# Introduction ot Quantum Computing
# Session 4

```
kill(all);
load("eigen");
load("linearalgebra");
load("nchrpl"); /* for trace */
```

(%o0) *done*
(%o1) *C:/maxima-5.44.0/share/maxima/5.44.0/share/matrix/eigen.mac*
(%o2) *C:/maxima-5.44.0/share/maxima/5.44.0/share/linearalgebra/linearalgebra.mac*
(%o3) *C:/maxima-5.44.0/share/maxima/5.44.0/share/matrix/nchrpl.mac*

## 1 Mixed state and density matrix

Suppose we have a mixed state p1=1/4, |phi1>=|e1>, p2=3/4, |phi2>=|e2>

```
1/4·transpose(matrix([1,0]));3/4·transpose(matrix([0,1]));
```

(%o4) $\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$

(%o5) $\begin{pmatrix} 0 \\ \frac{3}{4} \end{pmatrix}$

The density matrix is p1 |phi1><phi1|+p2|phi2><phi2|

```
rho:1/4·transpose(matrix([1,0])).ctranspose(transpose(matrix([1,0])))
+3/4·transpose(matrix([0,1])).ctranspose(transpose(matrix([0,1])));
```

(%o6) $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$

```
rho.rho-rho;/* rho is not a pure state */
```

(%o7) $\begin{pmatrix} -\frac{3}{16} & 0 \\ 0 & -\frac{3}{16} \end{pmatrix}$

Expected value of a measure on a mixed state

```
Y:matrix([0,-%i],[%i,0]);
```

(%o8) $\begin{pmatrix} 0 & -\%i \\ \%i & 0 \end{pmatrix}$

```
mattrace(rho.Y);
```

(%o9) $0$

```
sum((rho.Y)[i,i],i,1,2);/*another way to compute the trace */
```

(%o10) $0$

Probability of measuring -1 with Y on rho

```
eigenvectors(Y);
```

(%o11) $[[[-1,1],[1,1]],[[[1,-\%i]],[[1,\%i]]]]$

Normalized eigenvector of eigenvalue -1

```
lm1:transpose(matrix([1,-%i])/sqrt(innerproduct([1,-%i],[1,-%i])));;/*eigenvect for eival -1*/
```

(%o12) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{\%i}{\sqrt{2}} \end{pmatrix}$

```
Pm1:lm1.ctranspose(lm1); /* we could also use the projection formula */
```

(%o13) $\begin{pmatrix} \frac{1}{2} & \frac{\%i}{2} \\ -\frac{\%i}{2} & \frac{1}{2} \end{pmatrix}$

```
Pm1.rho;
```

(%o14) $\begin{pmatrix} \frac{1}{8} & \frac{3\,\%i}{8} \\ -\frac{\%i}{8} & \frac{3}{8} \end{pmatrix}$

```
mattrace(%);/*probability*/
```

(%o15) $\frac{1}{2}$

Expected value of Z over rho

```
Z:matrix([1,0],[0,-1]);
```

(%o16) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

```
→ rho.Z;
```

(%o17) $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix}$

```
→ mattrace(%);
```

(%o18) $-\frac{1}{2}$

## *1.1* Partial trace

```
→ phi:1/sqrt(2)·transpose((matrix([1,0,0,1])));
```

(%o19) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

density matrix of a pure state

```
→ rho:phi.ctranspose(phi);
```

(%o20) $\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$

Partial trace of rho with respect to the second component (formula in slide 28)

```
r:kronecker_product(ident(2),matrix([1,0])).rho.kronecker_product(ident(2),transpose(matrix([1,0])))+
kronecker_product(ident(2),matrix([0,1])).rho.kronecker_product(ident(2),transpose(matrix([0,1])));
```

