Contagion and Systemic Risk in Financial Networks

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ABSTRACT

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The 2007-2009 financial crisis has shed light on the importance of *contagion* and *systemic risk*, and revealed the lack of adequate indicators for measuring and monitoring them. This dissertation addresses these issues and leads to several recommendations for the design of an improved assessment of systemic importance, improved rating methods for structured finance securities, and their use by investors and risk managers.

Using a complete data set of all mutual exposures and capital levels of financial institutions in Brazil in 2007 and 2008, we explore in chapter 2 the structure and dynamics of the Brazilian financial system. We show that the Brazilian financial system exhibits a complex network structure characterized by a strong degree of heterogeneity in connectivity and exposure sizes across institutions, which is qualitatively and quantitatively similar to the statistical features observed in other financial systems. We find that the Brazilian financial network is well represented by a directed scale-free network, rather than a *small world* network. Based on these observations, we propose a stochastic model for the structure of banking networks,

representing them as a directed weighted scale free network with power law distributions for in-degree and out-degree of nodes, Pareto distribution for exposures. This model may then be used for simulation studies of contagion and systemic risk in networks.

We propose in chapter 3 a quantitative methodology for assessing contagion and systemic risk in a network of interlinked institutions. We introduce the *Contagion Index* as a metric of the systemic importance of a single institution or a set of institutions, that combines the effects of both common market shocks to portfolios and contagion through counterparty exposures. Using a directed scale-free graph simulation of the financial system, we study the sensitivity of contagion to a change in aggregate network parameters: connectivity, concentration of exposures, heterogeneity in degree distribution and network size. More concentrated and more heterogeneous networks are found to be more resilient to contagion. The impact of connectivity is more controversial: in well-capitalized networks, increasing connectivity improves the resilience to contagion when the initial level of connectivity is high, but increases contagion when the initial level of connectivity is low. In undercapitalized networks, increasing connectivity tends to increase the severity of contagion. We also study the sensitivity of contagion to local measures of connectivity and concentration across counterparties –the *counterparty susceptibility* and *local network frailty*– that are found to have a monotonically increasing relationship with the systemic risk of an institution. Requiring a minimum (aggregate) capital ratio is shown to reduce the systemic impact of defaults of large institutions; we show that the same effect may be achieved with less capital by imposing such capital requirements only on systemically important institutions and those exposed to them.

In chapter 4, we apply this methodology to the study of the Brazilian financial system. Using the Contagion Index, we study the potential for default contagion and systemic risk in the Brazilian system and analyze the contribution of balance sheet size and network structure to systemic risk. Our study reveals that, aside from balance sheet size, the network-based local measures of connectivity and concentration of exposures across counterparties introduced in chapter 3, the counterparty susceptibility and local network frailty, contribute significantly to the systemic importance of an institution in the Brazilian network. Thus, imposing an upper bound on these variables could help reducing contagion. We examine the impact of various capital requirements on the extent of contagion in the Brazilian financial system, and show that targeted capital requirements achieve the same reduction in systemic risk with lower requirements in capital for financial institutions.

The methodology we proposed in chapter 3 for estimating contagion and systemic risk requires visibility on the entire network structure. Reconstructing bilateral exposures from balance sheets data is then a question of interest in a financial system where bilateral exposures are not disclosed. We propose in chapter 5 two methods to derive a *distribution* of bilateral exposures matrices. The first method attempts to recover the balance sheet assets and liabilities "sample by sample". Each sample of the bilateral exposures matrix is solution of a relative entropy minimization problem subject to the balance sheet constraints. However, a solution to this problem does not always exist when dealing with *sparse* sample matrices. Thus, we propose a second method that attempts to recover the assets and liabilities "in the mean". This approach is the analogue of the Weighted Monte Carlo method introduced by Avellaneda et al. (2001). We first simulate independent samples of the bilateral exposures matrix from a relevant prior distribution on the network structure, then we compute posterior probabilities by maximizing the entropy under the constraints that the balance sheet assets and liabilities are recovered in the mean. We discuss the pros and cons of each approach and explain how it could be used to detect systemically important institutions in the financial system.

The recent crisis has also raised many questions regarding the meaning of structured finance credit ratings issued by rating agencies and the methodology behind them. Chapter 6 aims at clarifying some misconceptions related to structured finance ratings and how they are commonly interpreted: we discuss the comparability of structured finance ratings with bond ratings, the interaction between the rating procedure and the tranching procedure and its consequences for the stability of structured finance ratings in time. These insights are illustrated in a factor model by simulating rating transitions for CDO tranches using a nested Monte Carlo method. In particular, we show that the downgrade risk of a CDO tranche can be quite different from a bond with same initial rating. Structured finance ratings follow path-dependent dynamics that cannot be adequately described, as usually done, by a matrix of transition probabilities. Therefore, a simple labeling via default probability or expected loss does not discriminate sufficiently their downgrade risk. We propose to supplement ratings with indicators of downgrade risk. To overcome some of the drawbacks of existing rating methods, we suggest a risk-based rating procedure for structured products. Finally, we formulate a series of recommendations regarding the use of credit ratings for CDOs and other structured credit instruments.

Keywords: bilateral exposures, collateralized debt obligation, contagion, copula, credit derivatives, credit rating, default clustering, default risk, domino effects, macro-prudential regulation, random graph, relative entropy, scale-free, small-world, systemic risk, structured finance, transition probabilities.

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New York City, March 2011.

To my parents Daad and Nadim

Chapter 1

Introduction

"The greatest tragedy would be to accept the refrain that no one could have seen this coming and thus nothing could have been done. If we accept this notion, it will happen again."

Phil Angelides, Chairman of the Financial Crisis Inquiry Commission.

1.1 Contagion, systemic risk and financial regulation

Failures of financial institutions are mainly due to two forms of financial distress: *illiquidity* and *insolvency*. *Default* occurs when an institution fails to fulfill an obligation such as a scheduled debt payment of interest or principal or the inability to service a loan. This typically happens when the reserves in short term (liquid) assets do not suffice to cover short term liabilities. Insolvency happens when the capital of an institution is reduced to zero while illiquidity occurs when reserves in liquid assets, such as cash and cash equivalents, are insufficient to cover short term liabilities. Illiquidity leads to default while, in principle, insolvency may not necessarily entail default as long as the institution is able to obtain financing to meet payment obligations. Nevertheless, in the current structure of the financial system where institutions are primarily funded through short-term debt, which must be constantly renewed, insolvent institutions would have great difficulties in raising liquidity as their assets lose in value. Indeed, renewal of short term funding is subject to the solvency and credit worthiness of the institution. Thus, in practice, insolvency leads to illiquidity which in turn leads to default.

Bank failures have led in the recent years to a *systemic crisis* and shed light on the importance of *systemic risk*. Systemic risk is a macro-level risk which can impair the stability of the entire financial system, as opposed to the risk of failure of an individual entity in the system. Systemic risk could occur as a consequence of an aggregate negative shock affecting all institutions in the system, such as a common exposure to a macroeconomic factor: economic output, unemployment, inflation; or a common exposure to fluctuations in interest rates, foreign exchange rates, drop in market prices, etc. Another source of systemic risk is the *contagion* of financial distress in the system. A failure of a financial institution may spread in a *domino* fashion throughout the financial system. In a period of financial distress, a bank may fail to pay all of its creditors in full and on time. These losses are written down from the creditor's balance sheet. If the losses were sufficiently large to exceed the creditor's capital, it would cause the creditor to default on its short term debt obligations. This, in turn, will reduce the capital of the creditor's creditors triggering a cascade of losses and defaults. This cascade may lead to a disruption of the entire network and a spillover of financial distress to the larger economy.

To design public policies that can efficiently prevent the fragility of institutions to be translated into a systemic failure, it is necessary to understand the potential causes of both individual failures and systemic risk.

Banking regulation has imposed requirements on the amount of capital a bank should hold to withstand potential losses. The most important requirement is to maintain a capital ratio, defined as the ratio of a bank's capital to its risk-weighted assets. Basel 2 accords (BIS, 2001) propose guidelines to calculate the risk weights in such a way that banks hold enough capital to sustain the three main sources of risk they are faced to: credit, market and operational. For regulatory purposes, capital has been divided in three tiers. Tier 1 capital is mainly composed of shareholder's equity and represents the main cushion for a bank to absorb losses. The risk-based capital guidelines are supplemented by requirements on these tiers of capital. For example, in the U.S. banks must have a Tier 1 capital ratio of at least 4%, and a combined Tier 1 and Tier 2 capital ratio of at least 8% (FDIC, 2009).

The traditional regulatory framework for determining systemically important institutions has been to rank the institutions in the system in terms of the size of their balance sheet. Institutions with the largest balance sheet size are declared "too big to fail". This traces back to Continental Illinois Bank in 1984 when regulators feared that the failure of the bank might trigger a systemic crisis. Continental suffered from several defaults in its energy loans portfolio, and losses from other non performing loans. As a result, Moody's downgraded Continental from the Aaa rating category. This has led to a massive withdrawal of funds, triggering a liquidity crisis. Bank regulators were concerned that the financial distress of Continental, which held $45 billion in assets, might spread to more than 1000 other banks which might also default if Continental defaults. As a result, Continental and ten other of the U.S. largest banks were considered too big to fail (Gup, 2004).

Nevertheless, the recent financial crises have showed that institutions with a relatively small balance sheet size can pose a significant risk of contagion to the system.

In 1998, the Federal Reserve of New York organized a rescue plan for Long Term Capital Management (LTCM) fearing that a liquidation of the hedge fund assets to cover its debts might trigger a contagion of financial distress to its many counterparties. LTCM's on balance sheet assets totalled around $125 billion, with a capital of $4 billion. However, it held off balance sheet assets on a notional of more than $1 trillion (Parkinson, 1999; Lowenstein, 2000). LTCM's major counterparties had closely monitored their bilateral positions but were not aware of LTCM's total off balance sheet leverage. The example of LTCM has underlined the importance of *interconnectedness* when assessing the systemic risk of financial institutions. *Size* alone, as measured by the balance sheet assets, is not a good indicator of systemic risk.

The interconnectedness of the financial system was also a major factor in the 2007-2009 financial crisis. The crisis was triggered by the collapse of the housing bubble in the U.S. in 2007 and the rise in interest rates that led several home owners to default on their mortgages and enter foreclosures. As a result, asset backed securities written on a collateral of residential mortgages, such as residential mortgage backed securities (RMBS) and collateralized debt obligations (CDOs), incurred severe losses that destabilized the financial strength of the banking institutions that issued them and impaired their ability to provide short-term interbank and commercial lending. Several major institutions either failed, were acquired by other institutions, or were subject to government bailout. These included Lehman Brothers, Merrill Lynch, Fannie Mae, Freddie Mac, Washington Mutual, Wachovia, and AIG. The shortage of liquidity and the stock market crash contributed to a rapid spread of financial distress from one institution to another in a *domino effect*. Between January 2008 and March 2011, 348 banks had failed in the U.S. and were taken over by the FDIC with 25 failures occurring in 2008, 140 in 2009 and 157 in 2010 (Federal Deposit Insurance Corporation, 2011).

Regulators have had great difficulties anticipating the impact of defaults partly due to a lack of visibility on the structure of the financial system as well as a lack of a methodology for monitoring systemic risk. The complexity of the contemporary financial systems has made it a challenge to define adequate indicators of systemic risk that could help in an objective assessment of the systemic importance of financial institutions and an objective framework for assessing the efficiency of *macro-prudential* policies.

During the past two years, macro-prudential regulation has attracted the attention of regulators. The Federal Reserve (Bernanke, 2009) has examined the possibility of creation of a systemic risk authority whose responsibility is to (1) monitor large or rapidly increasing exposures across institutions and markets, rather than only at the level of individual institutions, (2) assess the potential changes in the markets and products that could increase systemic risk, (3) assess the risk of contagion between financial institutions within and across markets, such as the mutual exposures of highly interconnected institutions, and (4) identify possible regulatory gaps.

Another important aspect of the recent crisis has been the misleading role of credit ratings in representing the credit worthiness of financial instruments. Credit ratings failed to reflect the credit worthiness of asset backed securities, such as RMBS and CDOs. Several tranches in the AAA rating category, which is designed to reflect a very good credit worthiness, were downgraded of several notches. Between 2000 and 2007, Moody's rated \$4.7 trillion in RMBS and \$736 billion in CDOs. By the end of 2008, more than 90% of both Aaa and Baa CDO tranches had been downgraded (FCIC, 2010). As of July 2008, Standard & Poor's had downgraded 902 tranches of RMBS and CDOs of asset-backed securities that had been originally rated AAA out of a total of 4,083 tranches originally rated AAA, 466 of those were downgraded to the junk category (Standard & Poor's, 2008).

These misconceptions of the risk profile of CDO tranches come from a common confusion between CDOs and corporate bonds. CDOs have gained in popularity as attractive low risk products that offer much higher returns than corporate bonds with same ratings. However, the complex structure of CDOs cautions against any simple interpretation of their ratings.

Rating downgrades have dramatically increased the cost of raising capital and debt for the institutions that issued these securities, and simultaneously increased their capital requirement. The current regulatory framework (Basel 2) leaves financial institutions a certain freedom in choosing the right methodology for assessing their market risk, operational risk etc. However, in the case of credit risk assessment the regulators have endorsed credit ratings as the sole criterion for judging whether a given instrument is investment-grade: current regulation restricts holdings of some asset managers to investment grade as certified by ratings. Such restrictions can trigger instabilities in event of downgrades, forcing massive sales of downgraded assets. This has been especially visible in the cases of downgrades of bond insurers or monolines such as AIG which triggered massive downgrades of all instruments insured by them.

All these issues have pointed to the necessity of implementing measures that would allow to accurately assess the systemic importance of financial institutions and the credit worthiness of financial instruments, and to design effective macro-prudential policies to limit the extent of contagion and systemic risk in the financial system. This has motivated five research questions which this thesis attempts to answer.

1.2 Motivation and outline of the thesis

1.2.1 Research objectives

This dissertation attempts to tackle some of the major issues raised by of the recent financial crises, by introducing a quantitative methodology to address the questions listed below.

(a) Studying the structure and empirical features of interbank networks, which could help understanding how institutions are interconnected.

(b) Defining adequate measures of systemic importance that account for the complex and interconnected structure of the financial system.

(c) Re-examining the impact of macro-prudential policies on contagion and systemic risk, in the light of network models.

(d) Developing methods for exploring the structure of counterpart networks in absence of complete information on exposures. This is especially relevant since in practice only balance sheet data -total assets and liabilities- may be disclosed.

(e) Understand the reasons behind the volatility of structured finance ratings, their shortcomings and propose a rating methodology for structured products which better reflects their downgrade risk.

1.2.2 Outline

In chapter 2, we analyze the network structure of the Brazilian financial system using a *complete* data set of interbank exposures and capital levels provided by the Brazilian Central Bank. The Brazilian financial system exhibits a complex heterogeneous network structure, with power law distributions (Zipf's law) for the in-degree and out-degree and Pareto distribution for the exposures. These properties are found to be stable across time, and with parameters similar to the ones observed in the study of the Austrian network (Boss et al., 2004). We find that the Brazilian financial network is well represented by a directed scale-free network, rather than a *small world* network. Based on these observations, we propose a stochastic model for the structure of banking networks, representing them as a directed weighted scale free network with power law distributions for in-degree and out-degree of nodes, Pareto distribution for exposures. This model may then be used for simulation studies of contagion and systemic risk in networks.

In chapter 3, we introduce a new methodology for assessing contagion and systemic risk in a network of interlinked institutions. We present a metric for the systemic importance of a set of financial institutions, the Contagion Index, defined as the expected loss to the network triggered by the default of this set of institutions when the system is subject to a market shock. The definition of this indicator takes into account both common market shocks to portfolios (correlation) and contagion through counterparty exposures (network effects). Using the Contagion Index and a simulated scale-free network, we study the sensitivity of contagion to several network parameters: the aggregate level of connectivity and concentration of exposures, the heterogeneity in degree distribution, the network size, and the influence of local measures of connectivity and concentration across counterparties, the *counterparty susceptibility* and *local network frailty*. Our choice of a scale-free graph is motivated by the observations in chapter 2 on the structure of the Brazilian financial system and the empirical study of Boss et al. (2004) of the Austrian interbank network. Based on analogies with epidemiology and peer-to-peer networks (Cohen et al., 2003; Madar et al., 2004; Huang et al., 2007), we study the impact of capital requirements on reducing contagion and systemic risk in the system, and discuss the efficiency of *targeted* capital requirements that consist in imposing more stringent requirements on the creditors of the most contagious institutions.

Chapter 4 applies this methodology to the study of the magnitude of contagion risk in the Brazilian financial system. Our study reveals that the risk of default contagion is significant in the Brazilian financial system. The Contagion Index of a financial institution in Brazil is found to have a strong positive relationship with the total size of its interbank liabilities, meaning that balance sheet size does matter when assessing systemic risk. However, size alone is not a good indicator for the systemic importance of financial institutions: the network-based local measures of connectivity and concentration of exposures across counterparties introduced in chapter 3 –the counterparty susceptibility and local network frailty– are shown to contribute significantly to the systemic importance of an institution in the Brazilian network. In line with the simulation study of chapter 3, targeted capital requirements are found to achieve the same reduction in systemic risk with lower requirements in capital for financial institutions.

The methodology we proposed in chapter 3 for estimating contagion and systemic risk requires visibility on the entire structure of the financial network. Nevertheless, in practice, bilateral exposures are most often not disclosed, and regulators should rely essentially on the aggregate financial information provided by balance sheets, such as the capital, total assets and total liabilities of each institution, to assess systemic importance. Estimating bilateral exposures from balance sheet data has attracted then a significant attention from regulators and researchers on systemic risk in the past few years. Several studies have proposed a Maximum Entropy based method for deriving an estimator of the bilateral exposures matrix (Sheldon and Maurer, 1998; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b;

Mistrulli, 2007; Toivanen, 2009), but the estimated network failed to reproduce the heterogeneity of real-world interbank networks. Instead of finding a *point estimator* of the bilateral exposures matrix, we propose in chapter 5 two methods to derive a *distribution* of sample bilateral exposures matrices, assuming a prior distribution on the network structure as for example the power-law in-degree and out-degree distributions and Pareto exposures distribution observed in the empirical studies of financial networks (chapter 2). The first method attempts to recover the balance sheet assets and liabilities "sample by sample". Each sample of the bilateral exposures matrix is solution of a relative entropy minimization problem subject to the balance sheet constraints. However, a solution to this problem does not always exist when dealing with *sparse* sample matrices. Thus, we propose a second method that attempts to recover the assets and liabilities "in the mean". This approach is the analogue of the Weighted Monte Carlo method introduced by Avellaneda et al. (2001). We first simulate independent samples of the bilateral exposures matrix from a relevant prior distribution on the network structure, then we compute posterior probabilities by maximizing the entropy under the constraints that the balance sheet assets and liabilities are recovered in the mean. The posterior distribution obtained for the bilateral exposures matrix allows to build a posterior distribution for the Contagion Index and study the effect of the network structure on the extent of contagion. Furthermore, this provides regulators with a tool to detect systemically important institutions when the network structure plays a significant role in the propagation of financial distress, the latter being not accounted for by a ranking which segregates institutions solely in terms of their balance sheet size.

Finally, chapter 6 aims at clarifying some misconceptions related to CDO ratings, their interpretation and their use. We first describe the rating approaches used by major agencies for CDOs and other structured credit products. Most rating agencies have been using static factor models which are slight variations on the Gaussian copula model allowing for intersector and intrasector correlations. Using this framework, we explore several issues related to CDO ratings. Given the leveraged nature of CDOs, the downgrade risk of a CDO tranche can be quite different from a bond with same initial rating. Therefore, a simple labeling via default probability or expected loss does not discriminate sufficiently their downgrade risk. We show that migration probabilities for CDO tranches are path-dependent and non-homogeneous in time, thus one can not derive cumulative transition probabilities by raising the one-year transition matrix to iterative powers as currently suggested by rating agencies (Standard & Poor's, 2009). We also show that migration probabilities for tranches with similar rating can vary from structure to structure even for the same underlying debt portfolio: two tranches with the same rating can have completely different transition probabilities. Thus, it is not reasonable to compute ratings migration probabilities by raising the one-year transition matrix to iterative powers. While default probability is an adequate representation of the default risk of a corporate bond with known recovery rate, we show that the probability to incur loss fails to account for the risk carried by CDO tranches and can not differentiate between tranches with different risk profiles. As a solution to some of the drawbacks of the current rating methodologies, we propose a risk-based rating system, based on a risk measure applied to the loss distribution of the tranche. We show that such a risk-based approach can lead to quite different ratings for CDO tranches. CDO tranche ratings require assumptions on the dependence structure of the default times and are thus exposed to model risk. We explore this model risk for CDO and CDO-squared tranches in a multisector factor model. We examine the effect of using copulae with tail dependence, such as the Cauchy copula, to incorporate scenarios with default clusters. We find that this allows to define ratings with a smaller migration volatility. Based on these findings, we present a set of recommendations for the design, interpretation and use of credit ratings for CDOs and other structured products.

1.3 Contribution

Our contribution builds on previous theoretical and empirical studies of contagion and systemic risk in financial networks, and studies of credit ratings, but also differs from them both in terms of the methodology used and in terms of the results obtained. We refer the reader to De Bandt and Hartmann (2000) and Upper (2011) for a comprehensive review of the literature on systemic risk and contagion in financial networks.

1.3.1 Network structure of financial systems

Understanding how institutions are connected among each other has attracted the attention of regulators and researchers in the past few years. A branch of the literature has studied the network structure of the financial system in specific countries: Furfine (2003) in the US, Upper and Worms (2004) in Germany, Ágnes Lublóy (2006) in Hungary, van Lelyveld and Liedorp (2006) in the Netherlands, Wells (2004) and Elsinger et al. (2006a) in Austria, Wells (2004) in the UK, Mistrulli (2007) in Italy. Chapter 2 complements this literature by studying the network structure of the Brazilian financial system. While most empirical studies are based on partial information on bilateral exposures between banks, our analysis is based on a *complete* data set of all bilateral exposures and capital levels in the Brazilian financial system at various dates in 2007 and 2008.

This study reveals several statistical features of the Brazilian financial system that could be compared to the findings of the empirical studies of other banking systems: Pareto distribution for the exposures, and Zipf's law for the in-degree and out-degree, with tail exponents comparable to the ones found in the study of the Austrian interbank network (Boss et al., 2004).

Based on these empirical findings, we propose a stochastic model of the interbank network: directed weighted scale free network with Pareto weights distribution and capital levels computed according to the current regulatory framework (Basel 2). In fact, most previous studies on simulated networks (Allen and Gale, 2000; Freixas et al., 2000; Nier et al., 2007; Battiston et al., 2009) have assumed a "simplistic" network structure, such as a complete, regular or Erdös-Renyi graph, that does not reproduce the empirical features of real-world banking systems. Moreover, they have assumed "simplistic" capital allocations, that fail to mimic the Basel requirements on capital, which nevertheless represents the major concern for regulators since it constitutes a cushion to sustain potential losses. Chapter 2 contributes to this branch of simulation studies by suggesting a modeling framework backed by the empirical study of the network structure of real-world financial systems.

1.3.2 Too interconnected to fail: contagion and systemic risk in financial networks

In chapter 3, we contribute to the literature on contagion and systemic risk in three major aspects: First, we propose a new methodology for assessing contagion and systemic risk in financial networks. Second, we use this methodology to study the sensitivity of contagion to various network parameters. Third, we study the efficiency of various macro-prudential policies in limiting the extent of contagion in the network. We elaborate more on these contributions as follows.

Contrarily to indicators of systemic risk purely based on market data (Acharya et al., 2010; Adrian and Brunnermeier, 2008), our metric of systemic importance make use of exposures to simulate stress scenarios, resulting in a forward-looking measure of systemic risk. The Contagion Index measures the magnitude of loss *conditional* to the default of a given institution instead of averaging across all institutions as in Elsinger et al. (2006a). We argue that these conditional measures provide a better assessment of risk in a heterogenous system where the sample average may be a poor statistic. With the exception of Elsinger et al. (2006a,b), all previous studies examine the sole knock-on effects of the sudden failure of a single bank by considering an idiosyncratic shock that targets a single institution in the system. Our study, on the contrary, shows that common market shocks to balance sheets may exacerbate contagion during a crisis, thus takes into account common and independent market shocks to balance sheets as well as counterparty risk through mutual exposures. We use a copula with tail dependence to model the joint distribution of market shocks, which allows to generate clusters of large-magnitude market shocks that could not be otherwise generated with the Gaussian copula which has been the market standard in risk management for the past decade.

The loss contagion mechanism is similar to the one presented in Furfine (2003); Upper and Worms (2004); Wells (2004); Ágnes Lublóy (2006); van Lelyveld and Liedorp (2006); Mistrulli (2007); Nier et al. (2007). When an institution defaults, the unrecovered portion of the exposures to the defaulted institution (assuming an exogenous recovery rate) are absorbed by its creditors, that can themselves default if they do not hold enough capital to sustain their losses. However, this "sequential" (Upper, 2011) contagion mechanism is very different from the market equilibrium approach of Eisenberg and Noe (2001); Elsinger et al. (2006a,b) defined by a clearing payment vector, in which banks can liquidate their assets leading to a proportional sharing of losses among counterparties (endogenous recovery rate). We argue that, since bankruptcy procedures are usually slow and settlements may take up several months to be effective, creditors cannot recover the residual value of the defaulting institution according to such a hypothetical clearing mechanism, and write down their entire exposure in the short-run, leading to a short term recovery rate of zero. This seems a more reasonable approach in absence of a clearing mechanism.

While studying empirically contagion in real-world networks is a very important exercise for central banks and regulators, it does not allow to analyze the influence of key features of the network on the contagion process since these are fixed in the data. Previous studies on simulated networks (Allen and Gale, 2000; Freixas et al., 2000; Nier et al., 2007; Battiston et al., 2009) provide a flexible framework for studying the sensitivity of contagion to a change in various parameters, such as the level of connectivity, concentration and network structure. However, all these studies have assumed a "simplistic" network structure, such as a complete, regular or Erdös-Renyi graph, and "simplistic" capital levels, that do not reproduce the empirical features of real-world banking systems. Our study is based on a scale-free simulation of the interbank network with degree and exposures distributions similar to the ones observed in the Brazilian and Austrian networks, and capital levels determined according to Basel 2 accords. Thus, it provides a more realistic framework for analyzing the sensitivity of contagion and systemic risk to network parameters.

Our study also complements the existing literature by studying the contribution of network-based local measures of connectivity and concentration to systemic risk. Previous studies on simulated network structures have examined the contribution of aggregate measures of connectivity and concentration such as increasing the probability that two nodes are connected in an Erdös-Rényi graph, or increasing the number of nodes in the system (Battiston et al., 2009; Nier et al., 2007). However, they fail to detect the impact of connectivity and concentration *locally* around a single institution in the network. We thus introduce the *counterparty susceptibility* and *local network frailty* that measure respectively the susceptibility of the creditors of an institution to a potential default of the latter and the fragility of the entire network in the event of default of this institution. We find that the two measures can explain significantly default contagion.

We also contribute to the literature by introducing *targeted capital requirements* as macro-prudential strategies. We find that they require less capital, to achieve the same level of contagion and systemic risk, than the classical strategy consisting in imposing aggregate capital ratios on all institutions in the network.

On the results side, we find that contagion is very sensitive to a change in the network structure and the level of connectivity and concentration. We compare the extent of contagion in networks with different degree and exposures distributions, and find that more heterogeneous networks are more resilient to contagion. We also observe, in line with Nier et al. (2007) and Battiston et al. (2009), a trade-off phenomenon when increasing connectivity in the network between increasing the potential channels for the propagation of financial distress and the stabilizing benefit of risk sharing. The direction of the results is similar to Nier et al. (2007): in well-capitalized networks, increasing connectivity is found to increase significantly contagion up to a certain threshold above which a further increase in connectivity leads to a decrease in the extent of contagion. However, in undercapitalized networks, increasing connectivity makes the network more prone to contagion whatever the initial level of connectivity is. All the above observations point to the need of using realistic network structures in simulation studies to reduce the bias of the estimate of contagion and systemic risk.

1.3.3 Systemic risk in banking systems: the case of Brazil

Chapter 4 contributes to the empirical literature on contagion and systemic risk in banking systems in four main aspects.

First, it reveals that the Contagion Index of institutions in the Brazilian financial system exhibits a heavy-tailed distribution, with most institutions presenting a negligible risk and few of them posing a serious risk of contagion to the network. This cautions against the use of measures of systemic risk that do not segregate between different institutions in the network, for example by averaging systemic risk across all institutions as in Elsinger et al. (2006a).

Second, it reveals the importance of *contagion* that has been dismissed in the previous empirical literature (Sheldon and Maurer, 1998; Furfine, 2003; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b; Mistrulli, 2007). Our study reveals that the risk of default contagion is significant in the Brazilian financial system. In contrast with Elsinger et al. (2006a), we find that scenarios with contagion are more frequent than those without contagion when grouped by number of fundamental defaults, precisely when the number of fundamental defaults is at least 3. In fact, we find that market shocks can play an essential role in propagating default across the network. They are found to increase the proportion of *contagious exposures* which are exposures that transmit default in all scenarios of market shocks.

Most of the previous empirical studies on contagion in interbank networks are based on partial information on the bilateral exposures, and estimate missing exposures with a Maximum Entropy method (Sheldon and Maurer, 1998; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b; Degryse and Nguyen, 2007). However, the Maximum Entropy method is found to underestimate the possibility of default contagion (Mistrulli, 2007; van Lelyveld and Liedorp, 2006). Our study, by making use of a complete data set avoids this caveat.

Third, this study sheds light on the importance of the interconnectedness of the financial system. It measures the marginal contribution of balance sheet size to contagion, and the marginal contribution of the local measures of connectivity and concentration introduced in chapter 3, leading to the conclusion that interconnectedness or the network structure can have a significant impact on systemic risk. This highlights the need of visibility on the entire network structure to properly measure systemic risk.

Finally, we contribute to the literature by studying the impact of the targeted capital requirements introduced in chapter 3 on the extent of contagion in the Brazilian financial system. Such macro-prudential policies have been completely neglected in the literature in favor of micro-prudential policies such as imposing homogeneous capital requirements regardless of the level of interconnectedness of the institutions in the system.

1.3.4 Reconstruction of interbank networks

Chapter 5 introduces a new methodology to address the problem of reconstructing interbank networks when only aggregate balance sheet information is available, such as the total assets, total liabilities, and capital levels of each institution in the network.

Previous empirical studies of contagion and systemic risk in interbank networks (Sheldon and Maurer, 1998; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b; Mistrulli, 2007; Toivanen, 2009) have attempted to solve the problem of reconstructing bilateral exposures given only aggregate balance sheet data by formulating it as a *matrix balancing* problem (Schneider and Zenios, 1990) where the objective is to, given a prior matrix $\mathbf{x_0}$, find a matrix $\mathbf{x}$ that is as close as possible to $\mathbf{x_0}$ and satisfies a set of linear constraints (*balancing constraints*) on its rows and columns. The constraints are set to recover the balance sheets total assets and labilities. Due to the lack of information on the network structure, these studies have considered a *uniform* prior matrix $\mathbf{x_0}$ in which the assets of each institution in the network are distributed across its counterparties proportionally to their respective liabilities. As a result, the reconstructed network is *complete* in the sense that each institution is exposed to all other institutions in the network, which fails to reproduce the heavy-tailed in-degree, out-degree and exposures distributions of real-world banking systems, such as the Brazilian system studied in chapter 2.

Instead, the approach we propose, whose goal is to find a *distribution* of bilateral exposures matrices $\mathbf{x}^{(1)}, ..., \mathbf{x}^{(M)}$ given a relevant prior distribution on the network structure, allows to generate *sparse* matrices $\mathbf{x}^{(1)}, ..., \mathbf{x}^{(M)}$. That is, the networks represented by these sample bilateral exposures matrices all exhibit a scale-free degree distribution which is more realistic than the complete network structure proposed in previous studies.

In fact, any arbitrary choice of a prior matrix $\mathbf{x_0}$ would lead to an arbitrary reconstructed network that might be very different from the real unknown network. Hence, building a distribution of bilateral exposures matrices instead of estimating one particular matrix is necessary to avoid "artificially" creating information on the network structure. This allows to build a posterior distribution of any measure of systemic importance that requires knowledge of the network, such as the Contagion Index, and examine the impact of the network structure on the magnitude of contagion in the system.

In networks that are prone to contagion, we find that the posterior mean of the Contagion Index does not estimate well the Contagion Index of the real network. The Contagion Index is very sensitive to the network structure. This implies that a knowledge of aggregate levels of assets, liabilities, and capital levels does not suffice to accurately estimate the Contagion Index. Therefore, there is a need to disclose large bilateral exposures in order to have an accurate assessment of contagion and systemic risk.

Nevertheless, the posterior mean of the Contagion Index is found to mimic the same ranking as the Contagion Index of the real network, while the liabilities size does not mimic this ranking. Thus, the reconstruction methodology we presented could be used as a tool to detect systemically important institutions instead of a ranking which is based on a sole knowledge of the the balance sheets assets and liabilities.

1.3.5 A closer look at credit ratings of CDO tranches

Credit ratings assigned by rating agencies to structured credit products have played an important role in the development of the structured credit market, which has been cast into the limelight by the recent credit crisis. Several critics of rating methodologies have focused on the drawbacks of the Gaussian copula used to model the dependence structure between the assets default times (Donnelly and Embrechts, 2010). However, very few studies have examined the issues that do not arise from the use of the Gaussian copula model but are related to the specific characteristics of structured products.

Chapter 6 studies the particularities of credit rating methodologies when applied to structured products such as CDO tranches and CDOs of ABS. Using the Gaussian copula framework, we explore several issues related to ratings of CDO tranches.

Credit ratings have been attributed by major rating agencies to structured credit products based on the criterion of default probability or expected loss of these instruments (Standard & Poor's, 2007). These metrics are very poor indicators of the credit risk of such complex instruments which leads to situations of "rating arbitrage" where high ratings can be attributed to "high-yield" instruments which carry in fact a lot of risk (Cont and Jessen, 2011).

We show that the default probability fails to segregate tranches with different risk profiles. Since a rating based on default probability depends solely on the subordination level of the tranche, two tranches with completely different risk profiles can nevertheless have the same rating. For example, default probability can not differentiate between a tranche $6\% - 7\%$ and a tranche $6\% - 100\%$ while the loss of the latter is always smaller relatively to the initial notional of the tranche. We propose instead a risk-based rating procedure, based on a risk measure applied to the loss distribution of the tranche.

We show that the downgrade risk of a CDO tranche can be quite different from a bond with same initial rating. CDO tranches are levered products, meaning that a slight deterioration in the credit quality of the underlying obligors can lead to downgrades of several notches at the level of a CDO tranche. We also show, that credit ratings of CDO tranches are non Markov processes and non-homogeneous processes. Nevertheless, rating agencies still assume until today that ratings transitions are Markov and homogeneous, suggesting to compute transition probabilities by raising the annual transition matrix to the iterative powers (de Servigny and Jobst, 2007). Therefore, a simple labeling via default probability or expected loss does not discriminate sufficiently their downgrade risk. We propose to supplement ratings with indicators of downgrade risk.

Finally, we also examine the *model risk* to which credit ratings are exposed. In line with previous studies on the sensitivity of credit ratings of CDO tranches to modeling assumptions (Fender and Kiff, 2004; Meng and Sengupta, 2009; Wojtowicz, 2011), this study shows that credit ratings of CDO tranches are extremely sensitive to the correlation, recovery, and assets default rates parameters used to model the joint distribution of default times. The Gaussian copula has been criticized in the recent crisis for its failure to generate scenarios with default clusters (Donnelly and Embrechts, 2010). Several alternatives to the factor Gaussian copula model have been proposed in the literature, but always the context of pricing CDO tranches (Donnelly and Embrechts, 2010; Azizpour et al., 2010; Kalemanova et al., 2005; Duffie et al., 2009; Peng and Kou, 2009). We examine the effect of using a copula with tail dependence, such as the Cauchy copula, for tranching and rating CDO tranches. We find that this leads to much higher credit enhancement than required by the Gaussian copula, hence a smaller migration volatility of credit ratings.

Chapter 2

Network Structure of Financial Systems

This chapter is based on the paper "Network structure and systemic risk in banking systems" (Cont et al., 2010), which is a joint work with Professor Rama Cont and Doctor Edson Bastos e Santos, and the paper "Too Interconnected to Fail: Contagion and Systemic Risk in Financial Networks" (Cont and Moussa, 2010c), which is a joint work with Professor Rama Cont.

2.1 Introduction

The *interconnectedness*, or network structure, of the financial system has played a major role in the recent financial crisis. It has facilitated the propagation of financial distress from one institution to the rest of the system through bilateral exposures. Regulators have had great difficulties anticipating the impact of defaults partly due to a lack of both visibility and relevant indicators on the structure of the financial system. As a result, regulators have been concerned in the past few years in studying the structure and empirical features of interbank networks in order to better understand how institutions are interconnected.

Several studies have analyzed the empirical features of interbank networks in various countries: Furfine (2003) in the US, Upper and Worms (2004) in Germany, Ágnes Lublóy (2006) in Hungary, van Lelyveld and Liedorp (2006) in the Netherlands, Wells (2004) and Elsinger et al. (2006a) in Austria, Wells (2004) in the UK, Mistrulli (2007) in Italy. We refer the reader to Upper (2011) for a comprehensive survey. These studies have revealed the heterogeneity of interbank networks (see figure 2.1): asymmetric in- and out-degree distributions and Pareto distributions for exposures (Boss et al., 2004), multiple money center structure (Ágnes Lublóy, 2006; Müller, 2006) and tiered structures (Upper and Worms, 2004; Toivanen, 2009).

While most these empirical studies are based on partial information on bilateral exposures between banks, we study in this chapter the network structure of the Brazilian financial system using a *complete* data set of all bilateral exposures in the Brazilian financial system at various dates in 2007 and 2008. Our empirical findings are qualitatively and quantitatively similar to the statistical features observed in the Austrian financial system (Boss et al., 2004): power law distributions for the in-degree and the out-degree and Pareto distribution for the exposures, with tail exponents of the same order of magnitude in the two networks.

Therefore, modeling the financial system using simple network structures, such as the Erdös-Rényi model used in Nier et al. (2007), or the network with equal number of counterparties for all institutions used in Battiston et al. (2009), or the complete and incomplete structures in Allen and Gale (2000), fails to mimic the heterogeneity of financial systems.

Based on these observations, we propose a stochastic model for the structure of banking networks, representing them as a directed weighted scale free network with power law distributions for in-degree and out-degree of nodes, Pareto distribution for exposures. This model may then be used for simulation studies of contagion and systemic risk in networks.

2.1.1 Summary of main results

- The Brazilian financial system exhibits a complex heterogeneous network structure: the distribution of in-degrees, out-degrees and mutual exposures are found to be heavy-tailed, exhibiting power law (Zipf's law) tails with exponents between 2 and 3. Furthermore, these statistical regularities are shown to be stable across time.

- The network structure is qualitatively different from a small-world network. In particular, we observe many nodes with arbitrary small clustering coefficient.

- There is a positive association between the size of exposures (assets) and the number of debtors (in-degree) of an institution in the Brazilian financial system, and a positive association between the size of liabilities and the number of creditors (out-degree) of an institution. More (less) connected financial institutions have larger (smaller) exposures.

- The ratio of interbank exposures to Tier 1 capital exhibits a heterogenous distribution: most financial institutions hold much more Tier 1 capital than their interbank exposures, which means that they have a strong capacity to absorb losses. However, some institutions have interbank exposures more than a hundred times their Tier 1 capital. These institutions may present a

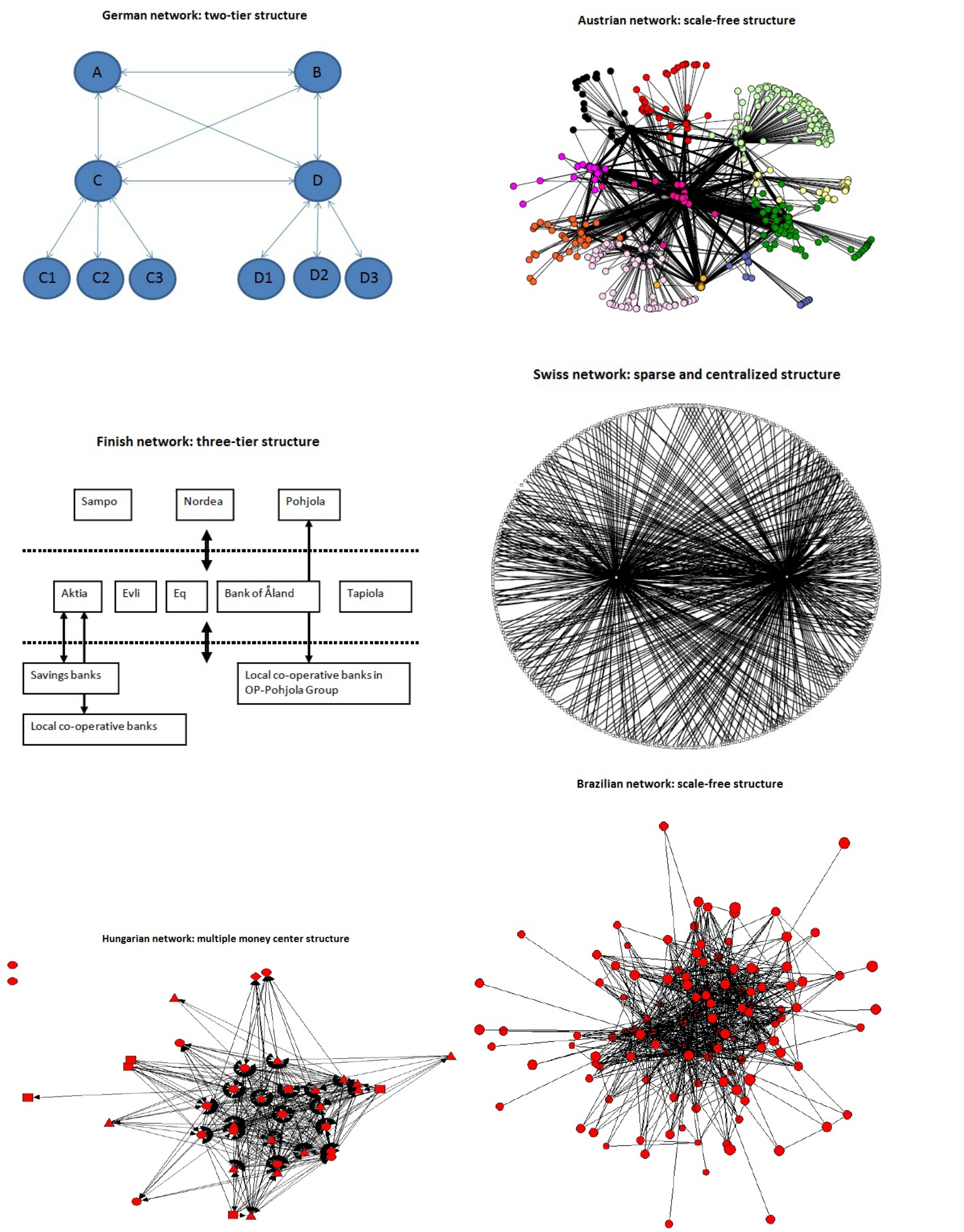


Figure 2.1: Network structures of real-world banking systems. Germany: two-tier structure (Upper and Worms, 2004), Austria: scale-free structure (Boss et al., 2004), Switzerland: sparse and centralized structure (Müller, 2006), Finland: three-tier structure (Toivanen, 2009), Hungry: multiple money center structure (Ágnes Lublóy, 2006), Brazil: scale-free structure (see chapter 4).

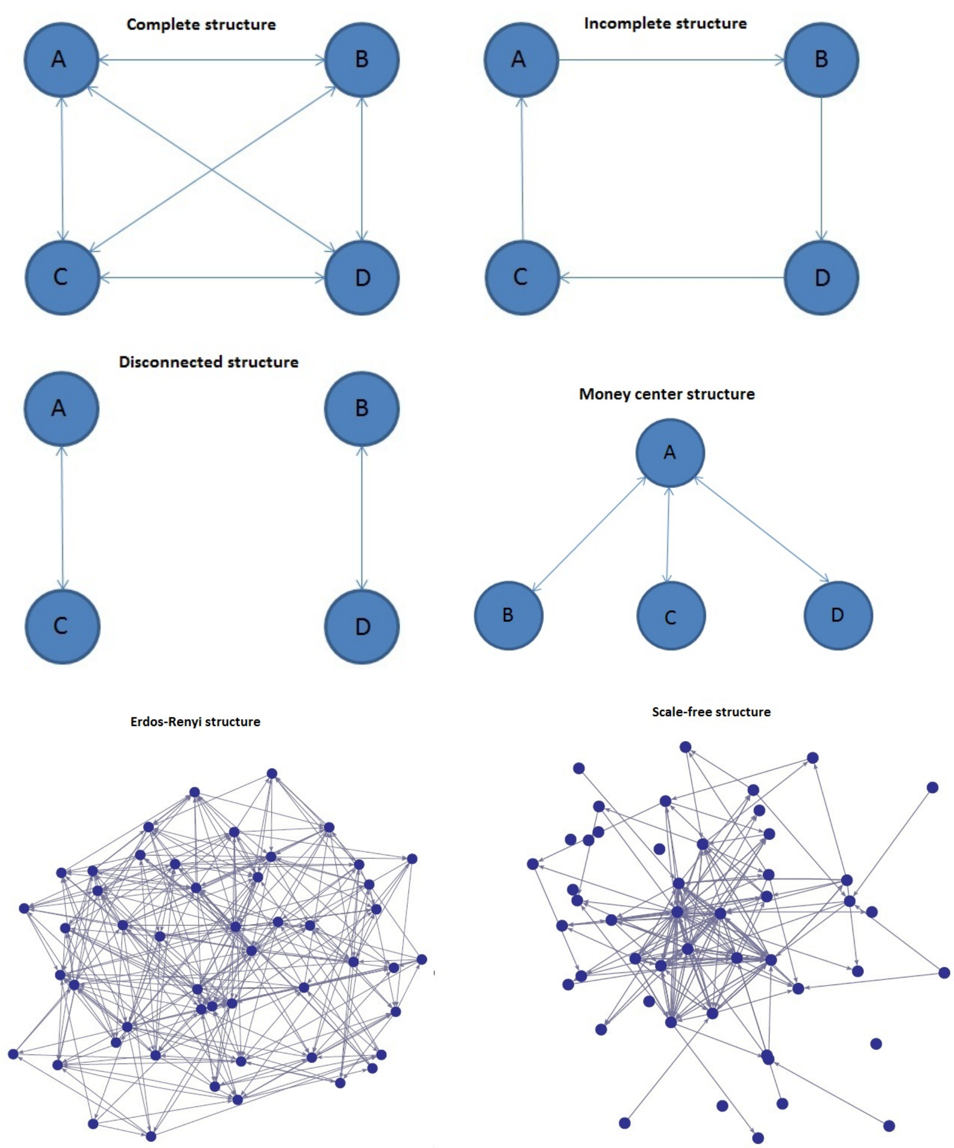


Figure 2.2: Graph structures used for modeling interbank networks: complete, incomplete and disconnected structures (Allen and Gale, 2000), money center structure (Freixas et al., 2000), Erdös-Rényi structure (Nier et al., 2007), scale-free structure (our contribution).

significant risk of default.

- Based on these empirical observations, we propose a realistic model of inter-bank networks: directed scale-free random graph with power law distributions for in-degree and out-degree of nodes, Pareto distribution for exposures and capital levels computed according to the current regulatory framework (Basel 2).

2.1.2 Outline

The chapter is organized as follows. In section 2.2, we present the network representation of the financial system. In section 2.3, we study various statistical features of the Brazilian financial system at different dates in 2007 and 2008. In section 2.4, we present a directed weighted scale-free network model of the financial system, generating power law distributions of the in-degree, out-degree and exposures. We propose such model as a framework for simulation studies on financial networks.

2.2 Network representation of financial systems

Counterparty relations in a financial system may be represented as a weighted directed graph, or a *network*, defined as a triplet $I = (V, E, c)$, consisting of

- a set V of financial institutions, whose number we denote by n,
- a matrix E of bilateral exposures: E_{ij} represents the exposure of node i to node j defined as the (mark-to-)market value of all liabilities of institution j to institution i at the date of computation. It is thus the maximal short term loss of i in case of an immediate default of j.

- $c = (c_i, i \in V)$ where c_i is the capital of the institution i, representing its capacity for absorbing losses.

Such a network may be represented as a graph in which nodes represent institutions and links represent exposures.

