0.1 Spectral decomposition of Pauli Matrix Y

```
load("eigen");
```

(%o38) C:/maxima-5.44.0/share/maxima/5.44.0/share/matrix/eigen.mac

```
Y:matrix([0,-%i],[%i,0]);
```

(%o39) $\begin{pmatrix} 0 & -\%i \\ \%i & 0 \end{pmatrix}$

```
eigenvalues(Y);
```

(%o40) [[−1,1],[1,1]]

```
eigenvectors(Y);
```

(%o41) [[[−1,1],[1,1]],[[[1,−%i]],[[1,%i]]]]

```
innerproduct([1,-%i],[1,-%i]);
```

(%o5) 2

```
e1:transpose(matrix(1/sqrt(2)·[1,-%i])); /*eigenvector for eigenvalue -1*/
e2:transpose(matrix(1/sqrt(2)·[1,%i])); /*idem for 1*/
```

(%o62) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{\%i}{\sqrt{2}} \end{pmatrix}$

(%o63) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{\%i}{\sqrt{2}} \end{pmatrix}$

```
Pm1:e1.ctranspose(e1);Pm1.Pm1-Pm1; /*Projection matrices for -1 and 1 */
P1:e2.ctranspose(e2);P1.P1-P1;
Pm1+P1;/*completeness relation*/
```

(%o409) $\begin{pmatrix} \frac{1}{2} & \frac{\%i}{2} \\ -\frac{\%i}{2} & \frac{1}{2} \end{pmatrix}$

(%o410) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(%o411) $\begin{pmatrix} \frac{1}{2} & -\frac{\%i}{2} \\ \frac{\%i}{2} & \frac{1}{2} \end{pmatrix}$

(%o412) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(%o413) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

```
Pm1b:(Y-1·ident(2))/(-1-1);P1b:(Y+1·ident(2))/(1-(-1)); /*projection matrices using the formula */
                                                     /* agree with the projection matrices using eigenvec
```

(%o70) $\begin{pmatrix} \frac{1}{2} & \frac{\%i}{2} \\ -\frac{\%i}{2} & \frac{1}{2} \end{pmatrix}$

(%o71) $\begin{pmatrix} \frac{1}{2} & -\frac{\%i}{2} \\ \frac{\%i}{2} & \frac{1}{2} \end{pmatrix}$

```
(-1)·Pm1+1·P1;Y; /*Spectral decomposition of Y */
```

(%o76) $\begin{pmatrix} 0 & -\%i \\ \%i & 0 \end{pmatrix}$

(%o77) $\begin{pmatrix} 0 & -\%i \\ \%i & 0 \end{pmatrix}$

```
Y.Y;
```

(%o80) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

```
(-1)^2·Pm1+1^2·P1;/* Y.Y using spectral decomposition */
```

(%o81) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

```
Y;P1-Pm1;  /*spectral decomp */
```

(%o44) $\begin{bmatrix} 0 & -\%i \\ \%i & 0 \end{bmatrix}$

(%o45) $\begin{bmatrix} 0 & -\%i \\ \%i & 0 \end{bmatrix}$

the function exp(%i*alpha*Y), using spectral decomposition

```
(cos(alpha)+%i·sin(alpha))·P1+(cos(alpha)-%i·sin(alpha))·Pm1;
/* exp(i%*alpha*Y)=sum_j exp(%i*alpha*lambda_j)*P_j */
```

(%o84)
$$\begin{pmatrix} \frac{\%i\ \sin(alpha)+\cos(alpha)}{2}+\frac{\cos(alpha)-\%i\ \sin(alpha)}{2} & \frac{\%i\ (\cos(alpha)-\%i\ \sin(alpha))}{2}-\frac{\%i\ (\%i\ \sin(alpha)+\cos(alpha))}{2} \\ \frac{\%i\ (\%i\ \sin(alpha)+\cos(alpha))}{2}-\frac{\%i\ (\cos(alpha)-\%i\ \sin(alpha))}{2} & \frac{\%i\ \sin(alpha)+\cos(alpha)}{2}+\frac{\cos(alpha)-\%i\ \sin(alpha)}{2} \end{pmatrix}$$

```
ratsimp(%);
```

(%o85)
$$\begin{pmatrix} \cos(alpha) & \sin(alpha) \\ -\sin(alpha) & \cos(alpha) \end{pmatrix}$$

The square root of Y would be sqrt(1)*P1+sqrt(-1)*Pm1

```
sqrtY:P1+%i·Pm1;
```

(%o414)
$$\begin{pmatrix} \frac{\%i}{2}+\frac{1}{2} & -\frac{\%i}{2}-\frac{1}{2} \\ \frac{\%i}{2}+\frac{1}{2} & \frac{\%i}{2}+\frac{1}{2} \end{pmatrix}$$

```
ratsimp(sqrtY.sqrtY);Y;
```

