ESSAYS ON FORMATION OF NETWORKS AND ITS APPLICATIONS

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Dedicated to my husband, Rahul.

We have toiled through this together.

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Pallavi Baral

ESSAYS ON FORMATION OF NETWORKS AND ITS APPLICATIONS

Abstract: Given the extent and value of social and financial interconnections today, the concept of a 'network' holds a lot of relevance. Economic entities may be viewed as part of a network, where their decisions affect those of their connections as well. This dissertation focuses on the analysis and prediction of these strategic decisions undertaken in different kinds of networks (e.g., interbank, inter-country, etc.), using tools from Game Theory and Microeconometrics. Each chapter presents either a theoretical or an econometric method for analyzing intra-network strategic interactions between economic agents and the ensuing evolution of new networks resulting from the choices that they make. The research presented here makes two contributions in this regard. The first pertains to the study of endogenous risk of contagion within an interbank network following an exogenous shock. Subsequent to such a shock, the nodes might re-evaluate their connections in order to prevent themselves from getting hit by it. A stochastic game of network formation is designed to model these decisions, which play a role in the rate of shock diffusion through the network. The game ultimately leads to a threshold condition in terms of the number of connections per node that could induce a state of complete collapse. The model also identifies network structures that are more resistant to widespread financial contagion. The second contribution is an econometric algorithm to create and analyze networks. Relative to the traditional methods based on Maximum Entropy, this copula based algorithm provides more flexibility for simulation of networks, given asymmetric data derived from certain data generating processes. The dissertation also discusses other innovative ways to analyze connections over networks using advanced econometric tools.

Contents

Chapter 1

Introduction to Networks in Economics

1.1 Introduction

The term 'network' has become more relevant than ever, thanks to people trying to *connect* on social forums such as Facebook or those trying to develop business connections through platforms such as LinkedIn. One of the important factors driving this is the fact that networks have emerged and exponentially engulfed us in whatever we do due to the growing interdependencies across stratas of social and financial interconnections. Thus, researchers must recognize their impacting presence and construct novel methods to account for 'network effects' while analyzing relevant research problems. Network theory can prove to be very useful in that regard as the scope of its applications has widened enormously. The areas in which network theory can be applied ranges from those in hard sciences like computer science, neurology and biology to those in social sciences like political science, sociology, psychology and economics, among others. Thus, it is hard to ignore the usefulness of this theory as a tool in modeling. Especially, in economics, it promises to be an extremely efficient toolkit for analyzing complex economic structures.

There has been a continuous development of equilibrium analyses within game theory based on networks since the path-breaking paper by Jackson and Wollinsky (1996, henceforth JW). Although there were earlier papers that talked about economic applications of network theory, JW threaded all these ideas together to establish a formal theory. Subsequently, a lot of research has taken place that has contributed to this literature. Since network theory combined with game theory promises to provide an excellent toolkit for a more efficient analysis of economic problems, the natural question that follows relates to the applicability of the theoretical work done already. To that extent, the notable papers that engage in answering this important question include Anderson, Wasserman and Crouch (1999, henceforth AWC), Mayer and Puller (2007, henceforth MP), Copic, Jackson and Kirman (2009, henceforth CJK)[1], Chapman (2010), Christakis, Fowler, Imbens and Kalyanaraman (2010, henceforth CFIK), among others.

This dissertation intends to add to this literature by introducing novel methods of modeling economic problems using networks. Each economic environment may be viewed as a network where the economic agents are nodes and the links between them are defined based on their strategies or nature of association with each other. Thus, if the strategies change or the the association changes, so does the network. In other words, the strategic interactions between the nodes of the network result in the formation of a new network. This strand of game theory has come to be known as *network formation game theory*. The chapters in this dissertation use tools from network formation game theory as well as econometrics in order to address different aspects of interesting economic problems that not only include theoretical modeling of complex economic interactions but also empirical analysis of connections that are a consequence of these interactions.

The remainder of this chapter has been divided into the following sections. Section 2 indulges in a brief literature survey to familiarize the reader with the recent as well as ongoing research in this area. Section 3 discusses important concepts in network formation games and section 4 provides an overview of the dissertation.

[1]The first draft version available is dated August, 2005.

1.2 Literature Review

We begin our discussion in this section with a very brief review of Jackson's book on social and economic networks. This is the first book of its kind and in its own right a pioneer in establishing network theory as an official sub-field of Microeconomics. In his book, Jackson diligently puts together all the definitions and analyses of behavior in networked societies as well as potential applications within economics. He also outlines ways in which network 'community structures' can be estimated, a subject that has drawn the attention of many researchers across the fields of political science, psychology and even physics. Jackson briefly touches upon the point that there is a natural tendency for nodes within a given network to form 'groups' or concentrate themselves in different parts of the distribution of nodes. There are different underlying processes, which can lead to the observed pattern of group formation. These are known as underlying 'community structures' of a network and are unobserved. Since these are not observed, researchers can infer which nodes should be grouped together by observing their interaction patterns. He suggests the use of Maximum Likelihood Estimation (MLE) methods to identify the community structure that has led to the observed network data. This is, in fact, based on CJK's paper (2009) on identifying community structures using MLE.

Their paper is a pioneer in introducing various methods that can be used to partition nodes into community structures based on network data. In particular, they discuss and investigate the method of identifying community structures based on MLE, which can also be used to rank community structures. The basic ideas underlying the likelihood approach for detecting community structures and the model on which it is based are as follows. CJK discuss the special case wherein links are either present between two nodes or not. So, they basically look at a directed network and also provide a description of the weights attached to each link. As a starting point, CJK consider a given set of nodes, which have some true underlying community structure which is, in fact, a partition of the nodes into groups. In other words, the communities of nodes can be thought of as groups of nodes that have some natural affinity for each other or some basic characteristics in common. Links between nodes are formed at random, but in a way that is dependent on the underlying community structure. The main feature of their model is the fact that a pair of nodes that lie in the same community are more likely to interact with each other than a pair of nodes that lie in different communities.

As observers, we do not directly observe the partition of nodes into communities nor do we observe the probabilities with which different nodes interact (are linked to each other). Instead, we simply observe the resulting network and use that to form an estimate of which underlying community structure and probability structure would be most likely to have generated the network. Their model is built around the idea of perceiving a community structure as a partition. In order to estimate the probability that two interacting nodes belong to a given partition or community, they use a binomial kind of likelihood function that they maximize to extract the estimated probabilities. They also use these to rank the communities and infer that nearby partitions have similar likelihoods.

They propose a grid search algorithm for finding the maximum-likelihood partition, which is as follows[2]: At the beginning they set up a grid of probability of 'in's' (of being in the partition) and probability of 'out's' (of not being in the partition). Then for each point on that grid they do the following. First they construct the pseudo community structure that maximizes the overall likelihood of observing the given data. Second, they search for the community structure that is closest to the pseudo community structure. Their results suggest that this will be close to an overall maximizer. In a second step, if the pseudo-community structure is not a community structure, then they search for the community structure that is closest to this pseudo community structure. In most applications, this can only be done approximately since the number of community structures close to a pseudo community structure grows exponentially in the number of nodes. They propose an ad-hoc algorithm to do this. In contrast to the logic behind identifying pseudo community structures and searching nearby for a community structure, for which they provide a strong foundation, the details of search nearby for a community structure, by itself, is not as well-founded. In particular, the approach they follow is described as follows. First, randomly

[2]The ensuing algorithm as well as the one that follows, have been quoted from the paper.

pick one of the nodes x that is involved in the most pairs in $\hat{\pi}$, a pseudo community structure. Then examine random subsets of x's extended neighborhoods in $\hat{\pi}$ to see which one requires the minimal number of added or deleted links in order to become a community. Specifically, starting with a parameter, say, δ in $[0, 1]$ and a pseudo community structure $\hat{\pi}$, let $MD(\hat{\pi})$ be the nodes with maximal degree (involved in the most pairs) in $\hat{\pi}$:

1. Uniformly at random pick a node $x \in MD(\hat{\pi})$.

2. Randomly pick a subset S of x's neighbors in $\hat{\pi}$.

3. With probability, δ, add a uniformly randomly chosen neighbor of a uniformly randomly chosen node in S (if it is not already in S), then begin step (3) again with the new set S. With probability $(1 - \delta)$ (or if there are no more nodes that are neighbors of any nodes in S that are not already in S), then proceed to step (4).

4. Make S into a community by adding all pairs of nodes in $S \cup \{x\}$ to $\hat{\pi}$ and deleting any pairs of nodes that involve only one node in $S \cup \{x\}$.

5. Iterate on steps 2 to 4 a number of times equal to the number of nodes. Select the choice of S that requires the fewest number of added and deleted pairs in step 5.

6. Discard the resulting $S \cup \{x\}$ from step (5) as a community, and repeat steps (1) to (5) on $\hat{\pi}$ restricted to the remaining nodes.

7. Stop when the remaining pseudo community structure in step (6) is a community structure (it may be empty).

They point out that their algorithm for finding the closest community structure to the pseudo community structure can be biased towards communities that are too small, depending on how S is chosen in step (2) above (for instance, by picking it by flipping a fair coin to decide if a given neighbor is in or out). Thus, they go ahead and add a third step where they consolidate two nodes that are in the same community obtained from the process described above. Then repeat the process on the consolidated network that now forces these two nodes to be in the same community. They keep iterating until either the resulting community structure consists of singletons, or the community structure of the consolidated problem has a lower likelihood than the previous community structure. Then, across the grid of probabilities, they look for the community structure and the probability combinations that maximize the overall likelihood. This interesting algorithm is one of the first ways to use maximum likelihood methods on grid searches to extract out the underlying community structure that must have, most likely, led to the observed network.

Another important paper that is fairly recent (May, 2010) is by CFIK. They develop and analyze an empirical model for strategic network formation, which can be estimated with data from a single, moderately large, network at a single point in time. Their network model involves a sequential process where, starting from an empty network, in each period a single randomly selected pair of agents has the opportunity to form a link until each pair of agents has met. Conditional on such an opportunity, a link will be formed if both agents view the link as beneficial to them. They base their decision on their own characteristics, the characteristics of the potential partner, and on features of the current state of the network, such as whether the two potential partners already have friends in common. One of their key assumption is that agents do not take into account possible future changes to the network when making their decision to form a link. They justify this naive assumption by arguing that this would avoid complications related to the presence of multiple equilibriums, and would greatly simplify the computational burden of analyzing these models. They use Bayesian Markov Chain Monte Carlo (MCMC) methods to obtain draws from the posterior distribution of interest. They apply their methods to a social network of 669 high school students with 4.6 friends, on average. Then they use the model to evaluate the effect of an alternative assignment to classes on the topology of the network. We will be using a restrictive version of the same dataset in order to carry out our analysis in one of the chapters in this dissertation.

Their MCMC algorithm consists of two parts. Given the preference parameters that they denote as θ, the observed data and an initial value for the sequence of meetings M, they use a Metropolis-Hastings step to update the sequence of meetings. In the second step, given the sequence of meetings, parameters and data they update the vector of parameters, again in a Metropolis-Hastings step. They point out that a key insight is that the likelihood function for the model with network effects would be easier to evaluate if we knew the history of opportunities, and, by implication, the history of the network formation. Snijders, Koskinen and Schweinberger (2010) use this insight in a setting with repeated observations on a network and a partly unknown continuous time stochastic process for meeting times. However, they focus on obtaining maximum likelihood estimates unlike using an MCMC approach.

Another recent paper on empirical applications to network data is the one by Chapman (2010). The paper investigates the underlying structure of a network of relationship between participants in the Canadian Large Value Payment System (LVTS) and tries to understand how liquidity is transferred through the system. The paper builds on the statistical model provided by CJK (2009) to estimate the most likely partition in the network of business relationships in the LVTS. Specifically, the paper estimates, from the LVTS transactions data, different 'communities' formed by the direct participants in the system. Using various measures of transaction intensity, Chapman uncovers communities of participants that are based on both transaction amount and their physical locations. More importantly, Chapman points out that these communities were not easily discernible in previous studies of LVTS data since previous studies did not take into account the network (or transitive) aspects of the data.

The research in this area is ongoing and still very new. A number of researchers have been attracted towards the investigation of observed networks as well as the evolution of networks from a given initial network. With this background, it might be useful to put together some of the major definitions from network theory as used in economics. It must be pointed out that although some of these definitions need not be directly used in the analyses provided in any of the chapters, the discussion will familiarize the reader about the underlying theory.

1.3 Translation of Networks into the Language of Economics

In economics, network formation game theory views any game as a network and the players as the nodes of the network. It harnesses the nice descriptive properties of network theory and its power to explain complex (simultaneous) strategic interactions between economic agents, in order to model an economic problem as a game.

In simple terms, a network can be viewed as a set of nodes (which could be individuals, firms, etc.) and the links that connects these nodes. For example, in a friendship network, the link between any two nodes could be that of a friend and so on. There are broadly two types of networks: directed and undirected. In a directed network, links are directed rays from one node to the other implying that between two nodes i and j, the relationship that i shares with j may not be the same as the one j shares with i. A quick example would be to assume that i is the father and j is the son. So, the link from i to j would be that of a father and from j to i would be that of a son. On the other hand, an undirected network is one that does not have directed links. That is, the relationships are symmetric - i and j share the same relationship with each other. An example would be: i and j are brothers. Note that the example did not assume i and j are friends, because it is possible to have a situation where i might consider j a friend, but not vice versa. Whereas, the relationship of 'brothers' is symmetric by nature. In order to make a distinction between connections in a directed network and those in an undirected network, we term them as *arcs* and *links*, respectively. Thus, arcs are simply directed links.

Formally, a directed network may be defined as follows.

Definition 1.1: (*Directed Networks*) (Page, 2009)

Given a node set N and an arc set A, a directed network G, is a nonempty, closed subset of $A\times(N\times N)$. The collections of all directed networks is denoted by $P_f(A\times(N\times N))$, where P_f stands for closed subsets. Thus, a directed network, $G \in P_f(A\times(N\times N))$, consists of a closed set of ordered pairs of the form $(a,(i,i'))$ where a is an arc type and (i,i') is an ordered pair of nodes.

A link or arc between two nodes is a description of the relationship that connects them. By convention, if two nodes are linked or connected by an arc, then that link takes the value 1 and 0 otherwise. Thus, we can represent it as a binary matrix that has 1 as an entry if nodes are connected and zero otherwise. This is known as an *adjacency matrix*. For example, consider the following set of connections: i_1 is connected to i_2 and i_3, i_2 is connected to i_3, i_3 is connected to i_2 and i_3 is connected to i_1. This set of connections can be represented as an adjacency matrix:

	i_1	i_2	i_3
i_1	0	1	1
i_2	0	0	1
i_3	1	1	0

This matrix represents the following network:

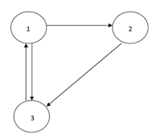

Figure 1.1: A Network

Note that since i_1 is not connected to itself, $(i_1, i_1)^{th}$ element in the matrix is 0. Similarly, for (i_2, i_2) and (i_3, i_3) the entries are 0. Moreover, since i_1 is connected to i_2 and i_3, i_2 is connected to i_3 and i_3 is connected to i_1 and i_2, we have that $(i_1, i_2)^{th}$, $(i_1, i_3)^{th}$, $(i_2, i_3)^{th}$, $(i_3, i_2)^{th}$ and $(i_3, i_1)^{th}$ elements assume value 1. Observe that since i_2 is not connected to i_1, the $(i_2, i_1)^{th}$ entry is 0 implying that this is a directed network. Otherwise, we would have had $(i_1, i_2) = (i_2, i_1) = 1$.

Next, we outline some more concepts related to the nodes and links. Formally, a network consists of the following:

N = finite set of nodes.

I = finite set of individuals (we could have $N = I$, as will be assumed in our analyses).

C = finite set of link/arc categories. This is the set of different types of links/arcs that are possible across the network. For example, in a friendship network one could have two categories: 'best friends' and 'friends'.

A_c = finite set of link/arc types of category c. These are the types of links/arcs that are available to the nodes, one of which defines the connection between them. For example, if a person extends a 'friend invite' to the another on Facebook and the invitee accepts the invitation, we have a 'friend' link between them (and not a 'best friend' link).

D = finite set of players (we could have $N = I = D$, as will be assumed here).

$P_\alpha(I)$ = collection of all nonempty subsets of I of size α. Then N can be redefined as $N = P_\alpha(I)$. For example, if we are looking at a bipartite network, we can set $\alpha = 2$.

$$A_L = \{0, 1\},\ L \in C$$

One way of representing a network is in the form of a function. First, denote by $\{i, j\}$ a typical element of $P_\alpha(I)$, where i and j are in I. Then define a link l_{ij} from i to j as follows:

$$l(i,j) = \begin{cases} 1 & \text{if } i \text{ has a directed link towards } j \\ 0 & \text{otherwise} \end{cases}$$

Then a network N is the union of the sets $g(0)$ and $g(1)$, as defined below.

$$g : \{0, 1\} \longrightarrow (N \times N)$$

Thus,

$$g(0) = \{(i, j) : i \text{ and } j \text{ are not connected.}\}$$

$$g(1) = \{(i, j) : i \text{ and } j \text{ are connected.}\}$$

Based on figure (1.1), we have:

$$g(0) = \{(i_1, i_1), (i_2, i_1), (i_2, i_2), (i_3, i_3)\}$$

$$g(1) = \{(i_1, i_2), (i_1, i_3), (i_2, i_3), (i_3, i_1), (i_3, i_2)\}.$$

A complete network is one where every node is connected to every other node directly[3]. Total number of connections for the complete version of the network above is given by $|P_\alpha(I)| \times |a|$ where $|P_\alpha(I)|$ is the cardinality of the set of subsets of I of size α each. In this network, $\alpha = 2$ and $|P_2(I)| = 6$. $|a|$ is the cardinality of the set of feasible arcs, which is assumed to be two - $\{$*connect or don't connect*$\}$. Thus, total number of combinations of connections possible, is given by $6 \times 2 = 12$.

The number of connections that are actually observed might also depend on what is known as the '*degrees of centrality*'. The term is a measure of how centrally a node is placed in the network. And centrality refers to the strength of a node's connections. It is useful to have this information as we can then rank all the nodes based on their degrees of centrality, an exercise that could prove to be useful for the analysis of certain economic problems that involve modeling the magnitude of network effects. Though there are different ways of calculating the degrees of centrality of the nodes of any given network, the one that will be illustrated here is one of the most widely used measures, known as the '*eigen vector*' centrality measure. The mechanism that this method uses to attach a degree of centrality to the nodes is as follows. First, assign to each node i a score, x_i. Then, let A be the adjacency matrix of the network. Hence, $A_{ij} = 1$ if i and j are connected and 0 otherwise. For the ith node, let the centrality score be proportional to the sum of the score of all nodes, which are connected to it. Hence,

$$x_i = \frac{1}{\lambda} \sum_{j \in M(i)} x_j = \frac{1}{\lambda} \sum_{j=1}^{N} A_{ji} x_j$$

where $M(i)$ is the set of nodes that are connected to the ith node, N is the total number

[3]If i is connected to j and i is connected to k who, in turn, is connected to j, then the former is referred to as a direct connection between i and j, while the latter is an indirect connection between the two.

of nodes and λ is a constant. In matrix form,

$$x = \frac{1}{\lambda} Ax \ \Rightarrow \lambda x = Ax$$

So, it incorporates the relative effects of the outgoing as well as incoming links for a node, through the adjacency matrix, A. Generally, an eigenvector solution will exist for many different eigenvalues, λ. However, the additional requirement that all entries in the eigenvector be positive implies (by Perron-Frobenius theorem which asserts that a real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector has strictly positive components) that only the greatest eigenvalue results in the desired centrality measure[4]. Thus, the first entry in the eigenvector would be x_1 or the degree of centrality of node 1, the second would be that of node 2 and so on. Let us look at the following example to understand this better. Consider the following adjacency matrix,

	i_1	i_2	i_3	i_4
i_1	0	1	1	0
i_2	1	0	1	1
i_3	0	1	1	1
i_4	1	0	1	1

that represents the following network in figure (1.2).

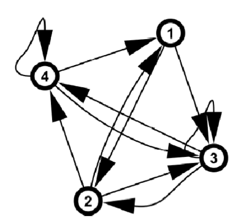

Figure 1.2: Another Network

[4]In fact, Perron-Frobenius theorem guarantees the existence of strictly positive eigen-vector for a stochastic matrix (with the elements of either the row or column summing to 1). However, as long as we have a matrix with all positive elements, we can normalize and apply the theorem.

The corresponding eigen vectors can be represented as a matrix where each column corresponds to an eigen vector of this adjacency matrix. The eigen vector matrix for this example is given by the following:

0.0563 + 0.3833i	0.0563 - 0.3833i	0.3822	-0.5774
0.4511 - 0.1354i	0.4511 + 0.1354i	0.5172	-0.5774
-0.6374	-0.6374	0.5648	0.5774
0.4511 - 0.1354i	0.4511 + 0.1354i	0.5172	0.0000

Here, the first column is the eigen vector corresponding to the first eigen value, the second column is the eigen vector corresponding to the second eigen value, and so on. The eigen values for our adjacency matrix are given by:

-0.4156 + 0.4248i	0	0	0
0	-0.4156 - 0.4248i	0	0
0	0	2.8312	0
0	0	0	0

Thus, observe that Perron-Frobenius theorem holds. There is a unique real positive eigen value, which is 2.8312 in this example. And the eigen vector corresponding to this eigen value, given by the third column of the eigen vector matrix above, is the only eigenvector that has all positive elements. This information can be interpreted as follows. Given the adjacency matrix, all nodes have a connection to i_3, thus i_3 has the highest degree of centrality and so on.

At this point, we have all the definitions that we need. However, we still need to outline the structure that will be used to build our model. So, we introduce that in the next section.

1.4 Econometrics of Network Structures

We discussed above that a network can be represented as a function. Extending this intuition further, let us now go ahead and define connections as random variables that are drawn from a distribution. So, we define a distribution of connections over a network. Let X be the sample space, that is, all possible connections. In our previous example $|X| = 12$. Thus, X is associated with the complete version of the observed network where each node has a direct *pairwise* connection with another node. Let E be the event that node i is connected to node j. Thus, $E \subset X$.

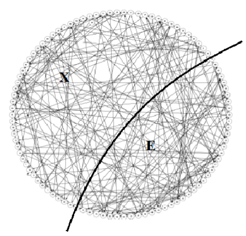

Figure 1.3: X-Sample Space; E-Events and $E \subset X$

Definition 1.2: (*Link*)

We define a random variable, l_{ij} as a mapping from the space of connections to a binary set $\{0, 1\}$. That is, $l_{ij} : X \longrightarrow \{0, 1\}$. Thus, if nodes i and j are connected, $l_{ij} = 1$ and 0 otherwise.

Recall, we define a discrete probability measure, P, as a real-valued function that satisfies,

1. $0 \leq P(E_k) \leq 1$
2. $P(E) = 1$, where $E = \cup_k E_k$
3. $P(\cup_k E_k) = \sum_k P(E_k)$ is E_k's are pairwise disjoint.

Thus, if L is a discrete random variable, $f(l) = P(L = l)$. In our setup then, $f(l_{ij}) = P(l_{ij} = 1)$ or $f(l_{ij}) = P(l_{ij} = 0)$. Therefore,

$$f_{ij}(1) = \text{probability that } (i, j) \text{ are connected}$$

$$f_{ij}(0) = \text{probability that } (i,j) \text{ are not connected } = 1 - f_{ij}(1)$$

Define the joint density function as follows: for random variables L_{ij} and $L_{i',j'}$, the joint density is defined as

$$f(l_{ij}, l_{i'j'}) = P(L_{ij} = l_{ij}, L_{i'j'} = l_{i'j'})$$

Thus,

$$f(l_{ij} = 1, l_{i'j'} = 1) = f_{ij,i'j'}(1,1)$$

where $f_{ij,i'j'}(1,1)$ = probability that both i and j as well as i' and j' are connected at the same time.

So, given that $|I|^2$ is the total number of connections possible (i.e., total number of pairwise connections allowing for self-loops), if we look at two distinct connections simultaneously, such that these events are independent, the probability must be the product of $\frac{1}{|I|^2}$ and $\frac{1}{|I|^2-1}$. That is,

$$f_{ij,i'j'}(1,1) = \frac{1}{(|I|^2)(|I|^2-1)}$$

where $|I|$ is the cardinality of the set of nodes and $|I|^2$ is the total number of possible connections in the network.

An example:

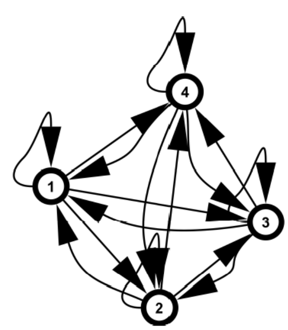

Figure 1.4: A Complete Network

In this figure, $|I| = 4$. So, total number of connections possible are $|I|^2 = 16$. Note that this figure is a complete network with 16 connections. So, $|I|^2$ is, in fact, the total number of connections possible in the complete version of the corresponding network.

If l_{ij} and $l_{i'j'}$ are associated with independent events, then the joint density is given by:

$$f(l_{ij}, l_{i'j'}) = g(l_{ij})h(l_{i'j'})$$

where $g(l_{ij})$ and $h(l_{i'j'})$ are individual densities of l_{ij} and $l_{i'j'}$ respectively. And if we do not have independence then,

$$f(l_{ij}, l_{i'j'}) = f(l_{ij})f(l_{i'j'}|l_{ij})$$

where

$$f(l_{ij}) = \sum_{l_{i'j'}} f(l_{ij}, l_{i'j'}) \,\text{and}\, f(l_{i'j'}|l_{ij}) \;=\; \frac{f(l_{ij}, l_{i'j'})}{f(l_{ij})}$$

Thus, $f(1,1) = f(1).f(1|1) =$ the probability that both (i,j) and (i',j') are connected is the product of the marginal probability that (i,j) are connected and the conditional probability that (i',j') are connected given that (i,j) are connected.

Let us define expectations now.

$$E[l_{ij}] = \text{expected connection between}\, i \,\text{and}\, j \;= \sum_{j=1}^{k} l_{ij} f(l_{ij})$$

Definition 1.3: (*Neighborhood*)

The neighborhood of a node i is defined by the function, $k(l_{ij}) : N_i \longrightarrow I$ where

$$N_i = \{\{l_{ij}\}_{j=1}^{n} : \textit{for a given node}\, i, l_{ij} = 1\} \textit{ and } N = \cup_i N_i$$

where I is an index set of nodes.

For example, one could think of this function as follows: $k(l_{ij}) = j.1(l_{ij} = 1)$. Thus, this function yields the list of nodes that constitute i's *neighborhood*. In other words, N_i is the definition of a neighborhood of a given node i. And $k(l_{ij})$ produces the list of all nodes that are neighbors to i. Also, $E(k(l)) = \sum_{j=1}^{n} k(l_{ij})f(l_{ij})$ is the *expected neighborhood* of node i.

Having defined neighborhood in this manner, we can now define a path between two nodes that are *not* connected directly.

Definition 1.4: (*Path*)

For a given node i, the path between i and some node $j \notin N_i$ can be defined as follows:

$$\mathbf{P_{in}} = \{\{(i', j')\}_{i'=j, j' \in I \setminus \{i,j\}}^{j' \to n} : \forall\, l_{ij} = 1 \, where\, i \neq j \; we \; have\, i' \in N_j, j' \in N_{i'}\}.$$

Thus, the arc between (i, j) precedes the arc between (i', j'). We say that the set above is empty if $\cap_{i'} N_{i'} = \phi$ for all $i' \neq i$, such that the condition above holds. Thus, a set of paths from i to n can be represented as $\cap_{i'} N_{i'}$ such that the above holds. The shortest path then is the $inf | \cap_{i'} N_{i'}|$, i.e., the infimum of the measure of the set of paths.

Finding the Shortest Path:

Suppose, we define $f(l_{ij})$ to be a binomial distribution, where p is the probability that a connection is formed and q is that it isn't. So,

$$p = P(l_{ij} = 1); \, q = P(l_{ij} = 0)$$

Now, if we want to look at the path from node i to s, consider the following.

x = number of successive connections that belong to $\mathbf{P_{in}}$ and have $l_{ij} = 1$, $j \leq s$. Let n be the total number of successive connections that can potentially constitute a path from i to s. Thus, this path exists if $n = x$. Therefore, the probability that this path exists is given by $p^x q^{n-x}$. The total number of possible paths then is given by $\frac{n!}{x!(n-x)!}$.

$$f(l_{is}) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

Suppose 1 and 2 are neighbors. Then, $f(l_{12}) = \frac{1!}{1!(0)!}p^1q^0 = p$. And if 1 and 2 are not connected then $f(l_{12}) = \frac{1!}{0!(1)!}p^0q^1 = q$. The shortest path between i and s is when $\frac{n!}{x!(n-x)!}p^xq^{n-x} \longrightarrow p^n$ where n is the minimum of the set of n's associated (respectively) with all possible paths between i and s, for which this true.

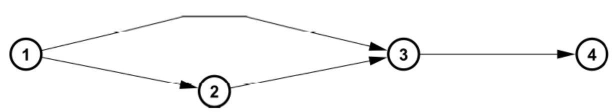

Figure 1.5: Shortest Path Between 1 and 4 is via 3

Shortest path: Consider Figure (1.5). $(i_1, i_3) \in N_1, (i_3, i_4) \in N_3$, so $n = 2$. And $x = 2$, since we have two successive $l'_{ij}s$ each equal to 1. Thus, the shortest path:

$$\frac{2!}{2!(0)!}p^2q^0 = p^2$$

Thus, this is one way of modeling the distribution of links. Here, we assume that links follow a binomial distribution and attempt to estimate the probability of two nodes being connected. Thereafter, we use those as weights to find the shortest path between two nodes that are not linked directly, but via at least one other node. This provides an interesting modeling tool to analyze various forms of contagion ranging from systemic failures in an interbank network to spread of a disease such as a swine flu through a friendship network. That is, depending on a node's position in the network, it may or may not survive the shock. So, an understanding of the network structure and the spread of connections over it, could enable us to predict and prevent a contagion by identifying the nodes that might lie on the path of such a shock diffusion.

1.5 Overview of the Dissertation Research

In order to carry forward such analyses, we must first start by outlining methods to analyze networks. The main contribution of this dissertation is to provide a set of theoretical and empirical tools to analyze complex economic problems within the framework of networks. This dissertation applies tools from network formation game theory as well as microeconometrics to model complex economic interactions as well as constructs empirical tools that enable advanced analysis of various networks including financial as well as social networks. For example, the latter can be very useful in fields such as marketing where consumer behavioral pattern recognition, which can be affected by the consumer's social network, plays a very important role in determining market segments and therefore, revenue earned.

This dissertation focuses on two aspects of these strategic decisions in different kinds of networks (e.g., interbank, inter-country, social, etc.)- analysis and prediction. Each chapter contributes towards this in the form of theoretical and econometric methods for analyzing strategic interactions between economic agents within different networks and the subsequent evolution of new networks resulting from the choices that the agents make. The research presented here makes two contributions in this regard. The first pertains to the study of endogenous risk of contagion within an interbank network following an exogenous shock to the system. Consequently, the nodes might re-evaluate the value derived out of their connections and might even wish to sever the ties in order to prevent themselves from getting hit by the shock. A stochastic game of network formation is designed to model these strategic decisions, which play a role in the rate at which the shock diffuses through the network. The game leads to a threshold condition in terms of the number of connections per node that could induce a state of complete collapse. The model also identifies network structures that are more resistant to widespread financial contagion. So, from a policy point of view, a ceiling on the number of connections per node may be put in place so that favorable network structures may be generated in order to avoid systemic spread of endogenous risk of contagion. The second contribution is an econometric algorithm to create and analyze networks. Relative to the traditional methods based on Maximum

Entropy, the copula based algorithm provides more flexibility for simulation of networks, given asymmetric data derived from certain data generating processes. The dissertation also discusses other innovative ways to analyze connections over networks using advanced econometric tools.

The remainder the dissertation has been divided as follows. Chapter 2 provides a network formation game theoretic model to analyze the endogenous risk of financial contagion within an interbank network and provides a potential solution in terms of policy recommendation. Chapter 3 provides a novel method of simulating network data, which expands the scope of techniques used for such simulations. It also provides a list of cases where the proposed method outperforms the more traditional method based on Maximum Entropy. Chapter 4 provides an intuitive way to model the spread of distribution of connections over a network using a Finite Mixture Model. A social network dataset is used to execute and illustrate the approach. Chapter 5 concludes.

Chapter 2

Endogenous Risk of Financial Contagion: A Network Formation Game

2.1 Introduction

The recent recession has provided a strong impetus for further research on the already popular topic of interbank contagion. A quick overview of the events that led to the climax of the crash may be outlined as follows.

1. **September 5, 2008** - Goldman Sachs downgraded Merrill Lynch's stock to "conviction sell" and warned of further losses from the company.

2. **September 8, 2008** - Lehman Brothers is reported to have been under severe liquidity pressures.

3. **September 14, 2008** - Meryll Lynch sold to Bank of America.

4. **September 15, 2008** - Lehman Brothers announced that it would file chapter 11 bankruptcy protection.

5. **September 16, 2008** - Barclays announced that it will acquire a major portion of Lehman Brothers.

6. **September 21, 2008** - Morgan Stanley and Goldman Sachs confirmed that they would become traditional bank holding companies, thus bringing an end to the era of investment banking in the US.

Despite the missing details (including the sale of Bear Stearns to J P Morgan Chase) in the outline above, it is a fact that once the system was hit by the shock, within a very short span of time the Wall street lost all of its investment banks (in the form they existed)! Such a rapid breakdown of the system was caused due to the intricate manner in which these banks were connected to each other. This event serves as the motivation underlying the ensuing analysis of the role of interconnections between these banks in facilitating the process of shock diffusion. The primary emphasis of this chapter is to find out the following - when a shock hits an intricate system, such as an interbank network, how do the dynamics of the system change? Does the positioning of the node in the network (i.e., how central a node is) matter? Can the nodes prevent the shock from hitting them by deciding to choose a particular position in the network, so that they don't come on the path of the shock diffusion?

A fair amount of work done in this area tries to model the phenomenon of shock propagation across banks using advanced econometric tools. Although a complete theoretical model that can explain such a contagion is still missing, issues that constitute one or more aspects of this problem have been addressed with the help of theory. For instance, there are papers on frenzies, manias, panics and crashes that employ the conventional game theoretic style of modeling. The seminal paper in modeling contagion was by Diamond and Dybvig (1983), where they introduced a simple interbank market model with deposit insurance from the consumers side. The model works under the assumption that not all depositors withdraw all the money that they have, at once. As such, a reshuffling of liquidity is possible by banks to deal with a bank run. This model was extended in the classic paper by Allen and Gale (2000) where they construct a model of interbank insurance. They view the interbank market as being divided into regions. So, if a particular region gets hit by a shock, that region can borrow resources from other regions that have not been hit and stabilize itself. Further, they find that the more complete the structure of the interbank market, the more insured the system is against shocks.

To the best of our knowledge, one of the first attempts made to study Allen and Gale's model in terms of an interbank network was by Ana Babus (2006). She constructed a similar model with an interbank network and arrived at similar conclusions as Allen and Gale. Another paper that is of interest is Amil Dasgupta's (2004) paper on financial contagion on capital connections where he uses Morris and Shin's (2000) equilibrium selection techniques to establish the optimal levels of interbank holdings during the situations when probability of contagion is high. A recent paper by Elliot, Golub and Jackson (2013) models contagion propagating through discontinuous changes in asset values triggered by financial failures. They illustrate aspects of the model with data on European debt cross-holdings. Cohen-cole, Kirilenko and Patacchini (2011) model systemic risk in a financial network using the methodological framework used in the social interactions literature and discuss the use of behavior-based models in the financial markets context and relate our approach to that used in the epidemiological literature. They make an interesting contribution in the form of a measure that differs from the existing approach in that it is dependent on the specific network architecture and is a function of the strategic behavior of agents in the system. They find that network patterns are closely related to profitability in the market and certain positions in the network are more valuable than others. Thus, an observed structure of the network may imply a very large impact of shocks. There are a number of such interesting papers on financial contagion that we would like to discuss but are prevented by the limited scope of this chapter. For more information on these, we refer the reader to the list of references included in this dissertation.

In most of the research done in this area, the evolution of an interbank network resulting from every step of shock diffusion (due to contagion) remains largely unexplored. Further, the role of an insurance agency, for example the central bank as an insurer, rather than banks insuring each other has not been clearly addressed. Also, what if interbank insurance is not a possibility? In that case, the interaction between lender and borrower banks is of paramount interest. In this chapter, we propose a model of network formation game to understand some of these issues.

Our research in this area is novel and unique as we develop a game theoretic model of the evolution of endogenous network dynamics. This is unlike the existing models that we discussed above, most of which take the network structure as given. Since we would like to analyze how changes in the connections of a given network impact the rate of shock diffusion and hence, the transformation of a given network to another, we cannot simply assume that the network under consideration is static[1]. Thus, we want to model each step of the network formation process resulting from every step of shock diffusion.

The model that we propose here builds on the model proposed by Page (2009) and constructs a stochastic game of default contagion. In this model, the dynamics that govern the diffusion of a contagion shock are determined endogenously, generated by interactions of the changing network structure which includes the strategic behavior of banks, a central bank's decisions and noise. In other words, we develop a *strategic network formation game* where the spread of contagion risk is endogenously determined by the decisions taken by nodes that are a part of the network itself. The spread of contagion is assumed to follow a Markov chain where the Markov transition probabilities are endogenously determined by allowing them to be functions of the nodes' strategies.

Further, we show (among other interesting results) that there are multiple equilibriums including ones that are characterized by a state where banks must request a central bank to bail them out and in turn, the central bank must decide how many banks to bailout (none, some or all - there exists a unique state corresponding to each case). Each of these equilibriums are represented by an equilibrium Markov supernetwork[2]. In this equilibrium Markov supernetwork, nodes represent specific configurations of the interbank loan network and the directed arcs connecting these various interbank networks represent the equilibrium

[1]Note that since the network is set of nodes and connections between them, if the connections between nodes change then so does the structure of the network. Hence, everytime, a connection between a pair of nodes changes, a network emerges.

[2]A supernetwork is defined as a *network of networks*. That is, the nodes in a supernetwork are networks as well.

Markov transition probabilities governing law of motion of transitioning from one network to another as a consequence of the equilibrium strategic behavior of banks and some noise. As such, the equilibrium Markov supernetwork may be viewed as one that simultaneously provides a network representation of the equilibrium network dynamics, as well as a snapshot of all the equilibrium interbank networks that are likely to evolve and persist.

We call these sets of equilibrium networks as the basins of attraction in the supernetwork. An attractive result of our model with finite number of basins, is that each basin of attraction constitutes only one equilibrium interbank network - with each network in a basin being a Nash network. This allows us to deduce the exact structure of the network that the process ends up in. Further, such an equilibrium Markov supernetwork allows us to be able to identify *tipping point networks* beyond which the only other possibility is a collapse of the system. Thus, this in turn provides a possibility to understand what restrictions must be imposed on the supernetwork structure by means of policies so that it might be altered to reduce the probability of the network process reaching a tipping point network on a path leading to an undesirable equilibrium.

The rest of the chapter has been divided into four sections. Section 2 outlines the setup and details of the model with subsections that lay out the primitives and related details. Section 3 discusses the main results. Section 4 provides and illustration of the model and section 5 concludes.

2.2 Setup

We are interested to study financial contagion within an interbank network. Since the interbank network consists of an intricate set of arcs between its nodes, which are not necessarily the same (e.g., lending and borrowing), we use directed networks to model this.

In this chapter, we define an interbank network as a set of banks and a central bank connected with each other in a certain manner. Thus, the players or nodes in our interbank network include a central bank or the Federal Reserve Bank (Fed) and a set of banks. We limit the functions of the banks to borrowing and lending within the interbank network.

On the other hand, the Fed, which is the central bank, has the power as well as authority to issue (or deny) loans or funds to the banks upon request (from these banks) - an action more widely known as 'bailouts'. So, Fed may be viewed as more of an insurance agency in this game, with the difference that, unlike traditional insurance agencies, it is not necessarily bound by a contract that obligates it to bailout the banks upon request.

Further, we are interested in looking at how an exogenous shock hits and propagates through the system as a result of the changing connections between nodes (thus, becoming an endogenous phenomenon). Therefore, we model it as a network formation game - as the network forms or evolves at each step given the strategy of the players.

We define a shock as an event that devalues the assets of a bank by more than a threshold percentage. When that event occurs for the first time in the game, we refer to it as 'the shock hits the system'. We assume that a shock can first hit the system through a borrower bank only. And when that happens, it is the consequent strategies employed by banks (borrowers and lenders) that decides the course of shock diffusion. By shock diffusion we mean the occurrence of a series of subsequent events, triggered by the initial event, that devalues the assets of other nodes in the network. Once the shock has diffused, the network we have may not be the same as the one we started with. Thus, the network evolves or forms as the process of diffusion progresses. The question then is, is there a pattern in terms of the spread of connections between nodes in the path of diffusion that these series of events follow? If yes, what is it governed by and can we control it?

2.2.1 The Network Formation Game

We design this *interbank contagion game* as a one-time multi-period game with two stages where the initial network is given to us. This network consists of lender and borrower type banks bound by what we term as a *debt contract* and a central bank that has passive connections to all banks, which may be activated if needed. The debt contract may be viewed as an agreement that binds the borrower to repay the borrowed amount to the lender bank at some point in the future.

The shock hits this initial network and continuously reforms the network as it diffuses. However, note that since it is a 'one-time' game, we assume that no new debt contracts can be issued. The changes within this network occur under the constraint that the existing debt contracts are either honored or defaulted upon. Alternatively, in case of a bailout request, the existing debt contracts may be honored once the borrower bank has been bailed out. But no new debt contracts may be issued at that stage either.

There are n banks and a central bank who are the players or the nodes in this game. They may be defined as follows.

Players (Nodes)

1. Banks - they borrow and lend within the interbank network.

2. Fed - the central bank that has the power as well as the authority to issue loans or funds to the banks

Mathematically, we introduce the following notations. The set of n banks is denoted by

$$B = \{b_1, \ldots, b_n\}_{1\times n}$$

where b_i refers to the ith bank. The Fed, on the other hand, is denoted by

$$F = \{f\}_{1\times 1}$$

We have two markets, namely, the interbank market and the Fed-bank market. The inter-bank node pairs are included in a set N_B defined as

$$N_B = B \times B$$

with a typical element given by (b_i, b_j) where b_i and b_j are the ith and jth banks in the network. The Fed-bank node pairs are given by

$$N_f = F \times B$$

with a typical element given by (F, b_j) where F is the Fed and b_j is the jth bank in the network.

Recall that the bank nodes are restricted to be either lenders or borrowers. Given this, each of the bank nodes ($b_i \in B$) is assumed to have started with endowments given by the following.

Endowments

1. Endowments of Lender Banks : e_{L_i} , a random endowment allotted by *nature* to each lender bank i. So, e_{L_i} need not be equal to e_{L_j}.

2. Endowments of Borrower Banks : e_{B_k} , a random endowment allotted by *nature* to each bank k. So, e_{B_k} need not be equal to e_{B_l}.

3. Endowments of the Fed : E_F (a constant)

We assume that we can partition the network into two parts - one where the nodes are lenders and the other where the nodes are borrowers. The connections, also known as arcs, of the interbank networks are directed from lenders to the borrowers at the beginning of the game. Thus, these initial directed arcs refer to the issuance and acceptance of a *debt contract* from the lender bank to the borrower bank, where debt contract is defined as earlier. Since no new debt contracts may be issued, lenders remain lenders and borrowers remain borrowers for the period of the game.

The strategy set available of the lenders, borrowers and the Fed, respectively, once the shock has hit may be defined as follows.

Strategy Space (Possible Arcs)

1. Banks:

   (a) Lender Banks - can {*enforce contracts, request Fed to bailout*}

   (b) Borrower Banks - can {*respond to enforced contracts, request Fed to bailout*}

   where '*enforce contracts*' refers to a request from the lender to the borrower for repayment of the loaned amount on or before the due time and '*respond to enforced* *contracts*' refers to the acceptance of the request and repayment of the loaned amount by the borrower. The strategy, '*request Fed to bailout*' refers to the situation where the nodes request the Fed to bail them out where 'bailout' is defined as earlier.

2. Fed - can {*bailout, not bailout*} upon receiving requests from the banks. So, Fed always accepts the bailout requests and then decides what to do.

Thus, mathematically, set of arcs for lender banks within the inter-bank market is given by

$$A_L = \{ \underbrace{a_{1l}}_{\text{enforce contracts}}, \underbrace{a_{2l}}_{\text{request fed to bailout}} \}$$

The set of arcs for lender banks within the inter-bank market is given by

$$A_B = \{ \underbrace{a_{1b}}_{\text{respond to enforced contracts}}, \underbrace{a_{2b}}_{\text{request fed to bailout}} \},$$

The set of arcs in the Fed-bank market is given by

$$A_f = \{ \underbrace{a_{1f}}_{\text{bailout}}, \underbrace{a_{2f}}_{\text{not bailout}} \}$$

So, a typical element of the interbank market may be denoted as

$$B_{\text{network}} = \{(a_{jl}, a_{kb}), (b_j, b_k)\} \in A_L \times A_B \times B \times B$$

A typical element of the fed-bank market may be denoted as

$$F_{\text{network}} = \{a_{if}, (f, b_j)\} \in A_f \times F \times B$$

Consider the strategy set of the nodes. Note that we start with an exogenously given network. So, the debt contracts have already been issued. As such, the formation of links on the basis of lending and borrowing is given to us. Thereafter, the shock hits. So, the strategy sets include the post shock options that the nodes have. Moreover, as mentioned earlier, since we are designing this as a one-time multi-period game, we are not allowing for the possibility of issuance of new contracts by the banks.

In order to respond to the lender's strategies, for example - *enforce contract*, the options that the borrower banks have are that of either honoring the debt contract, which would mean the following strategy (where the first entry is the lender's strategy and the second entry is the borrower's strategy) : {*enforce contracts, respond to enforced contracts*} or requesting the Fed to bailout it out, given by {*enforce contracts, request Fed to bailout*}. Similarly, when the lender requests Fed to bailout, the borrower could either choose to repay the debt or request Fed to bail it out as well. And the Fed, given all the requests, must decide whether to bailout or not and how many to bailout, if it must bailout some.

We also allow for the possibility of a node becoming '*insolvent*'. This refers to a situation where the Fed responds by '*not bailout*' to a bailout request initiated by a node. As per Dasgupta (2004), this has larger contagion effects as against a lender bank becoming insolvent.

There is a cost attached to initiating bailout requests to the Fed as well as receiving a bailout request by the Fed. These are given by:

Costs

1. Cost (to a bank) of requesting bailout : C (a constant).

2. Cost (to the Fed) of receiving a bailout request: 1.

So, the total cost to the Fed, incurred by receiving bailout requests, is equal to the total number of bailout requests.

We can now outline the stages of the game.

2.2.2 The Outline of the Game

Stage I

First, nature chooses a network with an exogenous probability that we take as given. This network consists of nodes that are banks (lenders and borrowers) and a central bank.

In this exogenous network, the lender banks and the borrower banks have been connected with directed arcs from the lender banks to the borrower banks, where each arc refers to a debt contract (as defined earlier). The Fed has passive connections to and from all banks that may be activated if required. Within this network then, nature chooses a 'lender-borrower' pair of nodes that are connected in the interbank network (and not the Fed-bank network) such that the borrower gets hit by a shock. Then, by the definition of a shock, we have that the value of this borrower's assets have been devalued by a certain (exogenous) percentage. Thus, this shock can be viewed as a *solvency shock* to the borrower. All nodes have the information that the shock has hit the system. However, the impact of the shock on the borrower's credit standing is not known. This forms the source of uncertainty in the model. And it is likely to have an impact on the borrower's relationship with its lender/s. Further, it also effects the expectations of the indirect connections of the pair of nodes[3] that have been hit by the shock, about the solvency statuses of their direct connections.

Given this, it is left to each of the banks to construct a measure to estimate their own levels of 'exposure'. We define this to be a function of their position in the network that they are a part of. A bank's position in the network is defined in terms of how central they are (degree of centrality). Further, their position in the network or centrality is determined by the share of their loaned or borrowed amount, which is defined as a percentage of total loans or borrowings in that network.

Once nature picks the first lender-borrower pair that are connected by a debt contract in the interbank network, the pair must decide what to do. And based on their choice of strategy the existing network gets modified. The way this happens is as follows.

The chosen lender-borrower pair play a simultaneous move game, where each one of them proposes their joint strategies. For example, the lender could propose that the strategies {*enforce contract, respond to contract*} be played, where the first one is the lender's strategy and the second one is the borrower's strategy. On the other hand, the borrower could, instead, propose {*enforce contract, request Fed to bailout*} be played, where the first

[3]Indirect links may be defined as the set of connections that the direct connections of the affected node possess, such that they are connected to the affected node through at least one node in between. Suppose i_1 is directly connected to i_2, that is, $i_1 \rightarrow i_2$ and i_2 is directly connected to i_3, so that $i_2 \rightarrow i_3$, and we have a 'linear' network - $i_1 \rightarrow i_2 \rightarrow i_3$, then i_1 and i_3 are indirectly connected.

one is the lender's strategy and the second one is the borrower's strategy. By doing this, the lender-borrower pair, whose turn it is to move, is not only proposing a new network, but also proposing the next lender-borrower pair whose turn it would be to move. For instance, by proposing the strategy {*enforce contract, respond to contract*}, the current lender-borrower pair is, in fact, proposing that the next lender-borrower pair may include at least one of the current lender or borrower, but not both as the borrower has repaid the loaned amount to the current lender. Thus, the potential next lender-borrower pair could be the current lender or borrower and one of its other direct links. As such, by proposing a strategy, the pair is indirectly also proposing the next possible pair that should make a move. Therefore, the proposal takes the form of a 'proposed network and the next lender-borrower pair'. Also note that a proposed strategy either modifies the current network by exactly one arc - the one shared by the current lender-borrower pair or does not modify it at all. The nodes that have not been chosen to make a move can only propose status quo, i.e., the current network and the current lender-borrower pair that have been chosen to make the move.

Notice that we refer to these as 'proposals', which they indeed are. These proposals are put forth to nature. And nature picks one of all possible proposals that can be made by the chosen lender-borrower pair (given that the others just propose status quo) in the network under shock, according to a law of transition. This results in a new network where the arc between the lender-borrower pair, whose turn it was to move, has either changed or remained the same (as it is always possible to propose the current status). Further, nature also chooses a lender-borrower pair (out of all the possible lender-borrower pairs given the possible proposals) within this new network, such that the borrower is hit by the shock. However, hereon, nature's probability of choosing this pair is drawn from the law of transition, hereon. Thus, the law of transition is in fact the joint probability of choosing the new network as well as the next lender-borrower pair that must make the move. This chosen (modified) network becomes the next step in the path of the shock diffusion. As will be seen in a following section, we propose a definition for the law of transition such that it is endogenous and takes into account the effects of strategic decisions made by the lender-borrower pair whose turn it is to move. Therefore, the law of transition is affected by the behavior of the nodes. Since the law of transition itself is endogenous as it is affected by the actions of the nodes that make the move, the shock diffusion process is endogenous as well.

On the Fed's part, as soon as a connection to the Fed is activated, the Fed-bank pair also becomes a part of the set of feasible pairs that can be chosen. And if chosen, they play the same simultaneous move game as described above. The only difference lies in the strategy set available to the Fed and the ones available to the lender and borrower nodes. And it still holds that based on the outcome of the game played by a Fed-bank pair, a new network is formed. To provide an example, the new network could represent a network where the request for bailout arc (going from a bank to the Fed) has been replaced by an arc from the Fed to the bank implying that the latter has been bailed out and so on.

A graphic illustration of the structure of this game is provided in the appendix for this chapter[4]. Please see figure (4.1).

Stage II

In the second stage then, we end up with the modified network that is a consequence of a series of strategic decisions made by the lender-borrower pairs in the interbank network and/or Fed-bank pairs in the Fed-bank network. We say *'and/or'* here, because there could be a situation where upon being hit by a shock, all debt contracts are honored as the shock diffuses and no bailout requests are ever initiated. In that case, the Fed-bank pairs do not belong to the feasible set of pairs and are never chosen. The network that emerges in this stage is one of our equilibriums. Each of these equilibriums is called a *basin of attraction*. Note that this game is then an infinite horizon stochastic game where once a basin of attraction is reached the process simply stays there forever. That is, in these basins of attraction, one can only propose the status quo. And each node keeps doing that forever!

So given that the law of transition leads nature to choose one network after another until we reach a basin of attraction, we have that a network of networks is created in the process. In this network of networks the connections take the form of the probabilities

[4] All graphs, tables and figures for this chapter have been provided in the corresponding appendix - Appendix 1.

drawn from the law of motion, which we will call *transition probabilities* as they define the chances of 'transitioning' from one network to the other. Further, note that the shock diffusion happens in this network of networks or the supernetwork where the transition probabilities connect the nodes (which are networks).

Having set up the outline of the model, we would like to find the answers to some questions. In particular, we would like to find out if there exist conditions that tend to lead the process of shock diffusion to a particular basin of attraction? If yes, what is the nature of such a basin - is it characterized by a complete or close to absolute collapse of the system? If so, is there a way to avoid reaching such a basin?

In the next sub-section, we introduce formal definitions of the concepts that we have loosely used in our description of the model above and our assumptions with regard to the model.

2.2.3 Definitions and Assumptions

In what follows, we denote a network as G such that $G = \{A \times (N \times N)\}$ where N is the set of nodes in the network and A is the arc set that contains all the arcs that connect the nodes within G.

A-1: (**Network Partition**)

We assume that the interbank network can be partitioned into a set of borrowers and a set of lenders. Further, since the game is designed as a 'one-time' game, we assume that role of lenders and borrowers continues to be the same in the interbank network. In other words, lenders continue to be lenders and borrowers continue to be borrowers until the end of the game and no new debt contracts can be issued in the interbank network.

Definition 2.1: (*Feasible Coalitions*) (Page, 2011)

Given a finite player set N, a feasible set of coalitions is a nonempty subset, F, of the collection of all coalitions $P(N)$

Recall that in our case, nature chooses a pair of two nodes each time - lender-borrower or Fed-bank. We call this pair of nodes to be a coalition.

A-2: (**The Feasible Set of Coalitions in the Interbank Market**)

$$\mathbf{F} = \{S \in P(N) : |S| = 2\} \textit{ where } P(N) \textit{ is a collection of subsets of } N.$$

That is, a coalition can be formed by only two nodes at a time. Further, we assume that only a lender-borrower node or Fed and any of the banks (actively connected to the Fed) can be part of a coalition. In particular, disconnected lender-borrower nodes or lender-borrower nodes that did not share a debt contract at the beginning of the game cannot be part of a coalition. And, in the lender-borrower network, since a connection can only be through debt contracts, lender-lender or borrower-borrower kind of connections cannot form a coalition.

Since we are looking at a directed network, we can define the network either in terms of arcs between nodes or nodes connected by the same arc. These sets are defined as follows where $G(a)$ is the set of nodes that are connected by arc type a, and $G(i, i')$ is the set of arcs that connects nodes i and i'.

Definition 2.2: (*Network Descriptors*) (Page, 2011)

For $G \in P_{f\in\mathbf{F}}(A \times N \times N)$,

$$G(a) = \{(i, i') \in N \times N : (a, (i, i')) \in G\}$$

and

$$G(i, i') = \{a \in A : (a, (i, i')) \in G\} \tag{2.1}$$

Thus in network G*,* $G(a)$ *is the set of node pairs connected by arc* [5] a *and* $G(i, i')$ *is the set of arcs from node* i *and* i'*. Further, we denote the cardinality of these sets as*

$$|G(i, i')| = |\{a \in A : (a, (i, i')) \in G\}| = \#\,\text{of elements in the set} \tag{2.2}$$

Also, the expression $|G^{+}(i)|$ denotes the number of outgoing arcs from $i \in N$ in the

[5] Recall, an arc is a strategy employed or action.

network G.

Next we define the rules of network formation.

Definition 2.3: (*Rules of Network Formation*)

The rules of network formation must specify for any more of the game which connections can be changed and which players can change them.

A-3: (**Jackson-Wollinsky Rules of Network Formation**)

We follow Jackson-Wollinsky (1996) rules of network formation with the following modification.

A Modification : Subtraction of a connection could be a unilateral or bilateral decision for the borrower. But it is only a bilateral decision for the lender. It is a unilateral decision for the Fed.

The intuition underlying this assumption is the idea that given the structure of our game, if a lender wants to sever ties with a borrower, it can either do so by enforcing the contract and asking the borrower to respond to this request, or it can request the Fed to bailout and expect the Fed to approve the request. In both these cases, the lender can't make a move unilaterally. On the other hand, a borrower, can always respond to an enforced contract (by repaying the borrowed amount) even when the lender eventually requests the Fed to bailout. This way, the borrower chooses to unilaterally sever the tie that it had with the lender in a 'positive' manner. And since the Fed responds to bailout requests, it's decision of not bailing out is only dependent on its own constraints (utility function) and the number of requests it gets. As such, given this if it decides not to bailout a bank, thus severing the ties with it, the bank cannot do much about it. So, the Fed can unilaterally subtract an arc.

A-4: (**Nature of Arcs**)

We assume the following about the type of arcs based on their direction.

1. An arc directed towards a borrower node, initiated from a lender node can mean two things[6]:

[6]One can easily deduce the interpretation of the arcs from the context of discussion. For example, in the

(a) There is a debt contract in place and the lender node has lent money to the borrower node.

(b) The lender bank has enforced the debt contract.

2. An arc directed towards a lender node, initiated from a borrower means that the borrower has paid the loan amount back to the lender.

3. An arc directed towards the Fed node, initiated from any bank node means that the bank node has requested bailout from the Fed.

4. An arc directed towards any node, initiated from the Fed means that the Fed has bailed out the node.

5. If a bailout request to the Fed has been responded by *no* directed arc from the Fed to the node that requested the bailout, then it means that the Fed decided not to bailout that node. Thus, that node is insolvent, though it is still part of the network.

So essentially, the arc labels are the strategies that the nodes employ as their response to the strategy employed by the each of the other nodes, including the two nodes that form the chosen coalition.

A-5: (**Approval or Refusal of Bail Out Requests**)

We assume that once a bailout request has been responded by an approval or a refusal, the node cannot request another bailout from the Fed for the duration of the game in the Fed-bank network.

A-6: (**Notations for Graphs**)

In the graphic illustrations provided in the appendix for this chapter, we use the following notations.

RC denotes 'Respond to Contract', EC denotes 'Enforce Contract', BO denotes 'Bailed Out' and NBO denotes 'Not Bailed Out'.

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initial network, the arcs are debt contracts. However, if it is one the networks that has emerged out of the diffusion process with a coalition's turn to move, then such an arc would mean that the lender has enforced the contract and so on. In situations where it is not clear, the definition of the arc has been explicitly mentioned.

Definition 2.4: (*Cardinality of Connections and Arc Feasibility*) (Page, 2011)

Suppose the feasible set of networks is given by

$$G_{nm} = \{G \in \mathbf{G} : \forall\,(i,i') \in N \times N, n(ii') \leq |G(i,i')| \leq m(ii')\}$$

where $n(.)$ *and* $m(.)$ *are nonnegative integer-valued functions such that for all node pairs* (i,i')*,* $n(ii') \leq m(ii')$ *(with* $0 < m(ii')$ *for some node pair), and where* $|G(i,i')|$ *is the cardinality of the set of arcs from node* i *to node* i' *in network* G*.*

In our model, since there may or may not exist a connection between two nodes and two nodes can be connected by only one directed arrow (in either direction) at a time, we have that $n(ii') = 0$ and $m(ii') = 1$. Therefore, $|G(i,i')| = 0$ or 1.

Next we introduce a set of definitions and assumptions that are related to the diffusion of the shock. Recall that in this model the shock diffusion is associated with the evolution of a new network and a new coalition is chosen in this network to make a move. And the choice of the network as well as the coalition is made by following a law of transition. Furthermore, this choice is only dependent on the previous network-coalition pair. As such, the process of shock diffusion has the Markov property, where the network-coalition pair serves as a particular state and the law of transition generates the transition probabilities. So, we have a finite Markov process of shock diffusion in this stochastic game.

Thus, a state, in fact, refers to a network-coalition pair here. We provide a formal definition as follows.

Definition 2.5: (*State*)

The state space is defined as the set $\Omega = \mathbf{G} \times \mathbf{F}$ *of all feasible network-coalition pairs. We denote a typical state as* $\omega = (G,S)$*.*

Here G refers to the network in ω and S refers to the coalition in G, which has been chosen to make a move. Also note that in our game, we have two kinds of networks. One is at the micro-level where the networks constitute the interbank network and the Fed-bank network. These are the networks where the strategic interactions take place. The other is where each of these networks take the form of nodes to generate a supernetwork. The supernetwork may be viewed as the network of states, where each node is a state, and hence consists of a network and a chosen coalition in that network. The strategic moves happen at the micro-level network, while the path of shock diffusion is viewed at the supernetwork level.

