. It provides insight into the stability of periodic waves under perturbations of the same period. It is computed using the derivative of the Evans function near $\mu = 0$.

2. The *modulational instability index* determines the presence of instabilities associated with long-wavelength perturbations. This index is derived from the characteristic polynomial of the monodromy matrix and reflects the geometric structure of the spectral problem. The orientation index can be expressed in terms of the Jacobian of the map $(a, E, c) \rightarrow (M, P, H)$, while the modulational instability index depends on the third-order derivative of the Evans function at the origin.

Both indices are expressed in terms of conserved quantities and their derivatives, offering a rigorous framework to study the interplay between the dynamics of the traveling wave and the spectral stability of the linearized operator.

Main Results

We can conclude that **if the orientation index is non-zero, the spectrum intersects the real axis, implying instability.**

Modulational instability arises if the discriminant $\Delta(f; u)$ of the Evans function satisfies $\Delta < 0$.

The geometric interpretation connects these indices to the structure of the null space of the linearized operator and the conserved quantities.

Concluding Remarks

These results bridge spectral and modulational stability theories, offering a rigorous framework for understanding periodic solutions of the gKdV equation. Future work could explore extensions to other nonlinear wave equations and their implications in physical systems.

3.3 Extensions to The Transverse Instability