(%o21) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

partial trace of a pure state is a mixed state

```
r.r-r;
```

(%o22) $\begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$

## *1.2* Computation of fidelity

```
rho1:1/3·1/2·matrix([1],[1]).matrix([1,1])+2/3·1/2·matrix([1],[%i]).matrix([1,-%i]);
```

(%o23) $\begin{pmatrix} \frac{1}{2} & \frac{1}{6}-\frac{\%i}{3} \\ \frac{\%i}{3}+\frac{1}{6} & \frac{1}{2} \end{pmatrix}$

```
ratsimp(mattrace(%));
```

(%o24) $1$

```
→ rho2:1/4·transpose(matrix([1,0])).ctranspose(transpose(matrix([1,0])))
+3/4·transpose(matrix([0,1])).ctranspose(transpose(matrix([0,1])));
```

(%o25) $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$

Fidelity of rho1, rho2 is trace(sqrt(sqrt(rho1).rho2.sqrt(rho1)))

for sqrt(rho1) we need the spectral decomposition of rho1

```
→ eigenvectors(rho1);
```

(%o26) $\left[\left[\left[-\frac{\sqrt{5}-3}{6},\frac{\sqrt{5}+3}{6}\right],\left[1,1\right]\right],\left[\left[\left[1,-\frac{2\sqrt{5}\,\%i+\sqrt{5}}{5}\right]\right],\left[\left[1,\frac{2\sqrt{5}\,\%i+\sqrt{5}}{5}\right]\right]\right]\right]$

eigenvectors of different eigenvalues are orthogonal, so we just
need to normalize them
to get an orthogonal basis

```
→ e1:transpose(matrix([1,-(2·sqrt(5)·%i+sqrt(5))/5]))
/sqrt(innerproduct([1,-(2·sqrt(5)·%i+sqrt(5))/5],[1,-(2·sqrt(5)·%i+sqrt(5))/5]));
```

(%o27) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-2\sqrt{5}\,\%i-\sqrt{5}}{5\sqrt{2}} \end{pmatrix}$

```
e2:transpose(matrix([1,+(2·sqrt(5)·%i+sqrt(5))/5]))
/sqrt(innerproduct([1,+(2·sqrt(5)·%i+sqrt(5))/5],[1,+(2·sqrt(5)·%i+sqrt(5))/5]));
```

(%o28)
$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{2\sqrt{5}\,\%i+\sqrt{5}}{5\sqrt{2}} \end{pmatrix}$$

```
Sqrtrho1:ratsimp(sqrt(-(sqrt(5)-3)/6)·e1.ctranspose(e1)+
    sqrt((sqrt(5)+3)/6)·e2.ctranspose(e2));
```

(%o29)
$$\begin{pmatrix} \frac{\sqrt{\sqrt{5}+3}\sqrt{6}+\sqrt{3-\sqrt{5}}\sqrt{6}}{12} & \frac{\sqrt{3-\sqrt{5}}\,(2\sqrt{5}\sqrt{6}\,\%i-\sqrt{5}\sqrt{6})+\sqrt{\sqrt{5}+3}\,(\sqrt{5}\sqrt{6}-2\sqrt{5}\sqrt{6}\,\%i)}{60} \\ -\frac{\sqrt{3-\sqrt{5}}\,(2\sqrt{5}\sqrt{6}\,\%i+\sqrt{5}\sqrt{6})+\sqrt{\sqrt{5}+3}\,(-2\sqrt{5}\sqrt{6}\,\%i-\sqrt{5}\sqrt{6})}{60} & \frac{\sqrt{\sqrt{5}+3}\sqrt{6}+\sqrt{3-\sqrt{5}}\sqrt{6}}{12} \end{pmatrix}$$

Check that the square root is correct:

```
ratsimp(Sqrtrho1.Sqrtrho1-rho1);
```

(%o30)
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Now we compute the square root of sqrt(rho1).rho2.sqrt(rho1)

```
prod:ratsimp(Sqrtrho1.rho2.Sqrtrho1);
```