The probability with which the *first* network-coalition pair is chosen is given by an exogenous product measure defined as follows.

Definition 2.6: (*Shock Probability Measure*) (Page, 2011)

Letting $B(\Omega) = B(\mathbf{G} \times \mathbf{F})$ *be the Borel* σ*-field generated by the metric* d_Ω*, we equip our state space* $(\Omega, B(\Omega))$ *with a probability measure*

$$\zeta = \nu \times \eta$$

where the probability measure η *on the node coalition pairs (*F*) is such that* $\eta(S) > 0$ *for all* $S \in \mathbf{F}$ *and* ν *is the probability measure on networks. In other words,* ζ *may be interpreted as an exogenous probability of nature choosing the first network-coalition pair as well that gets to make the move. That is, it gets hit by the shock by this probability and then the coalition decides on a strategy where a* shock *is defined as earlier.*

Now we can define the probability space as

$$(\Omega, B(\Omega), \zeta) = (\mathbf{G} \times \mathbf{F}, B(\mathbf{G} \times \mathbf{F}), \nu \times \eta)$$

a compact metric space with metric $d_\Omega = d(G, G') + d_F$ where $d(G, G')$ is the distance between two networks and $d_F(S, S')$ takes value 1 if it is coalition S's turn to move such that $S \neq S'$ and 0 otherwise. We define $d(G, G')$ in the following section. Because $\mathbf{G}$ is a compact metric space, $B(\mathbf{G} \times \mathbf{F}) = B(\mathbf{G}) \times B(\mathbf{F})$ where $B(\mathbf{F})$ is the set of all subsets of $\mathbf{F}$ (including the empty set).

A-7: (**Shock**)

1. For an event to be called a shock, we define the 'devaluation threshold' as an exogenous scalar $x \in [0, 1]$ (so, $x \times 100$ is in percentage terms). Thus, if the event results in the

devaluation of the assets of a node by more than $x\%$ then it is considered to be a shock for that node.

2. We assume that a shock can only hit a borrower bank in the interbank network.
3. In the Fed-bank network, since the lender is, in fact, a borrower requesting Fed for money, a lender can be hit by the shock as well[7].
4. We also assume that at a time the shock can hit only one bank directly and therefore, at a time only one coalition is directly affected. This assumption helps us to carve out the path of shock diffusion at a micro level.

A-8: (**Collapse**)

A *state of collapse with bailout* refers to a network where none of the debt contracts issued by lender banks are honored by the borrowers and bailout request approval arcs from the Fed (in response to the bailout request arcs to the Fed) are active for all nodes irrespective of whose turn it is to move.

A *state of collapse without bailout* refers to a network where none of the debt contracts issued by lender banks are honored by the borrowers and no arcs are directed from the Fed (in response to the bailout request arcs to the Fed) as the Fed refuses to bail out any node, irrespective of whose turn it is to move.

Note that the transition to the next state depends on the actions of the current coalition in the current network. The action of each member of the coalition that makes a move takes the form of a proposal of the next network-coalition pair. These proposals have to be feasible so that they do not violate the rules of network formation or the assumptions that we have made. All proposals that satisfy these conditions are collected in a set that we call as the feasible set of proposals and denote it as $\Phi_d(G, S)$ or $\Phi_d(\omega)$. We then, have the following definition.

Definition 2.7: (*Proposal Constraint*)

[7]Recall, that our assumption that a shock can only hit borrowers. Also, the assumption that the lenders continue to be lenders and borrowers continue to be borrowers so that no new debt contracts are issued is only applicable to the interbank network not the Fed-bank network.

Suppose the current status quo is $\omega = (G, S) \in \mathbf{G} \times \mathbf{F}$. *For each player* ($d \in N$), *the proposal constraint mapping is such that, for all* ω

$$(a)\ G \in \Phi_d(G, S) \text{ for all } d \in S$$

and

$$(b)\ \{G\} = \Phi_d(G, S) \text{ if } d \notin S$$

Thus, under (a) each player d in each state has the option of proposing that the status quo network be maintained and under (b) if the player is not part of the status quo coalition, then the status quo is the *only* network proposal available to that player. Moreover, if network $G' \in \Phi_d(G, S)$ is proposed by player $d \in S$, then under the rules of network formation and our assumptions, it must be feasible for player d, working alone or together with some coalition S, to change the status quo network G to the proposed network G'.

We will denote by $\mathbf{\Phi}(.)$ the aggregate constraint correspondence,

$$\omega \longrightarrow \mathbf{\Phi}(.) := \Pi_{d \in N} \Phi_d(\omega)$$

The proposal constraint defines a part of the rules of network formation as it specifies for each player, who is a member of the coalition, whose turn it is to move and what are the proposals that the coalition can pick from.

A-9: (**Proposal Constraint**)

Each node and one of its direct connections (a coalition) that is supposed to make a move must propose the next network (by the choice of their strategy) and the coalition under the following conditions

1. abide by the rules of network formation

2. proposed network-coalition pair must belong to the feasible set of network-coalition pairs, $\Phi_d(\omega)$, for each node d and hence, also abide by the assumptions. For instance,

members of the coalitions that haven't been chosen can only propose status quo. As such, for those members, $\Phi_d(\omega) = \omega$.

3. The proposal constraint also imposes on the lender nodes that if their borrowers have already initiated an arc to the Fed with a request for bailout, then they will not be able to enforce their contract with respect to these borrowers. So, they can only propose states where they are either proposing the current state or proposing that the lender-borrower coalition requests Fed to bailout both.

The third assumption is important, in particular. It is a *fairness* assumption that is intended to incorporate the fact that nodes are not unreasonable and if a borrower is already requesting the Fed to bailout, then it is an indication to the lender of the borrower's solvency condition. As such, the lender is expected to prefer initiating a connection to the Fed for bailout over asking the borrower to pay back by choosing to enforce the contract.

A-10: (**Continuity of the Constraint Mappings**)

All constraint correspondences, $\Phi_d(G, S)$ are such that

1. *for all states* $\omega = (G, S)$,

   (a)

   $$G \in \Phi_d(G, S)$$

   (b)

   $$\{G\} = \Phi_d(G, S) \forall\, d \notin S,$$

2. $\Phi_d(.)$ has a closed graph,

$$Gr\Phi_d(.) = \{(\omega, G) : G \in \Phi_d(\omega)\}.$$

Also,

$$\omega \mapsto \Phi(\omega) = \Pi_{d \in D} \Phi_d(\omega)$$

In our interbank network a directed arc from a lender to a borrower could mean either that the former has lent money to the borrower or a contract has been enforced. Either ways then, the higher the number of outgoing arcs from a lender, the more the number of banks that it has lent out money to. Since these arcs are not weighted, we assume that the higher the number of arcs, the higher the amount of loan issued by the node. We use this idea to define the term '*centrality*' here.

Definition 2.8: (*Degree of Centrality*)

We define the degree of centrality of a node, i as

$$d_{i,G} := \frac{\mid G^{+}(i) \mid}{\sum_i \mid G^{+}(i) \mid}$$

where $\mid G^{+}(i) \mid$ is the cardinality of the set $G(i,i') \; \forall \, i'$ ((i.e, the total number of outgoing arcs from node i in network G)) and $\sum_i \mid G^{+}(i) \mid$ is the total number of outgoing arcs in network G.

This plays an important role in determining how '*exposed*' a bank is to the shock. In other words, centrality is a measure of the share of i's amount lent as a percentage of the total loaned amount in the network. Note that, since no arc goes out of a borrower node this number is 0 for that node except for the case when it is requesting the Fed for a bailout. In that case, it increases the borrowers centrality and hence *exposure* because it 'exposes' itself to the mercy of the Fed. If the Fed does not bail it out, there is a chance that it will no longer survive. Further, this value is also positive for the borrower if it has an arc directed towards the lender, implying repayment of the the loaned amount. This makes the borrower more central in the network.

Definition 2.9: (*Exposure*)

The exposure level for bank $\{i\}$ is given by a increasing, continuous and differentiable function defined as:

$$X_i : d_i \longrightarrow [0,1]$$

such that $\frac{dX_i}{dd_i} > 0$.

In case of a coalition, $s \in \mathbf{F}$, the exposure levels for the members of the coalition may be denoted by X_s and $\{X_i, X_j\} = X_s$ where $\{i, j\} \in s$.

A-11: (**Exposure**)

In our model, we define the exposure function as follows:

$$X_i = d_i \exp(-d_i) \tag{2.3}$$

where d_i is the degree of centrality and is defined as above. Note that since $d_i \in [0, 1]$, therefore, in that range X_i is increasing in d_i. This also implies that $X_i \in [0, e^{-1}]$[8]. So, the definition of X_i still holds as it lies within $[0, 1]$. Note that this function is continuous as well as differentiable.

Definition 2.10: (*Payoffs*) (Page, 2011)

We define the payoffs for bank $d \in N$ by the following function:

$$u_{d \in N}(.,.) : \Omega \times \Omega \to [-M, M]$$

Thus, if the current state is $\omega = (G, S)$ and if players propose m-tuple of states $(G_N, S_{G_N}) := ((G_d)_{d \in N}, S_{G_d}) \in \mathbf{\Phi}(\omega)$ then player $\bar{d}$'s payoff is given by

$$r_{\bar{d}}(\omega, (G_N, S_{G_N})) := r_{\bar{d}}(\omega, ((G_{\bar{d}}, S_{G_{\bar{d}}}), (G_{-\bar{d}}, S_{G_{-\bar{d}}})))$$

A-12: (**The Return Function**)

We assume that the return function only depends on the current state and use the following functional forms. In what follows, l stands for a lender, b stands for a borrower

[8]At $d_i = 0, X_i = 0$ and at $d_i = 1, X_i = e^{-1}$. It can be verified that in this range X_i is positive as well as increasing in d_i. However, note that around $d_i = 1$, the change in exposure relative to change in degree of centrality is not much. This is because, if a node has degree of centrality close to 1, its centrality cannot increase by much more as it is already quite central and therefore, it makes sense intuitively that beyond this point the level of exposure also does not change by too much as it is already quite exposed. We use this particular functional form in order to be able to capture this intuition, rather than, say, $d_i \exp(d_i)$, where the increase would have been exponential. So, at $d_i = 1$, $\frac{dX_i}{dd_i} = 0$, indicating that the exposure can't change any further. So, we have that X_i is only *increasing* in d_i not strictly so.

and F stands for the Fed.

For a lender that belongs to a coalition whose turn it is to move,

$$u_l(\omega) = \sum_{l \neq F} (2|G(b,l)| - |G(l,b)|) - C|G(l,F)| + |G(F,l)| + \mathbf{1}_{l \in S} - 2C.\mathbf{1}_{A_{Fl=\,Not\,Bailout}} \quad (2.4)$$

where $\mathbf{1}_{l \in S}$ is an indicator function that takes the value 1 if l belongs to the coalition that makes the move. $\mathbf{1}_{A_{Fl=\,Not\,Bailout}}$ is an indicator function that takes the value 1 if the Fed does not bailout the lender l and 0 otherwise. This means that not getting bailed out adds to the dis-utility by twice the cost of requesting a bailout. Whereas, if the lender was bailed out, it just incurs C that is reflected in the term $C|G(l,F)|$. For a borrower belongs to a coalition whose turn it is to move,

$$u_b(\omega) = \sum_{b \neq F} (2|G^+_{bl}| - |G^+_{lb}|) - C|G^+_{bF}| + |G^+_{Fb}| + \mathbf{1}_{b \in S} - 2C.\mathbf{1}_{A_{Fb=\,Not\,Bailout}} \quad (2.5)$$

Here, $\mathbf{1}_{A_{Fb=\,Not\,Bailout}}$ has a similar interpretation for borrowers as $\mathbf{1}_{A_{Fl=\,Not\,Bailout}}$ has for lenders.

For a lender that belongs to a coalition whose turn it is *not* to move,

$$u_l(\omega) = \sum_{l \neq F} (|G(b,l)| - |G(l,b)|) - C|G(l,F)| + |G(F,l)| + \underbrace{\mathbf{1}_{l \in S}}_{0} - 2C.\mathbf{1}_{A_{Fl=\,Not\,Bailout}} \quad (2.6)$$

Note that this functional form resembles the one above except the first term which was given by $2|G(b,l)|$ earlier rather that just $|G(b,l)|$. So, the utility by being able to choose was doubled in the case where l belonged to the coalition that was supposed to make the move.

For a borrower belongs to a coalition whose turn it is not to move,

$$u_b(\omega) = \sum_{b \neq F} (|G(b,l)| - |G(l,b)|) - B|G(b,F)| + |G(F,b)| + \underbrace{\mathbf{1}_{b \in S}}_{0} - 2C.\mathbf{1}_{A_{Fb=\,Not\,Bailout}} \quad (2.7)$$

Again, we have a similar interpretation for this form where $2|G(b,l)|$ is replaced by $|G(b,l)|$.

The Fed's Payoff

$$u_f(\omega) = \sum_{l \neq F} (|G(F,l)| - |G(l,F)|) + \sum_{b \neq F} (|G(b,F)| - |G(F,b)|) + \mathbf{1}_{l,F \in S} + \mathbf{1}_{b,F \in S} \qquad (2.8)$$

Notice that Fed gets dis-utility from each request for bailout and gets utility if it can approve the request, so that the term $|G(F,l)| = 1$ for that the Fed-bank coalition in that state. The intuition is, the Fed does not want the economy to be unstable. As such, it would prefer to be able to bail everyone out. However, it can't necessarily do it given it's limited endowment, proposal constraint and rules of network formation. In other words, whether the process would arrive at the state where the Fed is able to bailout every bank depends on the law of transition. Further, such a state need not be desirable because it is, essentially, a state of collapse. Moreover, in order to be in a state where it can propose the state where it bails out all, the Fed would have already incurred maximum dis-utility induced by the maximum number of bailout requests. So, ultimately, it depends on the discounted payoffs, as one would see in a following section.

A-13: (**Continuity of the Utility Functions**)

u_d is a continuous function. However, it need not be differentiable as our state space is discrete.

Given all the definitions above, we can now formally outline the game that the nodes of the network play in every state. Recall that in the game played by the nodes, their action takes the form of a network-coalition proposal for the next state. Further, the proposals made by each of these nodes are suggestions that each node makes about the actions that they must undertake as a coalition. For example, one of the members of the coalition may suggest that both the members play $\{EC, EC\}$ whereas the other member may suggest that they play $\{EC, R\}$. We assume that they make these suggestions simultaneously. Thus, each lender-borrower pair chooses from the set $\{(EC, EC), (EC, R), (R, EC), (R, R)\}$ where the first strategy corresponds to a lender and the second to a borrower in a chosen coalition $\{S_i \in \mathbf{F}\}$ where $\mathbf{F}$ is the set of feasible coalitions. Similarly, each Fed-bank coalition chooses from the set $\{(R, BO), (R, NBO)\}$ and $\{S_i \in \mathbf{F}\}$ where $\mathbf{F}$ is the set of feasible coalitions.

Again, these strategies generate networks with coalitions chosen therein. Thus, in fact, these correspond to states which are network-coalition pairs. As such, it is just simpler to assume that each node's strategies take the form of proposing the next state by each member of the network according to the proposal constraint (or feasible set of proposals for each node). Therefore, at every period, the coalitions which have not been chosen to make a move, propose status quo state. And the coalition whose turn it is to move plays the following game where strategies are state proposals.

Suppose the current state is given by ω, where it is a particular coalition's ($\{i, j\} \in S$) turn to move. Further, suppose that the feasible proposals by each member of the coalition belong to the set $\{\omega, \omega', \omega''\}$. That is, their proposals can be one of ω, ω' or ω''. Then, at every period, the members of the coalition whose turn it is to move play the following game:

TABLE A : ILLUSTRATION OF THE STATE GAME

		$j \in S$	$j \in S$	$j \in S$
		ω	ω'	ω''
$i \in S$	ω	$\{u_i(\omega), u_j(\omega)\}$	$\{u_i(\omega), u_j(\omega')\}$	$\{u_i(\omega), u_j(\omega'')\}$
$i \in S$	ω'	$\{u_i(\omega'), u_j(\omega)\}$	$\{u_i(\omega'), u_j(\omega')\}$	$\{u_i(\omega'), u_j(\omega'')\}$
$i \in S$	ω''	$\{u_i(\omega''), u_j(\omega)\}$	$\{u_i(\omega''), u_j(\omega')\}$	$\{u_i(\omega''), u_j(\omega'')\}$

where $u_i(.)$ is the value of utility that i gets by proposing the state that it proposes and $u_j(.)$ is the value of utility that j gets by proposing the state that it proposes.

We can find the Nash equilibrium of each of these games and verify that no node in the network of the given state has an incentive to deviate from the proposals, thus making them *equilibrium proposals.* Note that we don't need to worry about the deviations of nodes which belong to coalitions that have not been chosen to move, as they never deviate given that they are proposing the status quo - the only proposal they can make. However, we do need to verify that the equilibrium strategies of this game are robust to the *status quo* being proposed by the other members. For example, if there was one other player, k, left out of this coalition, then that player simply proposes ω. The game for each member of S and k would look like the following (we just consider i here as a representative for $\{i, j\} \in S$).

		$k \notin S$
		ω
$i \in S$	ω	$\{u_i(\omega), u_k(\omega)\}$
$i \in S$	ω'	$\{u_i(\omega'), u_k(\omega')\}$
$i \in S$	ω''	$\{u_i(\omega''), u_k(\omega')\}$

Thus, if node i has a dominant strategy, it doesn't matter what node k proposes.

From these we get our equilibrium proposals that we denote by $f_d^*(\omega)$ for each node d and each state ω.

Definition 2.11: (*Pure Stationary Strategy*)

A **pure stationary strategy** *for player $d \in N$ is a stationary Markov strategy $(\sigma_d, \sigma_d, \ldots, \sigma_d)$ such that for some function*

$$f_d : \Omega \rightarrow \mathbf{G} \times \mathbf{F}$$

with $f_d(\omega) \in \Phi_d(\omega)$ for all $\omega \in \Omega$,

$$\sigma_d(f_d(\omega)|\omega) = 1 \ for \ all \ \omega \in \Omega.$$

Thus, the strategy f_d maps from any given state ($\omega \in \Omega$) to the set of corresponding *feasible* state proposals ($\mathbf{G} \times \mathbf{F}$), for node d. f_d is said to be a stationary strategy, if for any state ω, the probability of choosing f_d is 1.

We denote equilibrium strategies as $f_d^*(\omega)$ and use $fd(\omega)$ to calculate our equilibrium Markov transition matrix that has been defined in the next section. However, before we can define it, we must introduce the notion of distances between any two networks as we use this in defining our transition probabilities.

2.2.4 The Metric and The 'Endogenous' Transition Probabilities

We introduce a metric to measure distance between any two networks. This is important as the definition of the transition probabilities used here are based on these distances.

The Notion of Distance Between Two Networks:

We use a distance function that 'counts' the number of changes required to be imposed on the current network to get the new network and adjusts for the consequent change in exposure levels. For example, in the graphs shown in figure (4.2), it takes 4 changes to get the new network (G_1) from the current network (G_0), which has been illustrated in figure (4.3). The dotted lines in these figures denote dormant connections, that indicate that those could be potentially activated as per the rules of network formation. Since a pair of nodes can only have one directed arc between them, in order to initiate a new arc (in the reverse direction), they have to remove the current arc, replace it with a dotted arc, remove the dotted arc in the reverse direction and replace it with a solid directed arc. Thus, there are 4 steps between the new network and the old.

In order to capture the aforementioned idea, we propose the following distance function to measure the distance between two networks:

$$d(G, G') = \sum_{(i,j) \in N \times N} |(\mid G(i,j) \mid - \mid G(j,i) \mid) - (\mid G'(i,j) \mid - \mid G'(j,i) \mid)| + \sum_{i} \mid X_i - X_i' \mid \tag{2.9}$$

and the distance between states is defined as

$$d_{\Omega}((G', S'), (G, S)) = d(G', G) + d_F(S', S) \tag{2.10}$$

where $d_F(S', S) = 1$ if $S' \neq S$ and 0 otherwise. So, we have that this distance is a function of the outdegrees[9]. However, recall that the actions undertaken by the nodes affect the number of outdegrees as well as the exposures. Thus, the outdegrees are, in turn, dependent on actions. In other words, the distance measure is also a function of the actions undertaken

[9]In network terminology, *outdegrees* refer to the number of connections going out of a node

by the nodes within the current network. Given this, we have $d(G, G') = d(G, G', f_N(\omega))$ where $f_N(\omega)$ stands for actions of the nodes in state ω consisting of network G. Therefore, we also have $d((G', S'), (G, S)) = d((G', S'), (G, S), f_N(\omega))$. However, for ease of notation, we suppress $f_N(\omega)$ unless it is required for clarity.

Note that this distance is a function of the outdegrees shared between all feasible coalitions in the current network and the same pairs in the next network. Thus, it includes outdegrees shared between the coalition (of the current state) which is making the proposal for the next state as well as the outdegrees shared between the coalition that is proposed for the next state. Given this, our distance measure is endogenous to the system. More explicitly, the distance measure not only depends on the decisions of the current coalitions (whether they choose to continue with their existing arcs or initiate new arcs subject to the rules of network formation and proposal constraints) but also accounts for the corresponding changes that determine the next network and the feasible set of coalitions. Thus, the distance measure depends on the strategic interactions between the members of the coalition, which, in turn, have an effect on the network formation process. So, the distance measure is endogenous and not independent of the decisions of the nodes within the current network that affect the evolution of the next network.

Lemma 2.1: *The following distance function is a metric.*

$$d(\omega, \omega') = d(G, G') + d_F(S, S')$$

where,

$$d_F(S, S') = 1 \ if \ S \neq S' \ \text{and} \ 0 \ \text{otherwise.}$$

and

$$d(G, G') = \sum_{(i,j) \in N \times N} |(| \, G(i,j) \, | - | \, G(j,i) \, |) - (| \, G'(i,j) \, | - | \, G'(j,i) \, |)| + \sum_i | \, X_i - X_i' \, |$$

where the first term is the sum over the net change in the connections between a pair of nodes $\{i, j\}$ in G and G'. The second term is the net change in exposure levels per *node.*[10]

Transition Probabilities

A formal definition of transition probabilities is as follows.

Definition 2.12: (*Transition Matrix*) (Page, 2011)

Given the profile of player proposals

$$(G_N, S_{G_N}) = ((G_d)_{d\in N}, (S_{G_d})_{d\in N}) := ((G_d, S_{G_d}), (G_{-d}, S_{G_{-d}}))$$

and given the current state, $\omega = (G, S)$*,* **nature** *then chooses the next state (i.e., the next network-coalition pair and the corresponding level of exposure given the choice of coalition) in accordance with the* **Markov transition law***,* $q(.|., (G_N, S_{G_N}))$*.*

Thus, given m-tuple player proposals (G_N, S_{G_N}) and current state ω, the probability with which nature chooses the next state ω' is given by $q(\omega'|\omega, (G_N, S_{G_N}))$.

In order to simplify notation, let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$, and let $H := \{1, 2, \ldots, M\}$ with typical elements k, l index states. Thus, $k \in H$ if and only if $\omega_k \in \Omega$; and for any nonempty subset E of Ω, $k \in H_E$ if and only if $\omega_k \in E$. We will often use ω_k and ω_l rather than $\omega(= \omega_k)$ and $\omega'(= \omega_l)$.

For each proposal m-tuple (G_N, S_{G_N}), let $Q(G_N, S_{G_N})$ be the resulting $M \times M$ Markov transition matrix. The typical entry in our transition matrix $Q(G_N, S_{G_N})$ has typical entry

$$q_{ij}(G_N, S_{G_N}) := q(\omega_j|\omega_i, (G_N, S_N))$$

where $q_{ij}(G_N, S_{G_N})$ is the probability that nature moves from state $\omega_k = (G_k, S_k)$ to state $\omega_l = (G_l, S_l)$ given player proposals (G_N, S_{G_N}).

Here, we define our transition probabilities as a function of the distances between networks as follows:

[10] All proofs have been gathered in the appendix for this chapter - Appendix 1.

$$q(\omega'|\omega, f_N(\omega)) = \begin{cases} \frac{d((G,S),(G',S'),f_N(G,S))}{1+\sum_{(G',S')} d((G,S),(G',S'),f_N(G,S))} & if\ \omega_i \neq \omega_j \\ \frac{1}{1+\sum_{(G',S')} d((G,S),(G',S'),f_N(G,S))} & if\ \omega_i = \omega_j \\ 0 & (if\ (G',S')\ \notin \Phi(.)) \end{cases} \tag{2.11}$$

where Φ is the set of admissible proposals. Note that $(G, S) = \omega$ and $(G', S') = \omega'$. Further, since $\omega' \in \Phi(\omega)$ is generated by actions of the nodes, we have $\omega'(f_d(\omega))$. Therefore, in equilibrium the state proposal, ω^*, by node d may be written as $\omega^*(f_d^*(\omega))$. So, the equilibrium transition matrix is given by

$$q(\omega'|\omega, f_N^*(\omega)) = Q(f_N^*) = \begin{cases} \frac{d(\omega,\omega')}{1+\sum_{\omega'} d(\omega,\omega')} & if\ \omega \neq \omega' \\ \frac{1}{1+\sum_{\omega'} d(\omega,\omega')} & if\ \omega = \omega' \\ 0 & (if\ \omega\ \notin \Phi(.)) \end{cases} \tag{2.12}$$

Since the actions take the form of state proposals, they affect the transition probabilities via the expression $d(\omega, \omega')$. Thus, the law of motion as defined above seems to be invariant to $f_N^*(\omega)$ as $d(\omega, \omega')$ has already incorporated it to generate the probabilities of all states that the action profiles of the nodes can lead to, including the Nash state proposal. Any deviation to a state that is not feasible given the action profiles has zero probability of transition, by definition. In other words, we do not have to worry about actions that result in infeasible states as that happens with probability 0. So, we have $q(\omega'|\omega, f_N^*(\omega)) = q(\omega'|\omega, f_N(\omega))$.

Recall that the distance function is endogenous to the network formation game. And by defining the transition probabilities as a function of these distances, we have, in fact, constructed *endogenous transition probabilities*. By doing this, we suggest that the network formation game follows a Markov chain governed by these endogenous probabilities. Now we define each node's problem.

The Bank's Problem

Given the definitions and assumptions above, each bank's (denoted by 'd') discounted payoff may be defined as follows:

$$V_d(\omega) = \max_{f_d \in \mathbf{\Phi_d}(\omega)} \left[u_d(\omega, (f_d, f_{-d}(\omega))) + \beta_d \textstyle\sum_{\omega' \in \Omega} V_d(\omega') q(\omega' | \omega, (f_d, f_{-d}(\omega))) \right] \tag{2.13}$$

where the return from future states is discounted (with a bank-specific discount rate) and $q(.|.)$ is the transition probability function.

The solution to this problem is given by

$$w_d^*(\omega) = (I - \beta.Q(f_N^*))^{-1} u_d(.) \tag{2.14}$$

Herings and Peeters (2004) show that because $Q(f_D^*)$ is a stochastic matrix, with rows nonnegative and summing to 1, it follows from Hadamard's Theorem that the $(I - \beta.Q(f_N^*))^{-1}$ exists and

$$(I - \beta_d.Q(f_N^*))^{-1} = \sum_{n=0}^{\infty} \beta_d^n Q^n(f_N^*) \tag{2.15}$$

The proof for the following can be found in Page (2009).

(w_N^*, f_N^*) solves the minimization problem

$$\min \sum_{d \in N} \sum_{i \in H} \left[w_d(\omega_i - r_d(\omega_i, f_N(\omega_i)) - \beta_d \textstyle\sum_{j \in H} w_d(\omega_k)) q(\omega_k | \omega_i; \omega_i, f_N(\omega_i)) \right] \tag{2.16}$$

And, the value function-strategy pair $(w_N^*(.), f_N^*(.))$ is an equilibrium. We illustrate and verify this with our ensuing example.

Lemma 2.2: $(w_d^*(.), f_N^*(.))$ exists and $w_d^*(\omega)$ is bounded.

2.3 Main Results

In this section we discuss the main results of this chapter. The proofs for each of them have been gathered in the appendix for this chapter. We apply finite-state Markov chain theory to our financial network and shock diffusion process to arrive at some interesting results. Also, recall that we defined the micro-level network and the supernetwork in an earlier section. And argued that the shock diffusion happens in the supernetwork where states (which are made of networks) are the nodes. The results presented here pertain to the supernetwork.

Let the integer-valued random variable, $T^*_{\omega_j}(.) : \Theta \longrightarrow \{0, 1, 2, \ldots\}$, given by

$$T^*_{\omega_j} = \min\{n \geq 1 : W^*_n = \omega_j\}$$

be the *hitting time* of network-coalition formation process $\{W^*_n\}_n$ for state $\omega_j \in \Omega$, and let

$$\rho^*_{ij} = \Pi\{T^*_{\omega_j} < \infty | W^*_0 = \omega_i\}$$

be the *probability* that the process $\{W^*_n\}_n$ reaches the state ω_j after leaving state ω_i at time zero in finite time.

We have the following result that is standard, from Markov chain theory.

Lemma 2.3: (A Standard Result in Markov Chain Theory) $\rho^*_{ij} > 0$ *if and only if* $q^n(\omega_i, \omega_j) > 0$ *for some* $n \in \aleph_+$.

This result implies that we have a positive probability of reaching a particular state ω_j from ω_i if the transition probability of getting to ω_j from ω_i in n steps is positive, where n is a positive natural number. This makes sense intuitively because if $q^n(\omega_i, \omega_j) = 0$, the process may never (or with extremely low probability) transit to ω_j from ω_i to begin with, leave alone reaching there in finite time! This yields the following corollary which will serve useful in interpreting some of our results.

Corollary 2.1: *If* $q_{ii} = 0$*, then* $\rho_{ii} = 0$.

Corollary 1 is a direct consequence of Lemma 3 and indicates that if a state is not closed[11], then the probability of reaching there in finite time and staying there is 0.

Next, we establish the following Lemma that provides a condition under which, for states ω_i, ω_j and ω_k, such that $q_{ij} > 0$ and $q_{ik} > 0$, the probability of reaching, say, ω_j from ω_i in finite time is higher than that of reaching ω_k from ω_i in finite time.

[11]A state, ω_i, is said to be closed if there does not exist any other state, ω_j (not equal to ω_i), such that $q_{ij} > 0$. Thus, $q_{ii} = 1$.

Lemma 2.4: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* (as per definition (2.12)) with state space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$. Suppose there exists a state ω_j with the probability of transition from the current state ω_i given by $q_{ij} > 0$. Also, suppose that the transition probability of any other state ω_k is given by, $q_{ik} \geq 0$. Further, if $q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n}$ then the following condition holds:*

$$\Pi\{T_{\omega_j}^{*k} < \infty | W_0^* = \omega_i\} > \Pi\{T_{\omega_j}^{*k} < \infty | W_0^* = \omega_i\}$$

Lemma 2.4 serves as an important starting point for the results that we derive. Recall that our transition probabilities are a function of the distance between states. These distances depend on the distances between networks, which in turn depend on the net change of outdegrees of nodes from the current to the new network and net change in exposures. Thus, the condition $q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n}$ implies that at the nth ($n < \infty$) step the normalized ratio of net change in the outdegrees and exposure from ω_i to ω_j is higher than that from the net change in outdegrees and exposure from ω_i to ω_k, at the very first step. So, the process is more likely to reach ω_j in finite time than it is likely to reach ω_k. This implies that the higher the net change in outdegrees and net change in exposure from the current state to a state which can be transitioned to, in finite number of steps, the higher are the chances that the process would tend to go there in finite time.

In other words, the probability of reaching the state with higher net degree of centrality and exposure from a given initial state is greater than the probability of reaching a state with lower net degree of centrality and exposure from the same initial state given the Markov transition probabilities in finite time. Recall that these transition probabilities are endogenous and incorporate the diffusion of the shock.

This lemma leads us to our first theorem which establishes a *threshold condition* such that as long as it holds, the process always tends to reach a state with highest net change in exposure and net change in outdegrees.

Theorem 2.1: *Suppose we have $q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n}$ for all ω_i, ω_j and ω_k with $n = 1$. Further, let*

$\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then all states, with a typical state denoted by ω_r, that lie on the path of the process are such that for all other states, ω_m, given the previous state, ω_l, with $q_{lm} \geq 0$ and $q_{lr} \geq 0$ such that $q_{lm} < \frac{q_{lr}}{1+q_{lr}}$, we must have $q_{lm} < q_{lr}$. Further, if ω_m and ω_r are basins of attraction, and the state leading to them is given by ω_l, it would still be the case that $q_{lm} < q_{lr}$. In other words, if this 'threshold condition' condition holds at every step, the process always has a tendency to move to a state with higher net change in exposure and centrality.

Intuitively, this result gives us the condition under which from every step of the Markov process of shock diffusion, the process tends to go to the state with a higher net change in exposure and net change in outdegrees. And, given this happens at every single step, the basin of attraction that the process finally reaches must be one which has the highest net change in exposure and net change in outdegrees among the available options of states (which may include other basins of attraction) at that point in the path of shock diffusion. Further, this point in the path of shock diffusion, which is essentially a state, may be viewed as a *tipping point* and the corresponding network may be referred to as a *tipping point network*. The significance of this result lies in the fact that this may lead the process to a worse basin when a better basin may be available, as will be illustrated in our example in the next section.

The following lemma follows directly from the definition of transition probabilities.

Lemma 2.5: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then $q_{ik} < q_{ij}$ if and only if $d(\omega_i, \omega_k) < d(\omega_i, \omega_j)$ for $\omega_k \neq \omega_j$.*

The next theorem holds a lot of importance as it captures the essence of the model here and establishes that with endogenous transition probabilities, the equilibrium is no longer a completely 'random' outcome. The randomness is attributed to the fact that the realization of a particular basin of attraction is based on the law of transition followed by nature in order to pick the next state. However, the law of transition defined by the transition probabilities are, in turn, a function of the actions of the nodes in the micro-level network. Hence, the endogeneity of these probabilities ensures that the equilibrium chosen by nature is not a completely random choice. In other words, the actions of the nodes over the course of shock diffusion lead the Markov process to a state from where it can transition to a particular equilibrium state chosen by nature according to the law of transition, given a set of possible equilibriums.

Theorem 2.2: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in* (2.12)*. Further, let ω_{cb} denote the* state of collapse with bailout*. Then, for any ω_i and ω_j, such that $q_{icb} > 0$ and $q_{ij} > 0$, we must have $q_{icb} > q_{ij}$ implying that the transition probability of going to the state of complete collapse with bailout is higher than that of going to any other state from a given state ω_i from where ω_{cb} and ω_j are both reachable.*

Let there be a transient state that leads to some set of states one of which is the state of collapse with bailout. That is, the probability of transition to any of these states from the transient state is positive. This result establishes the fact that then we must have the probability of transition to the state of collapse with bailout to be the highest of all the states that nature could pick as a next state given this transient state as the current state. In other words, once we reach this transient state, irrespective of what is proposed, nature would pick the state of collapse with bailout with the highest probability.

The relationship between the structure of networks (i.e., the spread of outdegrees) and the transition probability is of crucial importance in the context of the analysis here. In fact, from one of the steps in the proof of the theorem above, we get the following corollary, which establishes that the association between the distances of two networks is sufficient to deduce the association between the corresponding transition probabilities.

Corollary 2.2: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (*2.12*). Then, if we have that for states ω_i, ω_j and ω_k such that $d(G, G_k) > d(G, G_j) > 0$ and $q_{ik} > 0$ and $q_{ij} > 0$, we must have $q_{ik} > q_{ij}$. So, $d(G, G_k) > d(G, G_j) > 0$ is sufficient to deduce that $q_{ik} > q_{ij}$ such that $q_{ik} > 0$ and $q_{ij} > 0$.*

Extending the discussion on theorem 2, we add that not only does nature choose the state of collapse with bailout with the highest probability, but also this state is a basin of attraction. The following theorem states this formally.

Theorem 2.3: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then if the condition $q_{ij} < q_{ic}$ holds for all ω_i including $\omega_i = \omega_c$, then state of collapse is a basin of attraction. That is, $\rho_{cc} = 1$.*

The underlying idea is that the state of complete collapse is a closed state. It is also finite given that each state contains only one network with finite number of nodes. And every finite closed state, ω_{ii}, is recurrent (that is, $q_{ii} = 1$ implying $\rho_{ii} = 1$). This is a direct implication of Lemma 2.4 and corollary 2.1. Note that idea of the proof for this theorem can be applied to show that there are other basins of attraction where, for example, Fed decides to bailout some nodes while not bailout the others. This theorem suggests that state of complete collapse is also an *equilibrium* where the process is likely to reach with positive probability.

However, it is important to note that what we want to emphasize here is as long as the threshold condition holds, the process tends to reach, with the highest probability, the state of complete collapse with bailout given any other basin that can be reached with a positive probability from the current state.

This result is crucial because it gives us a condition based on our endogenous transition probabilities for the Markov process of shock diffusion under which we may not be able to avoid going to an equilibrium that is not the most desirable. For example, if all are bailed out by the Fed, at the end of the day everyone is content; however, the price that one pays in terms of going through the increased exposure and the uncertainties of whether one would survive or not, is not worth it especially given that a node earns dis-utility by requesting Fed to bail it out. On the Fed's side, every time the Fed receives a bailout request, it earns a dis-utility as it is an indication of the impact of the shock on the network. Note that, in our model, since interbank insurance (where new debt contracts may be issued and potentially unaffected banks can bailout the affected banks) is not allowed, the Fed is under a lot more pressure if it must do something to stabilize the economy. As such, if all banks have active requests for bailout to the bank, it must mean that all are affected. In such a situation, the question arises if it is in Fed's best interest to be able to bail every one out if possible given that it has a limited endowment E_F. It is possible that bailing everyone out is not a desirable policy for the Fed given this resource constraint. As such, the Fed might want to be at an equilibrium where it bails only some of the banks out. As mentioned above, in this game there exist basins of attraction where Fed can choose to bailout only a few. But the threshold condition puts a lower probability on reaching any of those equilibriums.

So, in order to lead the process to a more desirable equilibrium, one must, loosely speaking, work backwards through the expected path of shock diffusion from the desired final state to a set of potential initial states. In other words, if we could impose restrictions on the initial network structures itself, then we could have a threshold condition that leads the process to a more desirable equilibrium. We provide an example.

For this example, please see figure (4.4). Consider the case of the initial network being G_0. Then, in order to get to a state with network G_6, one must first be in the state that contains G_2. However, note that from G_0 one could either go to G_2 or G_3. The distances between these networks is given by

$$d(G_0, G_2) = 4.6065, d(G_2, G_6) = 4.3679,\ d(G_0, G_3) = 6.3679,\ d(G_3, G_5) = 4 \tag{2.17}$$

Given that $d(G_0, G_3) > d(G_0, G_2)$, by corollary 2.2 it is sufficient to deduce that $q_{03} > q_{02}$. And from G_3 the only state that one could go to, given the proposal constraint and rules of network formation, is a state with G_5 which is the network where all bailout requests are active. So, between proposing the current state and G_5, the probability of transition to the state with G_5 given our endogenous transition probabilities is higher. And if the threshold condition holds, this is the case in every step - that is, the state with higher net exposure and net change in outdegrees has a higher probability of transition.

On the other hand, had we restricted the structure of the initial network to be G_1, then we would have the following

$$d(G_1, G_4) = 4.3679, d(G_1, G_7) = 4.73576 \tag{2.18}$$

Note that by doing so, we have reduced the possibility of going to a state with G_4, for instance, which corresponds to G_5 in the previous example, to be the same as going to another state that is much better, namely, G_7. And given that the distance between states is primarily dependent on the distance between networks, the difference in transition probabilities will not be enough to induce the threshold condition that would ensure a higher probability of transition to G_4 instead of G_7 as was the case earlier.

One might argue that the initial state is chosen exogenously by nature. However, recall that nature chooses this state from the set of available states. If we can impose a structure on network formations on the interbank network so that the threshold condition just can't hold, then we reduce the possibility of choosing a network which has a structure that could lead the process to an undesirable basin. Further, such an imposition may lead to a situation where even when the threshold condition is being satisfied, it goes to a better equilibrium. Because, note that the transition probabilities that the threshold condition uses, are a function of the *net change* of exposures and outdegrees. It is indifferent to the direction of change.

However, in this game as the outdegrees from lenders increase (which makes the network more exposed), the endogeneity of the transition probabilities incorporates this and the process moves towards the more exposed states with higher probabilities. So, we must create a network structure where the outdegrees for lenders are not at the maximum, but at an optimal level that depends on the nature of equilibrium one would like the process to eventually reach.

The following corollary just extends our discussion until now to the case where from the current state one might have access to more than one basin of attraction. Then, given our threshold condition, the process would end up in the basin with higher net change in exposure and net change in outdegrees. And given this, the probability that the process reaches this basin in finite time is higher than the probability of it reaching any other basin thus accessible, in finite time.

Corollary 2.3: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Suppose that ω_k and ω_j are both basins of attraction. Then under the threshold condition $q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n}$ for $n = 1$ for some state ω_i, we have $q_{ik} < q_{ij}$ and $\rho_{ik} < \rho_{ij}$*

We have spoken much about the various equilibriums and how to potentially get there. However, another important feature of these equilibriums or basins of attraction, which are basically networks, in this game is that they are singleton. In other words, if the process reaches an equilibrium, we know exactly what kind of network we have. In order to establish this, we use the following result from Markov chain theory applied to our model presented here.

Lemma 2.6: *If ω_x is a recurrent state and ω_x leads to ω_y (i.e., $q_{xy} > 0$) then ω_y is recurrent and $\rho_{xy} = \rho_{yx} = 1$.*

Using this, we show that all our basins of attraction, given the primitives of our model are singleton sets.

Theorem 2.4: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then all basins of attraction are singleton sets.*

This is an attractive feature of the model because, we know the structure of the network exactly, if we know which basin we are in. Going back to our discussion above, this is helpful in imposing restrictions on the initial networks that can potentially evolve to be the desired equilibrium state with the required network, should a shock hit the system. If we didn't have these basins to be singletons, it would be very difficult to impose any kind of structure in the sense that we talked about. This is because, then we would not be sure as to which state, and hence network, in the basin we would end up in or would like to end up in. And even if we ended up in the desired state with the desired network, we would not stay there as the process would move around within that basin, which would have other states containing other networks (since we don't have a singleton basin)!

Finally, the following result shows that it is always possible to partition this super-network - the network of networks into transient states and basins of attraction or recurrent states.

Theorem 2.5: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in* 2.12. *Then the state space can be decomposed into a finite number of disjoint basins of attraction and a disjoint transient sets. In particular, this decomposition is given by.*

$$\Omega = \{\cup_{i=1}^{N} H_i\} \cup T \tag{2.19}$$

where H_i refers to the ith basin and T refers to the transient sets.

In the following section we illustrate our model by means of an example.

2.4 An Example

Let there be three banks and the Fed - i_1, i_2 i_3 and F. By assumption, the interbank network may be partitioned into set of lender banks and set of borrower banks. Thus, lenders are not borrowers at the same time and borrowers are not lenders at the same time. Further suppose that the initial network given to us is G_0 in Figure (4.5). So, we have that both i_1 and i_2 are lenders. i_3 is the only borrower.

Let the endowment of i_1 be $e_{L_1} = 10$, i_2 be $e_{L_2} = 15$ and that of the borrower, i_3 be $e_{B_1} = e_B = 8$ (since there is only one borrower we may denote it, simply, as e_B). Also, let the amount lent to i_3 by i_1 be 2.5 and that lent by i_2 be 3. Let the cost of bailout, C, be 5.

Given the rules of network formation and game described earlier, we can have 46 possible networks (including the ones that evolve as the game proceeds). These have been illustrated in Figure (4.5). In order to prevent the graphs from being cluttered, we have removed the dotted arcs or the arcs that represent potential connections that are inactive.

Given these networks we have a list of possible states. The supernetwork is formed using these states. Please see figure (4.6). All the possible states (network-coalition pairs) have been summarized in table (1).

Recall that, given the rules of network formation, the network-coalition pairs propose the next network-coalition pair. Out of the set of proposals made, nature then chooses which to pick according to the law of transition. The choice of proposal made by each member of the network belongs to their respective proposal constraints. And the strategy is chosen as a best response to the others' chosen strategies. Note that we allow for each of the members of the coalition to propose different states according to what maximizes their payoffs. It does not affect the fact that nature ultimately chooses which will be the next state and the next state may not be either of the ones proposed by the coalition (if proposed more than one). Thus, the law of transition solves the conflict of choices between members of a coalition. The list of possible proposals per state have been given in table (2).

Next, we calculate the payoffs for each of the states. We use the functional forms for the return functions provided in $(2.4), (2.5), (2.6), (2.7)$ and (2.8). These payoffs have been reported in table (3). Given these, members of the coalition whose turn it is to move plays the game illustrated in table (A). For example, consider the state ω_1 where it is coalition $\{i_1, i_3\} \in S_1$ which has to make a move, with possible state proposals given by the set $\{\omega_1, \omega_3, \omega_4, \omega_5\}$. Then i_1 and i_3 play the game in Table (B).

Table B: THE STATE GAME

		$i_3 \in S_1$	$i_3 \in S_1$	$i_3 \in S_1$	$i_3 \in S_1$
		ω_1	ω_3	ω_4	ω_5
$i_1 \in S_1$	ω_1	$u_{i_1}(\omega_1), u_{i_3}(\omega_1)$	$u_{i_1}(\omega_1), u_{i_3}(\omega_3)$	$u_{i_1}(\omega_1), u_{i_3}(\omega_4)$	$u_{i_1}(\omega_1), u_{i_3}(\omega_5)$
$i_1 \in S_1$	ω_3	$u_{i_1}(\omega_3), u_{i_3}(\omega_1)$	$u_{i_1}(\omega_3), u_{i_3}(\omega_3)$	$u_{i_1}(\omega_3), u_{i_3}(\omega_4)$	$u_{i_1}(\omega_3), u_{i_3}(\omega_5)$
$i_1 \in S_1$	ω_4	$u_{i_1}(\omega_4), u_{i_3}(\omega_1)$	$u_{i_1}(\omega_4), u_{i_3}(\omega_3)$	$u_{i_1}(\omega_4), u_{i_3}(\omega_4)$	$u_{i_1}(\omega_4), u_{i_3}(\omega_5)$
$i_1 \in S_1$	ω_5	$u_{i_1}(\omega_5), u_{i_3}(\omega_1)$	$u_{i_1}(\omega_5), u_{i_3}(\omega_3)$	$u_{i_1}(\omega_5), u_{i_3}(\omega_4)$	$u_{i_1}(\omega_5), u_{i_3}(\omega_5)$

which yields,

		$i_3 \in S_1$	$i_3 \in S_1$	$i_3 \in S_1$	$i_3 \in S_1$
		ω_1	ω_3	ω_4	ω_5
$i_1 \in S_1$	ω_1	$(0, -1)$	$(0, 2)$	$(0, -6)$	$(0, -5)$
$i_1 \in S_1$	ω_3	$(3, -1)$	$(3, 2)$	$(3, -6)$	$(3, -5)$
$i_1 \in S_1$	ω_4	$(0, -1)$	$(0, 2)$	$(0, -6)$	$(0, -5)$
$i_1 \in S_1$	ω_5	$(-5, -1)$	$(-5, 2)$	$(-5, -6)$	$(-5, -5)$

The other nodes can only propose ω_1 and hence the payoffs would be $\{i_2, i_F\} = \{-1, 0\}$. Thus, the Nash equilibrium for this game is $\{f_1^*(\omega_1), f_2^*(\omega_1), f_3^*(\omega_1), f_F^*(\omega_1)\} = \{\omega_3, \omega_1, \omega_3, \omega_1\}$.

Let us check if any node has an incentive to deviate. Note that irrespective of who plays what, i_1's dominant strategy is to propose ω_3. Similarly, i_3 best response to i_1's proposal is to propose ω_3 as well. If i_3 proposes ω_3, then i_1's best response is to propose ω_3. And this is also the best response given that the other nodes are proposing ω_1. Note that since these passive coalitions (those whose turn it is *not* to move) only propose status quo, they anyways have no incentive to deviate as they cannot deviate given the proposal constraint. Thus, no one has an incentive to deviate and $\{f_1^*(\omega_1), f_2^*(\omega_1), f_3^*(\omega_1), f_4^*(\omega_1)\} = \{\omega_3, \omega_1, \omega_3, \omega_1\}$ is indeed a Nash equilibrium.

Therefore, the Nash equilibrium for each state is a vector containing each node's Nash proposal[12]. These equilibrium proposals have been given in table (4).

Now, let us calculate the transition probabilities according to the equilibrium transition law given by (2.12). In order to do so, we need the degree of centrality for each node given the network. Recall, we define degree of centrality as

$$d_{i,G} := \frac{| G^+(i) |}{\sum_i | G^+(i) |} \tag{2.20}$$

Using the above, we can now calculate the degrees of centrality for each node in a given network. A sample of these are provided in table (5)[13].

[12]Recall that each proposal takes the form of a state. Thus, it is a vector of proposed states such that none of the nodes have an incentive to deviate from their respective proposals.

[13]Only a sample is provided as it is more for the purpose of illustration. The main goal is to calculate the

Recall that the exposure function was defined as $X_i = d_i \exp(-d_i)$. Note that exposure is conditional on the network, so the exposure values will be dependent on the network. Using the d_i above, we can then calculate the exposures for each node, for each of the networks. In order avoid reporting tedious numbers, we straight away report the transition probabilities.

As mentioned earlier, we calculate the transition probabilities using the function provided in (2.12). Since we have 70 possible states reported in table 1, instead of providing the entire matrix, we provide the probabilities for state transitions which are positive. These have been reported in table (6) in the appendix. In order to check that these are indeed equilibrium transition probabilities, we must have that the discounted payoffs resulting from the Nash strategies of the nodes in the coalition whose turn it is to move, are greater than those resulting from any other feasible strategies available to them, in a given state. Further, note that strategies take the form of state proposals. Therefore, the transition probabilities change as the state proposals change for a given state. Given our transition probabilities, we can show that the Nash strategies do yield the highest payoffs. Consider ω_1 where it is $S_1 = \{i_1, i_3\}$'s turn to move. Their available strategies are given by $\{\omega_1, \omega_3, \omega_4, \omega_5\}$. For i_1, the discounted payoffs corresponding to each of these strategies is given as follows.

$$\omega_1 : u_{i_1}(\omega_1) + \beta_{i_1}.Q(.|\omega_1, f_{i_1}(\omega_1), f^*_{-i_1}(\omega_1)).w^*_{i_1}$$

where the first term is the state contingent payoff to i_1 and the second terms is the discounted payoff to i_1 if it proposes ω_1 for the next state. Inserting the values, we get -1.2577. Now for i_1's Nash strategy, ω_3, we have

$$\omega_3 : u_{i_1}(\omega_1) + \beta_{i_1}.Q(.|\omega_3, f^*_{i_1}(\omega_1), f^*_{-i_1}(\omega_1)).w^*_{i_1}$$

transition probabilities, which have been provided for all state transitions. The degrees of centrality for all players in all states as well as distances between all possible network transitions (based on state transitions) have been gathered in an appendix which is available upon request. Also, in this sample, we have included the number of self-loops in our calculation as well. However, ignoring the self-loops in calculating the transition probabilities will not affect their order. It is up to the reader to verify this. Self loops were not included in the calculation of the transition probabilities reported here.

where the second term yields the discounted payoff to i_1 if it proposes ω_3. Inserting the values, get -0.7398 which is greater than -1.2577. Similarly, we can calculate the payoffs to i_1 if it proposes ω_4 and ω_5. These are -1.3139 and -3.0284, respectively. Note that, the highest payoff for i_1 is associated with its Nash strategy for ω_1 given by ω_3. In the same way, we can check that this also holds for i_3. In order to verify that these are indeed equilibrium transition probabilities, one can make other state-strategy comparisons as well. We leave it up to the reader to check those.

Next, using the equilibrium proposals and the solution to the bank's problem defined in (2.13) where we use the equilibrium Markov transition matrix given in table (6), we can calculate the equilibrium payoffs, w_d^* for each player d. These have been reported in table (7).

Note that once the shock has hit the system, the joint payoffs are maximized if either the borrower that was hit by the shock repays all the debt (ω_9, ω_{10}) or if the Fed bails out all banks (ω_{66}). If the system reaches ω_9 or ω_{10}, then the process infinitely stays there. This is true for any basin of attraction. However, whether the process reaches a basin of attraction would depend on the transition probabilities. Note that in this supernetwork, under our threshold condition, the basin of attraction that we would *most likely* reach is ω_{66} which is the state of collapse with bailouts. Thus, theorem 2 holds.

Although, ω_{66} yields positive payoff for all nodes, the system pays a price in terms of going through the process of diffusion. If the process would have ended up in either ω_9 or ω_{10} then the borrower would have been better off. Note the payoff for the Fed is 0 in ω_9 and ω_{10}. This is because, the borrower repays the loan and the Fed is not active in these networks. So, Fed gets 0. However, if the process goes to some other state where there are bailout requests, the Fed derives utility or dis-utility (depending on the corresponding r_f) by responding to them. The question then remains, how many banks should the Fed bail out and in that sense, which network does the coalition involving the Fed choose to propose? In this game, the Fed does the best by bailing out all the banks.

The ergodic measures corresponding to this game have been provided in table 8.

These put a measure of 1 on all basins[14].

2.5 Conclusion

To the best of our knowledge, this the first attempt to study shock diffusion through a model of network formation game. We show that the process of shock diffusion is dependent on the strategic moves made by the nodes within the network. In our opinion, this is a novel way to model the endogenous risk of contagion in a financial network. We develop a network evolution based approach to analyze the impact of a shock (at a micro-level) on the nodes in an interbank market, which could potentially lead to contagion. We provide general results that allow for the process to reach a state of collapse with a positive probability among other possible equilibriums. The results of this chapter hold significant importance in financial analysis as well as have strong policy implications. Once the strategic interactions within an intricate network such as an interbank network are understood, policy makers can impose the most efficient network structure by means of regulation to minimize the probability of a complete collapse of the system. In particular, the policy can take the form of a restriction on the outdegrees of lenders and hence, a ceiling on how many borrowers a bank can have at a given point in time.

This model of financial contagion uses endogenous Markov transition matrix to show that given a certain threshold condition, we can direct the Markov process governed by this transition matrix to a particular basin of attraction. In our case, it turns out that given our law of transition, the process is led to (with highest probability) a basin that is characterized as a state of complete collapse with bailouts. The networks leading to this basin can be called the *tipping point networks.* Although, the payoffs are positive in this state, it is not a desirable basin as a basin where no bailout requests are initiated and yet the borrower is able to payoff the debt is available. As such, if we can impose bilateral connections within an interbank network such that the threshold condition leads the process to a better basin, then every one is better off. Although, it may not be possible to have regulations at a micro level, but if the rules of network formation could regulate the number of outdegrees

[14] Codes for this chapter have been included in the appendix titled 'Codes'.

(number of borrowers or amount lent), then the exposure as well as centrality levels could be controlled. This would minimize the state space, which would mean that a number of suboptimal basins can be removed, thereby enhancing the chances of being in a better state rather than a state of collapse (with or without bail out).

Chapter 3

Simulating Networks: A Copula Based Approach

3.1 Introduction

The use of network theory to study complex economic structures has gained momentum in recent years. However, the application of this method tends to find a severe obstacle in the lack of 'ready-to-use' bilateral level data that is essential for an in-depth network analysis of socio-economic problems. This constraint often leads economists to focus on the small number of datasets available (e.g., ICPSR - Add Health Dataset), limiting the variety of empirical exercises that can expand our understanding of networks. In particular, interbank networks, such as the ones studied in the previous chapter, face a strong scarcity of publicly (and frequently at a regulatory level as well) bilateral level data. This strongly impairs any analysis that aims to predict or explain how these connections are formed, consequently limiting any valuable insight that could inform the policy making process.

This has drawn the attention of researchers in the past, who have proposed different methods to simulate bilateral connections using available information such as the total number of connections per node and so on. For example, Upper and Worms (2004) address this issue by using Maximum Entropy[1] to simulate bilateral connections within the Ger-

[1]A list of formal definitions for all technical terms from the literature that have been used in this chapter

man interbank market using data on total assets and liabilities. The standard Maximum Entropy simulation technique is based on the assumption that each node maximizes the diversification of its connections. Therefore, it equally divides the total number or weight of connections among all nodes and then executes an iterative proportional fitting procedure such as the RAS algorithm proposed by Schneider (1990) to re-balance the matrix[2]. However, Mistrulli (2010) provides a description of the shortcomings of Maximum Entropy as a simulation method for these connections. This serves as the motivation for our research here.

In this chapter, we propose an alternate simulation methodology that uses a *copula* based algorithm to simulate bilateral connections. This method differs from Upper and Worms (2004) in the manner it treats the input matrix of connections as well as the application of the RAS algorithm for matrix re-balancing. The proposed method treats all connections differently based on the values assigned by the copula and generates a stochastic matrix, which is then multiplied by the total sum of columns of the matrix. This provides an estimate of the bilateral values and imposes the required 'structure'[3] on the connections, which is likely to enhance the quality of simulation.

One of the main contributions of this research is the simulation based comparison between our copula based method and the Maximum Entropy approach. We outline conditions under which the copula based method outperforms the Maximum Entropy approach, making it a more effective method of simulation. Further, we find numerical evidence that the degree of asymmetry in the data may affect the goodness of fit of both the copula based method[4] as well as Upper and Worms (2004) Maximum Entropy method, which is considered to be the benchmark for our analysis. The investigation of an association between the degree of asymmetry in the data and the performance of any of the simulation algorithms

have been gathered in the appendix for this chapter - Appendix 2.

[2]Matrix re-balancing is required to ensure that the sum of the individual elements amount to the original overall value of the connections.

[3]By structure, we mean the spread of connections across the network. By re-balancing, we not only ensure that the sum of (sum of rows) and sum of (sum of columns) are the same, but also capture the potential asymmetry in the spread of the connections by weighting it appropriately (achieved by the re-balancing procedure).

[4]Copulas provide the flexibility required to deal with asymmetries in the data. Therefore, they serve as an attractive tool that could enhance the fit of a simulation in cases where the data reveals potential asymmetries.

is novel and unique.

The copula based algorithm proposed here is similar to the one introduced in Raschke et al. (2010) that generates random networks. However, unlike Raschke et al. (2010), it is not a random network generation process. While they propose an algorithm for a predetermined degree association, our method estimates a dependence parameter using a Maximum Likelihood Estimation (MLE), which is then used to impose a degree association on the network generation process, rather than assuming it. Thereafter, the network is constructed through a sequential search[5], where the probabilities of a pair of nodes getting connected are generated by the copula fitted to the data. Also, we are concerned with directed networks whereas their model generates a random undirected network. As such, the degree of asymmetry in the distribution of the connections plays an important role in our research. This is one of the main differences between our contributions.

Another method for simulation of bilateral connections was proposed by Fernandez-Vazquez (2010), which uses cross-entropy technique for this purpose. The estimation process proposed in his paper considers, as a point of departure, another matrix of bilateral connections that is adjusted until it optimizes the Kullback-Leibler divergence from the true matrix of bilateral connections and is at the same time, consistent with some partial information available - on row and column sums - of the target matrix. His paper proposes the use of a composite Cross-Entropy approach that allows for introducing a mixture of two types of a priori information, i.e., two possible matrices to be included as point of departure in the estimation process.

A version of the algorithm proposed here, borrows his idea of potentially having a prior[6] that is a combination of two separates priors. However, the method of construction of these priors is different. Fernandez-Vazquez (2010) suggests that information from one (or more) earlier period(s) should be used as the prior(s). In situations where we may not have access to time series data, this suggestion does not help. As such, we propose that a copula-based prior may be used instead. As will be discussed in one of the following sections, such a

[5]Raschke et al. refer to the use of copula based methods in the context of sequential search of bilateral connections in their paper. However, they do not use it as their paper concentrates on random network generation with *a priori* degree-degree distribution (also referred to as the degree association.).

[6]Here, the term 'prior' is used to refer to an 'initial guess'.

prior promises to be a useful tool in situations where the network is generated by a mixture of two or more distributions. Our method borrows the idea of constraint optimization of the divergence measure from Fernandez-Vazquez (2010). However, the divergence measure that we use here is not the Kullback-Leibler measure. Instead, we use an 'error measure' that is directly based on the distance between the true and the estimated bilateral connections. Besides, the method does not use the cross entropy technique either. A formal definition of the error measure that is used as well as the justification for the need to use it rather than the cross entropy technique is provided in an ensuing section.