We define the *in-degree* $k_{in}(i)$ of a node $i \in V$ as the number of its debtors and *out-degree* $k_{out}(i)$ the number of its creditors:

$$k_{in}(i) = \sum_{j \in V} 1_{\{E_{ij}>0\}} \quad k_{out}(i) = \sum_{j \in V} 1_{\{E_{ji}>0\}}, \tag{2.1}$$

The *degree* $k(i)$ of a node i is defined as $k(i) = k_{in}(i) + k_{out}(i)$ and measures its connectivity.

Although all institutions in the network are not banks, we will refer to the exposures as "interbank" exposures for simplicity. We denote $A(i)$ the interbank assets of financial institution i, and $L(i)$ its interbank liabilities:

$$A(i) = \sum_{j \in V} E_{ij} \quad L(i) = \sum_{j \in V} E_{ji}, \tag{2.2}$$

A stylized balance sheet is presented in table 2.1.

Assets	**Liabilities and Shareholders' Equity**
Interbank Assets A_i	Interbank Liabilities L_i
Other Assets	Other Liabilities
	Equity Capital $c(i)$

Table 2.1: The general structure of a balance sheet.

2.3 The Brazilian financial system: a complex network

2.3.1 Data and consolidation procedure

The Brazilian financial system encompasses 2400 financial institutions chartered by the Brazilian Central Bank and grouped into three types of operation: Type I are banking institutions that have commercial portfolios, Type III are institutions that are subject to particular regulations, such as credit unions, and Type II represent all other banking institutions. Despite their reduced number (see table 2.2), financial institutions of Type I and II account for the majority (about 98%) of total assets in the Brazilian financial system (see table 2.3). We therefore consider in the Brazilian data set only Type I and Type II financial institutions which is a very good proxy for the Brazilian financial system. Most of the financial institutions belong to a conglomerate (75% of all financial institutions of Type I and II). Consequently, it is quite meaningful to analyze the financial system from a consolidated perspective where financial institutions are classified in groups that are held by the same shareholders. Only banking activities controlled by the holding company are considered in the consolidation procedure. The accounting standards for consolidation of financial statements were established by Resolutions 2,723 and 2,743, BCB (2000a,b), and they are very similar to IASB and FASB directives. If we regard financial institutions as conglomerates, the dimension of the exposures matrices reduces substantially, see table 2.2 for the number of financial conglomerates in the Brazilian financial system after the consolidation procedure.

Type	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**	**Dec-08**
Multiple Bank	135	135	135	136	139	139	140
Commercial Bank	20	20	21	20	20	18	18
Development Bank	4	4	4	4	4	4	4
Savings Bank	1	1	1	1	1	1	1
Investment Bank	17	17	17	18	18	18	17
Consumer Finance Company	51	52	51	56	55	55	55
Security Brokerage Company	113	107	114	107	107	107	107
Exchange Brokerage Company	48	46	48	46	46	45	45
Security Distribution Company	132	135	133	133	136	136	135
Leasing Company	40	38	41	37	36	36	36
Real Estate Credit Company and Savings and Loan Association	18	18	18	18	18	17	16
Mortgage Company	6	6	6	6	6	6	6
Development Agency	12	12	12	12	12	12	12
Total Banking Institutions of Type I and II	**597**	**591**	**601**	**594**	**598**	**594**	**592**
Credit Union	1.461	1.465	1.460	1.466	1.460	1.457	1.453
Micro-financing Institution	54	52	54	48	46	45	47
Total Banking Institutions Type III	**2.112**	**2.108**	**2.115**	**2.108**	**2.104**	**2.096**	**2.092**
Non-Banking Institutions	**332**	**329**	**333**	**324**	**317**	**318**	**317**
Total Banking and Non-Banking Institutions	**2444**	**2.437**	**2.448**	**2.432**	**2.421**	**2.414**	**2.409**

Table 2.2: Number of financial institutions by type of operation for the Brazilian financial system. Source: Sisbacen.

Assets in Billions USD	**Jun-07**	%	**Dec-07**	%	**Mar-08**	%	**Jun-08**	%	**Sep-08**	%	**Dec-08**	%
Banking - Type I	1,064.8	87.1	1,267.7	87.8	1,366.9	87.9	1,576.0	87.7	1,433.2	88.0	1,233.6	87.5
Banking - Type II	129.6	10.6	142.7	9.9	152.7	9.8	179.4	10.0	160.1	9.8	148.3	10.5
Banking - Type I and II	**1,194.5**	**97.7**	**1,410.4**	**97.7**	**1,519.6**	**97.7**	**1,755.4**	**97.7**	**1,593.2**	**97.8**	**1,382.0**	**98.0**
Banking - Type III	17.7	1.5	21.5	1.5	23.7	1.5	28.3	1.6	24.1	1.5	19.1	1.4
Non-Banking	10.4	0.9	12.8	0.9	12.5	0.8	14.4	0.8	11.4	0.7	9.3	0.7
Total Financial System	**1,222.6**	**100.0**	**1,444.8**	**100.0**	**1,555.8**	**100.0**	**1,798.1**	**100.0**	**1,628.8**	**100.0**	**1,410.4**	**100.0**
Number of Conglomerates	**Jun-07**	%	**Dec-07**	%	**Mar-08**	%	**Jun-08**	%	**Sep-08**	%	**Dec-08**	%
Banking - Type I	102	5.4	101	5.4	101	5.4	101	5.4	103	5.5	101	5.4
Banking - Type II	32	1.7	32	1.7	32	1.7	33	1.8	34	1.8	35	1.9
Banking - Type I and II	**134**	**7.1**	**133**	**7.1**	**133**	**7.1**	**134**	**7.2**	**137**	**7.3**	**136**	**7.3**
Banking - Type III	1,440	76.8	1,440	77.0	1,436	77.0	1,441	77.0	1,442	76.9	1,438	77.0
Non-Banking	302	16.1	298	15.9	297	15.9	296	15.8	296	15.8	294	15.7
Total Financial System	**1,876**	**100.0**	**1,871**	**100.0**	**1,866**	**100.0**	**1,871**	**100.0**	**1,875**	**100.0**	**1,868**	**100.0**

Table 2.3: Representativeness of Brazilian financial institutions in terms of total Assets and number. The total assets were converted from BRL (Brazilian Reals) to USD (American Dollars) with the following foreign exchange rates (BRL/USD): 1.9262 (Jun-07), 1.7713 (Dec-07), 1.7491 (Mar-08), 1.5919 (Jun-08), 1.9143 (Sep-08), and 2.3370 (Dec-08). Source: Sisbacen.

These exposures, reported at six dates (June 2007, December 2007, March 2008, June 2008, September 2008 and November 2008) cover various sources of risk:

1. fixed-income instruments (certificate of deposits and debentures);
2. borrowing and lending (credit risk);
3. derivatives (including OTC instruments such as swaps);
4. foreign exchange and,
5. instruments linked to exchange-traded equity risk.

Derivatives positions were taken into account at their market prices when available, or at fair value when a model-based valuation was required.

The data set also gives the Tier I and Tier 2 capital of each institution, computed according to guidelines provided in Resolution 3,444 BCB (2007a) of the Brazilian Central Bank, in accordance with the Basel 1 and 2 accords. Tier 1 capital is composed of shareholder equity plus net income (loss), from which the value of redeemed preferred stocks, capital and revaluation of fixed assets reserves, deferred taxes, and non-realized gains (losses), such as mark-to-market adjustments from securities registered as available-for-sale and hedge accounting are deducted. Tier 2 capital is equal to the sum of redeemed preferred stocks, capital, revaluation of fixed assets reserves, non-realized gains (losses), and complex or hybrid capital instruments and subordinated debt. We shall focus on Tier 1 capital as a measure of a bank's capacity to absorb losses in the short term.

Financial conglomerates in Brazil are subject to minimum capital requirements. The required capital is a function of the associated risks regarding each financial institution's operations, whether registered in their balance sheets (assets and liabilities) or not (off-balance sheet transactions), as defined in Resolution 3,490, BCB (2007b). The required capital is computed as $c_r = \delta \times$ Risk Base where the $\delta = 11\%$ is the so-called *Basel Index* and the risk base is the sum of credit exposures weighted by their respective risk weights, foreign currency and gold exposures, interest rate exposures, commodity exposures, equity market exposures, and operational risk exposures. It is important to highlight that the exposures considered in the computation of the *risk base* include not only interbank exposures but also exposures to all counterparties.

2.3.2 A heterogeneous network

2.3.2.1 Distribution of connectivity

Casual inspection of the Brazilian interbank network reveals the existence of nodes with widely differing connectivity. Figure 2.3 shows the Brazilian interbank network in December 2007. It is observed to have a heterogeneous and complex structure, some highly connected institutions playing the role of "hubs" while others are at the periphery.

Table 2.4 presents some descriptive statistics of key network variables: the number of debtors (in-degree), number of creditors (out-degree), exposures, relative exposures (ratio of the exposure of institution i to institution j to the capital of i), and distance between two institutions (nodes) in the network.

This observation is confirmed by further analyzing the data on in-degrees and out-degrees of nodes. Figures 2.4 and 2.5 show, respectively, the double logarithmic plot of the empirical complementary cumulative distribution for the in-degree $\hat{\mathbb{P}}(K_{in} \geqslant k)$ and out-degree $\hat{\mathbb{P}}(K_{out} \geqslant k)$ for $k \geqslant 1$. We notice that the tails of the distributions exhibit a linear decay in log-scale, suggesting a power law tail. We refer the reader to appendix B for a brief introduction to power law distributions.

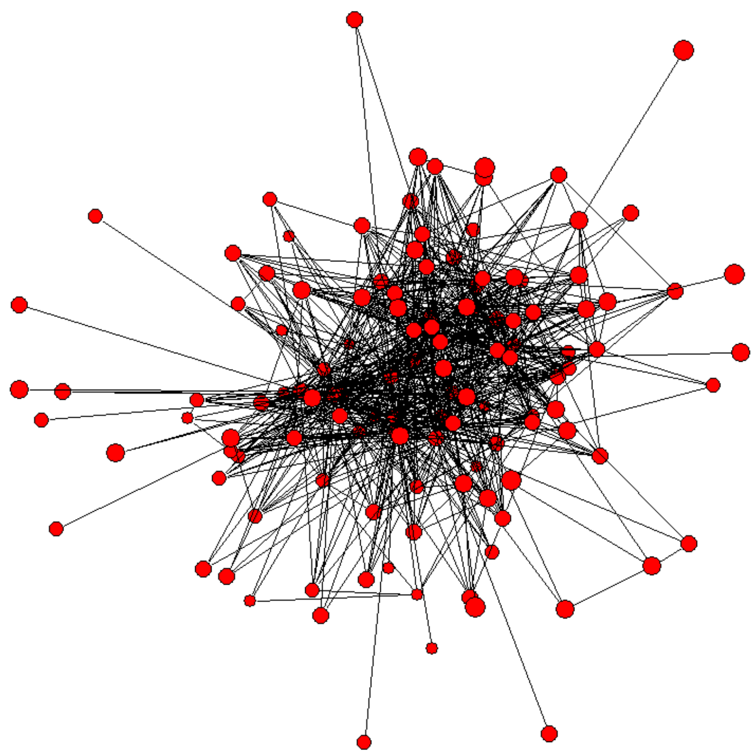


Figure 2.3: Brazilian interbank network, December 2007. The number of financial conglomerates is $n = 125$ and the number of links in this representation at any date does not exceed 1200.

In-Degree	Jun-07	Dec-07	Mar-08	Jun-08	Sep-08	Nov-08
Mean	8.56	8.58	8.75	8.98	8.99	7.88
Standard Deviation	10.84	10.86	10.61	11.15	11.32	11.02
5% quantile	0	0	0	0	0	0
95% quantile	30.50	29.30	30.45	31	32	30.60
Maximum	54	54	51	57	60	62
Out-Degree	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**
Mean	8.56	8.58	8.75	8.98	8.99	7.88
Standard Deviation	8.71	8.82	9.02	9.43	9.36	8.76
5% quantile	0	0	0	0	0	0
95% quantile	26	26	27.90	29.25	30.20	27.40
Maximum	36	37	39	41	39	44
Exposures (in billions of BRL)	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**
Mean	0.07	0.05	0.05	0.05	0.05	0.08
Standard Deviation	0.77	0.32	0.32	0.30	0.38	0.54
5% quantile	0.00	0.00	0.00	0.00	0.00	0.00
95% quantile	0.20	0.17	0.17	0.18	0.19	0.35
Maximum	23.22	9.89	9.90	9.36	12.50	15.90
Relative Exposures ($E_{ij}/c(i)$)	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**
Mean	0.23	0.20	0.04	0.04	0.03	0.05
Standard Deviation	1.81	1.62	0.16	0.17	0.06	0.21
5% quantile	0.00	0.00	0.00	0.00	0.00	0.00
95% quantile	0.70	0.59	0.20	0.21	0.16	0.18
Maximum	49.16	46.25	4.57	5.17	0.69	6.02
Distance	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**
Mean	2.42	2.42	2.38	2.38	2.33	2.35
Standard Deviation	0.84	0.85	0.84	0.82	0.77	0.78
5% quantile	1	1	1	1	1	1
95% quantile	4	4	4	4	3	4
Maximum (Diameter)	5	6	6	6	5	6

Table 2.4: Descriptive statistics of the number of debtors (in-degree), number of creditors (out-degree), exposures, relative exposures (ratio of the exposure of institution i to institution j to the capital of i), and distance between two institutions (nodes) in the network.

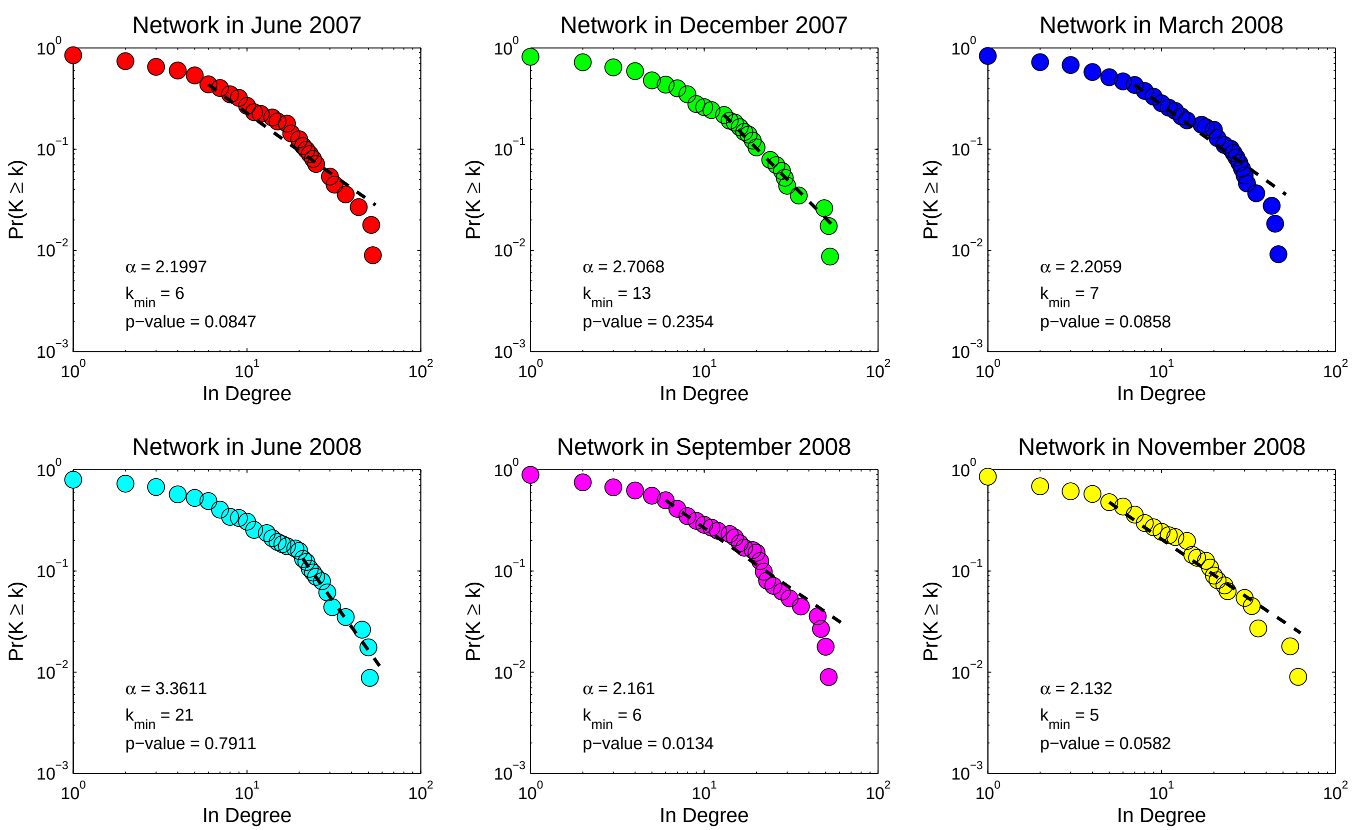


Figure 2.4: Brazilian interbank network: distribution of in-degree.

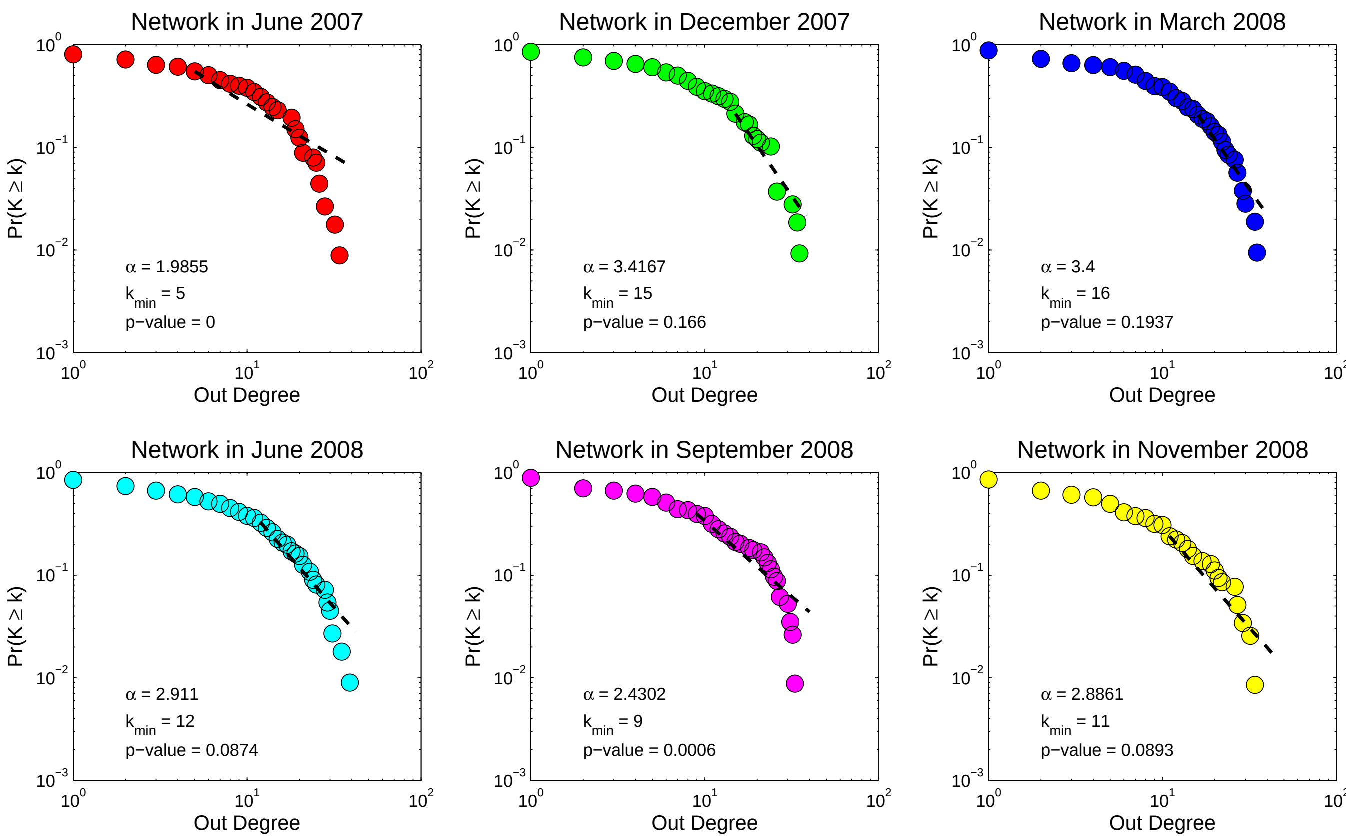


Figure 2.5: Brazilian interbank network: distribution of out-degree.

This observation is confirmed through semiparametric tail estimates. We refer the reader to appendix B for details on the estimation procedure for the tail exponent α and tail threshold k_{min}. The estimates are shown in Table 2.5 for the in-degree, out-degree and degree distributions. Maximum likelihood estimates for $\hat{\alpha}$ range from 2 to 3. The results are similar to the findings of Boss et al. (2004) for the Austrian network.

In-Degree	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**	**Mean**
$\hat{\alpha}$	2.19	2.70	2.20	3.36	2.16	2.13	2.46
$\hat{\sigma}(\hat{\alpha})$	0.48	0.46	0.47	0.53	0.47	0.44	0.48
$\hat{k}_{in,min}$	6	13	7	21	6	5	9.7
Out-Degree	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**	**Mean**
$\hat{\alpha}$	1.98	3.41	3.40	2.91	2.43	2.88	2.83
$\hat{\sigma}(\hat{\alpha})$	0.63	0.59	0.48	0.43	0.41	0.49	0.51
$\hat{k}_{out,min}$	5	15	16	12	9	11	11.3
Degree	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**	**Mean**
$\hat{\alpha}$	2.61	3.37	2.29	2.48	2.27	2.23	2.54
$\hat{\sigma}(\hat{\alpha})$	0.52	0.47	0.48	0.41	0.43	0.35	0.44
$\hat{k}_{min}$	17	34	12	15	12	10	16.7
Exposures*	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**	**Mean**
$\hat{\alpha}$	1.97	2.22	2.23	2.37	2.27	2.52	2.27
$\hat{\sigma}(\hat{\alpha})$	0.02	0.60	0.21	0.69	0.38	0.98	0.48
$\hat{E}_{min}$	39.5	74.0	80.0	101.7	93.4	336.7	120.9

*values in millions of BRL (Brazilian Reals)

Table 2.5: Statistics and maximum likelihood estimates for the distribution of in/out degree: tail exponent α, tail threshold for in-degree $k_{in,min}$, out-degree $k_{out,min}$, degree k_{min}, and exposures E_{min}.

We test the goodness-of-fit of the power law tails for in-degree, out-degree and degree via the one-sample Kolmogorov-Smirnov test with respect to a reference power law distribution. The results in figures 2.4 and 2.5 provide evidence for the power law hypothesis at the 1% significance level.

2.3.2.2 Stationarity of degree distributions

The precise pattern of exposure across institutions may vary a priori in time: it is therefore of interest to examine whether the large scale structure of the graph, as characterized by the cross-sectional distributions of in- and out-degrees, is stationary, that is, may be considered as time-independent. Comparing quantiles of the degree distributions at different dates (figure 2.6) shows that the empirical distribution of the degree, in-degree and out-degree are in fact stable over time, even though the observations span the turbulent period of 2007-2008. This is confirmed by a two-sample Kolmogorov-Smirnov test for consecutive dates, which produces p-values all greater than 0.6, suggesting that the null hypothesis that the samples are drawn from the same distribution cannot be rejected.

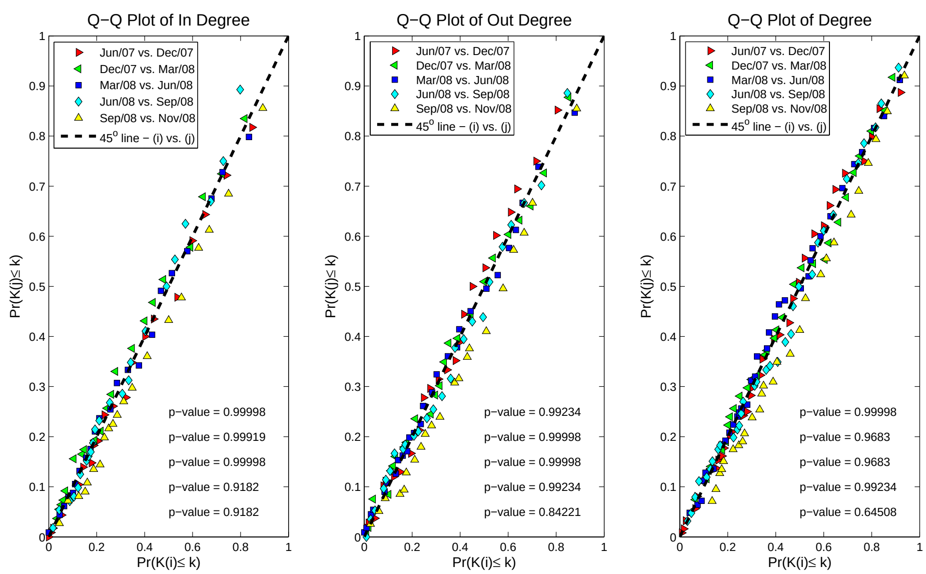


Figure 2.6: Scatterplot of the the empirical cumulative distributions at consecutive dates for the degree, in-degree and out-degree in the Brazilian interbank network.

2.3.2.3 Heterogeneity of exposure sizes

The distribution of interbank exposures is also found to be heavy-tailed, with Pareto tails. Figure 2.7 shows the existence of a linear decay in the tail of the double logarithmic plot for the empirical distribution of exposure sizes.

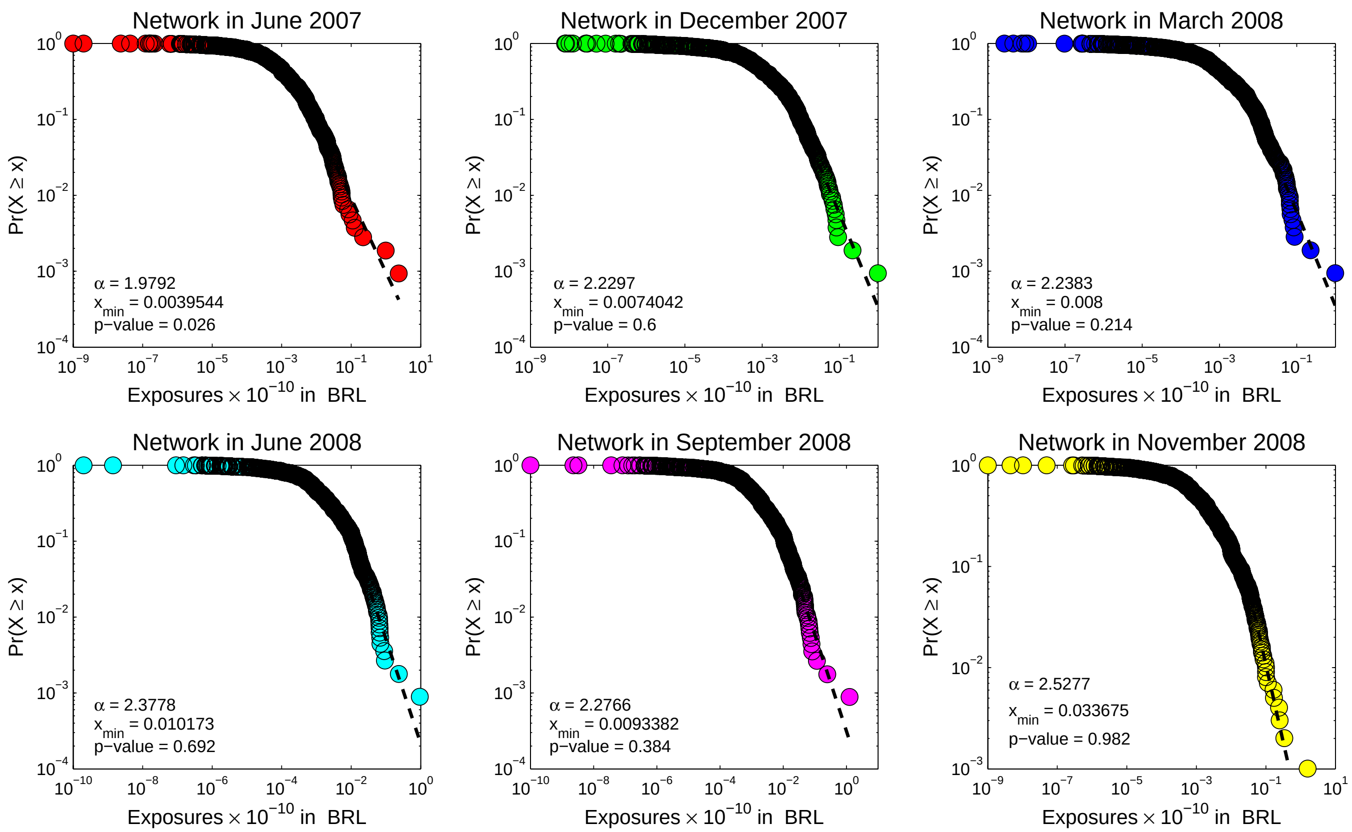


Figure 2.7: Brazilian interbank network: distribution of exposures in BRL.

The semi-parametric estimates of the tail exponent α and the tail cutoff k_{min} for the distribution of exposures are shown in Table 2.5. Note that an interbank asset for an institution is an interbank liability for its counterparty, thus, the distribution of interbank liability sizes is the same. The only difference is how these exposures are allocated among the financial institutions in the network. Figure 2.7 shows evidence for Pareto tails in the exposure distributions at all dates.

It is interesting to measure the sizes of these exposures in terms of each institutions' (Tier 1) capital. The linear regression of the interbank assets size against the Tier 1 capital gives a positive slope smaller than 1, indicating that financial institutions in Brazil have on average sufficient Tier 1 capital to cover their interbank exposures. Figure 2.8 shows that in June 2007 the ratio of interbank exposures to Tier 1 capital exhibits a heterogenous distribution: most financial institutions hold much more Tier 1 capital than their interbank exposures, which means that they have a strong capacity to absorb losses. However, some institutions have interbank exposures more than a hundred times their Tier 1 capital. Thus, these ones can be very fragile to losses and may present a significant risk of default. We will see in chapter 4 that these fragile nodes are counterparties of the five most systemic institutions in the network and play a significant role in the propagation of default across the network.

Model: $A = \beta_0 + \beta_1 c + \epsilon$

Coefficients	**Standard error**	**t-statistic**	R^2
$b_0 = -0.00$	0.07	-0.00	36%
$b_1 = 0.60$ **	0.07	8.37	

* significant at 5% confidence level

** significant at 1% confidence level

Table 2.6: Linear regression of interbank assets to Tier 1 capital in the Brazilian network in June 2007.

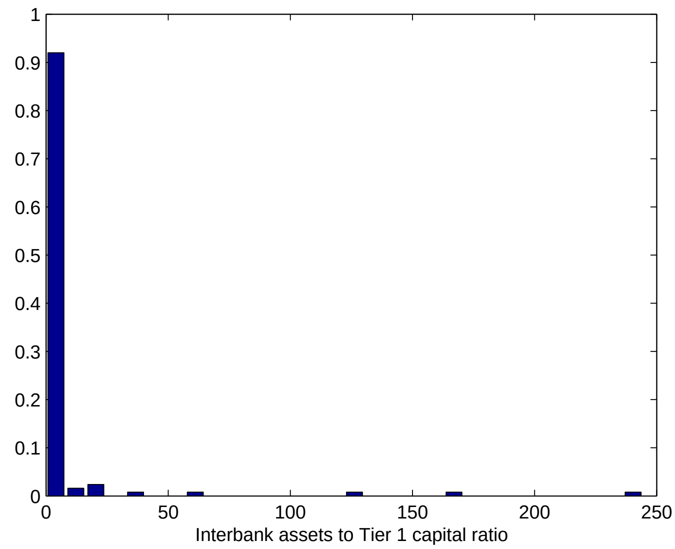


Figure 2.8: Distribution of the ratio of interbank assets to Tier 1 capital in the Brazilian network in June 2007.

2.3.2.4 Relation between exposure size and connectivity

Another interesting observation is that more (less) connected financial institutions have larger (smaller) exposures. We investigate the relationship between the number of debtors (in-degree) $k_{in}(i)$ of a node i and its average exposure size $A(i)/k_{in}(i)$ and also examine the relation between the number of creditors (out-degree) $k_{out}(i)$ and the average liability size $L(i)/k_{out}(i)$ and between $k(i)$ and $A(i)/k(i)$ by computing the Kendall tau coefficient that measures the statistical dependence (rank correlation) for each of these pairs. Table 2.7 displays the Kendall tau coefficients $\tau_{Kendall}$ and their respective p-values. The results show that the in-degree and the average interbank asset size, as well as the out-degree and the average interbank liability size, show positive dependence.

k_{in} **vs.** A/k_{in}	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**
$\tau_{Kendall}$	0.28	0.25	0.22	0.26	0.24	0.21
(p-value)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
k_{out} **vs.** L/k_{out}	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**
$\tau_{Kendall}$	0.27	0.28	0.31	0.32	0.34	0.30
(p-value)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
k **vs.** A/k	**Jun-07**	**Dec-07**	**Mar-08**	**Jun-08**	**Sep-08**	**Nov-08**
$\tau_{Kendall}$	0.24	0.24	0.21	0.23	0.23	0.23
(p-value)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)

Table 2.7: Brazilian interbank network: Kendall $\tau_{Kendall}$ coefficients for in-degree k_{in} vs. average interbank asset A/k_{in}, out-degree k_{out} vs. average interbank liability L/k_{out}, and degree k vs. average exposure A/k.

2.3.2.5 Clustering

The clustering coefficient of a node i in the network is defined as the the proportion of links between its neighbors divided by the number of links that could possibly exist between them (Watts and Strogatz, 1998).

Definition 1 (Clustering coefficient)**.** *We define $\mathcal{N}(i)$ the neighborhood of an institution (node) i in the network as the set of its immediate creditors and debtors, that is*

$$\mathcal{N}(i) = \{j \in V; E_{ij} > 0 \quad or \quad E_{ji} > 0\} \tag{2.3}$$

We also define $\mathcal{W}(i)$ the set of bilateral exposures between the creditors and debtors of institution i, that is

$$\mathcal{W}(i) = \{E_{jk}; j \in \mathcal{N}(i) \quad and \quad k \in \mathcal{N}(i)\} \tag{2.4}$$

The clustering coefficient of node i is then given by

$$C(i) = \frac{|\mathcal{W}(i)|}{|\mathcal{N}(i)|\,(|\mathcal{N}(i)| - 1)} \tag{2.5}$$

This ratio, between 0 and 1, tells how connected among themselves the neighbors of a given node are. In complete graphs, all nodes have a clustering coefficient of 1 while in regular lattices the clustering coefficient shrinks to zero with the degree.

A property often discussed in various networks is the *small world* property (Watts and Strogatz, 1998) which refers to networks where, although the network size is large and each node has a small number of direct neighbors, the distance between any two nodes is very small compared to the network size. Boss et al. (2004) report that in the Austrian interbank network any two nodes are on average 2 links apart, and suggest that the Austrian interbank network is a small-world. However, a small graph diameter does not characterize the small world property: indeed, complete networks are not small worlds and have diameter one. Another signature of the small world property is that, while the diameter is bounded or slowly increasing with the number of nodes, the *clustering coefficient* of nodes remain bounded away from zero (Cont and Tanimura, 2008). In the Brazilian financial system, we observe nodes with an arbitrary small clustering coefficient across all time periods (Figure 2.9). This absence of uniform clustering shows that the Brazilian financial system is not a small world network.

Figure 2.9 shows the relationship between the local clustering coefficient and number of degrees for the Brazilian interbank network. The negative slope of the plots shows that financial institutions with few connections (small degree) have counterparties that are very connected to each other (large clustering) while financial institutions with many connections (large degree) have counterparties with sparsely connected neighbors.

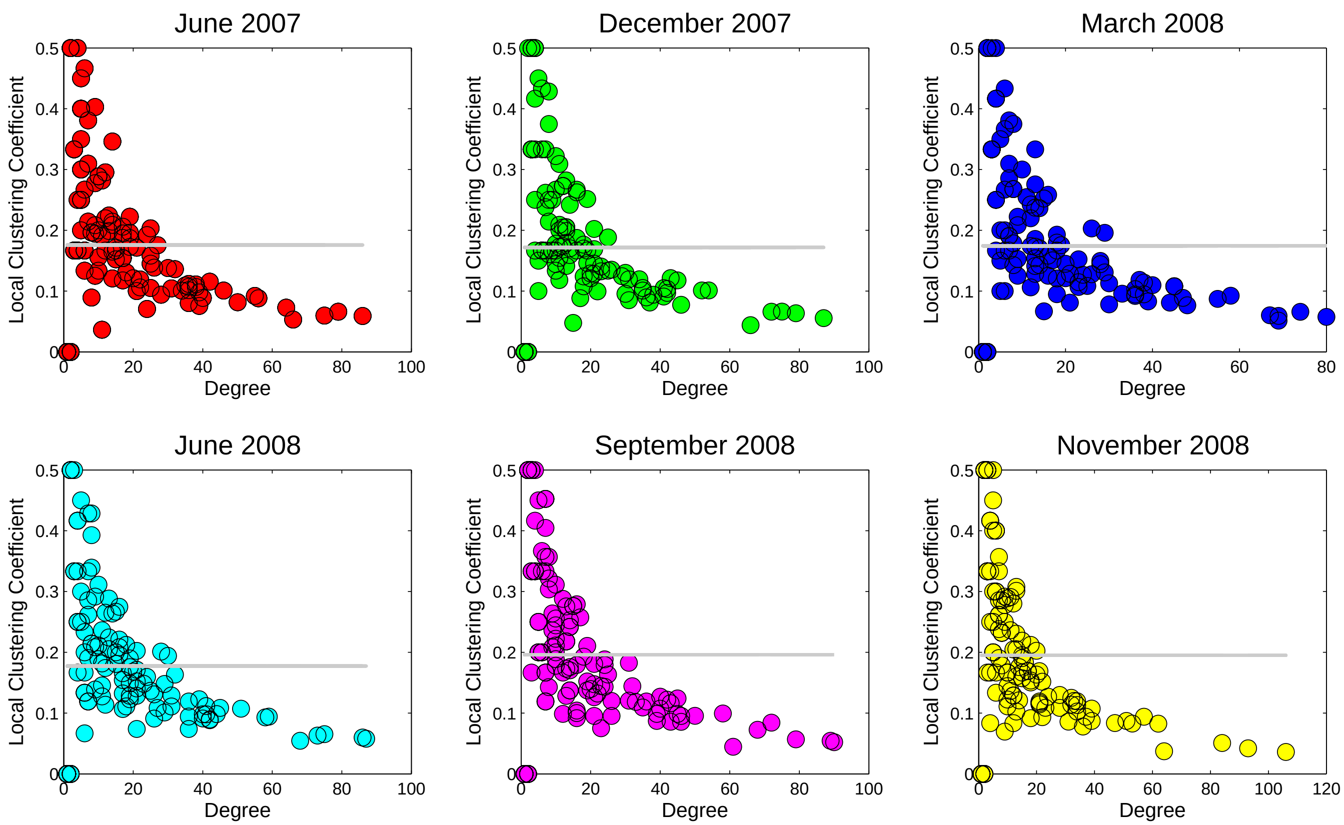


Figure 2.9: Degree vs. clustering coefficient for the Brazilian interbank network. The grey line is the average clustering coefficient.

2.4 Guidance to modeling financial systems in simulation studies

Empirical studies of interbank networks show that the in-degree and out-degree distributions follow power laws instead of the Poisson distributions of the classical Erdös-Rényi like models. That is the fraction of nodes in the network having more than k incoming links decays as $k^{-x_{in}}$ for large values of k and the fraction of nodes in the network having more than k outgoing links decays as $k^{-x_{out}}$ for large values of k. This means that there are very few banks with many creditors (or debtors) while there are many with just a few creditors (or debtors). The analysis of the Austrian and Brazilian interbank networks have shown that in both networks the scaling exponents for the in-degree and out-degree are respectively about 2 and 3, which appears to be a general characteristic of real world interbank networks. They also find a power law distribution of the size of exposures with a decaying exponent about 1.9. Furthermore, these statistical regularities are shown to be stable across time. Mistrulli (2007) finds that the Italian interbank market is not a complete market and exhibits the multiple money center structure described in Freixas et al. (2000), in which some banks trade with a bank the money center while they do not trade with each other.

Yet, most of the simulation studies on interbank networks have considered simple network structures. Allen and Gale (2000) and Freixas et al. (2000) consider a *complete* structure of claims in which every bank has symmetric linkages with all other banks in the economy, and an *incomplete* structure where banks have linkages only to few other neighboring banks. Freixas et al. (2000) also study the case of a *money center* structure where small banks are linked to the bank at the center but not among each others. Their model encompasses only four "regions" representing consolidations of banks, categories, or geographic regions; which ignores the heterogeneity of banks within the same region. Battiston et al. (2009) consider a *regular* graph where each bank has the same number of partners and exposures are equally distributed among creditors, which conflicts with the strong degree and exposures heterogeneity observed in real-world banking systems. Nier et al. (2007) first study the case of an Erdös-Rényi graph in which all banks have the same probability of being connected to another bank in the network, then extend it to a tiered structure where banks are grouped in two categories- large and small- and large banks have higher probability of being connected. Although the tiered structure is a more realistic representation of the financial system, the two-group model fails to generate scale-free degree distributions. Thus, there is a need of a model which allows a stronger heterogeneity in the network structure. As far as we know, Amini et al. (2010, 2011) is the only simulation study that models the financial network as a scale-free graph.

We propose in this section a model of the financial system that reproduces the statistical features observed in the empirical study of financial networks. Specifically, we suggest to model the financial system as a directed scale-free random graph that grows according to a preferential attachment process, generating power law distributions in the tails of the in-degree and out-degree. We construct a weighted version of the graph presented in Bollobás et al. (2003), which is a generalization to directed graphs of the original model developed by Barabasi and Albert (2002).

The preferential attachment model is just one example of simulating scale-free graphs, other models could be of course used instead of the preferential attachment model, as long as they generate power law distributions for the in-degree and out-degree of the network.

2.4.1 Interbank networks as directed scale-free graphs

Barabasi and Albert (2002) propose a preferential attachment mechanism to explain the appearance of the power-law degree distribution for nondirected scale-free graphs. Bollobás et al. (2003) introduce a model for directed scale-free graphs, such as the banking network, that grow with preferential attachment depending on the in- and out- degrees. We implement the directed scale-free graph proposed by Bollobás et al. (2003) where vertices are added at discrete steps according to a preferential attachment process: a new institution entering the financial system is more likely to establish financial links with the heavily connected institutions. The generating algorithm is the following:

Let $\alpha, \beta, \gamma, \delta_{in}$ and δ_{out} be non-negative real numbers such that $\alpha + \beta + \gamma = 1$. Starting from an initial graph G_0, we transit from graph G_t at step t to graph G_{t+1} at step $t+1$ according to the following procedure:

- With probability α, add a new node v with a link from v to an existing node ω selected with probability:

$$\mathbb{P}(\omega = \omega_i) = \frac{k_{in}(\omega_i) + \delta_{in}}{t + \delta_{in}|G(t)|} \tag{2.6}$$

- Else, with probability β, select independently an existing node v with probability:

$$\mathbb{P}(v = v_i) = \frac{k_{out}(i) + \delta_{out}}{t + \delta_{out}|G(t)|} \tag{2.7}$$

  and add a link from v to an existing node ω chosen with probability:

$$\mathbb{P}(\omega = \omega_i) = \frac{k_{in}(\omega_i) + \delta_{in}}{t + \delta_{in}|G(t)|} \tag{2.8}$$

- Else, with probability $\gamma = 1 - \alpha - \beta$, add a new node ω with a link from an existing node v to ω where v is selected with probability:

$$\mathbb{P}(v = v_i) = \frac{k_{out}(v_i) + \delta_{out}}{t + \delta_{out}|G(t)|} \tag{2.9}$$

Bollobás et al. (2003) show that as the size of the network increases to infinity the preferential attachment procedure described above generates power laws in the in-degree and out-degree distributions. Their model outerperforms the original scale-free graph models (Barabasi and Albert, 2002) in the sense that it allows different exponents for the in-degree and out-degree, which is required to reproduce the empirical findings of real-world interbank networks.

We specify the parameters δ_{in} and δ_{out} to match the features of the Austrian and Brazilian banking networks: a scaling exponent of 2 for the in-degree and 3 for the out-degree. We specify the parameters α and β to match a target average degree a of the network.

Proposition 1 (Expected average degree)**.** *Let N_j be the number of vertices in the network at the j^{th} iteration of the preferential attachment procedure described above and define the stopping time:*

$$\tau = inf\{j \geq 1; N_j = n\} \tag{2.10}$$

Then, the expected average degree of the network G_τ with n vertices is equal to $\frac{2}{\alpha+\gamma}$

Proof. $\{N_j\}$ is a Markov chain with:

$$\mathbb{P}(N_{j+1} = N_j + 1) = \mathbb{P}(\text{adding a new link at step } j+1) = \alpha + \gamma \tag{2.11}$$

and,

$$\mathbb{P}(N_{j+1} = N_j) = \mathbb{P}(\text{not adding a new link at step } j+1) = 1 - \alpha - \gamma = \beta \tag{2.12}$$

Let $\mathfrak{F}_j$ be the σ algebra generated by N_1, ..., N_j.

$$\begin{aligned}
E[N_{j+1} - (j+1)(\alpha+\gamma)|\mathfrak{F}_j] &= E[N_{j+1}|N_j] - (j+1)(\alpha+\gamma) \\
&= N_j\mathbb{P}(N_{j+1} = N_j) \\
&+ (N_j+1)\mathbb{P}(N_{j+1} = N_j + 1) - (j+1)(\alpha+\gamma) \\
&= N_j\beta + (N_j+1)(\alpha+\gamma) - (j+1)(\alpha+\gamma) \\
&= N_j - j(\alpha+\gamma)
\end{aligned}$$

Thus, $N_j - j(\alpha+\gamma)$ is a martingale.

By Optional Sampling applied to the bounded stopping time $\tau \wedge t$:

$$E[N_{\tau\wedge t}] = E[\tau \wedge t(\alpha+\gamma)] \tag{2.13}$$

Letting t go to infinity, by Monotone Convergence $E[N_{\tau\wedge t}]$ goes to $E[N_\tau]$ and $E[\tau \wedge t(\alpha+\gamma)]$ goes to $E[\tau(\alpha+\gamma)]$. Therefore,

$$E[N_\tau] = E[\tau(\alpha+\gamma)] = n \tag{2.14}$$

Since at each iteration, the final number of links is τ. Moreover, since at each iteration of the preferential attachment procedure a new link is created, the sum of the degrees of network participants is incremented by 2 at each iteration (one for the in-degree and one for the out-degree). Then, the sum of the degrees of network participants after τ iterations is 2τ and the average degree of the network is $\frac{2\tau}{n}$. Thus, the expected average degree of the network with n vertices is equal to $\frac{2}{\alpha+\gamma}$.

$\square$

We choose then α that satisfies:

$$\frac{1}{\alpha} = \frac{1}{\gamma} = a \quad \text{and} \quad \beta = 1 - 2\alpha \tag{2.15}$$

2.4.2 Modeling capital and exposures

Capital

To ensure financial stability, banks are required to hold a cushion for credit, market and operational risks, as suggested for example in Basel 2 accords (BIS, 2001; Duellmann, 2006). Capital constitutes a cushion to sustain potential losses. Nevertheless, most simulation studies have considered a simplistic distribution if capital. Nier et al. (2007) assume that the capital of an institution - to which they refer as "net worth"- is a fixed proportion of its total assets. However, this does not entirely agree with Basel accords that require a cushion of capital to cover credit risk, calculated as a fixed proportion of the risk weighted assets, and cushions to cover market and operational risks. We propose a stylized specification of capital for the institutions in the network following BIS (2001) and Duellmann (2006). We assign for each node i in the network a capital $c(i)$ that can cover a proportion θ of its market and credit risks (so we do not require a cushion for operational risk):

$$c(i) = \theta \times (RWA_{CrD} + 12.5VaR_{MkR}(i)) \tag{2.16}$$

where RWA_{CrD} is the sum of risk-weighted assets for credit risk and VaR_{MkR} is the value at risk for market risk.

θ varies according to the specific regulation of the country where the banks operate. For example, in the U.S. banks are required a capital ratio $\theta = 8\%$, while in Brazil banks are required $\theta = 11\%$.

The cushion for market risk is estimated as the 99% 10 days value-at-risk of the total net exposure:

$$VaR_{MkR}(i) = \Phi^{-1}(0.99)\sqrt{10}\sigma(|\sum_{j\in V} E_{ij} - \sum_{j\in V} E_{ji}|), \tag{2.17}$$

where σ is the daily volatility of the total net exposure, typically σ is about 1%.