(%o417)
$$\begin{pmatrix} 0 & -\%i \\ \%i & 0 \end{pmatrix}$$

(%o418)
$$\begin{pmatrix} 0 & -\%i \\ \%i & 0 \end{pmatrix}$$

0.1 SVD

```
norm(v):=sqrt(innerproduct(v,v));/*norm of vector*/;
```

(%o428) $norm(v):=\sqrt{innerproduct(v,v)}$

```
A:matrix([3,0,%i],[1+%i,-1,0]);
```

(%o86) $\begin{pmatrix} 3 & 0 & \%i \\ \%i+1 & -1 & 0 \end{pmatrix}$

the singular values are the square root of the eigenvalues of Adagger.A or A.Adagger

```
H1:ratsimp(ctranspose(A).A);
```

(%o158) $\begin{pmatrix} 11 & \%i-1 & 3\,\%i \\ -\%i-1 & 1 & 0 \\ -3\,\%i & 0 & 1 \end{pmatrix}$

```
H2:ratsimp(A.ctranspose(A));
```

(%o159) $\begin{pmatrix} 10 & 3-3\,\%i \\ 3\,\%i+3 & 3 \end{pmatrix}$

```
eigenvalues(H1);eigenvalues(H2);
```

(%o160) [[0,12,1],[1,1,1]]
(%o161) [[12,1],[1,1]]

```
Sigma:matrix([sqrt(12),0,0],[0,1,0]);
```

(%o162) $\begin{pmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

We take H1:ratsimp(ctranspose(A).A) and compute the eigenvectors. We normalize them

```
evH1:eigenvectors(H1);
```

(%o392) $[[[0,1,12],[1,1,1]],[[[1,\%i+1,3\,\%i]],[[0,1,-\frac{\%i+1}{3}]],[[1,-\frac{\%i+1}{11},-\frac{3\,\%i}{11}]]]]$

```
[v3,v2,v1]:[evH1[2][1][1],evH1[2][2][1],evH1[2][3][1]];
/*v1 is eigenvector for eigenvalue 12; v2 for eigenvalue 1; v3 for 0*/
```

(%o393) $[[1,\%i+1,3\,\%i],[0,1,-\frac{\%i+1}{3}],[1,-\frac{\%i+1}{11},-\frac{3\,\%i}{11}]]$

```
v1:v1/norm(v1);
v2:v2/norm(v2);
v3:v3/norm(v3);
```

(%o394) $[\frac{\sqrt{11}}{2\sqrt{3}},-\frac{\%i+1}{2\sqrt{3}\sqrt{11}},-\frac{\sqrt{3}\,\%i}{2\sqrt{11}}]$

(%o395) $[0,\sqrt{\frac{3}{11}},-\frac{\%i+1}{\sqrt{11}}]$

(%o396) $[\frac{1}{2\sqrt{3}},\frac{\%i+1}{2\sqrt{3}},\frac{\sqrt{3}\,\%i}{2}]$

```
V:transpose(matrix(v1,v2,v3));
```

(%o364) $$\begin{pmatrix} \frac{\sqrt{11}}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} \\ -\frac{\%i+1}{2\sqrt{3}\sqrt{11}} & \sqrt{\frac{3}{11}} & \frac{\%i+1}{2\sqrt{3}} \\ -\frac{\sqrt{3}\,\%i}{2\sqrt{11}} & -\frac{\%i+1}{\sqrt{11}} & \frac{\sqrt{3}\,\%i}{2} \end{pmatrix}$$

```
ratsimp(ctranspose(V).V);
```

(%o365) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

We have V, a unitary matrix.

To compute U, we compute ui=(1/sqrt(lambda_i))*A.vi for all lambda_i=0
We complete the base, if necessary, with an orthonormal base of eigenvectors of the eigenvalue 0 for the matrix A.ctranspose(A)

```
u1:ratsimp(transpose(A.transpose(v1)/sqrt(12)));
u2:ratsimp(transpose(A.transpose(v2)/sqrt(1)));
/* u1 is eigenvector for eigenvalue 12. u2 for eigenvalue 1.*/
```

(%o400) $\begin{pmatrix} \frac{3}{\sqrt{11}} & \frac{\%i+1}{\sqrt{11}} \end{pmatrix}$

(%o401) $\begin{pmatrix} -\frac{\%i-1}{\sqrt{11}} & -\frac{3}{\sqrt{11}} \end{pmatrix}$

```
U:transpose(matrix(u1[1],u2[1])); /*U is a unitary matrix */
ratsimp(ctranspose(U).U);
```

(%o405) $\begin{pmatrix} \frac{3}{\sqrt{11}} & -\frac{\%i-1}{\sqrt{11}} \\ \frac{\%i+1}{\sqrt{11}} & -\frac{3}{\sqrt{11}} \end{pmatrix}$

(%o406) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

```
→ ratsimp(U.Sigma.ctranspose(V));A;
```