We also provide a brief discussion on the performance of Maximum Entropy and our proposed methods relative to a generic random network generation method, in the context of sparse networks. The underlying intention is to examine the robustness of the methods presented here, in such extreme networks that are very spread out and do not have too many connections. The eMID[7] dataset is used for this application.

In the remainder of the chapter, section 2 discusses the proposed methodology. Section 3 discusses the Monte Carlo Simulations as well as the main findings. Section 4 illustrates the applications of the methodology to datasets from Bank of International Settlements (BIS) and eMID. Section 5 concludes.

3.2 Simulation of Bilateral Connections Within A Network

For the purposes of analysis in this chapter, we use numerical methods such as Monte Carlo simulations to implement and verify the proposed methodology. The simulated data takes the form of a matrix where each entry denotes the connection between two nodes. As such, this data may be viewed as an adjacency matrix of bilateral connections. However, unlike the traditional adjacency matrix which is binary with 1's reflecting existence of a connection and 0's reflecting the absence of it, we allow the entries to be any non-negative real numbers in this analysis. This is done to accommodate complex systems such as an interbank network where rather than simply being interested in who is connected with whom, a researcher might be interested in the '*value*' of the connection (e.g., amount borrowed, lent, etc.).

[7]eMID is the only electronic market for Interbank Deposits in the Euro Area and US.

We also apply the proposed methods to real data from the BIS that provides inter-country bilateral connections of financial institutions, where the connections are debt exposures. This dataset is relatively concentrated and has a pronounced pattern in the spread of connections. The association between the performance of the methods being compared and the level of asymmetry in the data is illustrated using this dataset,. The importance of the role of asymmetry in network data was pointed out by Cocco, et. al. (2009) who found that large banks tend to be net borrowers, while small banks tend to be net lenders in the market (also see Furfine (1999), Ho and Saunders (1985) for evidence on the Fed Funds market). This finding implies that there may be 'big' and 'small' players in the interbank or inter-country networks that have a larger or smaller (respectively) impact on the overall pattern of the spread of connections. Thus, in order to define what is meant by asymmetry in this context, a discussion on the structure underlying the data constructed here is warranted (e.g., possible concentration of connections in a particular region of the network and so on).

The structure of the data construction is based on the Core-Periphery model[8], which assumes that there are certain key nodes in the network (the core) that have a relatively larger impact in the way connections are formed and the way connections are sustained. The rest of the nodes are more 'peripheral' and have less of an impact on the manner in which dependencies arise within the network. For example, in an inter-bank network with connections defined as debt exposures, the core could be defined as the primary/major borrowers as they would have the highest exposures. Based on the core-periphery model, the adjacency matrix presented here is partitioned into four blocks per bilateral interactions: *Core-Core (CC), Core-Periphery (CP), Periphery-Core (PC), Periphery-Periphery (PP)* where C is the set of *core* nodes and P is the set of *periphery* nodes.

For example, suppose we have a generic adjacency matrix where the entries are non-negative real numbers[9] given by the following table.

[8]Craig and von Peter (2010) provide an interesting application of this model to the German interbank market.

[9]This discussion can easily be extended to the case where the adjacency matrices are binary with 1's and 0's as entries.

A Generic Adjacency Matrix

Nodes	n_1	...	n_i	...	n_d	Sum of Rows
n_1	a_{11}	...	a_{1i}	...	a_{1d}	$a_{1.}$
...	...	...	...	...	...	...
n_i	a_{i1}	...	a_{ii}	...	a_{id}	$a_{i.}$
...	...	...	...	...	...	...
n_d	a_{d1}	...	a_{di}	...	a_{dd}	$a_{d.}$
Sum of Columns	$a_{.1}$	...	$a_{.j}$	...	$a_{.d}$	S_G

Now, let that the entire network be partitioned into blocks {Block 1, Block 2, Block 3, Block 4} = {CC, CP, PC, PP } (e.g, figure (1) in the appendix for this chapter[10]).

Further, let the set of connections per block be denoted as $\{A_{cc}, A_{cp}, A_{pc}, A_{pp}\}$ respectively. Then, a block l may be defined as the set $\{A_{block_l}, (N_{block_l}, N_{block_l})\}$ where N_{block_l} denotes the set of nodes that belong to block l and A_{block_l} refers to the set of connections that connect each $n_i, n_j \in N_{block_l}$. A typical element of block l is denoted by $\{a_{ij}, (n_i, n_j)\}$. Formally, we have

$$A_{block_l} = \{a_{i,j} \,\forall\, (n_i, n_j) \in N_{block_l} \times N_{block_l}\}$$

So, the set A_{block_l} refers to the set of connections within the lth block. The cardinality of the set is expressed as $|A_{block_l}|$. Similarly, if the entire adjacency matrix, call it, G, was treated as one block we would have,

$$A_G = \{a_{i,j} \,\forall\, (n_i, n_j) \in N \times N\}$$

The total number of connections within a block l is then given by

$$S_{block_l} = \sum_i \sum_j |a_{ij}| \quad \forall\ a_{i,j} \in A_{block_l},\ (i,j) \in N_{block_l} \times N_{block_l}$$

[10]All graphs, tables and figures for this chapter have been provided in the corresponding appendix - Appendix 2.

If A_{block_l} is binary, then this sum gives us the total number of connections that exist within the block. However, if it consists of real numbers implying that the connections are weights, then $|A_{block_l}|$ gives the total weight of connections within the block l.

Similarly, the total of connections or total weights (depending on the interpretation of the connections)for the entire adjacency matrix, G is given by

$$S_G = \sum_i \sum_j |a_{ij}| \quad \forall \; a_{i,j} \in A_G, \; (i,j) \in N \times N$$

The share of a block refers to the ratio of total number of connections (or total weight of connections) per block to the total number of connections (or total value of the weights of all connections put together) in the given adjacency matrix. Formally, the share per block may be defined as

Definition 3.1: (*Share of a Block*)

$$Share_{block_l} = \frac{S_{block_l}}{S_G}$$

where S_{block_l} *and* S_G *are defined as earlier.*

In this chapter, the share per block serves as a measure of asymmetry across blocks. As the shares become equal across blocks, the data is said to be more symmetric and as the shares become unequal across blocks, the data becomes more asymmetric. It is important to point out that the intra-block asymmetries also play an important role in deciding the degree of asymmetries. However, since they are not observable, they cannot be measured explicitly. On the other hand, the inter-block asymmetries can be calculated if information on the 'marginal sums' of the adjacency matrix is available. Marginal sums refer to the sums of connections across rows and columns of an adjacency matrix. When information about the exact spread of connections is unavailable, these may be used to infer the underlying network structures.

The Marginals of an Adjacency Matrix

Nodes	n_1	$\dots$	n_i	$\dots$	n_d	Sum of Rows
n_1	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$a_{1.}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
n_i	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$a_{i.}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
n_d	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$a_{d.}$
Sum of Columns	$a_{.1}$	$\dots$	$a_{.j}$	$\dots$	$a_{.d}$	S_G

where we only know the values of $a_{i.}$'s and $a_{.j}$'s.

Formally,

Definition 3.2: (*The Marginal Sums*)

The marginal sums are defined as

$$a_{i.} = \sum_j a_{ij} \quad a_{.j} = \sum_i a_{ij}$$

We would like to extract out the joint distribution of connections that generate these marginal sums. Our proposed copula based methodology addresses this issue.

There are various functional forms of copulas that are available for modeling, which also include Archimedean copulas (namely, Gumbel, Clayton, Frank, etc.). The choice of copula depends on the nature of dependency that the researcher would like to model. For example, if one wants to model data with high upper tail dependencies, a Gumbel copula might be more appropriate than, say, a Clayton copula that captures the lower tail dependencies much better. The simulation method proposed here uses bivariate Archimedian copulas, where the random vectors that serve as inputs to the copula function are the vectors of sums of rows ($a_{i.}$) and sums of columns ($a_{.j}$).

3.2.1 Proposed Methodology

The aim is to estimate the association between unobserved bilateral connections given the marginals (i.e., $a_{i.}$'s and $a_{.j}$'s) and using that to simulate the adjacency matrix (i.e., the

a_{ij}'s) given the estimated joint distribution over connections. In order to achieve this, the proposed algorithm that fits a copula to the data works as follows. Before we proceed, it should be pointed out that copulas must satisfy Sklar's theorem. According to the Sklar's theorem, marginal distributions that serve as an input to the copula function must be continuous. Therefore, the discrete marginals of an adjacency matrix have to be approximated using a nonparametric method. For this purpose, we use the kernel density estimation method. It serves as an attractive tool because it allows us to choose from a variety of kernel functions, which include uniform, normal, etc. It also allows the specification of a bandwidth parameter that can desirably smooth the estimated density.

For an adjacency matrix of size $N \times N$ (i.e, with N rows and columns), the kernel density for its marginal sum of rows denoted by $f(a_{i.}) = \sum_j a_{ij}$ at $a_{i.} = a_0$ is defined as

$$f(\hat{a_{i.}}) = \frac{1}{N.h} \sum_{j=1}^{N} K(\frac{a_{i.} - a_0}{h}) \tag{3.1}$$

where $K(.)$ is the kernel function that places greater weight on $a_{i.}$ points that are closer to a_0. f is the vector of density values evaluated at the points in a_0. h is a bandwidth parameter.

The kernel density estimate for the marginal sum of columns (denoted by $a_{.j}$) can be defined in a similar manner.

This estimate computes a probability density estimate of the sample in each of the vectors $a_{i.}$ and $a_{.j}$.

Now the steps involved in fitting a copula to the data can be described as follows.

Preliminary Analysis:

1. *Plot the densities of the marginal sums of rows and columns (i.e, the available data).*

   Here, we use a normal kernel density to estimate each one of the sum of rows and columns of the given adjacency matrix for the purposes of plotting. For a given adjacency matrix, a normal or Gaussian kernel function for the marginal sum of rows is given by

$$f(\hat{a_{i.}}) = \frac{1}{\sqrt{2\pi}} e^{(\frac{a_{i.}-a_0}{h})^2} \tag{3.2}$$

A normal kernel function for the marginal sum of columns (denoted by $a_{.j}$) can be defined in a similar manner.

We use a bandwidth parameter that is a function of the number of points in each of $a_{i.}$ and $a_{.j}$. Then, we evaluate the density at 100 equally spaced points that cover the range of the data in $a_{i.}$. Thereafter, the estimated densities (i.e., the $\hat{f}$'s) are plotted.

2. *Based on these plots of the marginals, infer the nature of distribution of the marginals (e.g., perhaps, it is a mixture or it is not).*

   This is more for a visual analysis of the data and to justify a prior for the structure of the data and the choice of copula. Further, since the inputs are sums of columns and rows, super imposing these plots would give the researcher an idea of whether the peaks in the sum of rows correspond to the peaks in sum of columns, which might reflect some pattern of dependencies between these.

Fitting a Copula:

1. *Transform the discrete marginals into being uniformly distributed (using kernel density estimation), which is required for it to be able to serve as an input into the copula function.*

   For a given adjacency matrix, a uniform kernel function for the marginal sum of rows is given by

$$f(\hat{a_{i.}}) = \frac{1}{2}.\mathbf{1}_{|\frac{a_{i.}-a_0}{h}|\leq 1} \tag{3.3}$$

   A uniform kernel function for the marginal sum of columns (denoted by $a_{.j}$) may be similarly defined.

2. *Thereafter, fit a copula to the transformed data using MLE that estimates the dependency parameter.*

For this step, based on the plots of the marginals, the researcher must form a prior for the potential dependencies of the data and pick the functional form of the copula that seems most appropriate (see step 2 of the preliminary analysis). This may turn out to be a difficult task, as no 'thumb rule' exists that can be followed to choose the 'correct' copula function. On the brighter side, in network based data, such as spread of connections contained in an adjacency matrix, one is more likely see dependence patterns that resemble upper tail dependencies or lower tail dependencies. Thus, an appropriate Archimedean copula such as Gumbel or Clayton may be used.

Next, the MLE undertaken in this step makes a parametric assumption that the copula is a function of some dependence parameter, θ. Thus, in this case, the copula function depends on $\theta, a_{i.}$ and $a_{.j}$. The aim is estimate the θ in order to deduce the nature of dependence using the MLE method. So, if the copula distribution function is given by $C(a_{i.}, a_{.j}; \theta)$, the density can be easily derived by taking the first derivative of this function with respect to $a_{i.}$ and $a_{.j}$. This density is denoted as $c_\theta(a_{i.}, a_{.j})$ and can be estimated under a full parametric assumption. MLE is used to get an estimator of θ. In order to derive it, the following log-likelihood is maximized.

$$\ln L(\theta | a_{i.}, a_{.j}) = \sum_{k=1}^{N} c_\theta(\hat{F}_1(a_{ik}), \hat{F}_2(a_{kj})) \tag{3.4}$$

for some $N^{1/2}$-convergent estimates of the marginal CDFs, $\hat{F}_1(.)$ and $\hat{F}_2(.)$.

Thus, if one would like to use a Gumbel copula, the c_θ would be replaced by the Gumbel density with the estimates of uniform marginal CDFs as an input.

3. *This estimate of the dependency paramter, $\hat{\theta}$, is then used to generate a matrix of cumulative probabilities.*

   These probabilities give us the cummulative distribution of the likelihood of connections in the adjacency matrix.

This forms the starting point of the algorithm proposed here. There are two approaches that are proposed: direct and indirect.

3.2.2 METHOD 1: *Direct Method* Using Copula

This method continues on from the earlier algorithm that describes the method to fit a copula to the data. This is a direct method and is defined as follows.

1. *Begin by rescaling the matrix of probabilities derived from the copula estimation (step* 3 *of fitting a copula) into a stochastic matrix.*

   Since the probabilities derived from fitting the copula are cumulative, they need to be rescaled so that they add up to 1 and we have a stochastic matrix of probabilities.

2. *Then, apply the RAS algorithm to the matrix to re-balance it.*

   This re-balancing is required in order to be able to use the RAS algorithm which is an iterative proportional fitting procedure for estimating the bilateral connections (or a_{ij}'s) of the adjacency matrix such that the marginal sums remain unchanged and the estimated matrix decomposes into an outer product. So, the RAS algorithm uses these probabilities to form the bilateral connections.

   Since the RAS algorithm applies the probabilities generated by the copula function with the estimated dependency parameter, the formation of connections is not random. As such, it is expected that the matrix generated by RAS will imitate the true dependence structure and hence, the true spread of bilateral connections.

   In order to assess the accuracy of the matrix simulated using the RAS algorithm and copula based probabilities, a distance-based error measure is adopted, which aggregates the inconsistencies between the estimated adjacency matrix and the observed one.

3. *Then calculate the error measure.*

   Suppose the true value of a connection is represented as a_{ij} and the one estimated as $\hat{a}_{ij}$. Then the error measure is defined as the distance between $\hat{a}_{ij}$ and the actual a_{ij}. The following expression for the error measure is derived by aggregating the error of simulation across all nodes and normalizing by the total value of all connections.

Definition 3.3: (*Total Absolute Error Measure*)

$$\epsilon = \frac{\sum_i \sum_j |\hat{a}_{ij} - a_{ij}|}{\sum_i \sum_j a_{ij}}. \tag{3.5}$$

where $\hat{a}_{ij}$ is the simulated a_{ij}.

The underlying idea is that when the estimator produces a guessed value that lies either above or below the actual one, an error is incurred. The larger is the error measure, the larger will be the inconsistency between the simulated and the true matrix. Even though in the current form this measure weights each inconsistency identically, a weighted or quadratic measure can be obtained in a straightforward manner. It is also worth noting that since the measure is normalized, when comparing the results obtained with different estimated methods the improvements obtained by choosing one method instead of another always relates to the overall exposure of the system.

As had been mentioned, Fernandez-Vazquez (2010) uses the cross entropy technique to check for divergence of the estimates. However, the cross entropy fitness measure has a major disadvantage. While it produces a positive number in cases where exposures are overestimated, thus accurately reflecting the prediction error, it produces a negative number when exposures are underestimated, thus biasing the overall assessment of the fitness of the model. Therefore, in this analysis, it has been replaced by the aforementioned absolute value based error measure that is not only immune to this bias but also allows for more flexibility in treating simulation errors (e.g., quadratic functional forms).

Next, to compare the matrix simulated using the proposed direct method and the one generated by the traditionally used Maximum Entropy method, the next steps are followed.

4. *Then estimate the adjacency matrix using the Maximum Entropy method.*

5. *Calculate the error measure.*

6. *Compare both error measures.*

7. *The fit with the lower error measure is a better fit.*

The choice of copula varies according to the prior regarding the dependence structure of the data and the kind of restrictions that the researcher might want to impose. Also, the order of Maximum Entropy estimation and the copula estimation does not matter. So, the Maximum Entropy estimation could have been conducted first and the corresponding error measure stored for the subsequent comparison with the error measure derived from the copula based method, instead.

If it is observed that a particular copula does not do very well compared to the Maximum Entropy approach, the decision of using that copula must be re-evaluated. Instead, another copula that might have potential for a better fit may be used. Alternatively, even a combination of copulas (as will be outlined in the next methodology) may be used. It is also possible that the data is too symmetric across all blocks so that the Maximum Entropy actually does perform better because of the nature of its estimation method. We will numerically show this to be true. We also find that too of much of asymmetry in the data makes both these methods extremely sensitive to the share of the core with respect to the value of total connections. A slight increase in the share of the core can lead to a better performing copula and a slight decrease could lead to a better performing Maximum Entropy approach. These are discussed in the light of simulations presented in the following sections. Before proceeding to those, we present the alternate methodology that is more involved and uses the idea of combining two or more copulas as an initial guess.

3.2.3 METHOD 2: Hybrid of Copula as a Prior (*Indirect Method)*

As mentioned above, if using a single copula does not yield a good fit, one could use a combination of two or more copulas. For example, if the kernel density of any (or both) of the marginals seems to be bi-modal, it is believed that a mixture of copulas might fit better. This forms the basis for the methodology that is developed next.

This method builds upon Fernandez-Vazquez's (2010) cross-entropy model but with significant differences, which have already been discussed. Unlike Fernandez-Vazquez (2010), the proposed method seeks to minimize the following objective function:

$$\sum |G - Q_0| \tag{3.6}$$

subject to

$$R.P = B \tag{3.7}$$

where G is the matrix to be estimated with typical element $\{a_{ij}\}$, Q_0 is the initial guess with typical element $\{q_{ij}\}$, R is the matrix of restrictions imposed on G, B is the target vector and M is a scalar that serves as an optimization heuristic. The system of linear equations that serve as restrictions are given by

1. $\sum_j p_{ij}.\tilde{a_j} = \tilde{a_i}$

   where $p_{ij} \in P$, $\tilde{a_j} = \frac{\sum_i a_{ij}}{\sum_i \sum_j a_{ij}}$ and $\tilde{a_i} = \frac{\sum_j a_{ij}}{\sum_i \sum_j a_{ij}}$.

2. $\sum_i p_{ij} = 1$

3. $p_{ii} = 0 \; \forall \, i$.

The initial guess Q_0 serves as the prior. Thus, the term *prior* is used here in a *non-Bayesian* sense to mean just an initial guess.

Rather than using just one copula as a prior, a new prior that is termed as a '*hybrid*' of copulas is constructed. Here we make a distinction between a mixture and a hybrid of copulas. A hybrid of copulas is different from the mixture because it assigns the percentage of a_{ij}'s being drawn from each copula rather than referring to a weighted convex combination of the copulas. An illustration has been included as a part of the algorithm.

The algorithm for the indirect method of simulation is given by the following.

1. *First estimate the adjacency matrix using the Maximum Entropy method.*

2. *Calculate the error measure from this method.*

   Again, the total absolute error measure described earlier is used to calculate the error measure.

3. *Now, suppose it is believed that the data is a combination of Gumbel and Clayton in the ratio of x and $1-x$ respectively. Then, use the direct copula estimation method to fit a Gumbel copula and a Clayton copula to the data, separately.*

4. *Using these estimates construct Q_0 as follows.*

   (a) *First insert values extracted from the Gumbel estimates in $x\%$ of the number of cells in Q_0.*

   (b) *Then for the $(1-x)\%$ use the Clayton estimates. So, for example, if one is looking at a $5 \times 5 Q_0$ matrix with, say, 60% of values from Gumbel. Then take the first 15 (0.6×25) estimates provided by Gumbel, i.e., the first 3 rows and 5 columns from the matrix of Gumbel estimates derived using the RAS algorithm. Use them to be the first 3 rows and 5 columns of Q_0. And then the last 2 rows and 5 columns, are taken from the last 2 rows and 5 columns of the Clayton estimates*[11] *derived from using the RAS algorithm.*

5. *Once Q_0 has been constructed, conduct a constrained minimization of the objective function subject to the constraints as defined above.*

6. *Thereafter, extract the Q^* that is the minimizer.*

7. *Then calculate the error measure of the Q^* and compare it to the one derived from Maximum Entropy.*

8. *The better performer will have a lower error measure.*

As mentioned earlier, in order to implement these methodologies and verify the conditions under which these methods perform better than our benchmark, Maximum Entropy

[11] We could have used the more conventional notion of a mixture. In other words, the proportion of Gumbel was given by x, then we have

$$Q_0 = x.Gumbel + (1-x).Clayton$$

where Gumbel refers to the matrix of a_{ij}'s estimated using the Gumbel copula and Clayton refers to the one estimated out of Clayton. However, some preliminary simulations showed that this sort of a mixture yielded error measures that were always between the ones resulting from separately fitting the copulas that were being mixed (e.g., Gumbel and Clayton). This makes sense intuitively. Therefore, using a mixture does not add value to the exercise.

method, Monte Carlo (MC) simulation technique is used. This is presented in the next section.

3.3 Monte Carlo Simulations

3.3.1 The Data Generating Process

The main hypothesis that led to the construction of the copula based approach was the possibility of an association between the level of asymmetry in the data and the performance of the simulation algorithms. This motivates the manner in which we simulate the data here.

The simulations focus on two aspects of asymmetry - *share of blocks and size of blocks*. If there is a particular block that has a higher share of connections or higher value of connections relative to the rest, then the spread of connections is said to be asymmetric. On the other hand, if all blocks account for almost equal shares, then the data is said to be relatively symmetric. At this point, it is important to make the distinction between the share of a block and the size of a block. While the description of the share of a block has already been provided, the size of a block is defined to be the number of nodes (out of total number of nodes) that form the block. For example, if 4 nodes out of a total number of 10 nodes form the core, then the size of core-core (CC) is 0.4 and so on. In our ensuing discussion, we state it clearly whenever we are using the share of a block as against the size of a block (and vice-versa) as a measure of asymmetry. Most often, in order to view the impact of one over the other, we only change the one we are interested in (unless mentioned otherwise) and keep the other fixed.

To generate a matrix with asymmetric shares, with the assumed block structure, a random number generator is used. Recall, that the analysis here is based on the Core-Periphery model. Given this, it is assumed that the CC block has higher share of connections even when its size is small. Further, rather than simulating binary adjacency matrices, positive entries are simulated in the cell of adjacency matrices. For example, in case of an adjacency matrix of an interbank network, the entries would be amount loaned or borrowed, rather than 1's and 0's. Thus, it makes sense to analyze the generic case.

The idea is to simulate higher values in the CC block and then impose an asymmetric structure by adding positive random numbers in all entries below the diagonal. Two positive scalars are introduced, that are called '*inter*' and '*intra*' which indicate the levels of inter-block asymmetry and the intra-block asymmetry, respectively. These parameters are used to introduce the desired level of asymmetry to the data in the simulation.

For example, keeping the *inter* constant, if the parameter *intra* is increased, then the asymmetry of the data would increase by an increased share of connections within one (or more) block(s). More explicitly, the role of these parameters may be interpreted as follows. The higher the value of the *inter*, the higher is the value of the connections across the blocks (e.g., in the context of an interbank network where connections are debt exposures, higher is the value of amount borrowed by nodes in the core from nodes in the periphery) for a given level of intra-block asymmetry. Thus, in order to increase overall asymmetry, the value of *inter* must be lowered. On the other hand, the higher is the *intra*, the higher is the asymmetry of the connections within blocks (e.g., in the context of interbank network where connections are debt exposure, higher is the amount borrowed by i from j vis-*á*-vis the amount borrowed by j from i in a given block). Thus, to increase overall asymmetry, the value of *intra* for a given *inter* must be increased.

Therefore, in the simulations, these numbers are allowed to change. That way the magnitude of these additional perturbations can be controlled to create a more asymmetric matrix or a more symmetric matrix, as desired.

The primary finding of this numerical analysis is the following. The difference between the error measures obtained from the Maximum Entropy procedure vis-*á*-vis the proposed copula based methodology (direct or indirect or both) increases as the original matrix becomes more asymmetric. In other words, as the level of asymmetry in the data increases, the method proposed here does better. On the other hand, in smaller networks, there seems to be a threshold level of asymmetry beyond which Maximum Entropy starts to do better. This is because, as the asymmetry in smaller matrices is increased by too much, the connections span over the entire matrix. Thus, the spread of connections across blocks, in fact, becomes more symmetric. As such, up to a certain level as asymmetry increases, the copula based method does better and beyond that Maximum Entropy seems to do better. These differences are more closely related to the structure of the data. A discussion on the types of data structures, given the data generating processes, for which the copula based method outperforms the Maximum Entropy approach follows.

We use two different DGPs. The first one, DGP 1, imposes the periphery-periphery (PP) block to have 0's as entries so that the share of PP is 0 with all other entries nonnegative. The second one, DGP 2, imposes that the PP block has positive entries, although its share remains low in comparison to the CC block[12].

3.3.2 Results from Monte Carlo Simulations Using Both Methods in Large Networks

In this section, we report the results from a set of simulations, which were conducted using a Gumbel copula in order to determine conditions under which the direct method based on the Gumbel copula outperforms the traditional Maximum Entropy approach.

Simulation results for the indirect method, where a prior was constructed using a hybrid matrix with 90% of values taken from a Gumbel copula and 10% of values taken from a Clayton copula, have not been reported here as they were very similar to those obtained with the direct method based large networks' simulations. However, the application of the indirect method is illustrated in the context of the smaller networks, where in a number of scenarios the hybrid of copulas taken as a prior does much better than either of Maximum Entropy or the direct method that uses a Gumbel copula.

The simulations for large networks generate 1000, 100×100 adjacency matrices using both the DGPs and vary the intra block asymmetry while keeping the inter block asymmetry fixed. Figure (2) displays the results of the described simulation where a Gumbel copula is used for the first DGP. The x-axis denotes the level of intra block asymmetry and the y-axis denotes the mean difference in the error measures of Maximum Entropy and our copula based method. The core size for these simulations is fixed at 20. The *inter* parameter is

[12] Codes for this chapter have been included in the appendix titled 'Codes'.

fixed at 100 and *intra* is variable.

Note that such a small core size implies that the relative number of large players in the network is small. This, in turn, results in lower shares of the CC block compared to that of the other blocks put together. As such, varying intra block asymmetry does not change the asymmetry in the CC block enough to show any significant gain in the performance of the proposed direct methodology. Further, the fact that PP is imposed to be 0 implies that the majority of the shares are attributed to CP and PC blocks, which represent across-blocks connections . Thus, the impact of asymmetry in CC gets crowded out by that of these two blocks, thus creating a balancing effect. As such, one observes that the error measure for the copula based direct approach is higher and therefore, the difference between the error measure of Maximum Entropy and the direct method is negative.

Figure (3) displays the effect of a change in the size of the network with the same core size.

Next, the simulations are repeated with the same set of parameters, but with the second DGP. Figure (4) illustrates the findings.

Note that an upward trend in the mean difference of the error measures of the two methods (Maximum Entropy and Gumbel based method) is observed. Further, the series of the mean differences of the error measures is positive throughout. This implies that the mean error measure for the copula based direct method is strictly lower than that of the mean error measure of Maximum Entropy. The upward trend can be attributed to the increasing intra block asymmetry of the small core size and hence, a larger periphery size. Although, lower values are imposed in PP as compared to CC, with a low number of nodes in the core, the size of the periphery automatically increases. And by increasing the intra block asymmetry, the asymmetry within PP is increased as well. So, the low impact of CC is reinforced by the high impact of PP in the same direction, resulting in an upward trend.

However, if the values were reduced to be even lower, yet positive, one could get a downward trend as shown in figure (5).

For reasons described above (for the case where PP was 0), the effects of the CC and PP blocks that are supposed to make the data more asymmetric are mitigated by the effects of CP and PC blocks. However, the fact that the PP block is no longer 0 helps the case, and this is evident from the fact that due to this, the level of intra block asymmetry increases by enough for the direct method to outperform Maximum Entropy, which is confirmed by the fact that mean difference is positive throughout.

This difference in mean error measures is measured on an overall exposure basis, i.e., when asymmetry is very large the error measure obtained with the copula based direct procedure produces a smaller sum of inconsistencies that amount to about $2.4 - 2.5\%$ of the total exposure. This accuracy gain in the values of the cell entries in an adjacency matrix may not sound significant, however, one must consider that when a network such as an interbank network is being simulated, where the entries in an adjacency matrix are in billions of dollars, a 2% gain is still in millions of dollars! And an error that large can severly bias the inferences made about the interbank relationships.

Next the size of the core is increased to 40. Again, an upward trend is observed, for reasons similar to what was discussed before.

Similarly, using these two DGPs one can fix the intra block asymmetry and change the parameter *inter* and use the direct copula-based method with 20 nodes in the core out of a total of 100 nodes to see how it performs as inter block asymmetry is changed. Please see figure (6) generated with the first DGP.

Note that, the mean difference is strictly negative in this case, implying that with PP values set at 0, the core size is just not big enough to generate enough asymmetry for the direct method to perform better. This changes when the other DGP that generates positive values in PP is used, thus supporting the already high values in CC. Please see figure (7).

As seen in the figure, although the mean difference is positive implying that the direct approach does better, there is a distinct downward trend implying that as the inter-block asymmetry[13] is increased, it is observed that the mean difference in error measures starts to go down. This is because, with increase in symmetry by increasing *inter*, Maximum Entropy starts doing better. This is unlike the case where the copula based direct method

[13] This is a misnomer really because as *inter* is increased, the symmetry is essentially increasing by spreading the connections across blocks.

did better as *intra* was increased, thereby increasing the share of core and hence, asymmetry within blocks as well as overall asymmetry of the data in the matrix.

Next, the size of the core is increased to 40. This is presented in figure (8). Note that there continues to be a slight downward trend, however, throughout, Maximum Entropy does better. The explanation for this is as follows. In a large network such as the one with 100 nodes, a core size of 40% is huge. Further, given this, when intra block asymmetry is fixed and the inter block connections are increased by increasing *inter*, such a big core size does not help as it is not asymmetric enough[14]. In fact, the inter-block parameter has a smoothing effect on the asymmetries within this block as more connections are *inter* block rather than *intra*. This is why, Maximum Entropy tends to do better.

In order to check if these findings hold for smaller networks, some more simulations were carried out using the MC method. The results are discussed in the next section.

3.3.3 Results from Monte Carlo Simulations for Both Methods in Smaller Networks

In this section, we broadly discuss the results from a set of copula based simulations for smaller networks with a minimum of 5 nodes and a maximum of 21 nodes in the network. We compare the differences in error measures of the methods under consideration rather than mean differences. Only an overview of the results are provided because the findings for larger networks are imitated by those of the smaller networks as well. In other words, the level of asymmetry in the data continues to matter.

Three distinct datasets were created for the purpose of the current analyses. The first one was generated from the first DGP - an extremely asymmetric spread of points were generated, where we restricted the PP values to be equal to 0. The other two data sets were generated using the second DGP but with different restrictions on the values of the PP data points. For example, the second data set allowed the randomly generated values

[14]Suppose, we have a 5×5 matrix of nodes with 4 nodes in the core. Then, fixing *intra* and changing *inter* makes the 4×4 part of the matrix symmetric to the remainder of the matrix. Essentially, the entire matrix is getting symmetric then as 4×4 constitutes 80% of the matrix. Therefore, it cannot create enough asymmetry for the copula based methods to do better.

of the PP data points to range between 50 and 100. And, the third dataset allowed these values to be between 500 and 1000.

Broadly, it was observed that at very high levels of asymmetry both copula and Maximum Entropy approaches are extremely sensitive to the size as well as the share of the core. On repeated simulations, it was observed that, for certain configurations of the data, a minor increase in the share of the core from, say, 96% to 97% would result in the estimates from the copula approach (Gumbel) performing better than those derived from the Maximum Entropy approach and vice-versa if the share of the core is decreased.

The outline of results for each variation of the DGPs have been reported below.

1. PP is 0

In this case, the data was simulated such that the PP values were set to 0. Also, here it was assumed that $inter = 10$ and $intra = 100$. Table (A) displays the results for this set of simulations where the share of the core varies given the constant size of the core (i.e., number of nodes in the core remains fixed while total number of nodes increases).

We observe mixed results. This is because, as the number of nodes increases, the overall shares also change affecting the performance of the methods. One would observe that here, for the most part, as long as the share of the core lies around 80% and between 88% and 92%, the copula (either of Gumbel or the hybrid) does better. Since the share of periphery is 0, both the approaches are very sensitive to the share of the core. With CC share of around 90%, the copula approach (hybrid based) does better, while with a lower CC share or a higher CC share, the Maximum Entropy seems to do better. With a lower CC share, the shares of the other blocks tend to increase, thus approaching a balanced spread. Recall, this was the case in the simulations done with 20 core nodes in a network of 100 nodes. With a higher core size, the CC block accounts for the majority of the shares and as it approaches 100%, it almost does away with the asymmetry across blocks by pushing the shares of other blocks to 0 (thereby not having any impact on the asymmetries). In this case, only the level of intra-block asymmetry matters. By fixing it at a certain level, reducing the inter-block asymmetry would lead to a better performance by Maximum Entropy approach. This is helped by the fact that the PP block accounts for 0% of the share, by construction.

We obtain similar results when the core size is varied keeping the share of core fixed[15].

2. PP between 50 **and** 100

We performed similar analyses by changing the values of the PP data points and allowing them to be between 50 and 100. The value of *inter* is fixed at 10 and the values of *intra* were varied. Please see figure (9). By doing this, the levels of intra block asymmetry were changed (from high to low) while keeping the inter block asymmetry fixed. And in order to concentrate on the impact of this change, the size of the core was fixed at 3 nodes out of 5. The share of the core varies with the changing values of *intra* . In this case, the copula approach (mostly the indirect method) always yields better results[16]. Note that the value of *inter* is small. As such, the inter block asymmetry is already quite high. So, although the values in the periphery are not as high, its effect is strengthened by the already existent asymmetry induced by the lower *inter*. This also overrides the impact of reducing *intra*, which would have otherwise reduced asymmetry as well. Thus, it is again observed that as asymmetry in terms of the share of core increases, the copula approach does much better.

On the other hand, if the value of *inter* is increased to 100 thus reducing the inter-block asymmetry, one gets mixed results when *intra* is increased to 100 (thereby increasing intra-block asymmetry) at the same time, since they seem to offset each other. However, if *inter* is fixed at 100 and one starts reducing *intra* instead, one would still get that the copula based estimator performs better for the most part. This implies that the reduction in *intra* reduces intra-block asymmetry but not enough to approach the threshold level of symmetry beyond which Maximum Entropy does better until around *intra* = 90. Figure (10) shows these results for both of the direct as well as the indirect methods.

A similar exercise is undertaken with changed values of the periphery points - these values are increased further. A discussion follows.

3. PP between 500 **and** 1000

[15] These results have not been reported here in order to avoid repetition.

[16] Note that EM of ME $-$ EM of Hybrid Copula Method > 0 implying that EM of Hybrid Copula Method is lower.

First, the value of *intra* was varied while keeping *inter* constant at 10 for this scenario. Figure (11) shows the results for this. Note that even as the values of *intra* are reduced, meaning that the asymmetry is reduced (but not quite to make it symmetric), keeping the number of nodes in the core (3) and the total number of nodes (5) the same, we observe that the copula approach (direct or indirect) *always* does better than the Maximum Entropy approach. The same exercise is carried out with the value of *inter* changed to 100. These results are shown in Figure (12).

Again, it is observed that the copula based approach does much better. In fact, it is the hybrid copula based approach that outperforms the rest. Thus, with a small number of nodes and a relatively significant core size, as the 'symmetry' is increased by increasing *inter* and reducing *intra* (without making it completely symmetric), a situation for a complex set of asymmetries is created, where the inter block asymmetry is high and the intra block asymmetry is low (but not low enough to match the level of the inter block asymmetry) making it asymmetric overall. Given this, as one would expect, the copula based approach does perform better[17].

Finally, table (B) lists the dependence parameters estimated using Gumbel copula as well as Clayton copula along with the corresponding Kendall's τ with PP between 50 and 100. And table (C) lists it for the case with PP between 500 and 1000. Kendall's τ is a statistic that measures the rank correlation between two variables. In our context, these are the marginal sums of rows ($a_{i.}$) and columns ($a_{.j}$).

3.4 Applications to Networks: Dense and Sparse

3.4.1 An Application to Dense Network: BIS Data

To illustrate the copula based methodologies, we use data on cross-border bilateral debt exposures published by the Bank of International Settlements (BIS) in its International Consolidated Banking Statistics database. Before proceeding, a quick look at figure (13)

[17]Simulations for almost all possible scenarios of asymmetric data structures using varied combinations of *inter* and *intra* have been performed. These have been gathered in a working paper that can be made available upon request.

would give the reader an idea about the structure of the BIS data.

This unique source provides a bilateral description of exposures at a country level. In this chapter, these data are used on an ultimate risk basis (URB), i.e., the exposure is shown by residency of the ultimate debtor. This dataset has been used frequently to assess the resilience of the cross-border financial network to the risk of contagion (see Hattori and Suda (2007), McGuire and Tarashev (2008), Espinosa-Vega and Solé (2010)) and is freely available at the BIS website[18].

Given that the BIS data exemplifies the core-periphery structure and is the only publicly available dataset where bilateral exposures can be observed, we use it to test the copula based method. It is found that the copula approach produces a lower error measure (1.1662) as compared to the one obtained with the standard Maximum Entropy approach (1.1849). Thus, we obtain a precision 'gain' that amounts to roughly 2% of total exposures. Given that the Gumbel copula is asymmetric and heavy tailed, it is not surprising to find that the copula approach produces *closer estimates* in the CC and CP block whiles performing worst in the PP block. Please see Tables (1) and (2).

The following illustration is based on the core-periphery model and divides the BIS data, which is in the form of an adjacency matrix, into four blocks: core-core (CC), core-periphery (CP), periphery-core (PC) and periphery-periphery (PP). Consider the top and bottom 5 countries in terms of overall exposures as shown in Table (3).

It is easy to see that the cross-exposures established between the top countries are more substantial than the ones maintained in the remaining blocks. Moreover, the matrix is asymmetric, i.e., peripheral countries tend to lend more to the core than the other way around (this fact has also been observed by Cocco (2009) for the Portuguese interbank market). Finally, the block or sub-matrix that describes the relations between peripheral countries is quite sparse. Thus, in order to analyze the case of sparse matrices, more data is simulated. The results of these simulations have been discussed in the following section.

Next the copula based methods are applied and compared to the Maximum Entropy method in reference to the BIS data. However, we have a constraint because the BIS data

[18]For a more detailed description see McGuire and Wooldridge (2005)

comes with a certain level of asymmetry that cannot be changed. Therefore, we use subsets of the dataset to create variations in the asymmetry levels for the purpose of illustration here. This doesn't change the data in any way. We assume these subsets to be the networks (or sub-networks). For example, we start with the first 6 nodes and pretend as if those and their interconnections constitute the entire network. Then, we consider the first 7 nodes and do the same. Thereafter, the first 8, 10 and so on, until the entire dataset (all 21 nodes) is exhausted. This allows us to consider sub-networks with different levels of asymmetry. To this extent, the following configurations (from top left to the right end) of the BIS data are used: $6 \times 6, 7 \times 7, 8 \times 8, 10 \times 10, 11 \times 11, 13 \times 13, 15 \times 15, 18 \times 18$ and the full dataset, 21×21. The exposure matrices for all of these cases are simulated using both Maximum Entropy method as well as both the copula-based methods. The results are summarized in the next section.

3.4.2 Results

The BIS data seems to have 5 major nodes, that can be termed as the core nodes. Thus, the CC block would be a 5×5 matrix. Both Gumbel as well as Clayton copulas are used to do the copula based simulations for all the scenarios. The error measures for all the cases may be found in table (4). Using information in table (4), one can infer that until the 10×10 case, the Gumbel copula based direct method does much better than the Maximum Entropy estimator. As we approach 11×11, the share of the core continuously reduces to almost 60% as the core size remains fixed and the number of total nodes increase. This results in the reduction of asymmetry and hence, Maximum Entropy starts to perform better. It can be seen that as we approach 21×21, the shares of the blocks approach equality. Therefore, one would expect that Maximum Entropy does better. However, as the data gets more asymmetric, beyond a threshold level of asymmetry, the Gumbel copula fits the data much better. The exposure matrix of the BIS data is also simulated using the indirect method with the hybrid of copulas as a prior. Table (4) contains the results for this as well. It can be observed that up to 10×10, Gumbel outperforms the hybrid copula and thereafter, Maximum Entropy outperforms it.

Table (5) lists the various dependence parameter estimates corresponding to each of the copulas and the cases along with their respective Kendall's τ. Further, table (6) reports the confidence intervals for the dependence parameter estimates and the corresponding log-likelihoods[19]. The Kendall's τ for both the Gumbel copula and Clayton copula have been reported in the following table.

	Gumbel	Clayton		Gumbel	Clayton
6×6	0.2273	0.3660	11×11	0.6455	0.6609
7×7	0.3664	0.4861	13×13	0.6840	0.7227
8×8	0.5457	0.5984	15×15	0.6603	0.7281
10×10	0.6387	0.6600	18×18	0.6819	0.6981
21×21	0.6831	0.7242			

Thus, it turns out that as one approaches complete symmetry, the Maximum Entropy approach does better. However, as one makes the data asymmetric, after a threshold level, the copula based approach performs much better. This is represented in figure (15). Two cases are presented here, one that displays the performance of the copula based methods against Maximum Entropy in terms of asymmetry in the share of the core and the other in terms of asymmetry in the size of the core. The corresponding R^2 indicating the correlation of the relative performance of the copula based methods and the asymmetries have also been reported. In particular, the correlation between the difference in error measures (of Maximum Entropy and the direct method) and the asymmetry in share of the core is 0.5079 whereas the same in case of the indirect method is 0.9295. For the case with asymmetry in core sizes, the R^2 is observed to be equal to 0.5772 and 0.8084 for the differences in error measures between Maximum Entropy and the direct and indirect methods, respectively.

3.4.3 An Application to Sparse Networks: eMID Data

To assess how the methods compared in this chapter would perform when the network is sparse, a sample of eMID data comprised of 43 trading days recorded on the last two months

[19] Andrew Patton's code from the copula toolbox has been integrated into the code used here, in order to calculate the log-likelihoods. Source: http://public.econ.duke.edu/ ap172/. Last accessed on October 20, 2012.

of 2011 is used. eMID is the Italian based European reference electronic market for liquidity trading. This platform provides anonymized data for euro-denominated unsecured interbank transactions. Althought it, recently, experienced a strong decline, eMID accounted for 17% of overall Euro area unsecured money market turnover in 2006 (see ECB[20](2007)). Several papers have been written using this unique dataset (e.g, Iori et al. (2007), Iori et al. (2008), Cohen-Cole et al. (2011) and Gabrieli (2011)).

Before we begin the discussion, it must be established that the eMID data represents a sparse network of interbank trading. Figure (16) shows how many nonzero entries there are for each trading day. Notice that this number is lower than the number of active trading days throughout the period under analysis, and much lower than the total number of possible nonzero entries ($(n-1).n$). Thus, this network is taken to be sparse. Given that this network is sparse, one can now analyze how each method performs given this particular structure.

One would expect that the process of simulating bilateral connections in a sparse network to be a difficult one since the adjacency matrix is populated with a substantial number of zero entries. Nevertheless, a better understanding on how exposures are distributed in these cases may prove to be rewarding in the context of highly concentrated structures.

To provide a better description on the estimation of this particular type of networks, the analysis is expanded to include an additional methodology based on random networks. The justification for doing this lies in the fact that in sparse networks, the pattern of the spread of connections is not pronounced. Given this, one would expect repeated simulation of random network generation where the nodes that are matched with some exogenous probabilities, may eventually yield the desired network. Thus, it would be interesting to compare the performance of Maximum Entropy as well as the direct method[21] to that of the random network generation method. The idea underlying this tool is that nodes are matched randomly until their exposures (incoming and outgoing) sum to the known

[20]European Central Bank

[21]The results for comparisons with the indirect method have not been reported because due to the sparse nature of the network, a lower bound for the minimization problem could not be found for this method.

(desired) values. The algorithm used to generate these networks uses a pseudo-random number generator based on a uniform distribution to select the pairs of nodes that can potentially give rise to an interbank exposure. After the pairs are obtained, the effective creation of the connection is determined by the outcome of a binomial random variable. Using a pseudo-random number generator based on a binomial distribution with success probability equal to the share of overall exposures, we impose that banks with a higher proportion establish successful connections first. This process insures that all nodes are matched.

Figure (17) shows that the random network based method is always outperformed by the two main methods analyzed previously in this chapter. Having said that, it turns out that all methods perform very poorly with respect to sparse networks.

The error measure reaches almost the value of 2, implying that the sum of all individual deviations from the true values amount to almost the double of total exposures. Even though in such a structure overall exposures may be considerably lower than the ones expected to be found in a complete network, it is, nevertheless, an indicator of the difficulty of this problem.

Moreover, while both Maximum Entropy and Gumbel Copula based methods show very similar results, random network based methodology exhibits a much poorer performance. Therefore, further investigation of this topic is required.

3.5 Conclusion

The research presented here constructs a methodology that is based on copulas to simulate the adjacency matrices of networks. Further, the conditions under which the copula based approaches outperform the traditionally used approach based on Maximum Entropy in the context of both large and small networks are numerically established and verified. With the help of simulations as well as applications to real data it is shown that there exists an association between the level of asymmetry in the data and the performance of the Maximum Entropy based as well as both the copula based approaches. In particular, it is observed that in an interval of threshold levels of asymmetry, the copula based methods always perform better than the Maximum Entropy method. In general it is observed that, given our DGPs, the higher the asymmetry (upto a threshold) the better is the performance of the copula based simulations as compared to the one based on Maximum Entropy. In particular, if the CC blocks and PP blocks do not complement each other (e.g, where PP = 0), the Maximum Entropy performs better. However, as soon as we have PP values to be positive, our method performs better.

Illustrations applying the copula based methods to sparse as well as dense networks have also been provided. For the dense network, given by the BIS data, one finds that beyond the threshold level of asymmetry (such that the data becomes more symmetric), Maximum Entropy does better, while as the data gets more asymmetric the copula-based method does better. Further, in case of the sparse network, the copula based direct method performs as well as Maximum Entropy. And both these methods outperform the method of random network generation which is contrary to the belief that a random network generation based algorithm is appropriate for sparse matrices' simulations. This has been verified with the eMID data based simulations, which represent a sparse network leading to the conclusion that sparse networks are not easy to handle.

To the best of our knowledge, this is the first attempt to study and compare the performance of such simulation based algorithms in light of asymmetries in the data. Further, ideas from Raschke et al. (2010) and Fernandez-Vazquez (2010) have been efficiently applied to create new methodologies in order to simulate adjacency matrices of networks when information on only the sum of rows and sum of columns is available.

However, the copula based methods presented here also suffer from similar shortcomings as Maximum Entropy. When applied to real data, a lack of information on the actual bilateral connections, can create an identification problem. Further, in such situations, one cannot judge the performance based on an error measure because the error measure cannot be calculated to begin with. In order to address this, a variety of simulations are presented here with the intention of outlining the variations in the relative performances of these methods given different data structures.

The copula based method has another drawback. Although it performs as well as the Maximum Entropy procedure and better than the random network generation method, it does not yield the best fit when applied to sparse matrices. Thus, a future direction of this research could include construction of a methodology to address the case of sparse matrices and performing a similar comparative analysis as provided in this chapter. Further, this analysis may also be extended to include the effects of other network features such as clustering, etc. on the performance of the simulations. For example, if certain nodes, across blocks, tend to cluster then does it affect the level of asymmetry in the data and thereby, affect the performance of the estimators and so on.

Besides adding to the set of simulation techniques, this approach also aims to contribute towards extending the toolkit of financial stability evaluation that allows an analyst to accommodate different assumptions with respect to the dependency exhibited by the network. The merits of the method proposed in this chapter are based on the fact that systemic risk affects private entities directly and strongly. The copula based methodology suppresses the data limitations required by these stress testing exercises, such that a better founded risk management analysis can be carried out by the private sector. Under these circumstances, the copula based methodology promises to be a valuable contribution to stress testing exercises based on the familiar risk management tools that copulas are. That apart, this methodology can also be useful in markets/countries where the data granularity required by network models is simply not available. For example, this methodology, when applied to an interbank network, could provide a better understanding of how contagion propagates within that network as it demands bilateral linkages level data. Further, this could serve as a significant addition to risk measurement methods and pricing via, say, Credit Defaults Swaps (Fischer, 2012) pricing models based on networks.

Chapter 4

Modeling Distribution over Connections: A Finite Mixture Model

4.1 Introduction

Each of the chapters above has contributed a novel method of modeling networks, one in the form of a theoretical application and the other as an econometric tool for network analysis. In this chapter, we continue by introducing a set of simple econometric methods to model distribution over connections within a network. The nature of the ensuing discussion is experimental and can be extended to include higher degrees of complexities by using more advanced econometric techniques. The purpose of this chapter is to illustrate the scope of econometrics in empirical analysis of networks, which still remains largely unexplored. In particular, we are interested to know if we can predict the probabilities of link formation given a network? Also, we would like to explore the possibility that a spread of connections in a network could be a mixture of two or more distributions?

This is of economic interest because it would allow us to model and predict the probabilities with which economic agents may connect with each other within a given network. Further, a model that can answer these questions can be applied to analyze interesting questions in areas such as marketing, where an understanding of market segmentation is important. For example, from the consumer purchase or transaction level data, one could identify clusters reflecting particular market segments. Based on the spread of connections across and within these clusters, probabilities of connections between the consumers and the firms as well as between the consumers within a market segment may be predicted using mixture models or distribution cluster analysis[1], respectively. This, in turn, would be useful to target advertising as well as sales of the products offered by firms.

In this chapter, we use a friendship network to establish that standard techniques from econometrics such as finite mixture models and discrete choice models can be efficiently used to model distribution over connections. Irrespective of the type of networks, these methods can be appropriately applied to model link formation or connections across nodes as well as the nodes' underlying preferences that lead these network formations. To that extent, the variable of interest here is the 'link' or the 'connection' between nodes. Consequently, we are also interested in the factors that affect the probability of link formation or the existence of a connection such as the node's neighborhood, its degree of centrality, etc. For example, in a friendship network, these factors may include variables such as a node's gender, race, ethnicity, how influential a node is and so on. Using an interesting high school friendship social network dataset, we illustrate various ways of modeling the complex connections between nodes that constitute such a network.

In the remainder of the chapter, section 2 discusses the details of the data. Section 3 applies econometric tools to model distribution over connections. Section 4 reports the results. Section 5 concludes.

4.2 Empirical Estimation of Network Characteristics

Our aim is to model the manner in which links are formed between nodes within any network, which can be used to predict future link formations and therefore, the path of network

[1]Although distribution cluster analysis and finite mixture models may seem similar in spirit, they differ significantly in their implementation and what they achieve. For example, distribution cluster analysis assumes that each cluster of nodes is drawn from the same distribution. On the other hand, a finite mixture of distributions is an instrument to model the distributions that potentially generate these clusters.

'evolution'. This analysis can prove to be useful in the context of marketing networks (as discussed earlier) as well as banking networks. In particular, in case of the latter, if we can make such predictions, it is likely that we can control the diffusion of a random (bad) shock to the system.

There are two aspects to our analysis, both of which enable us to predict the probabilities of connections between nodes within a given network. First, we use various discrete choice and count data regression methods to model the spread of connections as a dependent variable affected by factors that include network descriptors. The discrete choice models are driven by the notion of an underlying unobserved latent variable. Here, we assume that the latent variable represents nodes' preferences over connections. So, we are likely to see a link between nodes only if a node derives a positive value out a connection, given its preferences. A more detailed discussion on this follows in a latter section.

Second, since networks have a natural way to partition themselves based on affinity forming clusters or groups, we model the spread of connections as a mixture of finite number of distributions.

Definition 4.1: (*Finite Mixture of Distributions*)

For a given finite set of probability density functions $f_1(x), f_2(x), \ldots, f_n(x)$ *or the corresponding cummulative distribution functions* $F_1(x), F_2(x), \ldots, F_n(x)$ *and weights* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *where* $\lambda_i \geq 0$ *and* $\sum_{i=1}^{n} \lambda_i = 1$, *the finite mixture of distributions is denoted as either the density,* f, *or the distribution function,* F *and defined as the following sum:*

$$f(x) = \sum_{i=1}^{n} \lambda_i f_i(x)$$

or

$$F(x) = \sum_{i=1}^{n} \lambda_i F_i(x)$$

respectively.

For this study, we use the Adolescent Health (Add Health) data[2] containing infor-

[2] Inter-University Consortium For Political and Social Research (ICPSR) has published the data under National Longitudinal Study of Adolescent Health (Add Health), 1994 − 2002 undertaken by Harris and Udry (principal investigators).

mation on social networks from over eighty different high school friendship networks. These networks exhibit variation in numbers of nodes, density of links as well as homophily. This allows us to analyze the possible distribution of links by estimating the probabilities of connections between nodes in this network.

4.2.1 The Data

In this section, we describe the variables that will be used for our analysis. We have the following definitions that will be useful in defining the variables explained later[3]:

- Node: Unique member of a network.
- Tie : Nomination of j as a friend of i.
- Dyad : Pair of tied nodes.
- Symmetric Dyad : Dyad in which respondent's nomination of alter is reciprocated.
- Asymmetric Dyad : Dyad in which respondent's nomination of alter is not reciprocated.
- Geodesic : The shortest path between two nodes.
- Out-Degree : Number of alters nominated by respondent.
- In-Degree : Number of alters who nominate respondent.
- Respondent's Send-Network : Respondent and the set of alters nominated by respondent.
- Respondent's Receive-network : Respondent and the set of alters who nominate respondent.
- Respondent's Send- and Receive-Network : The union of respondent's send-network and respondent's receive network.
- Total Friendship Network : Complete school-level friendship matrix.

[3]These have been quoted from the codebook for this dataset.

- Categorical Attribute : An attribute which divides a population into mutually exclusive sub-populations. Categorical attributes used here are sex, race, grade, and age.

- Trait : A characteristic which defines a sub-population in terms of a categorical attribute. Examples are male, female, black, white, 8th graders, 16 year olds, etc.

The race and ethnicity questions in the Add Health study allowed respondents to chose multiple racial and ethnic backgrounds. In order to calculate respondent-network heterogeneity measures and school-level segregation indexes, Harris and Udry created a categorical race/ethnicity variable, using the following logic. Respondent was classified as:

- WHITE if he/she marked white as his/her only race and did not claim a Hispanic background

- BLACK if he/she marked black as his/her only race and did not claim a Hispanic background

- HISPANIC if he/she claimed a Hispanic background, regardless of racial background

- ASIAN if he/she marked Asian as their only race and did not mark Hispanic background

- OTHER all other responses

If any of the four specific racial/ethnic groups comprised less than 2 percent of the total population of a particular school, that group was recoded to OTHER for all school-level segregation indexes. Network measures were not calculated for schools with low response rates or for individual respondents with problematic identification numbers. Beyond these cases, particular network measures were missing for certain respondents, due to substantive or mathematical reasons. For example, the female saliency index is missing for those students who attend all male schools.

Basic Network Descriptors

- In-degree (IDGX2)

  The number of times respondent is nominated by other students in the school.

$$IDGX2_i = \sum_j X_{ji}$$

  Where:

  $\sum_j X_{ji}$ = the sum of the ith column of the total friendship network X

- Out-degree (ODGX2)

  The number of times respondent is nominated by other students in the school.

$$ODGX2_i = \sum_j X_{ij}$$

  Where:

  $\sum_j X_{ij}$ = the sum of the i^{th} column of the total friendship network X. Students were allowed to nominate up to 10 people. If a student skipped this entire section, $ODGX2 = 0$.

- Bonacich centrality, $\beta = 0.1$ (BCENT10X)

  Respondent's centrality, weighted by the centrality of those to whom he/she sends ties (Bonacich, 1987).

$$BCENT10X(\alpha, \beta)_i = \alpha(I - \beta X)^{-1} X.e$$

  Where:

  α = a scaling vector

  β = power weight (here = 0.1)

  I = identity matrix

  X = total friendship network

  e = column of ones. If $ODGX2 = 0$, then $BCENT10X = 0$.

- Reach (REACH)

  Maximum number of nodes that the respondent can reach in the total friendship network.

$$REACH = \sum_j B_{ij}$$

  Where:

  B is the reachability matrix of X such that:

  D = geodesic distance matrix of the total friendship network X and

$$B_{ij} = 1\, if\, D_{ij} > 0$$

  If $ODGX2 = 0$, then $REACH = 0$.

- Proximity Prestige (PRXPREST)

  Measures the prestige of respondent relative to the number of people who can reach respondent (Wasserman and Faust $1994 : 203 - 205$).

$$PRXPREST = \frac{\frac{I_i}{(g-1)}}{\sum_j \frac{d(n_j,n_i)}{I_i}}$$

  Where:

  I_i = influence domain of i, which is equal to the number of nodes who can reach i.

  g = number of nodes in X

  $d(n_j, n_i)$ = length of the geodesic distance between actor j to actor i. If $IDGX2 = 0$, then PRXPREST is missing.

- Influence domain (INFLDMN)

  Number of nodes who can reach the respondent.

$$INFLDMN_i = \sum_j B_{ji}$$

  Where: B is the reachability matrix of X

such that:

$D =$ geodesic distance matrix of the total friendship network X

and

$B_{ij} = 1$ if $D_{ij} > 0$. If $IDGX2 = 0$ then $INFLDMN = 0$.

- Respondent has a best male friend (HAVEBMF)

  Dummy variable indicating whether the respondent nominated a male friend in the school as his/her best friend.

$$HAVEBMF_i = 0, \text{ respondent did not nominate a male best friend}$$

$$HAVEBMF_i = 1, \text{ respondent nominated a male best friend}$$

- Respondent has a best female friend (HAVEBFF)

  Dummy variable indicating whether respondent nominated a female friend in the school as his/her best friend.

$$HAVEBFF_i = 0, \text{ respondent did not nominate a female best friend.}$$

$$HAVEBFF_i = 1, \text{ respondent nominated a female best friend.}$$

- Best male friend reciprocates (BMFRECIP)

  Dummy variable indicating whether the person respondent nominated as his/her best male friend nominated respondent as a friend.

$$BMFRECIP_i = 0, \text{ respondent's best friend did not reciprocate a nomination.}$$

$$BMFRECIP_i = 1, \text{ respondent's best friend reciprocated with a nomination.}$$

  If respondent has no best male friend, or if the best male friend did not complete the questionnaire, BMFRECIP is missing.

- Best male friend reciprocates as best friend (BMFRECBF)

  Dummy variable indicating whether the person respondent nominated as his/her best male friend nominated respondent as his/her best friend.

  $$BMFRECIP_i = 0, \text{ respondent's best friend did not reciprocate as best friend.}$$

  $$BMFRECIP_i = 1, \text{ respondent's best friend reciprocated as best friend.}$$

  If respondent has no best male friend, or if the best male friend did not complete the questionnaire, BMFRECBF is missing.

- Best female friend reciprocates (BFFRECIP)

  Dummy variable indicating whether the person respondent nominated as his/her best female friend nominated respondent as any type of friend.

  $$BFFRECIP_i = 0, \text{ respondent's best friend did not reciprocate as best friend.}$$

  $$BFFRECIP_i = 1, \text{ respondent's best friend reciprocated as best friend.}$$

  If respondent has no best female friend, or if the best female friend did not complete the questionnaire, BFFRECIP is missing.

- Best female friend reciprocates as best friend (BFFRECBF)

  Dummy variable indicating whether the person respondent nominated as his/her best female friend nominated respondent as her best friend.

  $$BFFRECIP_i = 0 \text{ respondent's best friend did not reciprocate as best friend}$$

  $$BFFRECIP_i = 1 \text{ respondent's best friend reciprocated as best friend}$$

  If respondent has no best female friend, or if the best female friend did not complete the questionnaire, BFFRECBF is missing.