(%o31)
$$\begin{pmatrix} -\frac{\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}-6}{24} & -\frac{2\,\%i-1}{12} \\ \frac{2\,\%i+1}{12} & \frac{\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}+6}{24} \end{pmatrix}$$

```
eigenvectors(prod);
```

(%o32)
$$\left[\left[\left[-\frac{\sqrt{6}-3}{12},\frac{\sqrt{6}+3}{12}\right],[1,1]\right],\left[\left[\left[1,\frac{\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\,(2\,\%i+1)-4\sqrt{6}\,\%i-2\sqrt{6}}{10}\right]\right],\left[\left[1,\frac{\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\,(2\,\%i+1)+4\sqrt{6}\,\%i+2\sqrt{6}}{10}\right]\right]\right]\right]$$

```
e1:transpose(matrix([1,(sqrt(3-sqrt(5))·sqrt(sqrt(5)+3)·(2·%i+1)-4·sqrt(6)·%i-2·sqrt(6))/10]))
    /sqrt(innerproduct([1,(sqrt(3-sqrt(5))·sqrt(sqrt(5)+3)·(2·%i+1)-4·sqrt(6)·%i-2·sqrt(6))/10],
        [1,(sqrt(3-sqrt(5))·sqrt(sqrt(5)+3)·(2·%i+1)-4·sqrt(6)·%i-2·sqrt(6))/10]));
```

(%o33)
$$\begin{pmatrix}\frac{\sqrt{5}}{\sqrt{12-\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\sqrt{6}}}\\ \frac{\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\,(2\,\%i+1)-4\sqrt{6}\,\%i-2\sqrt{6}}{2\sqrt{5}\sqrt{12-\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\sqrt{6}}}\end{pmatrix}$$

```
e2:transpose(matrix([1,(sqrt(3-sqrt(5))·sqrt(sqrt(5)+3)·(2·%i+1)+4·sqrt(6)·%i+2·sqrt(6))/10]))
/sqrt(innerproduct([1,(sqrt(3-sqrt(5))·sqrt(sqrt(5)+3)·(2·%i+1)+4·sqrt(6)·%i+2·sqrt(6))/10]
    ,[1,(sqrt(3-sqrt(5))·sqrt(sqrt(5)+3)·(2·%i+1)+4·sqrt(6)·%i+2·sqrt(6))/10]));
```

(%o34)
$$\begin{pmatrix}\frac{\sqrt{5}}{\sqrt{\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\sqrt{6}+12}}\\ \frac{\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\,(2\,\%i+1)+4\sqrt{6}\,\%i+2\sqrt{6}}{2\sqrt{5}\sqrt{\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\sqrt{6}+12}}\end{pmatrix}$$

```
SqrtProd:sqrt(-(sqrt(6)-3)/12)·e1.ctranspose(e1)+sqrt((sqrt(6)+3)/12)·e2.ctranspose(e2);
```

(%o35)
$$\left(\begin{matrix}\frac{\sqrt{5}\sqrt{\sqrt{6}+3}}{2\sqrt{3}\,(\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\sqrt{6}+12)}+\frac{\sqrt{5}\sqrt{3-\sqrt{6}}}{2\sqrt{3}\,(12-\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\sqrt{6})} & \\ \frac{\sqrt{\sqrt{6}+3}\,(\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\,(2\,\%i+1)+4\sqrt{6}\,\%i+2\sqrt{6})}{4\sqrt{3}\,(\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\sqrt{6}+12)}+\frac{\sqrt{3-\sqrt{6}}\,(\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\,(2\,\%i+1)-4\sqrt{6}\,\%i-2\sqrt{6})}{4\sqrt{3}\,(12-\sqrt{3-\sqrt{5}}\sqrt{\sqrt{5}+3}\sqrt{6})} & \frac{\sqrt{\sqrt{6}+3}\,(-4\sqrt{6}\,\%i+\sqrt{3-\sqrt{5}}\sqrt{\ldots}}{}\end{matrix}\right.$$