(%o407) $\begin{pmatrix} 3 & 0 & \%i \\ \%i+1 & -1 & 0 \end{pmatrix}$

(%o408) $\begin{pmatrix} 3 & 0 & \%i \\ \%i+1 & -1 & 0 \end{pmatrix}$

0.2 Another example of SVD

```
→ M:matrix([1,1],[0,0],[%i,%i]);
```

(%o419) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ \%i & \%i \end{pmatrix}$

```
→ H1:ctranspose(M).M;H2:M.ctranspose(M);
```

(%o422) $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

(%o423) $\begin{pmatrix} 2 & 0 & -2\,\%i \\ 0 & 0 & 0 \\ 2\,\%i & 0 & 2 \end{pmatrix}$

```
→ eivals(H1);eivals(H2) /*eivals is an alias for eigenvalues */;
```

(%o425) $[[0,4],[1,1]]$

(%o426) $[[0,4],[2,1]]$

```
Sigma:matrix([2,0],[0,0],[0,0]);
```

(%o432) $\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

```
eiv:eigenvectors(H1); /*V*/;
```

(%o427) $[[[0,4],[1,1]],[[[1,-1]],[[1,1]]]]$

```
[v2,v1]:[eiv[2][1][1],eiv[2][2][1]]; /*v1 is the eigenvector for eigenvalue 4*/
v1:v1/norm(v1);v2:v2/norm(v2);
```

(%o434) $[[1,-1],[1,1]]$

(%o435) $[\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}]$

(%o436) $[\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}]$

```
V:transpose(matrix(v2,v1));
ratsimp(ctranspose(V).V); /* check V unitary */
```

(%o438) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

(%o439) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

```
u1:transpose(ratsimp(M.ctranspose(v1)/2)); /*u1=A.v1/sqrt(lambda1})*/
                                            /* no more nonzero eigenvalues */
```

(%o443) $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{\%i}{\sqrt{2}} \end{pmatrix}$

We need to complete U with an orthonormal base of the eigenvectors corresponding to zero

```
eiu:eigenvectors(H2);
```

(%o445) $[[[0,4],[2,1]],[[[1,0,-\%i],[0,1,0]],[[1,0,\%i]]]]$

```
[u2,u3]:[eiu[2][1][1],eiu[2][1][2]];
```

(%o446) $[[1,0,-\%i],[0,1,0]]$

```
[u2,u3]:gramschmidt([u2,u3]);
```

(%o447) $[[1,0,-\%i],[0,1,0]]$

```
u2:u2/norm(u2);
u3:u3/norm(u3);
```

(%o448) $[\frac{1}{\sqrt{2}},0,-\frac{\%i}{\sqrt{2}}]$

(%o449) $[0,1,0]$

```
U:transpose(matrix(u1[1],u2,u3));
ratsimp(ctranspose(U).U); /* check unitary */
```

(%o451) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{\%i}{\sqrt{2}} & -\frac{\%i}{\sqrt{2}} & 0 \end{pmatrix}$

(%o452) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

```
U.Sigma.ctranspose(V);/*svd decomposition, we recover M */
```

(%o453) $\begin{pmatrix} 1 & -1 \\ 0 & 0 \\ \%i & -\%i \end{pmatrix}$

0.1 Tensor product of matrices

```
TensorProduct(A,B):=block(
[mA:matrix_size(A)[1],mB:matrix_size(B)[1],
    nA:matrix_size(A)[2],nB:matrix_size(B)[2],
    salida],
    salida:apply('matrix,makelist(makelist(0,i,1,nA·nB),j,1,mA·mB)),
    for i:0 thru mA-1 do
        for j:0 thru nA-1 do
            for h:1 thru mB do
                for k:1 thru nB do
                    salida[i·mB+h,j·nB+k]:A[i+1,j+1]·B[h,k],
    return(salida)
);
```

(%o7) *TensorProduct(A,B):=block*

```
M:matrix([1,2,3],[4,5,6]);
```

(%o8) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

```
TensorProduct(M,ident(3));
```

(%o9)
$$\begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 3 \\ 4 & 0 & 0 & 5 & 0 & 0 & 6 & 0 & 0 \\ 0 & 4 & 0 & 0 & 5 & 0 & 0 & 6 & 0 \\ 0 & 0 & 4 & 0 & 0 & 5 & 0 & 0 & 6 \end{pmatrix}$$

```
u:transpose(matrix([2,%i,3]));
v:transpose(matrix([1+%i,0,2]));
```

(%o4)
$$\begin{pmatrix} 2 \\ \%i \\ 3 \end{pmatrix}$$

(%o5)
$$\begin{pmatrix} \%i+1 \\ 0 \\ 2 \end{pmatrix}$$

```
TensorProduct(u,v);
```

(%o11)
$$\begin{pmatrix} 2\,(\%i+1) \\ 0 \\ 4 \\ \%i\,(\%i+1) \\ 0 \\ 2\,\%i \\ 3\,(\%i+1) \\ 0 \\ 6 \end{pmatrix}$$