The cushion for credit risk is calculated as the risk-weighted-assets to absorb the unexpected losses (UL) for all individual credit claims in the portfolio:

$$RWA_{CrR} = 12.5 \sum_{j\in V} UL_j \tag{2.18}$$

We refer the reader to appendix C, BIS (2001) and Duellmann (2006) for the computation details of the regulatory capital.

Exposures

Once the graph is constructed we allocate exposures (weights) to each link drawn from a Pareto distribution with parameter $\alpha = 1.9$. That is, the exposures have the same power exponent as the one observed in the Austrian and Brazilian networks. A simple assumption would be to consider the exposures independent. Nevertheless, this might not be very realistic since very connected institutions tend to have larger exposures. For example, we observed in the Brazilian financial system a significant positive dependence between the average exposure size and the number of its debtors, and a positive dependence between the average liability size and the number of creditors. Thus, a more realistic framework would be to correlate exposures sizes and connectivity when modeling interbank networks.

Chapter 3

Too Interconnected to Fail: Contagion and Systemic Risk in Financial Networks

This chapter is based on the paper "Too Interconnected to Fail: Contagion and Systemic Risk in Financial Networks" (Cont and Moussa, 2010c), which is a joint work with Professor Rama Cont.

3.1 Introduction

The complexity of the contemporary financial systems has made it a challenge to define measures of contagion and systemic risk that account for all potential sources of risk an institution is faced to: credit, market, operational, liquidity, interest rate, foreign exchange, just to name a few. Network models have provided a practical framework for studying contagion in financial systems, in which nodes represent financial institutions and weighted links represent their bilateral exposures. Several studies have examined the extent of contagion in real-world interbank networks: Furfine (2003) in the US, Upper and Worms (2004) in Germany, Ágnes Lublóy (2006) in Hungary, van Lelyveld and Liedorp (2006) in the Netherlands, Wells (2004) and Elsinger et al. (2006a) in Austria, Wells (2004) in the UK, Mistrulli (2007) in Italy. We refer the reader to Upper (2011) for a comprehensive survey. Most of these studies (all with the exception of Elsinger et al. (2006a,b)) have considered the sole impact of an idiosyncratic shock on a single institution in the network, thus accounting for only one type of risk: credit risk through counterparty exposure. Nonetheless, in periods of financial crisis, the correlation of exposures could lead to a simultaneous shock on a large group of- if not all- institutions in the network leading to a wider propagation of financial distress in the system. It is then crucial to include the combined effects of both common market shocks to portfolios (correlation) and contagion through counterparty exposures (network effects).

We present in this study a metric of the systemic importance of a financial institution (or a group of institutions), the Contagion Index, defined as the expected loss to the network triggered by the default of this institution when the system is subject to a market shock. Contrarily to indicators of systemic risk purely based on market data (Acharya et al., 2010; Adrian and Brunnermeier, 2008), our metric of systemic importance make use of exposures, which represent potential losses, to simulate stress scenarios, resulting in a forward-looking measure of systemic risk. The Contagion Index measures the magnitude of loss *conditional* to the default of a given institution instead of averaging across all defaults as in Elsinger et al. (2006a). We argue that these conditional measures provide a better assessment of risk in a heterogenous system where the sample average may be a poor statistic. We consider *Default* as the event in which an institution's (Tier 1) capital is wiped out by losses (*insolvency*). In fact, the main reason an institution defaults is if it is unable to fulfill an obligation when it is due, such as a principal or interest payment, which typically happens when short term assets cannot cover short term liabilities, i.e. when the institution is in liquidity distress (*illiquidity*). We argue that insolvency leads in general to illiquidity which in turn leads to default.

The recent crisis has emphasized the importance of designing a measure of systemic risk that is able to capture scenarios with default clusters. We propose a statistical model of the dependence structure of market shocks that exhibits a tail dependence, meaning that the occurrence of a shock on the network increases the probability of possible future shocks. This allows to generate scenarios with *default clustering*, which would have not been possible with the market standard Gaussian copula model (Embrechts et al., 2001).

Using this methodology, we study the sensitivity of contagion and systemic to a change in various network parameters. We simulate a directed scale-free network according to the preferential attachment procedure introduced in chapter 2, with heavy-tailed exposures. This choice of a scale-free degree distribution and heavy-tailed exposures is motivated by the observations of Boss et al. (2004) on the structure of the Austrian network and our study of the statistical features of the Brazilian network. We refer the reader to chapter 2 for an analysis of the Brazilian network. Using the Contagion Index, we study the impact of several network features: the aggregate level of connectivity and concentration of exposures, the heterogeneity in degree distribution, the network size, the influence of local measures of connectivity and concentration across counterparties such as the *counterparty susceptibility* and *local network frailty*, the impact of capital requirements. We examine the sensitivity of contagion to the heterogeneity in degree and exposures distributions by comparing the severity of contagion in a scale-free versus an Erdös-Rényi network and by varying the distribution of exposures (Exponential, Pareto, etc.). We are thus led to revisit some of the conclusions in the previous literature (Allen and Gale, 2000;

Freixas et al., 2000; Nier et al., 2007; Battiston et al., 2009) on the influence of connectivity, concentration and network structure on the severity of contagion in other particular network structures.

3.1.1 Summary of main results

- The influence of connectivity on the severity of contagion is controversial: in well-capitalized networks, increasing connectivity is found to increase significantly contagion up to a certain threshold above which a further increase in connectivity leads to a decrease in the extent of contagion. However, in undercapitalized networks, increasing connectivity makes the network more prone to contagion whatever the initial level of connectivity is. This allows us to revisit the trade-off phenomenon described in previous studies between decreasing individual risk due to risk sharing and increasing systemic risk due to the propagation of financial distress.

- More heterogeneous network structures, in terms of both degree and exposures distributions, are found more resilient to contagion. Specifically, the severity of contagion in an Erdös-Rényi network is much greater than in a scale-free network with equal connectivity. Also, the severity of contagion is much greater in networks with less heterogenous distributions of exposures, such as an Exponential distribution or simply distributing equally exposures among creditors, than in a scale-free network with Pareto distribution of exposures.

- Local measures of connectivity and concentration of exposures across counterparties –*counterparty susceptibility* and *local network frailty*– are shown to contribute significantly to the systemic importance of an institution.

- Using the Contagion Index as a metric for systemic impact allows a comparative analysis of various capital requirement policies in terms of their (reduction in) systemic impact. While a floor on the aggregate capital ratio is shown to reduce the systemic impact of defaults of large institutions, imposing more stringent capital requirements- cap on the susceptibility- on the most systemic nodes and their counterparties is shown to be a more efficient procedure for immunizing the network against contagion.

3.1.2 Relation to the literature

Shock simulation and loss contagion

The loss contagion mechanism we introduce is similar to the one presented in Furfine (2003); Upper and Worms (2004); Wells (2004); Ágnes Lublóy (2006); van Lelyveld and Liedorp (2006); Mistrulli (2007); Nier et al. (2007). When an institution defaults, the unrecovered portion of the exposures to the defaulted institution (assuming an exogenous recovery rate) are absorbed by its creditors, that can themselves default if they do not hold enough capital to sustain their losses. However, this "sequential" (Upper, 2011) contagion mechanism is very different from the market equilibrium approach of Eisenberg and Noe (2001); Elsinger et al. (2006a,b) defined by a clearing payment vector, in which banks can liquidate their assets leading to a proportional sharing of losses among counterparties (endogenous recovery rate). We argue that, since bankruptcy procedures are usually slow and settlements may take up several months to be effective, creditors cannot recover the residual value of the defaulting institution according to such a hypothetical clearing mechanism, and write down their entire exposure in the short-run, leading to a short term recovery rate of zero. This seems a more reasonable approach in absence of a clearing mechanism.

With the exception of Elsinger et al. (2006a,b), all these studies examine the sole knock-on effects of the sudden failure of a single bank by considering an idiosyncratic shock that targets a single institution in the system. Upper and Worms (2004) estimate the scope of contagion by letting banks go bankrupt one at a time and measuring the number of banks that fail due their exposure to the failing bank. Sheldon and Maurer (1998) and Mistrulli (2007) also study the consequences of a single idiosyncratic shock affecting individual banks in the network. Furfine (2003) measures the risk that an exogenous failure of one or a small number of institutions will cause contagion. These studies fail to quantify the compounded effect of correlated defaults and contagion through network externalities. Our study, on the contrary, shows that common market shocks to balance sheets may exacerbate contagion during a crisis and ignoring them can lead to an underestimation of the extent of contagion in the network. We argue that, to measure adequately the systemic impact of the failure of a financial institution, one needs to account for the combined effect of correlation of market shocks to balance sheets and balance sheet contagion effects, the former increasing the impact of the latter. Our simulation-based framework takes into account common and independent market shocks to balance sheets, as well as counterparty risk through mutual exposures.

Sensitivity of contagion to network parameters

In line with Nier et al. (2007) and Battiston et al. (2009), we observe when increasing connectivity in the network a trade-off phenomenon between increasing the severity of contagion by increasing potential channels for the propagation of financial distress and the stabilizing benefit of risk sharing by dissipating losses among a larger number of counterparties. Specifically, we find similar results as in Nier et al. (2007): in well-capitalized networks, increasing connectivity is found to increase significantly contagion up to a certain threshold above which a further increase in connectivity leads to a decrease in the extent of contagion. However, in undercapitalized networks, increasing connectivity makes the network more prone to contagion whatever the initial level of connectivity is.

While contagion is found to be more likely to occur in incomplete and tiered (money center) network structures in Allen and Gale (2000) and Freixas et al. (2000), we find that heterogeneity in degrees and exposures improves the resilience of the network to contagion. The difference in results could be due to the difference in network structures since we consider a scale-free network instead of a four nodes graph, or to a difference in the contagion mechanism. In any case, this highlights again that the possibility and extent of contagion depend considerably on the precise structure of the network, hence the importance of using realistic models of financial networks when studying contagion.

Our study also complements the existing literature by studying the contribution of network-based local measures of connectivity and concentration to systemic risk. Previous studies on simulated network structures have examined the contribution of aggregate measures of connectivity and concentration such as increasing the probability that two nodes are connected in an Erdös-Rényi graph, or increasing the number of nodes in the system (Battiston et al., 2009; Nier et al., 2007). However, they fail to detect the impact of connectivity and concentration *locally* around a single institution in the network. We thus introduce the *counterparty susceptibility* and *local network frailty* that measure respectively the susceptibility of the creditors of an institution to a potential default of the latter and the fragility of the entire network in the event of default of this institution. We find that the two measures can explain significantly default contagion.

The impact of capital requirements in limiting the extent of systemic risk and default contagion has not been explored systematically in a network context. Based on analogies with epidemiology and peer-to-peer networks (Cohen et al., 2003; Madar et al., 2004; Huang et al., 2007), we discuss *targeted* capital requirements and show that targeting the creditors of the most contagious institutions is a more effective procedure –in terms of the total capital it requires for the same level of systemic risk– than increasing capital ratios for all institutions in the network.

3.1.3 Outline

The chapter is organized as follows. In section 3.2, we introduce a quantitative methodology for assessing contagion and systemic risk, and present a low-variance simulation procedure to calculate the Contagion Index. In section 3.3, we explore the impact of connectivity and concentration on systemic risk, and investigate the role of different institutional and local network characteristics which contribute to the systemic importance of financial institutions. In section 3.4, we study the impact of the size of the network on the level of contagion. In section 3.5, we examine the impact of the heterogeneity in network structure and exposures sizes. Finally, in section 3.6, we study the impact of various immunization procedures on the extent of default contagion and systemic risk.

3.2 Measuring the systemic impact of financial failures

3.2.1 Default mechanism

Changes in an institution assets value, such as fluctuations in market value or the failure to collect assets due to a financial distress of the counterparty, can impute losses to the institution. These losses are written down from the balance sheet and the capital is reduced by the amount of the loss incurred. A realistic valuation of exposures is then a critical factor in evaluating the financial condition and performance of financial institutions. For banks, loans given are considered as assets and if the debtor cannot pay back this loan, the new value is placed on the balance sheet, and the capital is reduced by the amount of the loss.

Default occurs when an institution fails to fulfill an obligation such as a scheduled debt payment of interest or principal or the inability to service a loan. This typically happens when the reserves in short term (liquid) assets do not suffice to cover short term liabilities. It is then important when modeling default to discuss the difference between the two major sources of financial distress: *illiquidity* and *insolvency*. Insolvency happens when the net worth of an institution is reduced to zero, i.e. losses exceed capital, while illiquidity occurs when reserves in liquid assets, such as cash and cash equivalents, are insufficient to cover short term liabilities. Illiquidity leads to default while, in principle, insolvency may not necessarily entail default as long as the institution is able to obtain financing to meet payment obligations. Nevertheless, in the current structure of the financial system where financial institutions are primarily funded through short-term debt, which must be constantly renewed, insolvent institutions would have great difficulties in raising liquidity as their assets lose in value. Indeed, renewal of short term funding is subject to the solvency and creditworthiness of the institution. Thus, in practice, insolvency leads to illiquidity which in turn leads to default.

Thus, in line with various previous studies, we define *default* as the event when the losses incurred by a financial institution render it insolvent.

3.2.2 Loss contagion

When a set of financial institutions default, they lead to an immediate writedown in value of all their liabilities to their creditors. These losses are imputed to the capital of the creditors, leading to a loss of E_{ji} for each creditor j of each defaulted institution. If this loss exceeds the creditor's capital i.e. $E_{ji} > c_j$ this leads to the insolvency of the institution j, which in turn may generate a new round of losses to the creditors of j. This *domino effect* may be modeled by defining a *loss cascade*, updating at each step the losses to balance sheets resulting from previously defaulted counterparties:

Definition 2 (Loss cascade)**.** *Consider an initial configuration of capital reserves* $(c_0(j), j \in V)$. *We define the sequence* $(c_k(j), j \in V)_{k \geq 0}$ *as*

$$c_{k+1}(j) = \max(c_0(j) - \sum_{\{i, c_k(i)=0\}} (1 - R_i)E_{ji}, 0), \tag{3.1}$$

where R_i *is an exogenous recovery rate at the default of institution* i. $(c_{n-1}(j), j \in V)$, *where* $n = |V|$ *is the number of nodes in the network, then represents the remaining capital once all counterparty losses have been accounted for. The set of insolvent institutions is then given by*

$$\mathbb{D}(c, E) = \{j \in V : c_{n-1}(j) = 0\} \tag{3.2}$$

Remark 1 (Fundamental defaults vs defaults by contagion)**.** *The set* $\mathbb{D}(c, E)$ *may be partitioned into two subsets*

$$\mathbb{D}(c, E) = \underbrace{\{j \in V : c_0(j) = 0\}}_{\text{Fundamental defaults}} \bigcup \underbrace{\{j \in V : c_0(j) > 0, \quad c_{n-1}(j) = 0\}}_{\text{Defaults by contagion}}$$

where the first set represents the initial defaults which trigger the cascade –we will refer to them as fundamental defaults– and the second set represents the defaults due to contagion.

The default of the initial set of failing institutions can therefore propagate to other participants in the network through the contagion mechanism described above. To measure the systemic importance of a set of institutions (say A) triggering the loss cascade, we introduce the *Default Impact* $DI(A)$ of A that measures the loss incurred by the network in the default cascade triggered by the default of the set of institutions A:

Definition 3 (Default Impact)**.** *The* Default Impact $DI(A, c, E)$ *of a set of financial institutions* $A \subset V$ *is defined as the total loss in capital in the cascade triggered by the default of* A*:*

$$DI(A, c, E) = \sum_{j \in V} c_0(j) - c_{n-1}(j), \tag{3.3}$$

where $(c_k(j), j \in V)_{k>0}$ *is defined by the recurrence relation* (3.1)*, with initial condition is given by*

$$c_0(j) = c(j) \quad \text{for} \quad j \notin A \quad \text{and} \quad c_0(j) = 0 \quad \text{for} \quad j \in A.$$

It is important to note that the Default Impact does not include the loss of the institutions triggering the cascade, but focuses on the loss this initial default inflicts to the rest of the networks: it thus measures the loss due to contagion.

Definition 4 (Total loss)**.** *The total loss* $L(A)$ *generated along the default cascade initiated by* A *is then given by summing the loss of capital of the defaulted firms and the loss of exposures across counterparties:*

$$\mathcal{L}(A) = \sum_{j \in A} c(j) + DI(A, c, E) = \sum_{j \in V} c(j) - c_{n-1}(j) \tag{3.4}$$

where $(c_k(j), j \in V)_{k>0}$ *is defined by the recurrence relation* (3.1)*, with initial condition is given by*

$$c_0(j) = c(j) \quad \text{for} \quad j \notin A \quad \text{and} \quad c_0(j) = 0 \quad \text{for} \quad j \in A.$$

Proposition 2 (Monotonicity of the total loss)**.** *The loss function* $\mathcal{L}$ *defines an increasing function on the subsets of* V*:*

$$A \subset B \subset V \Rightarrow \mathcal{L}(A) \leq \mathcal{L}(B). \tag{3.5}$$

Proof. Consider $A \subset B \subset V$. We denote $c_{k,A}(v)$ the sequence of capital defined by the recurrence relation (3.1), with initial configuration of capital given by

$$c_{0,A}(j) = c(j) \quad \text{for} \quad j \notin A \quad \text{and} \quad c_{0,A}(j) = 0 \quad \text{for} \quad j \in A. \tag{3.6}$$

and we denote $c_{k,B}(v)$ the sequence of capital defined by the recurrence relation (3.1), with initial configuration of given by

$$c_{0,B}(j) = c(j) \quad \text{for} \quad j \notin B \quad \text{and} \quad c_{0,B}(j) = 0 \quad \text{for} \quad j \in B. \tag{3.7}$$

We first prove that for every node v in the network, $c_{k,A}(v) \leq c_{k,B}(v)$ at each iteration k of the contagion cascades initiated by A and B:

- At step k=0: sets A and B are in default so for all $v \in A$:

  $$c_{0,A}(v) = c_{0,B}(v) = 0 \tag{3.8}$$

  and for all $v \notin A$:

  $$c_{0,A}(v) = c(v) \text{ but } c_{0,B}(v) \leq c(v) \tag{3.9}$$

  Thus, $c_{0,A}(v) \geq c_{0,B}(v)$ for every node v in the network.

- Induction step: assume that $c_{k,A}(v) \geq c_{k,B}(v)$ at the k^{th} iteration. Then,

$$\begin{aligned} c_{k+1,A}(v) &= \max(c_{0,A}(v) - \sum_{\{j, c_{k,A}(j)=0\}} (1-R_j)E_{v,j}, 0) & (3.10) \\ &\geq \max(c_{0,B}(v) - \sum_{\{j, c_{k,B}(j)=0\}} (1-R_j)E_{v,j}, 0) & (3.11) \\ &\geq c_{k+1,B}(v) & (3.12) \end{aligned}$$

Therefore, for $k = n-1, c_{n-1,A}(v) \geq c_{n-1,B}(v)$. This implies that $\mathcal{L}(A) \leq \mathcal{L}(B)$, that is the loss function $\mathcal{L}$ defines an increasing function on the subsets of V. □

The contagion mechanism described above is similar to the one presented in Furfine (2003); Upper and Worms (2004); Wells (2004); Mistrulli (2007) that also consider a deterministic default mechanism with an exogenous recovery rate. Since liquidation procedures are usually slow and settlements may take up several months to be effective, creditors cannot recover the residual value of the defaulting institution according to such a hypothetical clearing mechanism, and write down their entire exposure in the short-run, leading to a short term recovery rate of zero. In absence of a clearing mechanism, this approach seems more reasonable than the one proposed by Eisenberg and Noe (2001) which corresponds to a hypothetical situations where all portfolios are simultaneously liquidated. Finally, we note that this model does not capture medium- or long-term contagion: maintaining exposures constant over longer term horizons, as in (Elsinger et al., 2006a) is unrealistic since exposures and capital levels fluctuate significantly over such horizons.

3.2.3 Contagion Index of a set of financial institutions

Banking regulation requires banks to conduct stress tests on their minimum capital requirements in order to ensure adequate capital allocation levels to cover potential losses incurred during extreme events. Examples of scenarios that could be used are: economic downturns, market-place events, or decreased liquidity conditions. The bank must stress test its counterparty exposures including jointly stressing market and credit risk factors that could have unfavorable effects on the bank's ability to face such changes. It would be then interesting to introduce a metric of systemic importance that considers not only credit risk- such as the Default Impact- but also systemic events, such as market shocks that could affect the capital of all institutions at the same time. We propose a measure of the systemic importance of a financial institution that combines both market and credit risk factors: the *Contagion Index*.

We introduce correlated negative market shocks $\epsilon_i, i = 1..n$ that reduce the capital of all institutions in the network with a severity that depends on the credit worthiness of each institution: institutions with higher default probabilities will default more often when faced to market shocks. Each scenario of market shocks leads an initial set of institutions to default, and the default can propagate across the network through the loss cascade mechanism described in the previous section. We introduce the Contagion Index $CI(A, c, E)$ of a set of institutions A as a measure of the expected loss incurred by the network, conditional on the event that the set of institutions A has defaulted due to the market shock. Thus, while the Default Impact is a deterministic measure of the loss generated by an exogenous default of the set of institutions A, the Contagion Index is a measure of the expected loss generated by the failure of the set of institutions A in a stressed market, compounding the effects of both credit and market risks.

The computation of this index involves a model of market shocks ϵ affecting balance sheets. Different specifications –static or dynamic, factor-based or copula-based– are possible:

$$\epsilon_i = f_i(S, Z_i) \tag{3.13}$$

where S is a common factor and the Z_i's are IID random variables representing idiosyncratic shocks.

We refer the reader to appendix A for an overview of copulae.

Given a (statistical) model for the market shocks ϵ generating stress scenarios, we now define the

Definition 5 (Contagion Index). *The Contagion Index $CI(A, c, E)$ of the set of institutions $A \subset V$ is defined as its expected Default Impact in a market stress scenario:*

$$CI(A, c, E) = \mathbb{E}\left[DI(A, (c + \epsilon)_+, E) \middle| \, c(i) + \epsilon_i < 0, \forall i \in A \right] \tag{3.14}$$

$CI(A, c, E)$ is the expected loss –measured in terms of capital– inflicted to the network in the default cascade triggered by the initial default of A. The averaging is done over scenarios where the market shocks trigger the default of A. If the initial set of institutions defaulting due to the market shock does not trigger other institutions to fail, then the Contagion Index is zero: a non-zero value of this indicator therefore indicates that the set of institutions A is indeed a source of contagion.

3.2.4 Simulation procedure

The Contagion Index of a set A of nodes can be viewed as the expected Default Impact of the set A when drawing the market shocks from their conditional distribution given that the set A has defaulted, i.e. $c(k) + \epsilon_k < 0, \forall k \in A$:

$$CI(A, c, E) = \mathbb{E}\left[DI(A, (c + \tilde{\epsilon})_+, E)\right] \tag{3.15}$$

where $\widetilde{\epsilon}$ is drawn from the distribution of ϵ conditional on the event $c(k) + \epsilon_k < 0, \forall k \in A$.

Thus, we can compute an estimator of the Contagion Index of the set A by Monte-Carlo method:

For $j = 1..N$:

- draw samples $\tilde{\epsilon}_i(\omega_j), i = 1..n$ from the conditional distribution $\mathbb{P}(\epsilon_i \in .|c(j) + \epsilon_j < 0, \forall j \in A)$.
- run the default cascades initiated by the market shock triggering the failure of the set A of nodes and compute the Default Impact $DI(A, (c + \tilde{\epsilon}(\omega_j))_+, E)$ of A with the stressed capital levels $(c(i) + \tilde{\epsilon}_i(\omega_j))_+, i = 1..n$.

The Monte-Carlo estimator of the Contagion Index $\widehat{CI}$ is then given by:

$$\widehat{CI}(A,c,E) = \frac{1}{N}\sum_{j=1}^{N} DI(A,(c+\tilde{\epsilon}(\omega_j))_+,E) \tag{3.16}$$

Example: one factor α-stable copula model

In the following sections, we consider a one factor α-stable copula model for the market shocks,

$$\epsilon_i = \sigma_i F_\alpha^{-1} G_\alpha\left(\rho^{1/\alpha} S + (1-\rho)^{1/\alpha} Z_i\right) \tag{3.17}$$

where $S, Z_1, ..., Z_n$ are independent and identically distributed random variables with marginal cumulative density function G_α, and marginal probability density function g_α.

F_α is the marginal cumulative distribution of the $\epsilon_i's$, which we choose to be the conditional distribution of S given $S < 0$ in order to generate exclusively negative market shocks,

$$F_\alpha(x) = \frac{1}{G_\alpha(0)} G_\alpha(x) 1_{x<0} + 1_{x>0}. \tag{3.18}$$

σ_i is a scaling factor that allows to calibrate the magnitude of the market shock on institution i to its marginal default probability p_i:

$$\sigma_i = -\frac{c(i)}{F_\alpha^{-1}(p_i)} \tag{3.19}$$

The loss incurred by the network is a function of the correlated market shocks $\epsilon_i, i = 1..n$, so it is a function of the common factor S and the specific factors $Z_i, i = 1..n$. Therefore, the computation of the Contagion Index of an institution k

requires the knowledge of the joint distribution of S and $Z_i, i = 1..n$ conditional on the event $c(k) + \epsilon_k < 0$. Since the variables S and $Z_i, i = 1..n$ are independent, the Z_i's for $i \neq k$ are independent of the event $c(k)+\epsilon_k < 0$. In this case, the Contagion Index of all nodes $k = 1..n$ in the network can be calculated simultaneously using the same draws of idiosyncratic factors and just re-drawing the common factor S conditional on the event $c(k)+\epsilon_k < 0$. Hence, we only need to know the conditional distribution of S given $c(k) + \epsilon_k < 0$, that has a density function

$$f_\alpha(x|c(k) + \epsilon_k < 0) = \frac{1}{p_k} G_\alpha \left(\frac{G_\alpha^{-1} \circ F_\alpha \left(-\frac{c(k)}{\sigma_k} \right) - \rho^{1/\alpha} x}{(1-\rho)^{1/\alpha}} \right) g_\alpha(x) \tag{3.20}$$

Proof.

$$\begin{aligned} &P(S \leq s|c(i) + \epsilon_i < 0) \\ &\quad = \frac{1}{p_i} P\left(S \leq s, c(i) + \sigma_i \left(F_\alpha^{-1} G_\alpha \left(\rho^{1/\alpha} S + (1-\rho)^{1/\alpha} Z_i\right)\right) < 0\right) \\ &\quad = \frac{1}{p_i} P(S \leq s, Z_i < \frac{G_\alpha^{-1} F_\alpha \left(-\frac{c(i)}{\sigma_i}\right) - \rho^{1/\alpha} S}{(1-\rho)^{1/\alpha}}) \\ &\quad = \frac{1}{p_i} \int_{-\infty}^{s} G_\alpha(\frac{G_\alpha^{-1} F_\alpha \left(-\frac{c(i)}{\sigma_i}\right) - \rho^{1/\alpha} u}{(1-\rho)^{1/\alpha}}) g_\alpha(u) du \end{aligned}$$

Hence, the conditional probability density function of S given $c(i) + z_i < 0$ is:

$$f_\alpha(s|c(i) + \epsilon_i < 0) = \frac{1}{p_i} G_\alpha \left(\frac{G_\alpha^{-1} F_\alpha \left(-\frac{c(i)}{\sigma_i} \right) - \rho^{1/\alpha} s}{(1-\rho)^{1/\alpha}} \right) g_\alpha(s) \tag{3.21}$$

□

The algorithm to estimate the Contagion Index of all nodes $k = 1..n$ in the network is:

For $j = 1..N$:

- draw $Z_i(\omega_j), i = 1..n$ a copy of the idiosyncratic factors from their α-stable distribution G_α.
- For $k = 1..n$:
  - draw $S(\omega_j)$ from the conditional distribution $\mathbb{P}(S(\omega_j) \in .|c(k)+\epsilon_k(\omega_j) < 0)$.
  - compute the market shocks for all nodes i in the network:

$$\epsilon_i(\omega_j) = F_\alpha^{-1} G_\alpha \left(\rho^{1/\alpha} S(\omega_j) + (1-\rho)^{1/\alpha} Z_i(\omega_j)\right) \tag{3.22}$$

  - run the default cascades initiated by the market shock triggering the failure of node k and compute the Default Impact $DI(k, c+\epsilon(\omega_j), E)$ of k with the stressed capital levels $(c(i) + \epsilon_i(\omega_j))_+, i = 1..n$.

Both the gaussian and the cauchy copulae belong to the family of α-stable copulae: the gaussian corresponds to $\alpha = 2$ and the Cauchy to $\alpha = 1$.

Unless otherwise specified, the results presented in the below sections correspond to a scale-free network, simulated according to the preferential atachment procedure presented in chapter 2, with 400 nodes, average degree 10, in-exponent 2 and out-exponent 3. Exposures are iid and follow a Pareto distribution with parameter $\alpha = 1.9$. Market shocks are simulated from a Cauchy copula model with dependence parameter $\rho = 10\%$. We favor the Cauchy copula to the commonly used Gaussian copula in order to generate scenarios with clusters of shocks with large magnitude, that is possible only in the presence of tail dependence as in the Cauchy copula (Embrechts et al., 2001). The nodes are ranked in descending order of their interbank liability. The first 12% are assigned a default probability of 6 basis points, the second 13% are assigned a default probability of 33 basis points and the remaining ones are assigned a default probability of 79 basis points. In other words, we assume that institutions are willing to lend money to other institutions with a good credit rating, that is institutions with a small default probability. Thus, an institution with a high interbank liability should be assigned a small default probability. The Contagion Index is computed according to the simulation procedure described in section 3.2.4 with 1000 independent draws of the correlated market shocks.

3.2.5 Importance of correlated market shocks

Both the Default Impact and the Contagion Index exhibit heavy tailed distributions (see figure 3.1) indicating the existence of few institutions that present a high contagion risk to the financial system (up to 25% of the total capital in the network) while most institutions exhibit a small risk. Figure 3.2 shows the cross-sectional distribution of the ratio of the Contagion Index to the Default Impact. We observe that the Contagion Index may, for some nodes, significantly exceed (up to four times) the Default Impact. Thus correlated shocks to balance sheets seem to *amplify* contagion. This comes from the fact that market shocks reduce the capital available to financial institutions and render them more susceptible to default.

Exposures that are not covered by an adequate amount of capital to sustain their loss in the event of default constitute channels of contagion across the system. We will call such exposures *contagious exposures.*

Definition 6 (Contagious Exposure)**.** *An exposure of institution i to j is called* contagious *if it exceeds the capital of i, that is ,if $E_{ij} > c(i)$.*

If the link $i \to j$ represents a contagious exposure, the default of j leads to the default of i in all stress scenarios. Thus, the subgraph constituted of contagious exposures will be a primary support for the propagation of default cascades: the larger this subgraph, the larger the extent of contagion. In a stress scenario in which balance sheets are subjected to negative market shocks, new contagious exposures may appear, leading to a higher degree of contagion. In fact we find that the proportion of contagious exposures before applying market shocks is about 15% of the total number of bilateral exposures in the network, and their expected proportion when the common factor of the market shocks equals its 5% quantile is about 50% of the total number of bilateral exposures in the network (see figure 3.2). Hence, in periods of crisis (as simulated by the stress test) the proportion of contagious exposures increases considerably leading to a wider propagation of default. Thus, ignoring market shocks can lead to a serious underestimation of the extent of default contagion in the system. This motivates the use of the Contagion Index to assess the systemic risk of financial failures. The role of contagious exposures is further explored in Amini et al. (2010) from a theoretical point of view.

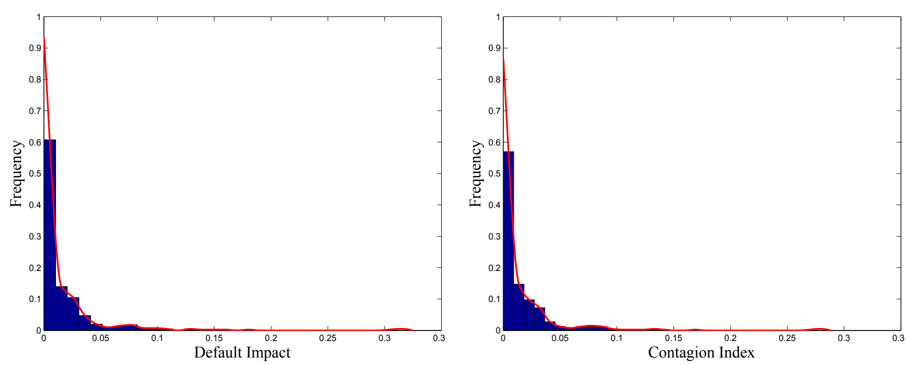


Figure 3.1: Distribution of the Default Impact and the Contagion Index: most institutions have a small Default Impact and Contagion Index, however, some can have an impact up to 25% of the total capital in the network.

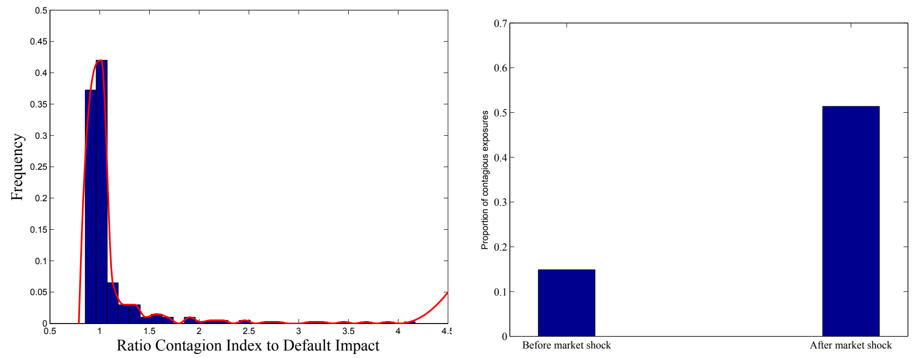


Figure 3.2: Distribution of the ratio of Contagion Index to Default Impact: the Contagion Index can exceed the Default Impact up to four times for some nodes. The proportion of contagious exposures before applying market shocks is about 15% of the total number of bilateral exposures in the network, and their expected proportion when the common factor of the market shocks equals its 5% quantile is about 50% of the total number of bilateral exposures in the network.

3.3 The impact of connectivity and concentration

3.3.1 Connectivity

Previous studies of default contagion in particular network structures have suggested that network connectivity as measured by the number of counterparty links, has an influence on the degree of contagion observed in the network. Allen and Gale (2000) show that the possibility of contagion depends strongly on the completeness of the structure of interregional claims and find that complete markets- in which banks are connected to all other banks in the network, which represents the maximum level of connectivity- are more resilient to contagion than incomplete markets. Babus (2006) investigate whether banks form networks in order to insure against the risk of contagion. They show that better connected networks are more resilient to contagion and find a connectivity threshold above which contagion does not occur. Nevertheless, other studies (Nier et al., 2007; Battiston et al., 2009) have identified two competing effects of increasing connectivity; on one hand interbank linkages can play the role of channels for the propagation of financial distress leading to an increase in the extent of contagion, while on the other hand they can dissipate losses among counterparties, thus reducing the marginal effect of losses on individual counterparties through risk sharing. Nier et al. (2007) show that the potential for knock-on defaults depends on the level of connectivity in the system: when the initial connectivity is low an increase in connectivity increases the risk of contagion, but when connectivity is already high, a further increase in connectivity tends to help dissipate losses across the system which renders it more resilient to contagion. They also find that in undercapitalized networks, increasing connectivity increases the risk of contagion whatever the initial level of connectivity is. Battiston et al. (2009) also investigate the effect of the network connectivity on the probability of individual defaults of financial institutions and find a converse result: when the initial connectivity is low an increase in connectivity improves the resilience of the network to contagion, but when connectivity is already high, a further increase in connectivity increases the risk of contagion.

While it is clear that the degree of connectivity in the network can have a significant effect on the occurrence and severity of contagion, the nature of this effect may depend strongly on the specific structure of the network. The studies we mentioned above consider simple network structures: network of four regions in Allen and Gale (2000), network of two regions in Babus (2006), Erdös-Rényi graph with 25 nodes in Nier et al. (2007) and a regular graph in Battiston et al. (2009). An interesting analysis would be to check whether the results hold in more complex network structures, such as the scale-free network described in chapter 2, that reproduce the heterogeneity in degree and exposures observed in real-world banking systems.

We consider a scale-free graph with 400 nodes and average degree 10, and we extend it, along multiple steps, by adding exposures between unconnected nodes randomly selected to achieve first an extended graph with an average degree of 15, then of 22 and 33. The "new" exposures are IID draws from a Pareto distribution with parameter $\alpha = 1.9$. We change the capital allocations to satisfy the minimal capital ratio θ, and update the default probabilities according to the new size of exposures. We first examine the case of a "well-capitalized" network in which banks are required a capital ratio of (at least) 8%. We then study the case of an "undercapitalized" network in which banks are required a capital ratio of only 4%. Similarly to Nier et al. (2007), we find (see figure 3.3) that in the well-capitalized network, the relationship between connectivity and the average Contagion Index follows a hump shaped pattern: when the initial connectivity is low, an increase in connectivity increases contagion, however when the initial connectivity is high, increasing connectivity reduces contagion. We also observe that in the undercapitalized network, the average Contagion Index is monotonically increasing with connectivity. We can also see this in the tail distribution of the Contagion Index.

Thus, linkages in well-capitalized networks play the role of "shock transmitters" or "shock-absorbers" (following the terminology of Nier et al. (2007)) depending on the initial level of connectivity in the network. When connectivity is low, creating new linkages increases the channels though which financial distress can propagate from one institution to another. In fact, since connectivity is low, the loss incurred per counterparty can still be significant relatively to its capital which may trigger default. However, when connectivity is high, creating new linkages help dissipating losses among a larger number of counterparties leading to a smaller loss per counterparty. This is the so-called benefit from risk sharing discussed in Battiston et al. (2009). There is no benefit from risk sharing in undercapitalized networks since even a small loss can exceed the capital of the counterparty leading it to default. In summary, our study on a simulated scale-free network reveals that increasing the connectivity of already well-connected and well-capitalized interbank networks tends to improve their resilience to contagion. However, if the network is not well-connected or well-capitalized, increasing connectivity tends to increase the risk of contagion.

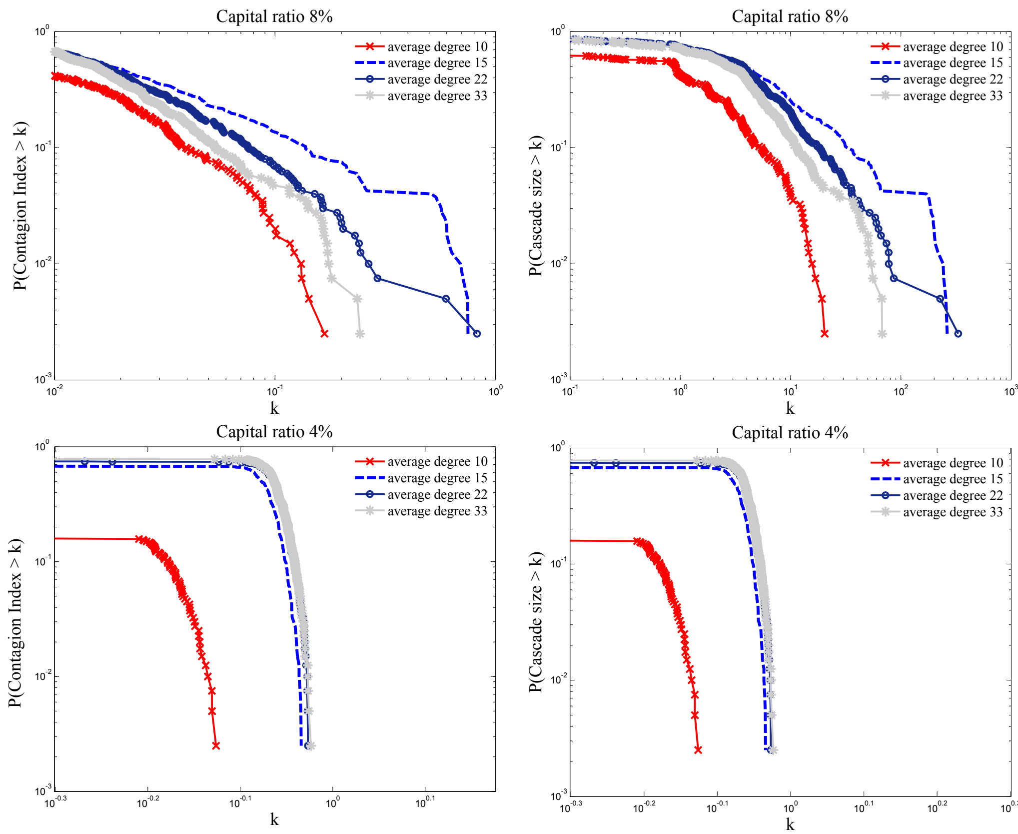


Figure 3.3: Relationship between connectivity and contagion in well-capitalized (capital ratio of 8%) and undercapitalized (capital ratio of 4%) systems. The figure shows the tail distributions of the Contagion Index and the cascade size at different levels of connectivity.

3.3.2 Concentration of exposures

Given that insolvency risk is related to the ratio of potential losses, resulting from exposures, to capital, the concentration of exposures across counterparties is a key factor in determining the extent of contagion. In a homogeneous network setting where assets are uniformly distributed across counterparties, concentration is related to connectivity: this is the setting used in previous simulation studies

Nier et al. (2007). However, in the context of heterogeneous exposures, a network with a higher level of connectivity may present a higher level of concentration in exposures if exposures are unevenly distributed across counterparties. In fact, empirical studies on the distribution of exposures show that exposures have a heavy-tailed empirical distribution which suggests a high degree of concentration.

Nier et al. (2007) find in the Erdös-Rényi framework that more concentrated systems are more prone to contagion. We also study the impact of concentration on the extent of contagion but in the scale-free framework. We compare the Contagion Index and the size of default cascades in a scale-free graph simulated by preferential attachment as described in chapter 2 with 400 nodes, in- and out-degree exponents respectively of 2 and 3 end exposures as follows:

- Pareto exposures: IID realizations of a Pareto distribution with parameter $\alpha = 1.9$.
- Exponential exposures: IID Exponential with intensity $\lambda = \frac{\alpha-1}{\alpha}$, thus guaranteeing equal exposures mean as the graph with Pareto exposures.
- Equal liability towards each creditor: the interbank liability of each institution in the graph with Pareto exposures is divided equally among its creditors, leading to a graph where each institution has the same interbank liability as in the graph with Pareto exposures but this liability is divided equally among all creditors.

Capital and default probabilities are updated in each of these networks according to the size of exposures. We note that the ratio of total capital to total exposures is about the same in the three networks (see table 3.1). We find that the probability of observing an institution with a large Contagion Index or triggering a large default cascade is smaller in the graph with Pareto exposures, followed by the graph with equal liability, then the graph with Exponential exposures. The same ranking is observed when measuring the average Contagion Index and average size of default cascades across the network. Therefore, contrarily to Nier et al. (2007), concentration of exposures seems to improve the resilience of the network to contagion.

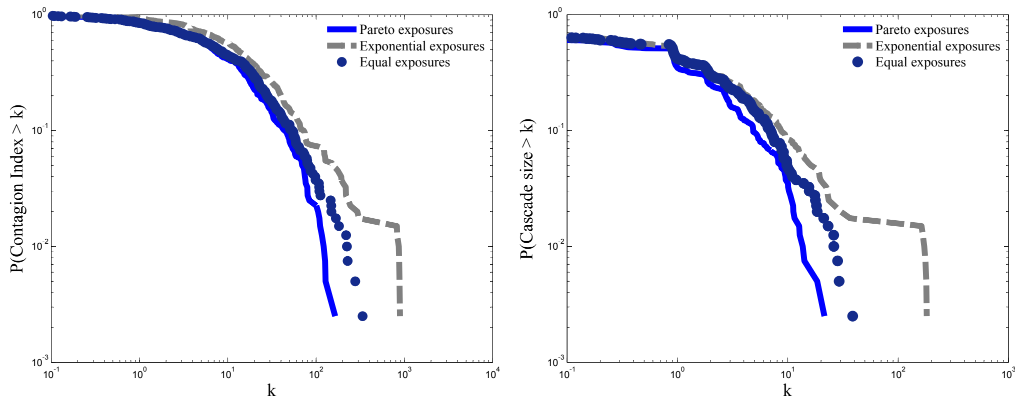


Figure 3.4: Relationship between exposures distribution and contagion. The figure shows the tail distributions of the Contagion Index and the cascade size in scale-free networks with Pareto, Exponential and equally distributed exposures.

Pareto exposures	Exponential exposures	Equal exposures
0.2584	0.2603	0.2537

Table 3.1: Ratio of total capital to total exposures in the networks with Pareto, Exponential and equally distributed exposures.

3.3.3 Local measures of connectivity and concentration: counterparty susceptibility and local network frailty

We have examined above the influence of connectivity and concentration on systemic risk and contagion at an aggregate level of the network. Having at our disposal the Contagion Index which measures the systemic impact of each institution in the network, we are able to identify the *local* characteristic of the counterparty network of institutions which pose a high systemic risk.

The analysis of contagion in the Brazilian interbank network reveals that size effects alone do not explain the magnitude of systemic impact and points to the possible contribution of interconnectedness, or network structure. We examine this in our simulated scale-free network. As shown in figure 3.5, the fifteen most systemic nodes are connected to each other, and many of their exposures are contagious exposures that propagate automatically default when it occurs. Thus, default might snowball among these very systemic nodes leading to a large extent of default contagion.

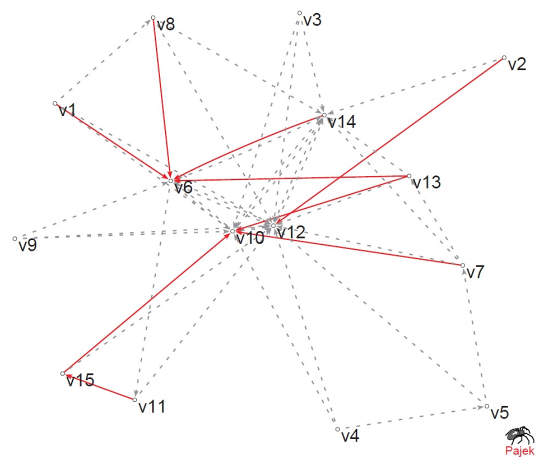


Figure 3.5: Subgraph of the 15 most systemic institutions in the network, contagious exposures are in red.

This motivates to define indicators which go beyond simple measures of connectivity such as the degree (or weighted degree). We define the following indicators which attempt to quantify the sensitivity of the counterparties of these nodes to their default:

Definition 7 (Susceptibility coefficient)**.** *The susceptibility coefficient of a node is the maximal fraction of capital wiped out by the default of a single counterparty.*

$$\chi(i) = \max_{j \neq i} \frac{E_{ij}}{c(i)}$$

A node with $\chi(i) > 100\%$ may become insolvent due to the default of a single counterparty. Counterparty risk management in financial institutions typically imposes an upper limit on this quantity.

Definition 8 (Counterparty susceptibility)**.** *The counterparty susceptibility $CS(i)$ of a node i is the maximal (relative) exposure to node i of its counterparties:*

$$CS(i) = \max_{j, E_{ji}>0} \frac{E_{ji}}{c(j)}$$

$CS(i)$ is thus a measure of the maximal vulnerability of creditors of i to the default of i.

Definition 9 (Local network frailty)**.** *The local network frailty $f(i)$ at node i is defined as the maximum, taken over counterparties exposed to i, of their exposure to i (in % of capital), weighted by the size of their interbank liability:*

$$f(i) = \max_{j, E_{ji}>0} \frac{E_{ji}}{c(j)} L(j)$$

Thus, local network frailty combines two risk components: the risk that the counterparty incurs due to its exposure to node i, and the risk that the (rest of the) network incurs if this counterparty fails. A large value $f(i)$ indicates that i is a node whose counterparties have large liabilities *and* are highly exposed to i.

For simplicity, we label the nodes in the network by their decreasing ranking in Contagion Index. A closer look at the creditors of the five most systemic institutions in the network shows that they are all counterparties to a common creditor, node 12, which is the twelfth most systemic node in the network. Table 3.2 shows that node 12 has much more creditors than most institutions in the network, and also a much greater interbank liability size, counterparty susceptibility and local network frailty. Moreover, 6 of the exposures to node 12 are contagious, meaning that they trigger default cascades in any shock scenario. Node 12 seems then to be a hub that diffuses default in the entire network. We also observe (see table 3.3) that the five most systemic nodes exhibit in general a high counterparty susceptibility and local network frailty.

	k_{out}	L	χ	CS	f
	25	79.58	2.55	15.95	67.74
Network median	3	5.46	2.06	4.92	6.55
90% quantile	13	30.06	5.53	1.33	35.76

Table 3.2: Analysis of node 12 (6 of its exposures are contagious).