- Respondent-centered Network Measures

  Respondent-centered networks are composed of the respondent and a set of nodes directly tied to the respondent. The set of nodes in a particular type of respondent network is defined in one of three ways: (1) those nodes nominated by respondent, the respondent send-network; (2) nodes nominating respondent, the respondent receive-network; or (3) the union of respondent's send- and receive-networks (the respondent send- and receive-network). All three types of respondent networks include ties sent from any node in the network to any other node in the respondent network. Thus, if the respondent sends ties to j and k, a tie from j to k is part of respondent's send-network (as are ties from k to j, j to respondent, and k to respondent). Harris and Udry have calculated sociometric characteristics, heterogeneity measures, and attribute means for each of the three types of respondent networks.

- Respondent-network heterogeneity measures for grade, race, and age.

  For each of the three types of respondent networks, they calculate a heterogeneity measure for three categorical attributes: grade, race and age. Three variables are associated with each network/attribute combination: a heterogeneity score, the proportion of all traits present in the school which are represented in the respondent network, and the number of nodes in the network used to calculate the heterogeneity score.

- Heterogeneity measures.

  This family of variables assesses the heterogeneity of an ego network with respect to the traits of a categorical attribute. The formula used to calculate respondent-network heterogeneity with respect to attribute A is:

  $$HETEROGENEITY_{iA} = 1 - \left[\sum_1^n (\tfrac{A_k}{en})^2\right]$$

  Where:

  $A =$ the categorical attribute

$A_k =$ the number of nodes with trait k in the respondent network

$en =$ the number of nodes in the respondent network with valid data on A.

$n =$ the total number of traits of A represented in the respondent network.

$HETEROGENEITY_{iA}$ is missing if respondent is the only member of the underlying respondent network, or if all members of the respondent network (including respondent) have missing data on attribute A. If all members of the respondent network who have valid data on attribute A share the same trait, $HETEROGENEITY_{iA} = 0$.

Proportion of possible traits represented in respondent network. This family of variables is a measure of the sheer diversity of the respondent network, with respect to the number of traits present in respondents school.

$$PROPORTION_{iA} = \frac{n_i}{n_A}$$

Where:

$A =$ the categorical attribute.

$n_i =$ the total number of traits of A represented in the respondent network.

$n_s =$ the total number of traits of A represented in respondent's school.

$PROPORTION_{iA}$ is missing if $HETEROGENEITY_{iA}$ is missing.

Number of cases used to calculate heterogeneity. This family of variables is a sheer count of the number of cases in the respondent network with valid data on attribute A.

$$No.\ of\ CASES_{iA} = en$$

Where:

$en =$ the number of nodes in the respondent network with valid data on A.

$No.\ of\ CASES_{iA}$ is missing if $HETEROGENEITY_{iA}$ is missing.

Variable names for heterogeneity and related measures:

		Attributes	
Type of Respondent Network	GRADE	RACE	AGE
RESPONDENT SEND-NETWORK	EHSGRD	EHSRC5	EHSAGE
	ERSNGRD	ESRNRC5	ERSNAGE
	NEHSGRD	NEHSRC5	NEHSAGE
RESPONDENT RECEIVE-NETWORK	EHRGRD	EHRRC5	EHRAGE
	ERRNGRD	ERRNRC5	ERRNAGE
	NEHRGRD	NEHRRC5	NEHRAGE
RESPONDENT SEND- AND RECEIVE-NETWORK	EHGRD	EHRC5	EHAGE
	ERNGRD	ERNRC5	ERNAGE
	NEHGRD	NEHRC5	NEHAGE

In the table above, within each cell, the first variable is the heterogeneity score, the second is the proportion of traits represented, and the third is the number of cases used to calculate heterogeneity.

Next, we describe methods for estimating probability of connections.

4.3 Estimation Methodology

We illustrate the uses of different discrete choice and count data models[4] to analyze various aspects of the prediction of probabilities of connection. The need to do this is justified by the fact that within a network, one may not only be interested in the probability of two nodes being connected, but also in predicting events such as the probability that two nodes are connected *given* the connection between another pair of nodes and so on. Further, the impact of factors such as the number of links that a node has, on the probability of connection between two nodes, can be of interest as well. Further note that there are two distinct types of connections - one that refers to the nominations extended by the respondent (sent arcs) and the other that refers to the nominations received by the respondent (received arcs). Each connection has a different set of factors affecting them as we will illustrate shortly. Given this, we carry out separate regressions to model probabilities of sending arcs and receiving arcs by the nodes. We also exemplify the case where the distribution of

[4]Codes for this chapter have been included in the appendix titled 'Codes'.

connections is a mixture of two distributions[5] by fitting a finite mixture model to this data.

4.3.1 Discrete Choice Models for Friendship Nominations

First, we illustrate the use of a bivariate probit model to estimate the association between HAVEBMF and HAVEBFF, BMFRECIP and BFFRECIP, BFFRECIP and BMFRECBF and relevant pairs across these. This exercise enables us to model the latent (unobservable) preferences of the node that lead it to nominate another node as a friend or reciprocate another node's nomination.

Consider the first pair - HAVEBMF and HAVEBFF. Let the latent variables y_1^* denote the respondent's preference level for a male best friend and y_2^* denote the respondent's preference level for a female best friend. Then our bivariate probit model specifies the observed outcomes to be

$$y_1 = \begin{cases} 1 & if\, y_1^* > 0 \\ 0 & if\, y_1^* \leq 0 \end{cases}$$

and

$$y_2 = \begin{cases} 1 & if\, y_2^* > 0 \\ 0 & if\, y_2^* \leq 0 \end{cases}$$

So, when the level of preference for a male best friend is positive, we observe y_1 and do not otherwise. Similarly, when the level of preference for a female best friend is positive we observe y_2 and do not otherwise.

Bivariate Probit for Having a Nomination

A bivariate probit (BVP) estimates the probabilities of HAVEBMF (y_1) and HAVEBFF (y_2) jointly. We might expect that the chances of a respondent having a male best friend nomination may not be independent of him/her choosing a female for a best friend nomination. This is because, there might be an unobserved latent variable such as gender preference that governs these probabilities. As such, a bivariate probit model seems appropriate. The

[5] All graphs, tables and figures for this chapter have been provided in the corresponding appendix - Appendix 3.

independent variables are attributes of the respondent's send-network as well as individual specific attributes such as his/her reachability (reach) and degree of centrality (bcent10x). This choice of regressors is justified by the fact that the respondent's nomination of a male or a female best friend, might depend on factors such as the nominee's race, age or even whether the nominee and the respondent are in the same class. Moreover, a respondent's prestige index as well as his/her influence domain might affect the number of nodes that pick him/her as their (best) friend, which in turn affects the probabilities of the respondent being chosen as a (best) friend by any of the other nodes. Further, in order to capture the effects of the respondent's own gender, we also introduce a *sexdummy* that takes value 1 if the respondent is a male and 0 otherwise.

Our pre-estimation checks reveal that HAVEBMF and HAVEBFF are negatively correlated with a coefficient of correlation, $\rho = -0.2686$. The sign of ρ can be explained by the fact that if the respondent prefers a male best friend to a female friend, it is less likely that he/she will nominate a female as a best friend. This is reflected in the regression as well. We regress $(HAVEBMF_i, HAVEBFF_i)$ on ehsgrd, ersngrd, ehsrc5, esrnrc5, ehsage, ersnage, bcent10x, reach and sexdummy, where the first six regressors are attributes from the respondent's send-network, bcent10x is the respondent's degree of centrality, reach is the number of nodes that the respondent can reach within the total network and the sexdummy that reflects the respondent's gender. The bivariate probit estimates are summarized in Table 1 and the results have been discussed in the following section.

Bivariate Probit for Receiving a Nomination

Similarly, for the second pair - BMFRECIP (y_1) and BFFRECIP (y_2), the latent variable may be defined as the nominees' preference for the respondent as a friend. This could depend on the respondent's gender, race, age, grade (whether they are in the same class), the degree of centrality of the respondent, the prestige index of the respondent as well as the influence domain. The correlation between BMFRECIP and BFFRECIP is 0.0557, which is near zero. One would expect these to be almost uncorrelated because the decisions of the respondent's nominees (of choosing the respondent as a friend) are likely to be independent of each other as the nominees themselves may not be connected to each other. However, since the correlation is not exactly zero, there might be some nominees who are connected to each other and as such, their decisions might be inter-dependent. We conduct a bivariate probit regression[6] of $(BMFRECIP_i, BFFRECIP_i)$ on ehrgrd, errngrd, ehrrc5, errnrc5, ehrage, errnage, bcent10x, prxprest, infldmn and sexdummy where the first six regressors are attributes from the respondent's receive-network, the seventh regressor is the respondent's degree of centrality, the eighth regressor is the prestige of the respondent relative to the number of nodes who can reach the respondent, the ninth regressor is the number of nodes that can reach the respondent within the total network and the last regressor is a sexdummy that reflects the respondent's gender.

Now before we repeat the procedure for the third pair, note the difference between (BMFRECIP and BMFRECBF) and (BFFRECIP and BFFRECBF). There could be two types of networks within this social network. One where a link is said to be formed when the following holds: *if the respondent's nominee for best friend chooses him/her as a friend as well*; and the other when: *if the respondent's nominee for best friend chooses him/her as a best friend as well*. In case of the former, our dependent variable of interest is BMFRECIP (not BMFRECBF) and BFFRECIP (not BFFRECBF). And in case of the latter, our dependent variables of interests are BMFRECBF (not BMFRECIP) and BFFRECBF (not BFFRECIP). So, the third pair really refers to the latter kind of network. The latent variable may still be defined as the nominees' preference for the respondent as a *best* friend. This could depend on the respondent's gender, race, age, grade (whether they are in the same class), the degree of centrality of the respondent, the prestige index of the respondent as well as the influence domain. Besides, it might also depend on whether the respondent has been nominated as a friend to begin with (BMFRECIP and BFFRECIP). This would capture the fact that even if the respondent did get nominated by a nominee as a friend, he/she may not get nominated as his/her nominee's best friend. For this purpose, we

[6]If the correlation between the dependent variables is 0, a bivariate probit is not required. Instead, we could execute two separate probit regressions for each dependent variable. However, we run a bivariate probit in order to ensure that the interdependencies leading to a positive correlation are not ignored. Moreover, we use STATA to execute the bivariate probit models, which runs two separate probits in case of uncorrelated dependent variables.

generate a new variable that counts the nominations for the respondent, irrespective of the nominees' gender. Thus, if the respondent has no nominantions, this variable takes value 0. If there is one nomination (either from a male nominee or a female nominee), it takes value 1. And if the respondent has nominations from both male as well as female nominees, it takes the value 2. We call this variable ANYFRIENDRECIP.

The correlation between BMFRECBF and BFFRECBF is 0.0014. If we add this new variable as a regressor, the hypothesis that the correlation between these two variables is 0, is rejected. However, if we exclude this regressor, then it is not rejected. This is fairly intuitive as the inclusion of this new regressor essentially links these two variables together, thus, making them dependent on each other. We regress $(BMFRECBF_i, BFFRECBF_i)$ on ehrgrd, errngrd, ehrrc5, errnrc5, ehrage, errnage, anyfriendrecip, bcent10x, prxprest, infldmn and sexdummy where the first six regressors are attributes from the respondent's receive-network, the seventh regressor is the new variable that we created, the eighth regressor is the respondent's degree of centrality, the ninth regressor is the prestige of the respondent relative to the number of nodes who can reach the respondent, the tenth regressor is the number of nodes that can reach the respondent within the total network and the last regressor is a sexdummy that captures the respondent's gender. The results of the regression with this additional regressor have been summarized in Table (13) and a discussion follows in the next section.

We can now predict the probabilities of the different types of connections. Here a connection refers to either a sent arc or a received arc[7]. So, the dependent variable in each of the bivariate probit regressions above, refers to a type of connection. The corresponding predicted probabilities have been reported in tables (2), (6), (10) and (14) respectively.

4.3.2 Count Data Models for Indegrees and Outdegrees

The total number of arcs sent by a node is defined as the outdegree of that node. And the total number of arcs received by the node is defined as the indegrees of that node. Thus, the total number of connections for a network is given by the sum outdegrees and indegrees of

[7]Recall, a directed connection is called an *arc*

all nodes in the network. In order to find the probabilities of having any type of connection, we must have a model that relates the total number of connections to factors that might affect it. Since the indegrees are an increasing function of BMFRECIP and BFFRECIP, the factors that affect indegrees include the variables that affect these. Similarly, the outdegrees depend on HAVEBMF and HAVEBFF. So, the factors affecting the latter may be used as explanatory variables for the out-degrees. We begin with two separate regressions for each of these to identify the statistically significant factors, which can then be used to choose explanatory variables for estimating the probabilities of total connections.

Given that indegrees and outdegrees are count data, we conduct a pre-estimation check for each one to find out whether a Poisson model or a Negative Binomial (2) (NB2) fits better. For the indegrees, the dependent variable is idgx2 and the explanatory variables are ehrgrd, errngrd, ehrrc5, errnrc5, ehrage, errnage, anyfriendrecip, bcent10x, prxprest, infldmn and sexdummy. The description of these variables continues to remain the same as mentioned earlier. We find that the Poisson overestimates the model by a small margin. This is confirmed by the test for overdispersion, which rejects the null hypothesis for equi-dispersion (see Table (15)). As such, the NB2 model seems to be a better choice. The results of the regression have been summarized in Table (16).

Next, we consider the outdegrees. We create another variable on the lines of ANYFRIENDRECIP and call it ANYFRIENDNOM. This variable takes the value 0 if the respondent has not nominated any one, 1 if the respondent has nominated either a male or a female as best friend and 2 if he/she has nominated both a male as well as a female as a best friend. Since, the earlier estimates already capture the gender effects, we are more concerned about the effect of overall connections sent out by the respondent, on his/her outdegree. The dependent variable here is odgx2 and the explanatory variables are ehsgrd, ersngrd, ehsrc5, esrnrc5, ehsage ersnage, anyfriendnom, bcent10x, reach and sexdummy. The description of these variables continues to remain the same as mentioned earlier. Again, we check for the fits of both a Poisson and a NB2. Although the test for overdispersion rejects the null hypothesis of equi-dispersion (see Table (17)), we use Poisson to estimate this model because the NB2 iterations fail to converge. This is, probably, because the magnitude of overdispersion is very close to 0, even though the test indicates that it is statistically significant. The results of the regression have been summarized in Table (18).

4.3.3 Finite Mixture Model for Connections

We can now propose a mixture model in order to estimate the probabilities of total connections. The intuition underlying the construction of a mixture model is outlined as follows. Recall, that the total number of connections is the sum of indegrees and outdegrees for a given node. We have already shown that the probability of a node (respondent) receiving or extending a connection can be estimated. Now, we would like to estimate the probability that any kind of connection forms between nodes. In order to do that we create a new variable known as TOTCON, which is the sum of IDGX2 and ODGX2.

A preliminary check reveals that the density for this variable is bimodal as can be seen in Figure (1).

Note that it seems like the density is a mixture of two (overlapping) densities. So, a mixture model with two components seems an appropriate choice for modeling. Again, given that we have count data, we consider Poisson and NB2 regressions. In order to check whether the mixture of two Poisson distributions fit better than the mixture of two NB2 distributions we check the AIC and BIC for both the models. The $(AIC, BIC) = (12638.72, 12770.39)$ for NB2 whereas $(AIC, BIC) = (12657.26, 12777.48)$ for Poisson. Since BIC penalizes more than AIC, we go for the model with a smaller BIC. In this case, both AIC as well as BIC are smaller in case of NB2. Therefore, we construct a two component mixture model with a mixture of NB2. The results have been summarized in Table (20).

4.3.4 Ordered and Multinomial Probit for Links

Next, we make a distinction between a connection and a link. A link is formed when a node receives an arc from the node it sent an arc to. Therefore, a link reflects bilateral reciprocation, whereas a connection is unilateral. Another question of interest is whether we can calculate the probabilities of link formation? Recall, that there are two forms of networks that may be defined within this social network. First, where simply a nomination by a respondent's nominee may be called a link and second, where only a best friend nomination by a respondent's nominee may be called a link. For the first case, we create our dependent variable, a *tie*, as $(HAVEBMF \times BMFRECIP) + (HAVEBFF \times BFFRECIP)$ which takes the value 0 if there is no link, 1 if there is a link with either a male or a female nominee and 2 if there is a link with both a male as well as a female nominee. Similarly, in case of the second model, we generate a tie as $(HAVEBMF \times BMFRECBF) + (HAVEBFF \times BFFRECBF)$, which follows the same argument as above in taking values $0, 1$ and 2.

Given this, it would be of interest to be able to rank nodes given the number of ties they have. Here we can specify a latent variable y_i^* that may be an unobserved measure of the preference for link formation of both the nodes. So, a node may have zero links if y_i^* is below a certain threshold, say, α_1; a node may have at least one link is $y_i^* > \alpha_1$ and a node may have two links if $y_i^* > \alpha_2$. We use ordered probit to model this. In case of the first network we assign $tiecon = 1$ if the $tie(connection) = 0$, $tiecon = 2$ if $tie(connection) = 1$ and $tiecon = 3$ if $tie(connection) = 2$. We call tie(connection) as *tiec*. Similarly, in the second network we assign $tieg = 1$ if the $tie(link) = 0$, $tieg = 2$ if $tie(link) = 1$ and $tieg = 3$ if $tie(link) = 2$. We call tie (link) as *tiel*. These for both the networks, have been summarized in Table (22). However, note that the α's here are not well-defined. The reason for this is there doesn't seem to be a natural cutoff that we can assume given the data. As such, another option is to estimate a multinomial probit and compare the categories with the base category of no connections or links. This regression would predict the probabilities that a category of, say, two connections, will be realized[8] as compared to a category of zero connections, if the coefficient of a given regressor for that category is positive.

4.4 Results

In this section, we analyze the results derived by regressions from the previous section.

[8]The reason we use the word 'realized' rather than 'chosen' is because, these regressors lead the nodes to make choices that finally lead to a connection being formed or realized.

4.4.1 Bivariate Probit Regressions

We begin by discussing our findings for the first bivariate probit regression used to estimate the joint model for the respondent's nominations. Table 1 summarizes the coefficients of all the regressors. We observe that both the dependent variables are strongly negatively correlated. The Wald test for zero correlation is rejected at 5% level of significance. Recall that this is what we found earlier in our pre-estimation checks and argued intuitively in favor of this observation. If we look at the coefficients of the bivariate probit, one would observe that there are only two regressors that are (strongly) statistically significant for the first regression (for HAVEBMF), which are bcent10x and the sexdummy. However, five out of nine regressors (besides the constant) are statistically significant for the second regression (for HAVEBFF), which are ehsrc5, esrnrcs, ehsage, bcent10x and sexdummy. These results imply that whether the respondent chooses to nominate a male best friend or not depends strongly on his/her own gender (coeff. of sexdummy = 0.87) as well as the respondent's centrality, weighted by the centrality of those to whom he/she sends arcs to (coeff. of bcent10x = 0.65). Note that this is what we suspected initially, when we claimed that the degree of centrality of a node plays an important role in the manner in which links are formed (or arcs are extended). So, if the degree of centrality is higher, it is more likely that the respondent will send across an arc to another node. This could be because a higher degree of centrality provides the respondent with higher confidence that his/her nomination will be reciprocated.

As against this, in the other regression, the probability of whether the respondent chooses a female best friend nomination is not only strongly affected by the degree of centrality and gender of the respondent, but also by the heterogeneity scores for race and age as well as the proportion of traits represented within the attribute of race in the send-network of the respondent (that includes the number of nodes the respondent can directly connect to). Observe that the coefficient of heterogeneity with respect to race is negative (-0.73). So, less heterogeneity in race and more in age (coeff. of heterogeneity in age = 0.58) of the respondent's connections as well as a higher proportion of traits represented within race (coeff. $= 0.74$)[9], implies a higher probability of the respondent of having a female best friend nomination. This could be because the respondents are adolescents, who are actively looking for romantic partners/relationships. As such, the males are more likely to choose females and vice-versa. This is reflected in the coefficient of the sexdummy which is negative (-0.87). So, we have that there is an underlying preference for the opposite sex that is driven by these factors.

Further, a test of joint significance of the regressors over both equations shows that only ehsrc5, ehsage, bcent10x and sexdummy are significant in the joint model. Also, Table (2) summarizes the predicted probabilities for this model. The marginal probabilities that $Pr(havebmf = 1)$ and $Pr(havebff = 1)$ are 0.74 and 0.84 respectively. Whereas, the the joint probability of $Pr(havebmf = 1, havebff = 1)$ is 0.58. Of all the joint probabilities, this is the highest. And of the joint probabilities $Pr(havebmf = 1, havebff = 0) = 0.16$ and $Pr(havebmf = 0, havebff = 1) = 0.26$, the latter is higher - that is, a female friend nomination is more likely. We also go ahead and estimate the average marginal effects for this joint model and summarize the results in Table (3. These are in line with our analysis above. Note that the average marginal effects of ehsrc5, esrnrc5, ehsage, bcent10x and sexdummy are statistically significant. A test for joint significance shows that ehsrc5, ehsage, bcent10x and sexdummy are significant across both models (see Table (4)). Overall, observe that the degree of centrality and within network attributes of heterogeneity play an important role in determining the send-network of a node. Thus, within the send-network of a node there is a 58% chance that the node would have extended friendship nominations to both males as well as females. However, if only one form of friendship nominations are extended, there is a higher chance that those will be extended to females. So, it is likely that the send-network constitutes of more female nominations on average.

The results for the second set of bivariate probit regressions with dependent variables, BMFRECIP and BFFRECIP, can be analyzed in a similar fashion. The Wald test for ρ equal to 0 is not rejected implying that there is not any significant correlation be-

[9]Lower heterogeneity in race and higher proportion of traits represented within the attribute of race leading to higher best female friend nominations might be a consequence of the fact that whether it is a best friend or romantic partner, it is more likely that the opposite sexes would match with same races rather than across races. This bias must have resulted in the observed signs of those coefficients.

tween the two, which confirms our conclusion from the pre-estimation checks. Recall that BMFRECIP and BFFRECIP refer to the status of reciprocation of a friendship nomination of a node by his/her nominees. These nominees may or may not know each other and as such, their decisions are likely to be independent of each other's decisions. Therefore, it is reasonable that the hypothesis '$\rho = 0$' is not rejected. Despite this, for reasons outlined above, we carry out a bivariate probit regression. The results for this joint estimation have been summarized in Table (5). Note that in the first probit regression, six out of ten regressors are statistically significant, which are ehrrc5(-0.96), errnrc5(1.1), bcent10x(0.25), prxprest(3.4), infldmn(0.001) and sexdummy(0.52). So, whether the respondent receives a friend nomination by his/her best male friend nomination is very strongly explained by the respondent's relative centrality measure, his/her prestige index and his/her gender. The respondent's influence domain also has a positive impact, however, the effect is very small. This implies that increase in the number of nodes that can reach the respondent will have an impact on the probability with which his/her best friend nomination nominates him/her as a friend - however, it will be very low in magnitude. Further, the heterogeneity score with respect to race has a negative impact on the probability of the respondent getting nominated whereas the proportion of traits within race have a strong positive impact. Thus, more heterogeneity with respect to race will affect the nature of connections (neighborhood) of the respondent. As the heterogeneity increases, the respondent's nominations are less likely to reciprocate and the former's neighborhood might end up as a bunch of sent arcs. So, his send- and receive- networks may almost overlap as this factor increases. On the other hand, for the second regression, we observe that prxprest(3.6), infldmn(0.01) and sexdummy(-0.84) are statistically significant. Interestingly, female reciprocative friend nominations are not explained very well by the relative degree of centrality or most of the heterogeneity attributes of the receive network of the respondent node. Moreover, the sexdummy has a negative impact on the probability, implying that if the node has received a nomination from his/her female friend nomination, it is likely that the respondent node is not a female[10]. So, except a preference for the opposite gender and the prestige index of

[10]This is in line with our earlier finding that in the send-network, a respondent prefers extending a nomination to a node of the opposite sex. Thus, since most of the other nodes are likely to be nominated

the respondent node, the female nominees do not seem to care much about the rest of the attributes while considering reciprocation. Also, the influence domain of the respondent node has a positive impact implying that an increase in the number of nodes that can reach the respondent node increased this probability.

The marginal probabilities that $Pr(bmfrecip = 1)$ and $Pr(bffrecip = 1)$ are 0.5 and 0.61 respectively. Whereas, the joint probability of $Pr(bmfrecip = 1, bffrecip = 1)$ is 0.31. Of all the joint probabilities, this the highest. Although, the joint probability $Pr(bmfrecip = 0, bffrecip = 1) = 0.30$ is very close (see Table (6)). Thus, if we compare these probabilities with our previous bivariate model and match, it would seem that this network (where a link is formed when the best friend nominated by the respondent reciprocates by a friend nomination) would have the respondents connected to female friends, mostly. This is in line with the conclusion earlier, which indicated that a female friend nomination is more likely. We also looked at the marginal effects, of which, ehrrc5, errnrc5, bcent10x, prxprest and infldmn have significant effects. Sexdummy does not have a significant marginal effect. Although in a joint test of significance ehrrc5, errnrc5, bcent10x, prxprest, infldmn and sexdummy were significant across both equations. These results have been summarized in Table (7) and Table (8), respectively.

Now, we will go ahead and analyze the third bivariate probit regression with dependent variables as BMFRECBF and BFFRECBF. Again, the Wald test for $\rho = 0$ is not rejected implying that there isn't any significant correlation between the two as was concluded from our pre-estimation checks. An argument similar to the case of BMFRECIP and BFFRECIP holds. The results for this estimation have been summarized in Table (9). We observe a number of similarities between the previous regression results and these. Note that in the first probit regression, six out of ten regressors are statistically significant, which are ehrrc5(-0.76), errnrc5(0.9), bcent10x(0.26), prxprest(1.9), infldmn(0.001) and sexdummy(0.5). These imply that whether the respondent receives a best friend nomination by his/her best male friend nomination is very strongly explained by the respondent's relative centrality measure, his/her prestige index and his/her gender. The respondent's

by the opposite sex, if they choose to reciprocate, it will be the case that they do so to a nomination of the opposite gender.

influence domain also has a positive impact, however, the effect is very small. We also find that the number of nodes that can reach the respondent continues to affect the probability with which his/her best friend nomination nominates him/her as a best friend - however, it is very low in magnitude. Moreover, the heterogeneity score with respect to race has a negative impact on the probability of the respondent getting nominated whereas the proportion of traits within race have a strong positive impact.

On the other hand, for the second regression, we observe that ehrgrd (.44), errngrd ($-.52$), errnage (.70), prxprest(2.26), infldmn(0.001) and sexdummy(-0.78) are statistically significant. Interestingly, we again find that female reciprocated best friend nominations are not explained very well by the relative degree of centrality or the heterogeneity attributes such as ehrrc5 and errnrc5 of the receive network of the respondent node, as was the case for the first regression. Moreover, the sexdummy has a negative impact on the probability, implying that if the node has received a nomination from his/her best female friend nomination, it is unlikely that the respondent node is a female. So, except a preference for the opposite gender, the prestige index of the respondent node, the heterogeneity score within a grade, the proportion of traits within grade as well as the proportion of traits within age, the female best friend nominees do not seem to care much about the rest of the attributes while considering to reciprocate. Also, the influence domain of the respondent node has a positive impact implying that an increase in the number of nodes that can reach the respondent node results in an increase in this probability.

The marginal probabilities that $Pr(bmfrecbf = 1)$ and $Pr(bffrecbf = 1)$ are 0.21 and 0.30 respectively. Whereas, the the joint probability of $Pr(bmfrecbf = 1, bffrecbf = 1)$ is 0.1 which is the lowest. On the other hand, the joint probability $Pr(bmfrecbf = 0, bffrecbf = 0) = 0.55$ is the highest (see Table(10)). Thus, if we compare these probabilities with our previous bivariate model and match, it would seem that this network (where a link is formed when the best friend nominated by the respondent reciprocates by a best friend nomination) would have the respondents connected mostly to female friends as had been concluded earlier. However, we might also observe with a fairly high chance that the respondent's best friend nominations (either male or female) do not reciprocate. We also looked at the marginal effects, of which, ehrrc5, errnrc5, bcent10x, prxprest and infldmn have significant effects. Sexdummy does not have a significant marginal effect. However, in a joint test of significance ehrgrd, errngrd, ehrrc5, errnrc5, bcent10x, prxprest, infldmn and sexdummy were significant across both equations. These results have been summarized in Table (11) and Table (12), respectively.

We also run the same bivariate probit regression as above with an additional regressor ANYFRIENDRECIP and find that this regressor is strongly statistically significant in both the bivariate probit regressions (Table (13)). So, as the number of overall nominations increase, the probability that a best male or a female friend nomination by the respondent will reciprocate also increases. The inclusion of this regressor here may not be useful, however, it serves as a crucial explanatory variable in one of our following models. With the additional regressor, the marginal probabilities that $Pr(bmfrecbf = 1)$ and $Pr(bffrecbf = 1)$ are 0.20 and 0.28, respectively. Whereas, the the joint probbaility of $Pr(bmfrecbf = 1, bffrecbf = 1)$ is 0.06. Of all the joint probabilities, this is the lowest. On the otherhand, the joint probability $Pr(bmfrecbf = 0, bffrecbf = 0) = 0.58$ is the highest (see Table (14)).

4.4.2 Count Regressions: NB2 and Poisson

The preliminary set of estimates have now been analyzed. Given that we have always been interested in studying the probability of link formation and hence, the distribution of links, we constructed a model for the in-degrees, the out-degrees and the total connections to study the distribution of links within a network. We first consider the model for in-degrees and conduct the test for overdispersion. The test shows significant overdispersion in the data (see Table (15)). Therefore, we ran a NB2 regression. The results of this have been summarized in Table (16). Observe that all of the regressors except errnage (proportion of traits within age) and sexdummy are significant. As such, these have strong explanatory powers with respect to the dependent variable, in-degrees. Also, note that the heterogeneity scores for race and grade have a negative impact on the probability of the number of in-degrees. Thus, number of arcs extended to the respondent is inversely related to these two attributes. Recall, the ehrrc5 (the score for race) also had a negative impact on BMFRECIP and BFFRECIP in the corresponding bivariate probit regression. Since, these are a function of the in-degrees, the negative sign is consistent.

Next, we test for overdispersion in the model with out-degrees and find that the null for equi-dispersion is strongly rejected (see Table (17)). As mentioned earlier, although a NB2 model will be appropriate, we do not achieve convergence. As such, we run a Poisson regression. The results have been summarized in Table (18). We observe that all regressors except ehsgrd, ersngrd and sexdummy are strongly significant. Thus, the heterogeneity scores of grade, the proportion of traits with respect to grade and sexdummy do not have good explanatory power with respect to the dependent variable.

4.4.3 Finite Mixture Models

We construct a 2-component mixture model to estimate total connections (TOTCON). The dependent variable is TOTCON, which is the sum of the in-degrees and the out-degrees of a node. So, the underlying latent variable is a combination of two different types of 'preferences' - one that defines the preference of a node over extending a friendship nomination to another node, and the other that defines the preference of other nodes over extending friendship nominations to a given node. So the node in focus, has control over only one set of preferences, while the other set of preferences is taken by it as given. Because of this, it makes sense that the distribution of links is bimodal as there are two sets of underlying latent variables in play here.

First, we check the fits for mixtures of both NB2 as well as Poisson (Table 19) and pick the one with the lower AIC and BIC. We find that the NB2 model has the desired properties. The results for the finite mixture of two-component NB2 model for total connections have been summarized in table (20). These indicate that for component 1 the only regressor that is not statistically significant is ehrc5. On the other hand for component 2 the regressors that are not statistically significant are ehage, ernage, anyfriendnom and sexdummy. The proportion of observations that belong to class I is about 48% and class II are about 53%. One could view these classes as components that have respondents with no reciprocated nominations and the ones that have respondents with reciprocated nominations. The first component has a mean of 9.5 whereas the second component has a mean of 12.5. The two classes have different responses to changes in regressors as well. These have been summarized in Table (21).

In component 1, the marginal effects of all but one (ehrc5) regressor is statistically significant. In component 2, on the other hand, we have that the marginal effects of ernage, anyfriendnom and sexdummy are not statistically significant. The fitted values distribution has been represented in Figures (2) and (3). The variances for both the components are very close (see Table (19)).

4.4.4 Ordered and Multinomial Probit

We will now discuss the ordered probit regressions and multinomial probit regressions for both links as well as connections, executed to analyze the probabilities of observing a link or connection that might belong to one of the three different groups, discussed in the previous section. The result for an ordered probit for links has been summarized in Table (23). Observe that only ehgrd, erngrd, ehage, bcent10x, prxprest and sexdummy are statistically significant. The latent tie (link) variable is decreasing in proportion of traits within grade, heterogeneity score of age and sexdummies, whereas it is increasing in the relative degree of centrality as well as the heterogeneity score of grade. The predicted probabilities of each of the three outcomes (number of links is 0, 1 or 2) has been summarized in Table (24).

The results for an ordered probit for connections has been summarized in Table (25). Observe that only ehrc5, ehage and infldmn are not statistically significant. The latent tie (connection) variable is decreasing is proportion of traits within grade, proportion of traits within race, proportion of traits within age, the influence domain and sexdummy, whereas it is increasing in the relative degree of centrality as well as the heterogeneity score of grade and prestige index. The predicted probabilities of each of the three outcomes (number of connections is 0, 1 or 2) has been summarized in Table (26).

The results for multinomial probit regressions for both connections and links have been reported in Tables (27) and (29), respectively. The multinomial probit regression for connections has six statistically significant regressors for group 1, which are erngrd, ehrc5, ernrc5, bcent10x, prxprest and sexdummy. The proportion of trait within grade (erngrd), the proportion of trait within race (ernrc5) as well as the sexdummy have a negative impact on probability of a connection belonging to group 2 (where tiec = 1) versus group 1 (where tiec = 0). The regression for group 3 (where tiec = 2) also has six statistically significant regressors, ehgrd, erngrd, ernrc5, bcent10x, reach and sexdummy. The proportion of trait within grade (erngrd), the proportion of trait within race (ernrc5) as well as the sexdummy have a negative impact on probability of a connection belonging to group 3 versus group 1 (where tiec = 0). The predicted probabilities have been summarized in Table (28).

The multinomial probit regression for links has five statistically significant regressors for group 1, which are ehgrd, erngrd, ehage, prxprest and sexdummy. The proportion of trait within grade (erngrd), the heterogeneity score for age (ehage) as well as the sexdummy have a negative impact on probability of a links belonging to group 2 (where tiel = 1) versus group 1 (where tiel = 0). The regression for group 3 (where tiel = 2) has six regressors, ehgrd, erngrd, bcent10x, prxprest, infldmn and sexdummy. The proportion of trait within grade (erngrd) and the sexdummy have a negative impact on probability of a link belonging to group 3 versus group 1 (where tiec = 0). The predicted probabilities have been summarized in Table (30).

4.5 Conclusion

This detailed empirical exercise is intended to serve as an illustration of how standard econometric tools can be efficiently used to predict the probability of connections between nodes. This analysis has shown that for a given network, the distribution of total connections can be a mixture of two or more distributions. The Add Health network data that we looked at, has served as an interesting starting point to analyze more complex interactions within nodes and the factors that affect or govern them. For example, we found that the degree of centrality of a node had strong explanatory powers in explaining the probabilities of tie formation. Moreover, other factors like heterogeneity attributes, prestige index, gender of the respondent and the centrality of a node, significantly affected the probabilities of link formation. Thus, this chapter explores the use standard discrete choice and finite mixture models to empirically estimate the distribution of the links within a network, and in the process the probability of link formation.

The empirical example presented here is simple and may be easily extended to model more complex interactions within a network. In fact, we have examples of models that use advanced econometric tools such as MCMC Metropolis-Hastings algorithm (CFIK) to estimate the posterior probabilities. However, the investigation of how links tend to be distributed over a network awaits to be explored in a more serious manner.

The data set that has been used here provides very rich information on high-school friendships. There could be various ways of perceiving these networks and their sub-networks. Within the large network, one could study sub-networks emerging out of race, ethnicity, age, grade as well as gender affinities. A richer analysis may be carried out by considering a layered or nested kind of a model, where we could have sub-categories and networks within those sub-categories that may be treated as a partition of the bigger network. In this way, we could study the underlying dynamics that result in certain patterns within a network.

In economics, we are always interested in trying to analyze the impact of behavior on economic and social decisions undertaken by agents. This friendship network may provide us with critical information about how 'groups' form within high school as a result of similar racial as well as ethnic attributes. A more detailed study may even help us explain the differences in educational levels and performances due to such classifications that eventually lead to disparity in economic classes. This exercise is just an introduction to the manner in which we can use network theory to study more involved economic problems. Although, on the face of it, the question of whether a respondent chooses a friend from the same race/ethnicity or not may seem to be unrelated to economics, there are deeper underlying questions, directly related to economic problems of interest, that can unravel themselves as we go on. This seems to be a strong justification for pursuing more detailed research/studies in this direction that will eventually lead to the creation of more advanced tools for network analysis, which will be very useful in modeling economic problems.

Conclusion

This dissertation has dealt with three different aspects of economic modeling networks that include a theoretical application, a set of simulation methodologies and a set of econometric applications. These contribute to the literature because they add to the existing tool set in a novel manner.

The use of network formation game theory in order to analyze financial contagion is not only creative in its approach, but also quite unique. It introduces a very efficient method to model complex interactions between economic agents that move simultaneously leading to alterations in the network structure. Thus, with each pair of nodes making a decision, a new network is formed. This process continues until no new network can be formed, which happens when no node has an incentive to change its decision - a situation, more popularly known as a Nash network or a basin of attraction. The contagion model identifies a threshold level of connections that is enough to induce a cascade of pairwise decisions that could lead the network formation process to evolve into a completely collapsed network. As such, it reiterates the importance of a ceiling on the number and nature of connections that any node should have at a given point in time, in order to avoid system wide impact of a shock. This result is interesting from a policy perspective as well.

The research presented here also appreciates the need to establish empirical method for network analysis within the realm of economics. To that extent, a set of simulation methodologies based on copulas have been introduced in order to simulate different kinds of networks. The primary contribution of this research is the performance based comparison between the copula based methods and the traditionally used method of Maximum Entropy. The scenarios where the former outperforms the latter have also been identified.

It is believed that these methodologies and comparisons add to the kind of insights that a researcher can draw from partially available data. In such circumstances, these methods reduce the handicap and help in reconstructing the network for meaningful analysis.

Since networks are generated by connections, the need to model those cannot be emphasized enough. The empirical application using Add Health data presented here, underscores the ease with which standard econometric tools may be used to model the distribution of connections over a network and thus, predict the probabilities of connections between nodes. Different ways to do this have been illustrated using this dataset with the intention of enlightening the readers about the potential applications in that direction.

To conclude, this dissertation provides a balanced set of tools, by means of theory and applications, that may be used to model networks to understand, investigate and deduce interesting inferences from complex economic problems. In that regard, the research presented in this dissertation may be viewed as an exclusive contribution to this area of study.

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Appendix 1: Proofs and Graphs in Chapter 2

Proofs

Lemma 2.1: The following distance function is a metric.

$$d(\omega, \omega') = d(G, G') + d_F(S, S')$$

where,

$$d_F(S, S') = 1 \ if \ S \neq S' \ and \ 0 \ otherwise.$$

and

$$d(G, G') = \sum_{(i,j)\in N\times N} |(\mid G(i,j) \mid - \mid G(j,i) \mid) - (\mid G'(i,j) \mid - \mid G'(j,i) \mid)| + \sum_i \mid X_i - X_i' \mid$$

where the first term is the sum over the net change in the connections between a pair of nodes $\{i, j\}$ in G and G'. The second term is the net change in exposure levels per node.

Proof of Lemma 2.1:

Note that the second term is a discrete metric by definition. As such, we must show that the first term is a metric. In order to do that, we proceed as follows.

By Jackson-Wollinsky rules of network formation, given the current state G, a proposed state G' must be such that

$$G' = G \setminus \{a, (i,j)\} \cup \{a', (i', j')\}$$

where $\{a, (i,j)\}$ is the set of connections being replaced by the set of connections given by $\{a', (i', j')\}$, respectively ((i', j') is the same pair of nodes in G' as (i,j) in G). This implies the following.

$$|G'| = |G| - M + N$$

where $|G|$ is $|\sum_i G^{+}(i)|$ and $|G'|$ is $|\sum_i G^{'+}(i)|$. M is the cardinality of the set $\{a, (i,j)\}$ for all $i \in G$ and N is the cardinality of the set $\{a', (i', j')\}$ for all $i' \in G'$ (note that the nodes don't change, so $i = i'$). Further, for any node $i \in G$ and $i' \in G'$ (which is essentially $i \in G'$) denoted by $G'(i)$, we have:

$$|G^{'+}(i)| = |G^{+}(i)| - c_i + k_i$$

where c_i is the number of outgoing arcs being removed from node i and k_i is the number of outgoing arcs being added to node i. Further,

$$|G^{'}(i,j)| = |G(i,j)| - c_{ij} + k_{ij}$$

where c_{ij} is the number of outgoing arcs from i to j in G being removed and k_{ij} is the number of outgoing arcs from i to j being added in G'. Also,

$$X_i = \left| \frac{|G^{+}(i)|}{|G|} e^{-\frac{|G^{+}(i)|}{|G|}} - \frac{|G^{'+}(i)|}{|G'|} e^{-\frac{|G^{'+}(i)|}{|G'|}} \right|$$

where we have replace d_i with its expression as given in definition (2.8) or expression (2.20), where $|G| = \sum_i |G^{+}(i)|$ and $|G'| = \sum_i |G^{'+}(i)|$.

To show that $d(G, G')$ is a metric.

1. $d(G, G') \geq 0$ as G and G' are finite.

Since,

$$\sum_{\{i,j\}\in N\times N} |(\mid G(i,j)\mid - \mid G(j,i)\mid) - (\mid G^{'}(i,j)\mid - \mid G^{'}(j,i)\mid)| \geq 0$$

$$\sum_{i\in G, i\in G'} \mid X_i - X_i' \mid \geq 0$$

and $d(G,G')$ is the sum of these, therefore, $d(G,G') \geq 0$.

2. $d(G,G') = 0$ if and only if $G = G'$.

   Now suppose that we have $d(G,G') = 0$ and $G \neq G'$. We have the following cases:

   (a) $c_{ij} = k_{ij}$ for all $\{i,j\}$.

   Note that this would mean that number of outgoing nodes for each pair of nodes remains the same in G and G'. This is possible only when $c_{ij} = k_{ij} = 0$ or $c_{ij} = k_{ij} = 1$. Either of which means that the number of connections removed from this pair is exactly the same the number of connections added to this pair! Again, that would mean, $\mid G^{'}(i,j)\mid = \mid G(i,j)\mid$ implying $\mid G^{'}(j,i)\mid = \mid G(j,i)\mid$. Thus, each pair of node is connected in the exact same way in G' as it is in G. This means that $G = G'$.

   For $d(G,G') = 0$, both the first and the second term should be zero. Since, each of these terms is an absolute value, each term must be equal to zero for $d(G,G') = 0$. Note that the first term will be zero if $c_{ij} = k_{ij}$ for all $\{i,j\}$. However, that would mean $G = G'$, which is a contradiction to our assumption. So, irrespective of the second term, as long as $G \neq G'$, we can't have $c_{ij} = k_{ij}$ for all $\{i,j\}$ and the first term will not be zero in this case.

   (b) $-c_{ij} + k_{ij} = -(c_{ji} - k_{ji})$ for all $\{i,j\}$[10].

   Alternatively, the first term could be zero if the above holds. However, if $(c_{ji} - k_{ji}) > 0 \Rightarrow c_{ji} > k_{ji}$ implying $-c_{ij} + k_{ij} < 0$. That is, $k_{ij} < c_{ij}$. $c_{ji} > k_{ji}$ can hold only when $c_{ji} = 1$ and $k_{ji} = 0$. Similarly, $k_{ij} < c_{ij}$ can hold only when

[10]We get this condition by expanding the expressions for $\mid G^{'}(i,j)\mid$ and $\mid G^{'}(j,i)\mid$ in terms of $\mid G(i,j)\mid$ and $\mid G(j,i)\mid$ respectively.

$c_{ij} = 1$ and $k_{ij} = 0$. These mean the number of connections from i to j removed is 1 and the number of connections from j to i is also 1. But this is not possible as two connected nodes can share an arc in one direction only - so connections can't be removed from both directions as there is only direction of connection! Similarly, if $(c_{ji} - k_{ji}) < 0$, by analogy, we can't have connections being added in both directions. So, either ways, this case is not feasible. So, irrespective of the second term, the first term can't be zero in this case. Hence, in this case $d(G, G') \neq 0$.

(c) $M = N$ implying $|G| = |G'|$, $c_i = k_i$.

The above condition implies that $|G'^+(i)| = |G^+(i)|$. Thus, $\frac{|G'^+(i)|}{|G'|} = \frac{|G^+(i)|}{|G|}$ This, in turn implies that for all i $|X_i - X_i'| = 0$. However, note that $c_i = k_i$ implies that $\sum_{j \in G} |\{a, (i, j)\}| = \sum_{j' \in G'} |\{a', (i', j')\}|$. This doesn't yield a zero for the first term unless for each $\{i, j\}$ we have $c_{ij} = k_{ij}$. But, we just saw above that this is possible only when $G = G'$, contrary to our assumption! As such, we still don't have $d(G, G') = 0$.

Now let us check if $d(G, G') = 0$ when $G = G'$. In this case, we have that $c_{ij} = k_{ij} = 0$ for all $\{i, j\}$ as no connections are being removed or added. Alternatively, we could also view it as $c_{ij} = k_{ij} = 1$ for all $\{i, j\}$ where the same connection is being removed and added and $c_{ji} = k_{ji} = 0$ for all $\{j, i\}$ where no connections are being removed or added. This implies that the first term is 0. Further, since $G = G'$, we have $M = N$ and $c_i = k_i$ for all i. Thus, the second term is 0 as well. So, we have that $d(G, G') = 0$ if and only if $G = G'$.

3. $d(G, G') = d(G', G)$.

We have this by symmetry of the absolute value function.

$$
\begin{aligned}
d(G, G') \quad = \quad & \sum_{\{i,j\} \in N \times N} |(| \ G(i,j) \ | - | \ G(j,i) \ |) - (| \ G^{'}(i,j) \ | - | \ G^{'}(j,i) \ |)| \\
& + \sum_{i} | \ X_i - X_i' \ |
\end{aligned}
$$

is the same as

$$
\begin{aligned}
d(G', G) \quad = \quad & \sum_{\{i,j\} \in N \times N} |(| G^{'}(i,j) | - | G^{'}(j,i) |) - (| G(i,j) | - | G(j,i) |)| \\
& + \sum_{i} | X_i' - X_i | \, .
\end{aligned}
$$

4. $d(G, G'') \leq d(G, G') + d(G', G'')$.

For ease of notation let us denote

$$| G(i,j) |= m, \; | G(j,i) |= n$$

$$| G^{'}(i,j) |= m', \; | G^{'}(j,i) |= n'$$

$$| G^{''}(i,j) |= m'', \; | G^{''}(j,i) |= n''$$

$$
\begin{aligned}
d(G, G'') \quad &= \quad \sum | (m-n) - (m''-n'') | + \sum | X_i - X_i'' | \\
&= \quad \sum | (m-n) - (m'-n') + (m'-n') - (m''-n'') | \\
&\quad\quad + \sum | X_i - X_i' + X_i' - X_i'' | \\
&\leq \quad \sum | (m-n) - (m'-n') | + \sum | (m'-n') - (m''-n'') | \\
&\quad\quad + \sum | X_i - X_i' | + \sum | X_i' - X_i'' | \\
&= \quad \sum | (m-n) - (m'-n') | + \sum | X_i - X_i' | \\
&\quad\quad + \sum | (m'-n') - (m''-n'') | + \sum | X_i' - X_i'' | \\
&= \quad d(G, G') + d(G', G'')
\end{aligned}
$$

Thus, $d(G, G')$ is a metric.

It is easy to verify that all four conditions (for being a metric) hold for $d(G, G') + d(S, S')$ since each term satisfies each of those conditions. As such, we have that $d(\omega, \omega')$ is a metric. □

Lemma 2.2: $(w_d^*(\omega), f_N^*(.))$ exists and $w_d^*(\omega)$ is bounded.

Proof of Lemma 2.2: The existence of $(w_d^*(\omega), f_N^*(.))$ can be proved by a direct application of Federgruen (1978).For each d, w_d^* is bounded by real number $\frac{M}{1-\beta_d}$ above and $\frac{-M}{1-\beta_d}$ below, by definition where $\beta_d \in (0,1)$ is the discount factor. $\square$

Lemma 2.3: (A Standard Result in Markov Chain Theory) $\rho_{ij}^* > 0$ *if and only if* $q^n(\omega_i, \omega_j) > 0$ *for some* $n \in \aleph_+$.

Proof of Lemma 2.3:

We have that $\{W_k^* = \omega_j\} \subset \{T_{\omega_j} < \infty\}$ *for some* $n \in \aleph_+$. Further, $\{T_{\omega_j} < \infty\} = \{W_n^* = \omega_j\}$ *for some* $n \in \aleph_+$. Then, by the increasing property of probability and Boole's inequality it follows that for all $n \in \aleph_+$ we have

$$q^k(\omega_i, \omega_j) \leq \rho_{ij} \leq \sum_{n=1}^{\infty} q^n(\omega_i, \omega_j) \quad \square \tag{4.1}$$

Corollary 2.1: *If* $q_{ii} = 0$, *then* $\rho_{ii} = 0$.

Proof of Corollary 2.1: It follows directly from the lemma above. $\square$

Lemma 2.4: *Let* $\{W_n^*\}$ *be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix* Q^* *(as per definition* (2.12)*) with state space* $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$. *Suppose there exists a state* ω_j *with the probability of transition from the current state* ω_i *given by* $q(\omega_i, \omega_j)(i.e., q_{ij})$. *Also, suppose that the transition probability of any other state* ω_k *is given by,* q_{ik}. *Further, if* $q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n}$ *meaning that at the* k*th* *(*$k < \infty$*) step the normalized ratio of net change in the outdegrees and exposure from* ω_i *to* ω_j *is higher than that from the net change in outdegrees and exposure from* ω_l *to* ω_k, *then the following condition holds:*

$$\Pi\{T_{\omega_j}^{*k} < \infty | W_0^* = \omega_i\} > \Pi\{T_{\omega_j}^{*k} < \infty | W_0^* = \omega_i\}$$

In other words, the probability of reaching the state with higher net degree of centrality and exposure from a given initial state is greater than the probability of reaching a state with lower net degree of centrality and exposure from the same initial state given the Markov transition probabilities in finite time, which incorporates the impact of the shock.

Proof of Lemma 2.4:

The proof for the first part follows directly from Lemma 2.3 using increasing property of probability and Boole's inequality. Note that the condition $q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n}$ implies that

$$\frac{q_{ik}}{1-q_{ik}} < q_{ij}^n$$

But by Lemma 2 we have for all n,

$$\rho_{ik} \leq \underbrace{\frac{q_{ik}}{1-q_{ik}}}_{\sum_{l=1}^{\infty} q_{ik}^l} < q_{ij}^n \leq \rho_{ij}$$

Formally,

$$\Pi\{T_{\omega_j}^{*k} < \infty | W_0^* = \omega_i\} > \Pi\{T_{\omega_k}^{*k} < \infty | W_0^* = \omega_i\} \quad \square \tag{4.2}$$

Theorem 2.1: *Suppose we have $q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n}$ for all ω_i, ω_j and ω_k with $n = 1$. Further, Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then all states, with a typical state denoted by ω_r, that lie on the path of the process are such that for all other states, ω_m, given the previous state, ω_l, , with $q_{lm} \geq 0$ and $q_{lr} \geq 0$ such that $q_{lm} < \frac{q_{lr}}{1+q_{lr}}$, we must have $q_{lm} < q_{lr}$. Further, if ω_m and ω_r are basins of attraction, and the state leading to them is given by ω_l, it would still be the case that $q_{lm} < q_{lr}$. In other words, if this 'threshold condition' condition holds at every step, the process always has a tendency to move to a state with higher net change in exposure and centrality.*

Proof of Theorem 2.1: In order to prove the theorem , we only need to prove the following condition.

$$q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n} \text{ implies that } q_{ik} < q_{ij} \text{ for } n = 1$$

Suppose not. Then, $q_{ik} < \frac{q_{ij}}{1+q_{ij}}$ implies that $q_{ik} \geq q_{ij}$.

1. $q_{ik} = q_{ij}$

Then, $q_{ik} < \frac{q_{ij}}{1+q_{ij}}$ and $q_{ik} = q_{ij}$ imply $q_{ij} < 0$! A contradiction since $q_{ij} \geq 0$.

2. $q_{ik} > q_{ij}$

   The given condition implies $q_{ik} + q_{ik}q_{ij} < q_{ij}$. However, since $q_{ik}q_{ij} \geq 0$, under the given condition, we can never have $q_{ik} > q_{ij}$!

   Thus, $q_{ik} < \frac{q_{ij}}{1+q_{ij}}$ can only imply $q_{ik} < q_{ij}$. And therefore, if the threshold condition holds at every step, the process does have a tendency to move to the state with higher net exposure and net centrality as q_{ij} is higher in that case compared to that of a state with lower net exposure and net centrality, by definition. □

Lemma 2.5: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then $q_{ik} < q_{ij}$ if and only if $d(\omega_i, \omega_k) < d(\omega_i, \omega_j)$ for $\omega_k \neq \omega_j$.*

Proof of Lemma 2.5:

Recall the definition of the transition probabilities. For the case where $\omega_m \neq \omega_n$, the transition probability has been defined as

$$q(\omega_m, \omega_n) = \frac{d(\omega_m, \omega_n)}{1 + \sum_{\omega'} d(\omega_m, \omega')}$$

Recall that $\sum_{\omega'} d(\omega_m, \omega') = \sum_{\omega' \neq \omega_m} d(\omega_m, \omega')$ as $d(\omega_m, \omega_m) = 0$ always. We have

$$\begin{aligned} q_{ik} &= \frac{d(\omega_i, \omega_k)}{1 + \sum_{\omega'} d(\omega_i, \omega')} \\ &< q_{ij} \\ &= \frac{d(\omega_i, \omega_j)}{1 + \sum_{\omega'} d(\omega_i, \omega')} \end{aligned}$$

Therefore,

$$\frac{d(\omega_i, \omega_k)}{1 + \sum_{\omega'} d(\omega_i, \omega')} < \frac{d(\omega_i, \omega_j)}{1 + \sum_{\omega'} d(\omega_i, \omega')}$$

implying

$$d(\omega_i, \omega_k) < d(\omega_i, \omega_j) \quad \square$$

Theorem 2.2: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Further, let ω_{cb} denote the* state of collapse with bailout. *Then, for any ω_i and ω_j, such that $q_{icb} > 0$ and $q_{ij} > 0$, we must have$q_{icb} > q_{ij}$ implying that the transition probability of going to the state of complete collapse with bailout is higher than that of going to any other state from a given state ω_i from where ω_{cb} and ω_j are both reachable.*

Proof of Theorem 2.2:

For a state of collapse with bail out denoted by ω_{cb}, we will show that

$$d(\omega_i, \omega_j) < d(\omega_i, \omega_{cb})$$

implies

$$q_{ij} < q_{icb}$$

for $\omega_i \neq \omega_{cb}, \omega_j$ and $\omega_{cb} \neq \omega_j$.

First we will show that the first term of the distance function is a multiple of 2. Recall that the distance function is given by

$$\sum_{(i,j)\in N\times N, G\neq G'} \mid (\mid G(i,j) \mid - \mid G(j,i) \mid) - (\mid G'(i,j) \mid - \mid G'(j,i) \mid) \mid$$

where $\mid G(i,j) \mid$ denotes the cardinality of outgoing arcs from i to j and $\mid (.) - (.) \mid$ denotes the absolute value of the net difference. Note that when we expand this, we will get the following for each pair of nodes i and j

$$\mid (\mid G(i,j) \mid - \mid G(j,i) \mid) - (\mid G'(i,j) \mid - \mid G'(j,i) \mid) \mid$$

$$+ \mid (\mid G(j,i) \mid - \mid G(i,j) \mid) - (\mid G'(j,i) \mid - \mid G'(i,j) \mid) \mid$$

This may be rewritten as

$$\mid (\mid G(i,j) \mid - \mid G'(i,j) \mid) + (\mid G'(j,i) \mid - \mid G(j,i) \mid) \mid$$

$$+ \mid (\mid G(j,i) \mid - \mid G'(j,i) \mid) + (\mid G'(i,j) \mid - \mid G(i,j) \mid) \mid$$

Thus, the above is in fact equal to $2\times \mid (\mid G(i,j) \mid - \mid G'(i,j) \mid) + (\mid G'(j,i) \mid - \mid G(j,i) \mid) \mid$ because we are looking at the absolute values. Using this, we have that the first term of the distance function is a multiple of 2.

In what follows we only consider the case for states from where it is possible to go to G^{cb} because if it is not a feasible proposal the probability of transition is 0. Also recall, our goal is to show that $q(\omega, \omega_{cb}) > q(\omega, \omega')$ for all ω, ω'. We will begin by showing that $d(\omega, \omega_{cb}) > d(\omega, \omega')$

Suppose, $d(G, G^{cb}) < d(G, G')$. Then,

$$d(G, G^{cb}) = \sum_{(i,j)\in N\times N, G\neq G^{cb}} \mid (\mid G(i,j) \mid - \mid G(j,i) \mid) - (\mid G^{cb}(i,j) \mid - \mid G^{cb}(j,i) \mid) \mid$$

This can be written as

$$\begin{aligned} d(G, G^{cb}) &= \sum_{(i,j)\in N\backslash F\times N\backslash F, G\neq G^{cb}} \mid (\mid G(i,j) \mid - \mid G(j,i) \mid) - (\mid G^{cb}(i,j) \mid - \mid G^{cb}(j,i) \mid) \mid \\ &\quad + \sum_{(i\in N, G\neq G^{cb})} \mid (\mid G(i,F) \mid - \mid G(F,i) \mid) - (\mid G^{cb}(F,j) \mid - \mid G^{cb}(F,i) \mid) \mid \end{aligned}$$

Since, only one coalition gets to move at a time, it must be the case that in order to get to G^{cb}, $(n-1)$ nodes have already arcs directed from the Fed towards them and only $\{n, F\}^{th}$ coalition must decide what to do. It essentially boils down to two choices then, either Fed bails out or doesn't. In case, the Fed does bail out, then for at least that coalition the link will be reversed and directed towards it. This means that, everything else the same

$$\sum_{(i\in N, G\neq G^{cb})} \mid (\mid G(i,F) \mid - \mid G(F,i) \mid) - (\mid G^{cb}(F,j) \mid - \mid G^{cb}(F,i) \mid) \mid = 4$$

whereas, if the Fed doesn't bailout, the link drops out yielding

$$\sum_{(i \in N, G \neq G^{cb})} \mid (\mid G(i,F) \mid - \mid G(F,i) \mid) - (\mid G^{cb}(F,j) \mid - \mid G^{cb}(F,i) \mid) \mid = 2$$

Thus, if $(n-1)$ nodes have been bailed out, then the probability of the nth node being bailed out is higher as well. Thus, yielding that given G, $d(G, G^{cb}) > d(G, G')$ for any G, G' such that $G' \in \Phi(\omega)$.

Next we will show that $q_{icb} > q_{ij}$ for any ω_i from where ω_j and ω_{cb} are reachable.

We already showed that $d(G, G')$ is a multiple of 2. Now, the term $|X - X'|$, which is between 0 and 1. Given that at each step, only one coalition gets to make the move, we have that at each step, there are at most 2 nodes whose centrality levels and hence, exposure levels change. Therefore, we have $\sum_{i=1}^{2} |X_i - X_i'|$ and the rest of the differences are 0. This implies that $\sum_{i=1}^{2} |X_i - X_i'| \in [0, 2e^{-1}]$.

For G, G' and G'' such that

$$d(G, G'') > d(G, G') \tag{4.3}$$

we should have

$$d(G, G'') + \underbrace{\sum_{i=1}^{2} |X_i - X_i''|}_{\in[0,2e^{-1}]} \geq d(G, G') + \underbrace{\sum_{i=1}^{2} |X_i - X_i'|}_{\in[0,2e^{-1}]} \tag{4.4}$$

Since, $d(G, G')$ for any G' is a multiple of 2, if $d(G, G'') > d(G, G')$ then $d(G, G'') - d(G, G') \geq 2$. Thus, the value of $\underbrace{\sum_{i=1}^{2} |X_i - X_i''|}_{\in[0,2e^{-1}]}$ for either X_i' or X_i'' is not big enough to reverse the inequality above.