So the fidelity is

```
ratsimp(mattrace(SqrtProd));
```

(%o36) $$\frac{\sqrt{3}\,\sqrt{\sqrt{6}+3}+\sqrt{3}\,\sqrt{3-\sqrt{6}}}{6}$$

```
%,numer;
```

(%o37) $0.8880738339771153$

Fidelity is a number between 0 and 1

```
rho1.rho2-rho2.rho1;
```

(%o38) $$\begin{pmatrix} 0 & \frac{\frac{1}{6}-\frac{\%i}{3}}{2} \\ -\frac{\frac{\%i}{3}+\frac{1}{6}}{2} & 0 \end{pmatrix}$$

## 1.3 another example with larger matrices

```
M1:matrix(
        [5/16,  0,  0,  1/8],
        [0, 3/16,   0,  0],
        [0, 0,  3/16,   0],
        [1/8,   0,  0,  5/16]
    );
```

(%o39)
$$\begin{pmatrix} \frac{5}{16} & 0 & 0 & \frac{1}{8} \\ 0 & \frac{3}{16} & 0 & 0 \\ 0 & 0 & \frac{3}{16} & 0 \\ \frac{1}{8} & 0 & 0 & \frac{5}{16} \end{pmatrix}$$

```
M2:matrix(
        [5/16,  0,  0,  0],
        [0, 3/16,   1/8,    0],
        [0, 1/8,    3/16,   0],
        [0, 0,  0,  5/16]
    );
```

(%o40)
$$\begin{pmatrix} \frac{5}{16} & 0 & 0 & 0 \\ 0 & \frac{3}{16} & \frac{1}{8} & 0 \\ 0 & \frac{1}{8} & \frac{3}{16} & 0 \\ 0 & 0 & 0 & \frac{5}{16} \end{pmatrix}$$

Fidelity tr(sqrt(sqrt(M1).M2.sqrt(M1)))

```
eigenvectors(M1);
```

(%o41) $[[[\frac{3}{16},\frac{7}{16}],[3,1]],[[[1,0,0,-1],[0,1,0,0],[0,0,1,0]],[[1,0,0,1]]]]$

```
e4:1/sqrt(2)·transpose(matrix([1,0,0,1]));
```

(%o42) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

eigenvectors of the same eigenvalue could be not orthogonal

```
gramschmidt([[0,0,1,0],[0,1,0,0],[1,0,0,-1]]);
```

(%o43) $[[0,0,1,0],[0,1,0,0],[1,0,0,-1]]$

```
e1:transpose(matrix([0,0,1,0]));
e2:transpose(matrix([0,1,0,0]));
e3:transpose(1/sqrt(2) ·matrix([1,0,0,-1]));
```

(%o44) $$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(%o45) $$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(%o46) $$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
sM1:sqrt(3/16) ·(e1.ctranspose(e1)+e2.ctranspose(e2)+e3.ctranspose(e3))+sqrt(7/16) ·e4.ctranspose(e4);
```

(%o47) $$\begin{pmatrix} \frac{\sqrt{7}}{8}+\frac{\sqrt{3}}{8} & 0 & 0 & \frac{\sqrt{7}}{8}-\frac{\sqrt{3}}{8} \\ 0 & \frac{\sqrt{3}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{4} & 0 \\ \frac{\sqrt{7}}{8}-\frac{\sqrt{3}}{8} & 0 & 0 & \frac{\sqrt{7}}{8}+\frac{\sqrt{3}}{8} \end{pmatrix}$$

```
ratsimp(sM1.sM1-M1);
```

(%o48)
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
AUX:ratsimp(sM1.M2.sM1);
```