Ranking	$\max_{j,E_{ji}>0} k_{out}(j)$	$\max_{j,E_{ji}>0} L(j)$	CS_i	f_i
1	25	79.58	3.46	97.96
2	25	79.58	2.55	203.62
3	25	79.58	10.49	189.37
4	25	79.58	2.20	20.78
5	25	79.58	3.42	29.37
Network median	21	58.29	1.33	6.55
90% quantile	25	79.58	4.92	35.76

Table 3.3: Analysis of the creditors of the five most contagious nodes.

We are thus led to investigate whether we could segregate systemically important institutions based on the measures of connectivity and centrality defined above. We classify institutions into those with a "high" Contagion Index (higher than 1% of the total network capital) and those with a "small" Contagion Index (smaller than 1% of the total network capital), according to their interbank liability, counterparty susceptibility and local network frailty. This can be achieved by conducting a logistic regression of the indicator of the Contagion Index being higher than 1% of the total network capital once on the the interbank liability and counterparty susceptibility, and once on the interbank liability and counterparty frailty. Figure 3.6 displays the decision boundaries at the probabilities 10% and 50% when observing once the interbank liability size and the counterparty susceptibility and once the interbank liability size and the local network frailty: a node outside the 10% decision boundary has an estimated probability of 10% to have a Contagion Index higher than 1% of the network capital; a node outside the 50% decision boundary has an estimated probability of 50% to have a Contagion Index higher than 1% of the network capital. We note that institutions with a high Contagion Index tend to have a large interbank liability, local network frailty and counterparty susceptibility.

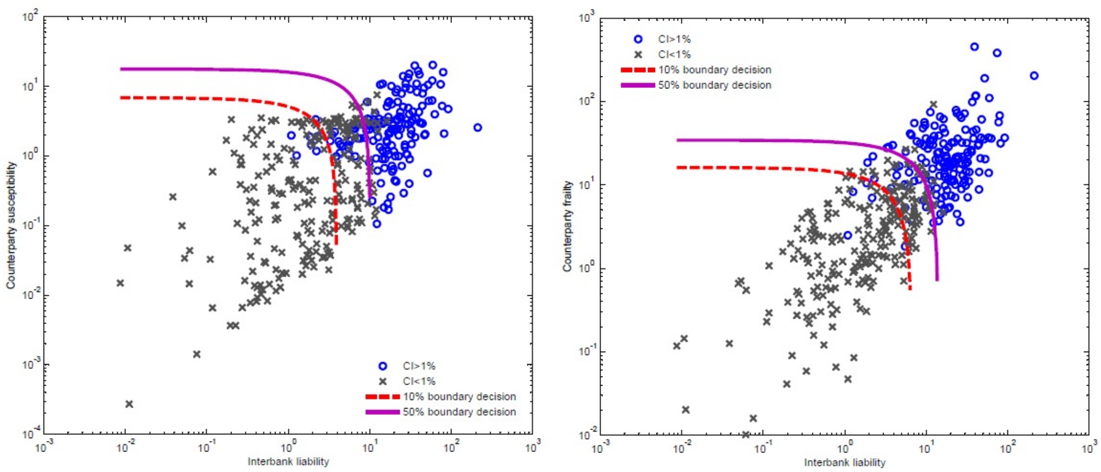


Figure 3.6: Counterparty susceptibility (left figure) and local network frailty (right figure) of the most systemic nodes (with a Contagion Index higher than 1% of the network capital) and the less systemic nodes (with a Contagion Index smaller than 1% of the network capital). Nodes above the 10% decision boundary have with 10% probability a Contagion Index higher than 1% of the network capital. The ones above the 50% decision boundary have with 50% probability a Contagion Index higher than 1% of the capital in the system.

The outputs of the logistic regression are summarized in table 3.4. We observe that the counterparty susceptibility and the local network frailty contribute significantly to the variability of the probability of observing a large Contagion Index: positive coefficients at the 1% significance level and a high adjusted pseudo-R^2 [1]. This indicates that not only the size of interbank liabilities matter in explaining a high

[1] The Adjusted Pseudo-R^2 in a logistic regression is defined as $1 - logL(M)/logL(0)((n-1)/(n-k-1))$ where $logL(M)$ and $logL(0)$ are the maximized log likelihood for the fitted model and the null model, n is the sample size and k is the number of regressors (Shtatland et al., 2002).

Contagion Index, but also the local network frailty and counterparty susceptibility.

We also test for the differences in median between the counterparty susceptibility of the institutions with a Contagion Index higher than 1% of the total network capital and the counterparty susceptibility of those with a Contagion Index smaller than 1% of the total network capital. The Wilcoxon signed-rank test rejects the hypothesis of equal medians at the 1% level of significance. The median of the counterparty susceptibility of the institutions with a high Contagion Index (2.48) is significantly higher than the median of the counterparty susceptibility of the institutions with a small Contagion Index (0.35). Similarly, the median of the local network frailty of the institutions with a high contagion index (19.53) is significantly higher than the median of the local network frailty of the institutions with a small contagion index (2.66).

3.4 The impact of the network size

Another variable that could have an impact on the level of contagion is the size of the network. To illustrate this, we consider various scale-free networks, simulated according to the preferential attachment procedure presented in chapter 2 with respective exponents for the in- and out-degree of 2 and 3, with different sizes ($n = 50, 100, 200, 300, 400$) but maintaining equal their average degree ($\overline{k} = 10$) and total assets ($\sum_{i \in V} A(i) = 1$). We find that larger networks are more resilient to contagion (see figure 3.7).

Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(L) + \beta_2 \log(CS)$		
Coefficients	**Standard error**	**Pseudo-R^2**
$\widehat{\beta_0}$ = -4.41**	0.53	84.55%
$\widehat{\beta_1}$ = 2.13**	0.25	
$\widehat{\beta_2}$ = 0.57**	0.15	
Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(L)$		
Coefficients	**Standard error**	**Pseudo-R^2**
$\widehat{\beta_0}$ = -4.69**	0.51	84.39%
$\widehat{\beta_1}$ = 2.32**	0.24	
Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(CS)$		
Coefficients	**Standard error**	**Pseudo-R^2**
$\widehat{\beta_0}$ = -0.40**	0.12	26.36%
$\widehat{\beta_1}$ = 0.94**	0.10	
Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(L) + \beta_2 \log(f)$		
Coefficients	**Standard error**	**Pseudo-R^2**
$\widehat{\beta_0}$ = -6.99**	0.79	91.69%
$\widehat{\beta_1}$ = 1.75**	0.25	
$\widehat{\beta_2}$ = 1.57**	0.26	
Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(f)$		
Coefficients	**Standard error**	**Pseudo-R^2**
$\widehat{\beta_0}$ = -4.78**	0.51	83.26%
$\widehat{\beta_1}$ = 2.16**	0.22	

* significant at 5% confidence level
** significant at 1% confidence level

Table 3.4: Marginal contribution to the Contagion Index.

3.5 The impact of degree heterogeneity

Very few studies on systemic risk in interbank networks have explored the role of the network heterogeneity (or tiering) in the propagation of financial distress. In

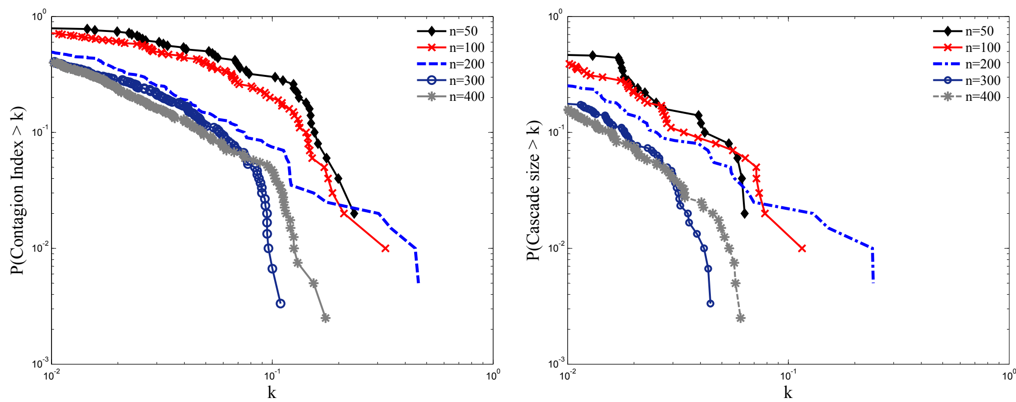


Figure 3.7: Relationship between network size and contagion in scale-free networks. The figure shows the tail distributions of the Contagion Index and the cascade size at different network sizes (n=50, 100, 200, 300, 400).

fact, most of the previous studies have mainly considered simple network structures such as the Erdös-Rényi like models used in Nier et al. (2007), or the regular graph with constant degree used in Battiston et al. (2009), or even simpler the four-regions and two-regions models used in Allen and Gale (2000) and Babus (2006). Nier et al. (2007) have explored simple tiered structure by separating the network into two groups- large banks and small banks- allowing a constant probability of being connected for large banks that is higher than the one for small banks. Their model is a generalization to tiered structures of the Erdös-Rényi but it does not reproduce the power law distributions of the in-degree and out-degree observed in real-world networks. We compare in this section the extent of systemic risk and default cascades in a scale-free graph and an Erdös-Rényi graph with equal connectivity (average degree) and same distribution of exposures. More precisely, we consider a scale-free network of $n = 400$ institutions and average degree $\overline{k} = 10$ and simulate an Erdös-Rényi network with 400 institutions, and probability of being connected $p = \frac{\overline{k}}{2(n-1)}$, thus guaranteeing the same average degree $\overline{k}$ as in the scale-free graph. Capital and default probabilities are assigned according to the size of exposures in each network. We first note that the ratio of total capital to total exposures is about the same in the two networks (see table 3.5) which means that any difference in the extent of contagion is not due to a difference in the capital available in the system but to a difference in network structure. We find (see figure 3.8) that the probability of observing an institution with a large Contagion Index or triggering a large default cascade is much higher in the Erdös-Rényi network. Also, the average Contagion Index and average size of default cascades is higher in the Erdös-Rényi network. Thus tiering seems to increase the resilience of the network to financial distress.

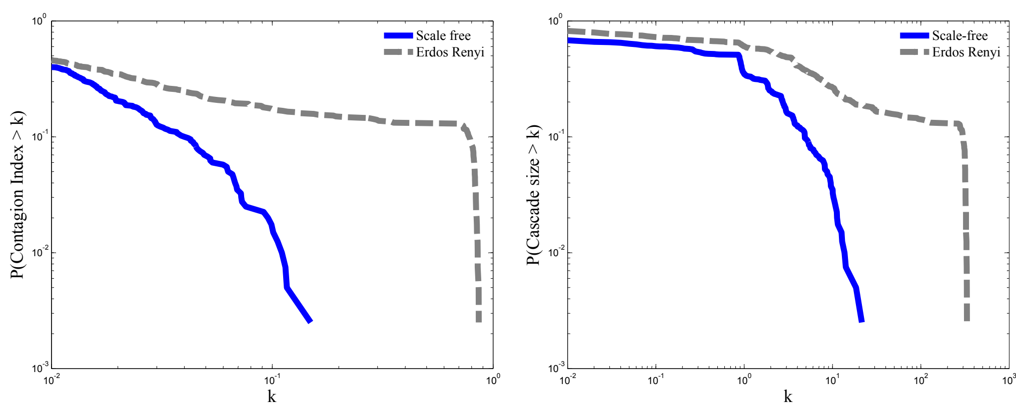


Figure 3.8: Relationship between network structure and contagion. The figure shows the tail distributions of the Contagion Index and the cascade size in a scale-free and Erdös-Rényi networks with equal connectivity (average degree).

Scale-free	**Erdös-Rényi**
0.2584	0.2443

Table 3.5: Ratio of total capital to total exposures in the scale-free and Erdös-Rényi networks.

3.6 The impact of targeted capital requirements

Capital requirements are a key ingredient of bank regulation: in the Basel Accords, a lower limit is imposed on the ratio of capital to (risk-weighted) assets. It is clear that globally increasing the capital cushion of banks will decrease the risk of contagion in the network, but given the heterogeneity of systemic importance, as measured by the Contagion index, it is not clear whether a *uniform* capital ratio for all institutions is the most efficient way of reducing systemic risk. Indeed, recent debate has considered the option of more stringent capital requirements on systemically important institutions. One idea, which we explore here, is to impose higher capital requirements on institutions whose position in the network plays a key role in the network's resilience to contagion.

Studies in epidemiology or the spread of viruses in peer-to-peer networks (Cohen et al., 2003; Madar et al., 2004; Huang et al., 2007) have explored similar problems in the context of immunization of heterogeneous networks to contagion. Madar et al. (2004) study various immunization strategies in the context of epidemic modeling. They show that in *random immunization* schemes, where nodes are randomly chosen and vaccinated, the whole population must get vaccinated to effectively control epidemic propagation. They propose instead a *targeted immunization* strategy that consists in vaccinating first the nodes with largest degrees. A third approach, called *acquaintance immunization* (Cohen et al., 2003; Madar et al., 2004), which consists in immunizing randomly selecting individuals as well as their acquaintances, is shown to perform better than random immunization, especially in scale-free networks.

Based on these analogies, we consider *targeted capital requirement* policy which consists in imposing capital requirements on the the 5% most systemic institutions in the network and their *creditors*: this aims at reducing the number of contagious links (see Definition 6) emanating from the most systemic institutions, since these links play a major role in contagion of default in the network Amini et al. (2010).

We consider two different policies for setting capital requirements:

- Minimum capital-to-exposure ratio: in this case, we require institutions to hold a capital larger than $\overline{c}$ that could cover at least a portion θ of their interbank exposures:

$$\overline{c}(i) = \max(c(i), \theta A(i)) \tag{3.23}$$

- Cap on susceptibility: Counterparty susceptibility (Definition 13) and local network frailty (Definition 14) are a significant source of systemic risk. Thus, preventing large values of counterparty susceptibility or network frailty from occuring can decrease systemic risk. This could be achieved by requiring that no exposure should represent more than a fraction γ of capital. In this case, a financial institution i is required to hold a capital larger than $\overline{c}$ given by:

$$\overline{c}(i) = \max(c(i), \frac{\max_{j\neq i}(E_{ij})}{\gamma}) \tag{3.24}$$

We compare the situations in which (i) these policies are applied to all financial institutions in the network (*non-targeted capital requirements*), (ii) they are applied only to the creditors of the 5% most systemic institutions (*targeted capital requirements*), by computing, in each case, the average of 5% largest Contagion Indexes (i.e. the 5% tail conditional expectation of the cross sectional distribution of Contagion Index) in the network.

The targeted acquaintance immunization is observed to be the most efficient strategy in reducing contagion in the network. Figure 3.9 shows that the targeted acquaintance immunization can achieve the same reduction in the contagion index and the size of default cascade while requiring less capital. Specifically, we observe that imposing a cap on the susceptibility of the creditors of the 5% most systemic institutions is the most efficient strategy, followed by the strategy consisting in imposing a floor on the capital ratio of the creditors of the 5% most systemic institutions. It is interesting also to note that imposing capital requirements on the institutions with the largest size is not optimal. This leads again to the conclusion that size is not the sole source of systemic risk, other factors also matter.

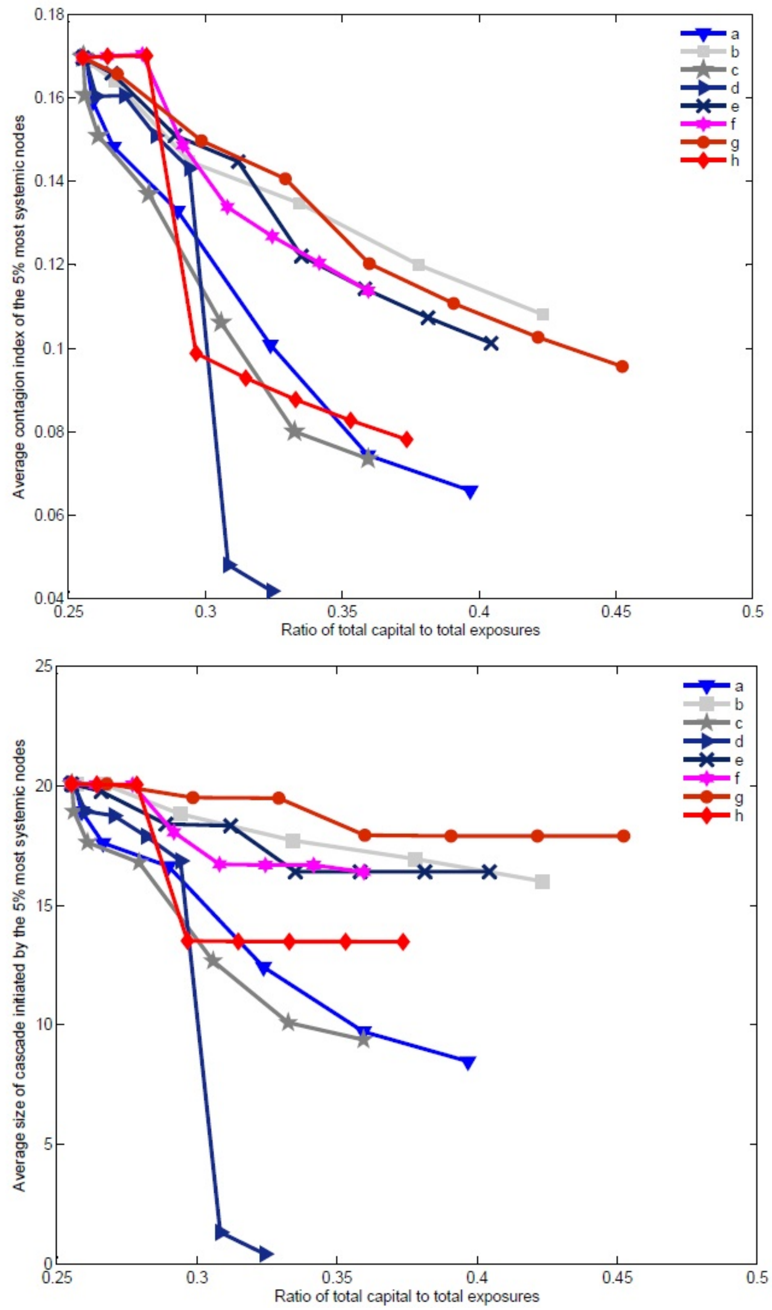


Figure 3.9: Impact of capital requirements: (a) imposing a floor on the capital ratio for all institutions in the network, (b) imposing a cap on the susceptibility for all institutions in the network, (c) imposing a floor on the capital ratio of the creditors of the 5% most systemic institutions, (d) imposing a cap on the susceptibility of the creditors of the 5% most systemic institutions, (e) imposing a floor on the capital ratio of the 5% institutions with the highest interbank liabilities, (f) imposing a cap on the susceptibility of the 5% institutions with the highest interbank liabilities, (g) imposing a floor on the capital ratio of the 5% institutions with the highest number of creditors, (h) imposing a cap on the susceptibility of the 5% institutions with the highest number of creditors.

Chapter 4

Systemic Risk in Banking Systems: the Case of Brazil

This chapter is based on the paper "Network structure and systemic risk in banking systems" (Cont et al., 2010), which is a joint work with Professor Rama Cont and Doctor Edson Bastos e Santos.

4.1 Introduction

We apply in this chapter the methodology introduced in chapter 3 to the study of the Brazilian financial system. Using a *complete* data set of interbank exposures and capital levels provided by the Brazilian Central Bank, we analyze the magnitude of contagion risk in the Brazilian financial system.

4.1.1 Summary of main results

Our study reveals several interesting features on the nature of systemic risk and default contagion in the Brazilian financial system:

- The systemic importance of institutions is quite heterogeneous: the cross-sectional distribution of the Contagion Index is found to be heavy-tailed. This implies that, while most financial institutions present only a negligible risk of contagion, a few of them may pose a significant risk of contagion.

- Ignoring the compounded effect of correlated market shocks and contagion via counterparty exposures can lead to a serious underestimation of contagion risk. Specifically, market shocks are found to increase the proportion of contagious exposures in the network, i.e. exposures that transmit default in all shock scenarios. We are thus led to question the conclusions of previous studies which dismissed the importance of contagion by looking at pure balance sheet contagion in absence of market shocks.

- Fundamental defaults due to market shocks are found to be the major source of aggregate losses for most periods. Nevertheless, contrarily to observations made in some previous studies, contagion is observed to be significant during periods of stress. This is explained by the fact that we measure the effect of contagion using conditional risk measures, whereas most previous studies examined cross-sectional averages, which underestimate the magnitude of contagion in a heterogeneous network.

- Balance sheet size matters when assessing systemic importance: the Contagion Index of a financial institution has a strong positive relationship with the total size of its interbank liabilities. However, size alone is not a good indicator for the systemic importance of financial institutions: network structure *does* matter when assessing systemic importance. Network-based local measures of connectivity and concentration of exposures across counterparties –*counterparty susceptibility* and *local network frailty*– are shown to contribute significantly to the systemic importance of an institution.

- Targeted capital requirements are found to be more efficient strategies to limit the extent of contagion in the Brazilian financial system than aggregate capital requirements.

4.1.2 Relation to the literature

While most of the empirical studies on systemic risk and default contagion in interbank networks have dismissed the importance of contagion (Sheldon and Maurer, 1998; Furfine, 2003; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b; Mistrulli, 2007), our study reveals that the risk of default contagion is significant in the Brazilian financial system. We show examples in which the expected loss resulting from the default of an institution can exceed up to forty times the size of its interbank liabilities and some defaults combined with common shocks can initiate up to four additional defaults. In contrast with Elsinger et al. (2006a), we find that scenarios with contagion are more frequent than those without contagion when grouped by number of fundamental defaults, when the number of fundamental defaults is at least 3. In fact, we find that market shocks can play an essential role in propagating default across the network. Specifically, we observe that the proportion of contagious exposures increases considerably when the system is subject to a market shock scenario, thus creating additional channels of contagion in the system. The Contagion Index, by compounding the effects of both market events and counterparty exposure, accounts for this phenomenon.

Our results do not necessarily contradict the findings of the previous empirical literature but present them in a different light. Most of the aforementioned studies use indicators *averaged* across institutions: we argue that, given the heterogeneity of the systemic importance across institutions, the sample average gives a poor representation of the degree of contagion and *conditional* measures of risk should be used. Also, most of these studies are based on a generous recovery rate assumptions whereby all assets of a defaulting bank are recovered at pre-default value, which is far from reality especially in the short term where recovery rates are close to zero in practice. With the exception of Elsinger et al. (2006a,b), all these studies measure the impact of the idiosyncratic default of a single bank, whereas we use the more realistic setting where balance sheets are subjected to correlated market shocks in default scenarios. Finally, we use a heavy-tailed model for generating the correlated shocks to balance sheets: we argue that this heavy-tailed model is more realistic than Gaussian factor models used in many simulation studies.

Another important aspect is the fact that most of the empirical studies are based on partial information on the bilateral exposures in the network, and estimate missing exposures with a maximum entropy method (Sheldon and Maurer, 1998; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b; Degryse and Nguyen, 2007). However, the maximum entropy method is found to underestimate the possibility of default contagion (Mistrulli, 2007; van Lelyveld and Liedorp, 2006). Our study, by making use of empirical data on all bilateral exposures, avoids this caveat.

Our study also complements the existing literature by studying the contribution of network-based local measures of connectivity and concentration to systemic risk. We find that the *counterparty susceptibility* and *local network frailty* introduced in chapter 3 can explain significantly default contagion.

We discuss the impact of *targeted capital requirements* such as the strategies introduced in chapter 3 and show that targeting the creditors of the most contagious institutions is a more effective strategy in terms of the total capital it requires to limit the extent of contagion in the Brazilian financial system.

4.1.3 Outline

The chapter is organized as follows. Section 4.2 applies the methodology introduced in chapter 3 to the Brazilian financial system. Section 4.3 investigates the role of different institutional and network characteristics which contribute to the systemic importance of Brazilian financial institutions. Section 4.4 analyzes the impact of capital requirements on these indicators of systemic risk and uses the insights obtained from the network model to examine the impact of *targeted* capital requirements which focus on the most systemic institutions and their counterparties.

4.2 Is default contagion a significant source of systemic risk?

Most empirical studies of interbank networks have pointed to the limited extent of default contagion (Sheldon and Maurer, 1998; Furfine, 2003; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b; Mistrulli, 2007). However, almost all these studies (with the exception Elsinger et al. (2006a,b)) examine the sole knock-on effects of the sudden failure of a single bank by an idiosyncratic shock, thus ignoring the compounded effect of both correlated market events and default contagion. A correlated market shock affecting the capital of all institutions in the network can considerably reduce the capital of the network, which makes it more vulnerable to potential losses and increases the likelihood of large default cascades. We explore in this section the extent of default contagion in the Brazilian financial system (section 4.2.1), and study the role market shocks have in generating channels of contagion across the network (section 4.2.2). We also analyze the contribution of *fundamental* defaults and *defaults by contagion* to the systemic risk of the network as whole (section 4.2.3). We compare the latter to the results obtained in Elsinger et al. (2006a) for the Austrian banking system.

4.2.1 Evidence for contagion

When studying the contagion risk an institution i may pose to the financial system, two interesting questions arise:

- How much would the financial system suffer if institution i fails?
- How many institutions in the system would become insolvent if institution i fails?

The Contagion Index provides an answer to the first question by measuring the contagion loss induced by the failure of institution i in a stressed market. The second question relates to the number of institutions that default by contagion in the cascade triggered by a default of institution i.

Definition 10 (Size of the default cascade)**.** *We define the size $\kappa(i, c, E)$ of the default cascade initiated by the default of institution i as the expected number of defaults by contagion generated when the system is subject to correlated market shocks given that the shock triggers the default of i.*

$$\kappa(i, c, E) = \mathbb{E}\left[\sum_{j=1}^{n} 1_{c(j)+\epsilon_j>0, c_{n-1}(j)=0} | c(i) + \epsilon_i < 0\right] \tag{4.1}$$

We find that the size of default cascades varies across the institutions that trigger the cascade: most institutions do not seem to generate other defaults due to contagion in the system (see figure 4.1), however some institutions can trigger up to 4 defaults which represents about 3% of the financial system. This means that domino effects should not be measure by averaging across the entire network: one should condition on the event of default of each individual institution in the system.

This presence of contagion is confirmed by comparing the Contagion Index of each institution to its interbank liabilities: a Contagion Index which exceeds the institution's interbank liabilities is a signature of contagion. As shown in figure 4.2 the Contagion Index can significantly exceed (up to forty times) the interbank liabilities for the most systemic nodes. This indicates that default contagion is a significant component of systemic risk for these systemically important institutions.

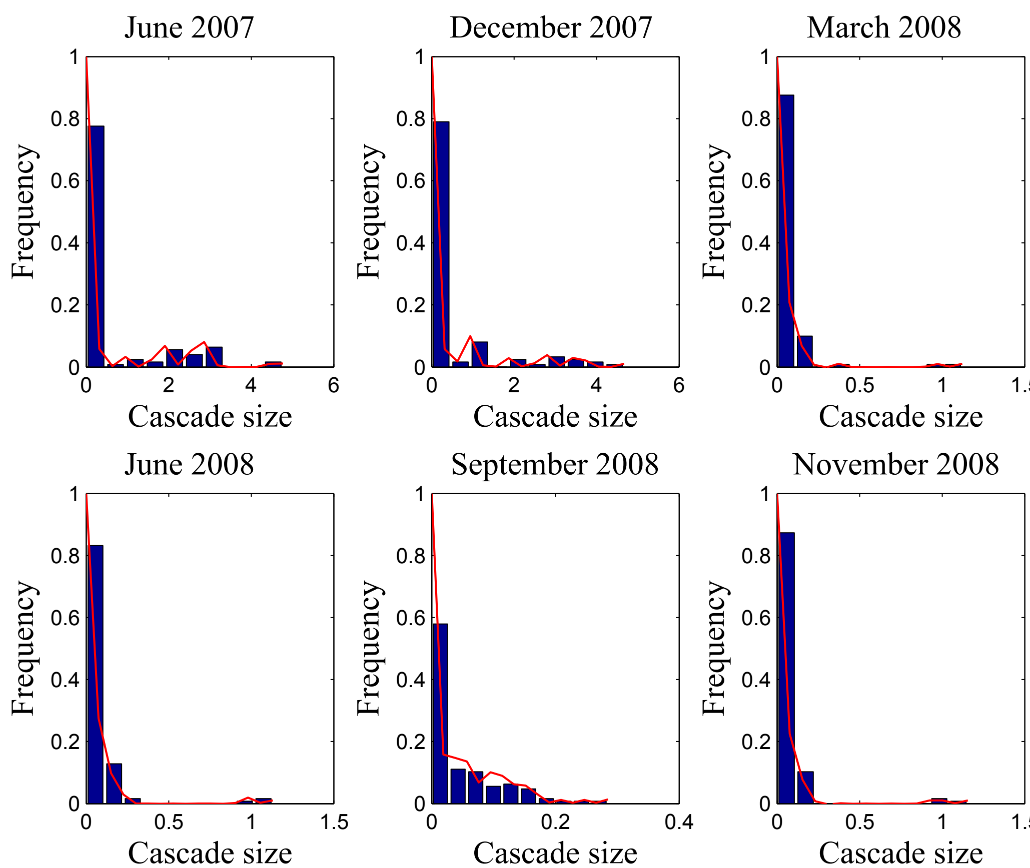


Figure 4.1: Distribution of the size of default cascade.

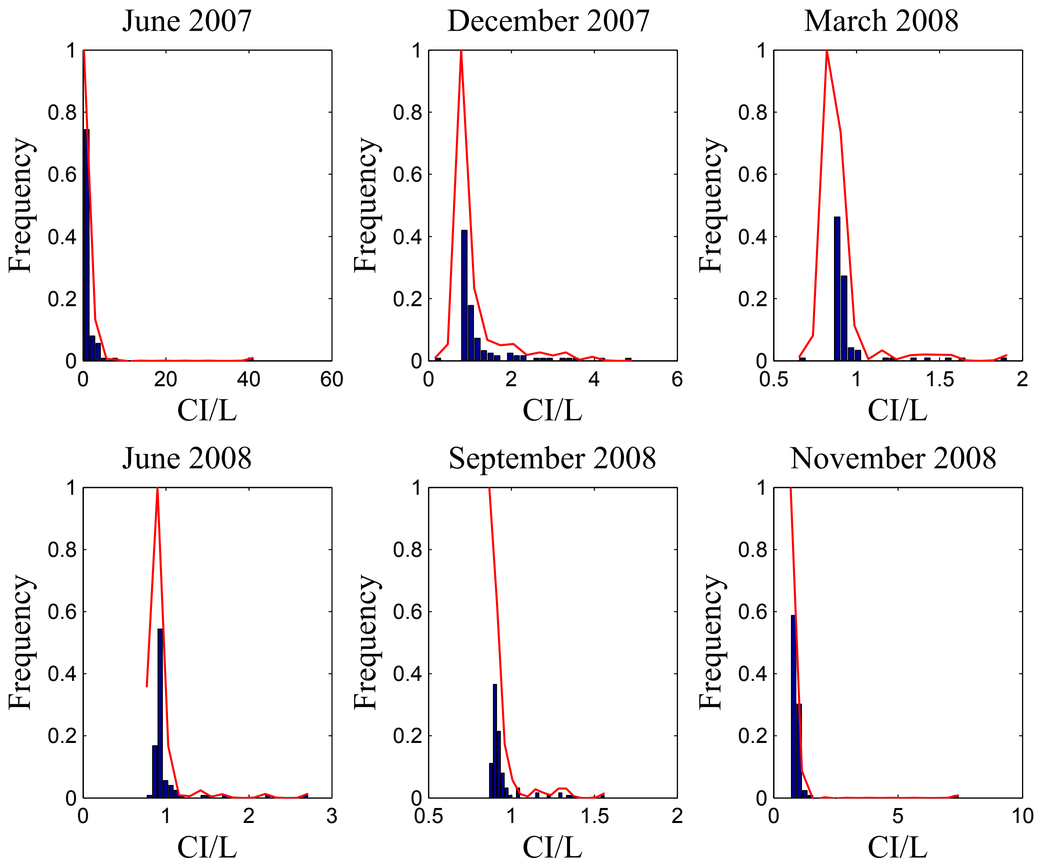


Figure 4.2: Ratio of the Contagion Index to the interbank liabilities: the Contagion Index can be up to forty times the size of interbank liabilities.

4.2.2 The role of correlated market shocks

The Default Impact and the Contagion Index exhibit heavy tailed distributions (see figures 4.4, 4.5) indicating the existence of few institutions that present a high contagion risk to the Brazilian financial system (up to 10% of the total capital of the network) while most institutions exhibit a small risk. We note that the probability of observing a large Default Impact and a large Contagion Index is the highest during June 2007 (see figure 4.3). This period corresponds to the appearance of the subprime mortgage crisis in the United States.

The cross-sectional distribution of the ratio of the Contagion Index to the Default

Impact (see figure 4.6) shows that the Contagion Index may, for some nodes, significantly exceed the Default Impact. Correlated shocks to balance sheets seem to amplify contagion in the Brazilian banking system. This confirms the findings of the study on a simulated network of chapter 3 on the role of market shocks in amplifying contagion.

We also find, similarly to the study on a simulated network presented in chapter 3, that market shocks tend to increase the proportion of *contagious exposures*, that are exposures not covered by an adequate amount of capital to sustain their loss in the event of default, thus they transmit default in all shock scenarios. We recall below the definition of such exposures.

Definition 11 (Contagious Exposure)**.** *An exposure of institution i to j is called* contagious *if its size exceeds the capital of i: $E_{ij} > c(i)$.*

Figure 4.7 shows the graph of contagious exposures (black) in the Brazilian network in June 2007, with, in red, the exposures that become contagious once a (particular) set of correlated market shocks is applied to balance sheets. Figure 4.8 presents the proportion of contagious exposures in the Brazilian system, their expected proportion under stress test scenarios, and their expected proportion in scenarios where the level of common downward shocks to balance sheets exceeds its 5% quantile. We clearly observe that market shocks may significantly increase the proportion of contagious exposures that play the role of channels for the transmission of financial distress across the network. Therefore, ignoring market risk when assessing contagion effects can lead to a serious underestimation of the extent of default contagion.

4.2.3 Fundamental losses vs losses by contagion

Elsinger et al. (2006a) distinguish *fundamental defaults* -due to exogenous market

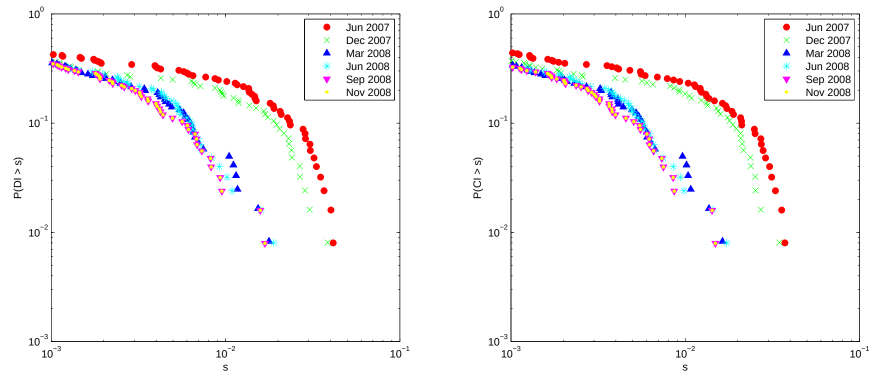


Figure 4.3: Brazilian interbank network: distribution of the default impact and the Contagion Index on the logarithmic scale. The highest probabilities of having a large Default Impact and a large Contagion Index are observed in June 2007.

shocks- from defaults by contagion and perform a simulation study of the respective contributions to systemic risk of fundamental defaults and contagion effects. In their study on the Austrian banking network, fundamental defaults are found to be more frequent than contagion effects, which leads them to conclude that the main source of systemic risk is the correlation among risk factors influencing balance sheets.

We conduct a similar analysis to study the contribution of default contagion to systemic risk, albeit with a different metric, the Contagion Index. We classify all simulated default events into those resulting from large market shocks and those resulting from contagion. We define the *expected loss* (EL) incurred by the institutions at the end of the default cascade when the system is subject to market shocks given that the common factor of the market shocks falls below its 5%-quantile level. We decompose the (expected) losses into losses resulting from fundamental market shocks L_F and those resulting from contagion $L_C = EL - L_F$:

$$EL = \sum_{v=1}^{n} \mathbb{E}\left[c(v) - c_{n-1}(v) | S < S_{0.05}\right] \qquad L_F = \sum_{v=1}^{n} \mathbb{E}\left[c(v) - c_0(v) | S < S_{0.05}\right] \quad (4.2)$$

where $c_0(v) = (c(v) + \epsilon_v)_+$ and $S_{0.05}$ is the 5%-quantile of S. Figure 4.9 shows that

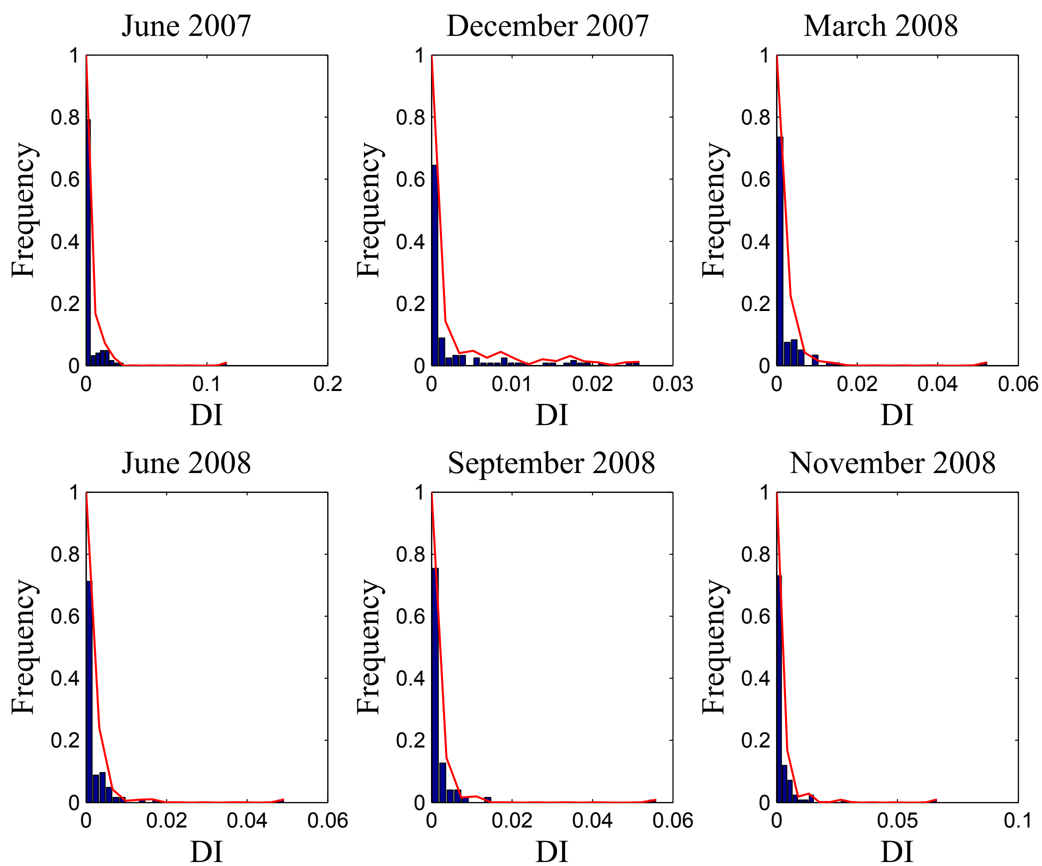


Figure 4.4: Distribution of the Default Impact. Most institutions have a small Default Impact, however, some can have an impact up to 10% of the total network capital in June 2007.

the losses due to fundamental defaults are significantly larger (by a factor of 10) than the loss due to contagion. However, the number of defaults by contagion (Figure 4.9, below) is comparable to the number of fundamental defaults especially in June 2007. Thus, although fundamental defaults seem to be a major source of systemic risk, one cannot neglect the impact of contagion. As in Elsinger et al. (2006a), we also compute the probabilities of occurrence of contagion and the expected number of defaults due to contagion grouped by the number of fundamental defaults. We find much more scenarios with contagion than in Elsinger et al. (2006a): for more than two fundamental defaults, the scenarios with contagion are more frequent than

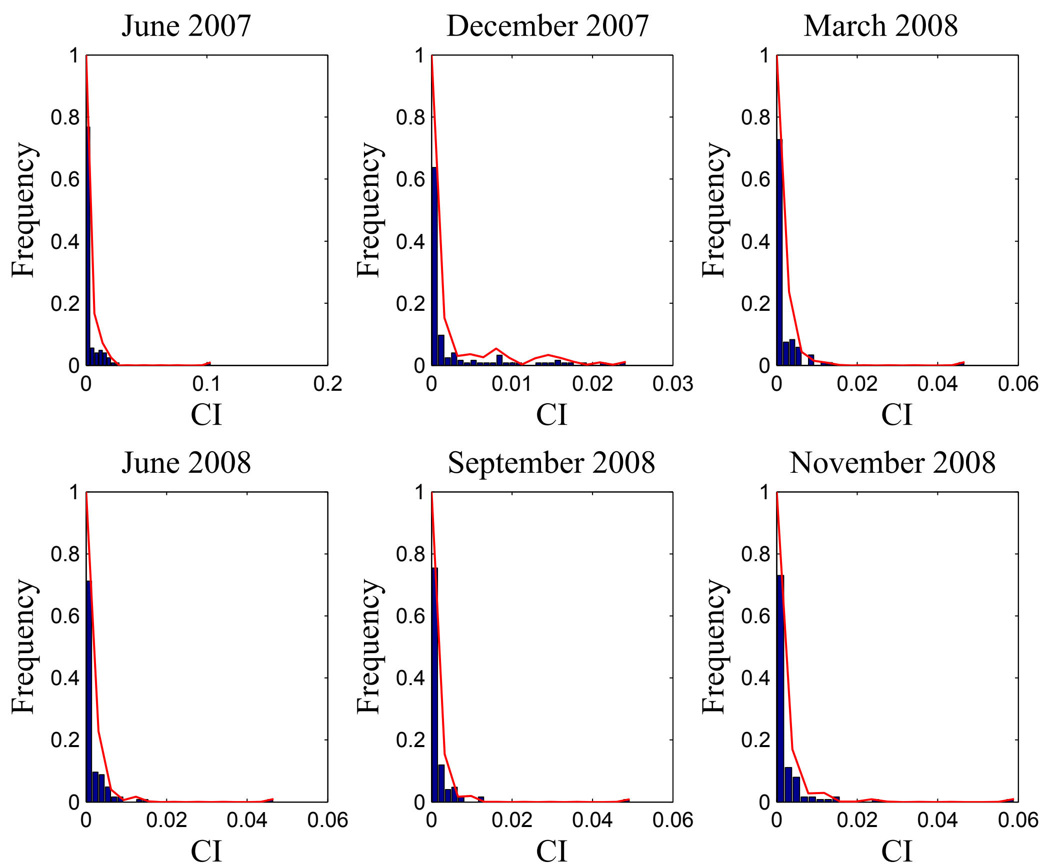


Figure 4.5: Distribution of the Contagion Index. Most institutions have a small Contagion Index, however, some can have an impact up to 10% of the total network capital in June 2007.

those without contagion. Thus, default contagion cannot be ignored.

Fundamental defaults	Scenarios with no contagion (%)	Scenarios with contagion (%)	Number of defaults by contagion
0	47.67	0.00	0.00
1	25.52	10.11	0.78
2	6.81	6.23	1.49
3	1.18	1.90	2.00
4	0.13	0.28	2.79
5	0.02	0.10	5.44
6 and more	0.00	0.05	10.52
Total	81.34	18.66	

Table 4.1: Probabilities of occurrence of contagion and expected number of defaults due to contagion, grouped by the number of fundamental defaults.

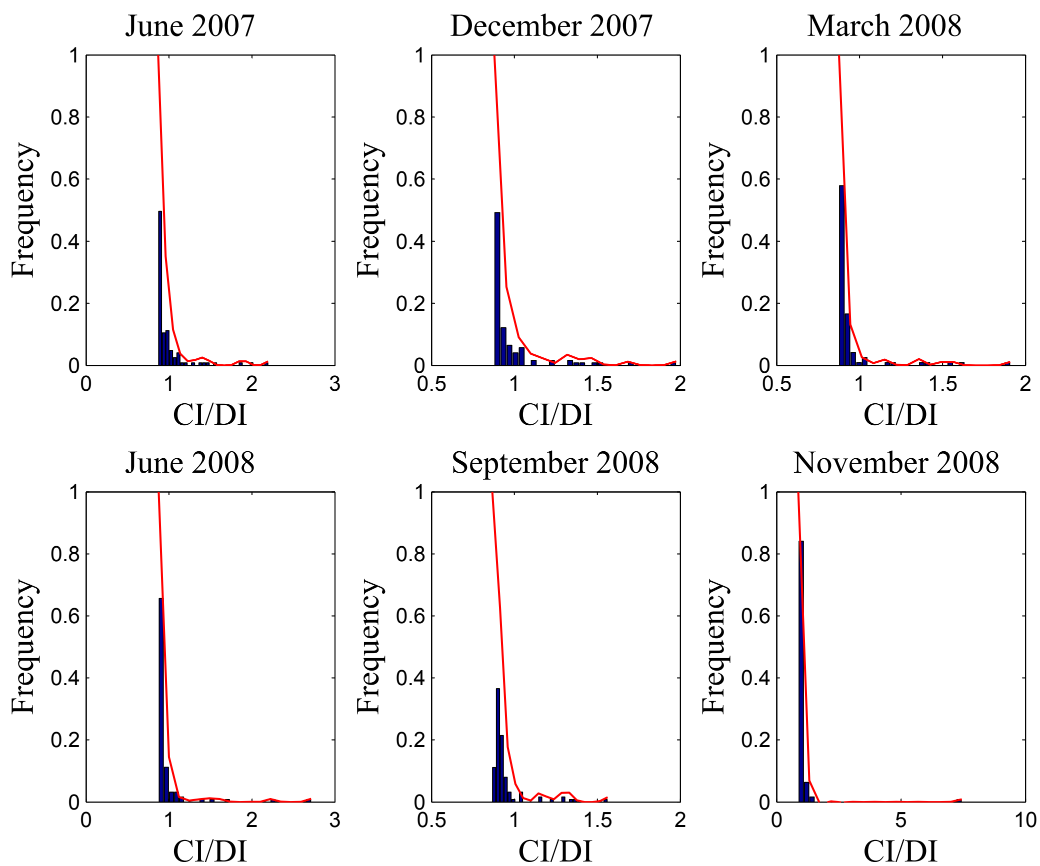


Figure 4.6: Default impact vs Contagion Index: the Contagion Index can be four times the Default Impact for some nodes.

4.3 What makes an institution systemically important?

Previous studies on contagion in financial networks (Allen and Gale, 2000; Battiston et al., 2009; Elsinger et al., 2006a; Nier et al., 2007) have examined how the network structure may affect the global level of systemic risk but do not provide metrics or indicators for localizing the source of systemic risk within the network. The ability to compute a Contagion Index for measuring the systemic impact of each institution in the network, enables us to locate the institutions which have the largest systemic

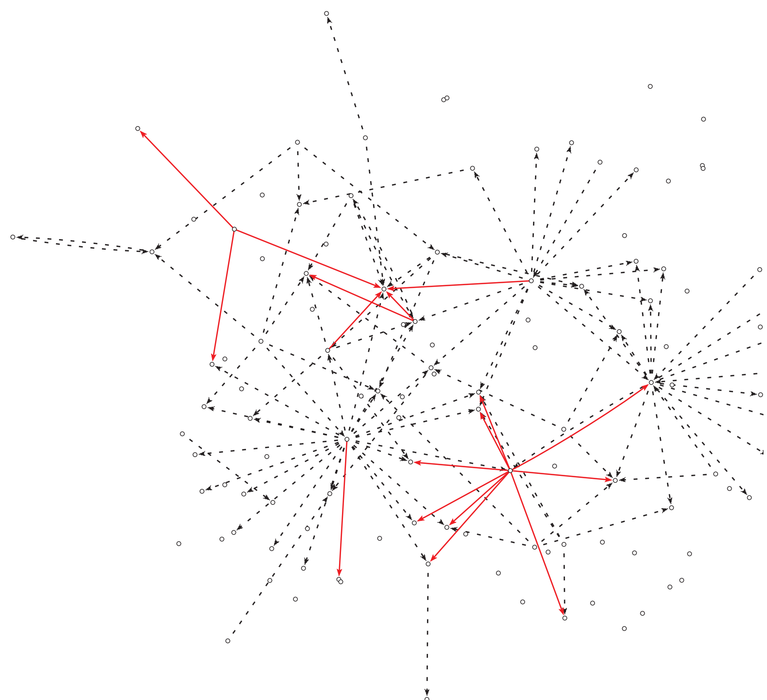


Figure 4.7: Network of contagious exposures before (dashed lines) and after (dashed and red lines) market shocks, June 2007.

impact and investigate their characteristics.

