A review of Burgard & Kjaer's Funding strategies, funding costs

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Burgard & Kjaer (B&K) [1] develop a derivative pricing PDE based on a hedging strategy that eliminates market, counterparty credit risk, but which does not necessarily eliminate all own credit risk. I review the development of the pricing PDE, suggest some amendments to their hedging strategy's specification and highlight an error in their conclusion even when using their hedging strategy specification.

[1] C. Burgard and M. Kjaer. Funding costs, funding strategies. Version 12.0, November 2013. URL `http://ssrn.com/abstract=2027195`

Correcting the B&K PDE

Background

A derivative is exposed to the three types of risks:

1. Market risk associated with the underlying asset S
2. Credit risk associated with the default of the counterparty C before the derivative's termination date.
3. Credit risk associated with the default of the issuer (say a bank) B before the derivative's termination date.

Hedging strategy

We can specify a self-financing, replicating[2] hedging strategy associated with the derivative as seen from the perspective of B:

[2] Replicating except perhaps at issuer default

1. δ units of the asset S financed by β_S units of cash from the repo market.
2. α_C units of the zero-recovery bond P_C issued by C financed by β_C units of cash from the repo market.
3. α_1 and α_2 units of the bonds P_1 and P_2 issued by B which have the recovery rates R_1 and R_2 respectively where without loss of generality, $R_1 < R_2$ (P_1 is more junior than P_2). These bonds are bought back/issued with the hedging strategy's cash surplus/shortfall.

The hedging strategy constituted with these instruments has the value process

$$\Pi^* = \delta S + \beta_S + \alpha_C P_C + \beta_C + \alpha_1 P_1 + \alpha_2 P_2 \tag{1}$$

If the derivative belongs to a netting setting defined by a Credit Support Annex (CSA), then the derivative will be supported by X

units of cash in a collateral pool where $X > 0$ where the derivative is in-the-money to the issuer. The issuer can use X to fund the cash needs of the hedging instruments if the collateral can be rehypothecated. As a result, the hedging strategy's value process inclusive of collateral, denoted by Π is:

$$\Pi = \Pi^* + X = \delta S + \beta_S + \alpha_C P_C + \beta_C + \alpha_1 P_1 + \alpha_2 P_2 + X \quad (2)$$
$$d\Pi = \delta dS + d\beta_S + \alpha_C dP_C + d\beta_C + \alpha_1 dP_1 + \alpha_2 dP_2 + dX \quad (3)$$

Note the specification of this hedging strategy is different to the one presented by B&K. In particular the hedging strategy represented by (2) contains the term $+X$ rather than $-X$ as in B&K. As we will see, this changes the resulting PDE representation of the derivative's price. [3]

[3] Even if we use the hedging strategy B&K specify I will demonstrate in a later section that it is impossible to arrive at B&Ks PDE.

The funding strategy of the asset S and the bond P_C outlined above can be represented as:

$$\delta S + \beta_S = 0 \quad (4)$$
$$\alpha_C P_C + \beta_C = 0 \quad (5)$$

Substituting (4) and (5) into (2), and using the fact that the hedging strategy replicates the derivative value process we find that:

$$\begin{aligned} \Pi &= \alpha_1 P_1 + \alpha_2 P_2 + X \\ &= -\hat{V} \end{aligned}$$

As a result we find the following funding constraint:

$$\hat{V} + X + \alpha_1 P_1 + \alpha_2 P_2 = 0 \quad (6)$$

Dynamics

B&K set out the following dynamics for elements of the hedging strategy:

$$dS = \mu S dt + \sigma S dW \quad (7)$$
$$d\beta_S = \delta(\gamma_S - q_S) S dt \quad (8)$$
$$dP_C = r_C P_C dt - P_C dJ_C \quad (9)$$
$$d\beta_C = -\alpha_C q_C P_C dt \quad (10)$$
$$dP_i = r_i P_i dt - \bar{R}_i P_i dJ_B \quad i = 1, 2 \quad (11)$$
$$dX = -r_X X dt \quad (12)$$

Note that:

1. γ_S and q_S denote the dividend income from and financing rate for S respectively;
2. q_C denotes the financing rate for P_C;
3. J_C and J_B are the independent jump processes denoting the default of C and B respectively;
4. B&K assume that there is zero basis between r_1 and r_2 so that $r_i = r + \hat{R}_i \lambda_B$;
5. $\bar{R}_i$ denotes the loss on the (pre)-default value of the bond P_i so that $\bar{R}_i = 1 - R_i$[4]; and
6. r_X denotes the rate accruing on the posted collateral X.