(%o49)
$$\begin{pmatrix} \frac{25}{256} & 0 & 0 & \frac{5}{128} \\ 0 & \frac{9}{256} & \frac{3}{128} & 0 \\ 0 & \frac{3}{128} & \frac{9}{256} & 0 \\ \frac{5}{128} & 0 & 0 & \frac{25}{256} \end{pmatrix}$$

```
eigenvectors(AUX);
```

(%o50)
$$[[[\frac{15}{256},\frac{3}{256},\frac{35}{256}],[2,1,1]],[[[1,0,0,-1]],[[0,1,1,0]],[[0,1,-1,0]],[[1,0,0,1]]]]$$

```
v1:1/sqrt(2)·transpose(matrix([1,0,0,-1]));
v2:1/sqrt(2)·transpose(matrix([0,1,1,0]));
v3:1/sqrt(2)·transpose(matrix([0,1,-1,0]));
v4:1/sqrt(2)·transpose(matrix([1,0,0,1]));
```

(%o51) $$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(%o52) $$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

(%o53) $$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

(%o54) $$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
sqrtAUX:sqrt(15/256)·(v1.ctranspose(v1)+v2.ctranspose(v2))+
sqrt(3/256)·v3.ctranspose(v3)+sqrt(35/256)·v4.ctranspose(v4);
```

(%o55)
$$\begin{pmatrix} \frac{\sqrt{35}}{32}+\frac{\sqrt{15}}{32} & 0 & 0 & \frac{\sqrt{35}}{32}-\frac{\sqrt{15}}{32} \\ 0 & \frac{\sqrt{15}}{32}+\frac{\sqrt{3}}{32} & \frac{\sqrt{15}}{32}-\frac{\sqrt{3}}{32} & 0 \\ 0 & \frac{\sqrt{15}}{32}-\frac{\sqrt{3}}{32} & \frac{\sqrt{15}}{32}+\frac{\sqrt{3}}{32} & 0 \\ \frac{\sqrt{35}}{32}-\frac{\sqrt{15}}{32} & 0 & 0 & \frac{\sqrt{35}}{32}+\frac{\sqrt{15}}{32} \end{pmatrix}$$

and the fidelity is

```
F:mattrace(sqrtAUX);
```

(%o56)
$$\frac{\sqrt{35}}{16}+\frac{\sqrt{15}}{8}+\frac{\sqrt{3}}{16}$$

```
F,numer;
```

(%o57) $0.9621310801927079$

But...

```
M1.M2-M2.M1;
```

(%o58)
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since they commute, we could use the formula in slide 33,
F=sum(sqrt(p[i]*q[i]),i,1,n), where p and q are the probabilities
for each state, or the eigenvalues for each density matrix:

```
→ eigenvectors(M1);eigenvectors(M2);
```

(%o59) $[[[\frac{3}{16},\frac{7}{16}],[3,1]],[[[1,0,0,-1],[0,1,0,0],[0,0,1,0]],[[1,0,0,1]]]]$

(%o60) $[[[\frac{1}{16},\frac{5}{16}],[1,3]],[[[0,1,-1,0]],[[1,0,0,0],[0,1,1,0],[0,0,0,1]]]]$

```
→ /* We pair the corresponding eigenvalues for a common eigenvector basis */
  /* The common eigenvector basis could be
     [1,0,0,1],[1,0,0,-1],[0,1,1,0],[0,1,-1,0] */
  p:[7/16,3/16,3/16,3/16];
  q:[5/16,5/16,5/16,1/16];
  F:sum(sqrt(p[i]·q[i]),i,1,4);
```

(%o61) $[\frac{7}{16},\frac{3}{16},\frac{3}{16},\frac{3}{16}]$

(%o62) $[\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{1}{16}]$

(%o63) $\frac{\sqrt{35}}{16}+\frac{\sqrt{15}}{8}+\frac{\sqrt{3}}{16}$

```
→ F,numer;
```

(%o64) 0.9621310801927079

As you can see, we get the same result...

***Warning: Can set maxima's working directory but cannot change it during the maxima session :***