We first investigate (section 4.3.1) the effect of the size, measured in terms of interbank liabilities or assets on the Contagion Index. Then we examine (section 4.3.2) the effect of network structure on the Contagion Index and define, following chapter 3, network-based indicators of connectivity *counterparty susceptibility* and *local network frailty*, which are then shown to be significant factors for contagion.

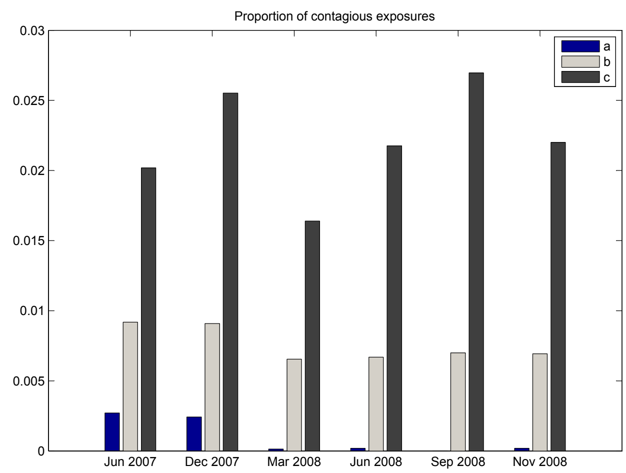


Figure 4.8: Proportion of contagious exposures (a) in the initial network, (b) averaged across market shock scenarios, (c) averaged across scenarios where common factor falls below 5% quantile level.

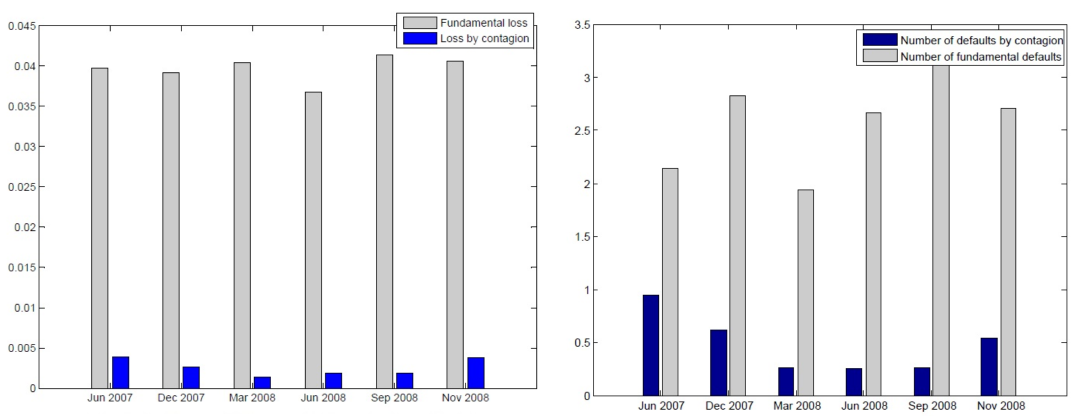


Figure 4.9: Fundamental loss vs loss by contagion when the system is subject to correlated market shocks, given that the common factor in the market shocks falls below its 5%-quantile. Left: fundamental loss vs loss by contagion in BRL. Right: expected number of fundamental vs contagious defaults.

4.3.1 The role of balance sheet size

Size is generally considered a factor of systemic importance. In our modeling approach, where losses flow in through the asset side and flow out through the liability side of the balance sheet, it is intuitive that, at least at the first iteration of the loss cascade, firms with large liabilities to other nodes will be a large source of losses for their creditors in case of default. Accordingly, interbank liabilities are highly correlated with any measure of systemic importance. A simple plot on the logarithmic scale of the Contagion Index against the interbank liability size reveals a strong positive relationship between the interbank liabilities of an institution in the Brazilian financial system and its Contagion Index (see figure 4.10). A linear regression of the logarithm of the Contagion Index on the logarithm of the interbank liability size supports this observation: interbank liabilities explains 96% of the cross-sectional variability of the Contagion Index.

Therefore, balance sheet size does matter, not surprisingly. However, the size of interbank liabilities does not entirely explain the variations in the Contagion Index across institutions: the interbank liability size does exhibit a strong positive relationship with the Contagion Index, but the ranking of institutions according to liability size does not correspond to their ranking in terms of systemic impact (see figure 4.10).

Model: $log(CI) = \beta_0 + \beta_1 log(L) + \epsilon$

Coefficients	**Standard error**	**t-statistic**	R^2
b_0 = -0.58	0.36	-1.41	96%
b_1 = 1.04**	0.02	51.75	

* significant at 5% confidence level

** significant at 1% confidence level

Table 4.2: Log-log cross-sectional regression of the Contagion Index (expressed in percentage of the total network capital) on the interbank liability in June 2007.

Table 4.3, where nodes are labeled according to their decreasing ranking in terms of the Contagion Index, shows that Node 5 has interbank liabilities less than the 90% quantile of the cross sectional interbank liability sizes. This suggests that factors other than size contribute to their systemic importance.

Plotting the Contagion Index against the interbank asset size (figure 4.11) shows

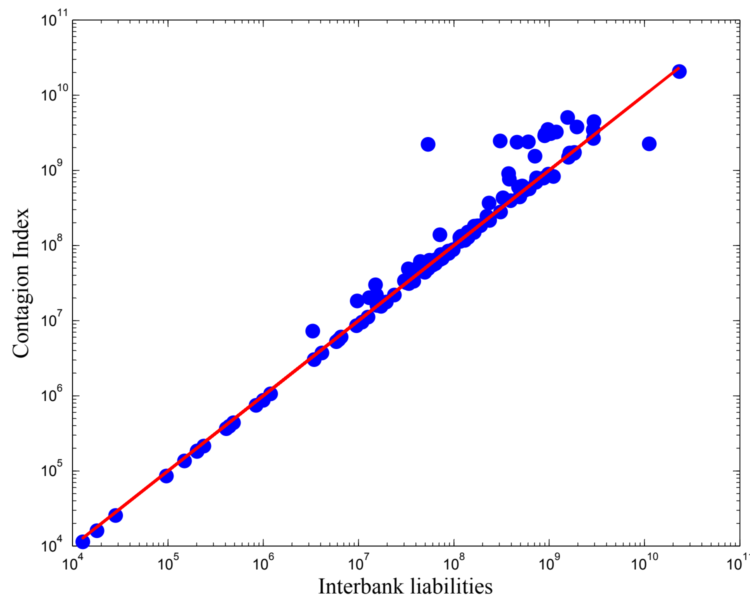


Figure 4.10: Scatterplot on the logarithmic scale of the Contagion Index versus the interbank liability size in June 2007.

Ranking	**Contagion index (billion BRL)**	**Number of creditors**	**Interbank liability (billion BRL)**
1	20.77	8	23.27
2	4.95	32	1.57
3	4.58	13	2.96
4	3.85	14	1.95
5	3.40	21	0.97
Network median	0.10	5	0.07
90%-quantile	2.45	21	1.11

Table 4.3: Analysis of the five most contagious nodes in June 2007.

that the contribution of the size of interbank assets to the Contagion Index is less significant. Note that interbank liabilities are not balanced with respect to interbank assets, due to deposits and other types of liabilities which are excluded from interbank liabilities.

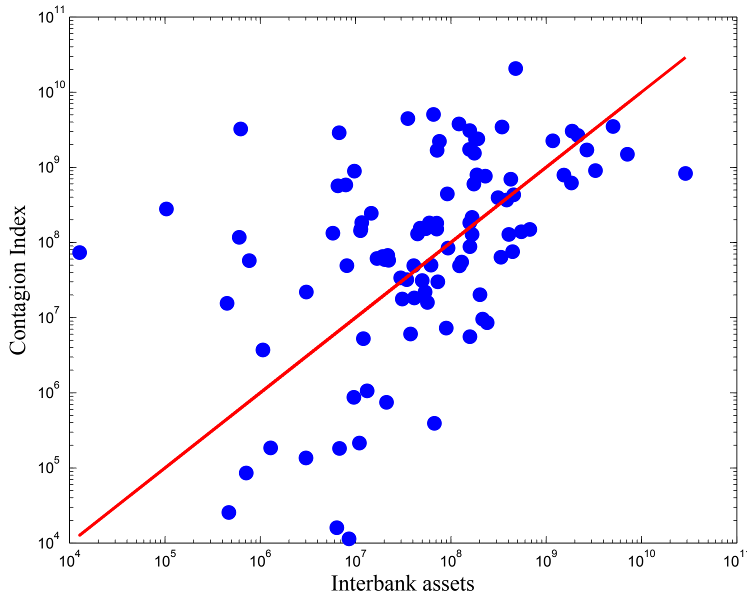


Figure 4.11: Scatterplot on the logarithmic scale of the Contagion Index versus the interbank assets size in June 2007.

Model: $log(CI) = \beta_0 + \beta_1 \log(A) + \epsilon$

Coefficients	Standard error	t-statistic	Adjusted R^2
$b_0 = 8.00$ **	1.99	4.02	22%
$b_1 = 0.58$ **	0.11	5.20	

* significant at 5% confidence level
** significant at 1% confidence level

Table 4.4: Log-log cross-sectional regression of the Contagion Index on the size of interbank assets in June 2007: $R^2 = 22\%$.

4.3.2 The role of network structure

Table 4.3 shows that, while the sheer size of liabilities of the node with the highest Contagion Index can explain its ranking, the four other most systemic nodes have liability sizes roughly in line with the network average, so size effects alone do not explain the magnitude of their systemic impact. This points to the possible contribution of interconnectedness, or network structure, in explaining the magnitude of their Contagion Index. As shown in figure 4.12 the five most systemic nodes are not very connected and just have few contagious exposures (in red) but, as shown in figure 4.13, their *creditors* are heavily connected and many of their cross-exposures are contagious exposures.

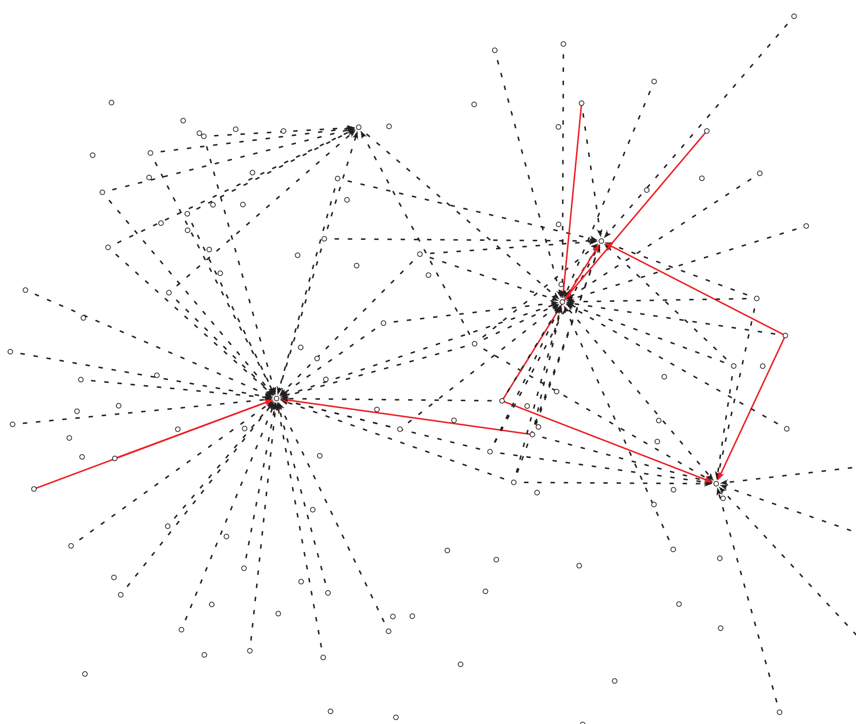


Figure 4.12: Subgraph of the five institutions with highest Contagion Index and their creditors in the network in June 2007. Non contagious exposures are dashed lines. Contagious exposures are full red lines.

This motivates to define indicators which go beyond simple measures of connectivity

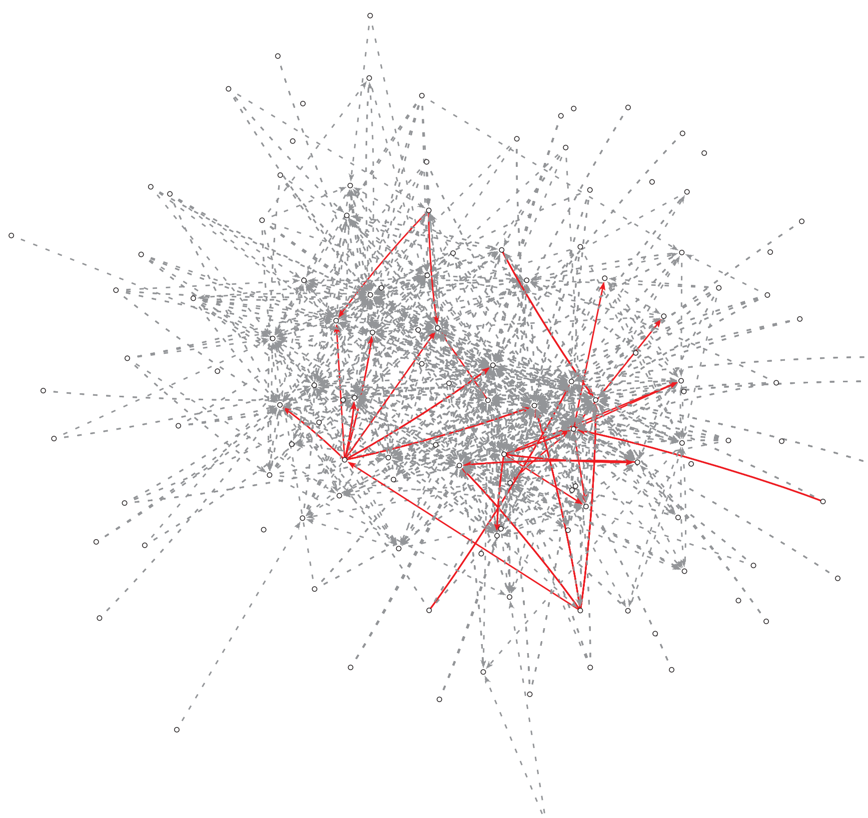


Figure 4.13: Subgraph of the five institutions with highest Contagion Index and their first and second-order neighbors in the network in June 2007. Non contagious exposures are dashed lines. Contagious exposures are full red lines.

such as the degree or the balance sheet size. Following chapter 3, we define the following indicators which attempt to quantify the sensitivity of the counterparties of these nodes to their default:

Definition 12 (Susceptibility coefficient)**.** *The susceptibility coefficient of a node* *is the maximal fraction of capital wiped out by the default of a single counterparty.*

$$\chi(i) = \max_{j \neq i} \frac{E_{ij}}{c(i)}$$

Definition 13 (Counterparty susceptibility)**.** *The counterparty susceptibility $CS(i)$ of a node i is the maximal (relative) exposure to node i of its counterparties:*

$$CS(i) = \max_{j, E_{ji}>0} \frac{E_{ji}}{c(j)}$$

Definition 14 (Local network frailty)**.** *The local network frailty $f(i)$ at node i is defined as the maximum, taken over counterparties exposed to i, of their exposure to i (in % of capital), weighted by the size of their interbank liability:*

$$f(i) = \max_{j, E_{ji}>0} \frac{E_{ji}}{c(j)} \quad L(j)$$

We refer the reader to chapter 3 for a detailed description of these indicators.

The analysis of the creditors of the five most systemic institutions in the Brazilian network in June 2007 (see table 4.5) indicates that the number of creditors and the size of interbank liabilities of the counterparties, as well as the counterparty susceptibility and local network frailty, can explain a high Contagion Index of a financial institution when the size of its interbank liabilities fails to explain. We observe that the five most systemic nodes have each at least one very connected counterparty with a large interbank liability size. They exhibit in general a high counterparty susceptibility and local network frailty.

Similarly to the study on a simulated network of chapter 3, we investigate in the Brazilian network in June 2007 whether we could rank systemically important institutions based on the measures of connectivity and centrality defined above. We

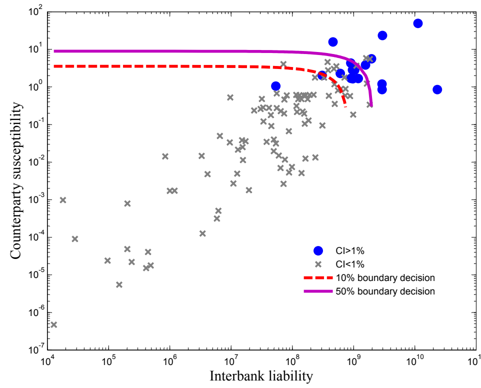
Ranking	$\max_{j,E_{ji}>0} k_{out}(j)$	$\max_{j,E_{ji}>0} L(j)$	$CS(i)$	$f(i)$
1	36	1.10	0.85	0.95
2	36	2.91	3.83	3.25
3	34	11.23	23.42	263.15
4	34	11.23	5.60	62.97
5	34	23.27	1.65	3.15
Network median	34	2.01	1.25	2.05
90%-quantile	36	11.23	3.04	6.89

Table 4.5: Analysis of the counterparties of the five most contagious nodes in June 2007. The counterparty interbank liability and local network frailty are expressed in billion BRL.

classify institutions into those with a high Contagion Index (higher than 1% of the total network capital) and those with a small Contagion Index (smaller than 1% of the total network capital), according to their interbank liability, counterparty susceptibility and local network frailty. This can be achieved by conducting a logistic regression of the indicator of the Contagion Index being higher than 1% of the total network capital once on the interbank liability and counterparty susceptibility, and once on the interbank liability and local network frailty. Figure 4.14 displays the decision boundaries at the probabilities 10% and 50% when observing once the interbank liability size and the counterparty susceptibility and once the interbank liability size and the local network frailty: a node outside the 10% decision boundary has an estimated probability of 10% to have a Contagion Index higher than 1% of the network capital; a node outside the 50% decision boundary has an estimated probability of 50% to have a Contagion Index higher than 1% of the network capital. We note that institutions with a high Contagion Index tend to have a large interbank liability, local network frailty and counterparty susceptibility.

The outputs of the logistic regression are summarized in table 4.6. We observe that the counterparty susceptibility and the local network frailty contribute significantly to the variability of the probability of observing a large Contagion Index: positive coefficients at the 1% significance level and a very high pseudo-R^2 [1].

[1]The Adjusted Pseudo-R^2 in a logistic regression is defined as $1 - logL(M)/logL(0)((n -$



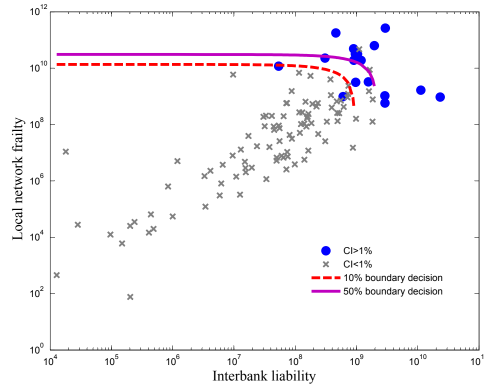


Figure 4.14: Counterparty susceptibility (left figure) and local network frailty (right figure) of the most systemic nodes (with a Contagion Index higher than 1% of the network capital) and the less systemic nodes (with a Contagion Index smaller than 1% of the network capital). Nodes above the 10% decision boundary have with 10% probability a Contagion Index higher than 1% of the network capital. The ones above the 50% decision boundary have with 50% probability a Contagion Index higher than 1% of the capital in the Brazilian system.

We also test for the differences in median between the counterparty susceptibility of the institutions with a Contagion Index higher than 1% of the total network capital and the counterparty susceptibility of those with a Contagion Index smaller than 1% of the total network capital. The Wilcoxon signed-rank test rejects the hypothesis of equal medians at the 1% level of significance. The median of the counterparty susceptibility of the institutions with a high Contagion Index (2.29) is significantly higher than the median of the counterparty susceptibility of the institutions with a small Contagion Index (0.06). Similarly, the median of the local network frailty of the institutions with a high Contagion Index (18.79 billion BRL) is significantly higher than the median of the local network frailty of the institutions with a small Contagion Index (0.02 billion BRL).

$1)/(n-k-1))$ where $logL(M)$ and $logL(0)$ are the maximized log likelihood for the fitted model and the null model, n is the sample size and k is the number of regressors.

Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(L) + \beta_2 \log(CS)$		
Coefficients	**Standard error**	**Adjusted Pseudo-R^2**
$\widehat{\beta_0}$ = -20.85**	7.96	93.46%
$\widehat{\beta_1}$ = 0.96*	0.39	
$\widehat{\beta_2}$ = 0.98*	0.40	
Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(L)$		
Coefficients	**Standard error**	**Adjusted Pseudo-R^2**
$\widehat{\beta_0}$ = -29.24**	7.11	94.54%
$\widehat{\beta_1}$ = 1.39**	0.34	
Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(CS)$		
Coefficients	**Standard error**	**Adjusted Pseudo-R^2**
$\widehat{\beta_0}$ = -1.46**	0.37	43.36%
$\widehat{\beta_1}$ = 1.31**	0.33	
Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(L) + \beta_2 \log(f)$		
Coefficients	**Standard error**	**Adjusted Pseudo-R^2**
$\widehat{\beta_0}$ = -43.20**	11.06	97.76%
$\widehat{\beta_1}$ = 1.05**	0.39	
$\widehat{\beta_2}$ = 0.97**	0.29	
Model: $logit(p(CI > 1\%)) = \beta_0 + \beta_1 \log(f)$		
Coefficients	**Standard error**	**Adjusted Pseudo-R^2**
$\widehat{\beta_0}$ = -21.32**	4.75	93.79%
$\widehat{\beta_1}$ = 0.95**	0.22	

* significant at 5% confidence level

** significant at 1% confidence level

Table 4.6: Marginal contribution of the interbank liabilities, counterparty susceptibility and local network frailty to the Contagion Index.

4.4 Does one size fit all? The case for targeted capital requirements

Controlling the risk of failures of financial institutions has been a major concern for financial regulation. Specifically, imposing a lower limit on the capital ratio has been the classical way to immunize institutions against contagion. Nonetheless, the observations made in the previous section highlight the fact that size alone does not explain contagion and point to the significant contribution of the counterparty susceptibility and local network frailty. Thus, imposing an upper bound on these variables could also help reducing systemic risk. We compare in this section the effects of two different policies for setting capital requirements:

- Minimum capital-to-exposure ratio: in this case, we require institutions to hold a capital larger than $\overline{c}$ that could cover at least a portion θ of their interbank exposures:

$$\overline{c}(i) = \max(c(i), \theta A(i)) \tag{4.3}$$

- Cap on susceptibility: Counterparty susceptibility (Definition 13) and local network frailty (Definition 14) are a significant source of systemic risk. Thus, preventing large values of counterparty susceptibility or network frailty from occuring can decrease systemic risk. This could be achieved by requiring that no exposure should represent more than a fraction γ of capital. In this case, a financial institution i is required to hold a capital larger than $\overline{c}$ given by:

$$\overline{c}(i) = \max(c(i), \frac{\max_{j \neq i}(E_{ij})}{\gamma}) \tag{4.4}$$

The study on a simulated network of chapter 3 have showed that *targeted capital requirements*, that consist in imposing higher capital requirements on institutions whose position in the network plays a key role in the network's resilience to contagion, can be a more efficient strategy to control the extent of contagion and systemic risk in a financial system. We check whether these findings hold in the Brazilian financial system.

We adopt the same approach as in chapter 3. We consider *targeted capital requirement* policy which consists in imposing capital requirements on the the 5% most systemic institutions in the network and their *creditors*.

We compare the situations in which capital requirements (i) are applied to all financial institutions in the network (*non-targeted immunization*), (ii) they are applied only to the creditors of the 5% most systemic institutions (*targeted acquaintance immunization*), (iii) they are applied only to the 5% most connected (out-degree) institutions (*targeted immunization*), (iv) they are applied only to the 5% institutions with the largest size (*targeted immunization*); by computing, in each case, the average of 5% largest Contagion Indexes i.e. the 5% tail conditional expectation of the cross sectional distribution of Contagion Index in the network.

Targeted capital requirements are observed to be more efficient in the sense that one can achieve the same reduction in systemic risk -in terms of the cross-sectional tail of the Contagion Index- with the same amount of capital, differently distributed across the network. Figure 4.15 shows that targeted capital requirements can achieve the same reduction in the size of default cascades while requiring less capital.

While it is clear that raising capital requirements reduces the number of defaults by contagion, the impact on the Contagion Index is the result of two competing effects. One has to bear in mind that increasing capital requirements will mainly increase the capital of the most fragile institutions, since those already well-capitalized satisfy the requirements without any additional capital. Thus, the proportion of total capital invested in fragile institutions increases, and consequently the Contagion

Index expressed in percentage of the total capital in the system may increase. In fact, we observe that the Contagion Index is decreasing when imposing these restrictions on the creditors of the 5% most systemic institutions (see figure 4.15), and globally decreasing when imposing these restrictions on all the institutions in the system. We also find that targeting the creditors of the most systemic nodes is a more efficient procedure to reduce the Contagion Index: for a same level of total capital the Contagion Index is smaller.

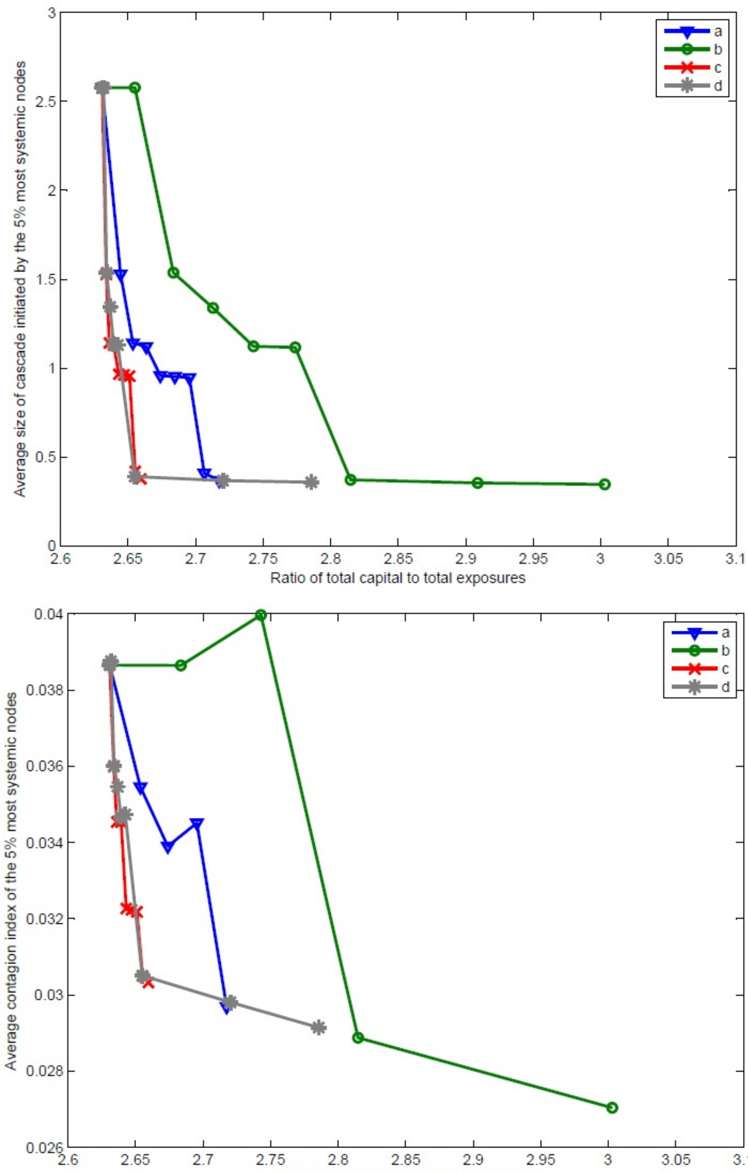


Figure 4.15: Comparison of various capital requirement policies: (a) imposing a floor on the capital ratio for all institutions in the network, (b) imposing a cap on the susceptibility for all institutions in the network, (c) imposing a floor on the capital ratio only for the creditors of the 5% most systemic institutions, (d) imposing a cap on the susceptibility only for the creditors of the 5% most systemic institutions.

Chapter 5

Reconstruction of Interbank Networks

This chapter is based on the paper "Reconstruction of interbank networks" (Cont and Moussa, 2010b), which is a joint work with Professor Rama Cont.

5.1 Introduction

In the two previous chapters, we have proposed a method for estimating contagion and systemic risk based on the knowledge of the entire matrix of bilateral exposures. However, in practice, bilateral exposures are most often not disclosed. Reconstructing bilateral exposures given only aggregate balance sheet data, such as the total assets and total liabilities of each institution, then becomes a question of interest.

A practical framework to address this problem is to model the financial network as a directed weighted graph with n nodes representing the financial institutions in the network, and weighted links representing their bilateral exposures which are summarized in a matrix $\mathbf{x}$. x_{ij} is an exposure of node i to node j, or in other words a liability of node j to node i. $a_i = \sum_{j=1}^{n} x_{ij}$ represents the total assets of institution i, and $l_j = \sum_{i=1}^{n} x_{ij}$ represents the total liabilities of institution j. Without loss of generality, we assume that the exposures are normalized so that $\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = 1$. This implies that the assets and liabilities are also normalized, that is $\sum_{i=1}^{n} a_i = 1$ and $\sum_{j=1}^{n} l_j = 1$.

The (normalized) assets and liabilities $(a_i, l_i)_{i=1,...,n}$ are observed in the accounting data of publicly traded firms. Given these observations, we are interested in *reconstructing* the bilateral exposures matrix $\mathbf{x}$ of the network. The reconstructed network should replicate the observed assets and liabilities. The reconstruction problem is then stated as follows.

Problem 1 (Reconstruction problem)**.** *Given a nonnegative vector* $\mathbf{a} = (a_i)_{i=1,...,n}$ *of the observed (normalized) assets and a nonnegative vector* $\mathbf{l} = (l_j)_{j=1,...,n}$ *of the observed (normalized) liabilities of the institutions in the network, determine a nonnegative* $n \times n$ *matrix* $\mathbf{x}$ *such that for all* $(i,j) \in \{1,...,n\}^2$,

$$x_{ii} = 0, \quad \textit{and}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad \sum_{i=1}^{n} x_{ij} = l_j \qquad \textit{(Balance sheet constraints)}$$

The reconstruction problem (Problem 1) involves $n^2 - n$ variables and $2n - 1$ equations, one degree of freedom being lost due the normalization of the assets and liabilities ($\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} l_j = 1$). Thus, for $n \geq 3$, the reconstruction problem is *ill-posed*. Therefore, if a solution to the reconstruction problem exists, it will not be unique. A possible way to ensure a unique solution is to chose a convex selection criterion and reformulate the reconstruction problem as a constrained convex optimization problem. A common choice for the selection criterion is the relative entropy, also called Kullback-Leibler divergence, that measures the distance of the candidate solution $\mathbf{x}$ to a prior matrix $\mathbf{x_0}$ (Csiszar, 1975):

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} \log \frac{x_{ij}}{x_{0ij}} \tag{5.1}$$

Similar problems have been studied in other fields of application under the name of *Matrix Balancing* problems (Schneider and Zenios, 1990). Given a matrix $\mathbf{x_0}$, the problem is to find a matrix $\mathbf{x}$ that is as close as possible to $\mathbf{x_0}$ and is *balanced* meaning that the row and column sums equal prespecified values (balancing constraints). The RAS algorithm (Schneider and Zenios, 1990) has been proposed to solve iteratively such problems. It has been shown that, if the balancing constraints are feasible, the RAS algorithm converges to the solution of the minimization of the relative entropy with respect to $\mathbf{x_0}$ subject to the balancing constraints (Schneider and Zenios, 1990; McDougall, 1999).

Empirical studies on systemic risk in financial networks (Sheldon and Maurer, 1998; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b; Mistrulli, 2007; Toivanen, 2009) have proposed reconstruction methods by minimizing the relative entropy with respect to the non-informative (uniform) prior $(a_i l_j)_{i,j=1,\ldots,n}$, subject to the constraints on the row and column sums. Most of them have used the RAS algorithm to solve this optimization. The choice of the non-informative prior is to reflect the lack of a priori information on the structure of the network. This leads to a reconstructed network where institutions are exposed as evenly as possible to each other. The reconstructed network is thus a complete graph that fails to exhibit the scale-free degree distributions observed in real-world interbank networks. Moreover, it has been noted that reconstructing bilateral exposures using the relative entropy with respect to the non-informative prior tends to introduce a bias when measuring systemic risk (Mistrulli, 2007). Upper (2011) suggests combining balance sheet data with other sources of data such as some information on the structure of the network (positions of the zeros, tiering, etc.) if available to reduce the bias.

A possible way to obtain a scale-free degree distribution for the reconstructed network is to minimize the relative entropy with respect to a *sparse* prior matrix that reflects our belief on the scale-free structure of the network. However, an arbitrary choice of such a prior matrix would lead to an arbitrary network where the existence or not of an exposure between two institutions is subjectively imposed by the prior. Thus, instead of considering a specific matrix $\mathbf{x_0}$ as prior, we assume in this chapter that the a priori knowledge about the network structure is given in the form of a *distribution*, such as the power law distributions of the in-degree and out-degree or the Pareto distribution of the exposures observed in the empirical studies of interbank networks (see Boss et al. (2004) for the Austrian network and chapter 2 for the Brazilian network). To incorporate the knowledge of these distributions, we consider a set of independent samples $\mathbf{x_0^{(1)}}, ..., \mathbf{x_0^{(M)}}$ (*set of beliefs*), drawn from the prior distribution on the network. Then, instead of finding a *point estimator* of the bilateral exposures matrix as in the literature, we derive a *distribution* $\left(\mathbf{x^{(1)}}, p_1\right), ..., \left(\mathbf{x^{(k)}}, p_M\right)$ of bilateral exposures matrices.

In section 5.2, we present a first approach to tackle this problem. We consider a prior distribution on the adjacency matrix $\mathbf{y} = 1_{\mathbf{x}>0}$ of the network, from which we simulate a set of independent samples $\mathbf{y^{(1)}}, ..., \mathbf{y^{(M)}}$ of adjacency matrices. Then, for each sample $\mathbf{y^{(k)}}$ we find the matrix $\mathbf{x^{(k)}}$ that solves the reconstruction problem stated in Problem 1. This leads to sample bilateral exposures matrices $\mathbf{x^{(1)}}, ..., \mathbf{x^{(M)}}$ each of them recovering the observed assets and liabilities. However, the existence of a solution to the reconstruction problem is not guaranteed in this case due to the sparse nature of the prior matrices. We propose then, in section 5.3, a second approach that is always feasible, in which we simulate a set of independent samples $\mathbf{x^{(1)}}, ..., \mathbf{x^{(M)}}$ of bilateral exposures matrices from a prior distribution on the bilateral exposures, and we weight them with probabilities (weights) $p_1, ..., p_M$ in such a way that the balance sheet constraints are satisfied *in the mean*, when averaging across all the sample bilateral exposures matrices. This is analogous to the Weighted Monte Carlo method introduced in Avellaneda et al. (2001) which is used to build and calibrate asset-pricing models for financial derivatives.

In section 5.4, we explain how to use this method to estimate a measure of systemic importance, such as the Contagion Index, when only aggregate levels of assets, liabilities and capital levels are known. In particular, we define the posterior distribution of the Contagion Index and its posterior mean. The posterior distribution of the Contagion Index allows to examine the impact of the network structure on the extent of contagion in the system.

Finally, in section 5.5, we assess the performance of this methodology in detecting systemically important institutions, vis à vis a ranking which segregates institutions solely in terms of their balance sheet size.

5.2 First approach: matching the balance sheet constraints sample by sample

5.2.1 Description of the method

In the first approach, we consider a prior distribution for the adjacency matrix $\mathbf{y}$ of the network. This prior distribution is chosen to reproduce the in- and out- degree distributions of real-world interbank networks. A possible choice is the preferential attachment model presented in section 2.4. We simulate M independent samples $\mathbf{y^{(1)}}, ..., \mathbf{y^{(M)}}$ of $\mathbf{y}$ from the assumed prior distribution.

We propose to build M independent samples $\mathbf{x^{(1)}}, ..., \mathbf{x^{(M)}}$ of bilateral exposures matrices that verify the constraints

$$x_{ij}^{(k)} = 0 \quad if \quad y_{ij}^{(k)} = 0, \quad \forall (i,j) \in \{1, ..., n\}^2 \tag{5.2}$$

which guarantees that the matrix $\mathbf{x^{(k)}}$ has the same (scale-free) in-degree and out-degree distribution as the matrix $\mathbf{y^{(k)}}$, and solve the reconstruction problem stated in Problem 1. The latter guarantees that the matrix $\mathbf{x^{(k)}}$ recovers the observed balance sheet assets and liabilities.

For each $k \in \{1, ..., M\}$, the problem of estimating the bilateral exposures matrix $\mathbf{x^{(k)}}$ of the network with adjacency matrix $\mathbf{y^{(k)}}$ involves $n^2 - n_k$ variables, where $n_k = card(\{(i,j) \in \{1, ..., n\}^2; y_{ij}^{(k)} > 0\})$, and $2n-1$ equations. If $n_k < n^2 - 2n + 1$, the problem is *ill-posed*.

To ensure uniqueness of a solution, when at least one exists, we use the relative entropy as a selection criterion (Csiszar, 1975; Egger and Engl, 2005). We reformulate the problem as a constrained convex optimization problem which is, for each $k \in \{1, ..., M\}$, find the matrix $\mathbf{x^{(k)}}$ that minimizes the relative entropy with respect to the non-informative prior matrix $\left(a_i l_j y_{ij}^{(k)}\right)_{ij}$ subject to the constraints 5.2 and the balance sheet constraints (Problem 1). The non-informative prior matrix reflects the lack of a priori knowledge on the distribution of bilateral exposures.

That is, for $k = 1, ..., M$, $\mathbf{x^{(k)}}$ solves the optimization problem

Problem 2.

$$\inf_{\mathbf{x} \in \mathcal{M}^+} H_k(\mathbf{x}) = \sum_{ij; y_{ij}^{(k)} > 0} x_{ij} \log \frac{x_{ij}}{a_i l_j} \tag{5.3}$$

subject to the constraints

$$\sum_{j; y_{ij}^{(k)} > 0} x_{ij} = a_i, \quad \forall i \in \{1, ..., n-1\} \tag{5.4}$$

$$\sum_{i;y_{ij}^{(k)}>0} x_{ij} = l_j, \quad \forall j \in \{1, ..., n-1\} \tag{5.5}$$

$$x_{ij}\left(1 - y_{ij}^{(k)}\right) = 0, \quad \forall (1, j) \in \{1, ..., n\}^2 \tag{5.6}$$

where $\mathcal{M}^+ = \left\{\mathbf{x} = (x_{ij})_{ij}\,; x_{ij} \geq 0 \quad \forall (i,j) \in \{1, ..., n\}^2 \quad \textit{and} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = 1\right\}$.

5.2.2 Solution to the optimization problem 2

To solve the constrained convex optimization problem 2, we introduce the Lagrangian:

$$L_k(\mathbf{x}, \lambda, \mu) = H(\mathbf{x}) - \sum_{i=1}^{n-1} \lambda_i \left[\sum_{j;y_{ij}^{(k)}>0} x_{ij} - a_i\right] - \sum_{j=1}^{n-1} \mu_j \left[\sum_{i;y_{ij}^{(k)}>0} x_{ij} - l_j\right]$$

for $\mathbf{x} \in \mathcal{M}^+$, and Lagrange multipliers $\lambda \in \mathbb{R}^{n-1}$ and $\mu \in \mathbb{R}^{n-1}$.

Problem 2 is then equivalent to the primal problem

Problem 3 (Primal problem)**.**

$$\inf_{\mathbf{x}\in\mathcal{M}^+} \sup_{(\lambda,\mu)\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} L_k(\mathbf{x}, \lambda, \mu) \tag{5.7}$$

We give a solution to the primal problem by using convex duality. The associated dual problem is

Problem 4 (Dual problem)**.**

$$\sup_{(\lambda,\mu)\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} \inf_{\mathbf{x}\in\mathcal{M}^+} L_k(\mathbf{x}, \lambda, \mu) \tag{5.8}$$

Proposition 3 (Condition for equivalence of primal and dual problems)**.** *If the system of linear equations,*

$$\sum_{j;y_{ij}^{(k)}>0} x_{ij} = a_i, \quad \forall i \in \{1, ..., n-1\}$$

$$\sum_{i;y_{ij}^{(k)}>0} x_{ij} = l_j, \quad \forall j \in \{1, ..., n-1\}$$

admits a positive solution, then the dual problem 4 admits a unique solution, which is also the unique solution of the primal problem 3.

Proof. Since $L_k(\mathbf{x}, \lambda, \mu)$ is strictly convex, and the constraints 5.24 and 5.25 are linear functions of λ, μ, Slater's condition applies (Boyd and Vandenberghe, 2004). That is, the primal and dual problems admit a same unique solution if there exists a λ, μ that satisfy the constraints 5.24 and 5.25. □

Proposition 4 (Solution to the dual problem 4). *For fixed Lagrange multipliers $(\lambda, \mu) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, the infimum of the Lagrangian in the variable $\mathbf{x} \in \mathcal{M}^+$ is given by*

$$\inf_{\mathbf{x}\in\mathcal{M}^+} L_k(\mathbf{x}, \lambda, \mu) = -\log Z_k(\lambda, \mu) + \sum_{i=1}^{n-1} \lambda_i a_i + \sum_{j=1}^{n-1} \mu_j l_j, \tag{5.9}$$

where,

$$Z_k(\lambda, \mu) = \sum_{ij;y_{ij}^{(k)}>0} a_i l_j \exp(\lambda_i + \mu_j) \tag{5.10}$$

$\inf_{\mathbf{x}} L_k(\mathbf{x}, \lambda, \mu)$ *is a convex function of λ and μ so its supremum can be found by gradient descent.*

The Jacobian of $Z_k(\lambda, \mu)$ is given by

$$\frac{\partial}{\partial \lambda_i} Z_k(\lambda, \mu) = \frac{1}{Z_k(\lambda, \mu)} \sum_{j;y_{ij}^{(k)}>0} a_i l_j \exp(\lambda_i + \mu_j) \tag{5.11}$$

$$\frac{\partial}{\partial \mu_j} Z_k(\lambda, \mu) = \frac{1}{Z_k(\lambda, \mu)} \sum_{i;y_{ij}^{(k)}>0} a_i l_j \exp(\lambda_i + \mu_j) \tag{5.12}$$

And its Hessian matrix

$$\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}Z_k(\lambda,\mu) = \frac{1}{Z_k(\lambda,\mu)^2}\left[\sum_{s;y_{is}^{(k)}>0} a_i l_s \exp(\lambda_i+\mu_s) \sum_{s;y_{js}^{(k)}>0} a_j l_s \exp(\lambda_j+\mu_s)\right]$$

$$\frac{\partial^2}{\partial\mu_i\partial\mu_j}Z_k(\lambda,\mu) = \frac{1}{Z_k(\lambda,\mu)^2}\left[\sum_{s;y_{si}^{(k)}>0} a_s l_i \exp(\lambda_s+\mu_i) \sum_{s;y_{sj}^{(k)}>0} a_s l_j \exp(\lambda_s+\mu_j)\right]$$

$$\frac{\partial^2}{\partial\lambda_i\partial\mu_j}Z_k(\lambda,\mu) =$$

$$\frac{1}{Z_k(\lambda,\mu)^2}\left[a_i l_j \exp(\lambda_i+\mu_j) - \sum_{s;y_{is}^{(k)}>0} a_i l_s \exp(\lambda_i+\mu_s) \sum_{s;y_{sj}^{(k)}>0} a_s l_j \exp(\lambda_s+\mu_j)\right]$$

Proof. We introduce $\mathbf{g}$ a perturbation of $\mathbf{x}$, $\mathbf{g}$ satisfies $\sum_{ij;y_{ij}^{(k)}>0} g_{ij} = 0$.

$$\frac{\partial}{\partial\epsilon}L_k(\mathbf{x}+\epsilon\mathbf{g},\lambda,\mu)|_{\epsilon=0} = \sum_{ij;y_{ij}^{(k)}>0} g_{ij}\left[\log\frac{x_{ij}}{a_i l_j} - \lambda_i - \mu_j\right] \tag{5.13}$$

Since $L_k(\mathbf{x},\lambda,\mu)$ is strictly convex, $\inf_{\mathbf{x}} L_k(\mathbf{x},\lambda,\mu)$ is attained if and only if

$$\frac{\partial}{\partial\epsilon}L_k(\mathbf{x}+\epsilon\mathbf{g},\lambda,\mu)|_{\epsilon=0} = 0. \tag{5.14}$$

That is, if and only if the vector $\left(\log\frac{x_{ij}}{a_i l_j} - \lambda_i - \mu_j\right)_{ij;y_{ij}^{(k)}>0}$ is orthogonal to the hyperplane

$$\mathcal{G}_k = \left((g_{ij})_{ij;y_{ij}^{(k)}>0};\ \sum_{ij;y_{ij}^{(k)}>0} g_{ij} = 0\right) \tag{5.15}$$

in the space $\mathbb{R}^{card(ij;y_{ij}^{(k)}>0)}$.

That is, if and only if there exists a constant $Z_k(\lambda,\mu)$ such that,

$$\log\frac{x_{ij}}{a_i l_j} - \lambda_i - \mu_j = \frac{1}{Z_k(\lambda,\mu)},\ \text{i.e. } x_{ij} = \frac{a_i l_j \exp(\lambda_i+\mu_j)}{Z_k(\lambda,\mu)} \tag{5.16}$$

Since $\sum_{ij;y_{ij}^{(k)}>0} x_{ij} = 1$, we get

$$Z_k(\lambda,\mu) = \sum_{ij;y_{ij}^{(k)}>0} a_i l_j \exp(\lambda_i+\mu_j) \tag{5.17}$$

It is easy to verify that

$$\inf_{\mathbf{x}\in\mathcal{M}^+} L_k(\mathbf{x},\lambda,\mu) = -\log Z_k(\lambda,\mu) + \sum_{i=1}^{n-1}\lambda_i a_i + \sum_{j=1}^{n-1}\mu_j l_j \tag{5.18}$$

□

5.2.3 Pros and cons

This method builds on the ideas proposed in the previous studies on reconstructing interbank networks (Sheldon and Maurer, 1998; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b; Mistrulli, 2007; Toivanen, 2009) but improves them in various aspects:

- These studies failed to reproduce the degree heterogeneity of interbank networks by estimating a bilateral exposures matrix that verifies the balance sheet constraints stated in Problem 1 and is as close as possible to a complete graph $(a_i l_j)_{i,j=1,..,n}$ that reflects the lack of a priori knowledge of the network structure. Our study, on the contrary, assumes a scale-free priori distribution on the network structure inspired from the empirical studies interbank network. Then, instead of computing a point estimator of the bilateral exposures matrix, we derive sample bilateral exposures matrices $\mathbf{x^{(1)}}, ..., \mathbf{x^{(M)}}$ each satisfying the balance sheet constraints and representing a network with scale-free in-degree and out-degree distributions.

- From the perspective of assessing systemic risk, which is the initial motivation of this study, this approach allows to compute any indicator of systemic importance –for example the Contagion Index introduced in chapter 3– in each of the estimated samples of the network. Thus, we can (1)-compute the sample mean as a point estimator of the Contagion Index and (2)-study the distribution of the Contagion Index across the different sample bilateral exposures matrices. Since all the samples recover the observed assets and liabilities, this allows to isolate the impact of the balance sheet size and the impact of the network structure on the extent of contagion.

However, this method suffers from some drawbacks:

- The existence of a solution to the reconstruction problem is not always guaranteed due to the sparse nature of the prior matrices. As an example, consider two institutions A and B with A having larger total liabilities l_A then the total assets a_B of institution B, that is $l_A > a_B$. If a sample adjacency matrix $\mathbf{y^{(k)}}$ attributes an out-degree of 1 to node A with a liability towards node B, then there is no way that node A recovers the observed liability l_A. This comes from the fact that since a liability of A towards B is an asset node B holds in A, a_B should be at least equal to l_A. We could handle this by keeping simulating samples $\mathbf{y^{(k)}}$ for which the constraints are feasible until having M of them (Accept-Reject method).

- This method involves M optimizations of dimension $2n-2$ each which might be prone to numerical issues.

- Finally, by generating sample bilateral exposures matrices as close as possible to the matrices $\left(a_i l_j y_{ij}^{(k)}\right)_{ij}$ for $k = 1, ..., M$, this method fails to reproduce the Pareto distribution of exposures observed in real-world interbank networks.