Now consider the term $d_F(S, S')$. Suppose, $S \neq S''$. Then, we have

$$d(G, G'') + \underbrace{\sum_{i=1}^{2} |X_i - X_i''|}_{\in[0,2e^{-1}]} + \underbrace{d_F(S, S'')}_{1} > d(G, G') + \underbrace{\sum_{i=1}^{2} |X_i - X_i'|}_{\in[0,2e^{-1}]} + \underbrace{d_F(S, S')}_{1\, or\, 0} \tag{4.5}$$

Suppose now that $S = S''$

$$d(G,G'') + \underbrace{\sum_{i=1}^{2} |X_i - X_i''|}_{\in[0,2e^{-1}]} + \underbrace{d_F(S,S'')}_{0} > d(G,G') + \underbrace{\sum_{i=1}^{2} |X_i - X_i'|}_{\in[0,2e^{-1}]} + \underbrace{d_F(S,S')}_{1\,or\,0} \tag{4.6}$$

Since, $\underbrace{\sum_{i=1}^{2} |X_i - X_i'|}_{\in[0,2e^{-1}]} < 1$, the right hand side of the equation above is going to be strictly less that 3. Further, we have $d(G,G'') > d(G,G')$. So, it must be the case that $d(G,G'') \geq 4$. If that were not true, then $d(G,G'') = 2$ would imply that $d(G,G') = 0$, in which case the inequality (15) holds trivially. Since $d(G,G'') > 4$, and the right hand side is less than 3, we have that $d(\omega,\omega'') > d(\omega,\omega')$.

Using this, and the fact that $d(G,G^{cb}) > d(G,G')$, we then have that

$$d(\omega,\omega_{cb}) > d(\omega,\omega') > 0.$$

As such, we have that for states ω_i, ω_j and ω_{cb} such that $q_{icb} > 0$ and $q_{ij} > 0$, we must have $q_{icb} > q_{ij}$ for all ω_i from where ω_{cb} is reachable and for all ω_j, which is reachable from ω_i. $\square$

Corollary 2.2: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then, if we have that for states ω_i, ω_j and ω_k such that $d(G,G_k) > d(\omega,\omega_j) > 0$ and $q_{ik} > 0$ and $q_{ij} > 0$, we must have $q_{ik} > q_{ij}$.*

Proof for Corollary 2.2: This corollary is a direct implication of a step in the proof of theorem 2 where we show that for networks G, G' and G'' the condition $d(G,G'') > d(G,G') > 0$ implies that $d(\omega,\omega'') > d(\omega,\omega') > 0$. This, by Lemma 2.5, in turn, implies that $q(\omega''|\omega,a) > q(\omega'|\omega,a)$.

Theorem 2.3: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then* *if the condition $q_{ij} < q_{ic}$ holds for all ω_i including $\omega_i = \omega_c$, then state of collapse is a basin of attraction. That is, $\rho_{cc} = 1$.*

Proof of Theorem 3: Suppose not. Then ω_{cb} is transient. Then, there must exist another state ω_j such that $q_{cbj} > 0$. By the definition of our endogenous transition probabilities, this would mean that $q_{cbj} > q_{cbcb}$. However, if for any ω_i and ω_j, such that ω_c is reachable from ω_i we have $q_{ij} < q_{ic}$, then this must hold for the state ω_c itself as well, since every state is reachable to itself. Thus, replacing i with a c we get $q_{cj} > q_{cc}$, which is a contradiction to the condition $q_{cj} > q_{cc}$. Thus, we must have $q_{cj} < q_{cc}$ for any ω_j reachable from ω_c.

Next, we will show that q_{cc} must, in fact, be equal to 1. If we have $q_{cj} < q_{cc}$, by the definition of our probabilities we must have $\frac{d(\omega_c,\omega_j)}{D} < \frac{1}{D}$ where D denotes the appropriate denominator. However, not that this is not possible, unless $d(\omega_c, \omega_j$ is 0! This implies that $q_{cc} = 1$ and there is no other ω_j that the process can go to from ω_c. $\square$

Lemma 2.6: *If ω_x is a recurrent state and ω_x leads to ω_y (i.e., $q_{xy} > 0$) then ω_y is recurrent and $\rho_{xy} = \rho_{yx} = 1$.*

Proof of Lemma 2.6: This can be proved directly by using the Markov and time homogeneous properties of our Markov process. From these properties, we have that $1 - \rho_{xx} \geq q_{xy}(1 - \rho_{yx}) \geq 0$. Since $\rho_{xx} = 1$, it must be that $\rho_{yx} = 1$ for this to hold. Further, there exists positive integers s and t such that for all $n \in \aleph$,

$$q_{yy}^{s+t+n} \geq q_{yx}^{s} . q_{xx}^{n} . q_{xy}^{t} \tag{4.7}$$

Summing over n on both sides yields $\sum_1^\infty q_{yy}^{s+t+n} = \infty$. Hence, ω_y is recurrent. We already have $\sum_1^\infty q_{xx}^n$ as ω_x is recurrent. Similarly, we can show that $\rho_{xy} = 1$. $\square$

Theorem 2.4:*Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then all basins of attraction are singleton sets.*

Proof of Theorem 2.4:

Suppose not. Then there exists at least one basin of attraction with more than one element. For convenience, say there exists a basin A with two elements - states ω_j and ω_k such that $\omega_j \neq \omega_k$.

Step 1: First we will show that $q_{jk} = q_{kj}$ where $q_{jk}, q_{kj} > 0$.

Suppose not. The, for $q_{jk} \neq q_{kj}$ we must have

$$\frac{d(\omega_j,\omega_k)}{1 + d(\omega_j,\omega_k) + d(\omega_j,\omega_j)} \neq \frac{d(\omega_k,\omega_j)}{1 + d(\omega_k,\omega_j) + d(\omega_k,\omega_k)}$$

Since $d(.)$ is a metric, by symmetry $d(\omega_j,\omega_k) = d(\omega_k,\omega_j)$. Therefore, the above inequality must imply that $d(\omega_j,\omega_j) \neq d(\omega_k,\omega_k)$. However, that can not be true given that $d(\omega_j,\omega_j) = d(\omega_k,\omega_k) = 0$! Therefore, we must have that $q_{jk} = q_{kj}$.

Step 2: From the proof of Lemma 2.6 we know that there exist positive integers s and t such that for all $n \in \aleph$

$$q_{kk}^{s+t+n} \geq q_{kj}^{s} q_{jj}^{n} q_{jk}^{t}$$

Using $q_{jk} = q_{kj}$ and $q_{jj} = q_{kk}$, we have that

$$q_{kk}^{s+t} \geq q_{jk}^{s+t} \quad \Rightarrow \quad q_{kk} \geq q_{jk}$$

implying

$$\frac{1}{1 + d(\omega_k,\omega_j) + d(\omega_k,\omega_k)} \geq \frac{d(\omega_j,\omega_k)}{1 + d(\omega_j,\omega_k) + d(\omega_j,\omega_j)}$$

Thus, $1 \geq d(\omega_j,\omega_k)$. However, recall that $d(\omega_j,\omega_k)$ is given by

$$\underbrace{\sum \mid (\mid G(i,j) - G(j,i) \mid) - (\mid G'(i,j) - G'(j,i) \mid) \mid}_{\geq 0} + \underbrace{\sum \mid X_i - X_i' \mid}_{\geq 0} + \underbrace{d_F(S,S')}_{=1, S \neq S'} > 1$$

Therefore, $1 \geq d(\omega_j,\omega_k)$ and the above hold simultaneously only when $\omega_j = \omega_k$. Else, $\rho_{kj} \neq 1$ or $\rho_{jk} \neq 1$. Both imply that under the endogenous transition probabilities, we can only have singleton basins of attraction. □

Theorem 2.5: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and* *coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Then the state space can be decomposed into a finite number of disjoint basins of attraction and a disjoint transient sets. In particular, this decomposition is given by.*

$$\Omega = \{\cup_{i=1}^{N} H_i\} \cup T \tag{4.8}$$

where H_i refers to the ith basin and T refers to the transient sets.

Proof of Theorem 2.5: This follows directly from Theorem 2.2 where we showed that the basins of this network formation game are singletons. As such, it is not possible to decompose them further. Thus, the set $\{\cup_{i=1}^{N} H_i\}$ contains all the singleton basins, and the rest of states are the transient states.

Corollary 2.3: *Let $\{W_n^*\}$ be an endogenous Markov process of the financial network and coalition formation governed by equilibrium transition matrix Q^* as defined in (2.12). Suppose that ω_k and ω_j are both basins of attraction. Then under the threshold condition $q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n}$ for $n = 1$ for some state ω_i, we have $q_{ik} < q_{ij}$ and $\rho_{ik} < \rho_{ij}$*

Proof of Corollary 2.3: The result that $q_{ik} < \frac{q_{ij}^n}{1+q_{ij}^n}$ with $n = 1$ implies $q_{ik} < q_{ij}$ follows directly from Lemma 5. Further, the threshold condition for $n = 1$ implies

$$\frac{q_{ik}}{1 - q_{ik}} < q_{ij}$$

That implies,

$$\rho_{ik} \leq \sum_{l=1}^{\infty} < q_{ij} \leq \rho_{ij}$$

where we have used the inequality in the proof of Lemma 2.3 that holds for all n, hence for $n = 1$ as well. Thus, we have that $q_{ik} < \frac{q_{ij}}{1+q_{ij}}$ implies $\rho_{ik} < \rho_{ij}$. Note that it is a special case of Lemma 2.4 (with $n = 1$). $\square$

Tables

TABLE 1: ALL POSSIBLE STATES

$\omega_1 = (G_0, S_1)$	$\omega_2 = (G_0, S_2)$	$\omega_3 = (G_1, S_1)$	$\omega_4 = (G_2, S_1)$
$\omega_5 = (G_3, S_1)$	$\omega_6 = (G_9, S_2)$	$\omega_7 = (G_{10}, S_2)$	$\omega_8 = (G_{11}, S_2)$
$\omega_9 = (G_4, S_1)$	$\omega_{10} = (G_4, S_2)$	$\omega_{11} = (G_5, S_2)$	$\omega_{12} = (G_6, S_2)$
$\omega_{13} = (G_7, S_2)$	$\omega_{14} = (G_8, S_1)$	$\omega_{15} = (G_8, S_2)$	$\omega_{16} = (G_{12}, S_1)$
$\omega_{17} = (G_{13}, S_1)$	$\omega_{18} = (G_{14}, S_1)$	$\omega_{19} = (G_{15}, S_3)$	$\omega_{20} = (G_{16}, S_4)$
$\omega_{21} = (G_{17}, S_5)$	$\omega_{22} = (G_{18}, S_5)$	$\omega_{23} = (G_{19}, S_5)$	$\omega_{24} = (G_{20}, S_3)$
$\omega_{25} = (G_{20}, S_4)$	$\omega_{26} = (G_{21}, S_4)$	$\omega_{27} = (G_{22}, S_3)$	$\omega_{28} = (G_{23}, S_4)$
$\omega_{29} = (G_{24}, S_5)$	$\omega_{30} = (G_{25}, S_4)$	$\omega_{31} = (G_{26}, S_3)$	$\omega_{32} = (G_{27}, S_3)$
$\omega_{33} = (G_{28}, S_3)$	$\omega_{34} = (G_{29}, S_4)$	$\omega_{35} = (G_{30}, S_5)$	$\omega_{36} = (G_{42}, S_5)$
$\omega_{37} = (G_{20}, S_4)$	$\omega_{38} = (G_{26}, S_3)$	$\omega_{39} = (G_{19}, S_5)$	$\omega_{40} = (G_{23}, S_4)$
$\omega_{41} = (G_{23}, S_5)$	$\omega_{42} = (G_{35}, S_3)$	$\omega_{43} = (G_{35}, S_4)$	$\omega_{44} = (G_{35}, S_5)$
$\omega_{45} = (G_{36}, S_3)$	$\omega_{46} = (G_{36}, S_4)$	$\omega_{47} = (G_{36}, S_5)$	$\omega_{48} = (G_{37}, S_3)$
$\omega_{49} = (G_{37}, S_4)$	$\omega_{50} = (G_{37}, S_5)$	$\omega_{51} = (G_{38}, S_1)$	$\omega_{52} = (G_{39}, S_2)$
$\omega_{53} = (G_{40}, S_3)$	$\omega_{54} = (G_{40}, S_4)$	$\omega_{55} = (G_{41}, S_4)$	$\omega_{56} = (G_{41}, S_5)$
$\omega_{57} = (G_{42}, S_3)$	$\omega_{58} = (G_{42}, S_4)$	$\omega_{59} = (G_{43}, S_3)$	$\omega_{60} = (G_{43}, S_4)$
$\omega_{61} = (G_{44}, S_4)$	$\omega_{62} = (G_{44}, S_5)$	$\omega_{63} = (G_{45}, S_3)$	$\omega_{64} = (G_{45}, S_5)$
$\omega_{65} = (G_{cc}, S_0)$	$\omega_{66} = (G_{cb}, S_6)$	$\omega_{67} = (G_{24}, S_4)$	$\omega_{68} = (G_{18}, S_3)$
$\omega_{69} = (G_{40}, S_5)$	$\omega_{70} = (G_{41}, S_3)$		

where S_0 refers to any coalition, $S_1 = \{b_1, b_3\}, S_2 = \{b_2, b_3\}, S_3 = \{b_1, F\}, S_4 = \{b_2, F\}, S_5 = \{b_3, F\}$ and $S_6 = \{b_i, F\}$. Note that we don't need to include a coalition for all combinations of players as except those included here, the others are never chosen by the rules of network formation and the game.

TABLE 2: ALL POSSIBLE PROPOSALS FOR EACH STATE

States	Proposals
ω_1	ω_1, ω_3, ω_4, ω_5
ω_2	ω_2, ω_6, ω_7, ω_8
ω_3	ω_3, ω_{10}, ω_{11}, ω_{12}
ω_4	ω_4, ω_{13}
ω_5	ω_5, ω_{15}
ω_6	ω_6, ω_9, ω_{16}, ω_{17}
ω_7	ω_7, ω_{18}
ω_8	ω_8, ω_{14}
ω_9	ω_9
ω_{10}	ω_{10}
ω_{11}	ω_{11}, ω_{15}
ω_{12}	ω_{12}, ω_{22}, ω_{23}
ω_{13}	ω_{13}, ω_{20}
ω_{14}	ω_{14}, ω_{19}, ω_{20}, ω_{21}, ω_{33}, ω_{34}, ω_{35}
ω_{15}	ω_{15}, ω_{19}, ω_{20}, ω_{21}, ω_{33}, ω_{34}, ω_{35}
ω_{16}	ω_{16}, ω_{41}, ω_{44}
ω_{17}	ω_{17}, ω_{24}, ω_{27}, ω_{29}, ω_{41}
ω_{18}	ω_{18}, ω_{24}, ω_{27}, ω_{29}, ω_{41}
ω_{19}	ω_{19}, ω_{22}, ω_{23}, ω_{25}, ω_{26}
ω_{20}	ω_{20}, ω_{24}, ω_{27}, ω_{29}, ω_{41}
ω_{21}	ω_{21}, ω_{28}, ω_{30}, ω_{31}, ω_{32}
ω_{22}	ω_{22}, ω_{43}, ω_{55}
ω_{23}	ω_{23}, ω_{46}, ω_{65}
ω_{24}	ω_{24}, ω_{44}, ω_{65}
ω_{25}	ω_{25}, ω_{44}, ω_{65}
ω_{26}	ω_{26}, ω_{47}, ω_{56}
ω_{27}	ω_{27}, ω_{50}, ω_{58}
ω_{28}	ω_{28}, ω_{48}, ω_{65}
ω_{29}	ω_{29}, ω_{42}, ω_{57}
ω_{30}	ω_{30}, ω_{45}, ω_{53}
ω_{31}	ω_{31}, ω_{46}, ω_{65}
ω_{32}	ω_{32}, ω_{49}, ω_{54}

States	Proposals
ω_{33}	ω_{33}, ω_{37}, ω_{39}, ω_{60}, ω_{64}
ω_{34}	ω_{34}, ω_{38}, ω_{40}, ω_{59}, ω_{62}
ω_{35}	ω_{35}, ω_{61}, ω_{63}, ω_{67}, ω_{68}
ω_{36}	ω_{36}
ω_{37}	ω_{37}, ω_{36}, ω_{50}
ω_{38}	ω_{38}, ω_{47}, ω_{56}
ω_{39}	ω_{39}, ω_{49}, ω_{54}
ω_{40}	ω_{40}, ω_{45}, ω_{53}
ω_{41}	ω_{41}, ω_{48}, ω_{65}
ω_{42}	ω_{42}
ω_{43}	ω_{43}
ω_{44}	ω_{44}, ω_{51}, ω_{52}
ω_{45}	ω_{45}
ω_{46}	ω_{46}
ω_{47}	ω_{47}
ω_{48}	ω_{48}
ω_{49}	ω_{49}
ω_{50}	ω_{50}
ω_{51}	ω_{51}, ω_{10}
ω_{52}	ω_{52}, ω_{9}
ω_{53}	ω_{53}
ω_{54}	ω_{54}
ω_{55}	ω_{55}
ω_{56}	ω_{56}
ω_{57}	ω_{57}
ω_{58}	ω_{58}
ω_{59}	ω_{59}, ω_{66}, ω_{69}
ω_{60}	ω_{60}, ω_{66}, ω_{69}
ω_{61}	ω_{61}, ω_{66}, ω_{70}
ω_{62}	ω_{62}, ω_{66}, ω_{70}
ω_{63}	ω_{63}, ω_{58}, ω_{66}
ω_{64}	ω_{64}, ω_{58}, ω_{66}
ω_{65}	ω_{65}
ω_{66}	ω_{66}
ω_{67}	ω_{67}, ω_{42}, ω_{57}
ω_{68}	ω_{68}, ω_{43}, ω_{55}
ω_{69}	ω_{69}
ω_{70}	ω_{70}

TABLE 3: PAYOFFS FOR ALL POSSIBLE STATES

$(i_1, i_2, i_3,\ F)$

$\omega_1 = (0, -1, -1, 0)$	$\omega_2 = (-1, 0, -1, 0)$	$\omega_3 = (3, -1, 2, 0)$	$\omega_4 = (0, -1, -6, -1)$
$\omega_5 = (-5, -1, -5, -2)$	$\omega_6 = (-1, 3, 2, 0)$	$\omega_7 = (-1, 0, -6, -1)$	$\omega_8 = (-1, -5, -5, -2)$
$\omega_9 = (3, 1, 5, 0)$	$\omega_{10} = (1, 2, 5, 0)$	$\omega_{11} = (1, 0, -3, -1)$	$\omega_{12} = (1, -4, -2, -2)$
$\omega_{13} = (-1, -4, -5, -2)$	$\omega_{14} = (-4, -5, -4, -3)$	$\omega_{15} = (-5, -4, -4, -3)$	$\omega_{16} = (0, 1, -3, -1)$
$\omega_{17} = (-4, 1, -2, -2)$	$\omega_{18} = (-4, -1, -5, -2)$	$\omega_{19} = (-9, -5, -5, -1)$	$\omega_{20} = (-5, -9, -5, -1)$
$\omega_{21} = (-5, -5, -9, -1)$	$\omega_{22} = (-10, -5, 3, 1)$	$\omega_{23} = (-10, -5, -9, 0)$	$\omega_{24} = (-9, -10, -5, 0)$
$\omega_{25} = (-10, -9, -5, 0)$	$\omega_{26} = (-10, 2, -5, 1)$	$\omega_{27} = (2, -10, -5, 1)$	$\omega_{28} = (-5, -9, -10, 0)$
$\omega_{29} = (-5, -10, 3, 1)$	$\omega_{30} = (-5, 2, -10, 1)$	$\omega_{31} = (-9, -5, -10, 0)$	$\omega_{32} = (2, -5, -10, 1)$
$\omega_{33} = (2, -5, -5, 0)$	$\omega_{34} = (-5, 2, -5, 0)$	$\omega_{35} = (-5, -5, 3, 0)$	$\omega_{36} = (1, -9, 1, 2)$
$\omega_{37} = (1, -9, -5, 0)$	$\omega_{38} = (-9, 1, -5, 0)$	$\omega_{39} = (1, -5, -9, 0)$	$\omega_{40} = (-5, 1, -9, 0)$
$\omega_{41} = (-5, -10, -9, 0)$	$\omega_{42} = (-9, -10, 2, 2)$	$\omega_{43} = (-10, -9, 2, 2)$	$\omega_{44} = (-10, -10, 3, 2)$
$\omega_{45} = (-9, 1, -10, 2)$	$\omega_{46} = (-10, 2, -10, 2)$	$\omega_{47} = (-10, 1, -9, 2)$	$\omega_{48} = (2, -10, -10, 2)$
$\omega_{49} = (1, -9, -10, 2)$	$\omega_{50} = (1, -10, -9, 2)$	$\omega_{51} = (-7, -10, 5, 0)$	$\omega_{52} = (-10, -7, 5, 0)$
$\omega_{53} = (2, 1, -10, 0)$	$\omega_{54} = (1, 2, -10, 3)$	$\omega_{55} = (-10, 2, 2, 3)$	$\omega_{56} = (-10, 1, 3, 3)$
$\omega_{57} = (2, -10, 2, 3)$	$\omega_{58} = (1, -9, 2, 3)$	$\omega_{59} = (2, 1, -5, 2)$	$\omega_{60} = (1, 2, -5, 2)$
$\omega_{61} = (-5, 2, 2, 2)$	$\omega_{62} = (-5, 1, 3, 2)$	$\omega_{63} = (2, -5, 2, 2)$	$\omega_{64} = (1, -5, 3, 2)$
$\omega_{65} = (-10, -10, -10, 0)$	$\omega_{66} = (2, 2, 3, 4)$	$\omega_{67} = (-5, -9, 1, 0)$	$\omega_{68} = (-9, -5, 1, 0)$
$\omega_{69} = (1, 1, -9, 2)$	$\omega_{70} = (-9, 1, 1, 2)$		

where S_0 refers to any coalition, $S_1 = \{b_1, b_3\}, S_2 = \{b_2, b_3\}, S_3 = \{b_1, F\}, S_4 = \{b_2, F\}, S_5 = \{b_3, F\}$ and $S_6 = \{b_i, F\}$. Note that we don't need to include a coalition for all combinations of players as except those included here, the others are never chosen by the rules of network formation and the game.

TABLE 4: EQUILIBRIUM PROPOSALS

	i_1	i_2	i_3	F
ω_1	$f_1^*(\omega_1) = \omega_3$	$f_2^*(\omega_1) = \omega_1$	$f_3^*(\omega_1) = \omega_3$	$f_F^*(\omega_1) = \omega_1$
ω_2	$f_1^*(\omega_2) = \omega_2$	$f_2^*(\omega_2) = \omega_6$	$f_3^*(\omega_2) = \omega_6$	$f_F^*(\omega_2) = \omega_2$
ω_3	$f_1^*(\omega_3) = \omega_3$	$f_2^*(\omega_3) = \omega_3$	$f_3^*(\omega_3) = \omega_{10}$	$f_F^*(\omega_3) = \omega_3$
ω_4	$f_1^*(\omega_4) = \omega_4$	$f_2^*(\omega_4) = \omega_4$	$f_3^*(\omega_4) = \omega_{13}$	$f_F^*(\omega_4) = \omega_4$
ω_5	$f_1^*(\omega_5) = \omega_{15}$	$f_2^*(\omega_5) = \omega_5$	$f_3^*(\omega_5) = \omega_{15}$	$f_F^*(\omega_5) = \omega_5$
ω_6	$f_1^*(\omega_6) = \omega_6$	$f_2^*(\omega_6) = \omega_6$	$f_3^*(\omega_6) = \omega_9$	$f_F^*(\omega_6) = \omega_6$
ω_7	$f_1^*(\omega_7) = \omega_7$	$f_2^*(\omega_7) = \omega_7$	$f_3^*(\omega_7) = \omega_{18}$	$f_F^*(\omega_7) = \omega_7$
ω_8	$f_1^*(\omega_8) = \omega_8$	$f_2^*(\omega_8) = \omega_{14}$	$f_3^*(\omega_8) = \omega_{14}$	$f_F^*(\omega_8) = \omega_8$
ω_9	$f_1^*(\omega_9) = \omega_9$	$f_2^*(\omega_9) = \omega_9$	$f_3^*(\omega_9) = \omega_9$	$f_F^*(\omega_9) = \omega_9$
ω_{10}	$f_1^*(\omega_{10}) = \omega_{10}$	$f_2^*(\omega_{10}) = \omega_{10}$	$f_3^*(\omega_{10}) = \omega_{10}$	$f_F^*(\omega_{10}) = \omega_{10}$
ω_{11}	$f_1^*(\omega_{11}) = \omega_{11}$	$f_2^*(\omega_{11}) = \omega_{11}$	$f_3^*(\omega_{11}) = \omega_{11}$	$f_F^*(\omega_{11}) = \omega_{11}$
ω_{12}	$f_1^*(\omega_{12}) = \omega_{12}$	$f_2^*(\omega_{12}) = \omega_{12}$	$f_3^*(\omega_{12}) = \omega_{22}$	$f_F^*(\omega_{12}) = \omega_{12}$
ω_{13}	$f_1^*(\omega_{13}) = \omega_{13}$	$f_2^*(\omega_{13}) = \omega_{13}$	$f_3^*(\omega_{13}) = \omega_{13}$	$f_F^*(\omega_{13}) = \omega_{13}$
ω_{14}	$f_1^*(\omega_{14}) = \omega_{33}$	$f_2^*(\omega_{14}) = \omega_{14}$	$f_3^*(\omega_{14}) = \omega_{35}$	$f_F^*(\omega_{14}) = \omega_{14}$
ω_{15}	$f_1^*(\omega_{15}) = \omega_{15}$	$f_2^*(\omega_{15}) = \omega_{34}$	$f_3^*(\omega_{15}) = \omega_{35}$	$f_F^*(\omega_{15}) = \omega_{15}$
ω_{16}	$f_1^*(\omega_{16}) = \omega_{16}$	$f_2^*(\omega_{16}) = \omega_{16}$	$f_3^*(\omega_{16}) = \omega_{44}$	$f_F^*(\omega_{16}) = \omega_{16}$
ω_{17}	$f_1^*(\omega_{17}) = \omega_{27}$	$f_2^*(\omega_{17}) = \omega_{17}$	$f_3^*(\omega_{17}) = \omega_{29}$	$f_F^*(\omega_{17}) = \omega_{17}$
ω_{18}	$f_1^*(\omega_{18}) = \omega_{27}$	$f_2^*(\omega_{18}) = \omega_{18}$	$f_3^*(\omega_{18}) = \omega_{19}$	$f_F^*(\omega_{18}) = \omega_{18}$
ω_{19}	$f_1^*(\omega_{19}) = \omega_{19}$	$f_2^*(\omega_{19}) = \omega_{19}$	$f_3^*(\omega_{19}) = \omega_{19}$	$f_F^*(\omega_{19}) = \omega_{22}, \omega_{26}$
ω_{20}	$f_1^*(\omega_{20}) = \omega_{20}$	$f_2^*(\omega_{20}) = \omega_{24}$	$f_3^*(\omega_{20}) = \omega_{20}$	$f_F^*(\omega_{20}) = \omega_{27}, \omega_{29}$
ω_{21}	$f_1^*(\omega_{21}) = \omega_{21}$	$f_2^*(\omega_{21}) = \omega_{21}$	$f_3^*(\omega_{21}) = \omega_{21}$	$f_F^*(\omega_{21}) = \omega_{30}, \omega_{32}$
ω_{22}	$f_1^*(\omega_{22}) = \omega_{22}$	$f_2^*(\omega_{22}) = \omega_{22}$	$f_3^*(\omega_{22}) = \omega_{22}$	$f_F^*(\omega_{22}) = \omega_{55}$
ω_{23}	$f_1^*(\omega_{23}) = \omega_{23}$	$f_2^*(\omega_{23}) = \omega_{23}$	$f_3^*(\omega_{23}) = \omega_{23}$	$f_F^*(\omega_{23}) = \omega_{46}$
ω_{24}	$f_1^*(\omega_{24}) = \omega_{24}$	$f_2^*(\omega_{24}) = \omega_{24}$	$f_3^*(\omega_{24}) = \omega_{24}$	$f_F^*(\omega_{24}) = \omega_{44}$
ω_{25}	$f_1^*(\omega_{25}) = \omega_{25}$	$f_2^*(\omega_{25}) = \omega_{25}$	$f_3^*(\omega_{25}) = \omega_{25}$	$f_F^*(\omega_{25}) = \omega_{44}$
ω_{26}	$f_1^*(\omega_{26}) = \omega_{26}$	$f_2^*(\omega_{26}) = \omega_{56}$	$f_3^*(\omega_{26}) = \omega_{26}$	$f_F^*(\omega_{26}) = \omega_{56}$
ω_{27}	$f_1^*(\omega_{27}) = \omega_{27}$	$f_2^*(\omega_{27}) = \omega_{27}$	$f_3^*(\omega_{27}) = \omega_{58}$	$f_F^*(\omega_{27}) = \omega_{58}$
ω_{28}	$f_1^*(\omega_{28}) = \omega_{28}$	$f_2^*(\omega_{28}) = \omega_{28}$	$f_3^*(\omega_{28}) = \omega_{28}$	$f_F^*(\omega_{28}) = \omega_{48}$
ω_{29}	$f_1^*(\omega_{29}) = \omega_{29}$	$f_2^*(\omega_{29}) = \omega_{29}$	$f_3^*(\omega_{29}) = \omega_{57}$	$f_F^*(\omega_{29}) = \omega_{57}$
ω_{30}	$f_1^*(\omega_{30}) = \omega_{30}$	$f_2^*(\omega_{30}) = \omega_{30}$	$f_3^*(\omega_{30}) = \omega_{30}$	$f_F^*(\omega_{30}) = \omega_{45}$
ω_{31}	$f_1^*(\omega_{31}) = \omega_{31}$	$f_2^*(\omega_{31}) = \omega_{31}$	$f_3^*(\omega_{31}) = \omega_{31}$	$f_F^*(\omega_{31}) = \omega_{46}$
ω_{32}	$f_1^*(\omega_{32}) = \omega_{32}$	$f_2^*(\omega_{32}) = \omega_{32}$	$f_3^*(\omega_{32}) = \omega_{32}$	$f_F^*(\omega_{32}) = \omega_{54}$

	i_1	i_2	i_3	F
ω_{33}	$f_1^*(\omega_{33}) = \omega_{33}$	$f_2^*(\omega_{33}) = \omega_{33}$	$f_3^*(\omega_{33}) = \omega_{33}$	$f_F^*(\omega_{33}) = \omega_{60}, \omega_{64}$
ω_{34}	$f_1^*(\omega_{34}) = \omega_{34}$	$f_2^*(\omega_{34}) = \omega_{34}$	$f_3^*(\omega_{34}) = \omega_{34}$	$f_F^*(\omega_{34}) = \omega_{59}, \omega_{60}$
ω_{35}	$f_1^*(\omega_{35}) = \omega_{35}$	$f_2^*(\omega_{35}) = \omega_{35}$	$f_3^*(\omega_{35}) = \omega_{44}$	$f_F^*(\omega_{35}) = \omega_{63}$
ω_{36}	$f_1^*(\omega_{36}) = \omega_{36}$	$f_2^*(\omega_{36}) = \omega_{36}$	$f_3^*(\omega_{36}) = \omega_{44}$	$f_F^*(\omega_{36}) = \omega_{36}$
ω_{37}	$f_1^*(\omega_{37}) = \omega_{37}$	$f_2^*(\omega_{37}) = \omega_{36}$	$f_3^*(\omega_{37}) = \omega_{37}$	$f_F^*(\omega_{37}) = \omega_{36}$
ω_{38}	$f_1^*(\omega_{38}) = \omega_{38}$	$f_2^*(\omega_{38}) = \omega_{38}$	$f_3^*(\omega_{38}) = \omega_{38}$	$f_F^*(\omega_{38}) = \omega_{56}$
ω_{39}	$f_1^*(\omega_{39}) = \omega_{39}$	$f_2^*(\omega_{39}) = \omega_{39}$	$f_3^*(\omega_{39}) = \omega_{39}$	$f_F^*(\omega_{39}) = \omega_{54}$
ω_{40}	$f_1^*(\omega_{40}) = \omega_{40}$	$f_2^*(\omega_{40}) = \omega_{45}$	$f_3^*(\omega_{40}) = \omega_{40}$	$f_F^*(\omega_{40}) = \omega_{45}$
ω_{41}	$f_1^*(\omega_{41}) = \omega_{41}$	$f_2^*(\omega_{41}) = \omega_{41}$	$f_3^*(\omega_{41}) = \omega_{41}$	$f_F^*(\omega_{41}) = \omega_{48}$
ω_{42}	$f_1^*(\omega_{42}) = \omega_{42}$	$f_2^*(\omega_{42}) = \omega_{42}$	$f_3^*(\omega_{42}) = \omega_{42}$	$f_F^*(\omega_{42}) = \omega_{42}$
ω_{43}	$f_1^*(\omega_{43}) = \omega_{43}$	$f_2^*(\omega_{43}) = \omega_{43}$	$f_3^*(\omega_{43}) = \omega_{43}$	$f_F^*(\omega_{43}) = \omega_{43}$
ω_{44}	$f_1^*(\omega_{44}) = \omega_{44}$	$f_2^*(\omega_{44}) = \omega_{44}$	$f_3^*(\omega_{44}) = \omega_{44}, \omega_{51}, \omega_{52}$	$f_F^*(\omega_{44}) = \omega_{44}$
ω_{45}	$f_1^*(\omega_{45}) = \omega_{45}$	$f_2^*(\omega_{45}) = \omega_{45}$	$f_3^*(\omega_{45}) = \omega_{45}$	$f_F^*(\omega_{45}) = \omega_{45}$
ω_{46}	$f_1^*(\omega_{46}) = \omega_{46}$	$f_2^*(\omega_{46}) = \omega_{46}$	$f_3^*(\omega_{46}) = \omega_{46}$	$f_F^*(\omega_{46}) = \omega_{46}$
ω_{47}	$f_1^*(\omega_{47}) = \omega_{47}$	$f_2^*(\omega_{47}) = \omega_{47}$	$f_3^*(\omega_{47}) = \omega_{47}$	$f_F^*(\omega_{47}) = \omega_{47}$
ω_{48}	$f_1^*(\omega_{48}) = \omega_{48}$	$f_2^*(\omega_{48}) = \omega_{48}$	$f_3^*(\omega_{48}) = \omega_{48}$	$f_F^*(\omega_{48}) = \omega_{48}$
ω_{49}	$f_1^*(\omega_{49}) = \omega_{49}$	$f_2^*(\omega_{49}) = \omega_{49}$	$f_3^*(\omega_{49}) = \omega_{49}$	$f_F^*(\omega_{49}) = \omega_{49}$
ω_{50}	$f_1^*(\omega_{50}) = \omega_{50}$	$f_2^*(\omega_{50}) = \omega_{50}$	$f_3^*(\omega_{50}) = \omega_{50}$	$f_F^*(\omega_{50}) = \omega_{50}$
ω_{51}	$f_1^*(\omega_{51}) = \omega_{10}$	$f_2^*(\omega_{51}) = \omega_{51}$	$f_3^*(\omega_{51}) = \omega_{10}$	$f_F^*(\omega_{51}) = \omega_{51}$
ω_{52}	$f_1^*(\omega_{52}) = \omega_{52}$	$f_2^*(\omega_{52}) = \omega_9$	$f_3^*(\omega_{52}) = \omega_9$	$f_F^*(\omega_{52}) = \omega_{52}$
ω_{53}	$f_1^*(\omega_{53}) = \omega_{53}$	$f_2^*(\omega_{53}) = \omega_{53}$	$f_3^*(\omega_{53}) = \omega_{53}$	$f_F^*(\omega_{53}) = \omega_{53}$
ω_{54}	$f_1^*(\omega_{54}) = \omega_{54}$	$f_2^*(\omega_{54}) = \omega_{54}$	$f_3^*(\omega_{54}) = \omega_{54}$	$f_F^*(\omega_{54}) = \omega_{54}$
ω_{55}	$f_1^*(\omega_{55}) = \omega_{55}$	$f_2^*(\omega_{55}) = \omega_{55}$	$f_3^*(\omega_{55}) = \omega_{55}$	$f_F^*(\omega_{55}) = \omega_{55}$
ω_{56}	$f_1^*(\omega_{56}) = \omega_{56}$	$f_2^*(\omega_{56}) = \omega_{56}$	$f_3^*(\omega_{56}) = \omega_{56}$	$f_F^*(\omega_{56}) = \omega_{56}$
ω_{57}	$f_1^*(\omega_{57}) = \omega_{57}$	$f_2^*(\omega_{57}) = \omega_{57}$	$f_3^*(\omega_{57}) = \omega_{57}$	$f_F^*(\omega_{57}) = \omega_{57}$
ω_{58}	$f_1^*(\omega_{58}) = \omega_{58}$	$f_2^*(\omega_{58}) = \omega_{58}$	$f_3^*(\omega_{58}) = \omega_{58}$	$f_F^*(\omega_{58}) = \omega_{58}$
ω_{59}	$f_1^*(\omega_{59}) = \omega_{66}$	$f_2^*(\omega_{59}) = \omega_{59}$	$f_3^*(\omega_{59}) = \omega_{59}$	$f_F^*(\omega_{59}) = \omega_{66}$
ω_{60}	$f_1^*(\omega_{60}) = \omega_{60}$	$f_2^*(\omega_{60}) = \omega_{66}$	$f_3^*(\omega_{60}) = \omega_{60}$	$f_F^*(\omega_{60}) = \omega_{66}$
ω_{61}	$f_1^*(\omega_{61}) = \omega_{61}$	$f_2^*(\omega_{61}) = \omega_{66}$	$f_3^*(\omega_{61}) = \omega_{61}$	$f_F^*(\omega_{61}) = \omega_{66}$
ω_{62}	$f_1^*(\omega_{62}) = \omega_{62}$	$f_2^*(\omega_{62}) = \omega_{62}$	$f_3^*(\omega_{62}) = \omega_{66}$	$f_F^*(\omega_{62}) = \omega_{66}$
ω_{63}	$f_1^*(\omega_{63}) = \omega_{66}$	$f_2^*(\omega_{63}) = \omega_{63}$	$f_3^*(\omega_{63}) = \omega_{63}$	$f_F^*(\omega_{63}) = \omega_{66}$
ω_{64}	$f_1^*(\omega_{64}) = \omega_{64}$	$f_2^*(\omega_{64}) = \omega_{64}$	$f_3^*(\omega_{64}) = \omega_{66}$	$f_F^*(\omega_{64}) = \omega_{66}$
ω_{65}	$f_1^*(\omega_{65}) = \omega_{65}$	$f_2^*(\omega_{65}) = \omega_{65}$	$f_3^*(\omega_{65}) = \omega_{65}$	$f_F^*(\omega_{65}) = \omega_{65}$
ω_{66}	$f_1^*(\omega_{66}) = \omega_{66}$	$f_2^*(\omega_{66}) = \omega_{66}$	$f_3^*(\omega_{66}) = \omega_{66}$	$f_F^*(\omega_{66}) = \omega_{66}$
ω_{67}	$f_1^*(\omega_{67}) = \omega_{67}$	$f_2^*(\omega_{67}) = \omega_{67}$	$f_3^*(\omega_{67}) = \omega_{67}$	$f_F^*(\omega_{67}) = \omega_{57}$
ω_{68}	$f_1^*(\omega_{68}) = \omega_{68}$	$f_2^*(\omega_{68}) = \omega_{68}$	$f_3^*(\omega_{68}) = \omega_{68}$	$f_F^*(\omega_{68}) = \omega_{55}$
ω_{69}	$f_1^*(\omega_{69}) = \omega_{69}$	$f_2^*(\omega_{69}) = \omega_{69}$	$f_3^*(\omega_{69}) = \omega_{69}$	$f_F^*(\omega_{69}) = \omega_{69}$
ω_{70}	$f_1^*(\omega_{70}) = \omega_{70}$	$f_2^*(\omega_{70}) = \omega_{70}$	$f_3^*(\omega_{70}) = \omega_{70}$	$f_F^*(\omega_{70}) = \omega_{70}$

TABLE 5: SAMPLE DEGREE OF CENTRALITY FROM EXAMPLE

	i_1	i_2	i_3	i_F
G_0	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
G_1	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
G_2	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
G_3	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
G_9	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
G_{10}	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
G_{11}	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
G_4	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{1}{6}$
G_5	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{1}{7}$
G_6	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
G_7	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
G_8	$\frac{3}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$
G_{12}	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{1}{7}$
G_{13}	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
G_{14}	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

So, we have $d_i \in [0, 1]$.

Table 6: Equilibrium Transition Probabilities

(ω, ω') Transition Probability		(1,1)	(2,2)	(1,3)	(3,3)	(1,4)	(4,4)	(1,5)	(5,5)
		0.0697	0.0697	0.3211	0.0603	0.1651	0.1667	0.4441	0.1667
(2,6)	(6,6)	(2,7)	(7,7)	(2,8)	(8,8)	(6,9)	(9,9)	(52,9)	(3,10)
0.3211	0.0603	0.1651	0.1667	0.4441	0.1667	0.3239	1.0000	0.6667	0.3239
(10,10)	(51,10)	(3,11)	(11,11)	(3,12)	(12,12)	(4,13)	(13,13)	(8,14)	(14,14)
1.0000	0.6667	0.1872	0.1667	0.4286	0.1029	0.8333	0.1171	0.8333	0.0363
(5,15)	(11,15)	(15,15)	(6,16)	(16,16)	(6,17)	(17,17)	(7,18)	(18,18)	(14,19)
0.8333	0.8333	0.0363	0.1872	0.0717	0.4286	0.0512	0.8333	0.0512	0.1223
(15,19)	(19,19)	(13,20)	(14,20)	(15,20)	(20,20)	(14,21)	(15,21)	(21,21)	(12,22)
0.1223	0.0528	0.8829	0.1223	0.1223	0.0528	0.1223	0.1223	0.0528	0.5637
(19,22)	(22,22)	(12,23)	(19,23)	(23,23)	(17,24)	(18,24)	(20,24)	(24,24)	(19,25)
0.2959	0.1027	0.3333	0.1777	0.0990	0.1716	0.1716	0.1716	0.0990	0.1777
(25,25)	(19,26)	(26,26)	(17,27)	(18,27)	(20,27)	(27,27)	(21,28)	(28,28)	(17,29)
0.0990	0.2959	0.1027	0.2929	0.2929	0.2959	0.1027	0.1716	0.0990	0.2995
(18,29)	(20,29)	(29,29)	(21,30)	(30,30)	(21,31)	(31,31)	(21,32)	(32,32)	(14,33)
0.2995	0.2959	0.1027	0.2959	0.1027	0.1716	0.0990	0.2959	0.1027	0.1989
(15,33)	(33,33)	(14,34)	(15,34)	(34,34)	(14,35)	(15,35)	(35,35)	(36,36)	(37,36)
0.1989	0.0535	0.1989	0.1989	0.0535	0.1989	0.1989	0.0535	1.0000	0.5382
(33,37)	(37,37)	(34,38)	(38,38)	(33,39)	(39,39)	(34,40)	(40,40)	(16,41)	(17,41)
0.1802	0.1003	0.1802	0.1003	0.1802	0.1003	0.1802	0.1003	0.5187	0.1848
(18,41)	(20,41)	(41,41)	(29,42)	(42,42)	(67,42)	(22,43)	(43,43)	(68,43)	(16,44)
0.1848	0.1716	0.0990	0.3460	1.0000	0.3459	0.3459	1.0000	0.3459	0.4096
(24,44)	(25,44)	(44,44)	(30,45)	(40,45)	(45,45)	(23,46)	(31,46)	(46,46)	(26,47)
0.5677	0.5677	0.0802	0.3460	0.3615	1.0000	0.5677	0.5677	1.0000	0.3460
(38,47)	(47,47)	(28,48)	(41,48)	(48,48)	(32,49)	(39,49)	(49,49)	(27,50)	(37,50)
0.3615	1.0000	0.5677	0.5677	1.0000	0.3460	0.3615	1.0000	0.5514	0.3615
(50,50)	(44,51)	(51,51)	(44,52)	(52,52)	(30,53)	(40,53)	(53,53)	(32,54)	(39,54)
1.0000	0.4599	0.3333	0.4599	0.3333	0.5514	0.5382	1.0000	0.5514	0.5382
(54,54)	(22,55)	(55,55)	(68,55)	(26,56)	(38,56)	(56,56)	(29,57)	(57,57)	(67,57)
1.0000	0.5514	1.0000	0.5514	0.5514	0.5382	1.0000	0.5514	1.0000	0.5514
(27,58)	(58,58)	(63,58)	(64,58)	(34,59)	(59,59)	(33,60)	(60,60)	(35,61)	(61,61)
0.3460	1.0000	0.3562	0.3562	0.1909	0.0959	0.1909	0.0959	0.1909	0.0959
(34,62)	(62,62)	(35,63)	(63,63)	(33,64)	(64,64)	(23,65)	(24,65)	(25,65)	(28,65)
0.1909	0.0959	0.1909	0.0959	0.1909	0.0959	0.3333	0.3333	0.3333	0.3333
(31,65)	(41,65)	(65,65)	(59,66)	(60,66)	(61,66)	(62,66)	(63,66)	(64,66)	(66,66)
0.3333	0.3333	1.0000	0.5479	0.5479	0.5479	0.5479	0.5479	0.5479	1.0000
(35,67)	(67,67)	(35,68)	(68,68)	(59,69)	(60,69)	(69,69)	(61,70)	(62,70)	(70,70)
0.1802	0.1027	0.1802	0.1027	0.3562	0.3562	1.0000	0.3562	0.3562	1.0000

TABLE 7: EQUILIBRIUM PAYOFFS

	w_1^*	w_2^*	w_3^*	w_F^*
ω_1	-1.2587	-4.5144	-3.6105	-0.1258
ω_2	-2.0963	-4.2359	-3.6159	-0.1254
ω_3	2.2598	-3.0841	2.2211	-0.1078
ω_4	-1.3139	-10.8806	-10.6830	-1.1955
ω_5	-8.0335	-6.2059	-8.8239	-2.2915
ω_6	-1.7237	-1.1944	1.9738	-0.1005
ω_7	-3.2576	-8.4735	-10.2655	-1.1806
ω_8	-3.3854	-11.2182	-8.8239	-2.2915
ω_9	5	2.5	10	0
ω_{10}	1.6667	5	10	0
ω_{11}	-1.6049	-5.0948	-6.6420	-1.2745
ω_{12}	-5.1944	-9.2232	-3.9360	-1.9440
ω_{13}	-3.6789	-17.5850	-9.1025	-2.1064
ω_{14}	-6.4791	-10.1928	-7.4125	-3.0393
ω_{15}	-7.4938	-9.1706	-7.4125	-3.0393
ω_{16}	-3.7440	-11.0867	-6.5269	-0.9174
ω_{17}	-6.1214	-13.1889	-5.1052	-1.9308
ω_{18}	-6.1214	-15.2523	-8.1841	-1.9308
ω_{19}	-15.5937	-9.3079	-7.7163	-0.9252
ω_{20}	-7.0974	-23.3122	-8.0861	-0.9260
ω_{21}	-7.1470	-9.5582	-18.8462	-0.9307
ω_{22}	-16.6667	-8.5424	5.0541	1.2737
ω_{23}	-16.6667	-8.8207	-18.9479	0.1274
ω_{24}	-14.9168	-21.5883	-6.4031	0.1156
ω_{25}	-15.9580	-20.5252	-6.4031	0.1156
ω_{26}	-16.6672	3.5658	-6.8091	1.2738
ω_{27}	2.7096	-24.4482	-9.7726	1.2507
ω_{28}	-6.7318	-23.9369	-20.0000	0.1274
ω_{29}	-6.6122	-25.0012	5.0543	1.2738
ω_{30}	-6.6122	3.5658	-20.0008	1.0881
ω_{31}	-15.6254	-8.8207	-20.0000	0.1274
ω_{32}	2.7096	-8.5427	-20.0008	1.2738
ω_{33}	2.5493	-9.4326	-7.9931	0.0969
ω_{34}	-7.0933	3.2165	-7.9249	0.0939
ω_{35}	-7.0852	-9.5001	4.2629	0.0980

	w_1^*	w_2^*	w_3^*	w_F^*
ω_{36}	1.6667	-22.5000	2.0000	2.2222
ω_{37}	1.6667	-23.0770	-8.1229	0.2020
ω_{38}	-15.6249	2.5000	-6.9897	0.2624
ω_{39}	1.6667	-8.7954	-18.9472	0.2624
ω_{40}	-6.7212	2.5000	-18.9472	0.0812
ω_{41}	-6.7318	-25.0000	-18.9479	0.1274
ω_{42}	-15	-25	4	2.2222
ω_{43}	-16.6667	-22.5	4	2.2222
ω_{44}	-13.6696	-15.5786	7.9165	2.0162
ω_{45}	-15.0000	2.5	-20	2.2222
ω_{46}	-16.6667	5	-20	2.222
ω_{47}	-16.6667	2.5	-18	2.2222
ω_{48}	3.3333	-25	-20	2.2222
ω_{49}	1.6667	-22.5000	-20	2.2222
ω_{50}	1.6667	-25	-18	2.2222
ω_{51}	-7.5641	-10	10	0
ω_{52}	-10	-7.5	10	0
ω_{53}	3.3333	2.5	-20	0
ω_{54}	1.6667	5	-20	3.3333
ω_{55}	-16.666667	5	4	3.3333
ω_{56}	-16.6667	2.5	6	3.3333
ω_{57}	3.3333	-25	4	3.3333
ω_{58}	1.6667	-22.5	4	3.3333
ω_{59}	2.7065	2.5000	-8.0436	2.2837
ω_{60}	1.6667	3.5610	-8.0436	2.2837
ω_{61}	-7.0420	3.5610	3.0503	2.2837
ω_{62}	-7.0420	2.5000	4.1007	2.2837
ω_{63}	2.7065	-9.5355	3.4245	2.3237
ω_{64}	1.6667	-9.5355	4.4748	2.3237
ω_{65}	-16.6667	-25	-20	0
ω_{66}	1.6667	2.5	2	3.3333
ω_{67}	-6.6121	-23.9343	2.9459	0.2634
ω_{68}	-15.6238	-8.5424	2.9459	0.2634
ω_{69}	1.6667	2.5000	-18.0000	2.2222
ω_{70}	-15.0000	2.5000	2.0000	2.2222

Table 8: Ergodic Measures

(1,9) 0.0182	(2,9) 0.1602	(3,9) 0.0029	(4,9) 0.0571	(5,9) 0.0147	(6,9) 0.4146	(7,9) 0.0570	(8,9) 0.0147	(9,9) 1.0000	(11,9) 0.0147
(13,9) 0.0571	(14,9) 0.0147	(15,9) 0.0147	(16,9) 0.2206	(17,9) 0.0570	(18,9) 0.0570	(19,9) 0.0591	(20,9) 0.0571	(24,9) 0.3150	(25,9) 0.3150
(44,9) 0.5000	(52,9) 1.0000	(1,10) 0.1371	(2,10) 0.0413	(3,10) 0.3476	(4,10) 0.0571	(5,10) 0.0147	(6,10) 0.0699	(7,10) 0.0570	(8,10) 0.0147
(10,10) 1.0000	(11,10) 0.0147	(13,10) 0.0571	(14,10) 0.0147	(15,10) 0.0147	(16,10) 0.2206	(17,10) 0.0570	(18,10) 0.0570	(19,10) 0.0591	(20,10) 0.0571
(24,10) 0.3150	(25,10) 0.3150	(44,10) 0.5000	(51,10) 1.0000	(1,36) 0.0128	(2,36) 0.0112	(3,36) 0.0047	(5,36) 0.0235	(8,36) 0.0235	(11,36) 0.0235
(14,36) 0.0235	(15,36) 0.0235	(33,36) 0.1139	(36,36) 1.0000	(37,36) 0.5982	(1,42) 0.0380	(2,42) 0.0553	(3,42) 0.0061	(4,42) 0.1204	(5,42) 0.0304
(6,42) 0.0555	(7,42) 0.1217	(8,42) 0.0304	(11,42) 0.0304	(13,42) 0.1204	(14,42) 0.0304	(15,42) 0.0304	(17,42) 0.1217	(18,42) 0.1217	(20,42) 0.1204
(29,42) 0.3856	(35,42) 0.0734	(42,42) 1.0000	(67,42) 0.3855	(1,43) 0.0548	(2,43) 0.0145	(3,43) 0.1166	(5,43) 0.0304	(8,43) 0.0304	(11,43) 0.0304
(12,43) 0.2423	(14,43) 0.0304	(15,43) 0.0304	(19,43) 0.1204	(22,43) 0.3855	(35,43) 0.0734	(43,43) 1.0000	(68,43) 0.3855	(1,45) 0.0170	(2,45) 0.0148
(3,45) 0.0062	(5,45) 0.0311	(8,45) 0.0311	(11,45) 0.0311	(14,45) 0.0311	(15,45) 0.0311	(21,45) 0.1204	(30,45) 0.3856	(34,45) 0.0765	(40,45) 0.4018
(45,45) 1.0000	(1,46) 0.0530	(2,46) 0.0141	(3,46) 0.1127	(5,46) 0.0295	(8,46) 0.0295	(11,46) 0.0295	(12,46) 0.2341	(14,46) 0.0295	(15,46) 0.0295
(19,46) 0.1182	(21,46) 0.1141	(23,46) 0.6301	(31,46) 0.6301	(46,46) 1.0000	(1,47) 0.0170	(2,47) 0.0148	(3,47) 0.0062	(5,47) 0.0311	(8,47) 0.0311
(11,47) 0.0311	(14,47) 0.0311	(15,47) 0.0311	(19,47) 0.1204	(26,47) 0.3856	(34,47) 0.0765	(38,47) 0.4018	(47,47) 1.0000	(1,48) 0.0361	(2,48) 0.0791
(3,48) 0.0058	(4,48) 0.1141	(5,48) 0.0290	(6,48) 0.1261	(7,48) 0.1227	(8,48) 0.0290	(11,48) 0.0290	(13,48) 0.1141	(14,48) 0.0290	(15,48) 0.0290
(16,48) 0.3521	(17,48) 0.1227	(18,48) 0.1227	(20,48) 0.1141	(21,48) 0.1141	(28,48) 0.6301	(41,48) 0.6301	(48,48) 1.0000	(1,49) 0.0170	(2,49) 0.0148
(3,49) 0.0062	(5,49) 0.0311	(8,49) 0.0311	(11,49) 0.0311	(14,49) 0.0311	(15,49) 0.0311	(21,49) 0.1204	(32,49) 0.3856	(33,49) 0.0765	(39,49) 0.4018
(49,49) 1.0000	(1,50) 0.0560	(2,50) 0.0827	(3,50) 0.0080	(4,50) 0.1919	(5,50) 0.0401	(6,50) 0.0865	(7,50) 0.1897	(8,50) 0.0401	(11,50) 0.0401
(13,50) 0.1919	(14,50) 0.0401	(15,50) 0.0401	(17,50) 0.1897	(18,50) 0.1897	(20,50) 0.1919	(27,50) 0.6145	(33,50) 0.0765	(37,50) 0.4018	(50,50) 1.0000
(1,53) 0.0261	(2,53) 0.0228	(3,53) 0.0095	(5,53) 0.0479	(8,53) 0.0479	(11,53) 0.0479	(14,53) 0.0479	(15,53) 0.0479	(21,53) 0.1919	(30,53) 0.6145
(34,53) 0.1139	(40,53) 0.5982	(53,53) 1.0000	(1,54) 0.0261	(2,54) 0.0228	(3,54) 0.0095	(5,54) 0.0479	(8,54) 0.0479	(11,54) 0.0479	(14,54) 0.0479
(15,54) 0.0479	(21,54) 0.1919	(32,54) 0.6145	(33,54) 0.1139	(39,54) 0.5982	(54,54) 1.0000	(1,55) 0.0873	(2,55) 0.0232	(3,55) 0.1858	(5,55) 0.0485
(8,55) 0.0485	(11,55) 0.0485	(12,55) 0.3861	(14,55) 0.0485	(15,55) 0.0485	(19,55) 0.1919	(22,55) 0.6145	(35,55) 0.1170	(55,55) 1.0000	(68,55) 0.6145
(1,56) 0.0261	(2,56) 0.0228	(3,56) 0.0095	(5,56) 0.0479	(8,56) 0.0479	(11,56) 0.0479	(14,56) 0.0479	(15,56) 0.0479	(19,56) 0.1919	(26,56) 0.6145
(34,56) 0.1139	(38,56) 0.5982	(56,56) 1.0000	(1,57) 0.0606	(2,57) 0.0881	(3,57) 0.0097	(4,57) 0.1919	(5,57) 0.0485	(6,57) 0.0885	(7,57) 0.1940
(8,57) 0.0485	(11,57) 0.0485	(13,57) 0.1919	(14,57) 0.0485	(15,57) 0.0485	(17,57) 0.1940	(18,57) 0.1940	(20,57) 0.1919	(29,57) 0.6145	(35,57) 0.1170
(57,57) 1.0000	(67,57) 0.6145	(1,58) 0.0476	(2,58) 0.0628	(3,58) 0.0096	(4,58) 0.1204	(5,58) 0.0481	(6,58) 0.0543	(7,58) 0.1190	(8,58) 0.0481
(11,58) 0.0481	(13,58) 0.1204	(14,58) 0.0481	(15,58) 0.0481	(17,58) 0.1190	(18,58) 0.1190	(20,58) 0.1204	(27,58) 0.3856	(33,58) 0.0795	(35,58) 0.0795

(58,58) 1.0000	(63,58) 0.3940	(64,58) 0.3940	(1,65) 0.0736	(2,65) 0.0854	(3,65) 0.0730	(4,65) 0.1340	(5,65) 0.0516	(6,65) 0.1046	(7,65) 0.1390
(8,65) 0.0516	(11,65) 0.0516	(12,65) 0.1375	(13,65) 0.1340	(14,65) 0.0516	(15,65) 0.0516	(16,65) 0.2067	(17,65) 0.1390	(18,65) 0.1390	(19,65) 0.1388
(20,65) 0.1340	(21,65) 0.1340	(23,65) 0.3699	(24,65) 0.3699	(25,65) 0.3699	(28,65) 0.3699	(31,65) 0.3699	(41,65) 0.3699	(65,65) 1.0000	(1,66) 0.0827
(2,66) 0.0723	(3,66) 0.0302	(5,66) 0.1514	(8,66) 0.1514	(11,66) 0.1514	(14,66) 0.1514	(15,66) 0.1514	(33,66) 0.2445	(34,66) 0.2445	(35,66) 0.2445
(59,66) 0.6060	(60,66) 0.6060	(61,66) 0.6060	(62,66) 0.6060	(63,66) 0.6060	(64,66) 0.6060	(66,66) 1.0000	(1,69) 0.0179	(2,69) 0.0157	(3,69) 0.0065
(5,69) 0.0328	(8,69) 0.0328	(11,69) 0.0328	(14,69) 0.0328	(15,69) 0.0328	(33,69) 0.0795	(34,69) 0.0795	(59,69) 0.3940	(60,69) 0.3940	(69,69) 1.0000
(1,70) 0.0179	(2,70) 0.0157	(3,70) 0.0065	(5,70) 0.0328	(8,70) 0.0328	(11,70) 0.0328	(14,70) 0.0328	(15,70) 0.0328	(34,70) 0.0795	(35,70) 0.0795
(61,70) 0.3940	(62,70) 0.3940	(70,70) 1.0000							