[4] $R_C = 0$ by definition

When we substitute the dynamics of (7) – (12) into (3) and collecting the risk terms associated with dS, dJ_B and dJ_C as well as the drift terms dt we find that:

$$\begin{aligned}
d\Pi &= \delta dS + d\beta_S + \alpha_C dP_C + d\beta_C + \alpha_1 dP_1 + \alpha_2 dP_2 + dX \\
&= \delta dS + \delta(\gamma_S - q_S) S dt + \alpha_C (r_C P_C dt - P_C dJ_C) - \alpha_C q_C P_C dt \\
&\quad + \alpha_1 (r_1 P_1 dt - \bar{R}_1 P_1 dJ_B) + \alpha_2 (r_2 P_2 dt - \bar{R}_2 P_2 dJ_B) - r_X X dt \\
&= \delta dS - \alpha_C P_C dJ_C - (\alpha_1 \bar{R}_1 P_1 + \alpha_2 \bar{R}_2 P_2) dJ_B \\
&\quad + \ (\delta(\gamma_S - q_S) S + (r_C - q_C)\alpha_C P_C + \alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 \\
&\qquad - r_X X)\, dt
\end{aligned} \tag{13}$$

Then using the assumption that r_1 and r_2 have no basis (see the notes above) and denoting $P = \alpha_1 P_1 + \alpha_2 P_2$ and $P_D = \alpha R_1 P_1 + \alpha R_2 P_2$ we find that:

$$\begin{aligned}
\alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 &= \alpha_1 (r + \hat{R}_1 \lambda_B) P_1 + \alpha_2 (r + \hat{R}_2 \lambda_B) P_2 \\
&= rP + \lambda_B (P - P_D)
\end{aligned} \tag{14}$$

Then, substituting (14) into (13), we find that:

$$\begin{aligned}
d\Pi = {} & \delta dS - \alpha_C P_C dJ_C - (P - P_D) dJ_B \\
& + \ (\delta(\gamma_S - q_S) S + (r_C - q_C)\alpha_C P_C \\
& \quad + rP + \lambda_B (P - P_D) - r_X X)\, dt
\end{aligned} \tag{15}$$

PDE

As in B&K, the PDE associated with the derivative's value when exposed to counterparty and issuer risk:

$$
\begin{aligned}
d\hat{V} &= \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2}\sigma^2 S^2 \partial^2 \hat{V}_S dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C \\
&= \partial_S \hat{V} dS + \Delta \hat{V}_C dJ_C + \Delta \hat{V}_B dJ_B + \left(\partial_t \hat{V} + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} \right) dt \qquad (16)
\end{aligned}
$$

where $\Delta \hat{V}_B = g_B - \hat{V}$, $\Delta \hat{V}_C = g_C - \hat{V}$ and g_B and g_C denote the close-out values of the derivative when B and C default[5].

[5] See B&Ks paper for what they term "regular" close-out values

Net position

Let's now consider the change in the issuer's net value of the hedging portfolio (15) and the derivative (16):

$$
\begin{aligned}
d\hat{V} + d\Pi = &(\partial_S \hat{V} + \delta) dS + (\Delta \hat{V}_C - \alpha_C P_C) dJ_C + (\Delta \hat{V}_B - (P - P_D)) dJ_B \\
&+ \Big(\partial_t \hat{V} + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} + \delta(\gamma_S - q_S) S \\
&\quad + (r_C - q_C)\alpha_C P_C + rP + \lambda_B (P - P_D) - r_X X \Big) dt \qquad (17) \\
\triangleq\ &X dS + Y_C dJ_C + Y_B dJ_B + Z dt \qquad (18)
\end{aligned}
$$

Suppose that the issuer would like to perfectly hedge the P&L due to variations in S or J_C. Then from (17) we can see that this can be achieved by holding the following units of S and P_C respectively. [6]

[6] As there may be restrictions on the issuer's hedging strategy associated with its own default, we do not, for the moment, assume that the the issuer wishes to hedge its own default risk

$$
\begin{aligned}
\delta &= -\partial_S \hat{V} \qquad (19) \\
\alpha_C &= \frac{\Delta \hat{V}_C}{P_C} \triangleq \frac{g_C - \hat{V}}{P_C} \qquad (20)
\end{aligned}
$$