5.3 Second approach: matching the balance sheet constraints in the mean

5.3.1 Description of the method

The method we proposed in the previous section is not always feasible since in many cases, particulary those involving sparse sample adjacency matrices $\mathbf{y^{(1)}}, ..., \mathbf{y^{(M)}}$, there is no solution for the reconstruction problem. Thus, we propose in this section a method that is always feasible. The idea of this method is similar to the Weighted Monte Carlo method introduced by Avellaneda et al. (2001). It involves two steps:

1. First, we simulate M independent samples $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, ..., \mathbf{x}^{(M)}$ of bilateral exposures matrices, drawn from a relevant prior distribution on the network, such as a scale-free random network with power law distributions for the in-degree and out-degree and Pareto distribution for the exposures sizes. A possible way to generate these samples is for example by the preferential attachment model described in section 2.4.

2. Since these sample matrices do not achieve the balance sheet constraints, we weight them with probabilities (weights) $p_1, ..., p_M$ in such a way that the balance sheet constraints are achieved *in the mean* defined as follows.

   Definition 15. *(Balance sheet constraints in the mean)*

$$\mathbb{E}^{\mathbf{P}}\left[\sum_{j=1}^{n} x_{ij}\right] = \sum_{k=1}^{M} p_k \sum_{j=1}^{n} x_{ij}^{(k)} = a_i \tag{5.19}$$

$$\mathbb{E}^{\mathbf{P}}\left[\sum_{i=1}^{n} x_{ij}\right] = \sum_{k=1}^{M} p_k \sum_{i=1}^{n} x_{ij}^{(k)} = l_j \tag{5.20}$$

The problem of finding the weights $(p_1, ..., p_M) \in {\mathbb{R}_+}^M$ that verify the balance sheet constraints in the mean involves M variables and $2n - 2$ equations. Thus, for $M > 2n-2$, the problem is ill-posed. To ensure uniqueness of a solution we introduce a selection criterion that consists in minimizing the relative entropy with respect to the uniform prior distribution $\frac{1}{M},...,\frac{1}{M}$ subject to the balancing constraints in the mean. The choice of the uniform prior distribution is to reflect the lack of a priori knowledge on the network structure.

The reconstruction problem in this approach is then formulated as the following constrained convex optimization problem,

Problem 5 (Primal problem)**.**

$$\inf_{\mathbf{p}\in\mathbb{R}_+{}^M} H(\mathbf{p}) = \sum_{k=1}^{M} p_k \log Mp_k \tag{5.21}$$

subject to the constraints 5.19 and 5.20.

For sake of simplifying the notations, we define for $k = 1,...,M$ the vectors $C(k), D(k)$ by

$$C_i(k) = \sum_{j=1}^{n} x_{ij}^{(k)} - a_i \quad \text{for } i = 1..n-1 \tag{5.22}$$

$$D_j(k) = \sum_{i=1}^{n} x_{ij}^{(k)} - l_j \quad \text{for } j = 1..n-1 \tag{5.23}$$

The balance sheet constraints are then given by

$$C_i(k) = 0 \quad \text{for } i = 1..n-1 \tag{5.24}$$

$$D_j(k) = 0 \quad \text{for } j = 1..n-1 \tag{5.25}$$

and the balance sheet constraints in the mean are given by

$$\mathbb{E}^{\mathbf{P}}\left[\mathbf{C_i}\right] = \sum_{k=1}^{M} p_k C_i(k) = 0 \quad \text{for } i = 1..n-1 \tag{5.26}$$

$$\mathbb{E}^{\mathbf{p}}\left[\mathbf{D_j}\right] = \sum_{k=1}^{M} p_k D_j(k) = 0 \quad \text{for } j = 1..n-1 \tag{5.27}$$

Proposition 5 (Posterior mean of the bilateral exposures matrix)**.** *The posterior mean* $\widehat{\mathbf{x}}$ *of the bilateral exposures matrix* $\mathbf{x}$*,*

$$\widehat{\mathbf{x}} = \mathbb{E}^{\mathbf{p}}\left[\mathbf{x}\right] = \sum_{k=1}^{M} p_k \mathbf{x}^{(\mathbf{k})} \tag{5.28}$$

satisfies the balance sheet constraints of the reconstruction problem.

The matrix $\widehat{\mathbf{x}}$ would be a natural candidate for a point estimator of the bilateral exposures matrix. However, it might fail to represent a network with power law in-degree and out-degree distributions, but would represent a complete graph. Thus, $\widehat{\mathbf{x}}$ suffers from the same drawback as the previous studies (Sheldon and Maurer, 1998; Upper and Worms, 2004; Wells, 2004; Elsinger et al., 2006a,b; Mistrulli, 2007; Toivanen, 2009). Our contribution is, on the contrary, to provide a distribution of bilateral exposures matrices as opposed to a point estimator.

5.3.2 Solution to the optimization problem 5

To solve the constrained convex optimization problem 5 we introduce the Lagrangian

$$L(\mathbf{p}, \lambda, \mu) = H(\mathbf{p}) - \sum_{i=1}^{n-1} \lambda_i \mathbb{E}^{\mathbf{p}}[\mathbf{C_i}] - \sum_{j=1}^{n-1} \mu_j \mathbb{E}^{\mathbf{p}}[\mathbf{D_j}]$$

for $\mathbf{p} \in \mathbb{R}_+^M$, and Lagrange multipliers $\lambda \in \mathbb{R}^{n-1}$ and $\mu \in \mathbb{R}^{n-1}$.

We give a solution to the primal problem by using convex duality. The associated dual problem is

Problem 6 (Dual problem)**.**

$$sup_{(\lambda,\mu)\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} \inf_{\mathbf{p}\in\mathbb{R}_+^M} L(\mathbf{p},\lambda,\mu) \tag{5.29}$$

Proposition 6 (Condition for existence of a solution and equivalence of primal and dual problems)**.** *If the* $(2n-2)\times 1$ *null vector is in the convex envelop of* $(C_1(k),...,C_{n-1}(k),D_1(k),...,D_{n-1}(k))_{k=1,...,M}$ *then the dual problem 6 admits a unique solution, which is also the unique solution of the primal problem 5.*

Proof. Since $H(\mathbf{p})$ is a strictly convex function, and the constraints 5.19 and 5.20 are linear functions of $\mathbf{p}$, Slater's condition applies (Boyd and Vandenberghe, 2004). That is, the primal and dual problems admit a same unique solution if there exists a vector $\mathbf{p}$ that satisfies the constraints 5.19 and 5.20, i.e. the $(2n-2)\times 1$ null vector is in the convex envelop of the samples of the constraints. □

Proposition 7 (Solution to the dual problem 6)**.** *For fixed Lagrange multipliers* $(\lambda,\mu)\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$*, the supremum of the Lagrangian in the variable* $\mathbf{p}\in\mathbb{R}_+^M$ *is given by*

$$\inf_{\mathbf{p}\in\mathbb{R}_+^M} L(\mathbf{p},\lambda,\mu) = -\log Z(\lambda,\mu) \tag{5.30}$$

where,

$$Z(\lambda,\mu) = \frac{1}{M}\sum_{k=1}^{M}\exp(\sum_{i=1}^{n-1}\lambda_i C_i(k) + \sum_{j=1}^{n-1}\mu_j D_j(k)) \tag{5.31}$$

$\inf_{\mathbf{p}\in\mathbb{R}_+^M} L(\mathbf{p},\lambda,\mu)$ *is a convex function of* λ *and* μ *so its supremum can be found using classical convex optimization methods.*
The Jacobian of $\log Z(\lambda,\mu)$ *is given by*

$$\frac{\partial}{\partial\lambda_i}\log Z(\lambda,\mu) = \mathbb{E}^{\mathbf{p}}[\mathbf{C_i}] = \sum_{k=1}^{M} p_k C_i(k) \tag{5.32}$$

$$\frac{\partial}{\partial\mu_j}\log Z(\lambda,\mu) = \mathbb{E}^{\mathbf{p}}[\mathbf{D_j}] = \sum_{k=1}^{M} p_k D_j(k) \tag{5.33}$$

And its Hessian matrix

$$\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}\log Z(\lambda,\mu) = \mathbf{Cov^p}(\mathbf{C_i},\mathbf{C_j}) = \sum_{k=1}^{M} p_k C_i(k)C_j(k) - \sum_{k=1}^{M} p_k C_i(k)\sum_{k=1}^{M} p_k C_j(k)$$

$$\frac{\partial^2}{\partial\mu_i\partial\mu_j}\log Z(\lambda,\mu) = \mathbf{Cov^p}(\mathbf{D_i},\mathbf{D_j}) = \sum_{k=1}^{M} p_k D_i(k)D_j(k) - \sum_{k=1}^{M} p_k D_i(k)\sum_{k=1}^{M} p_k D_j(k)$$

$$\frac{\partial^2}{\partial\lambda_i\partial\mu_j}\log Z(\lambda,\mu) = \mathbf{Cov^p}(\mathbf{C_i},\mathbf{D_j}) = \sum_{k=1}^{M} p_k C_i(k)D_j(k) - \sum_{k=1}^{M} p_k C_i(k)\sum_{k=1}^{M} p_k D_j(k)$$

Proof. We introduce a perturbation $\mathbf{q}$ that verifies $\sum_{k=1}^{M} q_k = 0$.

$$\frac{\partial}{\partial\epsilon}L(\mathbf{p}+\epsilon\mathbf{q},\lambda,\mu)|_{\epsilon=0} = \sum_{k=1}^{M} q_k\left[\log Mp_k - \sum_{i=1}^{n-1}\lambda_i C_i(k) - \sum_{j=1}^{n-1}\mu_j D_j(k)\right] \tag{5.34}$$

At a given λ and μ, a solution to the inner minimization is

$$\widetilde{p}_k = \frac{1}{MZ(\lambda,\mu)}\exp(\sum_{i=1}^{n-1}\lambda_i C_i(k) + \sum_{j=1}^{n-1}\mu_j D_j(k)) \tag{5.35}$$

with,

$$Z(\lambda,\mu) = \frac{1}{M}\sum_{k=1}^{M}\exp(\sum_{i=1}^{n-1}\lambda_i C_i(k) + \sum_{j=1}^{n-1}\mu_j D_j(k)) \tag{5.36}$$

$\square$

Which implies,

$$L(\widetilde{\mathbf{p}},\lambda,\mu) = -\log Z(\lambda,\mu) \tag{5.37}$$

5.3.3 Pros and cons

This method is always feasible as long as the number M of simulated sample bilateral exposures matrices is chosen sufficiently large so that the $(2n-2)\times 1$ null vector is in the convex envelop of $(C_1(k),...,C_{n-1}(k),D_1(k),...,D_{n-1}(k))_{k=1,...,M}$.

This method guarantees that the constraints are satisfied in the mean, that is by averaging across all samples of the bilateral exposures matrix. Moreover, the networks associated to the sample bilateral exposures matrices exhibit heavy-tailed degree *and* exposures distributions as observed in real-world interbank networks.

On the computational side, this method requires only one optimization of dimension $2n-2$ to estimate the a posteriori weights.

Therefore, we recommend this approach as an estimation method of the posterior distribution of the bilateral exposures matrix.

5.4 Application: computing the Contagion Index

Several measures of systemic importance requiring the knowledge of the entire network structure have been proposed in the literature on default contagion and systemic risk in financial networks. We explain in this section how the estimation methodology we presented above could be used to compute these indicators when only the total assets, total liabilities and capital level of each institution in the network are known. As an example, we consider the Contagion Index introduced in chapter 3.

We estimate the posterior distribution $\left(\mathbf{x}^{(\mathbf{1})}, p_1\right), ..., \left(\mathbf{x}^{(\mathbf{M})}, p_M\right)$ of the bilateral exposures matrix according to the estimation procedure described in section 5.3.

The posterior distribution of the Contagion Index of an institution i in the network is defined as follows.

Definition 16 (Posterior distribution of the Contagion Index). *The posterior distribution of the Contagion Index of an institution i in the network with known assets*

$\mathbf{a}$*, liabilities* $\mathbf{l}$*, and capital levels* $\mathbf{c}$ *is*

$$\left(CI(i, \mathbf{c}, \mathbf{x^{(1)}}), p_1\right), ..., \left(CI(i, \mathbf{c}, \mathbf{x^{(M)}}), p_M\right) \tag{5.38}$$

where $(\mathbf{x^{(1)}}, p_1), ..., (\mathbf{x^{(1)}}, p_M)$ *is the posterior distribution of the bilateral exposures matrix determined according to the estimation method proposed in section 5.3, and* $CI(i, \mathbf{c}, \mathbf{x^{(k)}})$ *is the Contagion Index of institution* i *in the network with bilateral exposures matrix* $\mathbf{x^{(k)}}$ *and capital levels* $\mathbf{c}$*.*

A natural estimator $\widehat{CI}$ of the Contagion Index is its posterior mean, defined as

Definition 17 (Posterior mean of the Contagion Index)**.** *The posterior mean of the Contagion Index of an institution* i *in the network with known assets* $\mathbf{a}$*, liabilities* $\mathbf{l}$*, and capital levels* c *is*

$$\widehat{CI} = \mathbb{E}^{\mathbf{p}}[CI(i, \mathbf{c}, \mathbf{x})] = \sum_{k=1}^{M} p_k CI(i, \mathbf{c}, \mathbf{x^{(k)}}) \tag{5.39}$$

where $\left(CI(i, \mathbf{c}, \mathbf{x^{(1)}}), p_1\right), ..., \left(CI(i, \mathbf{c}, \mathbf{x^{(M)}}), p_M\right)$ *is the posterior distribution of the Contagion Index of institution* i*.*

The posterior distribution helps studying the effect of the network structure on the extent of contagion in the system, as well as identifying "worse-case" scenarios of network structures that lead to an increase of the risk of contagion. This is an important exercise for regulators to control the architecture of the financial network in such a way to limit contagion and systemic risk.

5.5 Detecting systemically important institutions

To illustrate this method, we consider a directed-scale free network of 20 nodes and average degree of 6, simulated according to the preferential attachment model presented in section 2.4, with power law (Zipf's law) distributions of the in-degree, out degree and Pareto distribution of exposures with respective tail exponents 2.1, 3, and 1.9. The graph illustrating this network is presented in figure 5.1. Then, we suppose that only the total assets, total liabilities and capital level of each institution in the network are known and we derive the posterior distribution of the bilateral exposures matrix given only these observations. We simulate $M = 5000$ independent sample bilateral matrices , and find their posterior distribution $(\mathbf{x^{(1)}}, p_1), ..., (\mathbf{x^{(1)}}, p_M)$ in such a way to recover the total assets and liabilities in the mean. A zoom on the 200 largest posterior probabilities is presented in figure 5.2.

To check the success of the method in recovering the assets and liabilities in the mean, we compare the posterior mean of the total assets of each institution in the reconstructed network, to its total assets in the real network. We find a very good fit of the balance sheet assets and liabilities in the mean as shown in figure 5.3.

In the following sections we derive the posterior distribution of the Contagion Index. We examine the examples of a network in which there is no room for much contagion, and the example of a network where contagion can have a significant extent. This will allow us to study the effect of both the balance sheet size and the network structure on the Contagion Index.

Example 1

We first consider a network in which institutions have a capital ratio of 11%. We derive the posterior distribution of the Contagion Index and its posterior mean. We find that the Contagion Index can significantly exceed the liabilities (see figure 5.4) indicating the existence of contagion in the network.

We also find that the posterior mean of the Contagion Index does not estimate very well the Contagion Index of the real network (see table 5.1). However, it has the

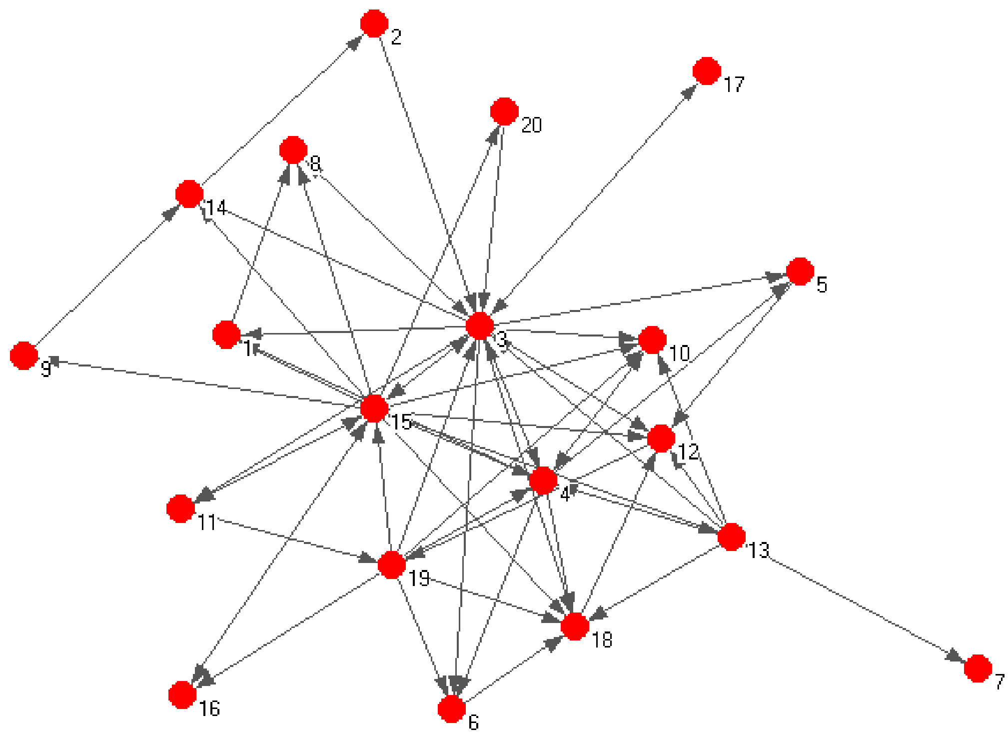


Figure 5.1: Scale-free graph of 20 nodes and average degree of 6.

same ranking as the Contagion Index of the real network. Therefore, the posterior mean could be used to detect systemically important institutions in the network by looking at their ranking in terms of the posterior mean of the Contagion Index. It is interesting to note that the posterior mean tends to underestimate the Contagion Index for the most systemic institutions, however, it is a good estimator of the Contagion Index of the least systemic ones. This is due to the fact that the least systemic institutions do not trigger in general additional rounds of default when they fail. Thus, the loss they transmit to the network is limited to their balance sheet liabilities. Since the reconstruction method recovers the interbank liabilities in the mean, the posterior mean of the Contagion Index is a good estimator of its true

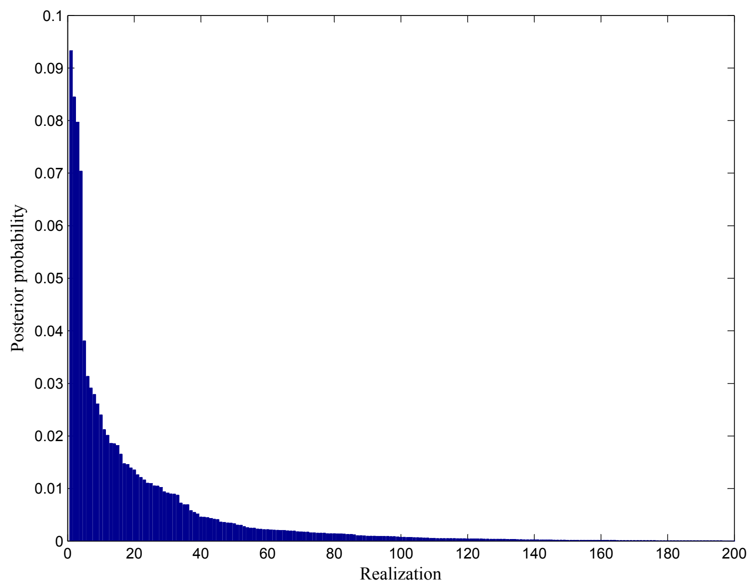


Figure 5.2: Bar plot of the 200 largest a posteriori probabilities, the realizations being labeled by their decreasing rank in the a posteriori probability.

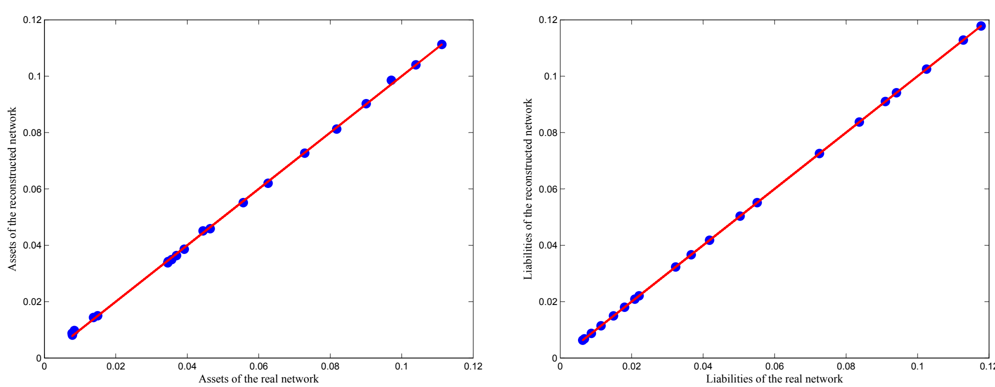


Figure 5.3: Goodness of fit: assets and liabilities in the real network versus assets and liabilities in the reconstructed network.

value. However, the most systemic institutions trigger additional rounds of defaults, thus the loss they transmit to the network exceeds their balance sheet liabilities and propagates across the network in a domino fashion. Thus, it depends on the *interconnectedness* or specific structure of the network. Then the posterior mean may not be a good estimator of the Contagion Index in this case. This highlights the significant effect of the network structure on the extent of contagion. In fact, we can observe this effect by examining the posterior distribution of the Contagion Index. Figure 5.5 presents the posterior distribution of the Contagion Index for the two most and two least systemic institutions. We find that the Contagion Index of the two most systemic institutions exhibits a larger variance than the Contagion Index of the two least systemic ones, which confirms the previous observations. Finally, it is interesting to note that for the systemically important institutions the posterior mean of Contagion Index follows the same ranking as the Contagion Index of the real network, but does not follow the same ranking as the liabilities, which indicates that estimating the Contagion Index according to the methodology we presented in the previous sections provides additional information than the balance sheet size to detect systemically important institutions.

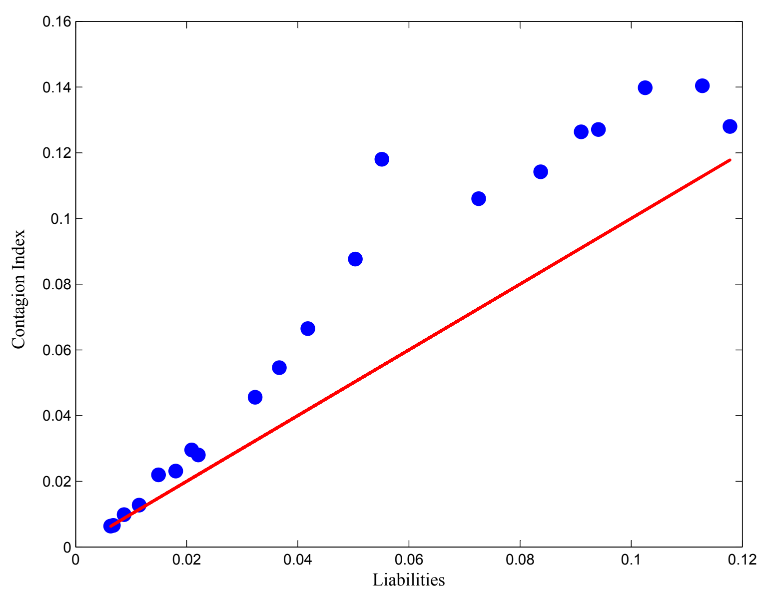


Figure 5.4: Evidence for contagion: interbank liabilities versus Contagion Index.

Node	1	2	3	4	5	6
Liability (%)	11.28	10.25	9.41	11.78	9.10	8.37
CI from true data (%)	27.64	26.78	26.64	26.01	9.58	8.60
Posterior mean $\widehat{CI}$ (%)	14.03	13.98	12.70	12.79	12.63	11.42
Standard error (%)	3.94	3.91	3.52	4.43	3.52	4.38
Node	15	16	17	18	19	20
Liability (%)	2.20	1.80	1.49	1.14	0.87	0.68
CI from true data (%)	2.00	1.71	1.34	1.08	0.78	0.72
Posterior mean $\widehat{CI}$ (%)	2.80	2.30	2.19	1.27	0.98	0.65
Standard error (%)	2.25	1.34	1.34	0.77	0.84	0.16

Table 5.1: Posterior mean of the Contagion Index of the six most and six least systemic nodes. The nodes are labeled by decreasing ranking in terms of their Contagion Index in the real network.

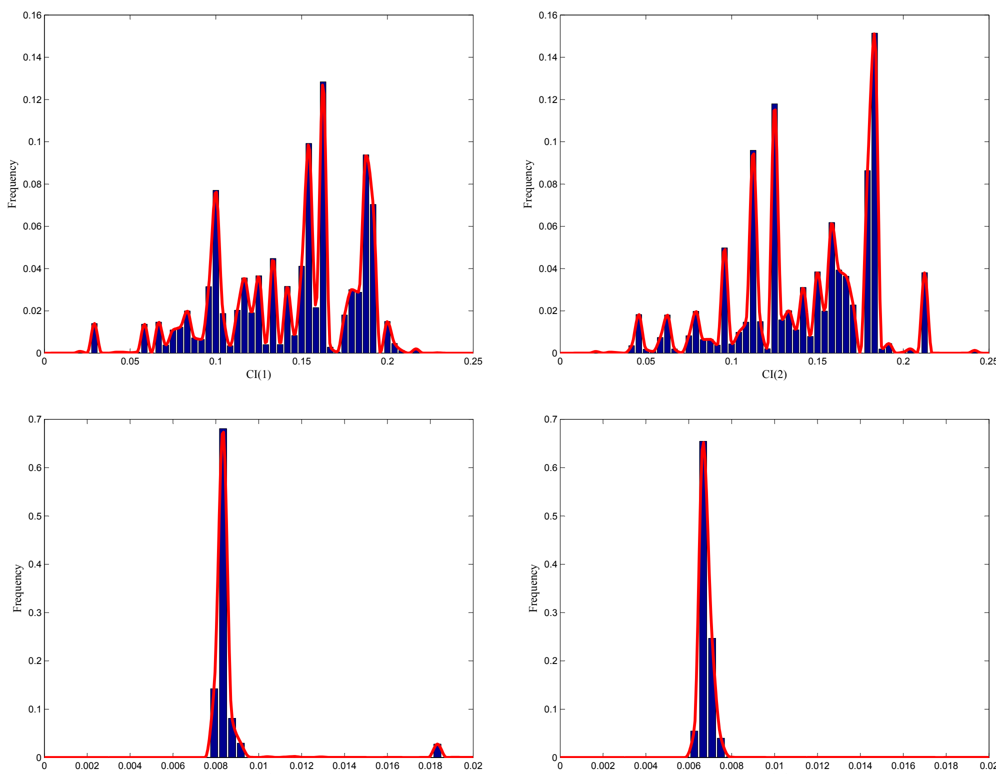


Figure 5.5: Posterior distribution of the Contagion Index of the two most and two least systemic nodes (in terms of their ranking in the real network).

Example 2

We consider now the case of a network in which institutions have a capital ratio of 30%. We do a similar analysis as in the previous example. We find in this case that the posterior mean is a very good estimator of the Contagion Index. This could be due to the fact that there is few contagion in the network (see figure 5.6), the Contagion Index is almost equal to the liabilities meaning that there is no additional rounds of defaults in the network. Since the liabilities are recovered in the mean, it is not so surprising that the posterior mean is a good estimator of the Contagion Index in this case.

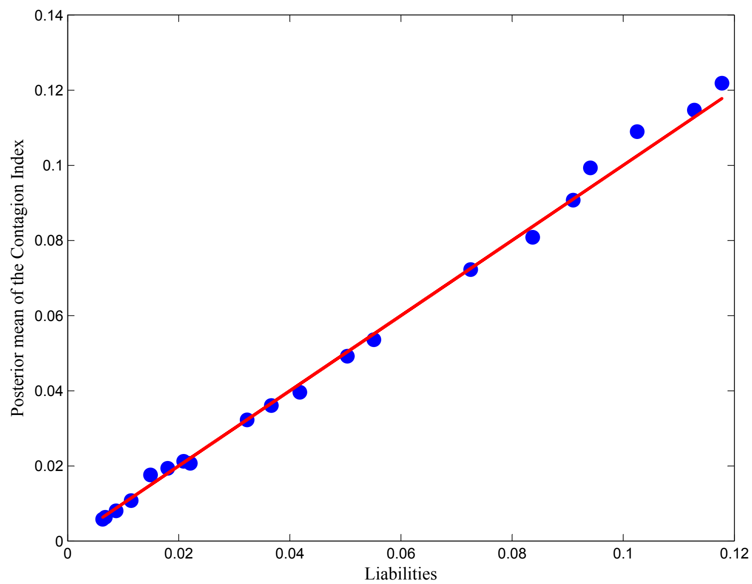


Figure 5.6: Evidence for lack of contagion: interbank liabilities versus Contagion Index.

Node	1	2	3	4	5	6
Liability (%)	11.78	11.28	10.25	9.41	9.10	8.37
CI from true data (%)	12.05	10.37	9.40	8.63	8.30	7.58
Posterior mean $\widehat{CI}$ (%)	12.18	11.46	10.89	9.93	9.07	8.08
Standard error (%)	3.33	2.12	2.63	2.39	2.12	1.75
Node	15	16	17	18	19	20
Liability (%)	1.80	1.49	1.14	0.87	0.68	0.63
CI from true data (%)	1.69	1.36	1.10	0.78	0.67	0.57
Posterior mean $\widehat{CI}$ (%)	1.93	1.75	1.07	0.80	0.63	0.58
Standard error (%)	0.93	1.03	0.08	0.05	0.01	0.01

Table 5.2: Posterior mean of the Contagion Index of the six most and six least systemic nodes. The nodes are labeled by decreasing ranking in terms of their Contagion Index in the real network.

5.6 Conclusion

This study leads to several observations:

In networks that are prone to contagion, the posterior mean of the Contagion Index does not estimate well the Contagion Index of the real network. This implies that a knowledge of aggregate levels of assets, liabilities, and capital levels does not suffice to accurately estimate the Contagion Index. Therefore, there is a need to disclose large bilateral exposures in order to have an accurate assessment of contagion and systemic risk.

Nevertheless, the posterior mean of the Contagion Index is found to follow the same ranking as the Contagion Index of the real network while the size of liabilities does not follow this ranking. Thus, the reconstruction methodology we presented in this study could provide regulators with a tool for detecting systemically important institutions given only aggregate information on their balance sheet assets, liabilities and capital levels. This methodology could perform better, particularly in networks where contagion may have a significant extent, than a ranking based on balance sheet size which does not account for the interconnectedness of the financial system. We refer the interested reader to chapters 3 and 4 for a statistical analysis of the sensitivity of contagion and systemic risk to the network structure.

Chapter 6

A Closer Look at Credit Ratings of CDO Tranches

This chapter is based on the paper "A Closer Look at Credit Ratings of CDO Tranches" (Cont and Moussa, 2010a), which is a joint work with Professor Rama Cont.

6.1 Introduction

Collateralized Debt Obligations (CDOs) are structured credit products referencing a portfolio of fixed income securities. The originator of the portfolio creates a Special Purpose Vehicle (SPV) that acquires the portfolio and issues notes to the investors in the form of *tranches*. An investor in a CDO tranche receives regular premium payments and must pay in return the losses generated by that tranche when default occurs in the reference portfolio. Each tranche has priorities concerning payments and losses: the more senior the tranche, the higher the priority on receiving payments and the lower the priority on incurring losses. Thus, more senior tranches are safer investments, as a result they receive smaller premium payments. The Equity tranche, also known as First-Loss tranche, is the first to absorb the losses of the portfolio and the last to receive premium payments, followed by Mezzanine then Senior tranches. The latest may be affected only when the more junior tranches are completely lost. This prioritization of losses and payments, also called *subordination*, is a form of internal credit enhancement of the issued notes. More subordinated tranches serve as protective layers to the more senior tranches. The level of subordination, also called attachment level or credit enhancement level, is the determinant of the seniority of the tranche.

Various types of CDOs have been issued in the past decade, differing by the nature (vintage) of the assets pooled in their reference portfolio: loans, bonds, credit derivatives, structured finance products such as residential mortgages, and even other asset backed securities (ABS) such as residential mortgage backed securities (RMBS) or credit default swaps (CDS) or even CDO tranches.

CDOs have gained in popularity for a variety of reasons. For CDO issuers, tranching allows to create classes of securities whose rating is higher than the average rating of the underlying assets, through credit enhancement. This allows issuers to pay smaller spreads than they would pay if they had to sell the underlying assets individually. Furthermore, selling part of their assets to the SPV allows them to free their balance sheets which leads to lower capital requirements and enables them to acquire new assets. CDOs play also the role of risk transfer instruments that transfer credit and market risks from the issuer to the investor. For CDO investors, the "tranching" permits satisfaction for different risk appetites: investors seeking high payoff may leverage their credit risk and returns by investing in the Equity tranches while risk-averse investors seeking a lower risk profile may invest in the Mezzanine and Senior tranches. Moreover, the pooling of well-diversified assets reduces idiosyncratic risk which can be appealing to investors.

However, the massive downgrades witnessed in the 2007-2009 crisis have shed light on the complexity of the risk embedded in CDO tranches. The relationship between individual default events in the underlying pool of assets and the nominal loss of a given CDO tranche is complex due to both the complexity of the collateral and the prioritization of payments. CDO tranches exhibit a risk profile that sensibly differs from those of other categories of defaultable fixed-income assets. They have been nevertheless rated by the main rating agencies on a scale similar to the one used for corporate bonds, which can be misleading to the uninformed investor who could tend to believe that a similar rating for a CDO tranche and a corporate bond would indicate a similar risk profile. The complex structure of CDOs cautions against any simple interpretation of their ratings. Accordingly, different rating scales should be used for CDOs and corporate bonds.

Credit ratings are labels issued by the rating agencies as an indication of the credit risk of debt securities. For example Standard & Poor's issues the labels AAA, AA+, AA, AA-, A+ , etc. and Moody's issues the labels Aaa, Aa1, Aa2, Aa3, A1, etc. Securities rated above BBB- (S&P) or Baa1 (Moody's) are called *investment grade* securities and those rated below BBB- or Baa1 are called *speculative* or *high yield* or *junk* securities. Securities with very deteriorated credit quality are not rated (NR category). Rating agencies use different measures of risk to produce their ratings. Ratings issued by Standard & Poor's and Fitch measure the probability that the tranche incurs a loss -sometimes referred to as *default probability*[1]- and depends only on the subordination level of the CDO tranche, while ratings issued by Moody's measure the expected loss. This measure is compared to given cut-

[1]This can be confusing since this is a measure of the probability of the tranche incuring a loss and not the probability of the tranche being completely lost. In the case of Bonds, incurring a loss is a default event so the term *default probability* is accurate, however it is not in the case of structured finance securities.

offs (also called *rating quantiles*) according to which a rating category (label) is attributed. Until 2007 the cut-offs used for CDO tranches were the same as the one used for corporate bonds that were determined from historical data. Since the onset of the crisis in 2007, rating agencies defined cut-offs specifically for CDO tranches based on both historical data and a quantitative and qualitative analysis (de Servigny and Jobst, 2007).

This study aims at clarifying some misconceptions related to CDO ratings, their interpretation and their use. We first describe the rating approaches used by major agencies for CDOs and other structured credit products. Most rating agencies have been using static factor models which are slight variations on the Gaussian copula model allowing for intersector and intrasector correlations. Using this framework, we explore the particularities of credit rating methodologies when applied to structured products such as CDO tranches and CDOs of ABS.

6.1.1 Summary of main results

Our study leads to several interesting insights on the dynamics of credit ratings for CDO tranches.

- Given the leveraged nature of CDOs, the downgrade risk of a CDO tranche can be quite different from a bond with same initial rating. Therefore, a simple labeling via default probability or expected loss does not discriminate sufficiently their downgrade risk.
- The interaction between the rating threshold and the structuring procedure can cause new issues to have tranches structured "at the limit" of rating categories and thus a high probability of downgrading.
- Migration probabilities for tranches with similar rating can vary from struc-

ture to structure even for the same underlying debt portfolio: two tranches with the same rating can have completely different transition probabilities.

- We show that migration probabilities for CDO tranches are path-dependent and non-homogeneous in time.

- While default probability is an adequate representation of the default risk of a corporate bond with known recovery rate, we show that the probability to be hit by default fails to account for the risk carried by CDO tranches and can not differentiate between tranches with different risk profiles, while a risk measure applied to the loss distribution can. Based on these findings, we suggest that different rating scales should be used for corporate bonds and CDO tranches as the latter carry a more complex default risk.

- As a solution to some of the drawback of the current rating methodologies, we propose a risk-based rating system, based on a risk measure applied to the loss distribution of the tranche. We show that such a risk-based approach can lead to quite different ratings for CDO tranches.

- Estimating default probabilities of CDO tranches requires a statistical model for the co-dependence structure of the default times. The rating of a CDO arbitrarily assumes a given model, unlike the rating of a corporate bond which is determined by the accounting data of the issuing company. We show in particular that a change in the correlation assumption can lead to multiple downgrades of senior tranches.

- The Gaussian copula commonly used to model the joint distribution of the underlying assets default times fails to generate scenarios with default clusters. Other copulae, such as the Cauchy, can model scenarios with default clustering, thus leading to additional credit enhancement than required with the Gaussian copula. This improves the stability of credit ratings.

In light of these findings, we present a set of recommendations for the design, interpretation and use of credit ratings for CDOs and other structured products.

6.1.2 Relation to the literature

Credit ratings assigned by rating agencies to structured credit products have played an important role in the development of the structured credit market, which has been cast into the limelight by the recent credit crisis. Several critics of rating methodologies have focused on the drawbacks of the Gaussian copula used to model the dependence structure between the assets default times (Donnelly and Embrechts, 2010). However, very few studies have examined the issues that do not arise from the use of the Gaussian copula model but are related to the specific characteristics of structured products.

We study in this chapter the particularities of credit rating methodologies when applied to structured products such as CDO tranches and CDOs of ABS. Using the Gaussian copula framework, we explore several issues related to ratings of CDO tranches.

Credit ratings have been attributed by major rating agencies to structured credit products based on the criterion of default probability or expected loss of these instruments (Standard & Poor's, 2007). These metrics are very poor indicators of the credit risk of such complex instruments which leads to situations of "rating arbitrage" where high ratings can be attributed to "high-yield" instruments which carry in fact a lot of risk (Cont and Jessen, 2011).

We show that the default probability fails to segregate tranches with different risk profiles. Since a rating based on default probability depends solely on the subordination level of the tranche, two tranches with completely different risk profiles can nevertheless have the same rating. For example, default probability can not differentiate between a tranche $6\% - 7\%$ and a tranche $6\% - 100\%$ while the loss of the latter is always smaller relatively to the initial notional of the tranche. We propose instead a risk-based rating procedure, based on a risk measure applied to the loss distribution of the tranche.

We show that the downgrade risk of a CDO tranche can be quite different from a bond with same initial rating. CDO tranches are levered products, meaning that a slight deterioration in the credit quality of the underlying obligors can lead to downgrades of several notches at the level of a CDO tranche. We also show, that credit ratings of CDO tranches are non Markov processes and non-homogeneous processes. Nevertheless, rating agencies still assume until today that ratings transitions are Markov and homogeneous, suggesting to compute transition probabilities by raising the annual transition matrix to the iterative powers (de Servigny and Jobst, 2007). Therefore, a simple labeling via default probability or expected loss does not discriminate sufficiently their downgrade risk. We propose to supplement ratings with indicators of downgrade risk.

Finally, we also examine the *model risk* to which credit ratings are exposed. In line with previous studies on the sensitivity of credit ratings of CDO tranches to modeling assumptions (Fender and Kiff, 2004; Meng and Sengupta, 2009; Wojtowicz, 2011), this study shows that credit ratings of CDO tranches are extremely sensitive to the correlation, recovery, and assets default rates parameters used to model the joint distribution of default times. The Gaussian copula has been criticized in the recent crisis for its failure to generate scenarios with default clusters (Donnelly and Embrechts, 2010). Several alternatives to the factor Gaussian copula model have been proposed in the literature, but always the context of pricing CDO tranches (Donnelly and Embrechts, 2010; Azizpour et al., 2010; Kalemanova et al., 2005; Duffie et al., 2009; Peng and Kou, 2009). We examine the effect of using a copula with tail dependence, such as the Cauchy copula, for tranching and rating CDO tranches. We find that this leads to much higher credit enhancement than required by the Gaussian copula, hence a smaller migration volatility of credit ratings.

6.1.3 Outline

The chapter is organized as follows. In section 6.2, we briefly review the performance of CDO tranches by vintage category in the 2007-2009 crisis. In section 6.3, we present the rating methodology of CDO tranches referencing bonds and loans, and the one for CDO tranches referencing asset backed securities, emphasizing the complexity of the loss distribution of the latter products. In section 6.4, we study the impact on the tranching procedure of a change in rating criteria such as changing rating cut-offs or requiring additional stress-tests. In section 6.5, we examine the leverage carried in CDO tranches with respect to a change in the underlying assets credit worthiness, and the sensitivity of CDO and CDO-squared tranches to the correlation and dependence structure of the underlying assets. In section 6.6, we study the dynamics of credit ratings, in particular their path-dependence, non-homogeneity and dependence on the specific structure of the CDO tranche. In section 6.7, we investigate the use of tail-related risk measures in the rating procedure. Finally, in section 6.8, we explore the effects of using a copula with tail dependence on credit ratings of CDO tranches.

6.2 An empirical look at CDO ratings

CDO tranches have been the most attractive instruments of the past decade, providing satisfaction for both issuers and investors as discussed in the introduction. Their volume has grown significantly in the 2001-2006 period (see figure 6.1), reaching more than \$500 billion of new issuances in 2006, way above the \$78.5 billion issued in 2001 (SIFMA, 2010). Moreover, the 2001-2006 period has witnessed a major increase in the complexity of the assets composing the underlying collateral. In 2001, about 40% of the collateral of issued CDO tranches is composed of investment grade bonds and only 1% of structured finance securities. In 2006, the picture is reversed. Only 4% of the collateral is composed of investment grade bonds while 60% is composed of structured finance securities (SIFMA, 2010). However, since the onset of the credit crisis in 2007, the volume of CDOs has dramatically shrunk: about \$61.8 billion of new issuances in 2008, \$4.3 billion in 2009 and \$8 billion in 2010 (see figure 6.1). Also, the proportion of structured finance securities has been decreasing in favor of investment grade bonds. In 2010, 60% of the collateral of issued CDO tranches is composed of investment grade bonds and only 20% of structured finance securities. Figures 6.2 and 6.3 show respectively the global issuance of CDOs by type of collateral respectively in the pre- and post-crisis periods.

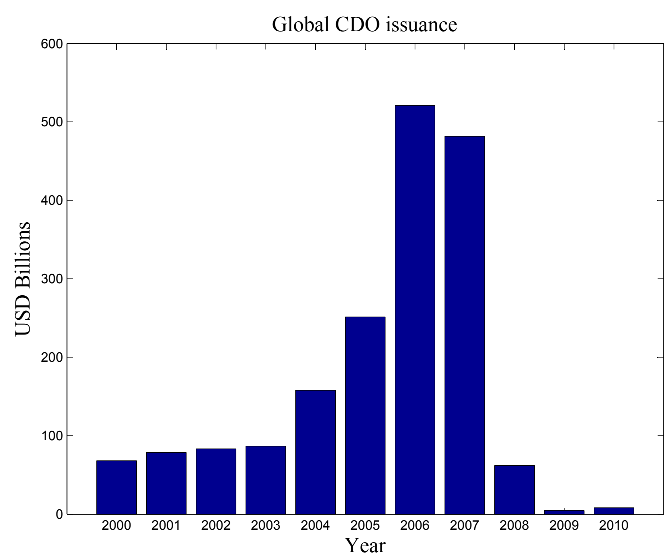


Figure 6.1: Global CDO issuance. Source: SIFMA (2010).

This shift has been mainly triggered by the rising delinquencies on the subprime residential mortgage backed securities, and the widening spreads of investment

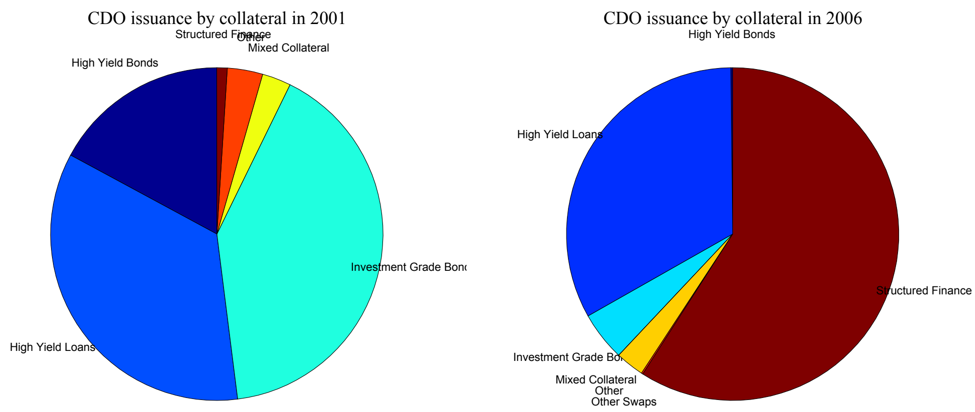


Figure 6.2: CDO issuance by type of collateral in the pre-crisis period 2001-2006. Source: SIFMA (2010).

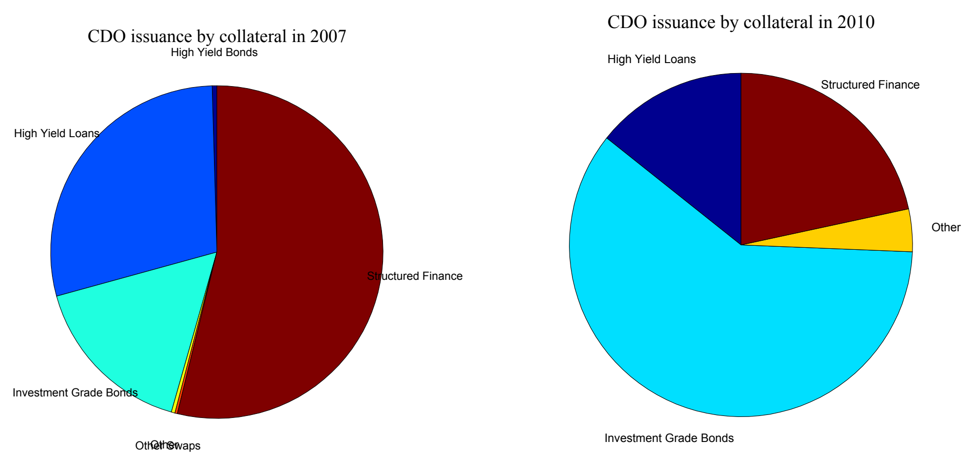


Figure 6.3: CDO issuance by type of collateral in the crisis and post-crisis period 2007-2010. Source: SIFMA (2010).

grade CDO tranches. The deterioration in the credit quality of subprime residential mortgage backed securities has induced a dramatic decline in the credit quality of CDO tranches backed by RMBS. Even AAA (Aaa)-rated tranches, which are designed to be safe and high yield investments, have incurred severe losses. The rating categories Aaa and AAA are the highest one issued by the corresponding rating agencies Moody's and Standard & Poor's and should reflect a very good credit quality. In fact, according to Moody's "Obligations rated Aaa are judged to be of the highest quality, with minimal credit risk" and according to Standard & Poor's "An obligor rated AAA has extremely strong capacity to meet its financial commitments". Nonetheless, in 2007 35.3% of outstanding CDO tranches have been downgraded by Moody's. Specifically, 47.8% of Aaa tranches across all vintages have experienced downgrades, with 86.4% in the ABS vintage (Newman et al., 2008). On average, a CDO tranche was downgraded of 5.5 notches with 13 notches for ABS CDOs (Newman et al., 2008). By the end of 2008, more than 90% of both Aaa and Baa CDO tranches were downgraded. In total, $564 billion worth of CDOs have been downgraded since January 2007 (FCIC, 2010). Overall, rating agencies downgraded $1.9 trillion of mortgage backed securities. These observations suggest that credit ratings fail to accurately assess the credit risk of structured finance securities.