GRAPHS

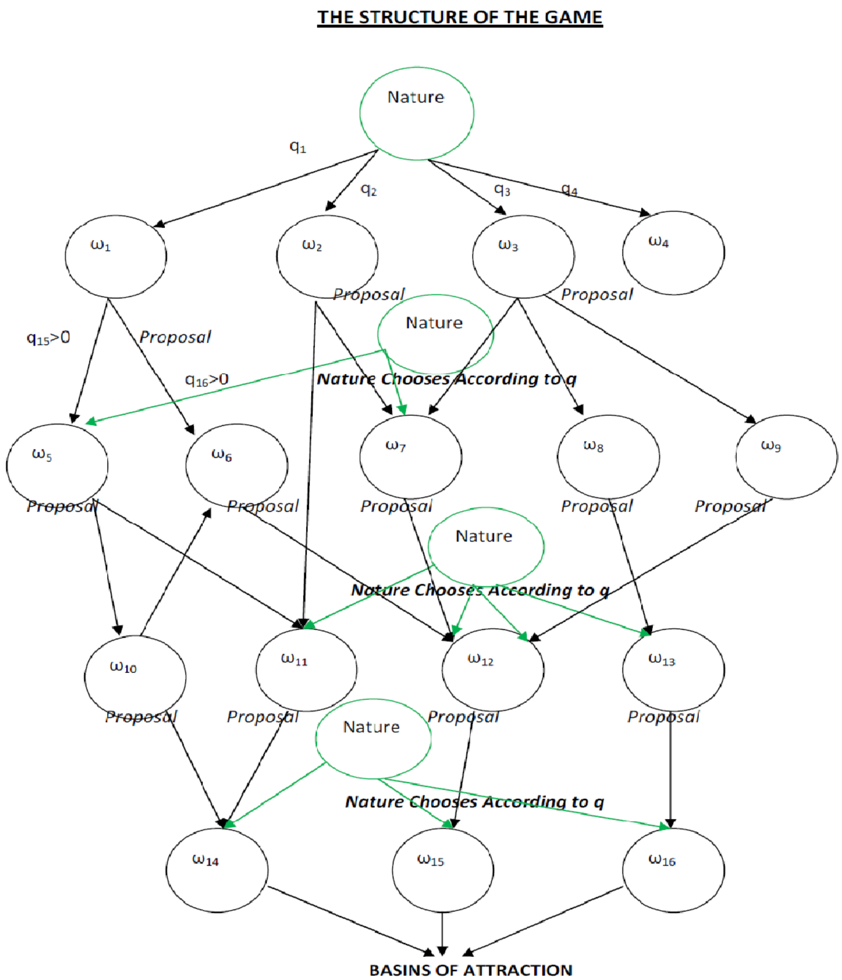


Figure 4.1: The Game

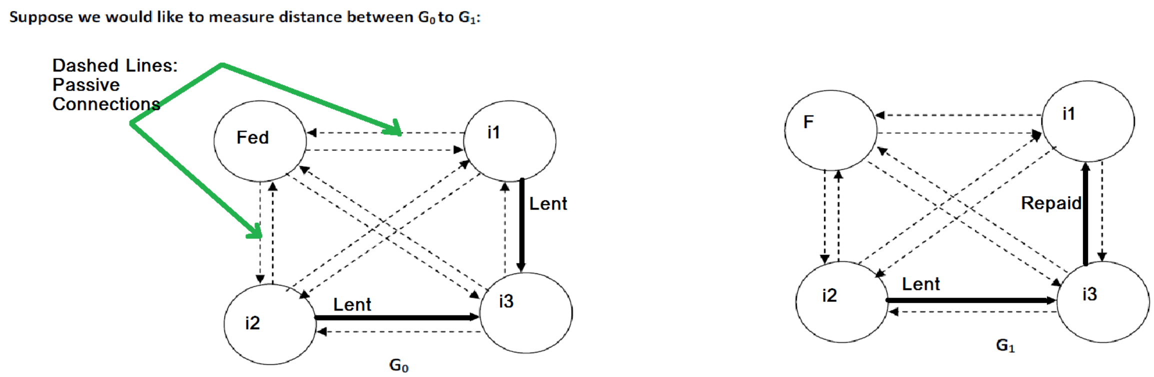


Figure 4.2: An Example of Measuring Distance Between Networks

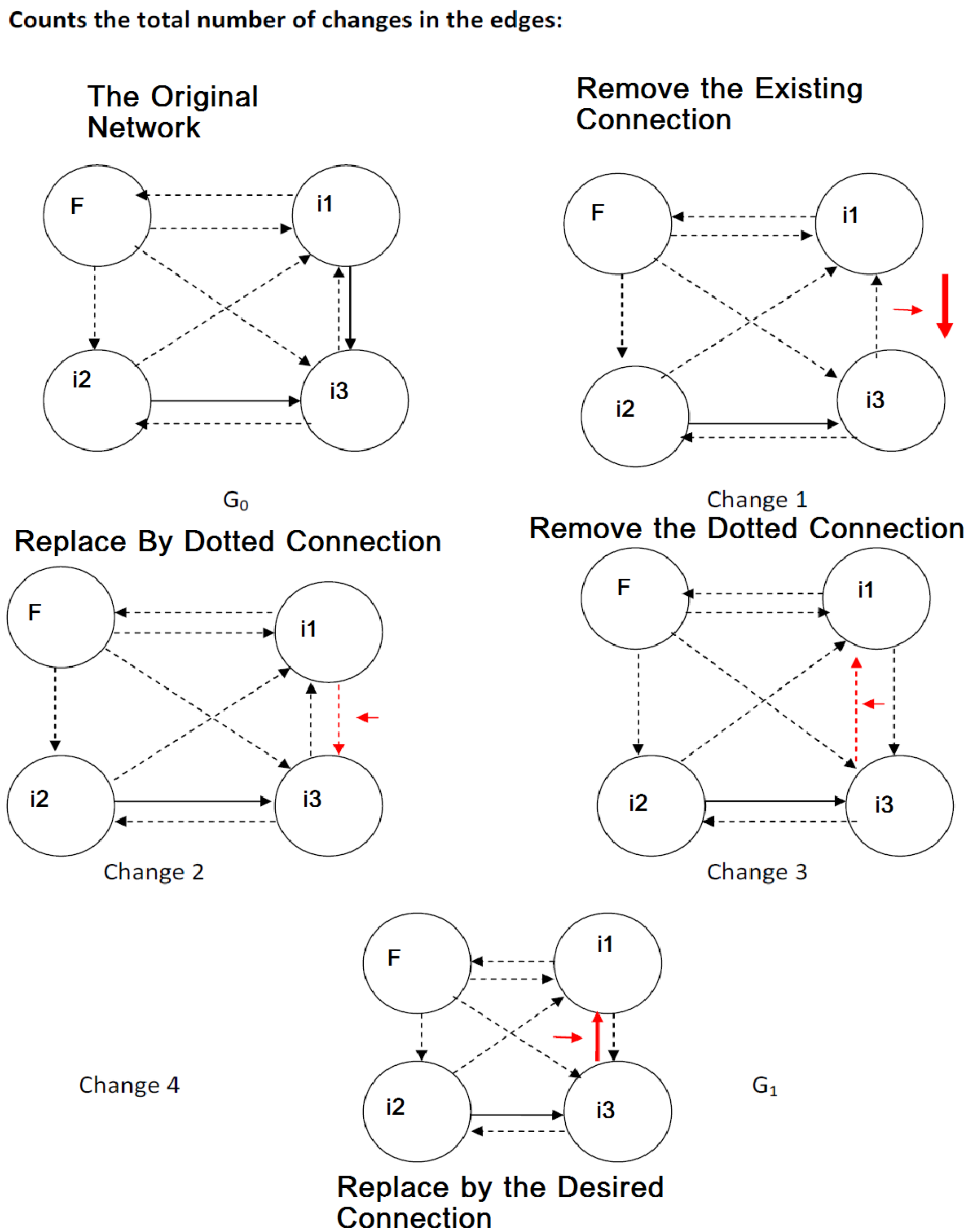
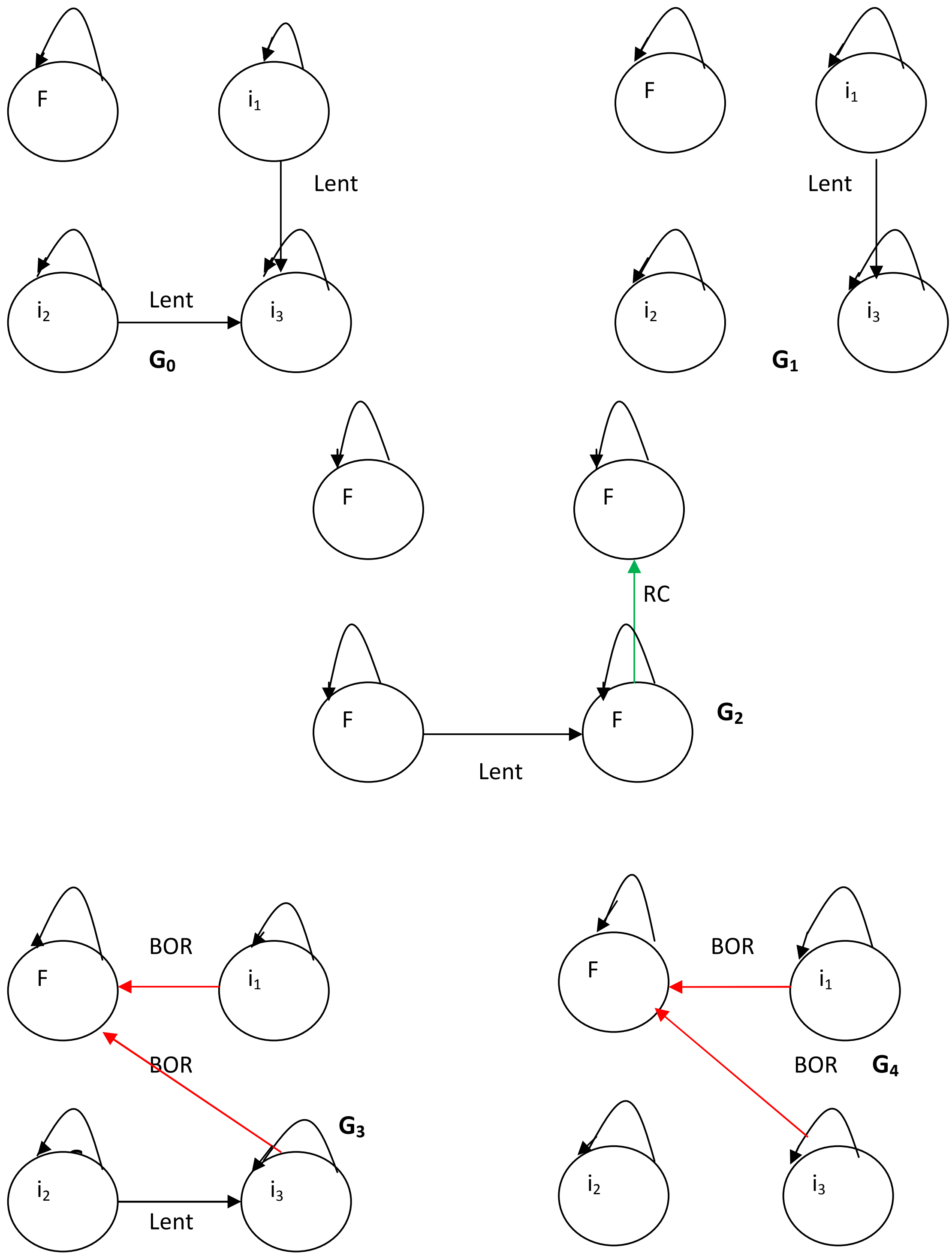
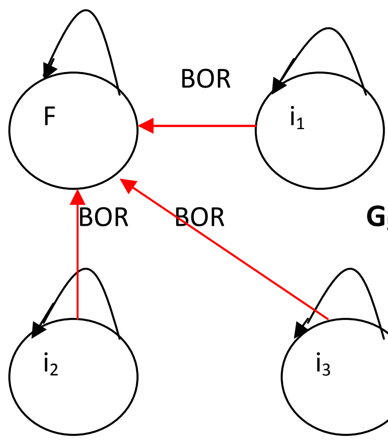


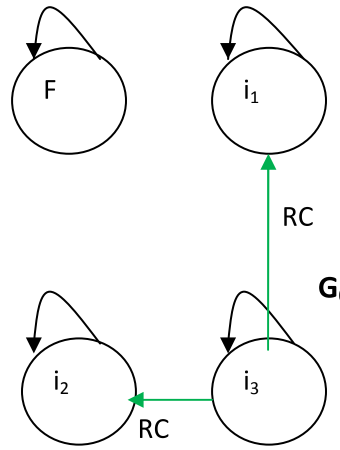
Figure 4.3: The Idea Underlying Measuring Distance Between Networks

Figure 4.4: An Example

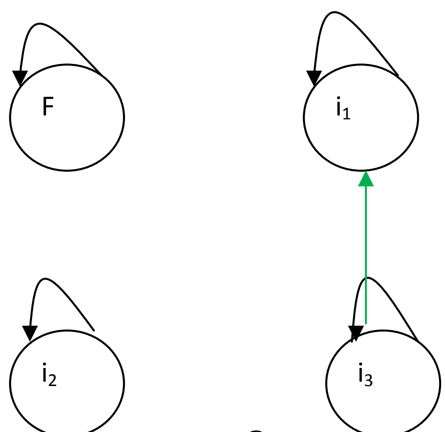




G_5



G_6

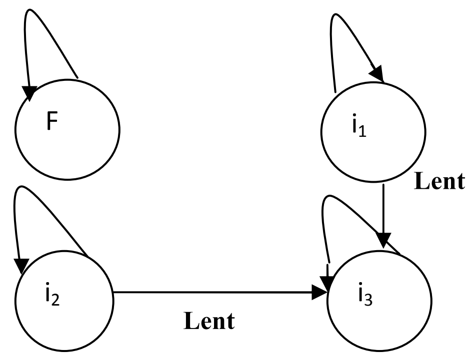


G_7

Figure 4.5: The Network Space

F: Fed, i_1: Bank 1, i_2: Bank 2, i_3: Bank 3

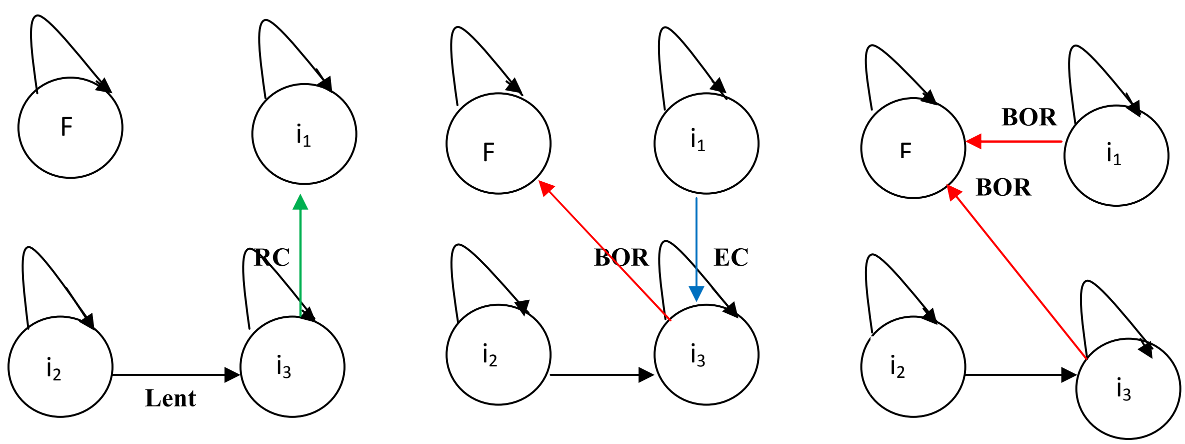
G_0



Let the initial network be G_0

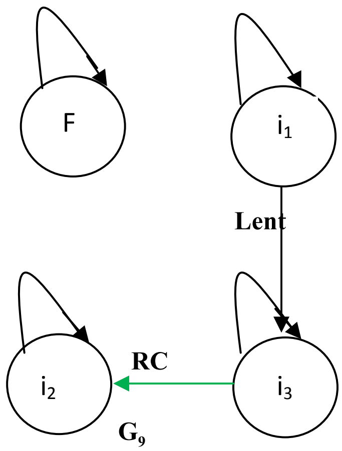
Legend for the following – EC: Enforced Contract, RC: Respond to Contract, BOR: Bailout Request, BO: Bailed Out, NBO : Not BO

If S_1 is chosen to move first, these are the networks that the coalition can propose:

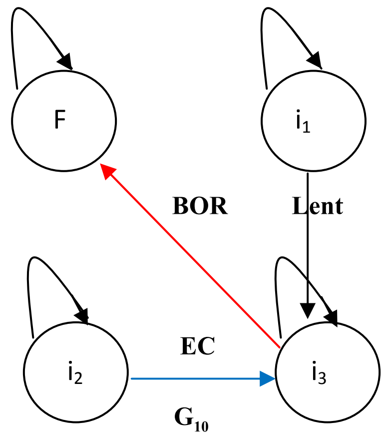


G_1 **G_2** **G_3**

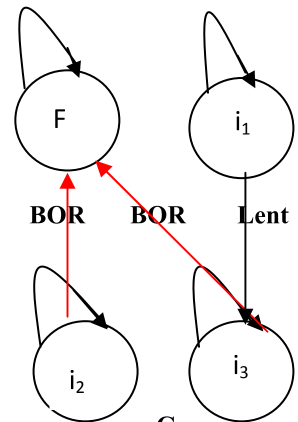
If S2 is chosen to move first, these are the networks that the coalition can propose:



G_9

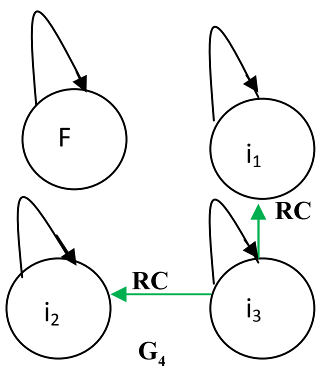


G_{10}

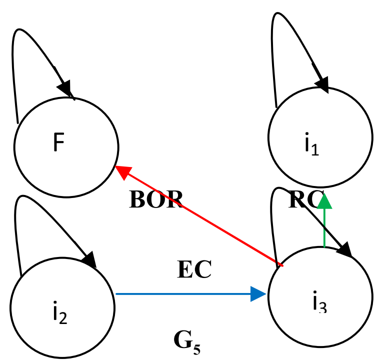


G_{11}

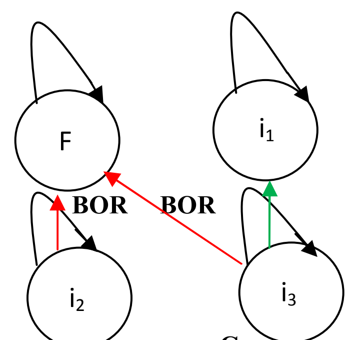
Given S_1 was chosen first and recommended that S_2 moves next, S_2 can make the following proposals:



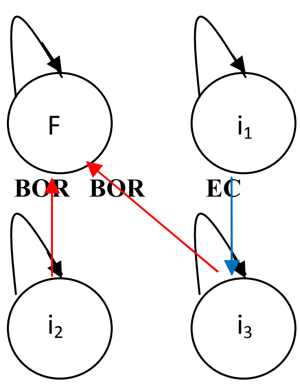
G_4



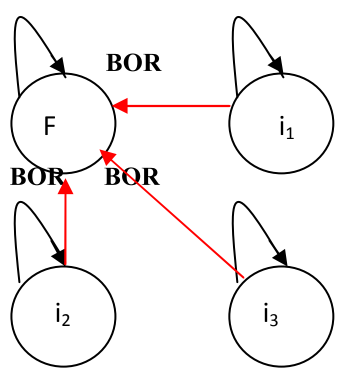
G_5



G_6

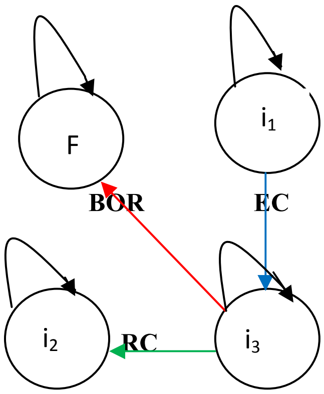


G_7

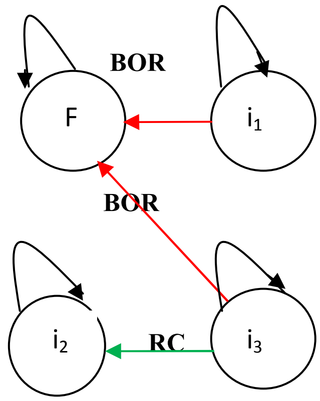


G_8

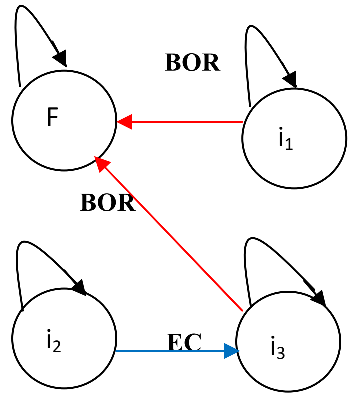
Given S_2 was chosen first and recommended that S_1 move next, S_1 can make the following



G_{12}

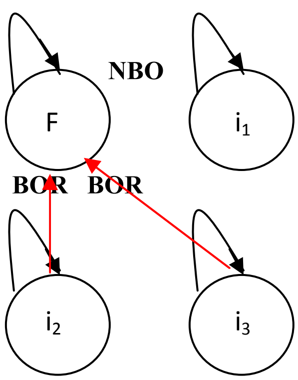


G_{13}

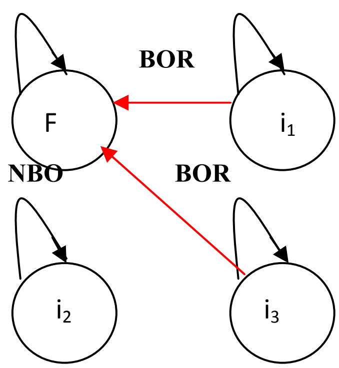


G_{14}

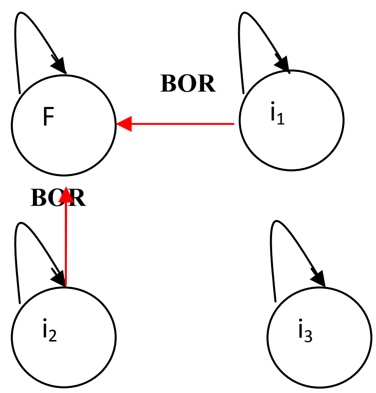
Based on these networks, we have the following set where the Fed and nodes form coalition pairs making a move conditional on what happened in the previous period, that is, in G_8.



G_{15}



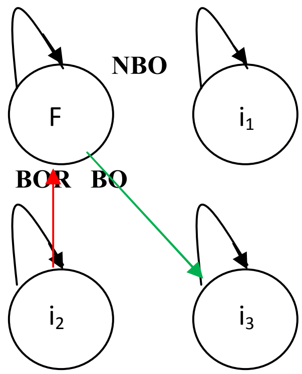
G_{16}



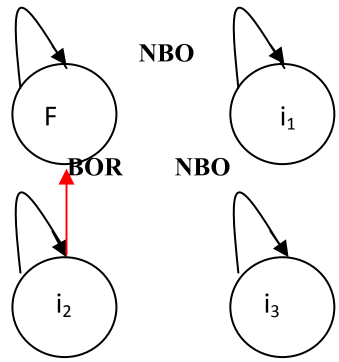
G_{17}

Here 'S(i, F)' refers to the coalition formed by node i and the Fed, F. And 'N' stands for Not BO.

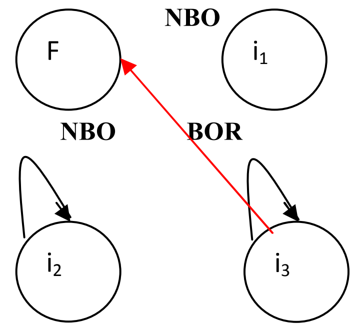
This was the case for No Bailout. But we could have the case where banks have been bailed out. This represented below.



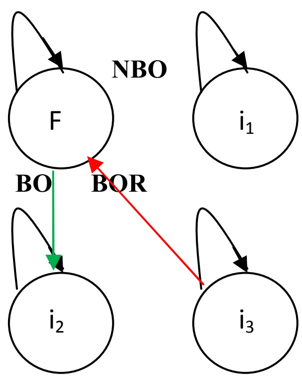
G_{18}



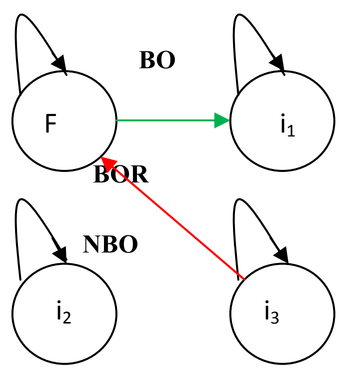
G_{19}



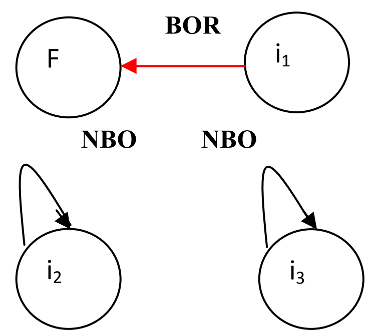
G_{20}



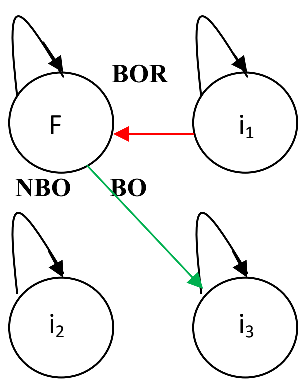
G_{21}



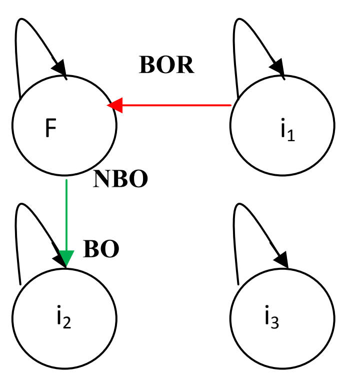
G_{22}



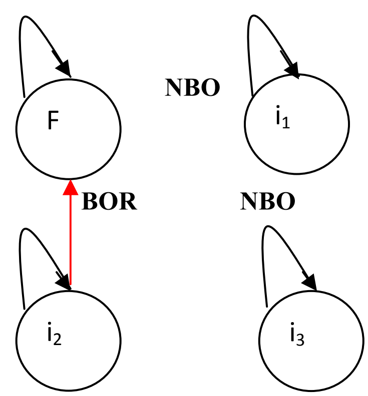
G_{23}



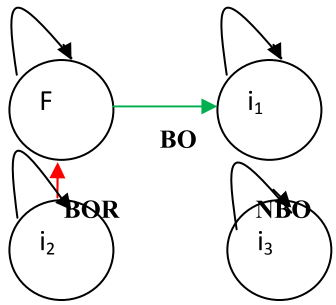
G_{24}



G_{25}

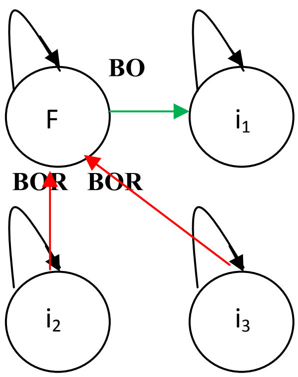


G_{26}

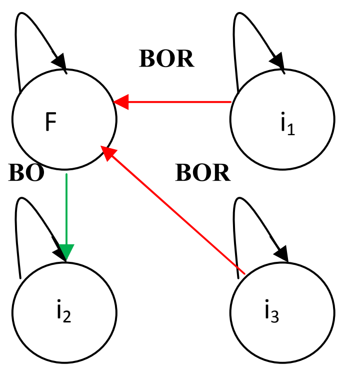


G_{27}

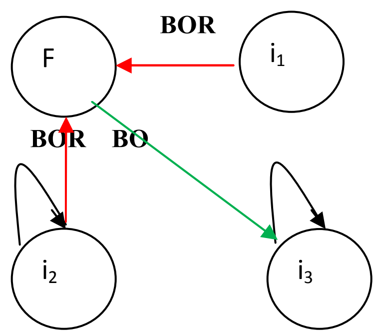
Case for Bailouts after G_8 has happened.



G_{28}



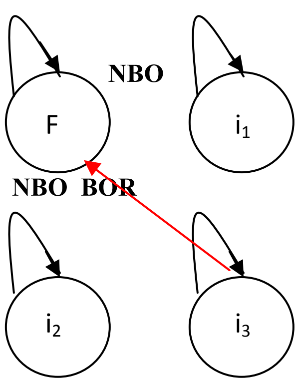
G_{29}



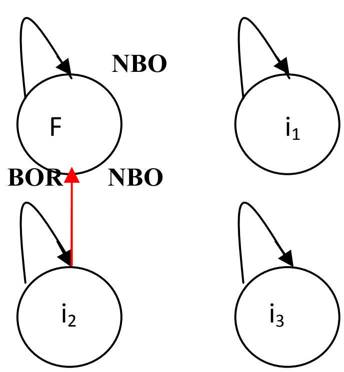
G_{30}

Here 'S(i, F)' refers to the coalition formed by node i and the Fed, F. And 'B' stands for BO.

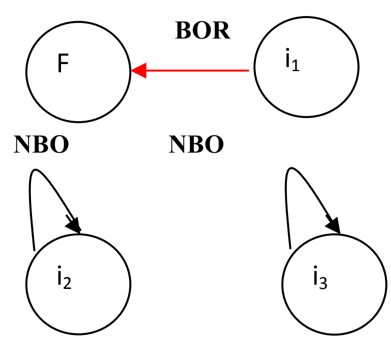
We also have the following networks,



G_{31}



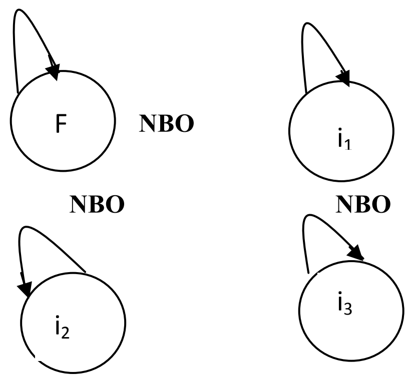
G_{32}



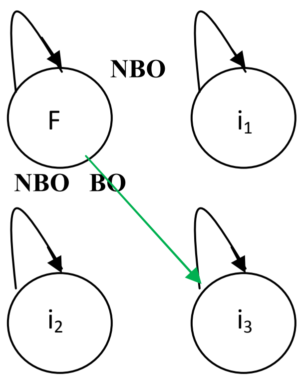
G_{33}

State of Complete Collapse without Bailout:

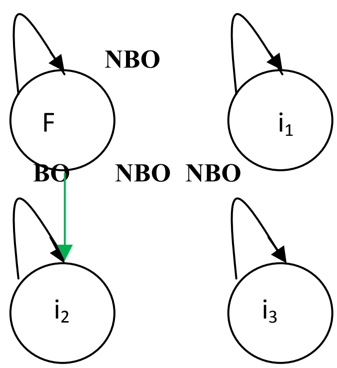
G_{cc}



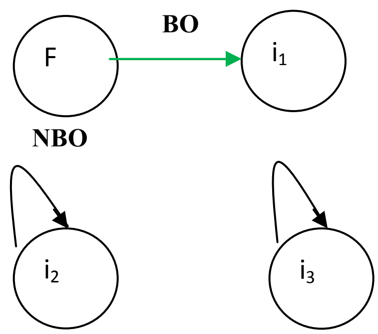
However, note that it is also possible, that the Fed bails out only one of the nodes.



G_{35}

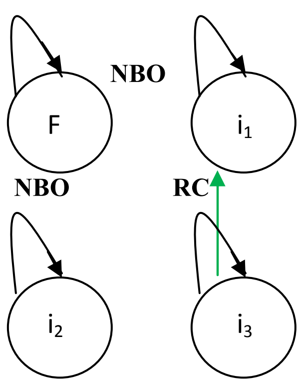


G_{36}

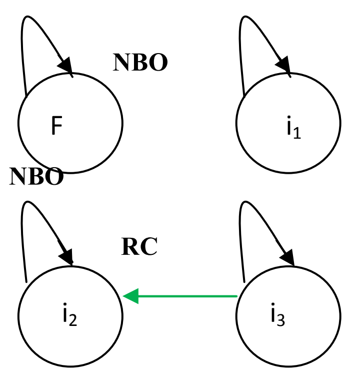


G_{37}

If we are at $G_{B3(3, F)}$ then we can transit to one of the following where either S(3, 1) or S(3, 2) are chosen:



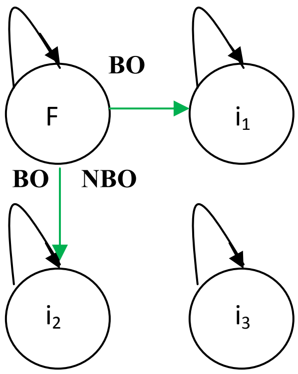
G_{38}



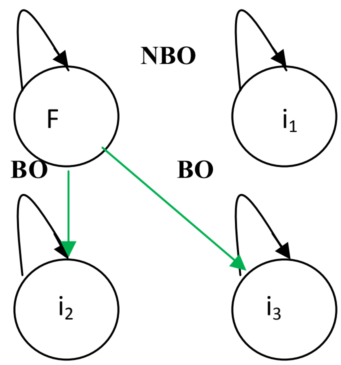
G_{39}

From here then, one could end up in G_4.

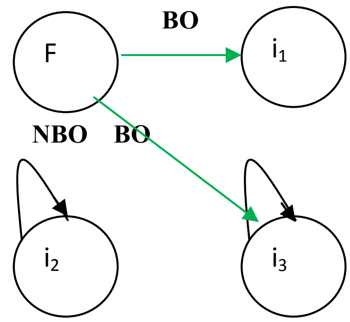
Alternatively, we could have the following networks for the case where Fed bails the banks out eventually.



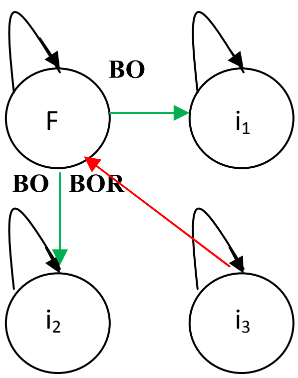
G_{40}



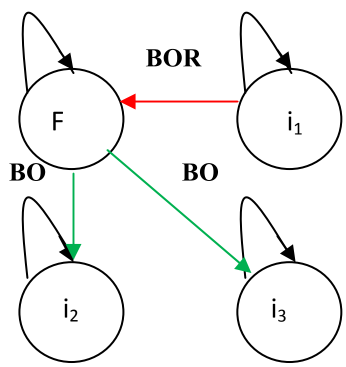
G_{41}



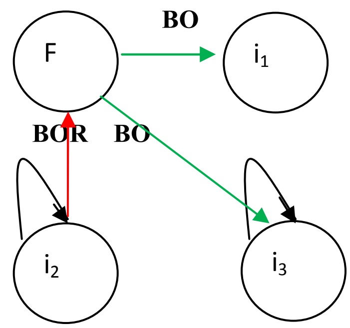
G_{42}



G_{43}



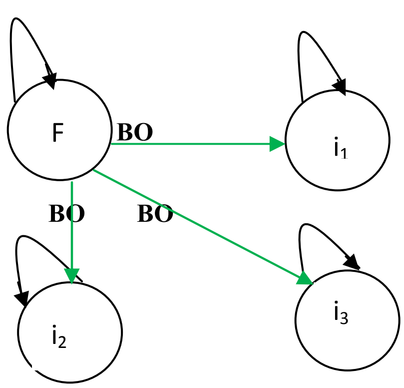
G_{44}

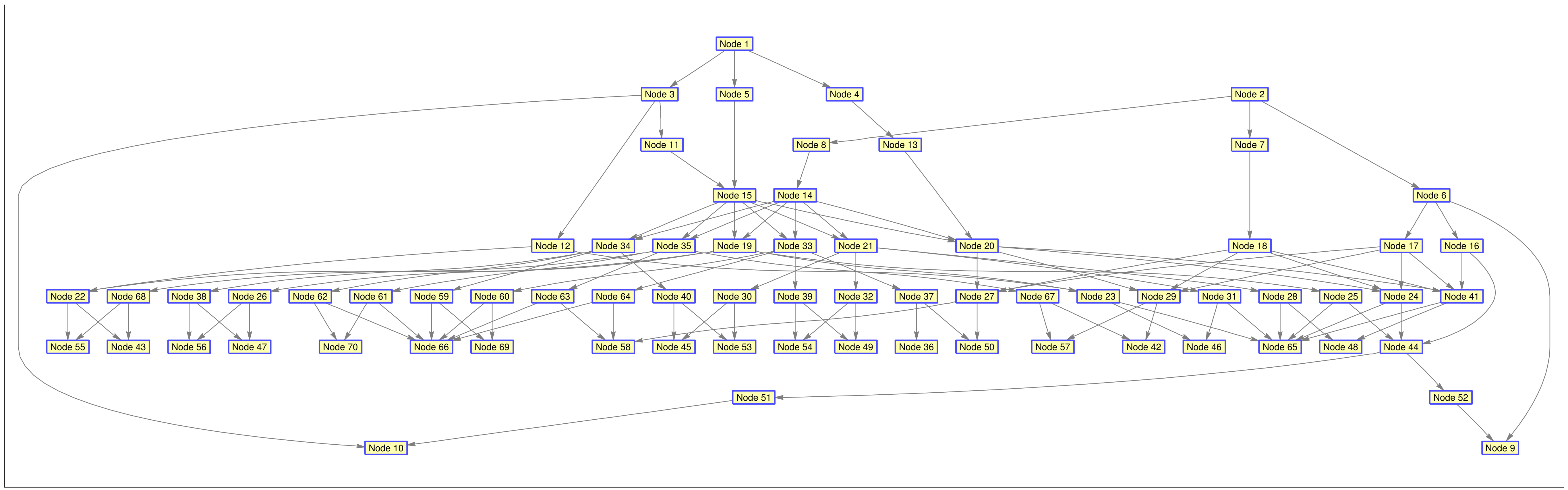


G_{45}

State of Complete Collapse With Bailouts:

G_{cB}




Node 1
Node 3
Node 5
Node 4
Node 2
Node 11
Node 8
Node 13
Node 7
Node 15
Node 14
Node 6
Node 12
Node 34
Node 35
Node 19
Node 33
Node 21
Node 20
Node 18
Node 17
Node 16
Node 22
Node 68
Node 38
Node 26
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Node 61
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Node 24
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Node 55
Node 43
Node 56
Node 47
Node 70
Node 66
Node 69
Node 58
Node 45
Node 53
Node 54
Node 49
Node 36
Node 50
Node 57
Node 42
Node 46
Node 65
Node 48
Node 44
Node 51
Node 52
Node 10
Node 9

Appendix 2: Definitions, Graphs and Tables in Chapter 3

Appendix - Formal Definitions

Now, let G denote the entire network that is under consideration.

Definition 3.1**:** (*A Network*)

Given the definitions above, a network may be defined as the set $G = A \times (N \times N)$ with a typical element given by $\{a_{ij}, (n_i, n_j)\}$ where nodes n_i and n_j are connected with a_{ij}.

Definition 3.2**:** (*A Connection*)

$$a_{ij} : X \to \Re_+$$

And, a normalized set of a_{ij}'s then may be denoted by

$$\tilde{a_{ij}} : X \to [0, 1]$$

For the binary case,

$$a_{ij} : X \to \{0, 1\}$$

For random variable, A_{ij}, define $f_{ij}(a_{ij}) = P(A_{ij} = a_{ij})$ where $f_{ij}(0)$ implies the probability that n_i and n_j are not connected. Thus, $1 - f_{ij}(0)$ denotes the probability that the nodes are connected with a_{ij}, which is the realized value of the random variable A_{ij}. Note that when $a_{ij} \neq 0$, $a_{ij} > 0$ as we are operating in $\Re_+$. Then, the joint density function over the connections between node pairs (n_i, n_j) and (n_k, n_l) as follows.

Definition 3.3**:** (*Joint Density of Connections*)

For random variables A_{ij} and A_{kl} , the joint density is defined as

$$f(a_{ij}, a_{kl}) = P(A_{ij} = a_{ij}, A_{kl} = a_{ij})$$

This refers to the probability of nodes (n_i, n_j) and nodes (n_k, n_l) not connected at the same time. So, $1 - f(a_{ij}, a_{kl})$ refers to the probability that they are connected at the same time. Since we have the marginals, we can use that as a means to estimate this joint density. In particular, we are interested in

$$f(a_{ij}, a_{ji}) = P(A_{ij} = a_{ij}, A_{ji} = a_{ji})$$

subject to the observed $a_{i.}, a_{j.}, a_{.i}$ and $a_{.j}$. This brings in the need to use a *copula* in order to estimate the joint distribution under the restriction.

Definition 3.4**:** (*Maximum Entropy Probability Distribution*)

Let X be a discrete random variable with distribution given by $P(X = x_i) = p_i$. Then the entropy of X is defined as

$$H(x) = -\sum_{i \geq 1} p(i) \log p(i)$$

And for X being a continuous random variable with probability density $p(x)$, the entropy of X may be defined as

$$H(x) = -\int_{-\infty}^{\infty} p(x) \log p(x)$$

where $p(x) \log p(x) = 0$ if $p(x) = 0$.

A *maximum entropy probability distribution* is a distribution whose entropy is at least as large as that of all other members of a specified class of distributions.

As already mentioned, this has been a popular method for estimation of joint densities given the marginals. In order to verify the goodness of fit, the most widely used measure is known as Kullback-Leibler Divergence which measures the distance between a given probability distribution **P** and the estimated or model probability distribution **Q**.

Definition 3.5**:** (*Kullback-Leibler Divergence*)

The Kullback-Leibler (K-L) divergence is expressed as

$$D_{KL}(P \parallel Q) = \sum_i p_i \ln(\frac{p_i}{q_i})$$

Thus, it is a measure of the closeness between the prior and posterior distributions.

Another concept that is widely used to identify the true distribution given the estimated distribution is known as cross-entropy. Cross entropy ($H(P,Q)$) between two distributions P and Q is a measure of the average information required to identify the true distribution P given the distribution Q.

Definition 3.6**:** (*Cross Entropy*)

Formally,

$$H(P,Q) = H(P) + D(P\|Q)$$

where $H(P)$ is the entropy of P and $D(P\|Q)$ is the K-L divergence of Q from P.

The cross-entropy problem that Fernandez-Vazquez (2010) defines may be stated as follows.

$$\min_P D(P\|Q) = \sum_i \sum_i p_{ij} \ln(\frac{p_{ij}}{q_{ij}})$$

subject to

$$\sum_{i=1}^{N} p_{ij} = 1 \; \forall j = 1, \ldots, M$$

$$\sum_{j=1}^{M} p_{ij} v_j = y_i \ \forall i = 1, \ldots, N$$

where v_j is the marginal sum of rows and y_i is the marginal sum of columns. Please see figure (20).

Definition 3.7**:** (*Copula*)

Let $X_i = [X_1, X_2, \ldots, X_m]$ be a random vector. Assume that its marginal CDFs (i.e., $F_i(x) = P(X_i \leq x)$) are continuous functions. Further, these CDFs may be transformed to uniform margins by applying the probability integral transform to each element of the random vector. Let the transformed vector be denoted as $[U_1, U_2, \ldots, U_m]$. Thus we have that

$$[U_1, U_2, \ldots, U_m] = [F_1(X_1), F_2(X_2), \ldots, F_m(X_m)]$$

A *copula* of the random vector X_i is defined as the joint cumulative distribution of the transformed random vector $[U_1, U_2, \ldots, U_m]$. That is,

$$C[U_1, U_2, \ldots, U_m] = P(U_1 \leq u_1, \ldots, U_m \leq u_m)$$

Formally then, a d-dimensional copula of a d-dimensional random vector X_i is a joint distribution function of the uniform marginals derived by transforming X_i and is defined as follows:

$$C : [0,1]^d \rightarrow [0,1]$$

Definition 3.7.1**:** (*Gumbel Copula*)

The Gumbel Copula belongs to the extreme values family. And for a bivariate case may be expressed as

$$C_G(U_1, U_2) = exp(-[(-\ln U_1)^\theta + (-\ln U_2)^\theta])^{\frac{1}{\theta}}$$

where the parameter θ controls the strength of dependence. The upper tail dependence for this copula is given by $2 - 2^{-\frac{1}{\theta}}$. And the lower dependency is 0.

Definition 3.7.2**:** (*Frank Copula*)

A bivariate Frank Copula may be expressed as

$$C_F(U_1, U_2) = -\frac{1}{\theta}.[\ln(1 + \frac{(e^{(-\theta.U_1)} - 1).(e^{(-\theta.U_2)} - 1)}{e^{-\theta} - 1})]$$

where the parameter θ controls the strength of dependence.

Definition 2.7.3**:** (*Clayton Copula*)

A bivariate Clayton Copula may be expressed as

$$C_C(U_1, U_2) = [U_1^{-\theta} + U_2^{-\theta} - 1]^{-\frac{1}{\theta}}.$$

where the parameter θ controls the strength of dependence. The upper tail dependency for this copula is 0. The lower tail dependency is given by $2^{-\frac{1}{\theta}}$.

Theorem 1**:** (*Sklar's Theorem*)

Sklar's Theorem states that an d-dimensional copula (or d-copula) is a function C from the unit d-cube $[0, 1]^d$ to the unit interval $[0, 1]$ which satisfies the following conditions:

1. $C(1, ..., 1, u_j, 1, ..., 1) = u_j$ for every $j \leq m$ and all $u_n \in [0, 1]$;

2. $C(u_1, ..., u_m) = 0$ if $u_j = 0$ for any $j \leq m$;

3. C is d-increasing.

For example, consider in a bivariate case, we have $C : [0, 1] \times [0, 1] \to [0, 1]$. This qualifies to be a copula is

1. $C(0, u) = C(u, 0) = 0$;

2. $C(1, u) = C(u, 1) = 1$;

3. $C[b_1, b_2] - C[a_1, b_2] - C[a_2, b_1] + C[a_1, a_2] \geq 0 \,\forall [a_1, b_1] \times [a_2, b_2] \subseteq [0, 1] \times [0, 1]$.

Appendix

Tables and Graphs

Table A – Mixed Results For ME and Copula Based Methods

No. of Core Nodes/Total No. of Nodes	EM Max Entropy	EM Gumbel	EM Clayton	EM Mixture [0.9*Gumbel +0.1*Clayton]	Asymmety in Share of Blocks [CC, PP, CP, PC] As % of total
3/6	0.0874	0.0843	0.6723	0.0893	[92.5652, 0 , 0.0656 , 7.3692]
3/6	0.0784	0.0772	0.6472	0.0749	[91.6254 , 0 , 0.0962, 8.2784]
3/7	0.0848	0.0926	0.6903	0.0769	[90.9727 , 0 , 0.1151, 8.9122]
3/7	0.0791	0.0808	0.6846	0.0582	[90.3807, 0, 0.1259, 9.4934]
3/10	0.0628	0.0635	0.6147	0.1015	[83.8531, 0 , 0.1660 , 15.9809]
3/10	0.1001	0.0999	0.6368	0.1433	[83.3109, 0 , 0.1531, 16.5360]
3/10	0.1036	0.0965	0.6153	0.1537	[82.7460 , 0 0.1933 , ,17.0607]
3/10	0.1215	0.1248	0.6395	0.1249	[81.6609, 0, 0.1827 , 18.1563]
3/10	0.1298	0.1175	0.6140	0.1427	[80.3947, 0 , 0.1865 , 19.4188]
3/10	0.0920	0.0870	0.6504	0.0887	[88.1364 , 0 , 0.1565 , 11.7071]

Table B: Estimates of the Dependence Parameters for the Copulas with the Corresponding Kendall's τ for PP between 50 and 100

No. of Core Nodes/Total No. of Nodes	Value of Inter	Value of Intra	Thetahat (Gumbel)	Depedence (Gumbel)	Kendall's τ (Gumbel)	Thetahat (Clayton)	Depedence (Clayton)	Kendall's τ (Clayton)
3/5	100	100	14.1086	0.9496	0.929121	39.1215	0.9824	0.951364
3/5	100	90	50.3752	0.9861	0.980149	118.4091	0.9942	0.98339
3/5	100	70	21.4195	0.9671	0.953314	34.2147	0.9799	0.944774
3/5	100	50	173.7848	0.9960	0.994246	182.7400	0.9962	0.989174
3/5	100	40	629.1111	0.9989	0.99841	437.8666	0.9984	0.995453
3/5	100	30	45.3172	0.9846	0.977933	106.6321	0.9935	0.981589
3/5	100	10	719.4131	0.999	0.99861	438.5051	0.9984	0.99546
5/10	100	10	1255.1	0.9994	0.999203	522.5210	0.9987	0.996187
2/11	100	10	4.9526	0.8498	0.798086	2.9444	0.7902	0.595502
5/11	100	10	14.6331	0.9515	0.931662	9.9174	0.9325	0.832178
10/11	100	10	254.6824	0.9973	0.996074	229.4014	0.9973	0.991357
3/5	10	100	101.7621	0.9932	0.990173	276.9699	0.9975	0.992831
5/10	10	100	355.6663	0.998	0.997188	522.2818	0.9987	0.996185
2/11	10	100	1.8917	0.5575	0.471375	0.732	0.6932	0.267936
5/11	10	100	7.0121	0.8961	0.857389	8.1523	0.9059	0.803
10/11	10	100	18.0962	0.9610	0.94474	40.0728	0.9624	0.952463

Table C: Estimates of the Dependence Parametes for the Copulas with the Corresponding Kendall's τ for PP between 500 and 1000

No. of Core Nodes/Total No. of Nodes	**Value of Inter**	**Value of Intra**	**Thetahat (Gumbel)**	**Depedence (Gumbel)**	**Kendall's τ (Gumbel)**	**Thetahat (Clayton)**	**Depedence (Clayton)**	**Kendall's τ (Clayton)**
3/5	100	100	84.7886	0.9918	0.988206	113.9990	0.9939	0.982758
3/5	100	90	152.1170	0.9954	0.993426	193.2851	0.9964	0.989759
3/5	100	70	44.564	0.9843	0.97756	28.9483	0.9763	0.935376
3/5	100	50	61.5970	0.9887	0.983765	41.7993	0.9888	0.954337
3/5	100	40	105.3679	0.9934	0.990509	65.2980	0.9934	0.970281
3/5	100	30	143.8296	0.9952	0.993047	106.2187	0.9935	0.981519
3/5	100	10	73.094	0.9905	0.986319	45.1904	0.9848	0.957618
5/10	100	10	83.147	0.9916	0.987973	64.9226	0.9894	0.970115
2/11	100	10	2.5841	0.6923	0.613018	1.9795	0.7046	0.497424
5/11	100	10	6.9283	0.8948	0.855664	4.4716	0.8564	0.690957
10/11	100	10	150.4943	0.9954	0.993355	229.4014	0.9954	0.991357
3/5	10	100	70.0074	0.9900	0.985716	56.2010	0.9901	0.965636
5/10	10	100	144.6470	0.9952	0.993087	203.8101	0.9952	0.990282
2/11	10	100	1.6375	0.4730	0.389313	0.3328	0.6549	0.142661
5/11	10	100	7.1525	0.8982	0.860189	4.9407	0.9076	0.711845
10/11	10	100	21.6136	0.9674	0.953733	41.9414	0.9684	0.954485

Table 1: **Exposure Matrix Estimated Using a Gumbel Copula Based Approach**

Country	US	GB	DE	FR	IT	FI	TR	SE	GR	CL
US	0	**779720,4**	**558819,6**	611760,2	**133285,8**	**3865,5**	4190,0	**89741,3**	**17140,1**	518,1
GB	350526,3	0	**405226,9**	434518,2	**99580,6**	**2899,2**	**3142,6**	**67136,0**	12851,6	**388,6**
DE	231484,2	**331229,5**	0	**267727,6**	**68123,7**	**2015,7**	**2184,9**	**46175,3**	8923,9	270,3
FR	**180278,4**	245604,8	**192644,8**	0	55578,7	**1692,3**	1834,2	38013,0	7473,8	227,0
IT	137937,9	**185739,1**	**146138,0**	150859,4	0	1332,1	1443,8	29666,1	5876,8	178,8
FI	31789,6	**40200,8**	**32086,0**	**32766,6**	11536,9	0	480,8	**8342,4**	1907,1	60,0
TR	24227,1	**30550,9**	**24398,2**	**24904,7**	8879,8	351,0	0	6452,3	**1501,6**	47,4
SE	**20085,1**	**25283,0**	20198,5	**20612,2**	7408,9	**298,3**	323,0	0	1272,6	40,4
GR	21878,4	**27564,6**	22017,3	22471,3	8044,7	320,9	347,5	5854,9	0	43,4
CL	**3891,1**	**4875,8**	3898,9	**3975,9**	1459,9	62,1	67,2	1073,7	262,4	0
Source: Authors' calculations										

Table 2: **Exposure Matrix Estimated Using Maximum Entropy**

Country	US	GB	DE	FR	IT	FI	TR	SE	GR	CL
US	0	748778,6	565554,4	606587,6	147321,4	4470,5	4856,8	101068,3	19796,5	606,9
GB	362811,9	0	395215,2	423889,6	102949,7	3124,0	3394,0	70627,5	13834,0	424,1
DE	231084,7	333274,7	0	269986,7	65571,4	1989,8	2161,7	44984,6	8811,2	270,1
FR	177848,2	256496,0	193732,0	0	50465,3	1531,4	1663,7	34621,2	6781,3	207,9
IT	133117,5	191984,5	145006,4	155527,2	0	1146,2	1245,3	25913,6	5075,7	155,6
FI	30451,8	43918,1	33171,5	35578,2	8640,8	0	284,9	5928,0	1161,1	35,6
TR	23212,4	33477,4	25285,6	27120,1	6586,6	199,9	0	4518,7	885,1	27,1
SE	18949,4	27329,2	20641,8	22139,5	5377,0	163,2	177,3	0	722,5	22,2
GR	20884,0	30119,2	22749,1	24399,7	5925,9	179,8	195,4	4065,4	0	24,4
CL	3738,2	5391,2	4072,0	4367,5	1060,7	32,2	35,0	727,7	142,5	0
Source: Authors' calculations										

Table 3: **Top and Bottom 5 Countries Cross Exposures**

Country	US	GB	DE	FR	IT	FI	TR	SE	GR	CL
US	**0**	**1095468**	**491136**	**529732**	**35143**	*379*	*4781*	*35410*	*5434*	*1558*
GB	**520953**	**0**	**465682**	**281450**	**46050**	*2324*	*2790*	*43516*	*13441*	*64*
DE	**175515**	**175165**	**0**	**260927**	**257408**	*2557*	*4152*	*77171*	*5185*	*55*
FR	**202470**	**269083**	**189401**	**0**	**42440**	*3434*	*1578*	*12794*	*2089*	*58*
IT	**35071**	**66387**	**162285**	**392577**	**0**	*711*	*439*	*1094*	*584*	*24*
FI	10400	3943	13773	7860	1198	***0***	***1***	***121991***	***2***	***2***
TR	19745	24682	18306	23873	4017	***0***	***0***	***234***	***30451***	***5***
SE	22778	19033	34244	13119	2745	***3405***	***166***	***0***	***24***	***8***
GR	7320	14060	26059	56740	4085	***27***	***107***	***145***	***0***	***0***
CL	7846	2948	4542	3318	813	***0***	***0***	***100***	***0***	***0***
Source: BIS - ultimate risk basis dataset										

Table 4: Effect of the Variable No. of Total Nodes on Goodness of Fit for BIS Data

No. of Core Nodes/Total No. of Nodes	EM Max Entropy	EM Gumbel	EM Clayton	EM Mixed	Share of Each Block [CC, CP, PC, PP] As a % of Total
5/6	0.2412	0.2343	0.4871	0.2548	[96.2623, 0, 1.6282, 2.1095]
5/7	0.2438	0.2416	0.4995	0.3356	[88.8449, 0.0307, 4.7542, 6.3702]
5/8	0.2988	0.2899	0.5260	0.3901	[82.9705, 0.1558, 6.5606, 10.3131]
5/10	0.3221	0.3199	0.5391	0.5146	[72.5383, 0.7122, 14.0374, 12.7121]
5/11	0.3783	0.3805	0.5724	0.5853	[62.5944, 1.7755, 19.4862, 16.1440]
5/13	0.3891	0.3903	0.5771	0.5387	[61.3118, 1.9016, 19.4595, 17.3270]
5/15	0.4044	0.4149	0.5873	0.6359	[55.7905, 2.5438, 20.7914, 20.8742]
5/18	0.4942	0.4991	0.6212	0.9048	[36.5073, 8.4333, 29.0447, 26.0147]
5/21	0.5185	0.5198	0.6350	0.9208	[34.5, 10.4128, 28.7384, 26.3488]

Table 5: Estimates of the Dependence Parametes for the Copulas with the Corresponding Kendall's τ for BIS Data with No. of Core Nodes = 5

BIS Matrices	Thetahat (Gumbel)	Upper Tail Depedence (Gumbel)	Kendall's τ (Gumbel)	Thetahat (Clayton)	Lower Tail Depedence (Clayton)	Kendall's τ (Clayton)
6 x 6	1.2941	0.2915	0.227262	1.1544	0.5486	0.365965
7 x 7	1.5782	0.4485	0.366367	1.8919	0.6932	0.486112
8 x 8	2.2012	0.6299	0.545702	2.9799	0.7925	0.598386
10 x 10	2.7681	0.7154	0.638741	3.8830	0.8365	0.660037
11 x 11	2.8209	0.7215	0.645503	3.8978	0.8371	0.660891
13 x 13	3.1645	0.7551	0.683994	5.2120	0.8755	0.722684
15 x 15	2.9434	0.7345	0.660257	5.3548	0.8786	0.728069
18 x 18	3.1435	0.7533	0.681883	4.6244	0.8608	0.698086
21 x 21	3.1552	0.7543	0.683063	5.2515	0.8028	0.724195

Table 6: Estimates of the Dependence Parametes for the Copulas with the Confidence Intervals and Estimated Log-Likelihoods for BIS Data with No. of Core Nodes = 5

BIS Matrices	Thetahat (Gumbel)	Confidence Intervals	Log-Likelihood	Thetahat (Clayton)	Confidence Intervals	Log-Likehood
6 x 6	1.2941	[0.3031, 2.2852]	-0.0683	1.1544	[-1.9515, 4.2603]	-0.0847
7 x 7	1.5782	[0.4358, 2.7206]	-0.1961	1.8919	[-2.4267, 6.2104]	-0.2944
8 x 8	2.2012	[1.1509, 3.2515]	-0.3832	2.9799	[0.1926, 5.7671]	-0.3815
10 x 10	2.7681	[1.7397, 3.7964]	-0.8012	3.8830	[0.7464, 7.0196]	-0.6181
11 x 11	2.8209	[1.8471, 3.7947]	-0.6923	3.8978	[0.9796, 6.8159]	-0.5735
13 x 13	3.1645	[2.1930, 4.1359]	-0.7508	5.2120	[2.0552, 8.3688]	-0.7002
15 x 15	2.9434	[1.9285, 3.9584]	-0.9934	5.3548	[2.1635, 8.5460]	-0.8309
18 x 18	3.1435	[2.2170, 4.0700]	-0.7621	4.6244	[2.1733, 7.0754]	-0.8273
21 x 21	3.1552	[2.2502, 4.0603]	-0.4675	5.2515	[2.9866, 7.5164]	-0.9151

Figure 1: Dividing a Network into Blocks

Nodes	Node 0	Node 1	Node 2	Node 3	Node 4	Sum of Rows
Node 0	0	1	1	0	0	2
Node 1	0	0	1	1	1	3
Node 2	0	1	0	0	0	1
Node 3	1	0	0	0	0	1
Node 4	1	1	1	0	0	3
Sum of Columns	2	3	3	1	1	10

BLOCK 1 BLOCK 2

BLOCK 3 BLOCK 4

Figure 2: Performance of Gumbel vs. ME under the First DGP

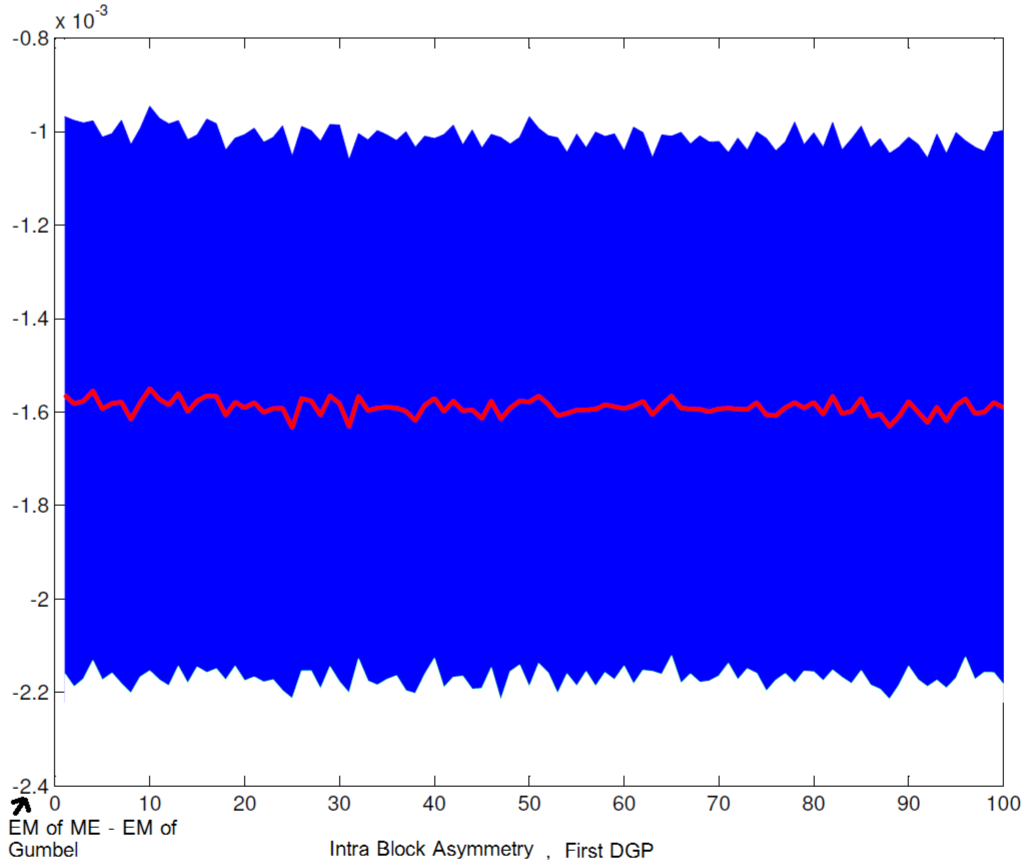


Figure 3: Performance of Gumbel vs. ME under the First DGP with Variable Network Size

