

# *A review of Burgard & Kjaer's Funding strategies, funding costs*

*Imanuel Costigan, Model Risk, Westpac Banking Corporation*

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Burgard & Kjaer (B&K) <sup>1</sup> derive a derivative pricing PDE from an issuer's perspective based on a hedging strategy that eliminates the issuer's market and counterparty credit risk, but which does not necessarily eliminate its own credit risk. I try to clarify some parts of the derivation and insodoing I will highlight one error in it.

<sup>1</sup> C. Burgard and M. Kjaer. Funding costs, funding strategies. Version 12.0, November 2013. URL <http://ssrn.com/abstract=2027195>

## *Correcting the B&K PDE*

### *Background*

A derivative is exposed to the three types of risks:

1. market risk associated with the underlying asset  $S$
2. credit risk associated with the default of the counterparty  $C$  before the derivative's termination date.
3. credit risk associated with the default of the issuer (say a bank)  $B$  before the derivative's termination date.

### *Hedging strategy*

B&K propose the following instruments that will form a self-financing, replicating<sup>2</sup> hedging strategy associated with the derivative as seen from the perspective of  $B$  and which will be used to specify the PDE that defines the derivative's value, denoted by  $\hat{V}$ :

<sup>2</sup> Replicating except perhaps at issuer default

1.  $\delta$  units of the asset  $S$  financed by  $\beta_S$  units of cash from the repo market.
2.  $\alpha_C$  units of the zero-recovery bond  $P_C$  issued by  $C$  financed by  $\beta_C$  units of cash from the repo market.
3.  $\alpha_1$  and  $\alpha_2$  units of the bonds  $P_1$  and  $P_2$  issued by  $B$  which have the recovery rates  $R_1$  and  $R_2$  respectively where without loss of generality,  $R_1 < R_2$  ( $P_1$  is more junior than  $P_2$ ). These bonds are bought back/issued with the hedging strategy's cash surplus/shortfall.

The hedging strategy constituted with these instruments has the value process

$$\Pi^* = \delta S + \beta_S + \alpha_C P_C + \beta_C + \alpha_1 P_1 + \alpha_2 P_2 \quad (1)$$

If the derivative belongs to a netting setting supported by a Credit Support Annex (CSA), then the derivative will be supported by  $X$  units of cash in a collateral pool where  $X > 0$  where the derivative is in-the-money to the issuer. The issuer can use  $X$  to fund the cash needs of the hedging instruments if the collateral can be rehypothecated. As a result, the hedging strategy's value process inclusive of collateral, denoted by  $\Pi$  is:

$$\Pi = \Pi^* + X = \delta S + \beta_S + \alpha_C P_C + \beta_C + \alpha_1 P_1 + \alpha_2 P_2 + X \quad (2)$$

$$d\Pi = \delta dS + d\beta_S + \alpha_C dP_C + d\beta_C + \alpha_1 dP_1 + \alpha_2 dP_2 + dX \quad (3)$$

Note the specification of this hedging strategy is different to the one presented by B&K. In particular the hedging portfolio in (2) contains the term  $+X$  rather than  $-X$  as in B&K. This has significant flow on effects from here on in as it changes the funding constraint formula in (6).

I also note that it is impossible to arrive at the PDE representation presented by B&K even if we use the hedging strategy they specify. I will demonstrate this in a later section of this note.

The funding strategy of the asset  $S$  and the bond  $P_C$  outlined above can be represented as:

$$\delta S + \beta_S = 0 \quad (4)$$

$$\alpha_C P_C + \beta_C = 0 \quad (5)$$

Substituting (4) and (5) into (2), and using the fact that the hedging strategy replicates the derivative value process we find that:

$$\begin{aligned} \Pi &= \alpha_1 P_1 + \alpha_2 P_2 + X \\ &= -(\hat{V} - X) \end{aligned}$$

As a result we find the following funding constraint:

$$\hat{V} + \alpha_1 P_1 + \alpha_2 P_2 = 0 \quad (6)$$

### *Dynamics*

B&K set out the following dynamics for elements of the hedging strategy:

$$dS = \mu S dt + \sigma S dW \quad (7)$$

$$d\beta_S = \delta(\gamma_S - q_S) S dt \quad (8)$$

$$dP_C = r_C P_C dt - P_C dJ_C \quad (9)$$

$$d\beta_C = -\alpha_C q_C P_C dt \quad (10)$$

$$dP_i = r_i P_i dt - \bar{R}_i P_i dJ_B \quad i = 1, 2 \quad (11)$$

$$dX = -r_X X dt \quad (12)$$

Note that:

1.  $\gamma_S$  and  $q_S$  denote the dividend income from and financing rate for  $S$  respectively;
2.  $q_C$  denotes the financing rate for  $P_C$ ;
3.  $J_C$  and  $J_B$  are the independent jump processes denoting the default of  $C$  and  $B$  respectively;
4. B&K assume that there is zero basis between  $r_1$  and  $r_2$  so that  $r_i = r + \hat{R}_i \lambda_B$ ;
5.  $\bar{R}_i$  denotes the loss on the (pre)-default value of the bond  $P_i$  so that  $\bar{R}_i = 1 - R_i$ <sup>3</sup>; and
6.  $r_X$  denotes the rate accruing on the posted collateral  $X$ .