An example: the Abacus 2007-AC1 synthetic CDO

Abacus 2007-AC1 is a $2 billion notional synthetic CDO issued by the SPV ABACUS 2007-AC1, Ltd. and backed by a portfolio of 90 Baa2-rated midprime and subprime RMBS each with a notional amount of $22.22 million and a 4.2-year average life, selected by ACA Management, LLC. Class A to class D notes (tranches) are issued with a projected average life of 3.9 to 4.9 years. Their capital structure is presented in tables 6.1 and 6.2. Goldman Sachs Capital Markets, an affiliate of Goldman, Sachs & Co., enters into a CDS with the issuer to buy protection on the reference portfolio losses related to the class A through class D notes. An obligor in the reference portfolio defaults in case of a writedown (book value overvalued compared to the market value) or a failure to pay principal. Figure 6.4 presents a diagram of the structure of the the Abacus 2007-AC1 deal. We refer the reader to Goldman Sachs (2007) for more details.

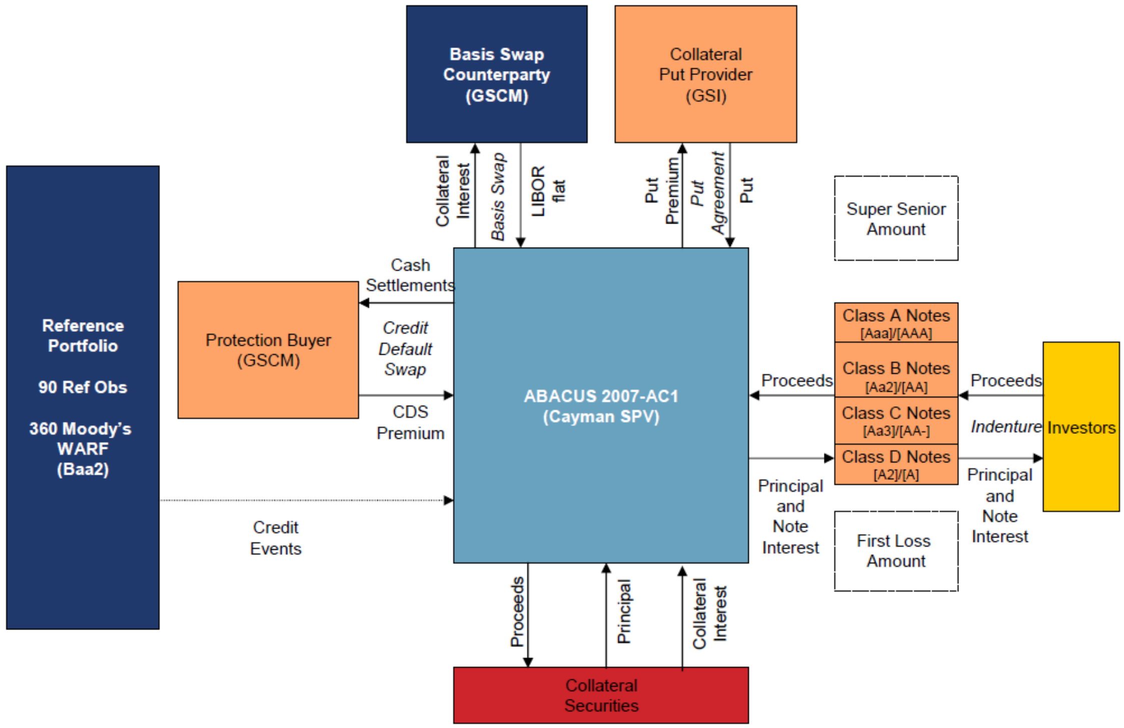


Figure 6.4: The Abacus 2007-AC1 deal. Source: Goldman Sachs (2007).

Although according to Moody's "Obligations rated Baa are subject to moderate credit risk", 83% of the RMBS in the ABACUS 2007-AC1 portfolio had been downgraded by October 24, 2007, and 17% were on negative watch. By January 29, 2008, 99% of the portfolio had been downgraded generating losses over $1 billion (Reisner et al., 2010). The Class A1 and A2 notes, initially rated Aaa by Moody's and AAA by Standard & Poor's, were nearly worthless within few months (see figures 6.5 and 6.6 for their rating history).

A closer look on the credit ratings of structured finance products, in particular CDO tranches, is then crucial to understand their meaning and dynamics.

Tranche	Initial tranche notional amount (million USD)	Rating Moody's	Rating S&P	Tranche size (%)	Tranche attachment (%)	Tranche detachment (%)	Projected WAL (yrs.)	Legal final
Super senior	1100	N/A	N/A	55	45	100	3.9	2037
Class A	480	Aaa	AAA	24	21	45	4.4	2037
Class B	60	Aa2	AA	3	18	21	4.6	2037
Class C	100	Aa3	AA-	5	13	18	4.7	2037
Class D	60	A2	A	3	10	13	4.9	2037
First loss	200	N/A	N/A	10	0	10	5.2	2037

Table 6.1: Capital structure of the Abacus 2007-AC1. Source: Goldman Sachs (2007).

Tranche	Initial tranche Principal Amount (million USD)	Initial tranche notional amount (million USD)	Rating Moody's	Rating S&P	Coupon	Average life (yrs.)
Class A1	50	200	Aaa	AAA	1.347	4.0
Class A2	142	280	Aaa	AAA	1.357	4.2

Table 6.2: Capital structure of the class A1 and A2 of Abacus 2007-AC1. Source: Goldman Sachs (2007).

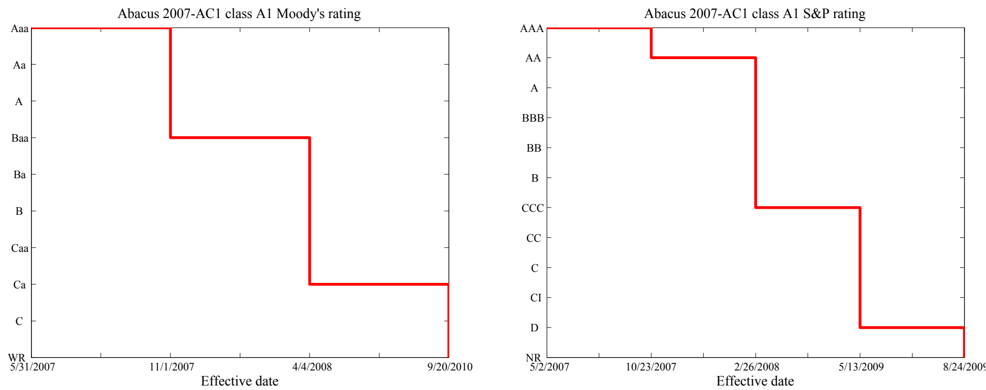


Figure 6.5: Rating history of the class A1 of Abacus 2007-AC1. Source: Bloomberg.

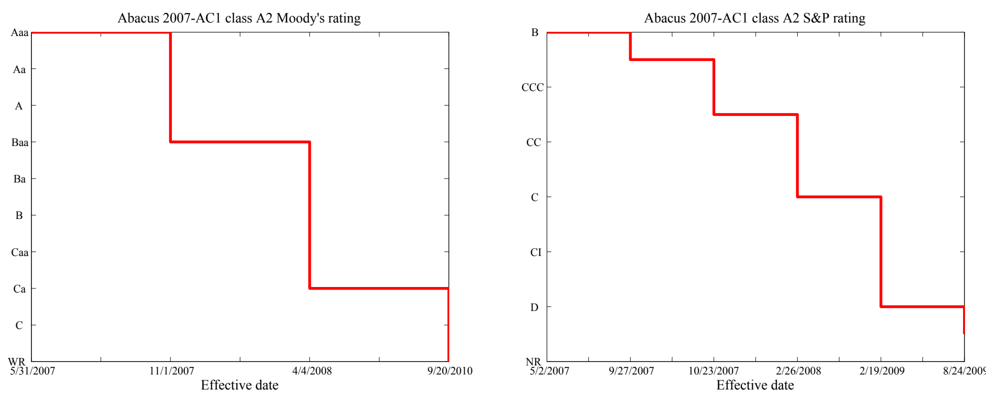


Figure 6.6: Rating history of the class A2 of Abacus 2007-AC1. Source: Bloomberg.

6.3 Rating methods for structured credit products

6.3.1 Credit ratings of CDO tranches referencing loans and bonds

Each CDO tranche is rated on the basis of an ordinal measure of the credit risk such as the probability that a tranche is hit by loss (Fitch and Standard & Poor's) or the expected loss of the tranche (Moody's). In order to assign a rating, this measure is compared to a defined cut-off. For example, according to Standard & Poor's cut-offs (Standard & Poor's, 2009), if the probability that a CDO tranche maturing in 7 years absorbs losses is between 0.144% and 1.069%, then this tranche is rated AA, and if this probability is between 1.069% and 2.780% then the tranche is rated A. Rating agencies often adjust these cut-offs to allow "more severe" or "more easy" rating scales.

We consider a hypothetical CDO with k tranches, with respective attachments $a_1, a_2, ..., a_k$ and detachments $a_2, a_3, ..., a_k, 100\%$, on a homogenous portfolio of n assets (bonds or loans) each with unit notional amount, horizon T, recovery rate R, loss given default l, and random default times $\tau_1, \tau_2, ..., \tau_n$. That is, at time τ_i, the default of asset i generates a loss of $(1-R)$ in the reference portfolio.

The portfolio total loss at time t -expressed in percentage of the total notional of the portfolio- is simply the sum of all losses generated by the defaults of the individual assets in the portfolio that have occurred prior to time t, that is,

$$L(t) = \frac{(1-R)}{n} \sum_{i=1}^{n} 1_{\tau_i \leq t} \tag{6.1}$$

The CDO tranche $a_j - a_{j+1}$ is affected only when the portfolio total loss $L(t)$ exceeds its attachment level a_j, and is completely lost if the loss exceeds its detachment level a_{j+1}. Thus, the loss absorbed by the tranche $a_j - a_{j+1}$ in proportion to its notional is at time t,

$$M_j(t) = \frac{(L(t)-a_j)^+ - (L(t)-a_{j+1})^+}{a_{j+1}-a_j} \tag{6.2}$$

This is the payoff of a Call Spread option on the total portfolio loss (see figure 6.7), which underlines the complex nature of the losses of CDO tranches and the high leverage they carry.

The credit rating of the tranche $a_j - a_{j+1}$ at a time t is determined by compar-

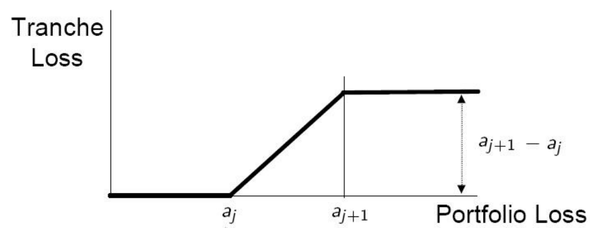


Figure 6.7: Tranche loss as a function of total default losses in portfolio. We recognize the nonlinear profile of a Call Spread.

ing a measure of the credit risk based on the distribution of $M_j(t)$ to given rating categories cut-offs. For example, Standard & Poor's ratings are based on the probability of the tranche having incurred a loss by time t, $\mathbb{P}(M_j(t) > 0)$, also called default probability, and Moody's ratings are based on the expected loss, $\mathbb{E}[M_j(t)]$. We shall focus in the following sections on the rating methodology based on the probability of incurring a loss as measure of credit risk.

Factor models for portfolio default losses

Since the default times are random, the losses of the reference portfolio are also random. The rating procedure requires then a statistical model to simulate the losses. The choice of the model has of course an impact on the outcomes of the rating procedure and there is an important model risk related to the agencies' assumptions of correlation, recovery rate, and assets default rates.

The dependence structure between the default times is specified as a copula $C(u_1, ..., u_n)$ defined as the joint cumulative distribution of $F_1(\tau_1), ..., F_n(\tau_n)$ where $F_1, ..., F_n$ are the respective cumulative marginal distributions of $\tau_1, ..., \tau_n$. One example of copula that has been extensively used in the financial industry is the

Gaussian copula introduced by Li (2000),

$$C^G_\rho(u_1, ..., u_n) = \Phi_R(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)), \tag{6.3}$$

where Φ_R is the joint cumulative density function of an n-dimensional Gaussian vector with correlation matrix R, and Φ is the cumulative density function of a standard Gaussian.

Currently most agencies have been using factor Gaussian copula models which are slight variations of the general Gaussian copula introduced by Li (2000) allowing for inter-and intra-sector correlations. The dependence structure between the assets in the reference portfolio is then driven by their common dependence to Gaussian systemic factors.

Default times are generally assumed identically distributed from an Exponential distribution with intensity λ. That is, the instantaneous conditional default rate (hazard rate) is a constant (equal to λ). The marginal distribution of the default times is then "memoryless".

One factor Gaussian copula model

The one factor Gaussian copula has been a market standard in risk management for its easy implementation and tractability. The one factor Gaussian copula model is,

$$\Phi^{-1}(F(\tau_i)) = \sqrt{\rho}S + \sqrt{1-\rho}Z_i, \tag{6.4}$$

Where F is the cumulative density function of an exponential with intensity λ. S is a standard Gaussian systemic factor and the $Z_i's$ are independent standard Gaussian idiosyncratic factors independent of S.

The probability that the i^{th} asset defaults before the horizon t conditional on S is,

$$p(S,t) = \mathbb{P}(\tau_i \leq t|S) = \phi(\frac{\phi^{-1}(F(t)) - \sqrt{\rho}S}{\sqrt{1-\rho}}) \tag{6.5}$$

Given S, the $\tau_i's$ are independent. Thus, conditional on S, the number of defaults follow a Binomial distribution, and the conditional probability that tranche $a_j - a_{j+1}$ incurs a loss is then,

$$\mathbb{P}(L(t) > a_j|S) = \sum_{i>a_j/l} \mathbb{P}(i \text{ assets default}|S) \tag{6.6}$$

$$= \sum_{i>a_j/l} \binom{n}{i} p(S,t)^n (1-p(S,t))^{n-i} \tag{6.7}$$

Therefore, the unconditional probability that tranche $a_j - a_{j+1}$ incurs a loss is,

$$\mathbb{P}(L(t) > a_j) = \int_{-\infty}^{+\infty} (\sum_{i>a_j/l} \binom{n}{i} p(s,T)^i (1-p(s,t))^{n-i})\phi(s)ds \tag{6.8}$$

When the number of obligors is large, the loss distribution can be approximated using a conditional law of large numbers. Conditionally on the factor S, the random variables $1_{\tau_i \leq t}, ..., 1_{\tau_n \leq t}$ are independent and identically distributed so by the strong law of large number, conditionally on the factor S,

$$\frac{L(t)}{n} \to lp(S,T) \quad as \quad n \to \infty \tag{6.9}$$

Which implies that,

$$\mathbb{P}(\frac{L(t)}{n} \leq lx) = \mathbb{P}(p(S,t) \leq x) \simeq \Phi(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(F(t))}{\sqrt{\rho}}) \tag{6.10}$$

Example 1 (Rating of a senior tranche)**.** *We consider a 7-years senior x%-100% tranche of an reference portfolio of 1000 BB rated obligors with average annual default probability* $p = 6.639\%$ *(according to S&P asset default rates), which corresponds to an intensity* $\lambda = -\frac{1}{T}\log(1-p) = 0.98\%$*. We choose as a statistical framework a one-factor Gaussian copula model with default correlation* $\rho = 30\%$

and recovery rate $R = 40\%$. The above computations yield that probability of losing 34% of portfolio in 7 years is less than 0.144%. S&P gives 7-years default probability thresholds for AAA tranches as 0.144%. Therefore, if we fix x=34% the senior tranche will be AAA.

Multifactor Gaussian copula model

One major drawback of the one factor Gaussian copula model is that it assumes an equal correlation ρ between all assets in the reference portfolio. However, in practice, assets that belong to a common industrial sector tend to be more correlated than assets in different sectors. This has motivated the extension of the one factor Gaussian copula model to the multifactor case.

We consider $k+1$ common factors $S_0, S_1, S_2, ..., S_k$ where S_0 is a global systemic factor representing for example the global economy and $S_1, S_2, ..., S_k$ are factors representing different industrial sectors. We assume that n_1 assets of the reference portfolio operate in sector S_1, n_2 in sector S_2, ..., n_k in sector S_k.

The Gaussian copula model in this case is,

$$\Phi^{-1}(F(\tau_i)) = \sqrt{\rho_0}S_0 + \sqrt{\rho_q - \rho_0}S_q + \sqrt{1-\rho_q}Z_i, \tag{6.11}$$

for all assets i in the sector S_q.

The correlation between two assets in the same sector S_q is ρ_q while the correlation between two assets in different sectors is ρ_0.
The probability that asset i in the sector S_q defaults before the horizon T, conditionally on the factors S_0 and S_q is,

$$p(S_0, S_q, T) = \mathbb{P}(\tau_i \leq T | S_0, S_q) = \Phi(\frac{\Phi^{-1}(F(T)) - \sqrt{\rho_0}S_0 - \sqrt{\rho_q - \rho_0}S_q}{\sqrt{1-\rho_q}}) \tag{6.12}$$

The probability that tranche $a_j - a_{j+1}$ incurs a loss conditionally to the factors

$S_0, S_1, S_2, ..., S_k$ is,

$$\sum_{i > a_j/l} \mathbb{P}(i \text{ assets default} | S_0, S_1, S_2, ..., S_k) \tag{6.13}$$

which is equal by independence of the $\tau_i' s$ conditionally to the factors $S_0, S_1, S_2, ..., S_k$ and their homogeneity within each sector to

$$p_{j,t|S_0,S_1,S_2,...,S_k} \tag{6.14}$$

$$= \sum_{m_1,..,m_k; l \sum_{q=1}^{k} m_q > a_j} \prod_{q=1}^{k} \binom{n_q}{m_q} p(S_0, S_q, t)^{m_q} (1 - p(S_0, S_q, t))^{n_q - m_q} \tag{6.15}$$

Therefore, the unconditional probability that the j^{th} tranche incurs a loss is

$$\mathbb{P}(L(t) > a_j) = \mathbb{E}[p_{j,t|S_0,S_1,S_2,...,S_k}] \tag{6.16}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} p_{j,t|s_0,s_1,s_2,...,s_k} \phi(s)\phi(s_1)...\phi(s_k) ds ds_1 ... ds_k \tag{6.17}$$

6.3.2 Credit ratings of CDO tranches referencing asset backed securities

The hypothetical CDO structure described above illustrates in fact a generic mechanism for enhancement by subordination (tranching) which has been repeatedly applied to various portfolios, leading to a zoology of structured products depending on the type of asset underlying the structure: loans (CLO), mortgages (CMO), etc. An important and widespread example is the one where the subordination mechanism is applied to a portfolio itself composed of senior tranches of other debt portfolios (called inner CDOs), giving rise to a so-called CDO-squared structure.

Rating a CDO referencing a portfolio of asset backed securities (ABS CDO) is a more complex exercise than rating a CDO of bonds or loans due to the higher complexity in the loss distribution of the reference portfolio. We shall describe the loss mechanism in the example of a CDO-squared, the other types of ABS CDOs follow a similar mechanism.

We consider a hypothetical CDO-squared with k tranches with respective attachments $a_1, ..., a_k$ referencing a portfolio of h inner senior CDO tranches with respective attachments $b_1, b_2, ..., b_h$ and detachments $c_1, c_2, ..., c_h$. The inner CDO tranches are themselves backed by homogenous portfolios of bonds. The tranche $a_j - a_{j+1}$ of the outer CDO incurs a loss $K_j(t)$ at time t when the total loss $H(t)$ of the reference portfolio of inner CDO tranches exceeds the attachment level a_j. The tranche is completely lost when the loss exceeds its detachment level a_{j+1}. That is,

$$K_j(t) = \frac{(H(t) - a_j)^+ - (H(t) - a_{j+1})^+}{a_{j+1} - a_j} \tag{6.18}$$

The total loss $H(t)$ of the portfolio underlying the outer CDO is then the sum of the losses incurred by each of the inner CDO tranches when default occurs in the pool of bonds underlying the inner CDO tranches. That is,

$$H(t) = \frac{1}{h} \sum_{r=1}^{h} M_r(t) \tag{6.19}$$

The loss distribution $M_r(t)$ of the inner CDO tranches is the same as the one described in section 6.3.1 (equation 6.2). There is no closed formula for the probability that tranche $a_j - a_{j+1}$ in the outer CDO is hit by loss. The rating is then done by nested Monte Carlo simulation (Lee, 2009).

6.4 Influence of rating criteria on the tranching procedure

6.4.1 Rating cut-offs

Ratings are assigned to the tranches of a CDO by comparing the probability that they are hit by loss to the cut-offs defined by the rating agency. Therefore, a change in the rating cut-offs may lead to a change in the ratings of CDO tranches. For example, upon the release of the new CDO criteria in October 2009, Standard & Poor's placed more than half a trillion dollars of outstanding CDO securities under review for possible downgrade (Adelson, 2010).

New issuances will have then to adjust their capital structure to the new cut-offs. A CDO issuer concerned in paying the lowest spreads to investors would issue the best possible rated tranches, starting from issuing the thickest tranche possible with a rating AAA, then BBB, BB, etc. Specifically, the attachment a_{AAA} of the senior AAA tranche is fixed in such a way that it is the lowest attachment a for which the probability of incurring a loss of the tranche $a-100\%$ is smaller than the AAA cut-off $p_{AAA,T}$ for the corresponding horizon T, that is,

$$a_{AAA} = \inf_{a\in[0,1]} \mathbb{P}(L(t) > a) < p_{AAA,T}, \tag{6.20}$$

More generally, considering a range of ratings $g_1, g_2, ..., g_k$ ordered by increasing credit worthiness with respective cut-offs $p_{g_1,T}, p_{g_2,T}, ..., p_{g_k,T}$ for the horizon T, the attachment of the tranche $a_j - a_{j+1}$ rated g_j at time t is then fixed in such a way that it is the lowest attachment a for which the probability of incurring a loss of the tranche $a - a_{j+1}$ is smaller than the cut-off $p_{g_j,T}$ for the horizon T, that is

$$a_j = \inf_{a\in[0,1]} \mathbb{P}(L(t) > a) < p_{g_j,T} \tag{6.21}$$

As an illustration, we compare the capital structure of the hypothetical CDO introduced in example 1 -issued in a way to maximize tranches credit ratings- when using the rating cut-offs of S&P CDO Evaluator versions 3.0 and 4.1. We note that the change in the rating cut-offs leads to different CDO structures (see tables 6.3 and 6.4). The AAA tranche structured according to the version 4.1 criteria has a larger credit enhancement (higher attachment) and consequently a smaller probability of incurring a loss. This change in rating criteria renders AAA tranches more resilient to defaults in the reference portfolio[2].

Attachment (%)	Rating	Default probability (%)	Rating cut-off (%)
34	AAA	0.134	0.144
24	AA	0.922	1.069
17	A	3.096	3.476
11	BBB	8.633	9.959
6	BB	21.496	24.709
3	B	40.545	43.347
2	CCC	52.055	67.529

Table 6.3: Capital structure of CDO tranches issued to maximize credit ratings using S&P CDO Evaluator version 4.1 rating cut-offs.

Attachment (%)	Rating	Default probability (%)	Rating cut-off (%)
31	AAA	0.250	0.285
26	AA	0.646	0.701
22	A	1.320	1.368
15	BBB	4.331	4.443
8	BB	14.765	15.110
4	B	32.686	32.903
1	CCC	69.292	73.396

Table 6.4: Capital structure of CDO tranches issued to maximize credit ratings using S&P CDO Evaluator version 3.0 rating cut-offs.

Definition 18 (Stylized CDO)**.** *In the following sections we call Stylized CDO the one with tranches defined by their attachment-detachment levels 2%-3%, 3%-6%,*

[2]In fact, according to Adelson (2010), "the (Standard & Poor's) updated CDO criteria produces a AAA credit enhancement of roughly 42% for a typical nine-year collateralized loan obligation backed by a well-diversified pool of B-rated credits, which is substantially higher than under the prior criteria"

6%-11%, 11%-17%, 17%-24%, 24%-34%, 34%-100%, with horizon 7 years, and referencing a portfolio of 1000 BB assets with default probability $p = 6.639\%$ *and recovery rate* $R = 40\%$.

6.4.2 Stress tests

The massive downgrades of AAA CDO and RMBS tranches observed at the end of 2008 have led rating agencies to adjust their rating criteria to avoid such scenarios. Aside from revising their rating cut-offs, Standard & Poor's has introduced supplemental stress tests to assess whether CDO tranches have sufficient credit enhancement to withstand simultaneous defaults in the reference portfolio (Standard & Poor's, 2009). Under the *largest obligor default test* a AAA rated tranche should have sufficient credit enhancement to, assuming a recovery rate of 5%, survive the defaults of each of the following combinations of underlying assets: the two largest obligors rated between AAA and CCC-, the three largest obligors rated between AA+ and CCC-, the four largest obligors rated between A+ and CCC-, the six largest obligors rated between BBB+ and CCC-, the eight largest obligors rated between BB+ and CCC-, the 10 largest obligors rated between B+ and CCC- and the 12 largest obligors rated between CCC+ and CCC-.

Then, the AAA tranche of the Sylized CDO (definition 18) should have sufficient credit enhancement to withstand the default of eight obligors with a 5% recovery rate. The loss generated by this default scenario is of 0.76%, which means that the attachment of the AAA tranche should exceed 0.76%. This is a trivial requirement in the simplistic example of a homogeneous reference portfolio, however it could imply significant credit enhancement when the reference portfolio is constituted of assets with heterogeneous notionals. As an example, consider a portfolio of BB rated obligors with the eight largest accounting for 40% of the total notional and the remaining accounting for 60%. Then the largest obligor default test requires the AAA tranche to survive a loss of 38%, thus leading to an additional 4% credit enhancement to the 34% attachment level of the AAA tranche.

6.5 Leverage effects and sensitivity to modeling assumptions

6.5.1 Leverage effects

Since the onset of the subprime crisis in 2007, credit risk models used by rating agencies and banks have been repeatedly criticized as having underestimated the default probability of many obligors. For instance, in the case of residential mortgage backed securities, most models failed to incorporate scenarios where housing prices -which represent the common factor affecting residential mortgages- would decline. Model coefficients were estimated using historical data which did not contain any scenario even remotely similar to the crisis we have witnessed in 2007-2009. In hindsight, this seems to have led to a serious underestimation of default risk for many categories of RMBS backed securities.

In fact, since the loss of a CDO tranche is a Call Spread on the total loss of the reference pool of assets, it should exhibit a high leverage effect compared to a portfolio of bonds or loans. Call Spread strategies are highly leveraged compared to direct investments in the underlying in the world of stocks or futures. Thus, a slight increase in the credit risk of the underlying obligors, due to the tension in the credit market, tends to be amplified by the CDO structure and could imply serious downgrades of the tranches. Figure 6.8 illustrates this phenomenon on the Stylized CDO: a slight decrease in the credit worthiness of the underlying assets (increase in their default probability) while maintaining their BB rating can imply a downgrade of the AAA-rated senior tranche of three notches. The other tranches are also downgraded of several notches.

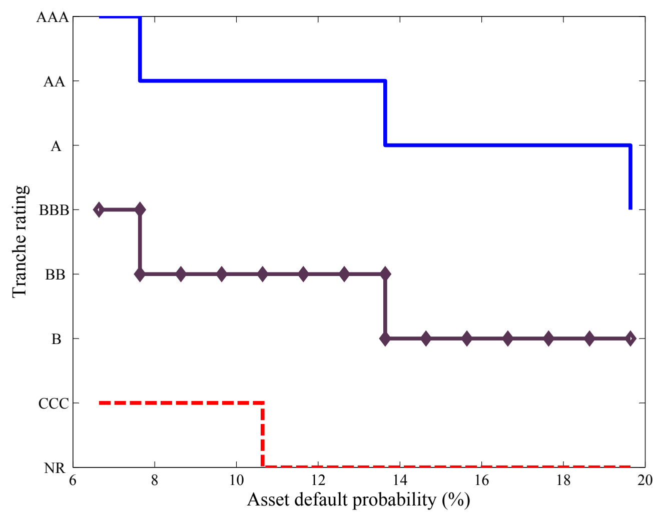


Figure 6.8: Sensitivity to a change in asset default rates: a decrease in the credit worthiness of the underlying assets while maintaining their BB rating can imply a downgrade of CDO tranches of several notches.

The leverage can be even more expressed in CDOs of asset backed securities, such as a CDO-squared. As an illustration, we consider a stylized CDO-squared referencing ten underlying inner A-rated CDO tranches each with attachment 17% and detachment 60%. The portfolio of bonds underlying the pool of inner CDOs is composed of 1000 BB-rated bonds belonging to five different industrial sectors. Each inner CDO is backed by two hundred bonds: five inner CDOs contain each two hundred bonds from the same sector, and the other five contain one hundred bonds from one sector and one hundred from another sector such that each bond is contained in two different inner CDOs. The capital structure of the CDO-squared issued to maximize credit ratings using S&P CDO Evaluator version 4.1 rating cut-offs, assuming a one factor copula model with correlation 30%, is presented in table 6.5. It is interesting to note the "little" credit enhancement needed to issue investment grade CDO-squared tranches, which made them be very popular products in the past decade.

Attachment (%)	Rating	Default probability (%)	Rating cut-off (%)
40	AAA	0.139	0.144
16	AA	0.951	1.069
2	A	3.086	3.476
0	BBB	4.948	9.959

Table 6.5: Capital structure of CDO-squared issued to maximize credit ratings using S&P CDO Evaluator version 4.1 rating cut-offs, assuming a one factor copula model with correlation 30%.

We change the credit quality of the inner tranches by varying their detachment level while maintaining constant their attachment level. That is, we change the "thickess" of the tranches without changing their credit rating since the latter depends only on the attachment level. We observe in figure 6.9 that this leads the outer tranches of the CDO-squared to downgrade. In fact, the "thickness" of the inner CDO tranches is a very important factor to examine when assessing the risk of CDO-squared structures. Contrarily to a bond, which can maximum lose the unrecovered portion $1 - R$ of its notional, an inner CDO tranche can be entirely wiped out. Thus, the "thinner" is the tranche, the larger is the proportion of notional lost at default.

6.5.2 Sensitivity to the dependence structure

Structured credit ratings are *more* sensitive to modeling assumptions than bonds and loans since they depend also on assumptions regarding *default correlation* or, more precisely, the co-dependence of default times. Yet such parameters are not directly observable and cannot be reliably estimated from historical data given that they pertain to rare events.

In the Gaussian copula framework detailed in section 6.3, the co-dependence structure is specified by a correlation. The higher the correlation between the obligors,

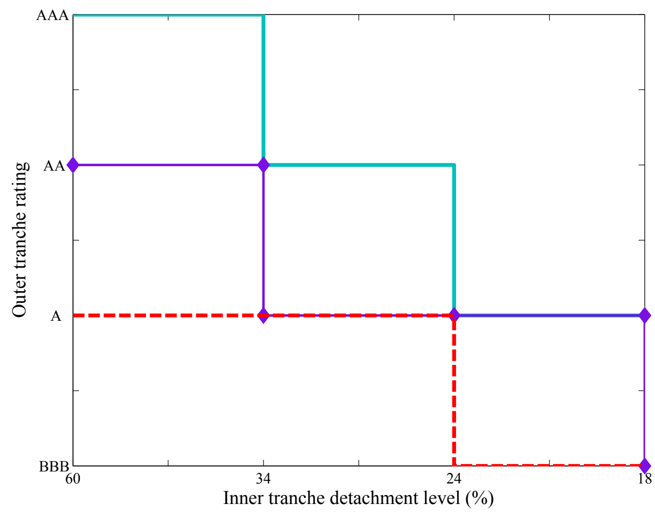


Figure 6.9: Sensitivity of three CDO-squared tranches to a change in the inner CDOs credit quality: a decrease in the thickness of the inner tranches while maintaining their A rating can imply a downgrade of the outer CDO tranches of several notches.

the more likely they are to default together or to survive together, and hence the more likely senior tranches are hit by default. It is reasonable to think that in a period of global crisis, rating agencies could revise their rating with a higher default correlation, since the most likely driver of default is the shared global market exposure. Such a statistical model revision can also drive multiple downgrades, as shown in figure 6.10 for the tranches of the Stylized CDO. It is interesting to note that senior tranches credit ratings have a negative sensitivity to correlation ("short correlation" by analogy with the pricing terminology) and the Equity tranche credit rating has a positive sensitivity to correlation ("long correlation" by analogy with the pricing terminology): when correlation increases, the assets in the underlying portfolio are more likely to default together, which increases the probability that the total portfolio loss exceeds the senior tranche attachment level; the assets are also more likely to survive together, which increases the probability of zero default i.e. the Equity tranche not hit by loss.

Since rating agencies use a multifactor Gaussian copula model to allow a greater

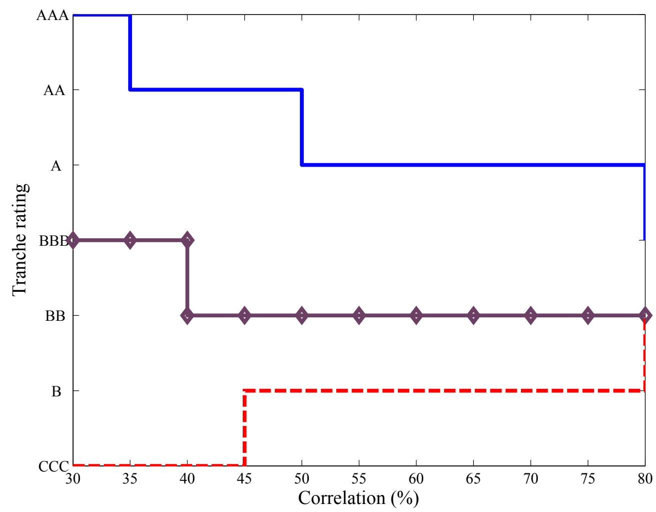


Figure 6.10: Sensitivity to a change in correlation: an increase in the correlation between the underlying assets can imply a downgrade of the senior tranche of several notches, and an upgrade of the equity tranche.

correlation between two pier firms in the same industrial sector, we consider an reference portfolio of 1000 BB assets belonging to five different sectors (200 assets in each sector) with average annual default probability $p = 6.639\%$, $R = 40\%$ recovery and unit notional. We choose the inter-sector correlation equal to 13.4% and the intra-sector correlation equal to 80% so that the average correlation is still 30%. We rate the Stylized CDO tranches in the six factors (one global common factor and five sectors) Gaussian copula framework with the above correlation parameters. We observe (see figure 6.11) that several tranches are downgraded when changing the dependence structure from the one factor to the multifactor case while maintaining the average correlation constant.

The impact of the correlation structure on credit ratings con be more revealed with more complex structured products such as CDOs-squared. Their two-layers structure is extremely sensitive to the correlation between the obligors in the portfolio underlying the inner CDO tranches: if the obligors are equally correlated (homogeneous correlation), the loss generated by a default event will be evenly distributed

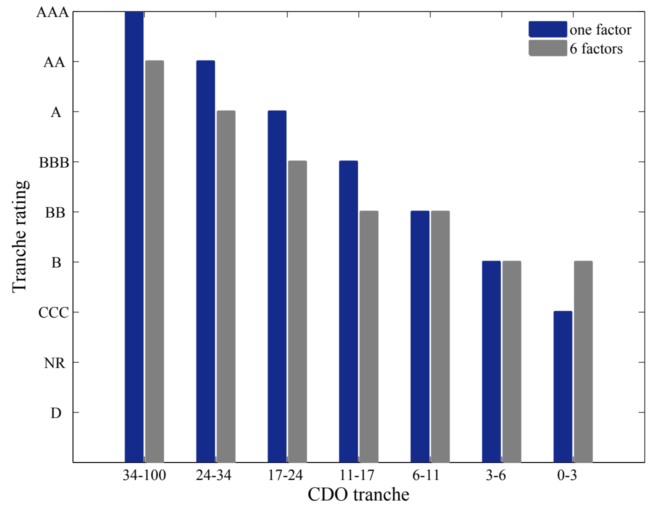


Figure 6.11: One factor versus multifactor copula model: a change in the dependence structure can have a significant impact on tranche ratings.

across the inner CDOs and most probably will not exceed their subordination level. In this case the loss will not affect the CDO-squared. However, if the obligors underlying each inner CDO are highly correlated and the cross correlation is low, the loss will concentrate on one inner CDO exceeding its subordination level and therefore the loss will flow into the outer CDO portfolio. Figure 6.12 illustrates this phenomenon.

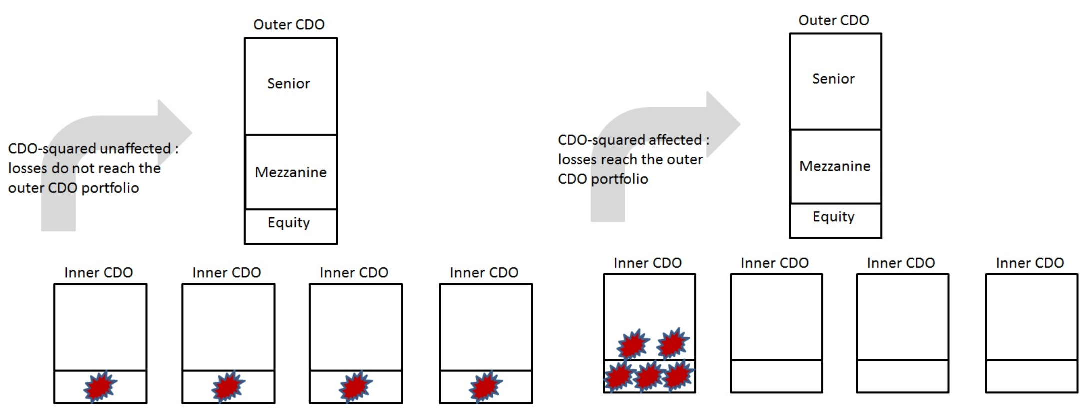


Figure 6.12: Default scenario of a CDO-squared backed by an reference portfolio with homogeneous versus heterogenous intra- and inter-sector correlation.

We consider the stylized CDO-squared introduced in the previous section. A change in the dependence structure from the one factor to the six factors model leads almost all tranches to downgrade (see figure 6.13).

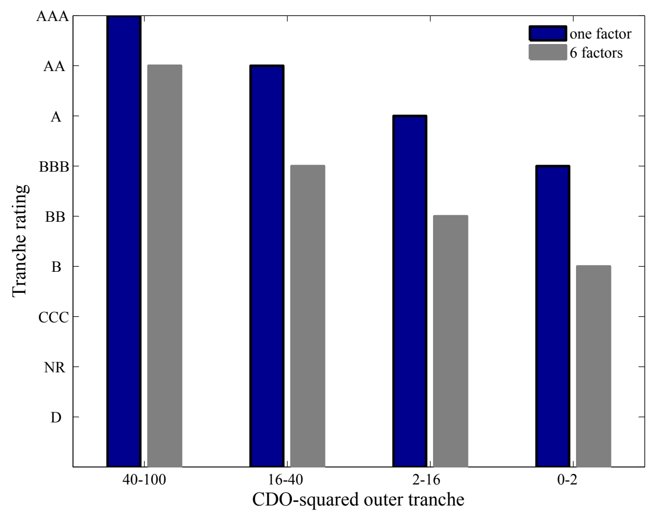


Figure 6.13: One factor versus multifactor copula model: a change in the dependence structure can have a significant impact on CDO-squared tranche ratings.

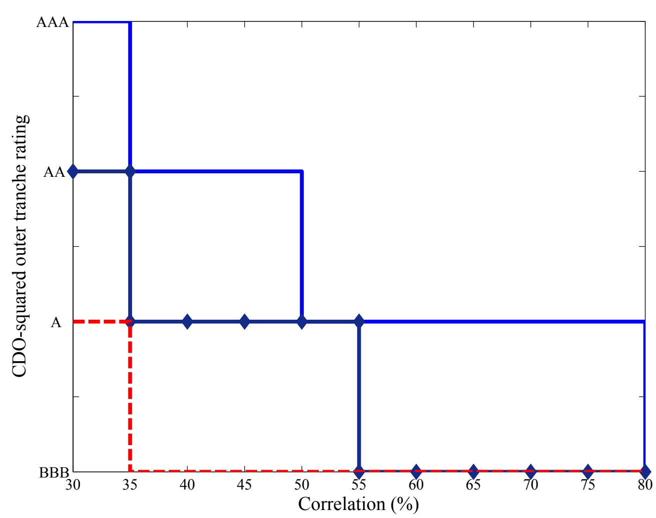


Figure 6.14: Sensitivity to a change in correlation: an increase in the correlation between the underlying assets can imply a downgrade of the outer CDO tranches.

6.6 Dynamics of credit ratings

6.6.1 Non-Markov dynamics of rating transitions

The loss distribution of a structured finance product, specifically a CDO tranche, varies with time and market conditions. Thus, credit ratings only reflect the credit worthiness perceived at a specific time. Factors influencing the loss distribution are the passage of time, the change of the credit quality of the underlying obligors, the change of the co-dependence structure of their default times, and the change in credit enhancement (attachment level). Since such products live for several years, it is necessary to update periodically their ratings during their lifetime. Rating agencies publish annual transition matrices to provide investors with a measure of the likelihood of possible downgrade or upgrade from a given rating category to another one. Specifically, they assume that the rating transitions follow a homogeneous Markov chain, so the cumulative transition probabilities are derived by raising the one-year transition matrix to iterative powers (Standard & Poor's, 2009).

To check this assumption, we calculate the 0-2 years and 2-4 years transition probabilities between the different rating categories by pooling all tranches of the Stylized CDO and averaging the proportion of tranches transiting from one rating category to the other. We describe as follows the algorithm to compute the probability of transition from a rating category g_1 at time t_1 to a rating category g_2 at time t_2 by nested Monte Carlo method with Importance Sampling. Since defaults are are events, Importance Sampling is necessary to reduce the variance of the Monte Carlo estimator[3]. Then, the Monte Carlo estimator of the probability of transition from

[3] In the case of a Gaussian copula: instead of simulating the common factor s of the Gaussian copula from a Standard Gaussian, we simulate it from a Gaussian distribution with shifted mean $-x$ to increase the probability of defaults. The weight of the Importance Sampling (Radon-Nikodym derivative for the change of measure from the shifted Gaussian to the standard Gaussian

g_1 to g_2 is,

$$\widehat{p}_{g_1,t_1\to g_2,t_2} = \frac{\sum_{i=1}^{M} N^i_{g_1,t_1\to g_2,t_2} w_i}{\sum_{i=1}^{M} N^i_{g_1,t_1} w_i}, \tag{6.22}$$

where M is the number of Monte Carlo simulations, $N^i_{g_1,t_1\to g_2,t_2}$ is the number of transitions from the rating category g_1 at time t_1 to the rating category g_2 at time t_2 in the i^{th} simulation, $N^i_{g_1,t_1}$ is the number of tranches rated g_1 at time t_1 and w_i is the weight of the Importance Sampling.

In each simulation $\tau_1^i, ..., \tau_n^i$ of the default times, $N^i_{g_1,t_1}$ and $N^i_{g_1,t_1\to g_2,t_2}$ are computed as follows:

1. Rate the CDO tranches at time t_1: count the number k_1 of defaults occurring before t_1 , then compute the default probabilities for each tranche using the outstanding subordination level $a_1 = \frac{a-k_1 l}{1-k_1 l}$, horizon $T - t_1$ and outstanding number of obligors $n - k_1$. Count the number of tranches $N^i_{g_1,t_1}$ in the rating category g_1 at time t_1.

2. Rate the CDO tranches at time t_2: count the number k_2 of defaults occurring before t_2 , then compute the default probabilities for each tranche using the outstanding subordination level $a_2 = \frac{a-k_2 l}{1-k_2 l}$, horizon $T - t_2$ and outstanding number of obligors $n-k_2$. Count the number of tranches $N^i_{g_1,t_1\to g_2,t_2}$ transiting from the rating category g_1 at time t_1 to the rating category g_2 at time t_2.

The two transition matrices are very different (see tables 6.6 and 6.7) which means that the transition probabilities do not depend only on the horizon but also on the current date. This violates the assumption of a homogeneous Markov chain. In fact, we find that rating transitions are path-dependent. The probability of downgrade

distribution) is $\exp(xs - x^2/2)$.

from the AA category between years 2 and 4 is of 5.28%, which is different from the probability (2.34%) of downgrade from the category AA between years 2 and 4 given that the current rating is AAA, which is also different from the probability (6.28%) of downgrade from the category AA between years 2 and 4 given that the current rating is AA. It is interesting to remark that these conditional probabilities disagree with the natural intuition that, since the obligors are correlated, we would expect to have a greater probability of downgrading if the tranche has already downgraded. The fact that we obtained the opposite phenomenon could be explained by the way the CDO is structured: when a tranche is downgraded from AAA to AA, the attachment point could be far from the cut-off of the AA rating category and therefore a default event may not provoke a downgrade from AA, however if the tranche is rated AA and remains AA, the attachment point is still near the cut-off of the AA rating category and so a single default event may provoke a downgrade.

Transition matrix 0-2 years

	AAA	AA	A	BBB	BB	B	CCC	NR	D
AAA	62.64	36.98	0.29	0.04	0.01	0.00	0	0	0.00
AA	0	82.32	16.45	0.95	0.15	0.03	0.01	0.00	0.05
A	0	0	80.39	17.66	1.18	0.39	0.14	0.01	0.19
BBB	0	0	0	79.13	16.89	2.24	0.67	0.12	0.91
BB	0	0	0	0	54.11	37.22	4.26	0.68	3.72
B	0	0	0	0	0	54.11	32.25	2.97	10.65
CCC	0	0	0	0	0	0	77.72	5.14	17.13
NR	0	0	0	0	0	0	0	0	100
D	0	0	0	0	0	0	0	0	100

Table 6.6: Transition matrix for the period 0-2 years, computed with the one factor Gaussian copula model with $\rho = 30\%$.

6.6.2 Is AAA=AAA?

CDO tranches with different attachment levels can have the same rating but different downgrade transitions. A tranche with a larger credit enhancement is less

Transition matrix 2-4 years									
	AAA	AA	A	BBB	BB	B	CCC	NR	D
AAA	100	0	0	0	0	0	0	0	0
AA	14.25	80.46	5.28	0	0	0	0	0	0
A	0	0.04	97.89	1.92	0.13	0	0	0	0
BBB	0	0	2.22	73.41	23.75	0.45	0.12	0.00	0.01
BB	0	0	0	0	90.85	5.36	2.84	0.27	0.67
B	0	0	0	0	6.37	56.31	31.45	1.49	4.35
CCC	0	0	0	0	0	0	65.24	10.07	24.67
NR	0	0	0	0	0	0	0	0	100
D	0	0	0	0	0	0	0	0	100

Table 6.7: Transition matrix for the period 2-4 years, computed with the one factor Gaussian copula model with $\rho = 30\%$.

vulnerable to losses in the reference portfolio than a tranche with the same rating category but a smaller credit enhancement. To illustrate this, we compare the one-year rating transitions of the senior tranche 34%-100% of the Stylized CDO to the one-year rating transitions of the tranche 40%-100% backed by the same reference portfolio. We find that, although both tranches have a current AAA rating, the tranche 40%-100% has a much greater likelihood (99.70%) of conserving the rating AAA at year 1 than the tranche 34%-100% (79.28%). Thus, the same rating category could reflect completely different future credit worthiness for different tranches.

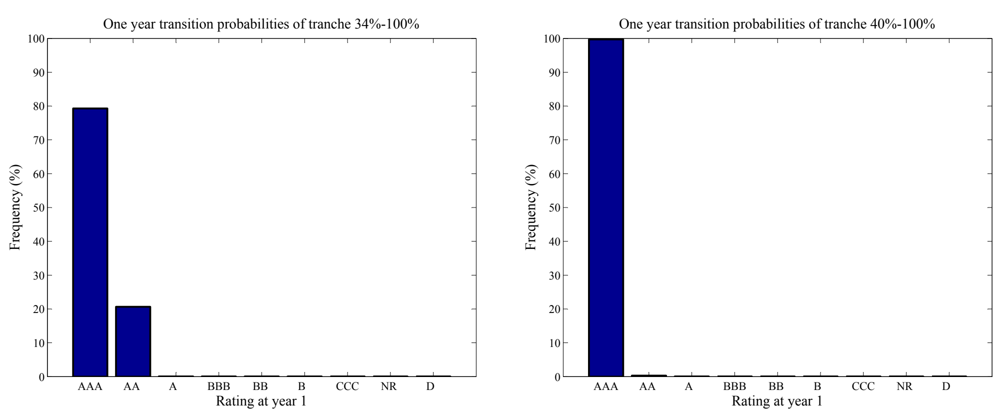


Figure 6.15: One year transition probabilities of the tranches 34%-100% and 40%-100%, both with current AAA rating category.