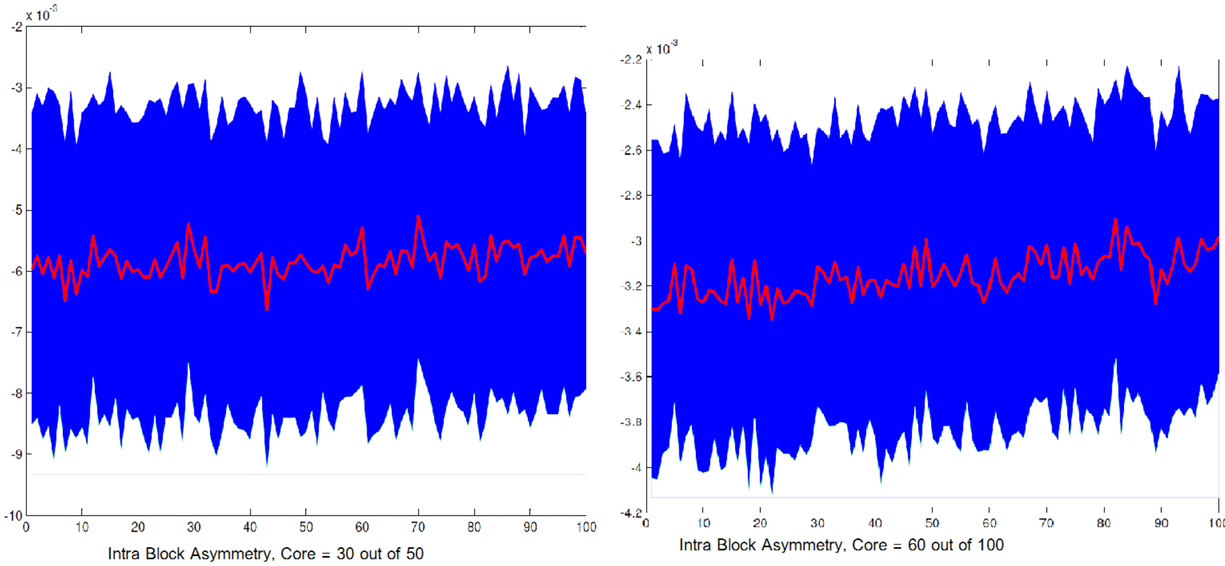


Figure 4: Performance of Gumbel vs. ME under the Second DGP

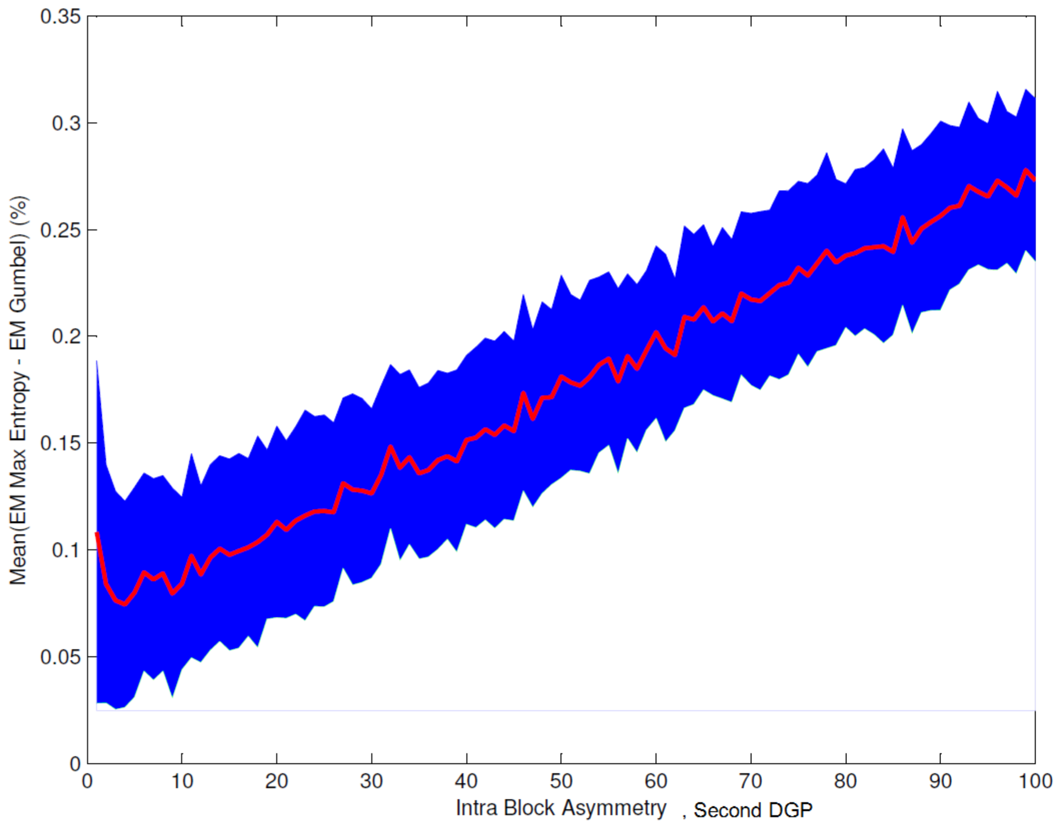


Figure 5: Performance of Gumbel vs. ME under the Second DGP with Lower PP Values

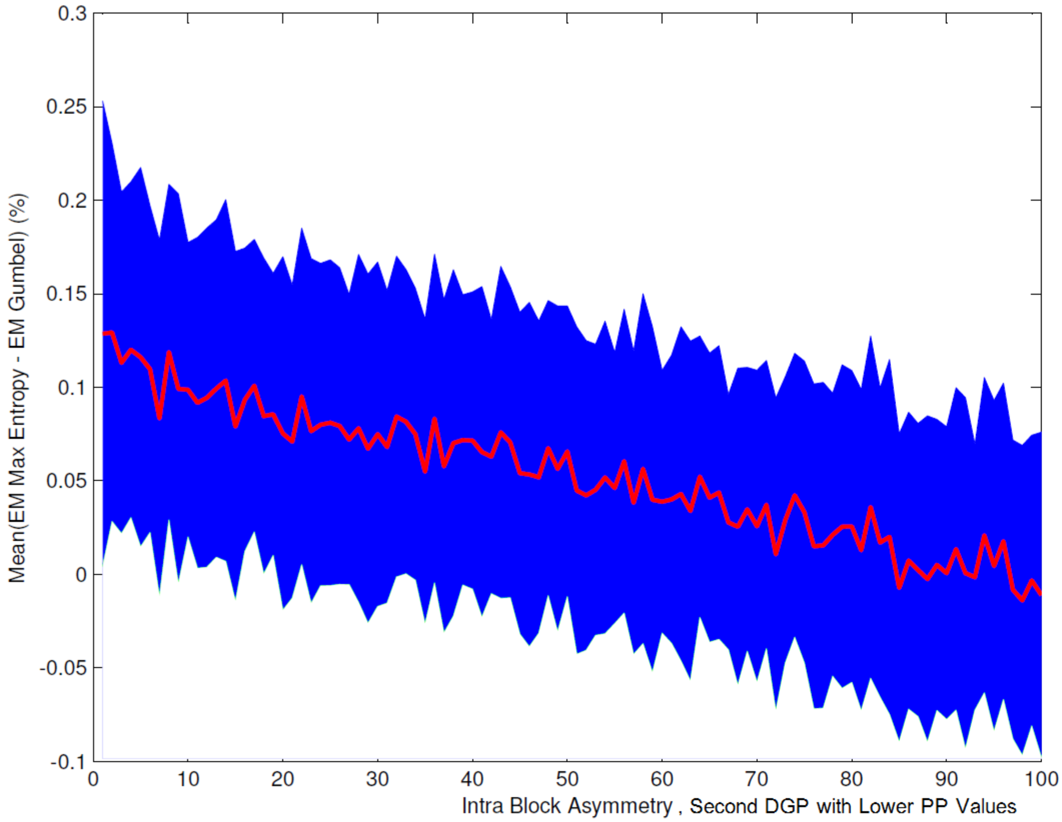


Figure 6: Performance of Gumbel vs. ME with Variable Inter Block Asymmetry Using First DGP

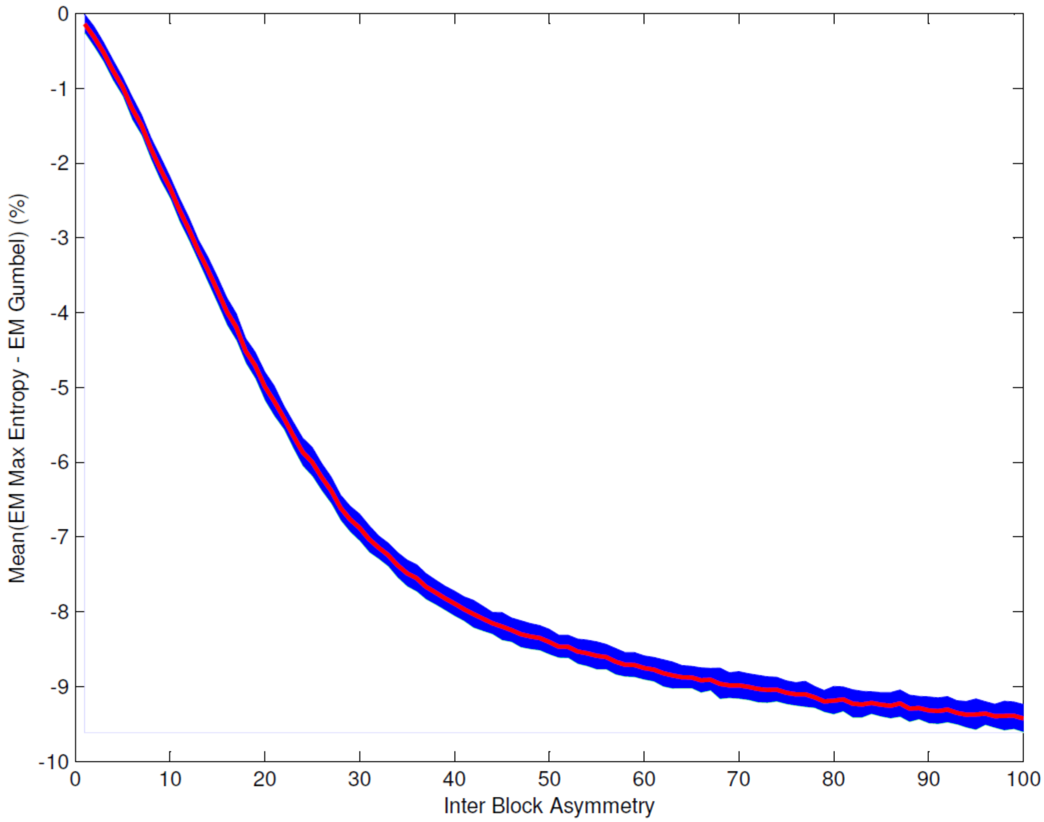


Figure 7: Performance of Gumbel vs. ME with Variable Inter Block Asymmetry Using Second DGP

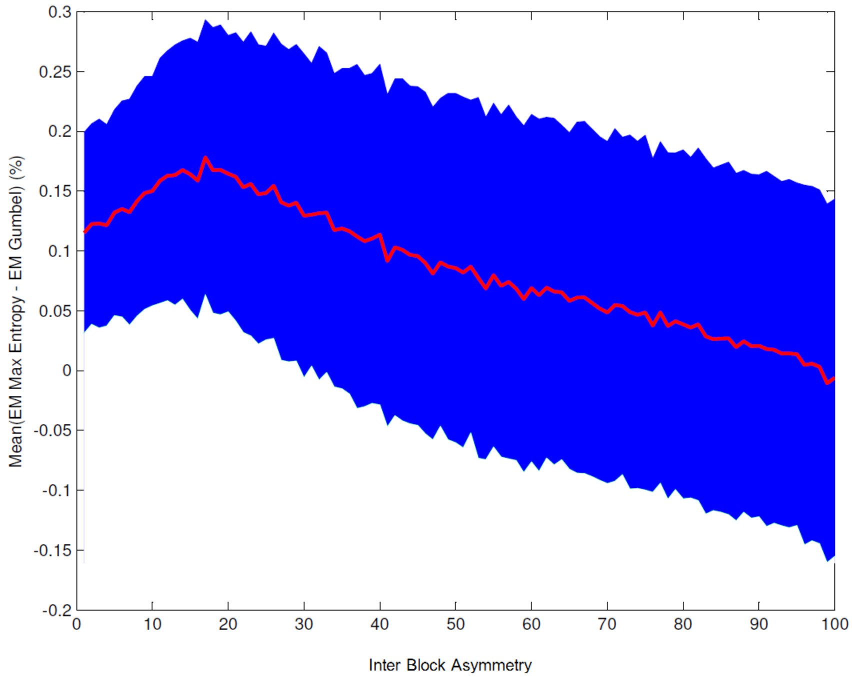


Figure 8: Performance of Gumbel vs. ME with Variable Inter Block Asymmetry Using Second DGP With Different Core Sizes

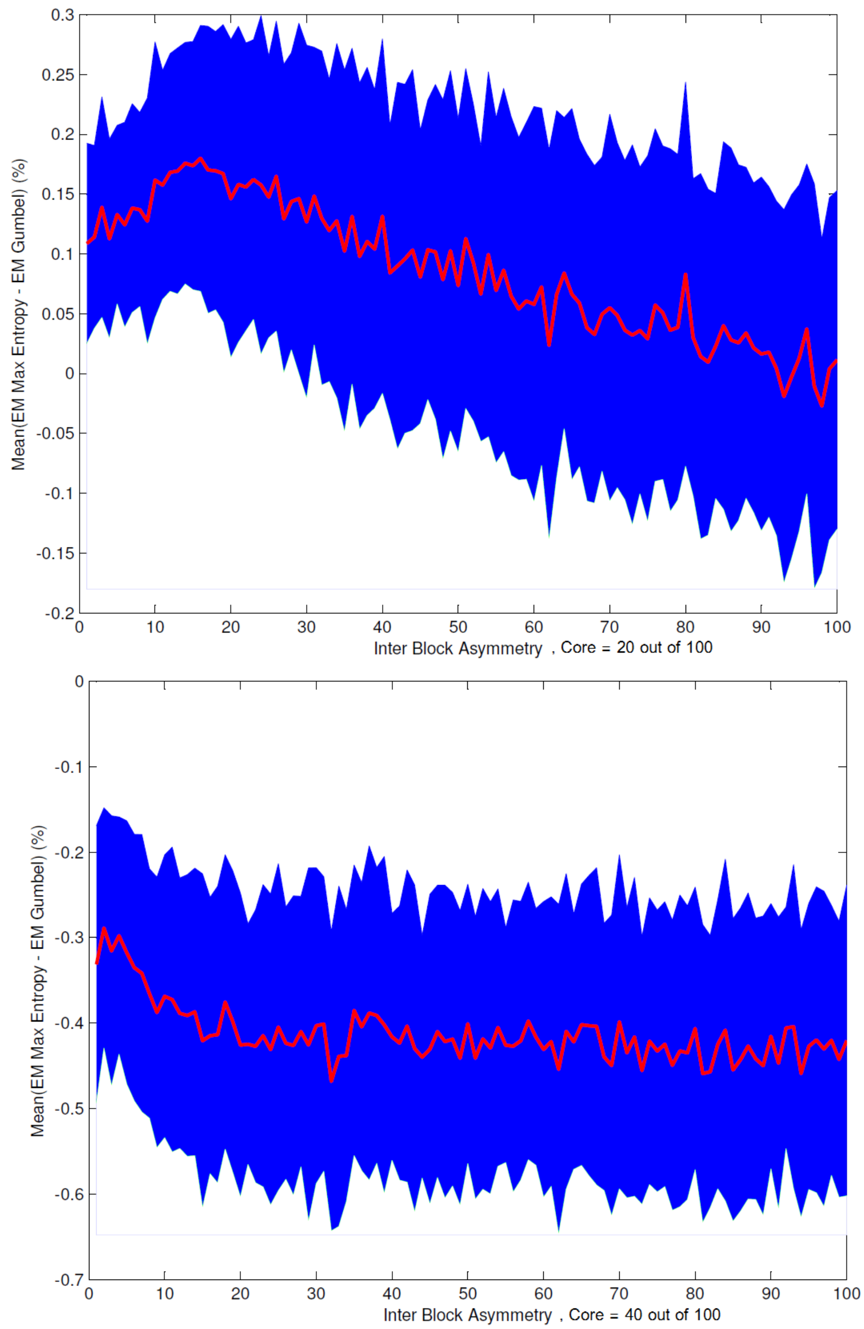


Figure 9: Impact of Variable Intra Block Asymmetry with Low Inter Block Asymmetry on the Performance of ME and Copula Based Estimators

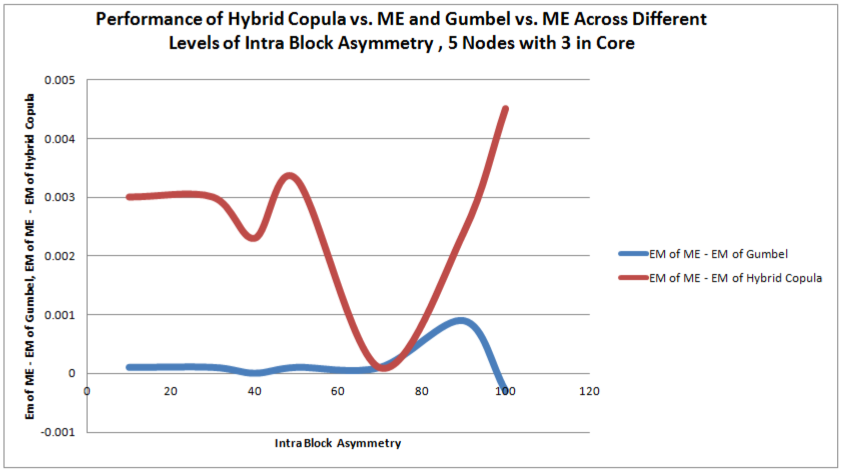
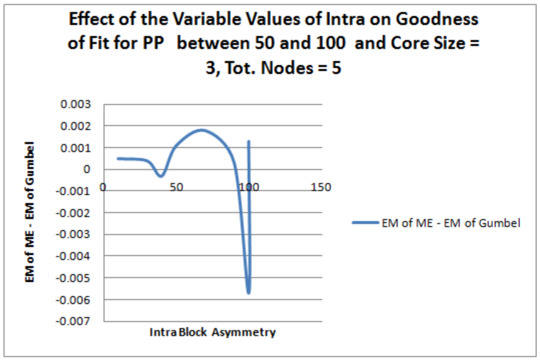
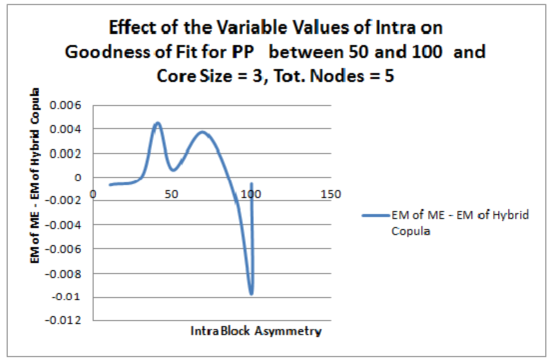


Figure 10: Impact of Variable Intra Block Asymmetry with High Inter Block Asymmetry on the Performance of ME and Copula Based Estimators

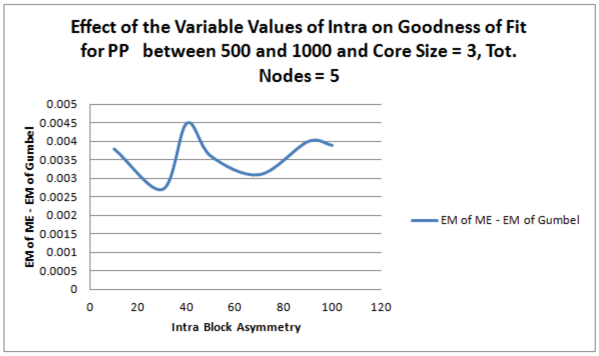


Direct Method

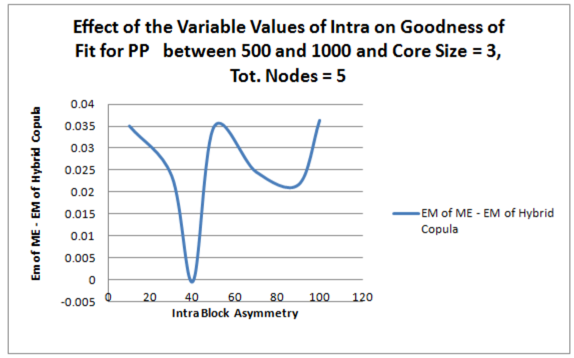


Indirect Method Using Hybrid Copula as a Prior

Figure 11: Impact of Variable Intra Block Asymmetry with Low Inter Block Asymmetry on the Performance of ME and Copula Based Estimators with PP between 500 and 1000

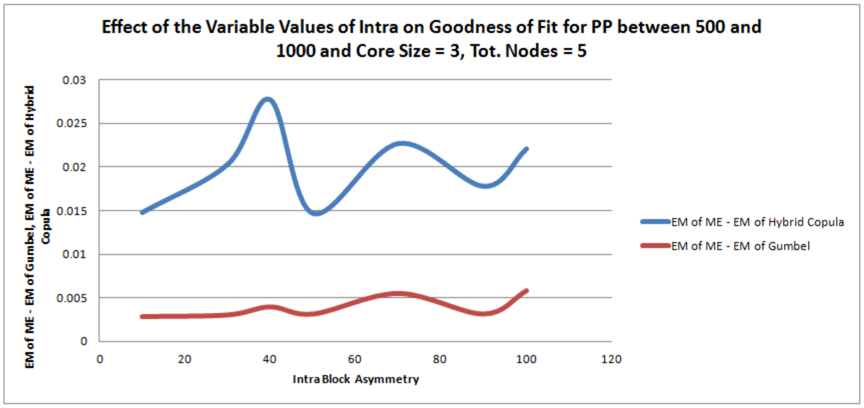


Direct Method



Indirect Method With Hybrid Copula as a Prior

Figure 12: Impact of Variable Intra Block Asymmetry with High Inter Block Asymmetry on the Performance of ME and Copula Based Estimators with PP between 500 and 1000



Graph for Performance of both Direct and Indirect Method Applied to a Small Network

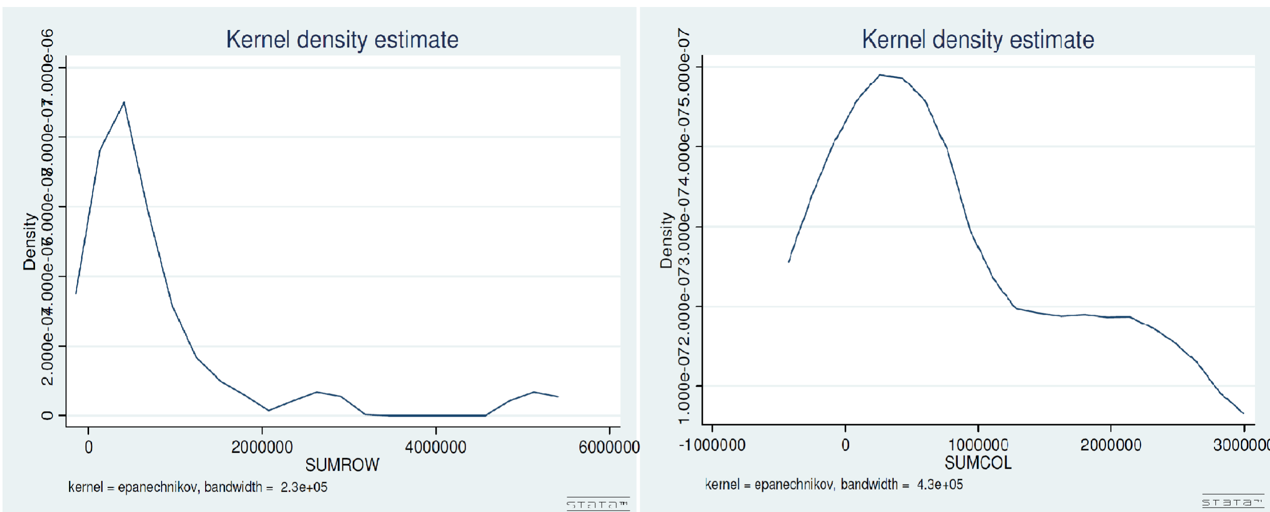
Figure 13: BIS Data

STRUCTURE OF THE BIS DATA

Countries	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	**Sum**
N_1	a_{11}	a_{12}	a_{13}	...	...	...	...	...	...	a_{110}	
N_2	a_{21}						CORE - PERIPHERY			:	
N_3	:		CORE-CORE							:	
N_4	:									:	
N_5	:									:	
N_6	:					a_{ii}	a_{ij}			:	
N_7	:	PERIPHERY-CORE					PERIPHERY-PERIPHERY			:	
N_8	:									:	
N_9	:									:	
N_{10}	a_{101}	a_{102}	a_{103}	...	...	...	...	...	...	a_{1010}	
Sum											

Assumption: $N_1 – N_6$ are core nodes

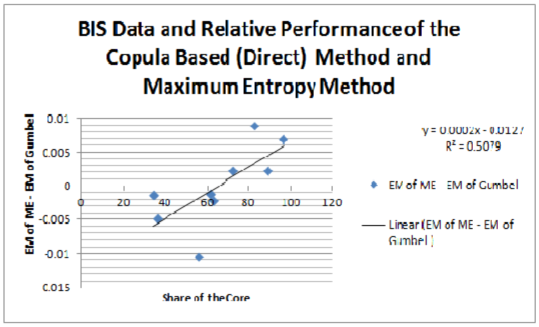
Figure 14: Densities of the Marginals of BIS Data



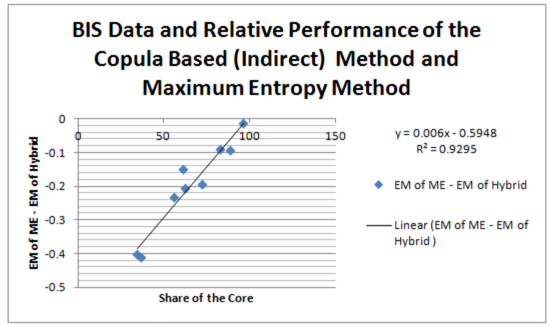
Density of the Marginals(Sum of Rows)

Density of the Marginals(Sum of Columns)

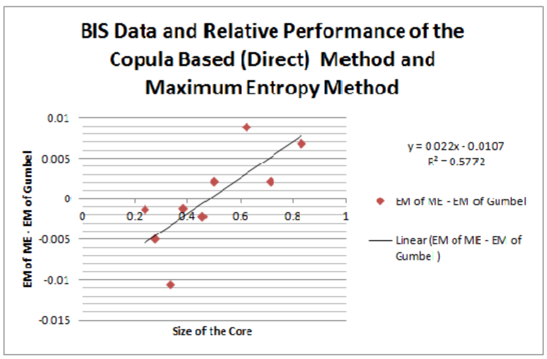
Figure 15: Performance of Copula Based Methods vs. Maximum Entropy with BIS Data



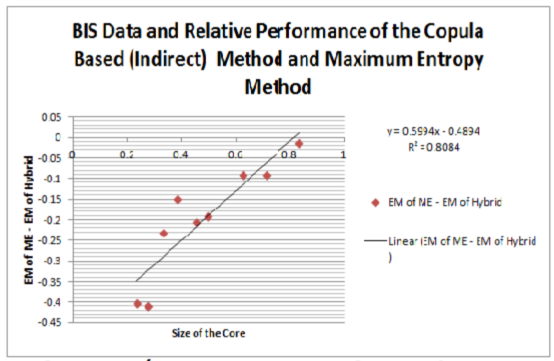
BIS Data (Asymmetry in Share of the Core) - Direct Method



BIS Data (Asymmetry in Share of the Core) - Indirect Method



BIS Data (Asymmetry in Size of the Core) - Direct Method



BIS Data (Asymmetry in Size of the Core) - Indirect Method

Figure 16: eMID Data Representation

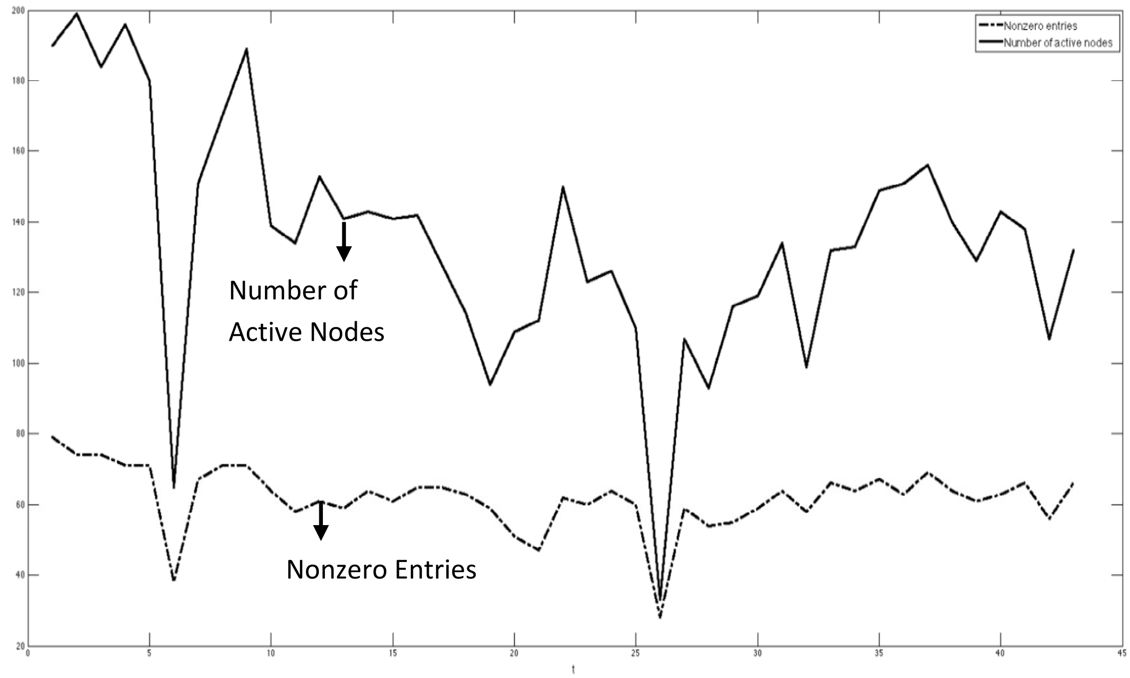
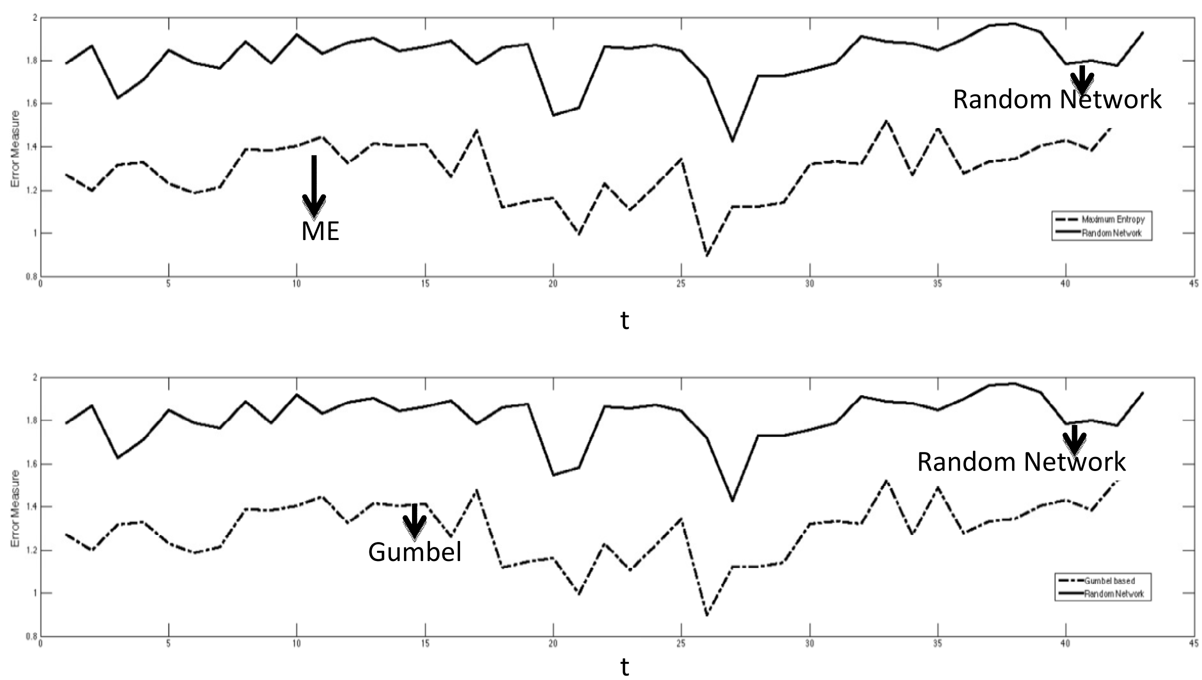


Figure 17: Comparison of Performances of the Methods using eMID Data



Appendix 3: Graphs and Tables in Chapter 4

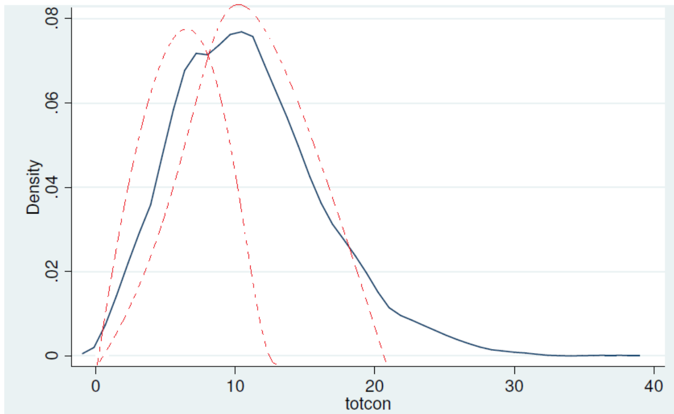


Figure 1: Density with two ‘humps’

Legend for Tables:

BVP: BIVARIATE PROBIT

AME: AVERAGE MARGINAL EFFECTS

NB2: NEGATIVE BINOMIAL (2)

OP: ORDERED PROBIT

MNP: MULTINOMIAL PROBIT

BF: BEST FRIEND

PROB.: PROBABILITIES

ADD.: ADDITIONAL

TABLE 1: BVP FOR RESPONDENT'S NOMINATIONS

Bivariate probit regression

Log pseudolikelihood = **-1900.1081**

Number of obs	=	2358
Wald chi2(18)	=	533.64
Prob > chi2	=	0.0000

	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
havebmf						
ehsgrd	.3513306	.1958324	1.79	0.073	-.0324939	.7351552
ersngrd	.1643735	.2125029	0.77	0.439	-.2521246	.5808715
ehsrc5	.0066003	.229671	0.03	0.977	-.4435467	.4567473
esrnrc5	.2739586	.29891	0.92	0.359	-.3118942	.8598114
ehsage	.0535035	.2183195	0.25	0.806	-.3743949	.4814019
ersnage	-.147507	.3279368	-0.45	0.653	-.7902514	.4952374
bcent10x	.6486769	.0635772	10.20	0.000	.5240679	.7732858
reach	3.99e-06	.0000842	0.05	0.962	-.000161	.0001689
sexdummy	.868785	.0643525	13.50	0.000	.7426564	.9949135
_cons	-.5455359	.1281253	-4.26	0.000	-.7966569	-.2944148
havebff						
ehsgrd	.0302075	.2206749	0.14	0.891	-.4023074	.4627225
ersngrd	.3791888	.2436432	1.56	0.120	-.0983431	.8567206
ehsrc5	-.7247413	.2670148	-2.71	0.007	-1.248081	-.201402
esrnrc5	.7353066	.3685065	2.00	0.046	.0130471	1.457566
ehsage	.5824804	.2447632	2.38	0.017	.1027533	1.062208
ersnage	-.3239719	.3755689	-0.86	0.388	-1.060073	.4121296
bcent10x	.6623688	.074138	8.93	0.000	.5170611	.8076766
reach	.0001098	.000095	1.16	0.248	-.0000765	.000296
sexdummy	-.8706745	.0690404	-12.61	0.000	-1.005991	-.7353579
_cons	.3502899	.1442875	2.43	0.015	.0674916	.6330882
/athrho	-4.53708	1.503271	-3.02	0.003	-7.483436	-1.590724
rho	-.9997708	.0006889			-.9999994	-.9202603

Wald test of rho=0: chi2(1) = **9.10917** Prob > chi2 = **0.0025**

TABLE 2: BVP FOR RESPONDENT'S NOMINATIONS - PREDICTED PROB.

Variable	Obs	Mean	Std. Dev.	Min	Max
havebmf	3010	.7421927	.4375	0	1
havebff	3010	.8259136	.3792467	0	1
biprob1	2358	.738392	.1648435	.3312134	.9992626
biprob2	2358	.8367962	.1397014	.3237986	.999876
biprob11	2358	.5751881	.2007149	.0233601	.9955496
biprob10	2358	.1632038	.1397014	.0001241	.6762013
biprob01	2358	.261608	.1648435	.0007375	.6687866
biprob00	2358	1.71e-09	7.22e-08	0	3.48e-06

Where:

- biprob1 is the marginal probability that $y_1 = 1$.
- biprob2 is the marginal probability that $y_2 = 1$.
- biprob11 is the joint probability that $(y_1, y_2) = (1, 1)$.
- biprob10 is the joint probability that $(y_1, y_2) = (1, 0)$.
- biprob01 is the joint probability that $(y_1, y_2) = (0, 1)$.
- biprob00 is the joint probability that $(y_1, y_2) = (0, 0)$

TABLE 3: AME FOR BVP OF THE RESPONDENT'S NOMINATIONS

```
Average marginal effects                          Number of obs     =       2358
Model VCE    : Robust

Expression   : Pr(havebmf=1,havebff=1), predict()
dy/dx w.r.t. : ehsgrd ersngrd ehsrc5 esrnrc5 ehsage ersnage bcent10x reach sexdummy
```

	dy/dx	Delta-method Std. Err.	z	P>|z|	[95% Conf.	Interval]
ehsgrd	.1054228	.0606869	1.74	0.082	-.0135214	.224367
ersngrd	.1254	.0687265	1.82	0.068	-.0093013	.2601014
ehsrc5	-.1491725	.0714873	-2.09	0.037	-.2892849	-.0090601
esrnrc5	.2305337	.1015367	2.27	0.023	.0315255	.4295419
ehsage	.1364838	.0657871	2.07	0.038	.0075435	.2654241
ersnage	-.1091341	.1080413	-1.01	0.312	-.3208912	.102623
bcent10x	.32106	.0191039	16.81	0.000	.283617	.3585029
reach	.000024	.0000256	0.94	0.348	-.0000262	.0000742
sexdummy	.0636799	.0173293	3.67	0.000	.0297151	.0976446

TABLE 4: OVERALL SIGNIFICANCE OF EHSRC5, EHSAGE, BCENT10X AND SEXDUMMY

```
 ( 1)  [havebmf]ehsrc5 = 0
 ( 2)  [havebff]ehsrc5 = 0

           chi2(  2) =    8.04
         Prob > chi2 =    0.0179
```

```
 ( 1)  [havebmf]ehsage = 0
 ( 2)  [havebff]ehsage = 0

           chi2(  2) =    6.85
         Prob > chi2 =    0.0326
```

```
 ( 1)  [havebmf]bcent10x = 0
 ( 2)  [havebff]bcent10x = 0

           chi2(  2) =  232.10
         Prob > chi2 =    0.0000
```

```
 ( 1)  [havebmf]sexdummy = 0
 ( 2)  [havebff]sexdummy = 0

           chi2(  2) =  259.63
         Prob > chi2 =    0.0000
```

TABLE 5: BVP FOR RESPONDENT BEING NOMINATED BY NOMINEES

Bivariate probit regression — Number of obs = 983; Wald chi2(20) = 235.27; Prob > chi2 = 0.0000

Log pseudolikelihood = -1181.4675

	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
bmfrecip						
ehrgrd	.2437613	.2750308	0.89	0.375	-.2952891	.7828118
errngrd	-.2883108	.2978246	-0.97	0.333	-.8720363	.2954148
ehrrc5	-.9634329	.3454776	-2.79	0.005	-1.640557	-.2863093
errnrc5	1.103092	.40672	2.71	0.007	.3059349	1.900248
ehrage	.3585654	.325191	1.10	0.270	-.2787972	.9959281
errnage	-.0591569	.5008711	-0.12	0.906	-1.040846	.9225324
bcent10x	.2469689	.0738848	3.34	0.001	.1021574	.3917805
prxprest	3.458521	.8169724	4.23	0.000	1.857284	5.059757
infldmn	.00048	.0001328	3.61	0.000	.0002197	.0007404
sexdummy	.5220559	.0844774	6.18	0.000	.3564832	.6876286
_cons	-1.5845	.2204173	-7.19	0.000	-2.01651	-1.15249
bffrecip						
ehrgrd	.0156266	.2859746	0.05	0.956	-.5448733	.5761265
errngrd	-.1298444	.3236337	-0.40	0.688	-.7641548	.504466
ehrrc5	-.3561528	.3536385	-1.01	0.314	-1.049272	.336966
errnrc5	.468512	.4205525	1.11	0.265	-.3557558	1.29278
ehrage	.0978136	.3425385	0.29	0.775	-.5735495	.7691767
errnage	.8485439	.5528273	1.53	0.125	-.2349777	1.932065
bcent10x	.1349264	.0747606	1.80	0.071	-.0116016	.2814544
prxprest	3.690909	1.047458	3.52	0.000	1.637929	5.743889
infldmn	.0004039	.0001359	2.97	0.003	.0001376	.0006703
sexdummy	-.8438508	.0861505	-9.80	0.000	-1.012703	-.6749989
_cons	-.6960922	.2301785	-3.02	0.002	-1.147234	-.2449506
/athrho	.0112178	.0570089	0.20	0.844	-.1005176	.1229532
rho	.0112174	.0570017			-.1001804	.1223374

Wald test of rho=0: chi2(1) = .03872 Prob > chi2 = 0.8440

TABLE 6: BVP FOR RESPONDENT BEING NOMINATED - PREDICTED PROB.

Variable	Obs	Mean	Std. Dev.	Min	Max
bmfrecip	1963	.5435558	.4982262	0	1
bffrecip	2312	.6271626	.483664	0	1
biprobq1	2262	.5037801	.1665067	.0634746	.9913036
biprobq2	2262	.6072472	.1946064	.0899726	.9997432
biprobq11	2262	.3085446	.1444934	.0135769	.9840146
biprobq10	2262	.1952355	.1203659	.0002523	.4527909
biprobq01	2262	.2987026	.1389579	.0086268	.5755479
biprobq00	2262	.1975173	.12437	4.50e-06	.7773798

Where:

- biprobq1 is the marginal probability that $y_1 = 1$.
- biprobq2 is the marginal probability that $y_2 = 1$.
- biprobq11 is the joint probability that $(y_1, y_2) = (1, 1)$.
- biprobq10 is the joint probability that $(y_1, y_2) = (1, 0)$.
- biprobq01 is the joint probability that $(y_1, y_2) = (0, 1)$.
- biprobq00 is the joint probability that $(y_1, y_2) = (0, 0)$

TABLE 7: AME FOR BVP OF THE RESPONDENT BEING NOMINATED

```
Average marginal effects                          Number of obs   =        983
Model VCE    : Robust

Expression   : Pr(bmfrecip=1,bffrecip=1), predict()
dy/dx w.r.t. : ehrgrd errngrd ehrrc5 errnrc5 ehrage errnage bcent10x prxprest infldmn sexdummy
```

	dy/dx	Delta-method Std. Err.	z	P>|z|	[95% Conf.	Interval]
ehrgrd	.0580887	.0815681	0.71	0.476	-.1017818	.2179591
errngrd	-.0891492	.0903423	-0.99	0.324	-.2662168	.0879184
ehrrc5	-.2836331	.1039479	-2.73	0.006	-.4873672	-.0798989
errnrc5	.3358977	.1229717	2.73	0.006	.0948776	.5769177
ehrage	.0991838	.0956618	1.04	0.300	-.0883099	.2866775
errnage	.1423786	.1568645	0.91	0.364	-.1650701	.4498272
bcent10x	.0807169	.0215599	3.74	0.000	.0384603	.1229735
prxprest	1.461061	.2580706	5.66	0.000	.9552521	1.96687
infldmn	.0001829	.0000399	4.59	0.000	.0001048	.000261
sexdummy	-.0366553	.0234497	-1.56	0.118	-.0826159	.0093053

TABLE 8: OVERALL SIGNIFICANCE OF THE SIX SIGNIFICANT REGRESSORS

```
 ( 1)  [bmfrecip]ehrrc5 = 0
 ( 2)  [bffrecip]ehrrc5 = 0

           chi2(  2) =     8.45
         Prob > chi2 =     0.0146
```

```
 ( 1)  [bmfrecip]errnrc5 = 0
 ( 2)  [bffrecip]errnrc5 = 0

           chi2(  2) =     8.21
         Prob > chi2 =     0.0165
```

```
 ( 1)  [bmfrecip]bcent10x = 0
 ( 2)  [bffrecip]bcent10x = 0

           chi2(  2) =    14.22
         Prob > chi2 =     0.0008
```

```
 ( 1)  [bmfrecip]prxprest = 0
 ( 2)  [bffrecip]prxprest = 0

           chi2(  2) =    30.71
         Prob > chi2 =     0.0000
```

```
 ( 1)  [bmfrecip]infldmn = 0
 ( 2)  [bffrecip]infldmn = 0

           chi2(  2) =    20.47
         Prob > chi2 =     0.0000
```

```
 ( 1)  [bmfrecip]sexdummy = 0
 ( 2)  [bffrecip]sexdummy = 0

           chi2(  2) =   135.41
         Prob > chi2 =     0.0000
```

TABLE 9: BVP FOR RESPONDENT BEING NOMINATED AS BEST FRIEND

```
Bivariate probit regression                       Number of obs   =       2262
                                                  Wald chi2(20)   =     397.94
Log pseudolikelihood = -2336.4455                 Prob > chi2     =     0.0000
```

	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
bmfrecbf						
ehrgrd	.3641307	.1934267	1.88	0.060	-.0149786	.7432401
errngrd	-.137427	.2095833	-0.66	0.512	-.5482027	.2733487
ehrrc5	-.7567466	.2473489	-3.06	0.002	-1.241542	-.2719516
errnrc5	.903496	.2929999	3.08	0.002	.3292267	1.477765
ehrage	-.173766	.2329195	-0.75	0.456	-.6302798	.2827478
errnage	.3453678	.3669258	0.94	0.347	-.3737936	1.064529
bcent10x	.2610394	.0525889	4.96	0.000	.1579671	.3641117
prxprest	1.887047	.6051447	3.12	0.002	.700985	3.073109
infldmn	.0002438	.0000958	2.55	0.011	.0000561	.0004315
sexdummy	.4950799	.0613853	8.07	0.000	.3747669	.6153929
_cons	-2.065557	.1475396	-14.00	0.000	-2.354729	-1.776384
bffrecbf						
ehrgrd	.4358656	.1859214	2.34	0.019	.0714663	.8002648
errngrd	-.5138729	.2101436	-2.45	0.014	-.9257469	-.101999
ehrrc5	-.123632	.2336063	-0.53	0.597	-.5814919	.3342279
errnrc5	-.0511817	.2870213	-0.18	0.858	-.6137331	.5113697
ehrage	-.0748734	.2172481	-0.34	0.730	-.5006718	.3509249
errnage	.6910839	.347656	1.99	0.047	.0096908	1.372477
bcent10x	.0651063	.0502462	1.30	0.195	-.0333744	.163587
prxprest	2.259699	.5461911	4.14	0.000	1.189184	3.330214
infldmn	.0001964	.0000879	2.23	0.025	.0000241	.0003686
sexdummy	-.7810609	.0600932	-13.00	0.000	-.8988415	-.6632803
_cons	-.8465648	.132567	-6.39	0.000	-1.106391	-.5867382
/athrho	-.0245785	.0421306	-0.58	0.560	-.1071529	.0579959
rho	-.0245735	.0421051			-.1067446	.057931

```
Wald test of rho=0:                 chi2(1) =  .340342    Prob > chi2 = 0.5596
```

<u>TABLE 10: PREDICTED PROBABILITIES FOR BVP IN TABLE 5</u>

Variable	Obs	Mean	Std. Dev.	Min	Max
bmfrecbf	3010	.1966777	.3975527	0	1
bffrecbf	3010	.2740864	.4461268	0	1
biprobb1	2262	.2068989	.1085178	.0208044	.830245
biprobb2	2262	.2930526	.1398193	.0418379	.885325
biprobb11	2262	.0550727	.0427419	.0022174	.67079
biprobb10	2262	.1518263	.09037	.0171782	.5130407
biprobb01	2262	.2379799	.1206245	.0389843	.4810516
biprobb00	2262	.5551212	.1172553	.0261319	.9021628

Where:

- biprobq1 is the marginal probability that $y_1 = 1$.
- biprobq2 is the marginal probability that $y_2 = 1$.
- biprobq11 is the joint probability that $(y_1, y_2) = (1, 1)$.
- biprobq10 is the joint probability that $(y_1, y_2) = (1, 0)$.
- biprobq01 is the joint probability that $(y_1, y_2) = (0, 1)$.
- biprobq00 is the joint probability that $(y_1, y_2) = (0, 0)$

TABLE 11: AME FOR BVP OF THE RESPONDENT BEING NOMINATED

```
Average marginal effects                          Number of obs   =       2262
Model VCE    : Robust

Expression   : Pr(bmfrecbf=1,bffrecbf=1), predict()
dy/dx w.r.t. : ehrgrd errngrd ehrrc5 errnrc5 ehrage errnage bcent10x prxprest infldmn sexdummy
```

	dy/dx	Delta-method Std. Err.	z	P>|z|	[95% Conf.	Interval]
ehrgrd	.053456	.0185524	2.88	0.004	.0170939	.0898181
errngrd	-.041635	.0201893	-2.06	0.039	-.0812053	-.0020648
ehrrc5	-.0630474	.0238375	-2.64	0.008	-.1097682	-.0163267
errnrc5	.0630631	.028748	2.19	0.028	.0067182	.1194081
ehrage	-.0173324	.0213714	-0.81	0.417	-.0592197	.0245548
errnage	.0677578	.0342314	1.98	0.048	.0006654	.1348502
bcent10x	.0231278	.005333	4.34	0.000	.0126752	.0335803
prxprest	.2770819	.0577005	4.80	0.000	.1639909	.3901729
infldmn	.0000299	9.15e-06	3.27	0.001	.000012	.0000479
sexdummy	-.0116978	.0053573	-2.18	0.029	-.0221979	-.0011976

TABLE 12: OVERALL SIGNIFICANCE OF THE SIX SIGNIFICANT REGRESSORS

```
( 1)  [bmfrecbf]ehrgrd = 0
( 2)  [bffrecbf]ehrgrd = 0

          chi2(  2) =     9.04
        Prob > chi2 =   0.0109

( 1)  [bmfrecbf]ehrrc5 = 0
( 2)  [bffrecbf]ehrrc5 = 0

          chi2(  2) =     9.55
        Prob > chi2 =   0.0084

( 1)  [bmfrecbf]bcent10x = 0
( 2)  [bffrecbf]bcent10x = 0

          chi2(  2) =    25.89
        Prob > chi2 =   0.0000

( 1)  [bmfrecbf]infldmn = 0
( 2)  [bffrecbf]infldmn = 0

          chi2(  2) =    11.31
        Prob > chi2 =   0.0035

( 1)  [bmfrecbf]errngrd = 0
( 2)  [bffrecbf]errngrd = 0

          chi2(  2) =     6.43
        Prob > chi2 =   0.0402

( 1)  [bmfrecbf]errnrc5 = 0
( 2)  [bffrecbf]errnrc5 = 0

          chi2(  2) =     9.62
        Prob > chi2 =   0.0081

( 1)  [bmfrecbf]prxprest = 0
( 2)  [bffrecbf]prxprest = 0

          chi2(  2) =    28.00
        Prob > chi2 =   0.0000

( 1)  [bmfrecbf]sexdummy = 0
( 2)  [bffrecbf]sexdummy = 0

          chi2(  2) =   232.00
        Prob > chi2 =   0.0000
```

TABLE 13: BVP (RESPONDENT BEING NOMINATED AS BF) WITH ADD. REGRESSOR

Bivariate probit regression — Number of obs = **2262**; Wald chi2(22) = 1030.15; Prob > chi2 = 0.0000

Log pseudolikelihood = **-1796.7474**

	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
bmfrecbf						
ehrgrd	.1616408	.222937	0.73	0.468	-.2753077	.5985893
errngrd	.1211247	.2488214	0.49	0.626	-.3665562	.6088056
ehrrc5	-.6251636	.2794113	-2.24	0.025	-1.1728	-.0775275
errnrc5	.7284765	.3255305	2.24	0.025	.0904484	1.366505
ehrage	-.3998157	.2642327	-1.51	0.130	-.9177022	.1180708
errnage	.1811208	.4206863	0.43	0.667	-.6434091	1.005651
bcent10x	-.0568755	.0622413	-0.91	0.361	-.1788663	.0651152
prxprest	-.5639459	.6706292	-0.84	0.400	-1.878355	.7504633
infldmn	.0000262	.0001068	0.25	0.806	-.0001831	.0002354
anyfriendr~p	1.317497	.0644978	20.43	0.000	1.191083	1.44391
sexdummy	.6816828	.0719323	9.48	0.000	.5406981	.8226674
_cons	-2.581754	.1601684	-16.12	0.000	-2.895678	-2.267829
bffrecbf						
ehrgrd	.2605665	.2065886	1.26	0.207	-.1443397	.6654726
errngrd	-.3367267	.2361427	-1.43	0.154	-.799558	.1261045
ehrrc5	.134707	.2591318	0.52	0.603	-.373182	.6425959
errnrc5	-.4198048	.321221	-1.31	0.191	-1.049386	.2097768
ehrage	-.2098207	.2370633	-0.89	0.376	-.6744562	.2548147
errnage	.5943113	.385291	1.54	0.123	-.1608452	1.349468
bcent10x	-.2227041	.0564247	-3.95	0.000	-.3332945	-.1121138
prxprest	.1676584	.586365	0.29	0.775	-.9815958	1.316913
infldmn	.0000254	.0000935	0.27	0.786	-.0001579	.0002088
anyfriendr~p	1.172194	.0542308	21.61	0.000	1.065903	1.278484
sexdummy	-.9011851	.0697023	-12.93	0.000	-1.037799	-.764571
_cons	-1.113862	.1355318	-8.22	0.000	-1.3795	-.848225
/athrho	-.525981	.044966	-11.70	0.000	-.6141128	-.4378492
rho	-.482303	.0345062			-.5470157	-.4118601

Wald test of rho=0: chi2(1) = 136.827 Prob > chi2 = 0.0000

TABLE 14: PREDICTED PROBABILITIES FOR BVP IN TABLE 7

Variable	Obs	Mean	Std. Dev.	Min	Max
bmfrecbf	3010	.1966777	.3975527	0	1
bffrecbf	3010	.2740864	.4461268	0	1
biprobp1	2262	.1995027	.2127613	.0021421	.8806283
biprobp2	2262	.2844397	.2338657	.0031841	.8970554
biprobp11	2262	.05916	.1230571	9.92e-08	.501088
biprobp10	2262	.1403426	.1273059	.0021417	.6347069
biprobp01	2262	.2252796	.1648997	.0031839	.5288234
biprobp00	2262	.5752177	.2887347	.0128121	.9803647

Where:

- biprobp1 is the marginal probability that $y_1 = 1$.
- biprobp2 is the marginal probability that $y_2 = 1$.
- biprobp11 is the joint probability that $(y_1, y_2) = (1, 1)$.
- biprobp10 is the joint probability that $(y_1, y_2) = (1, 0)$.
- biprobp01 is the joint probability that $(y_1, y_2) = (0, 1)$.
- biprobp00 is the joint probability that $(y_1, y_2) = (0, 0)$

TABLE 15: TEST FOR OVERDISPERSION (IDGX)

ystar	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
muhat	.0758009	.0086034	8.81	0.000	.0589296	.0926722

TABLE 16: NB2 REGRESSION ESTIMATES FOR IDGX2

Negative binomial regression		Number of obs =	2262
Dispersion = mean		Wald chi2(11) =	1448.75
Log pseudolikelihood = -5095.4888		Prob > chi2 =	0.0000

idgx2	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
ehrgrd	-.2322858	.0675101	-3.44	0.001	-.3646032	-.0999685
errngrd	.3826066	.0751797	5.09	0.000	.2352571	.5299562
ehrrc5	-.5105999	.0940984	-5.43	0.000	-.6950294	-.3261704
errnrc5	.9913493	.1131292	8.76	0.000	.76962	1.213079
ehrage	.562205	.0905892	6.21	0.000	.3846535	.7397565
errnage	.0090894	.1371257	0.07	0.947	-.259672	.2778508
anyfriendr~p	.1611918	.0175319	9.19	0.000	.1268299	.1955538
bcent10x	.2709699	.0186438	14.53	0.000	.2344287	.3075111
prxprest	2.97253	.3034277	9.80	0.000	2.377822	3.567237
infldmn	.0003688	.0000324	11.39	0.000	.0003054	.0004323
sexdummy	-.0197733	.021765	-0.91	0.364	-.062432	.0228854
_cons	-.1556891	.0593176	-2.62	0.009	-.2719494	-.0394288
/lnalpha	-2.873211	.1206104			-3.109603	-2.636819
alpha	.0565171	.0068166			.0446187	.0715886

TABLE 17: TEST FOR OVERDISPERSION (ODGX2)

```
regress ystart muhatt, noconstant noheader
```

ystart	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
muhatt	-.0850472	.0024111	-35.27	0.000	-.0897752	-.0803191

TABLE 18: POISSON REGRESSION FOR OUT-DEGREES (ODGX2)

```
Poisson regression                                Number of obs   =       2358
                                                  Wald chi2(10)   =    2359.77
                                                  Prob > chi2     =     0.0000
Log pseudolikelihood = -4630.7834                 Pseudo R2       =     0.1681
```

odgx2	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
ehsgrd	.0017433	.0352248	0.05	0.961	-.067296	.0707826
ersngrd	.042787	.0344219	1.24	0.214	-.0246786	.1102527
ehsrc5	-.3603842	.0433238	-8.32	0.000	-.4452973	-.2754711
esrnrc5	.5403445	.0462108	11.69	0.000	.4497731	.6309159
ehsage	.1486468	.0500615	2.97	0.003	.0505282	.2467654
ersnage	.2527206	.0594975	4.25	0.000	.1361076	.3693336
anyfriendnom	.1622508	.0130333	12.45	0.000	.1367061	.1877955
bcent10x	.4726983	.0181223	26.08	0.000	.4371793	.5082173
reach	.0000482	.000018	2.67	0.008	.0000128	.0000835
sexdummy	-.0123701	.0114036	-1.08	0.278	-.0347208	.0099806
_cons	.609169	.0305497	19.94	0.000	.5492928	.6690452

TABLE 19: YFIT FOR MIXTURE OF NB2 AND POISSON (TOTAL CONNECTIONS)

NB(2) YFIT

Variable	Obs	Mean	Std. Dev.	Min	Max
yfit1	2364	9.479173	3.506059	3.324896	25.79653
yfit2	2364	12.51985	3.958615	5.711755	39.36028

POISSON YFIT

Variable	Obs	Mean	Std. Dev.	Min	Max
yfit11	2364	9.212211	3.148003	3.846781	26.53814
yfit12	2364	13.97649	4.060923	6.701697	39.90377

TABLE 20: FINITE MIXTURE MODEL FOR TOTAL CONNECTIONS (NB2)