<sup>3</sup>  $R_C = 0$  by definition

When we substitute the dynamics of (7) – (12) into (3) and collecting the risk terms associated with  $dS$ ,  $dJ_B$  and  $dJ_C$  as well as the drift terms  $dt$  we find that:

$$\begin{aligned} d\Pi &= \delta dS + d\beta_S + \alpha_C dP_C + d\beta_C + \alpha_1 dP_1 + \alpha_2 dP_2 + dX \\ &= \delta dS + \delta(\gamma_S - q_S) S dt + \alpha_C (r_C P_C dt - P_C dJ_C) - \alpha_C q_C P_C dt \\ &\quad + \alpha_1 (r_1 P_1 dt - \bar{R}_1 P_1 dJ_B) + \alpha_2 (r_2 P_2 dt - \bar{R}_2 P_2 dJ_B) - r_X X dt \\ &= \delta dS - \alpha_C P_C dJ_C - (\alpha_1 \bar{R}_1 P_1 + \alpha_2 \bar{R}_2 P_2) dJ_B \\ &\quad + (\delta(\gamma_S - q_S) S + (r_C - q_C) \alpha_C P_C + \alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 \\ &\quad - r_X X) dt \end{aligned} \quad (13)$$

Then using the assumption that  $r_1$  and  $r_2$  have no basis (see the notes above) and denoting  $P = \alpha_1 P_1 + \alpha_2 P_2$  and  $P_D = \alpha R_1 P_1 + \alpha R_2 P_2$  we find that:

$$\begin{aligned} \alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 &= \alpha_1 (r + \hat{R}_1 \lambda_B) P_1 + \alpha_2 (r + \hat{R}_2 \lambda_B) P_2 \\ &= rP + \lambda_B (P - P_D) \end{aligned} \quad (14)$$

Then, substituting 14 into 13, we find that:

$$\begin{aligned} d\Pi = & \delta dS - \alpha_C P_C dJ_C - (P - P_D) dJ_B \\ & + (\delta(\gamma_S - q_S)S + (r_C - q_C)\alpha_C P_C \\ & + rP + \lambda_B(P - P_D) - r_X X) dt \end{aligned} \quad (15)$$

### PDE

B&K present the PDE associated with the derivative's value when exposed to counterparty and issuer risk:

$$\begin{aligned} d\hat{V} = & \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2} \sigma^2 S^2 \partial_S^2 \hat{V} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C \\ = & \partial_S \hat{V} dS + \Delta \hat{V}_C dJ_C + \Delta \hat{V}_B dJ_B + \left( \partial_t \hat{V} + \frac{1}{2} \sigma^2 S^2 \partial_S^2 \hat{V} \right) dt \end{aligned} \quad (16)$$

where  $\Delta \hat{V}_B = g_B - \hat{V}$ ,  $\Delta \hat{V}_C = g_C - \hat{V}$  and  $g_B$  and  $g_C$  denote the close-out values of the derivative when  $B$  and  $C$  default.

### Net position

Let's now consider the change in the issuer's net value of the hedging portfolio (15) and the derivative (16):

$$\begin{aligned} d\hat{V} + d\Pi = & (\partial_S \hat{V} + \delta) dS + (\Delta \hat{V}_C - \alpha_C P_C) dJ_C \\ & + (\Delta \hat{V}_B - (P - P_D)) dJ_B \\ & + \left( \partial_t \hat{V} + \frac{1}{2} \sigma^2 S^2 \partial_S^2 \hat{V} \right. \\ & \left. + \delta(\gamma_S - q_S)S + (r_C - q_C)\alpha_C P_C \right. \\ & \left. + rP + \lambda_B(P - P_D) - r_X X \right) dt \end{aligned} \quad (17)$$

$$\triangleq X dS + Y dJ_C + Z dt \quad (18)$$

Suppose that the issuer would like to perfectly hedge the P&L due to variations in  $S$  or  $J_C$ . Then from (17) we can see that this can be achieved by holding the following units of  $S$  and  $P_C$  respectively. <sup>4</sup>

$$\delta = -\partial_S \hat{V} \quad (19)$$

$$\alpha_C = \frac{\Delta \hat{V}_C}{P_C} \triangleq \frac{g_C - \hat{V}}{P_C} \quad (20)$$