6.6.3 Tranche-based rating transitions

Since the transition probabilities depend considerably on the structure of the CDO and its rating history and not only on the current ratings, one can not merge time series of rating transitions from different tranches to construct a global "CDO rating transition matrix" and transition probabilities should be simulated for each particular tranche. In this new representation, transition probabilities can not be historical since the matrices are specific to this particular CDO structure and historical data are therefore not available. Hence, transition probabilities should be simulated. As an example, we simulate as follows the 2-4 years transition matrices for the tranches 34%-100% and 24%-34% of the Stylized CDO (see tables 6.8 and 6.9). It is interesting to observe that the transition probability from AA to AA between years 2 and 4 is 50.61% for the tranche 34%-100% while it is 94.01% for the tranche 24%-34%.

Transition matrix 2-4 years for the tranche 34%-100%									
	AAA	AA	A	BBB	BB	B	CCC	NR	D
AAA	100	0	0	0	0	0	0	0	0
AA	47.16	50.61	2.22	00.01	0	0	0	0	0
A	0	0	21.65	55.96	22.39	0	0	0	0
BBB	0	0	0	0	10.43	29.36	28.51	4.71	26.98
BB	0	0	0	0	0	0	0	0	100
B	0	0	0	0	0	0	0	0	100
CCC	0	0	0	0	0	0	0	0	100
NR	0	0	0	0	0	0	0	0	100
D	0	0	0	0	0	0	0	0	100

Table 6.8: Transition matrix for the tranche $34\% - 100\%$ of the Stylized CDO for the period 2-4 years, computed with the one factor Gaussian copula model with $\rho = 30\%$.

Transition matrix 2-4 years for the tranche 24%-34%									
	AAA	AA	A	BBB	BB	B	CCC	NR	D
AAA	0	0	0	0	0	0	0	0	0
AA	0	94.01	5.99	0	0	0	0	0	0
A	0	0.05	91.40	08.07	0.48	0	0	0	0
BBB	0	0	0	3.04	72.92	12.40	10.99	0	0.66
BB	0	0	0	0	0	0	7.42	5.85	86.72
B	0	0	0	0	0	0	0	0	100
CCC	0	0	0	0	0	0	0	0	100
NR	0	0	0	0	0	0	0	0	100
D	0	0	0	0	0	0	0	0	100

Table 6.9: Transition matrix for the tranche $24\% - 34\%$ of the Stylized CDO for the period 2-4 years, computed with the one factor Gaussian copula model with $\rho = 30\%$.

6.7 Towards a risk-based rating methodology

Default probability does not perfectly reflect the credit risk of a financial security since it does not account for the whole distribution of the loss but only for the lower tail of the loss distribution (probability that the loss exceeds a certain threshold). The probability of loosing \$1 could be equal to the probability of losing \$1 million but it is clear that given default, the impact of the second default scenario is significantly greater than the first one. With corporate bonds, default scenarios have all the same impact since the loss is always equal to nominal×(1-recovery), then the default probability is a good risk measure. More precisely, since the default probability describes entirely the loss distribution of a bond with known recovery rate, any risk measure will be equivalent to the default probability and therefore the question of adequate choice of risk measure does not arise. However in the case of a CDO, default scenarios can be very different and the losses can be very different so that the default probability can not be sufficient to translate credit risk. For example, since default probability depends only on the subordination level, it can not differentiate between a tranche $6\% - 7\%$ and a tranche $6\% - 100\%$ while the loss of the latter is always smaller relatively to the initial notional of the tranche. If the total loss of the portfolio is 7%, an investor in the tranche $6\%-7\%$ loses the totality of his investment while an investor in the tranche $6\% - 100\%$ loses only about 1% of his investment. Other risk measures could therefore be more informative.

6.7.1 Measures of downside risk

We present in this section some of most commonly used risk measures and discuss their possible use as measures of credit risk in the rating procedure.

Value-at-Risk

VaR_α^j: Value-at-Risk at level α defined as the right-tail α-quantile of the loss distribution of the corresponding tranche $a_j - a_{j+1}$:

$$VaR_\alpha^j = \inf\{l \in R; \mathbb{P}(M_j(t) > l) \leq 1 - \alpha\} \tag{6.23}$$

While Value-at-Risk has broad applications in Risk Management, it seems to be unappropriate in the context of CDO ratings since it may fail to account for the severity of losses beyond the quantile threshold. Moreover, its threshold-based nature leads to awkward results:

- Value-at-Risk of all tranches whose detachment point is lower than the right-tail $\alpha-$quantile χ_α of the distribution of the portfolio total loss is equal to one. Suppose $a_{j+1} < \chi_\alpha$, then

$$\mathbb{P}(M_j(t) > 1) = 0 \tag{6.24}$$

$$\mathbb{P}(M_j(t) = 1) = \mathbb{P}(L(t) > a_{j+1}) > 1 - \alpha, \tag{6.25}$$

  Hence the Value-at-Risk VaR_α^j of the tranche $a_j - a_{j+1}$ is equal to 1.

- Value-at-Risk of all tranches whose attachment point is larger than the right-tail $\alpha-$quantile of the distribution of the total loss is equal to zero. Suppose $a_j > \chi_\alpha$, then

$$\mathbb{P}(M_j(t) > 0) = \mathbb{P}(L(t) > a_j) \leq 1 - \alpha, \tag{6.26}$$

  Hence the Value-at-Risk VaR_α^j of the tranche $a_j - a_{j+1}$ is equal to 0.

Thus, Value-at-Risk does not differentiate between tranches of different credit risk quality and can not be used as a measure of the risk quality of the tranches of a CDO.

Expected Shortfall

ES_α^j: Expected Shortfall at level α –also called Conditional Value-at-Risk or Tail Conditional Expectation– defined as the expected loss incurred by tranche $a_j - a_{j+1}$ given that the loss exceeds the right-tail α-quantile of the loss distribution that is the Value-at-Risk at level α:

$$ES_\alpha^j(t) = \mathbb{E}[M_j(t)|M_j(t) > VaR_\alpha^j] \tag{6.27}$$

Expected Shortfall suffers from a similar problem as the VaR_α^j since it gives the value one for all tranches whose detachment point is lower than the right-tail $\alpha-$ quantile of the distribution of the portfolio total loss. If α is large enough then this problem has a limited impact because in this case the right-tail $\alpha-$ quantile is small enough so that the tranches with $ES_\alpha = 1$ are the non-rated tranches.

Minvar

Another example is the so-called Minvar, introduced by Cherny and Madan (2006), defined as the expected maximum of N independent copies of the loss:

$$Minvar_N^j(t) = \mathbb{E}[\max\{M_j^1(t), M_j^2(t), ..., M_j^N(t)\}]$$

This can be interpreted as the expected worst loss for the investor in N independent scenarios.

The expected loss criterion used by Moody's is a particular example of Minvar with only one loss scenario (N=1). Thus, it fails to properly assess the impact of rare but extreme events. Instead, the $Minvar_N$ with N sufficiently large is able to capture the entire loss distribution including extreme values. Also, it is able to differentiate between tranches of different risk quality: the higher the risk of the tranche the higher the Minvar. Minvar appears then to be a more suitable measure of the credit worthiness of CDO tranches.

6.7.2 A risk-based rating method

We consider as an example a rating method based on the Minvar criterion. Rating cut-offs are subjective parameters decided by each rating agency and reflects in a certain way its "severeness". We convert the S&P cut-offs into Minvar so that they will give the same rating if applied to a corporate bond with recovery rate $R = 40\%$. For such a Bond the loss is 0 with probability $1 - p$ and $1 - R$ with probability p, hence the expected maximum of N independent copies of the loss is $1 - R$ if at least one of the losses is equal to $1 - R$ and 0 otherwise; which leads to:

$$Minvar_N(t) = (1 - R)(1 - (1 - p)^N) \tag{6.28}$$

The cut-offs obtained are presented in table 6.10. Using these cut-offs and the

Minvar rating criterion, we rate the Stylized CDO (see figure 6.16) in the one factor Gaussian copula framework with correlation $\rho = 30\%$. We observe that the $24\% - 34\%$ tranche originally rated AA with the default probability criterion is downgraded by 2 notches, and all other more junior tranches are downgraded to the NR category. The Minvar is a much more conservative measure of credit risk than the probability of incurring a loss.

Rating	Default probability rating cut-off (%)	$Minvar_{20}$ rating cut-off (%)
AAA	0.144	1.704
AA	1.069	11.605
A	3.476	30.429
BBB	9.959	52.638
BB	24.709	59.794
B	43.347	59.999
CCC	67.529	60.000

Table 6.10: Equivalent $Minvar_{20}$ rating cut-offs.

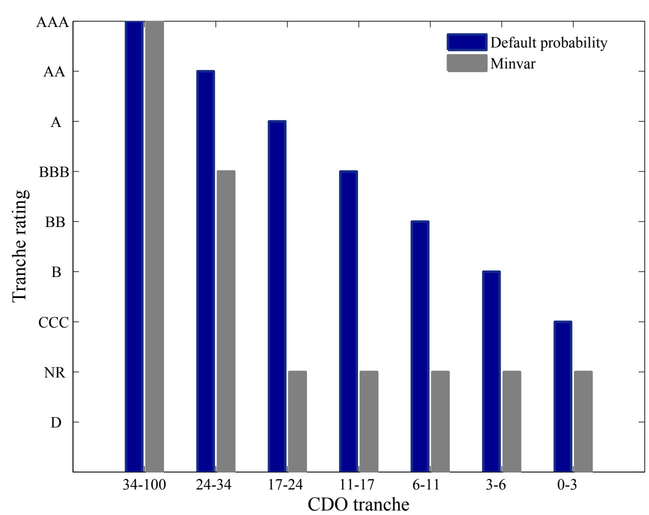


Figure 6.16: The impact of using the $Minvar$ as a risk measure for credit ratings: almost all tranches originally rated with the default probability criteria are downgraded when using $Minvar_{20}$.

6.8 Modeling default clustering

6.8.1 Modeling default clustering using copulae

The Gaussian copula model has been extensively criticized in the past few years mainly for its inability to generate scenarios with simultaneous defaults or default clusters. We refer the reader to appendix A for an overview of copulae. Several alternatives to the Gaussian copula have been suggested in the financial literature. Examples are the Gumbel, Clayton, Cauchy or t-copula models. A common characteristic of these models is that they allow a dependence in the lower tail of the distribution of default times. That is, the occurrence of one default increases the probability of occurrence of other defaults in the reference portfolio. We recall the definition of the coefficient of lower tail dependence defined in Embrechts et al. (2001):

Definition 19 (Coefficient of lower tail dependence)**.** *Let X_1 and X_2 be two continuous random variables with cumulative marginal distributions F_1 and F_2. Then, the coefficient of lower tail dependence between X_1 and X_2 is,*

$$\lambda_l(X_1, X_2) = \lim_{q \searrow 0} \mathbb{P}(X_2 \leq F_2^{-1}(q) | X_1 \leq F_1^{-1}(q)) \tag{6.29}$$

If $\lambda_l(X_1, X_2) > 0$ then X_1 and X_2 are said to show lower tail dependence. If $\lambda_l(X_1, X_2) = 0$ then X_1 and X_2 are said to be asymptotically independent in the lower tail.

Embrechts et al. (2001) show that the Gaussian copula does not present lower tail dependence ($\lambda_U(X, Y) = 0$) while the Student copula does. The lower tail dependence coefficient of the Student copula with ν degrees of freedom and with homogeneous correlation ρ is,

$$\lambda_l = 2\left(1 - F_{\nu+1}((\nu + 1)^{1/2} \frac{(1-\rho)^{1/2}}{(1+\rho)^{1/2}})\right), \tag{6.30}$$

where $F_{\nu+1}$ is the cumulative distribution function of a Student distribution with $\nu+1$ degrees of freedom.

The Cauchy copula is a particular case of the Student copula with one degree of freedom ($\nu = 1$). For example, the coefficient lower tail dependence of a Cauchy copula with correlation $\rho = 30\%$ is $\lambda_l = 0.4084$. This lower tail dependence is illustrated in figure 6.17 where we observe a clear upper-right and lower-left quadrant tail of the bivariate distribution of X_1 and X_2 when the dependence is modeled by a Cauchy copula (in both cases of Gaussian and Exponential marginal distributions). The interested reader is referred to Embrechts et al. (2001) for a broad overview on copulas.

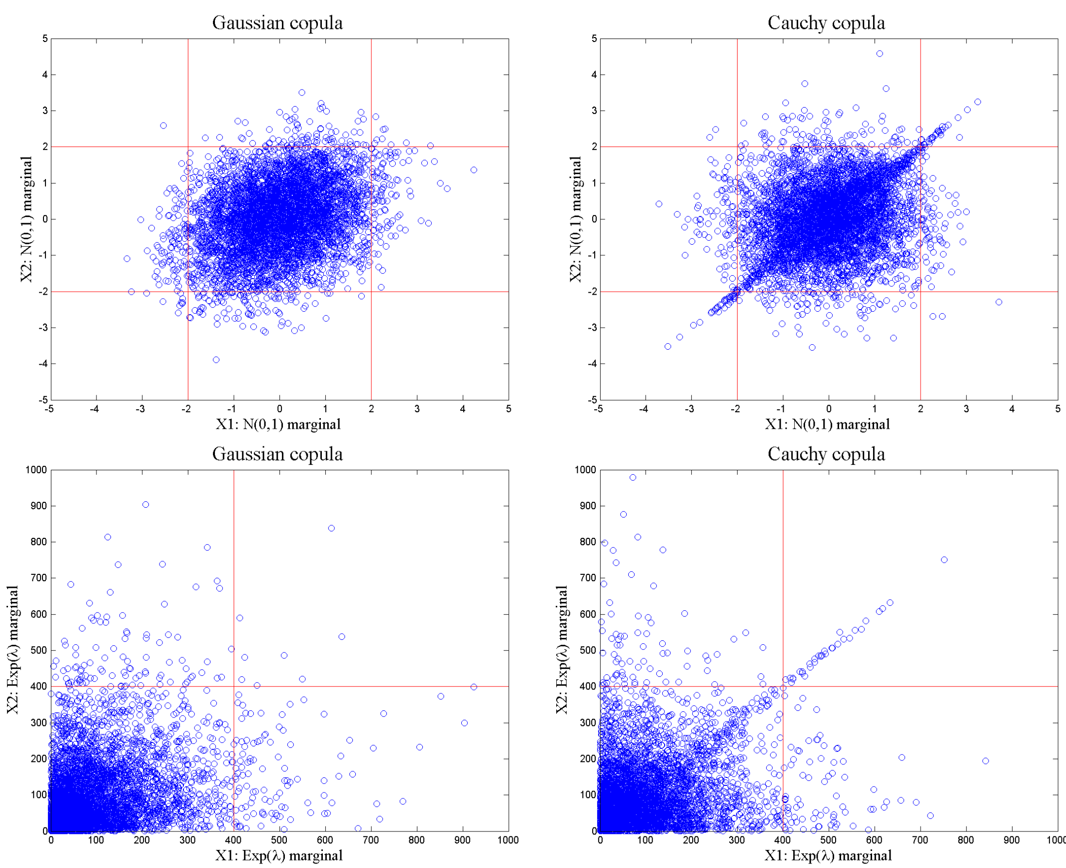


Figure 6.17: Tail dependence in the Gaussian and Cauchy copulae

6.8.2 Impact of using a Cauchy copula in rating outstanding CDO tranches

We study the impact of using a one factor Cauchy copula model -instead of the one factor Gaussian copula- for the joint default times on the credit ratings of CDO tranches:

$$G^{-1}(F(\tau_i)) = \rho S + (1-\rho)Z_i, \tag{6.31}$$

where $S, Z_1, ..., Z_n$ are IID standard Cauchy random variables, G is the cumulative distribution function of a standard Cauchy, and F the cumulative distribution function of an Exponential with parameter λ.

We rate the tranches of the Stylized CDO using this model with dependence parameter $\rho = 30\%$. We note (see figure 6.18) that the use of the Cauchy copula implies a downgrade of the senior tranches (originally rated with the Gaussian copula model with correlation parameter $\rho = 30\%$). In fact, since the Cauchy copula exhibits an lower tail dependence, it allows more scenarios with extreme events than the Gaussian copula, which explains the more severe rating of the senior tranches. For example, considering two obligors in the reference portfolio with respective default times τ_1 and τ_2 , the probability $\mathbb{P}(\tau_2 < 7|\tau_1 < 7)$ that the second obligor defaults before the 7 years horizon given that the first has defaulted is 29.94% when using a Cauchy copula with dependence parameter $\rho = 30\%$ and 18.30% when using a Gaussian copula with correlation parameter $\rho = 30\%$.

6.8.3 Impact of using a Cauchy copula in tranching a portfolio

The choice of the statistical model of the dependence between default times has also an influence on the tranching of the portfolio of underlying assets. We compare

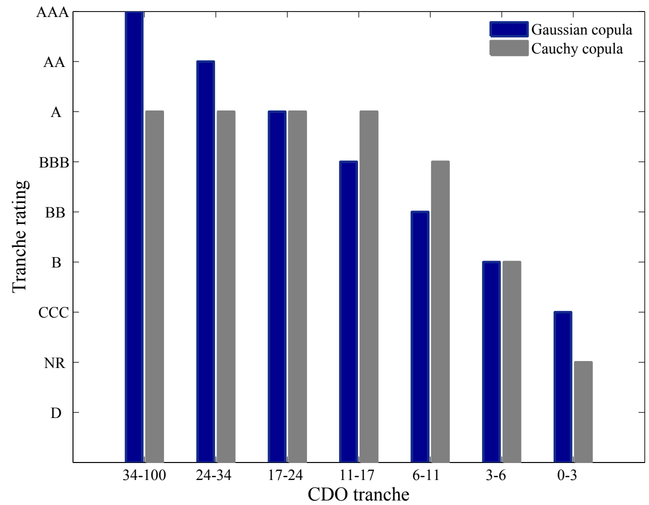


Figure 6.18: The impact of using a Cauchy copula for the joint default times.

the attachment levels obtained to maximize credit ratings when using the Cauchy copula versus the Gaussian copula. We observe (see table 6.11) that the Cauchy copula requires much more credit enhancement for the senior tranches than the Gaussian copula.

	Gaussian copula	Cauchy copula
Rating	Attachment level (%)	Attachment level (%)
AAA	34	60
AA	24	57
A	17	6
BBB	11	4
BB	6	3.3
B	3	3
CCC	2	2.6

Table 6.11: Influence of statistical model of the default times on the CDO structure: the Cauchy copula model requires additional credit enhancement for the senior tranches than the Gaussian copula.

The one-year transition probabilities of the tranche 60%-100% computed using the Cauchy copula model with dependence parameter $\rho = 30\%$ are displayed in figure 6.19. The tranche remains in one year in the AAA category with probability

99.78%.

Thus, the use in the tranching procedure of the Cauchy copula as statistical model of the dependence structure between default times allows defining senior CDO tranches with additional credit enhancement leading to a stability of credit ratings across time.

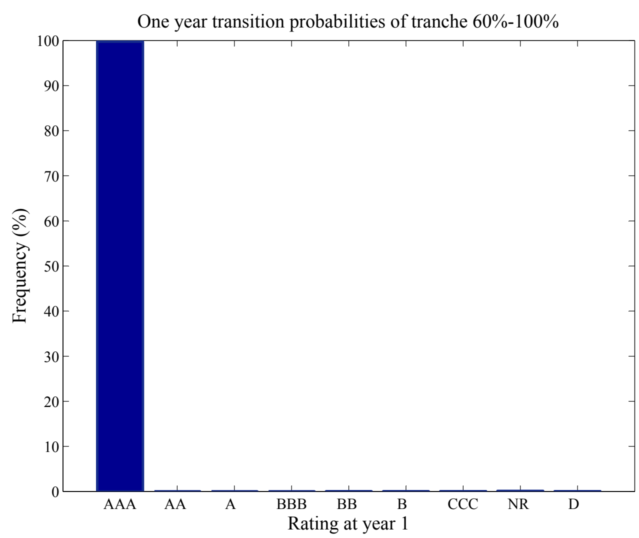


Figure 6.19: One-year transition probabilities of the AAA tranche 60%-100% computed using the Cauchy copula model with dependence parameter $\rho = 30\%$. The tranche remains in one year in the AAA category with probability 99.78%.

Copula models with lower tail dependence such as the Cauchy copula described above are able to capture default clustering in the sense that they increase the probability of future defaults given that one asset has already defaulted in the reference portfolio. However, factor models with continuous joint distributions of the default times fail to model simultaneous failures. Nonetheless, the 2007-2009 have underlined the importance of accounting for the possibility of simultaneous defaults, arising from a common exposure to market factors or from contagion effects through mutual exposures between the underlying assets. Other models have incorporated default clustering and contagion effects in the intensity of the default event (Peng and Kou, 2009; Duffie et al., 2009; Azizpour et al., 2010), as apposed to assuming a constant intensity.

Appendix A

A brief introduction to copulae

We present in this appendix a brief review of copulae. We refer the reader to Nelsen (1999); Embrechts et al. (2001); Schmidt (2008).

A.1 Definition and properties

Definition 20 (Copula)**.** *A copula $C(u_1, ..., u_n)$ is the multivariate joint distribution of random variables U_1,...,U_n defined on the cube $[0,1]^n$ with standard uniform marginal distributions.*

Property 1 (General properties)**.** *If C is a copula, then it satisfies the following properties:*

1. *$C(u_1, ..., u_n)$ is increasing in each of the components.*
2. *$C(u_1, ..., u_i, ..., u_n) = u_i \quad$ for all $\quad 1 \leq i \leq n$.*

3. *Rectangle inequality:*

$$\sum_{(i_1,...,i_n)\in\{1,2\}^n} (-1)^{i_1+...+i_n} C(u_{1,i_1}, ..., u_{n,i_n}) \geq 0$$

where,

$$a_i \leq b_i \quad \text{for all} \quad 1 \leq i \leq n \qquad u_{j,1} = a_j \qquad u_{j,2} = b_j$$

The reverse is also true: if a multivariate function C defined on the cube $[0, 1]^n$ satisfies the above properties then it is copula.

The main purpose of a copula is to separate the dependence structure of a random vector from its marginals.

Theorem 1 (Sklar's theorem)**.** *Consider a n-dimensional cumulative distribution F with marginals $F_1, ..., F_n$. There exists a copula C, such that*

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)) \quad \forall x_i \in [-\infty, \infty], i = 1, ..., n \tag{A.1}$$

If F_i is continuous for all $i = 1, ..., n$ then C is unique,

$$C(x_1, ..., x_n) = F(F_1^{-1}(x_1), ..., F_n^{-1}(x_n)) \tag{A.2}$$

Conversely, consider a copula C and univariate cumulative distributions $F_1, ..., F_n$, then F defined as in equation A.1 is a multivariate cumulative distribution with marginals $F_1, ..., F_n$.

Property 2 (Invariance under transformation)**.** *Consider a sequence of real-valued strictly increasing transformations $T_1, ..., T_n$ defined on $[-\infty, \infty]$, then the random variables $X_1, ..., X_n$ and $T_1(X_1), ..., T_n(X_n)$ have the same copula.*

Property 3 (Fréchet-Hoeffding bounds)**.** *Consider a n-dimensional copula C, then,*

$$\max(\sum_{i=1}^{n} u_i + 1 - n, 0) \leq C(u_1, ..., u_n) \leq \min(u_1, ..., u_n) \tag{A.3}$$

A.2 Important examples of copulae

Definition 21 (Gaussian copula)**.** *For a $n \times n$-correlation matrix Γ, the n-dimensional Gaussian copula is defined as*

$$C_\Gamma(u_1, ..., u_n) = \Phi_\Gamma(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)), \tag{A.4}$$

where Φ_Γ is the n-dimensional Gaussian distribution with correlation matrix Γ, and Φ is the univariate standard Gaussian cumulative distribution.

Definition 22 (Student copula)**.** *For a $n \times n$-correlation matrix Γ and ν degrees of freedom, the n-dimensional Student (or t) copula is defined as*

$$C_{\nu,\Gamma}(u_1, ..., u_n) = t_{\nu,\Gamma}(t_\nu^{-1}(u_1), ..., t_\nu^{-1}(u_n)), \tag{A.5}$$

where $t_{\nu,\Gamma}$ is the n-dimensional Student distribution with ν degrees of freedom and correlation matrix Γ, and t_ν is the univariate Student distribution with ν degrees of freedom.

Remark 2. *The Gaussian and Cauchy distributions are extreme cases of a Student distribution with respective degrees of freedom $\nu = \infty$ and $\nu = 1$.*

Definition 23 (α-stable distribution)**.** *A real-valued random variable X is said to be strictly α-stable if for any n independent copies $X_1, ..., X_n$ of X and constants $k_1, ..., k_n$, the random variable $k_1 X_1 + .. + k_n X_n$ has the same distribution as $(\sum_{i=1}^n |k_i|^\alpha)^{\frac{1}{\alpha}} X$.*

Example 2. *(Important examples of α-stable distributions)*

1. *The Gaussian distribution is 2-stable.*

2. *The Cauchy distribution is 1-stable.*

A.3 Tail Dependence

Definition 24 (Coefficient of upper tail dependence). *Let X_1 and X_2 be two continuous random variables with cumulative marginal distributions F_1 and F_2 and copula C. Then, the coefficient of upper tail dependence between X_1 and X_2 is,*

$$\lambda_u(X_1, X_2) = \lim_{q \nearrow 1} \mathbb{P}(X_2 > F_2^{-1}(q) | X_1 > F_1^{-1}(q)) \tag{A.6}$$

If $\lambda_u(X_1, X_2) > 0$ then X_1 and X_2 are said to show upper tail dependence. If $\lambda_u(X_1, X_2) = 0$ then X_1 and X_2 are said to be asymptotically independent in the upper tail.

Definition 25 (Coefficient of lower tail dependence). *Let X_1 and X_2 be two continuous random variables with cumulative marginal distributions F_1 and F_2 and copula C. Then, the coefficient of lower tail dependence between X_1 and X_2 is,*

$$\lambda_l(X_1, X_2) = \lim_{q \searrow 0} \mathbb{P}(X_2 \leq F_2^{-1}(q) | X_1 \leq F_1^{-1}(q)) \tag{A.7}$$

If $\lambda_l(X_1, X_2) > 0$ then X_1 and X_2 are said to show lower tail dependence. If $\lambda_l(X_1, X_2) = 0$ then X_1 and X_2 are said to be asymptotically independent in the upper lower tail.

Proposition 8 (Tail dependence for Gaussian copula). *A Gaussian copula has no tail dependence if the correlation is not equal to 1 or -1.*

Proposition 9 (Tail dependence for Student copula). *The bivariate Student distribution with ν degrees of freedom and correlation ρ has tail dependence provided that $\rho > -1$.*

$$\lambda_u = \lambda_l = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right) \tag{A.8}$$

Appendix B

Power law

We present in this appendix a brief introduction to power law distributions. We refer the reader to Clauset et al. (2009) and Clementi et al. (2006) for an overview of power law distributions and their use to fit empirical data.

B.1 Definition and properties

Definition 26 (Regularly varying distribution). *A regularly varying distribution is a distribution whose density (or probability mass) function is of the form*

$$p(x) = cL(x)x^{-1-\alpha}, \tag{B.1}$$

where c is a positive constant, $\alpha > 0$ and $L(x)$ is a slowly varying function, i.e. it satisfies

$$lim_{x\to\infty}\frac{L(tx)}{L(x)} = 1 \quad \forall t \in [-\infty, \infty] \tag{B.2}$$

The form of L controls the shape and finite extent of the lower tail.

Definition 27 (Continuous power law distribution). *If p is continuous, $L(x)$ is the constant function and there is a lower bound x_{min} from which the law holds, the* *power law distribution has a density function of the form:*

$$p(x) = \frac{\alpha}{x_{min}} \left(\frac{x_{min}}{x}\right)^{1+\alpha} \quad for \quad x \geq x_{min} \tag{B.3}$$

In this case, the distribution is called Pareto distribution.

Proposition 10 (First and second moments for the Pareto distribution)**.** *The expected value of a random variable X following a Pareto distribution with $\alpha > 1$ and lower bound x_{min} is*

$$\mathbb{E}[X] = \frac{\alpha x_{min}}{\alpha - 1} \tag{B.4}$$

The variance is for $\alpha > 2$,

$$\mathbf{V}(X) = \left(\frac{x_{min}}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2} \tag{B.5}$$

The variance is undefined for $\alpha \leq 2$.

Definition 28 (Discrete power law distribution)**.** *If p is defined on the integers, $L(x)$ is the constant function and there is a lower bound x_{min} from which the law holds, the power law distribution has a density function of the form:*

$$p(x) = \frac{x^{-1-\alpha}}{\zeta(\alpha, x_{min})} \quad for \quad x \geq x_{min} \tag{B.6}$$

where $\zeta(\alpha, x_{min})$ is the generalized or Hurwitz zeta function.
In this case, the distribution is called Zipf's law or zeta-distribution.

Definition 29 (Generalized Pareto distribution)**.** *The generalized Pareto distribution with shape parameter $\xi \neq 0$, scale parameter σ and location parameter μ, admits the probability density function*

$$f_{\xi,\sigma,\mu}(x) = \frac{1}{\sigma} \left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-1-\frac{1}{\xi}}, \tag{B.7}$$

for $x > \mu$ when $\xi > 0$, or for $\mu < x < -\frac{\sigma}{\xi}$ when $\xi < 0$.
For $\xi = 0$ the density is

$$f_{\sigma,\mu}(x) = \frac{1}{\sigma} \exp\left(\frac{x - \mu}{\sigma}\right) \tag{B.8}$$

Proposition 11. *If $\xi = 0$ and $\mu = 0$, the generalized Pareto distribution is equivalent to the exponential distribution. If $\xi > 0$ and $\mu = \frac{\sigma}{\xi}$, the generalized Pareto distribution is equivalent to the Pareto distribution with parameter $\alpha = \frac{1}{\xi}$ and lower bound $x_{min} = \frac{\sigma}{\xi}$.*

B.2 Maximum likelihood estimation of the tail exponent

Proposition 12 (Maximum likelihood estimator for α in the continuous case)**.** *Consider a sample $x_1, ..., x_n$ from a Pareto distribution with parameter α and lower bound x_{min}. Assuming x_{min} known, the maximum likelihood estimator for α is*

$$\widehat{\alpha} = n \left[\sum_{i=1}^{n} \ln \frac{x_i}{x_{min}} \right]^{-1} \tag{B.9}$$

Its standard error is

$$\sigma(\widehat{\alpha}) = \frac{\widehat{\alpha} - 1}{\sqrt{n}} \tag{B.10}$$

Proposition 13 (Maximum likelihood estimator for α in the discrete case)**.** *Consider a sample $x_1, ..., x_n$ from a Zipf's law with parameter α and lower bound x_{min}. Assuming x_{min} known, the maximum likelihood estimator for α is the one that maximized the log-likelihood*

$$\mathcal{L}(\alpha) = -n \ln \zeta(\alpha, x_{min}) - \alpha \sum_{i=1}^{n} x_i \tag{B.11}$$

with standard error

$$\sigma(\widehat{\alpha}) = \frac{1}{\sqrt{n \left[\frac{\zeta''(\widehat{\alpha}, x_{min})}{\zeta(\widehat{\alpha}, x_{min})} - \left(\frac{\zeta'(\widehat{\alpha}, x_{min})}{\zeta(\widehat{\alpha}, x_{min})} \right)^2 \right]}} \tag{B.12}$$

Clauset et al. (2009) derive an approximation of $\widehat{\alpha}$ as follow

$$\widehat{\alpha} \simeq n \left[\sum_{i=1}^{n} \ln \frac{x_i}{x_{min} - \frac{1}{2}} \right]^{-1} \tag{B.13}$$

which is in fact identical to the MLE for the continuous case except for the $-\frac{1}{2}$ in the denominator.

B.3 Fitting a power law to empirical data

Consider a data set $y_1, ..., y_n$. A log-log plot of the empirical cumulative distribution of $y_1, ..., y_n$ shows a linear decay in the tail.

We propose then to fit the tail with a power law distribution (Pareto if the y's are continuous real numbers and Zipf's law if the y's can only take discrete values). That is, we aim to estimate the lower bound y_{min} and tail exponent α of the power law.

A heuristic method to determine the lower bound is by visually identifying the threshold y_{min} above which the log-log plot of the empirical cumulative distribution exhibits a linear trend. Then, the parameter α is estimated by maximum likelihood using y_{min} as lower bound. This approach is clearly subjective and sensitive to the heuristic choice of the lower bound.

A more objective approach is proposed in Clauset et al. (2009), who suggest to select the lower bound y_{min} that maximizes a measure of goodness fit of the empirical cumulative distribution of the data set by a power law with parameter y_{min} and tail exponent $\widehat{\alpha}(y_{min})$ which is the maximum likelihood estimator of α using y_{min} as lower bound.

The goodness of fit is measured by the Kolmogorov-Smirnov (KS) statistic, which is the maximum between the empirical cumulative distribution of the data and the fitted model:

$$KS = \max_{y \geq y_{min}} |\widehat{F}_{y_{min}}(y) - F_{\widehat{\alpha}(y_{min}), y_{min}}(y)|, \tag{B.14}$$

where $\widehat{F}_{y_{min}}(y)$ is the empirical cumulative distribution of the observations with value of at least y_{min} and $F_{\widehat{\alpha}(y_{min}),y_{min}}(y)$ is the cumulative distribution of the power law with parameter $\widehat{\alpha}(y_{min})$ and lower bound y_{min}.

The estimate of the lower bound is then the value y_{min} that minimizes KS.

This method is based on the idea that a choice y_{min} below the true value would imply a poor fit of the tail since the lower end of the tail would not follow a power law, and a choice y_{min} above the true value would lead to a fit not as good as the one obtained when using y_{min} as a lower bound since the sample size would be smaller.

Then, using the Kolmogorov-Smirnov test, we check the goodness of fit by testing the hypothesis that the power law with parameter $\widehat{\alpha}(y_{min})$ and lower bound y_{min} generates the observed empirical cumulative distribution of the observations with value of at least y_{min}, that is

$$H_0 : \widehat{F}_{y_{min}}(y) = F_{\widehat{\alpha}(y_{min}),y_{min}}(y) \quad versus \quad H_1 : \widehat{F}_{y_{min}}(y) \neq F_{\widehat{\alpha}(y_{min}),y_{min}}(y) \tag{B.15}$$

Appendix C

Capital requirements under Basel 2

We present in this appendix the framework proposed by Basel 2 accords for assessing the total capital ratio required for each banking institution in the financial system. We refer the reader to BIS (2001) and Duellmann (2006) for more details.

The total capital ratio (TCR) is defined in Basel 2 accords as

$$TCR = \frac{TC - RD_{CrD} - Ded}{RWA_{CrD} + 12.5VaR_{MkR} + 12.5VaR_{OpR}} \tag{C.1}$$

where:

- TC: total capital consisting of Tier 1, Tier 2 and Tier 3 capital. Besides the 8% minimum for the TCR, banks also need to set a minimum of 4% for their Tier 1 ratio.
- RD_{CrD}: regulatory difference between expected loss and provisions for credit risk.

- Ded: deductions.
- RWA_{CrD}: risk-weighted assets for credit risk.
- VaR_{MkR}: value at risk for market risk.
- VaR_{OpR}: value at risk for operational risk.

The cushion for market risk is estimated as the 99% 10 days value-at-risk of the total net exposure:

$$VaR_{MkR} = \Phi^{-1}(0.99)\sqrt{10}\ SD(|\sum_{j=1}^{n} E_{ij} - \sum_{j=1}^{n} E_{j,i}|), \tag{C.2}$$

where SD denotes the Standard Deviation.

The cushion for credit risk is calculated as the risk-weighted-assets to absorb the unexpected losses for all individual credit claims in the portfolio:

$$RWA_{CrR} = 12.5 \sum_{i=1}^{n} UL_i \tag{C.3}$$

The losses caused by default fluctuate depending on the severity of default events. Financial institutions never know in advance the losses that they will experience but they can forecast the expected loss by estimating the proportion of institutions that might default within a given time horizon, multiplied by the outstanding exposure at default when assuming zero recovery rate. The *expected loss* (EL_i) of institution i in the network is evaluated as:

$$EL_i = \sum_{j=1}^{n} E_{ij} p_j \tag{C.4}$$

where p_j is the marginal default probability of institution j in the network.

Whereas the *expected loss* (EL) is the mean of the loss distribution, the *unexpected loss* (UL) is defined as the difference between an adverse 99.9%-percentile of the portfolio loss distribution and EL. Therefore, the evaluation of the unexpected loss requires a statistical model on the default of the portfolio. Basel 2 accords propose to use the *asymptotic single risk factor model* (ASRF) according to which the variation ΔE_{ij} of the asset value E_{ij} that institution i holds in institution j depends on a systemic (common) risk factor S and an idiosyncratic (specific) risk factor Z_i with an asset value correlation ρ :

$$\Delta E_{ij} = \sqrt{\rho} S + \sqrt{1-\rho} Z_i \tag{C.5}$$

Default occurs if the variation of E_{ij} falls below a default threshold κ_i. Assuming S to be distributed as a standard gaussian and the Z_i are iid standard gaussian random variables independent of S, the default probability conditional on S is then given by:

$$PD_i(S) = \mathbb{P}(\Delta E_{ij} < \kappa_i | S) = \Phi(\frac{\kappa_i - \sqrt{\rho} S}{\sqrt{1-\rho}}) \tag{C.6}$$

where $\Phi(x)$ is a standard normal cumulative density function.

The unexpected loss is computed as the difference between the conditional default probability given the 99.9%-percentile adverse movement of systematic risk factor and the marginal default probability, multiplied by the loss given default assuming zero recovery rate:

$$UL_i = \sum_{j=1}^{n} E_{ij}[\Phi(\frac{\Phi^{-1}(p_j) - \sqrt{\rho}\Phi^{-1}(1-0.999)}{\sqrt{1-\rho}}) - p_j] \tag{C.7}$$

Bibliography

Acharya, V., Pedersen, L., Philippon, T., and Richardson, M. (2010). Measuring systemic risk. Working paper.

Adelson, M. (2010). Big changes in Standard & Poor's rating criteria. Technical report, Standard & Poor's.

Adrian, T. and Brunnermeier, M. K. (2008). CoVaR. Staff Report 348, Federal Reserve Bank of New York.

Ágnes Lublóy (2006). Topology of the Hungarian large-value transfer system. Occasional Papers 57, Magyar Nemzeti Bank, Budapest.

Allen, F. and Gale, D. (2000). Financial contagion. *Journal of Political Economy*, 108:1–33.

Amini, H., Cont, R., and Minca, A. (2010). Resilience to contagion in financial networks. Working paper.

Amini, H., Cont, R., and Minca, A. (2011). Stress testing the resilience of financial networks. *International Journal of Theoretical and Applied Finance*, 14.

Avellaneda, M., Buff, R., Friedman, C., Grandchamp, N., Kruk, L., and Newman, J. (2001). Weighted monte carlo: A new technique for calibrating asset-pricing models. *International Journal of Theoretical and Applied Finance*, 4:1–29.

Azizpour, S., Giesecke, K., and Schwenkler, G. (2010). Exploring the sources of default clustering.

Babus, A. (2006). The formation of financial networks. Tinbergen institute discussion papers, Tinbergen Institute.

Barabasi, A.-L. and Albert, R. (2002). Emergence of scaling in random networks. *Science*, (286):509–512.

Battiston, S., Gatti, D. D., Gallegati, M., Greenwald, B. C., and Stiglitz, J. E. (2009). Liaisons dangereuses: Increasing connectivity, risk sharing, and systemic risk. Working Paper 15611, National Bureau of Economic Research.

BCB (2000a). Resolution 2,723, Brazilian Central Bank, Brazil. May 31th.

BCB (2000b). Resolution 2,743, Brazilian Central Bank, Brazil. June 28th.

BCB (2007a). Resolution 3,444, Brazilian Central Bank, Brazil. February 28th.

BCB (2007b). Resolution 3,490, Brazilian Central Bank, Brazil. August 29th.

Bernanke, B. S. (2009). Financial reform to address systemic risk. speech delivered at the at the council on foreign relations, Washington DC.

BIS (2001). Overview of the new basel capital accord. Consultative document, Bank for International Settlements.

Bollobás, B., Borgs, C., Chayes, J., and Riordan, O. (2003). *Directed scale-free graphs*. Society for Industrial and Applied Mathematics.

Boss, M., Elsinger, H., Lehar, A., and Summer, M. (2004). The network topology of the interbank market. *Quantitative Finance*, 4:677 – 684.

Boyd, S. and Vandenberghe, L. (2004). *Convex Optimization*. Cambridge University Press.

Cherny, A. and Madan, D. (2006). On measuring the degree of market efficiency. Technical report.

Clauset, A., Shalizi, C. R., and Newman, M. E. J. (2009). Power-law distributions in empirical data. Working Paper.

Clementi, F., Di Matteo, T., and Gallegati, M. (2006). The power-law tail exponent of income distributions. *Physica A*, 370:49.

Cohen, R., Havlin, S., and ben Avraham, D. (2003). Efficient immunization strategies for computer networks and populations. *Phys. Rev. Lett.*, 91(24):247901.

Cont, R. and Jessen, C. (2011). Constant proportion debt obligations (cpdos): modeling and risk analysis. *Quantitative Finance*.

Cont, R. and Moussa, A. (2010a). A closer look at credit ratings of CDO tranches. Financial engineering report, Columbia University.

Cont, R. and Moussa, A. (2010b). Reconstruction of interbank networks. Financial engineering report, Columbia University.

Cont, R. and Moussa, A. (2010c). Too interconnected to fail: contagion and systemic risk in financial networks. Financial engineering report, Columbia University.

Cont, R., Moussa, A., and e Santos, E. B. (2010). Network structure and systemic risk in banking systems.

Cont, R. and Tanimura, E. (2008). Small world graphs: characterization and alternative constructions. *Advances in Applied Probability*, 40(4):939–965.

Csiszar, I. (1975). I-divergence geometry of probability distributions and minimization problems. *The Annals of Probability*, 3(1):146–158.

De Bandt, O. and Hartmann, P. (2000). Systemic risk: A survey. Working Paper 35, European Central Bank, Frankfurt, Germany.

de Servigny, A. and Jobst, N. (2007). *The Handbook of Structured Finance*. McGraw-Hill.

Degryse, H. and Nguyen, G. (2007). Interbank exposures: An empirical examination of contagion risk in the belgian banking system. *International Journal of Central Banking*, pages 123–171.

Donnelly, C. and Embrechts, P. (2010). The devil is in the tails: actuarial mathematics and the subprime mortgage crisis. *ASTIN Bulletin*, 40(1):1–3.

Duellmann, K. (2006). Regulatory capital and basel II. Technical report, Deutsche Bundesbank.

Duffie, D., Eckner, A., Horel, G., and Saita, L. (2009). Frailty correlated default. *The Journal of Finance*, 64(5):2089–2123.

Egger, H. and Engl, H. W. (2005). Tikhonov regularization applied to the inverse problem of option pricing: convergence analysis and rates. *Inverse Problems*, 3(21):1027–1045.

Eisenberg, L. and Noe, T. H. (2001). Systemic risk in financial systems. *Management Science*, 47(2):236–249.

Elsinger, H., Lehar, A., and Summer, M. (2006a). Risk assessment for banking systems. *Management Science*, 52(9):1301–1314.

Elsinger, H., Lehar, A., and Summer, M. (2006b). Systemically important banks: an analysis for the european banking system. *International Economics and Economic Policy*, 3(1):73–89.

Embrechts, P., Lindskog, F., and McNeil, A. (2001). Modelling dependence with copulas and applications to risk management.

FCIC (2010). Credit ratings and the financial crisis. Technical report, Financial Crisis Inquiry Commission.

FDIC (2009). Appendix D to Part 225-Capital Adequacy Guidelines for Bank Holding Companies: Tier 1 Leverage Measure. Technical report, FDIC.

Federal Deposit Insurance Corporation (2011). Failed bank list. Technical report, Federal Deposit Insurance Corporation.

Fender, I. and Kiff, J. (2004). CDO rating methodology: Some thoughts on model risk and its implications. BIS Working Papers 163, Bank for International Settlements.

Freixas, X., Parigi, M., and Rochet, J. (2000). Systemic risk, interbank relations and liquidity provision by the central bank. *Journal of Money Credit and Banking*, 32(3):611–638.

Furfine, C. H. (2003). Interbank exposures: Quantifying the risk of contagion. *Journal of Money, Credit and Banking*, 35(1):111–128.

Goldman Sachs (2007). ABACUS 2007-AC1. Technical report.

Gup, B. E. (2004). *Too Big to Fail: Policies and Practices in Government Bailouts*. Praeger.

Huang, X.-L., Zou, F.-T., and Ma, F.-Y. (2007). Targeted local immunization in scale-free peer-to-peer networks. *J. Comput. Sci. Technol.*, 22(3):457–468.

Kalemanova, A., Schmidt, B., and Werner, R. (2005). The normal inverse gaussian distribution for synthetic CDO pricing.

Lee, Y. T. (2009). Impact of CDO tranches on economic capital of credit portfolios.

Li, D. (2000). On default correlation: A copula function approach.

Lowenstein, R. (2000). When genius failed: The rise and fall of long-term capital management.

Madar, N., Kalisky, T., Cohen, R., Ben-avraham, D., and Havlin, S. (2004). Immunization and epidemic dynamics in complex networks. *The European Physical Journal B*, 38:4269–276.

McDougall, R. (1999). Entropy theory and ras are friends. GTAP Working Papers 300, Center for Global Trade Analysis, Department of Agricultural Economics, Purdue University.

Meng, C. and Sengupta, A. N. (2009). CDO tranche sensivitivities in the gaussian copula model.

Mistrulli, P. E. (2007). Assessing financial contagion in the interbank market: Maximum entropy versus observed interbank lending patterns. Bank of Italy Research Paper Series - Temi di Discussione 641, Bank of Italy, Economic Research Department, Rome, Italy.

Müller, J. (2006). Interbank credit lines as a channel of contagion. *Journal of Financial Services Research*, 29(1):37–60.

Nelsen, R. B. (1999). *An introduction to copulas*, volume 139 of *Lecture Notes in Statistics*. Springer-Verlag.

Newman, D., Fabozzi, F. J. ., Lucas, D. J. ., and Goodman, L. S. . (2008). Empirical evidence on CDO performance. *The journal of fixed income*, 18(2):32–40.

Nier, E., Yang, J., Yorulmazer, T., and Alentorn, A. (2007). Network models and financial stability. *Journal of Economic Dynamics and Control*, 31:2033–2060.

Parkinson, P. M. (1999). Hedge funds, leverage, and the lessons of long-term capital management. Technical report, The Federal Reserve Board.

Peng, X. and Kou, S. (2009). Default clustering and valuation of collateralized debt obligations.

Reisner, L. L., Lench, K. R., and Muoio, R. A. (2010). Sec charges goldman sachs with fraud in structuring and marketing of CDO tied to subprime mortgages. Technical report, U.S. Securities and Exchange Commission.

Schmidt, T. (2008). Copulas and dependence measurement.

Schneider, M. H. and Zenios, S. A. (1990). A comparative study of algorithms for matrix balancing. *Opererations Research*, 38:439–455.

Sheldon, G. and Maurer, M. (1998). Interbank lending and systemic risk: An empirical analysis for switzerland. *Swiss Journal of Economics and Statistics*, 134(IV):685–704.

Shtatland, E., Kleinman, K., and Cain, E. (2002). One more time about r2 measures of fit in logistic regression. *NESUG 15 Proceedings*, pages 222–226.

SIFMA (2010). Global CDO market issuance data. Technical report, Securities Industry and financial Markets association.

Standard & Poor's (2007). CDOs: An introduction to CDOs and Standard & Poor's global CDO ratings. Technical report, Standard & Poor's.

Standard & Poor's (2008). 73 ratings lowered on 18 u.s. cdos of abs and 2 synthetic deals; $13.413 billion in issuance affected. Technical report, Standard & Poor's.

Standard & Poor's (2009). Update to global methodologies and assumptions for corporate cash flow and synthetic CDOs. Technical report, Standard & Poor's.

Toivanen, M. (2009). Financial interlinkages and risk of contagion in the finnish interbank market. Research discussion papers, Bank of Finland.

Upper, C. (2011). Simulation methods to assess the danger of contagion in interbank markets. *Journal of Financial Stability*, forthcoming.

Upper, C. and Worms, A. (2004). Estimating bilateral exposures in the german interbank market: Is there a danger of contagion? *European Economic Review*, 48(4):827–849.

van Lelyveld, I. and Liedorp, F. (2006). Interbank contagion in the dutch banking sector: A sensitivity analysis. *International Journal of Central Banking*.

Watts, D. J. and Strogatz, S. H. (1998). Collective dynamics of 'small-world' networks. *Nature*, 393(4):440–442.

Wells, S. (2004). Financial interlinkages in the united kingdom's interbank market and the risk of contagion. Working Paper 230, Bank of England, London.

Wojtowicz, M. (2011). CDOs and the financial crisis: Credit ratings and fair premia.