2 component Negative Binomial-2 regression

Number of obs = **2364**
Wald chi2(**18**) = 1362.08
Prob > chi2 = **0.0000**

Log pseudolikelihood = **-6593.2188**

totcon	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
component1						
ehgrd	-.3342331	.1174002	-2.85	0.004	-.5643332	-.104133
erngrd	.1891364	.094072	2.01	0.044	.0047585	.3735142
ehrc5	-.0408717	.1907293	-0.21	0.830	-.4146943	.3329509
ernrc5	.5415377	.1729469	3.13	0.002	.2025679	.8805074
ehage	.3154175	.1525624	2.07	0.039	.0164008	.6144342
ernage	.4034075	.2048152	1.97	0.049	.0019771	.8048378
anyfriendnom	.3895717	.0534421	7.29	0.000	.284827	.4943164
anyfriendr~p	.2679363	.0371556	7.21	0.000	.1951126	.3407599
sexdummy	-.0884332	.0359272	-2.46	0.014	-.1588493	-.0180171
_cons	.8080637	.1275176	6.34	0.000	.5581339	1.057994
component2						
ehgrd	-.2386658	.1060182	-2.25	0.024	-.4464577	-.0308738
erngrd	.393615	.088234	4.46	0.000	.2206796	.5665504
ehrc5	-1.294794	.2164278	-5.98	0.000	-1.718984	-.8706031
ernrc5	1.338415	.1763256	7.59	0.000	.9928233	1.684007
ehage	.1695107	.1344196	1.26	0.207	-.093947	.4329684
ernage	-.009964	.1500452	-0.07	0.947	-.3040471	.2841192
anyfriendnom	.0750464	.0415525	1.81	0.071	-.006395	.1564877
anyfriendr~p	.1841647	.0235324	7.83	0.000	.138042	.2302875
sexdummy	.0110732	.0307597	0.36	0.719	-.0492147	.0713612
_cons	1.702519	.1029107	16.54	0.000	1.500818	1.904221
/imlogitpi1	-.097838	.3836936	-0.25	0.799	-.8498635	.6541876
/lnalpha1	-4.835462	1.673446	-2.89	0.004	-8.115356	-1.555569
/lnalpha2	-3.490576	.2996852	-11.65	0.000	-4.077948	-2.903204
alpha1	.007943	.0132922			.0002989	.2110693
alpha2	.0304833	.0091354			.0169422	.0548472
pi1	.47556	.0956942			.2994615	.6579535
pi2	.52444	.0956942			.3420465	.7005385

TABLE 21: MARGINAL EFFECTS OF COMPONENTS 1 AND 2 AT MEAN

COMPONENT 1

	dy/dx	Delta-method Std. Err.	z	P>|z|	[95% Conf.	Interval]
ehgrd	-2.965366	1.054562	-2.81	0.005	-5.032269	-.8984632
erngrd	1.678046	.8543376	1.96	0.050	.0035753	3.352517
ehrc5	-.3626199	1.698492	-0.21	0.831	-3.691601	2.966361
ernrc5	4.804603	1.648252	2.91	0.004	1.574089	8.035117
ehage	2.798431	1.392602	2.01	0.044	.0689819	5.527881
ernage	3.579091	1.720011	2.08	0.037	.2079311	6.950252
anyfriendnom	3.456338	.4450641	7.77	0.000	2.584029	4.328648
anyfriendr~p	2.377171	.2877614	8.26	0.000	1.813169	2.941173
sexdummy	-.7845926	.3263423	-2.40	0.016	-1.424212	-.1449734

COMPONENT 2

	dy/dx	Delta-method Std. Err.	z	P>|z|	[95% Conf.	Interval]
ehgrd	-2.862819	1.287331	-2.22	0.026	-5.385941	-.3396973
erngrd	4.72145	1.113697	4.24	0.000	2.538643	6.904257
ehrc5	-15.53117	2.65615	-5.85	0.000	-20.73713	-10.32522
ernrc5	16.05442	2.101752	7.64	0.000	11.93506	20.17377
ehage	2.033297	1.604899	1.27	0.205	-1.112246	5.178841
ernage	-.1195188	1.800312	-0.07	0.947	-3.648066	3.409028
anyfriendnom	.9001885	.4912814	1.83	0.067	-.0627054	1.863082
anyfriendr~p	2.209074	.2720579	8.12	0.000	1.67585	2.742297
sexdummy	.1328245	.3693425	0.36	0.719	-.5910734	.8567225

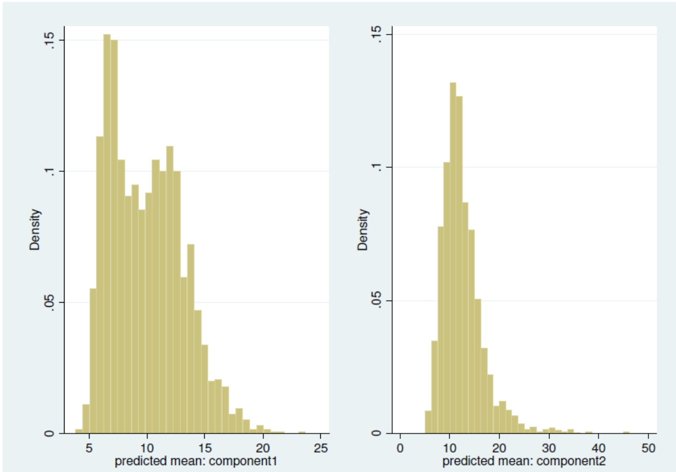


Figure 1: *
Figure 2: FITTED VALUES DISTRIBUTION, FMM2-NB2

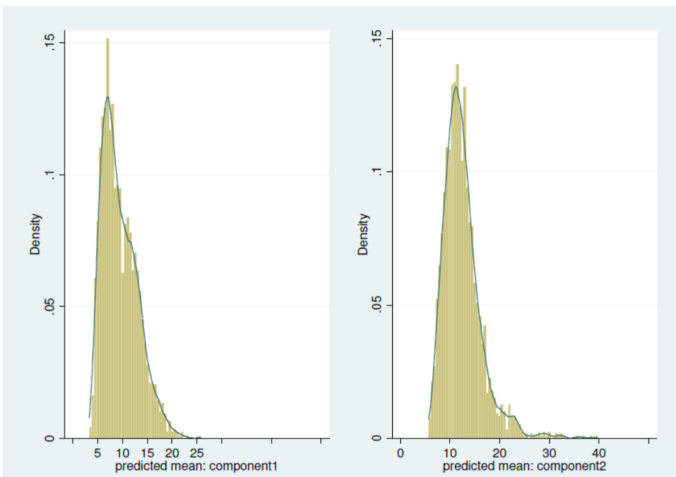


Figure 2: *
Figure 3: FITTED VALUES DISTRIBUTION WITH KDENSITY AND SCALED, FMM2-NB2

TABLE 22: TABLES FOR LINKS AND CONNECTIONS

TABULATE LINKS

tiel	Freq.	Percent	Cum.
0	1,756	58.34	58.34
1	1,091	36.25	94.58
2	163	5.42	100.00
Total	3,010	100.00	

TABULATE CONNECTIONS

tiec	Freq.	Percent	Cum.
0	939	31.20	31.20
1	1,625	53.99	85.18
2	446	14.82	100.00
Total	3,010	100.00	

TABLE 23: ORDERED PROBIT FOR LINKS

Ordered probit regression

Number of obs = 2264
Wald chi2(11) = 153.89
Prob > chi2 = 0.0000
Pseudo R2 = 0.0401

Log pseudolikelihood = -1862.3654

tieg	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
ehgrd	.5672102	.1691452	3.35	0.001	.2356918	.8987286
erngrd	-.5028972	.1598361	-3.15	0.002	-.8161703	-.1896242
ehrc5	.03613	.2088925	0.17	0.863	-.3732919	.4455518
ernrc5	-.2743106	.2277869	-1.20	0.228	-.7207648	.1721436
ehage	-.417555	.211867	-1.97	0.049	-.8328067	-.0023033
ernage	-.2359855	.2530187	-0.93	0.351	-.7318932	.2599221
bcent10x	.2135002	.0466161	4.58	0.000	.1221344	.304866
reach	.0001982	.0001868	1.06	0.289	-.000168	.0005644
prxprest	4.098408	.4741644	8.64	0.000	3.169063	5.027753
infldmn	-.0000408	.0002034	-0.20	0.841	-.0004394	.0003578
sexdummy	-.2282781	.0515678	-4.43	0.000	-.3293491	-.1272071
/cut1	.5241915	.1293264			.2707164	.7776666
/cut2	2.043233	.130041			1.788357	2.298108

TABLE 24: PREDICTED PROB. FROM OP FOR LINKS

Variable	Obs	Mean	Std. Dev.	Min	Max
p1oprobit	2264	.5552723	.118936	.0275448	.912096
p2obrobit	2264	.3882562	.0849772	.0858699	.5524358
p3probit	2264	.0564715	.0406318	.0020342	.6551006

TABLE 25: OP FOR CONNECTIONS

Ordered probit regression

Number of obs = 2264
Wald chi2(11) = 321.89
Prob > chi2 = 0.0000
Log pseudolikelihood = -2016.5259
Pseudo R2 = 0.0827

tiecon	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
ehgrd	.4290999	.1620864	2.65	0.008	.1114164	.7467835
erngrd	-.599094	.1547186	-3.87	0.000	-.9023369	-.2958512
ehrc5	.2140849	.2009621	1.07	0.287	-.1797935	.6079633
ernrc5	-.4544266	.2233279	-2.03	0.042	-.8921412	-.0167119
ehage	-.0314571	.1995676	-0.16	0.875	-.4226024	.3596882
ernage	-.5140753	.2425594	-2.12	0.034	-.9894829	-.0386677
bcent10x	.4847272	.0449536	10.78	0.000	.3966197	.5728347
reach	.000401	.0001797	2.23	0.026	.0000488	.0007532
prxprest	6.197098	.5051192	12.27	0.000	5.207083	7.187114
infldmn	-.0001922	.000194	-0.99	0.322	-.0005724	.0001879
sexdummy	-.1356263	.0496494	-2.73	0.006	-.2329373	-.0383153
/cut1	.397339	.1234293			.155422	.639256
/cut2	2.162213	.1268248			1.913641	2.410785

TABLE 26: PREDICTED PROB. FROM OP FOR CONNNECTIONS

Variable	Obs	Mean	Std. Dev.	Min	Max
l1oprobit	2264	.2808783	.1458691	.000202	.8654209
l2obrobit	2264	.563735	.0766038	.0379511	.6224595
l3probit	2264	.1553867	.1118566	.0020532	.9618469

TABLE 27: MNP FOR CONNNECTIONS

Multinomial probit regression

Number of obs = 2264
Wald chi2(22) = 355.84
Prob > chi2 = 0.0000

Log pseudolikelihood = -1980.4556

tiec	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
0	(base outcome)					
1						
ehgrd	.2842029	.2739842	1.04	0.300	-.2527964	.8212021
erngrd	-.7784158	.2595395	-3.00	0.003	-1.287104	-.2697277
ehrc5	.6606391	.3370126	1.96	0.050	.0001065	1.321172
ernrc5	-.9108065	.3706451	-2.46	0.014	-1.637258	-.1843555
ehage	-.6106688	.3560858	-1.71	0.086	-1.308584	.0872465
ernage	.0877437	.4147417	0.21	0.832	-.7251351	.9006224
bcent10x	.1947757	.076982	2.53	0.011	.0438938	.3456576
reach	.0005628	.0002898	1.94	0.052	-5.31e-06	.0011308
prxprest	6.77807	.8447873	8.02	0.000	5.122317	8.433823
infldmn	-.0004983	.000313	-1.59	0.111	-.0011118	.0001152
sexdummy	-.3266237	.0819286	-3.99	0.000	-.4872008	-.1660467
_cons	.2180081	.2093629	1.04	0.298	-.1923356	.6283518
2						
ehgrd	1.014148	.3496657	2.90	0.004	.3288162	1.699481
erngrd	-1.242651	.3279519	-3.79	0.000	-1.885424	-.5998768
ehrc5	.4226798	.4328307	0.98	0.329	-.4256527	1.271012
ernrc5	-.9344949	.4510196	-2.07	0.038	-1.818477	-.0505127
ehage	.0016173	.455014	0.00	0.997	-.8901936	.8934283
ernage	-.9241862	.5123462	-1.80	0.071	-1.928366	.0799939
bcent10x	1.067883	.093747	11.39	0.000	.8841424	1.251624
reach	.0003004	.000359	0.84	0.403	-.0004032	.001004
prxprest	14.13113	1.176082	12.02	0.000	11.82605	16.43621
infldmn	.0003698	.0004055	0.91	0.362	-.0004251	.0011646
sexdummy	-.2213637	.1047483	-2.11	0.035	-.4266666	-.0160607
_cons	-3.300479	.30039	-10.99	0.000	-3.889232	-2.711725

TABLE 28: PREDICTED PROB. FROM MNP FOR CONNNECTIONS

Variable	Obs	Mean	Std. Dev.	Min	Max
pmprobit1	2264	.2823023	.1298447	1.74e-06	.7960244
pmprobit2	2264	.5622119	.0905995	.063997	.7726806
pmprobit3	2264	.1554858	.1409963	.0000753	.9359996

TABLE 29: MNP FOR LINKS

Multinomial probit regression

Log pseudolikelihood = -1839.8592

Number of obs = 2264
Wald chi2(22) = 189.99
Prob > chi2 = 0.0000

tiel	Coef.	Robust Std. Err.	z	P>|z|	[95% Conf.	Interval]
0	(base outcome)					
1						
ehgrd	.6736139	.2572538	2.62	0.009	.1694056	1.177822
erngrd	-.6084476	.2435519	-2.50	0.012	-1.0858	-.1310947
ehrc5	.2687431	.3113101	0.86	0.388	-.3414135	.8788998
ernrc5	-.4919976	.335056	-1.47	0.142	-1.148695	.1647001
ehage	-.9812101	.332798	-2.95	0.003	-1.633482	-.3289381
ernage	.0022046	.3801796	0.01	0.995	-.7429338	.747343
bcent10x	.0845125	.0698358	1.21	0.226	-.0523632	.2213882
reach	.0003065	.0002848	1.08	0.282	-.0002516	.0008646
prxprest	4.763857	.7702146	6.19	0.000	3.254264	6.273449
infldmn	-.0002064	.0003068	-0.67	0.501	-.0008077	.0003948
sexdummy	-.3251784	.077775	-4.18	0.000	-.4776147	-.1727422
_cons	-.3622813	.2008096	-1.80	0.071	-.755861	.0312983
2						
ehgrd	1.17093	.4096893	2.86	0.004	.3679532	1.973906
erngrd	-.9979289	.3903133	-2.56	0.011	-1.762929	-.2329289
ehrc5	-.379253	.53561	-0.71	0.479	-1.429029	.6705234
ernrc5	-.2668943	.5354335	-0.50	0.618	-1.316325	.7825361
ehage	.1241987	.5510496	0.23	0.822	-.9558387	1.204236
ernage	-.9476264	.6097836	-1.55	0.120	-2.14278	.2475275
bcent10x	.6814498	.1059145	6.43	0.000	.4738612	.8890383
reach	-.0003631	.0003038	-1.19	0.232	-.0009586	.0002324
prxprest	9.364324	1.168687	8.01	0.000	7.07374	11.65491
infldmn	.0009702	.0003698	2.62	0.009	.0002454	.001695
sexdummy	-.4014296	.1254619	-3.20	0.001	-.6473305	-.1555287
_cons	-3.528241	.3529211	-10.00	0.000	-4.219954	-2.836529

TABLE 30: PREDICTED PROB. FROM MNP FOR LINKS

Variable	Obs	Mean	Std. Dev.	Min	Max
pmprobit11	2264	.5563628	.1127208	.0094643	.8786596
pmprobit21	2264	.3868376	.0781222	.1212562	.6386965
pmprobit31	2264	.0567995	.060556	.0000385	.5995376

Codes

Code for Chapter 2

Software Used: MATLAB

```
clear all
N = 70;
A = zeros(70, 70);

% Insert the Transition Probabilities - One could write a function for
this, but in this code it was inserted manually.

A(1, 1) = 0.0697; A(1, 3) = 0.3211; A(1, 4) = 0.1651 A(1, 5) = 0.4441;
A(2, 2) = 0.0697; A(2, 6) = 0.3211; A(2, 7) = 0.1651; A(2, 8) = 0.4441;
A(3, 3) = 1/16.57482; A(3, 10) = 5.3679/16.57482; A(3, 11) =
3.10346/16.57482; A(3, 12) = 7.10346/16.57482; A(4, 4) = 1/6; A(4, 13)
= 5/6; A(5, 5) = 1/6;A(5, 15) = 5/6;A(6, 6) = 1/16.57482; A(6, 9) =
5.3679/16.57482; A(6, 16) = 3.10346/16.57482; A(6, 17) =
7.10346/16.57482; A(7, 7) = 1/6; A(7, 18) = 5/6; A(8, 8) = 1/6; A(8,
14) = 5/6; A(9, 9) = 1; A(10, 10) = 1; A(11, 11) = 1/6; A(11, 15) =
5/6; A(12, 12) = 1/9.7168; A(12, 22) = 5.4778/9.7168; A(12, 23) =
3.239/9.7168; A(13, 13) = 1/8.5418; A(13, 20) = 7.5418/8.5418; A(14,
14) = 1/27.5344; A(14, 19) = 3.3672/27.5344; A(14, 20) =
3.3672/27.5344; A(14, 21) = 3.3672/27.5344; A(14, 33) = 5.4776/27.5344;
A(14, 34) = 5.4776/27.5344; A(14, 35) = 5.4776/27.5344; A(15, 15) =
1/27.5344; A(15, 19) = 3.3672/27.5344; A(15, 20) = 3.3672/27.5344;
A(15, 21) = 3.3672/27.5344; A(15, 33) = 5.4776/27.5344; A(15, 34) =
5.4776/27.5344; A(15, 35) = 5.4776/27.5344; A(16, 16) = 1/13.955; A(16,
41) = 7.239/13.955; A(16, 44) = 5.7165/13.955; A(17, 17) = 1/19.5144;
A(17, 24) = 3.348/19.5144; A(17, 27) = 5.7159/19.5144; A(17, 29) =
5.8443/19.5144; A(17, 41) = 3.6062/19.5144; A(18, 18) = 1/19.5144;
A(18, 24) = 3.348/19.5144; A(18, 27) = 5.7159/19.5144; A(18, 29) =
5.8443/19.5144; A(18, 41) = 3.6062/19.5144; A(19, 19) = 1/18.9478;
A(19, 22) = 5.606/18.9478; A(19, 23) = 3.3679/18.9478; A(19, 25) =
3.3679/18.9478; A(19, 26) = 5.606/18.9478; A(20, 20) = 1/18.9478; A(20,
24) = 3.348/19.5144; A(20, 27) = 5.606/18.9478; A(20, 29) =
5.606/18.9478; A(20, 41) = 3.348/19.5144; A(21, 21) = 1/18.9478; A(21,
32) = 5.606/18.9478; A(21, 31) = 3.348/19.5144; A(21, 28) =
3.348/19.5144; A(21, 30) = 5.606/18.9478; A(22, 22) = 1/9.7358; A(22,
43) = 3.3679/9.7358; A(22, 55) = 5.3679/9.7358; A(23, 23) = 1/10.1037;
A(23, 65) = 3.3679/10.1037; A(23, 46) = 5.7358/10.1037; A(24, 24) =
1/10.1037; A(24, 65) = 3.3679/10.1037; A(24, 44) = 5.7358/10.1037;
A(25, 25) = 1/10.1037; A(25, 65) = 3.3679/10.1037; A(25, 44) =
5.7358/10.1037; A(26, 26) = 1/9.735; A(26, 47) = 3.36787/9.735; A(26,
56) = 5.36787/9.735; A(27, 27) = 1/9.735; A(27, 58) = 3.36787/9.735
A(27, 50) = 5.36787/9.735; A(28, 28) = 1/10.1037; A(28, 65) =
3.3679/10.1037; A(28, 48) = 5.7358/10.1037; A(29, 29) = 1/9.735; A(29,
42) = 3.36787/9.735; A(29, 57) = 5.36787/9.735; A(30, 30) = 1/9.735;
A(30, 45) = 3.36787/9.735; A(30, 53) = 5.36787/9.735; A(31, 31) =
1/10.1037; A(31, 65) = 3.3679/10.1037; A(31, 46) = 5.7358/10.1037;
A(32, 32) = 1/9.735; A(32, 49) = 3.36787/9.735; A(32, 54) =
5.36787/9.735; A(33, 33) = 1/18.6896; A(33, 37) = 3.3672/18.6896; A(33,
39) = 3.3672/18.6896; A(33, 60) = 5.4776/28.6896; A(33, 64) =
5.4776/28.6896; A(34, 34) = 1/18.6896; A(34, 38) = 3.3672/18.6896;
A(34, 40) = 3.3672/18.6896; A(34, 59) = 5.4776/28.6896; A(34, 62) =
5.4776/28.6896; A(35, 35) = 1/18.6896; A(35, 67) = 3.3672/18.6896;
A(35, 68) = 3.3672/18.6896; A(35, 63) = 5.4776/28.6896; A(35, 61) =
```

```
5.4776/28.6896; A(36, 36) = 1; A(37, 37) = 1/9.9739; A(37, 50) =
3.606/9.9739; A(37, 36) = 5.3679/9.9739; A(38, 38) = 1/9.9739; A(38,
47) = 3.606/9.9739; A(38, 56) = 5.3679/9.9739; A(39, 39) = 1/9.9739;
A(39, 49) = 3.606/9.9739; A(39, 54) = 5.3679/9.9739; A(40, 40) =
1/9.9739; A(40, 45) = 3.606/9.9739; A(40, 53) = 5.3679/9.9739; A(41,
41) = 1/10.1037;A(41, 65) = 3.3679/10.1037; A(41, 48) = 5.7358/10.1037;
A(42, 42) = 1; A(43, 43) = 1; A(44, 44) = 1/12.4715; A(44, 51) =
5.7358/12.4715; A(44, 52) = 5.7358/12.4715; A(45, 45) = 1; A(46, 46) =
1; A(47, 47) = 1; A(48, 48) = 1; A(49, 49) = 1; A(50, 50) = 1; A(51,
51) = 1/3; A(51, 10) = 2/3; A(52, 52) = 1/3; A(52, 9) = 2/3; A(53, 53)
= 1; A(54, 54) = 1; A(55, 55) = 1; A(56, 56) = 1; A(57, 57) = 1; A(58,
58) = 1; A(59, 59) = 1/10.4328; A(59, 66) = 5.7164/10.4328; A(59, 69) =
3.7164/10.4328; A(60, 60) = 1/10.4328; A(60, 66) = 5.7164/10.4328;
A(60, 69) = 3.7164/10.4328; A(61, 61) = 1/10.4328; A(61, 66) =
5.7164/10.4328; A(61, 70) = 3.7164/10.4328; A(62, 62) = 1/10.4328;
A(62, 66) = 5.7164/10.4328; A(62, 70) = 3.7164/10.4328; A(63, 63) =
1/10.4328; A(63, 58) = 3.7164/10.4328; A(63, 66) = 5.7164/10.4328;
A(64, 64) = 1/10.4328;A(64, 58) = 3.7164/10.4328; A(64, 66) =
5.7164/10.4328; A(65, 65) = 1; A(66, 66) = 1; A(67, 67) = 1/9.7358;
A(67, 42) = 3.3679/9.7358; A(67, 57) = 5.3679/9.7358; A(68, 68) =
1/9.7358; A(68, 43) = 3.3679/9.7358; A(68, 55) = 5.3679/9.7358; A(69,
69) = 1; A(70, 70) = 1;

P = full(A); % Create the Markov Transition Probabilities Matrix

% Plot the Supernetwork%
bg = biograph(P)
h = view(bg)

S = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,
20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37,
38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55,
56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69 70];
% States in the Markov Chain

rs = [9 10 24 42 43 45 46 47 48 49 50 53 54 55 56 57 58 65 66, 69, 70];
% recurrent states
ts =[1 2 3 4  5 6 7 8 11 12 13 14 15 16 17 18 19 20 21 22 23 25 26
27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 44 51 52 59 60 61 62 63
64, 67, 68]  % transient states


mu= [0.5, 0.5, zeros(1, 68)]; % initial guess

rone = [0; -1; 3; 0; -5; -1; -1; -1; 3; 1; 1; 1; -1; -4; -5; 0; -4; -4;
-9; -5; -5; -10; -10; -9; -10; -10; 2; -5; -5; -5; -9; 2; 2; -5; -5; 1;
1; -9; 1; -5; -5; -9; -10; -10; -9; -10; -10; 2; 1; 1; -7; -10; 2; 1; -
10; -10; 2; 1; 2; 1; -5; -5; 2; 1; -10; 2; -5; -9; 1; -9];     % Return
Function for Node 1

rtwo = [-1; 0; -1; -1; -1; 3; 0; -5; 1;2;0;-4; -4; -5; -4; 1; 1;-1;-5;-
9; -5;-5;-5;-10; -9;2;-10;-9; -10;2;-5;-5; -5; 2; -5; -9; -9; 1; -5;
1; -10; -10; -9; -10; 1; 2; 1; -10; -9;-10; -10; -7; 1;2;2;1; -10;-
9;1;2; 2; 1; -5; -5; -10; 2; -9; -5; 1; 1];% Return Function for Node 2
```

```
rthree = [-1; -1; 2; -6; -5; 2; -6; -5; 5; 5; -3; -2; -5; -4; -4; -3;
-2; -5; -5; -5; -9; 3; -9; -5; -5; -5; -5; -10; 3; -10; -10; -10; -5;
-5; 3; 1; -5; -5; -9; -9; -9; 2; 2; 3; -10; -10; -9; -10; -10; -9; 5;
5; -10; -10; 2; 3; 2; 2; -5; -5; 2; 3; 2; 3; -10; 3; 1; 1; -9; 1];
% Return Function for Node 3

rf = [0; 0; 0; -1; -2; 0; -1; -2; 0; 0; -1; -2; -2; -3; -3; -1; -2; -2;
-1; -1; -1; 1; 0; 0; 0; 1; 1; 0; 1; 1; 0; 1; 0; 0; 0; 2; 0; 0; 0; 0; 0;
2; 2; 2; 2; 2; 2; 2; 2; 2; 0; 0; 0; 3; 3; 3; 3; 3; 2; 2; 2; 2; 2; 2; 0;
4; 0; 0; 2; 2]; % Return Function for the Fed Node

bone = 0.4;   % Node specific discount rates to calculate equilibrium
              % payoffs.
btwo = 0.6;
bthree = 0.5;
bf = 0.1;

I = eye(N);
l = 1-diag(P);               % jump rates/sojourn parameters

%Perturbation for non-singularity%
lambda = zeros(70, 1)

for i = 1:68
    if l(i, 1) < 0.000001
      lambda(i, 1) = 0.001
    else lambda(i, 1) = l(i, 1)
 end
end

D = diag(diag(P));         % diagonal matrix with the same diagonal as P
Q = inv(diag(lambda))*(P-D);      % make diagonal zero, normalize rows

Tmax = 80;

N     = length(S);                     % number of states
step  = Tmax/100;                      % standard time increment
T     = 0:step:Tmax;                   % vector of times

A     = diag(lambda)*(Q-eye(size(Q)));  % generator

Pstep = expm(A*step);             % this is P(step); matrix exponential

m     = mu;                            % distribution at each step
dist  = mu;                            % each row of dist is distn
                                       % at a time

for t=1:(length(T)-1),
  m    = m*Pstep;                      % step forward one time step
  dist = [dist; m];                    % add another row to m
end
```

```
% The graph generated by the following has not been included in this
dissertation.

for v=1:N,
  subplot(N,1,N-v+1);                       % one plot for each state
  plot(T,dist(:,v));                        % plot probability over time
  axis([0 Tmax 0 1]);
  ylabel(['P(X_{t} = ' num2str(S(v)) ')']);
end

subplot(N,1,1);
title('Probabilities of being in states 1, 2, 3, ... over time');

% Now compute the invariant/limiting distribution

eta = invariant(expm(A));
eta2 = invariant(Q)*diag(1./lambda);
eta2 = eta2/sum(eta2);                    % Normalize

vone = inv(I - bone*P)*rone;
vtwo = inv(I - btwo*P)*rtwo;
vthree = inv(I - bthree*P)*rthree;
vf = inv(I - bf*P)*(rf);
v = [vone vtwo vthree vf]; % The payoff matrix

% Invariant Probability Measure %

IP = invariant(P);

% Stationary Probability as a Limit

S = P^100000000000;

% Payoffs with Invariant Probabilities
o = ones(1, N);
c = (IP'*o)';
voneinv = inv(I - bone*c)*rone; % Similarly, payoffs with invariant
                                % probabilities for nodes 2, 3 and
                                % Fed may be calculated.

% Payoffs with Stationary Probabilities

vonestat =  (I - bone*S)*rone;      % Similarly, payoffs with
                                    % stationary probabilities for
                                    % nodes 2, 3 and Fed may be
                                    % calculated.


% Plot Paths to Basins of Attraction %

n=100;                % number of time steps to take
x=zeros(1,n+1);       % clear out any old values
t=0:n;                % time indices
```

```
x(1)=rando(mu);       % generate first x value (time 0, not time 1)

for i=1:n,
  x(i+1) = rando(P(x(i),:));
end

plot(t, x, '*');
axis([0 n 0 (length(mu)+1)]);

clear T;
clear x;              % clear out previous values

T(1) = 0;             % start times at 0
x(1) = rando(mu);     % generate first x value (time 0, not time 1)
i    = 1;

while T(i) < Tmax,
  T(i+1) = T(i)  - log(rand)/lambda(x(i));  % generate exponential r.v.
                                            % occasionally causes
                                            % errors; ignore them
  x(i+1) = rando(Q(x(i),:));          % use Q to make state transitions
  i=i+1;
end

cla;
hold on

for i=1:(length(T)-1),
  plot([T(i) T(i+1)], [S(x(i)) S(x(i))]);
end

%stairs(T,S(x));                  % sometime this is more appropriate
axis([0 Tmax 0 (length(mu)+1)]);
xlabel('Time');

%Generates Many Processes as above%
num=6;
for k=1:num,
  subplot(num,1,k);
  process;
end

% Plot Expected Number of Visits %

l = length(ts);
Q = P(ts,ts);   % pull out the matrix corresponding to transient states
L = P(ts,rs);   % transitions from transient to recurrent states

W = inv(eye(size(Q))-Q);    % compute the expected number of visits to
                            % states in t

m = sum(W')';                     % expected number of visits before
                                  % the end of the game
```

```
F(r,r) = eye(size(P(r,r)));

F(r,t) = zeros(size(L'));
F(t,r) = W*L;
F(t,t) = (W-eye(size(W)))*inv(diag(diag(W)));

clf;
 subplot(2,1,1);
 graph_matrix(1, N-19, W);            % N-10 may be replaced by size(t)
title('Graphical  representation  of  expected  number  of  visits  matrix
W');
ylabel('Initial state');
xlabel('Visited state');


subplot(2,1,2);
graph_matrix(0,N-1,F);
title('Graphical representation of probability of hitting matrix F');
ylabel('Initial state');
xlabel('Hit state');

fprintf('Expected length of game, as a function of initial state:')
[(1:(N-19))' m]


% Graphs of Transition Probabilities

mu=[0.5, 0.5, zeros(1,N-2)];  % set up initial distribution
mu(1)=1;                      % start with wealth 10 with probability 1

% the commands below change the color of the graphs

colormap(1-gray); % uncomment this command before printing a color
                  % graph
colormap('default'); % uncomment this command to see color graphs on
                     % screen
orient tall          % stretch graph vertically when printing
                     % the commands below display P and powers of P

x=0:1:N;          % column numbers
y=x;              % row numbers

subplot(3,2,1);   % 3 by 2 array of plots, this is plot # 1

pcolor(x, y,[[P zeros(N,1)]' zeros(N+1,1)]');   % display matrix P
axis ij;         % number the axes as for a matrix, not a regular graph
title('Graphical representation of transition matrix P');
xlabel('Final state');
ylabel('Initial state');

subplot(3,2,3);   % 3 by 2 array of plots, this is plot # 1
```

```
pcolor(x, y, [[P^2 zeros(N,1)]' zeros(N+1,1)]');
axis ij;
title('Graphical representation of P^2');
xlabel('Final state');
ylabel('Initial state');

subplot(3,2,5);

pcolor(x, y, [[P^4 zeros(N,1)]' zeros(N+1,1)]');
axis ij;
title('Graphical representation of P^{4}');
xlabel('Final state');
ylabel('Initial state');

subplot(3,2,2);

pcolor(x, y, [[P^20 zeros(N,1)]' zeros(N+1,1)]');
axis ij;
title('Graphical representation of P^{20}');
xlabel('Final state');
ylabel('Initial state');

subplot(3,2,4);


pcolor(x, y, [[P^225 zeros(N,1)]' zeros(N+1,1)]');
axis ij;
title('Graphical representation of P^{225}');
xlabel('Final state');
ylabel('Initial state');

subplot(3,2,6);

pcolor(x, y, [[P^2000 zeros(N,1)]' zeros(N+1,1)]');
axis ij;
title('Graphical representation of P^{2000}');
xlabel('Final state');
ylabel('Initial state');


%Plot the Supernetwork

bg = biograph(P)
h = view(bg)
set(h.Nodes(path),'Color',[1 0.4 0.4])
edges = getedgesbynodeid(h,get(h.Nodes(path),'ID'));
set(edges,'LineColor',[1 0 0])
set(edges,'LineWidth',1.5)
[dist, path, pred] = graphshortestpath(S, 1, 25)
```

Code for Chapter 3

Software Used: MATLAB

Let inter_ba, intra_ba, n, n1 be the inter block asymmetry, intra block asymmetry, number of total nodes and number of core nodes, respectively.

Algorithm for DGP #1

```
for i=1 until n1
    for j=1 until n1

  A(i,j)=rand*5000+5000;    % Core with exposures between 5000 and 10000
    end
end

for i=n1+1 until n
    for j=1 until n1
  A(j,i)=rand*inter_ba;     % Core-periphery exposures between 0 and
   end                      % inter_ba
 end

for i=1 until n1
    for j=n1+1 until n
        A(j,i)=rand*1;      % Periphery-core exposures between 0 and 1
    end
end

for i=1 until n
    A(i,i)=0;                            % Intra node exposures are null
end

for j=1 until n
    for i=j+1:n
        A(i,j)=A(j,i)+rand*intra_ba;     % Exposure matrix is asymmetric
    end
end

for i=n1+1 until n
    for j=n1+1 until n       % Periphery-periphery exposures are null
     A(i,j)=0;
    end
end
```

Algorithm for DGP #2

```
for i=1 until n1
    for j=1 until n1
A(i,j)=rand*500000+500000;        % Note that this number is larger than
    end                           % what we had for DGP #1
  end

for i=n1+1 until n
    for j=1 until n1

  A(j,i)=5000 + rand*inter_ba;
  end
end
```

```
for i=1 until n1
    for j=n1+1 until n
        A(j,i)=rand*500;    % Periphery-core exposures between 0 and 500
    end
end

for i=1 until n
    A(i,i)=0;
end

for j=1 until n
    for i=j+1:n
        A(i,j)=A(j,i)+rand*intra_ba;
    end
end
```

Note that we have removed the code

```
for i=n1+1 until n
    for j=n1+1 until n      % Periphery-periphery exposures are not null
     A(j,i)=0;
    end
end
```

Thus, DGP # 2 generates random positive numbers in the PP block. We do not impose whether these should be larger or small – we use the positive numbers generated by the code.

Note that in case of the smaller networks, we used this and modified it to

```
for i=n1+1 until n
   for j=n1+1 until n
    A(j,i)=rand*50 + 50; or rand*500 + 500 % Generate values between 50
   end                                     % and 100 or 500 and 1000
end
```

Algorithm for Direct Method

```
% Generate max entropy matrix
u=sum(A');
v=sum(A);
m=length(u);
n=length(v);
G=zeros(n,m);

for i=1:m
    for j=1:n
        if i ~= j
        G(i,j)=u(i)/((n-1)+(1e-10));
        end
    end
end


% Maximum Entropy
```

```
Max_entropy=RAS(G,u,v,0.00001);

% Goodness-of-fit Evaluation
em_max_entropy=error_measure(A,Max_entropy)

% Generate Copula Matrix
H=u;
E=v;

D = [H', E'];
w = H';
f = E';
x = ksdensity(w, w,'function','cdf'); % Transform the data into Copula
                                       % scale
y = ksdensity(f, f,'function','cdf');

scatterhist(x,y)
xlabel('x')
ylabel('y')

[xx, yy] = meshgrid(x, y);

% Fitting a Gumbel Copula
options=statset('MaxIter',1000000);
[paramhat,paramci] = copulafit('gumbel', [x y]);
j = -log(xx);
l = -log(yy);

C = exp(-(j.^(paramhat) + l.^(paramhat)).^(1/paramhat));

%Probabilities:
C;
lu=length(u);

for i = 1:lu
    C(i, i) = 0;
end
Z = C;

bilateral_estimates_stochastic=scale_matrix_stochastic(Z);

for i=1:m
    for j=1:n
       B(i,j)=bilateral_estimates_stochastic(i,j)*u(i);
    end
end

bilateral_estimates=RAS(B,u,v,0.001);
em_max_gumbel = error_measure(A,bilateral_estimates)

%Fit a Frank Copula
[paramhatone,paramcione] = copulafit('frank', [x y]);
r = -1/(paramhat);
```

```
m1 = exp(-(paramhat*xx)) - 1;
m2 = exp(-(paramhat*yy)) - 1;
e = exp(- paramhat) - 1;
frank = r*log(1 + (m1 + m2)/e);

F = frank
bilateral_estimates_stochastic_F=scale_matrix_stochastic(F);

for i=1:m
    for j=1:n
       F(i,j)=bilateral_estimates_stochastic_F(i,j)*u(i);
    end
end
frank_estimate=RAS(F,u,v,0.00001);
em_frank=error_measure(A,frank_estimate)

% Then compare error measures.
```

Algorithm for Indirect Method

```
% Mixture of Gumbel and Clayton Copulas

% Get the Gumbel estimated from the algorithm for direct method.

% Fit a Clayton Copula
[paramhattwo,paramcitwo] = copulafit('clayton', [x y]);

q = -1/(paramhattwo);
q1 = xx.^(-paramhattwo);
q2 = yy.^(-paramhattwo);
clayton = (q1 + q2 - 1).^q;

L = clayton

bilateral_estimates_stochastic_clayton=scale_matrix_stochastic(L);

for i=1:m
    for j=1:n
      L(i,j)=bilateral_estimates_stochastic_clayton(i,j)*u(i);
  end
end
clayton_estimate=RAS(L,u,v,0.00001);
em_max_clayton=error_measure(A,clayton_estimate)

% Define new variables
nq = length(qa);
g = 0.9;                 % proportion of Gumbel, 1 - g = 0.1, proportion
                         % of clayton


nq1 = g*nq ;             % Number of elements to be taken from qa
nq2 = (1 - g)*nq ;       % Number of elements to be taken from qa
```

```
% Initialize vector for mixture of gumbel and   clayton with 0.9*Gumbel
and 0.1* Clayton, i.e., first 90% are from Gumbel and last 10% are from
Clayton.

qstar = zeros(1, nq);

for r = 1:nq1
    qstar(1, r) = qa(1, r);
end

for s = nq1+1:nq2
    qstar(1, s) = ql(1, s)
end

% Optimization Problem via MC

% Parameters
szA=size(A);

 N=szA(1);              % Number of nodes in the network

 % Restrictions
 % Initialize restrictions matrix and independent term
   R=zeros(1,N*N);
 % R is the restriction matrix, and B is the target matrix.
 % For e.g., Ax = B where A = R. And in our case, x = p_{i,j}'s
 % in a matrix form.
B=0;

% First set of restrictions: \sum_j p_ij*v_j=y_i
  for i=1:N
      aux=zeros(N,N);
      for j=1:N
          aux(i,j)=v(j)/sum(v);
      end

        R=vertcat(R,roll_on(aux));
        B=vertcat(B,u(i)/sum(u));
  end
% Second set of restrictions*: \sum_i p_ij=1, these will be self
% imposed by the MC Method

 R=R(2:end,:);
 B=B(2:end);
 R=simp_matrix(R);

% MC
  guess=qstar
  M=1000000000000000000000000000;

  of=inf;
  iter=1;

while iter<10000
```

```
      l=length(guess);
            aux=guess;
      for i=1:l
          aux2=guess;
          aux2(i)=aux2(i)+rand*5;
          aux2=recov_matrix(aux2,N);
          aux2=generate_prior(aux2);
          aux=vertcat(aux,simp_matrix(roll_on(aux2)));
      end
      szaux=size(aux);
for k=1:szaux(1)
       objaux(k)=sum(abs(aux(k,:)'-qstar'))+sum(abs(R*aux(k,:)'-B))*M;
 if objaux(k)<of
              of=objaux(k);
              aux_star=aux(k,:);
 end
end

      guess=aux_star;
      iter=iter+1;
      clc
      fprintf('%g',iter/10000);
end

  objaux_star=sum(abs(aux_star'-qstar'))+sum(abs(R*aux_star'-B))*M

 SP=recov_matrix(aux_star,N);

  for j=1:m
    for i=1:n
       SP(i,j)=SP(i,j)*v(j);
    end
  end


em_max_sp=error_measure(A,SP)
em_max_entropy
em_max_gumbel
em_max_clayton
```

Algorithm for the Graphs

```
% Script that generates estimation distributions for the simulated
% matrix data
clear all
clc

ndatapoints=100;    % number of data points simulated for each set of
                    % parameters of asymmetry
nrepetitions=100;   % number that increments the asymmetry parameters
```

```
auxvec=zeros(ndatapoints,1);          % initialize the vector that
                                      % stores the simulated differences
                                      % between each method

auxvec_mean=zeros(nrepetitions,1);       % initialize vector that stores
                                         % the mean difference between
                                         % the methods
auxvec_stdev=zeros(nrepetitions,1);      % stdev of the Parameters of
                                         % the simulation

intra_ba=1;                         % intra block asymmetry initial point
inter_ba=100;                       % inter block asymmetry initial point
n=100;                                   % number of nodes
n1=20;                                   % number of core nodes

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 1. very high CP - 500000 + rand with low CC and PP (500 + rand)     %
%                                                                     %
% 2. very high PP - 500000 + rand with low CC and CP                  %
%                                                                     %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:nrepetitions
    for j=1:ndatapoints
auxvec(j)=dif_fct(inter_ba,intra_ba+(0.99*(10000/nrepetitions))*i,n,n1;
    end
    i/nrepetitions
    auxvec_mean(i)=mean(auxvec);
    auxvec_stdev(i)=std(auxvec);
end


x=1:nrepetitions;
index = x;

x_up=(auxvec_mean+auxvec_stdev)';
x_lo=(auxvec_mean-auxvec_stdev)';
baseLine = min(x_lo-0.000005);         % Baseline value for filling
                                       % under the curves

plot(x,100*x_up,'b',x,100*auxvec_mean,'r',x,100*x_lo,'b'); % Plot the
                                                           % first line
hold on; % fill(x,x_up,'b')

% Add to the plot
h1 = fill(x(index([1 1:end end])),[100*baseLine 100*x_up(index)
100*baseLine],'b','EdgeColor','none');
plot(x,100*x_lo,'g');                  % Plot the second line
h2 = fill(x(index([1 1:end end])),... % Plot the second filled polygon

         [100*baseLine 100*x_lo(index) 100*baseLine],...
         'w','EdgeColor','none');
     plot(x,100*auxvec_mean,'r','LineWidth',2)
```

Algorithm for the Functions Used

Some of these functions may not have been directly used, however, it was considered best to report those as well because they were used to verify the analyses presented in chapter 3.

```
% Function that returns the error between two matrices. Matrix A is
% the true matrix and B is the estimated one

function output=error_measure(A,B)

szmat=size(A);
aux=0;
tot=sum(sum(A));

for i=1:szmat(1)
    for j=1:szmat(2)
       aux=aux+abs(B(i,j)-A(i,j));
    end
end

% Function that returns the cross-entropy between two matrices. Matrix
% A is the true matrix and B is the estimated one

output=aux/tot;

function output=cross_entropy(A,B)

szmat=size(A);
aux=0;

for i=1:szmat(1)
    for j=1:szmat(2)
     aux=aux+log((0.001+B(i,j))/(A(i,j)+0.001));
    end
end

output=aux;

% Objective function
function output=dwp_min(x)
global Qa Qb

szQ=size(Qa);
N=szQ(1);
b=0:1/(N-1):1;
Qg=(1/N)*ones(N,N);

for i=1:N
    Qg(i,i)=0;
end
```

```
Qg=generate_prior(Qg);

% First block
fb=0;
fb_1=0;
gamma=zeros(N,1);

for j=1:N
    for i=1:N
       fb_1=fb_1+x(i+(j-1)*N)*error_measure(x(i+(j-1)*N),Qa(i,j));
    end
    for h=1:N
       gamma(j)=gamma(j)+b(h)*x((h-1)*N+N^2+j);
    end
       fb=fb+(1-gamma(j))*fb_1;
end

 % Second block
 sb=0;
 sb_1=0;

 for j=1:N
     for i=1:N
         sb_1=sb_1+x(i+(j-1)*N)*error_measure(x(i+(j-1)*N),Qb(i,j));
     end
    sb=sb+gamma(j)*sb_1;
 end

 % Third block
  tb=0;
  for j=1:N
      for h=1:N
       tb=tb+x(N^2+h+(j-1)*N)*error_measure(x(N^2+h+(j-1)*N),Qg(i,j));
      end
  end

  output=fb+sb+tb;

%RAS
% Source: SCHNEIDERM, . H., ANDS . A. ZENIOS1 (1990). A Comparative
% Study of Algorithms for Matrix Balancing. Opns. Res. 38, 439-455.
% function that returns the rebalanced matrix given:
% - A_0,the initial matrix A (e.g., the one obatined by dividing total
% exposures equaly);
% - u, vector of a priori sum of the rows;
% - v, vector of a priori sum of the columns;
% - tol, the tolerance assigned to the stopping rule.
function output=RAS(A_0,u,v,tol,max_iter)

% # of rows
m=length(u);
```

```
% # of columns
n=length(v);

% Step 0 (Initialization)
A_aux=A_0;
tol_effective=inf;
k=1;
while tol_effective>tol % k < max_iter ||
  % Step 1 (Row Scaling)
  for i=1:m
    if sum(A_aux(i,:))>0
    rho(i)=u(i)/(sum(A_aux(i,:))+1e-10);
    else
        rho(i)=0;
    end
end

for i=1:m
    for j=1:n
      A_aux2(i,j)=rho(i)*A_aux(i,j);
    end
end

% Step 2 (Column Scaling)
for j=1:n
    sigma(j)=v(j)/(sum(A_aux(:,j))+1e-10);
end


for i=1:m
    for j=1:n
      A_aux2(i,j)=sigma(j)*A_aux2(i,j);
    end
end

tol_effective=0;

for i=1:m
    for j=1:n
      tol_effective=tol_effective+abs(A_aux(i,j)-A_aux2(i,j));
    end
end
tol_effective;
k=k+1;
A_aux=A_aux2;
end
fprintf('converged after %d iterations',k);
output=A_aux;

% function that recovers from the simplified matrix
function output=recov_matrix(input,n)
aux=zeros(n,n);
k=0;
for i=1:n
```

```
     for j=1:n

        if i~=j
            k=k+1;
            aux(i,j)=input(k);
        end

    end
end
output=aux;

% Function that scales a matrix into a stochastic ones
function output=scale_matrix_stochastic(input)

szmat=size(input);
aux=zeros(szmat(1),szmat(1));
sum_rows=sum(input');
for i=1:szmat(1)
    for j=1:szmat(2)
        aux(i,j)=input(i,j)/(sum_rows(i)+1e-10);
    end
end
output=aux;

% function that simplifies the matrix to eliminate the diagonal
% elements
function output=simp_matrix(input)

szmat=size(input);
n=szmat(2);
aux=zeros(szmat(1),1);
k=0;
for j=1:n
    if j~=1+(sqrt(n)+1)*k
    aux=horzcat(aux,input(:,j));
    else
        k=k+1;
    end
end
output=aux(:,2:end);

% function to roll on constraint matrices
function output=roll_on(input)

szmat=size(input);
aux=0;
for i=1:szmat(1)
        aux=horzcat(aux,input(i,1:szmat(2)));
end
output=aux(2:end);
```

Code for Chapter 4

Software Used: STATA

```
clear all
insheet using "C:\Users\Pallavi\Desktop\ICPSR-QUESTIONNAIRE\network-
data - Baral.csv", comma

*Bivariate Probit
tabulate  havebmf havebff
correlate  havebmf havebff
biprobit  havebmf havebff ehsgrd ersngrd ehsrc5 esrnrc5 ehsage ersnage
bcent10x reach sexdummy, robust
predict biprob1, pmarg1
predict biprob2, pmarg2
predict biprob11, p11
predict biprob10, p10
predict biprob01, p01
predict biprob00, p00
summarize havebmf havebff biprob1 biprob2 biprob11 biprob10 biprob01
biprob00

*Test Statistical Significance of Regressors
test  bcent10x
test  reach
test  ehsgrd
test  ersngrd
test  ehsrc5
test  esrnrc5
test  ehsage
margins, dydx(*)

tabulate  bmfrecip bffrecip
correlate  bmfrecip bffrecip
biprobit   bmfrecip bffrecip ehrgrd errngrd ehrrc5 errnrc5 ehrage
errnage bcent10x prxprest infldmn sexdummy, robust
predict biprobq1, pmarg1
predict biprobq2, pmarg2
predict biprobq11, p11
predict biprobq10, p10
predict biprobq01, p01
predict biprobq00, p00
summarize bmfrecip bffrecip biprobq1 biprobq2 biprobq11 biprobq10
biprobq01 biprobq00

test ehrgrd
test errngrd
test ehrrc5
test errnrc5
test ehrage
test errnage
test bcent10x
test prxprest
test infldmn
test sexdummy
margins, dydx(*)
```

```
tabulate bmfrecbf bffrecbf


correlate bmfrecbf bffrecbf
biprobit bmfrecbf bffrecbf ehrgrd errngrd ehrrc5 errnrc5 ehrage errnage
bcent10x prxprest infldmn sexdummy, robust
predict biprobb1, pmarg1
predict biprobb2, pmarg2
predict biprobb11, p11
predict biprobb10, p10
predict biprobb01, p01
predict biprobb00, p00
summarize bmfrecbf bffrecbf biprobb1 biprobb2 biprobb11 biprobb10
biprobb01 biprobb00

test ehrgrd
test errngrd
test ehrrc5
test errnrc5
test ehrage
test errnage
test bcent10x
test prxprest
test infldmn
test sexdummy
margins, dydx(*)

*Predicted Probabilities (Bivariate Probit)
biprobit bmfrecbf bffrecbf ehrgrd errngrd ehrrc5 errnrc5 ehrage errnage
bcent10x prxprest infldmn anyfriendrecip sexdummy, robust
predict biprobp1, pmarg1
predict biprobp2, pmarg2
predict biprobp11, p11
predict biprobp10, p10
predict biprobp01, p01
predict biprobp00, p00
summarize bmfrecbf bffrecbf biprobp1 biprobp2 biprobp11 biprobp10
biprobp01 biprobp00

*Poisson/Negative Binomial(2) Regressions (In-Degrees and Out-Degrees)
quietly poisson idgx2 ehrgrd errngrd ehrrc5 errnrc5 ehrage errnage
anyfriendrecip bcent10x prxprest infldmn sexdummy, robust

predict muhat, n

quietly generate ystar = ((idgx2 -muhat)^2 - idgx2)/muhat
regress ystar muhat, noconstant noheader

nbreg idgx2 ehrgrd errngrd ehrrc5 errnrc5 ehrage errnage anyfriendrecip
bcent10x prxprest infldmn sexdummy, robust

quietly poisson odgx2 ehsgrd ersngrd ehsrc5 esrnrc5 ehsage ersnage
anyfriendnom bcent10x reach sexdummy, robust
```

```
predict muhatt, n

quietly generate ystart = ((odgx2 -muhatt)^2 - odgx2)/muhatt
regress ystart muhatt, noconstant noheader

poisson odgx2 ehsgrd ersngrd ehsrc5 esrnrc5 ehsage ersnage anyfriendnom
bcent10x reach sexdummy, robust

kdensity totcon

*Finite Mixture Model
quietly fmm totcon ehgrd erngrd ehrc5 ernrc5 ehage ernage anyfriendnom
anyfriendrecip sexdummy, vce(robust) components(2) mixtureof(negbin2)
predict yfit1, equation(component1)
predict yfit2, equation(component2)
summarize yfit1 yfit2

estat ic

*Plot Densities
quietly histogram yfit1, name(_comp_1, replace)
quietly histogram yfit2, name(_comp_2, replace)
quietly graph combine _comp_1 _comp_2

quietly histogram yfit1, name(_comp_1, replace) kdensity xtick(0(10)50)
quietly histogram yfit2, name(_comp_2, replace) kdensity xtick(0(10)25)
graph combine _comp_1 _comp_2

quietly fmm totcon ehgrd erngrd ehrc5 ernrc5 ehage ernage anyfriendnom
anyfriendrecip sexdummy, vce(robust) components(2) mixtureof(poisson)
predict yfit11, equation(component1)
predict yfit12, equation(component2)
summarize yfit11 yfit12

estat ic

quietly histogram yfit11, name(_comp_1, replace)
quietly histogram yfit12, name(_comp_2, replace)
quietly graph combine _comp_1 _comp_2

quietly histogram yfit11, name(_comp_1, replace) kdensity
xtick(0(10)50)
quietly histogram yfit12, name(_comp_2, replace) kdensity
xtick(0(10)25)
graph combine _comp_1 _comp_2

fmm totcon ehgrd erngrd ehrc5 ernrc5 ehage ernage anyfriendnom
anyfriendrecip sexdummy, vce(robust) components(2) mixtureof(negbin2)
margins, dydx(*) predict(eq(component1)) atmean
margins, dydx(*) predict(eq(component2)) atmean

fmm totcon ehgrd erngrd ehrc5 ernrc5 ehage ernage anyfriendnom
anyfriendrecip sexdummy, vce(robust) components(2) mixtureof(poisson)
margins, dydx(*) predict(eq(component1)) atmean
```

```
margins, dydx(*) predict(eq(component2)) atmean

generate tiecon = 1 if tiec==0
quietly replace tiecon = 2 if tiec ==1
quietly replace tiecon = 3 if tiec==2


*Ordered/Multinomial Probit
tabulate tiec

oprobit tiecon ehgrd erngrd ehrc5 ernrc5 ehage ernage bcent10x reach
prxprest infldmn sexdummy, robust
predict l1oprobit l2oprobit l3oprobit, pr

mprobit tiec ehgrd erngrd ehrc5 ernrc5 ehage ernage bcent10x reach
prxprest infldmn sexdummy, robust baseoutcome(0)

predict pmprobit1 pmprobit2 pmprobit3, pr

generate tieg = 1 if tiel==0
quietly replace tieg = 2 if tiel==1
quietly replace tieg = 3 if tiel==2

tabulate tiel

oprobit tieg ehgrd erngrd ehrc5 ernrc5 ehage ernage bcent10x reach
prxprest infldmn sexdummy, robust
predict p1oprobit p2oprobit p3oprobit, pr

mprobit tiel ehgrd erngrd ehrc5 ernrc5 ehage ernage bcent10x reach
prxprest infldmn sexdummy, robust baseoutcome(0)

predict pmprobit11 pmprobit21 pmprobit31, pr
```

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Research Interests

Primary	**Game Theory, Industrial Organization, Econometrics**
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Skills

Academic

- Game Theoretic/Econometric Modeling to **study Behavior in Financial and Social Networks**
- Applications to Marketing: **Targeted Advertising and Consumer Search Behavior**
- **Econometric Modeling** and **Time Series Analysis**
- **Statistical** and **Mathematical Methodologies**
- **Economic Theory** and **Quantitative Analysis**

Software: MATLAB, STATA, R, MS-Excel, C#(basic), SQL, LaTeX

Education

May 2013 (Expected) — **Ph.D. Economics - Indiana University, Bloomington, USA**
(GPA: 3.8, Academic Standing - Top 5%)

- Thesis: Essays on Formation of Networks and its Applications
- Committee: Frank H. Page, Jr. (Chair), Pravin K. Trivedi, Matthijs Wildenbeest, Filomena Garcia

2010 — M.A. Economics - Indiana University, Bloomington, USA

2007 — M.Sc./M.Phil. Economics - Indira Gandhi Institute of Development and Research (IGIDR), India

2003 — B.A. (Honors) Economics - University of Delhi, India

Working Papers

1. **Interactions of Strategic Behavior and Financial Network Structure: A Network Formation Game Model of Endogenous Risk of Contagion**

2. **Estimation of Bilateral Connections in a Network : Copula vs. Maximum Entropy** (with Jose Pedro Fique)

3. Social Networks, Targeted Advertising and Catalog Competition (with Frank H. Page, Jr. and Joana Resende)

Work in Progress

1. Empirical Investigation of a Friendship Network using Finite Mixture Models (Tentative title)
2. Speculative Bubbles in a Small World: An Information Transmission Mechanism
3. Strategic Interactions and Collusive Contracts within a Principal-Agent Network
4. Analyzing Bureaucratic Corruption: An Experiment (with Sudipta Sarangi and Bibhudatta Panda)

Awards, Fellowships and Honor Societies

Apr, 2012	Susan Thrasher Fellowship for Best Graduate Student 2012-13, Indiana University, USA
Apr, 2011	Best Paper Award, Jordan River Economics Conference 2011, Indiana University, USA
Fall, 2010-11	Research Assistantship (NSF funded research; under James Walker), Indiana University, USA
Oct 2011 - Present	Women in Science Program, Advisory Board Member, Indiana University, USA
Fall 2008 - Present	Department of Economics Graduate Scholarship, Indiana University, USA
Fall-Spr 2005	Junior Research Fellowship, Indira Gandhi Inst. of Dev. Research, India

Presentations at Conferences

Nov, 2012	'Estimation of Bilateral Exposures: A Copula Approach' at 1st Annual CIRANO Workshop on Networks in Trade and Finance, Montreal, Canada
Mar, 2012	'Strategic Behavior and Endogenous Risk of Contagion in a Financial Network - A Network Formation Game' at the Info-Metrics Workshop, American University, Washington D.C.
Mar, 2012 (Invited)	'Strategic Behavior and Endogenous Risk of Contagion in a Financial Network - A Network Formation Game' at the 'Networks and Development' Conference, Louisiana State University, Baton Rouge, LA
Apr, 2011	'Strategic Behavior and Endogenous Risk of Contagion in a Financial Network - A Network Formation Game' at the Jordan River Economics Conference, Indiana University, Bloomington, IN

Other Conferences Attended

Sept, 2012	NBER/NSF/CEME Conference in Mathematical Economics and General Equilibrium Theory, Indiana University, Bloomington, IN
Jun, 2011	Association of Public Economic Theory Meetings, Indiana University, Bloomington, IN

Research Experience at Indiana University

Research Assistant, Interdisciplinary Experimental Laboratory, Fall 2010-11

- NSF-Funded Research in Experimental Economics (conducting experiments and investigating behavioral patterns in different VCM and CPR environments using econometric methods on the observed data). (**Principal Investigators: Nobel Prize Winner Elinor Ostrom and James Walker**)

Teaching Experience at Indiana University

Spring, 2012	Financial Economics (Graduate Assistant, for Frank H. Page, Jr.)
	Economics of Labor Markets (Graduate Assistant, for Peter Olson)
Spring, 2011	Graduate Core Microeconomics II (Associate Instructor, for Frank H. Page, Jr.)
Summer, 2010	Intermediate Microeconomics (Full Teaching Responsibility)
Spring, 2010	Intermediate Microeconomics (Associate Instructor, for Michael Kaganovich)
Fall, 2009	Optimization Theory for Economic Analysis (Graduate Level - Associate Instructor, for Gerhard Glomm)
Spring, 2009	Principles of Microeconomics (Graduate Assistant, for Peter Olson)
Fall, 2008	Statistics for Business and Economics (Graduate Assistant, for Mary Beth Camp)

Research and Teaching Experience at IGIDR

Research Assistant, Fall 2005

- Institute funded research on Two-Sided Matching Models: Market Designs, Auctions, Options and Information; Assistance towards the organization of Summer School in Mathematical Finance and IGIDR-Hamburg Workshop on Law and Economics. (Principal Investigator: P. G. Babu)

Teaching Assistant, Spring 2006
Graduate Core Microeconomics II (Teaching Assistant, P. G. Babu)

Languages

Native	Oriya (fluent, written and verbal), Hindi (fluent, written and verbal)
Other	**English (fluent, written and verbal)**

References Available upon request.