When the holdings of S and P_C conform to these quantities, $X = Y = 0$, denoting $\lambda_C = r_C - q_C$ and $\mathcal{A}_t \hat{V} = \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} - (\gamma_S - q_S) S \partial_S \hat{V}$ and using the funding constraint in (6) along with the notation we introduced earlier for P.

$$
\begin{aligned}
Zdt = &\Big(\partial_t \hat{V} + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} - (\gamma_S - q_S) S \partial_S \hat{V} \\
&\quad + (r_C - q_C)(g_C - \hat{V}) + rP + \lambda_B (P - P_D) - r_X X \Big) dt \qquad (21) \\
= &\big(\partial_t \hat{V} + \mathcal{A}_t \hat{V} + \lambda_C (g_C - \hat{V}) \\
&\quad - r(\hat{V} + X) - \lambda_B (\hat{V} + X + P_D) - r_X X \big) dt
\end{aligned}
$$

Denoting $\tilde{\epsilon}_h = g_B + P_D + X$ we find that:

$$
\begin{aligned}
d\hat{V} + d\Pi &= \Upsilon_B dJ_B + Z dt \\
&= (\Delta \hat{V}_B - (P - P_D)) dJ_B \\
&\quad + \big(\partial_t \hat{V} + \mathcal{A}_t \hat{V} + \lambda_C (g_C - \hat{V}) \\
&\qquad - r(\hat{V} + X) - \lambda_B(\hat{V} + X + P_D) - r_X X\big)\, dt \\
&= (g_B - \hat{V} + (\hat{V} + X + P_D)) dJ_B \\
&\quad + \big(\partial_t \hat{V} + \mathcal{A}_t \hat{V} + \lambda_C (g_C - \hat{V}) \\
&\qquad - r(\hat{V} + X) - \lambda_B(\hat{V} + X + P_D) - r_X X\big)\, dt \\
&= \big(\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\qquad + \lambda_C g_C - \lambda_B P_D - (r + r_X + \lambda_B) X\big) + \tilde{\epsilon}_h dJ_B \\
&= \big(\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\qquad + \lambda_C g_C - \lambda_B(\tilde{\epsilon}_h - g_B - X) - (r + r_X + \lambda_B) X\big) + \tilde{\epsilon}_h dJ_B \\
&= \big(\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\qquad + \lambda_C g_C + \lambda_B g_B - (r + r_X) X - \lambda_B \tilde{\epsilon}_h\big) + \tilde{\epsilon}_h dJ_B
\end{aligned}
$$

Revised PDE representation

For the hedging strategy to perfectly replicate the derivative, we need the following conditions to hold:

$$
0 = \tilde{\epsilon}_h = g_B + P_D + X \tag{22}
$$

$$
\begin{aligned}
0 &= \partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\quad + \lambda_C g_C + \lambda_B g_B - (r + r_X) X - \lambda_B \tilde{\epsilon}_h
\end{aligned} \tag{23}
$$

This compares to the PDE presented by B&K:

$$
\begin{aligned}
0 &= \partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\quad + \lambda_C g_C + \lambda_B g_B - (r_X - r) X - \lambda_B \tilde{\epsilon}_h
\end{aligned} \tag{24}
$$

where $\tilde{\epsilon}_h = g_B + P_D - X$.

Problem with B&K PDE derivation

Let's return to my earlier claim that it is impossible to arrive at the PDE representation presented by B&K even if we use the hedging strategy they specify. B&K's hedging portfolio is specified as (I have highlighted the difference to my specification in red):

$$
\Pi = \delta S + \beta_S + \alpha_C P_C + \beta_C + \alpha_1 P_1 + \alpha_2 P_2 - X \tag{25}
$$

$$
d\Pi = \delta dS + d\beta_S + \alpha_C dP_C + d\beta_C + \alpha_1 dP_1 + \alpha_2 dP_2 - dX \tag{26}
$$

Importantly, the funding constraint (6) becomes:

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0 \tag{27}$$

The PDE remains unchanged from (16) so that following from (17) changes in the net position becomes:

$$\begin{aligned}
d\hat{V} + d\Pi = {} & (\partial_S \hat{V} + \delta)dS + (\Delta \hat{V}_C - \alpha_C P_C)dJ_C \\
& + (\Delta \hat{V}_B - (P - P_D))dJ_B \\
& + \bigg(\partial_t \hat{V} + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} \\
& \quad + \delta(\gamma_S - q_S)S + (r_C - q_C)\alpha_C P_C \\
& \quad + rP + \lambda_B(P - P_D) + r_X X \bigg) dt \qquad (28) \\
\triangleq {} & X dS + Y dJ_C + Z dt \qquad (29)
\end{aligned}$$

as $dX = -r_X X dt$ (as per (12)). This is different to the net position expression given by B&K in equation (47) of their working paper[7] (with the difference highlighted in red).