<sup>4</sup> As there may be restrictions on the issuer's hedging strategy associated with its own default, we do not, for the moment, assume that the issuer wishes to hedge its own default risk

When the holdings of  $S$  and  $P_C$  conform to these quantities,  $X = Y = 0$ , denoting  $\lambda_C = r_C - q_C$  and  $\mathcal{A}_t \hat{V} = \frac{1}{2} \sigma^2 S^2 \partial_S^2 \hat{V} - (\gamma_S - q_S) S \partial_S \hat{V}$

and using the funding constraint in (6) along with the notation we introduced earlier for  $P$ .

$$\begin{aligned}
Zdt &= \left( \partial_t \hat{V} + \frac{1}{2} \sigma^2 S^2 \partial_S^2 \hat{V} - (\gamma_S - q_S) S \partial_S \hat{V} \right. \\
&\quad \left. + (r_C - q_C)(g_C - \hat{V}) \right. \\
&\quad \left. + rP + \lambda_B(P - P_D) - r_X X \right) dt \\
&= (\partial_t \hat{V} + \mathcal{A}_t \hat{V} + \lambda_C(g_C - \hat{V}) \\
&\quad - r\hat{V} - \lambda_B(\hat{V} + P_D) - r_X X) dt
\end{aligned} \tag{21}$$

Denoting  $\tilde{\epsilon}_h = g_B + P_D$  we find that:

$$\begin{aligned}
d\hat{V} + d\Pi &= (\Delta \hat{V}_B - (P - P_D)) dJ_B \\
&\quad + (\partial_t \hat{V} + \mathcal{A}_t \hat{V} + \lambda_C(g_C - \hat{V}) \\
&\quad - r\hat{V} - \lambda_B(\hat{V} + P_D) - r_X X) dt \\
&= (g_B - \hat{V} + (\hat{V} + P_D)) dJ_B \\
&\quad + (\partial_t \hat{V} + \mathcal{A}_t \hat{V} + \lambda_C(g_C - \hat{V}) \\
&\quad - r\hat{V} - \lambda_B(\hat{V} + P_D) - r_X X) dt \\
&= \tilde{\epsilon}_h dJ_B + (\partial_t \hat{V} + \mathcal{A}_t \hat{V} + \lambda_C(g_C - \hat{V}) \\
&\quad - r\hat{V} - \lambda_B(\hat{V} + P_D) - r_X X) dt \\
&= (\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B) \hat{V} \\
&\quad + \lambda_C g_C - \lambda_B P_D - r_X X) + \tilde{\epsilon}_h dJ_B \\
&= (\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B) \hat{V} \\
&\quad + \lambda_C g_C - \lambda_B(\tilde{\epsilon}_h - g_B) - r_X X) + \tilde{\epsilon}_h dJ_B \\
&= (\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B) \hat{V} \\
&\quad + \lambda_C g_C + \lambda_B g_B - r_X X - \lambda_B \tilde{\epsilon}_h) + \tilde{\epsilon}_h dJ_B
\end{aligned} \tag{22}$$

### Revised PDE

For the hedging strategy to perfectly replicate the derivative, we need the following conditions to hold:

$$0 = \tilde{\epsilon}_h = g_B + P_D \tag{23}$$

$$\begin{aligned}
0 &= \partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B) \hat{V} \\
&\quad + \lambda_C g_C + \lambda_B g_B - r_X X - \lambda_B \tilde{\epsilon}_h
\end{aligned} \tag{24}$$

This compares to the PDE presented by B&K:

$$\begin{aligned}
0 &= \partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B) \hat{V} \\
&\quad + \lambda_C g_C + \lambda_B g_B - (r_X - r)X - \lambda_B \tilde{\epsilon}_h
\end{aligned} \tag{25}$$

where  $\tilde{\epsilon}_h = g_B + P_D - X$ .

### *Problem with B&K PDE formulation*

Let's return to my earlier claim that it is impossible to arrive at the PDE representation presented by B&K even if we use the hedging strategy they specify. B&K's hedging portfolio is specified as (I have highlighted the difference to my specification in red):

$$\Pi = \delta S + \beta_S + \alpha_C P_C + \beta_C + \alpha_1 P_1 + \alpha_2 P_2 - X \quad (26)$$

$$d\Pi = \delta dS + d\beta_S + \alpha_C dP_C + d\beta_C + \alpha_1 dP_1 + \alpha_2 dP_2 - dX \quad (27)$$

Importantly, the funding constraint (6) becomes:

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0 \quad (28)$$

The PDE remains unchanged from (16) so that following from (17) changes in the net position becomes:

$$\begin{aligned} d\hat{V} + d\Pi &= (\partial_S \hat{V} + \delta) dS + (\Delta \hat{V}_C - \alpha_C P_C) dJ_C \\ &\quad + (\Delta \hat{V}_B - (P - P_D)) dJ_B \\ &\quad + \left( \partial_t \hat{V} + \frac{1}{2} \sigma^2 S^2 \partial_S^2 \hat{V} \right. \\ &\quad \left. + \delta(\gamma_S - q_S) S + (r_C - q_C) \alpha_C P_C \right. \\ &\quad \left. + rP + \lambda_B(P - P_D) + r_X X \right) dt \end{aligned} \quad (29)$$

$$\triangleq X dS + Y dJ_C + Z dt \quad (30)$$

as  $dX = -r_X X dt$  (as per (12)). This is different to the net position expression given by B&K in equation (47) of their working paper<sup>5</sup> (with the difference highlighted in red).

Using B&K's funding constraint (28) and the hedge ratios for  $S$  (19) and  $P_C$  (20) we find that:

$$\begin{aligned} Z dt &= \left( \partial_t \hat{V} + \frac{1}{2} \sigma^2 S^2 \partial_S^2 \hat{V} - (\gamma_S - q_S) S \partial_S \hat{V} \right. \\ &\quad \left. + (r_C - q_C)(g_C - \hat{V}) \right. \\ &\quad \left. + rP + \lambda_B(P - P_D) + r_X X \right) dt \\ &= (\partial_t \hat{V} + \mathcal{A}_t \hat{V} + \lambda_C(g_C - \hat{V}) \end{aligned} \quad (31)$$

$$+ r(X - \hat{V}) + \lambda_B(X - \hat{V} + P_D) + r_X X) dt \quad (32)$$

And consequently, with  $\epsilon_h = g_B + P_D - X$ , we find that:

<sup>5</sup> C. Burgard and M. Kjaer. Funding costs, funding strategies. Version 12.0, November 2013. URL <http://ssrn.com/abstract=2027195>

$$\begin{aligned}
d\hat{V} + d\Pi &= (\Delta\hat{V}_B - (P - P_D))dJ_B \\
&\quad + (\partial_t\hat{V} + \mathcal{A}_t\hat{V} + \lambda_C(g_C - \hat{V}) \\
&\quad \quad + r(X - \hat{V}) + \lambda_B(X - \hat{V} + P_D) + r_X X) dt \\
&= (g_B - \hat{V} - (X - \hat{V} - P_D))dJ_B \\
&\quad + (\partial_t\hat{V} + \mathcal{A}_t\hat{V} + \lambda_C(g_C - \hat{V}) \\
&\quad \quad + r(X - \hat{V}) + \lambda_B(X - \hat{V} + P_D) + r_X X) dt \\
&= \epsilon_h dJ_B + (\partial_t\hat{V} + \mathcal{A}_t\hat{V} + \lambda_C(g_C - \hat{V}) \\
&\quad \quad + r(X - \hat{V}) + \lambda_B(X - \hat{V} - P_D) + r_X X) dt \\
&= (\partial_t\hat{V} + \mathcal{A}_t\hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\quad \quad + \lambda_C g_C + (X - P_D)\lambda_B + (r + r_X)X) + \epsilon_h dJ_B \\
&= (\partial_t\hat{V} + \mathcal{A}_t\hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\quad \quad + \lambda_C g_C + (g_B - \epsilon_h)\lambda_B + (r + r_X)X) + \epsilon_h dJ_B \\
&= (\partial_t\hat{V} + \mathcal{A}_t\hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\quad \quad + \lambda_C g_C + \lambda_B g_B + (r + r_X)X - \lambda_B \epsilon_h) + \epsilon_h dJ_B \quad (33)
\end{aligned}$$

For the hedging strategy to perfectly replicate the derivative, we need the following conditions to hold:

$$0 = \tilde{\epsilon}_h = g_B + P_D - X \quad (34)$$

$$\begin{aligned}
0 &= \partial_t\hat{V} + \mathcal{A}_t\hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\quad + \lambda_C g_C + \lambda_B g_B + (r + r_X)X - \lambda_B \epsilon_h \quad (35)
\end{aligned}$$

The difference between (35) and the one presented by B&K is that the coefficient of  $X$  is  $r + r_X$  rather than  $r_X - r$  (respectively) - the latter which would appear to have a more natural explanation. Unfortunately, the latter term does not follow from the specified hedging strategy.

## References

C. Burgard and M. Kjaer. Funding costs, funding strategies. Version 12.0, November 2013. URL <http://ssrn.com/abstract=2027195>.