[7] C. Burgard and M. Kjaer. Funding costs, funding strategies. Version 12.0, November 2013. URL http://ssrn.com/abstract=2027195

Using B&K's funding constraint (27) and the hedge ratios for S (19) and P_C (20) we find that:

$$\begin{aligned}
Z dt = {} & \bigg(\partial_t \hat{V} + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} - (\gamma_S - q_S) S \partial_S \hat{V} \\
& + (r_C - q_C)(g_C - \hat{V}) \\
& + rP + \lambda_B(P - P_D) + r_X X \bigg) dt \\
= {} & \big(\partial_t \hat{V} + \mathcal{A}_t \hat{V} + \lambda_C (g_C - \hat{V}) \qquad (30) \\
& + r(X - \hat{V}) + \lambda_B (X - \hat{V} + P_D) + r_X X \big)\, dt \qquad (31)
\end{aligned}$$

And consequently, with $\epsilon_h = g_B + P_D - X$, we find that:

$$\begin{aligned}
d\hat{V} + d\Pi &= (\Delta\hat{V}_B - (P - P_D))dJ_B \\
&\quad + \big(\partial_t\hat{V} + \mathcal{A}_t\hat{V} + \lambda_C(g_C - \hat{V}) \\
&\qquad + r(X - \hat{V}) + \lambda_B(X - \hat{V} + P_D) + r_X X\big)\, dt \\
&= (g_B - \hat{V} - (X - \hat{V} - P_D))dJ_B \\
&\quad + \big(\partial_t\hat{V} + \mathcal{A}_t\hat{V} + \lambda_C(g_C - \hat{V}) \\
&\qquad + r(X - \hat{V}) + \lambda_B(X - \hat{V} + P_D) + r_X X\big)\, dt \\
&= \epsilon_h dJ_B + \big(\partial_t\hat{V} + \mathcal{A}_t\hat{V} + \lambda_C(g_C - \hat{V}) \\
&\qquad + r(X - \hat{V}) + \lambda_B(X - \hat{V} - P_D) + r_X X\big)\, dt \\
&= \big(\partial_t\hat{V} + \mathcal{A}_t\hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\qquad + \lambda_C g_C + (X - P_D)\lambda_B + (r + r_X)X\big) + \epsilon_h dJ_B \\
&= \big(\partial_t\hat{V} + \mathcal{A}_t\hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\qquad + \lambda_C g_C + (g_B - \epsilon_h)\lambda_B + (r + r_X)X\big) + \epsilon_h dJ_B \\
&= \big(\partial_t\hat{V} + \mathcal{A}_t\hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\qquad + \lambda_C g_C + \lambda_B g_B + (r + r_X)X - \lambda_B\epsilon_h\big) + \epsilon_h dJ_B \qquad (32)
\end{aligned}$$

For the hedging strategy to perfectly replicate the derivative, we need the following conditions to hold:

$$\begin{aligned}
0 &= \tilde{\epsilon}_h = g_B + P_D - X \qquad (33) \\
0 &= \partial_t\hat{V} + \mathcal{A}_t\hat{V} - (r + \lambda_C + \lambda_B)\hat{V} \\
&\quad + \lambda_C g_C + \lambda_B g_B + (r + r_X)X - \lambda_B\epsilon_h \qquad (34)
\end{aligned}$$

The difference between (34) and the one presented by B&K is that the coefficient of X is $r + r_X$ rather than $r_X - r$ (respectively) - the latter which would appear to have a more natural explanation. Unfortunately, the latter term does not follow from the specified hedging strategy while the former is consistent with the alternative hedging strategy specification I outlined above.

References

C. Burgard and M. Kjaer. Funding costs, funding strategies. Version 12.0, November 2013. URL `http://ssrn.com/abstract=2027195`.