Intuition Behind Bayesian Structural Time-Series Models

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Contents

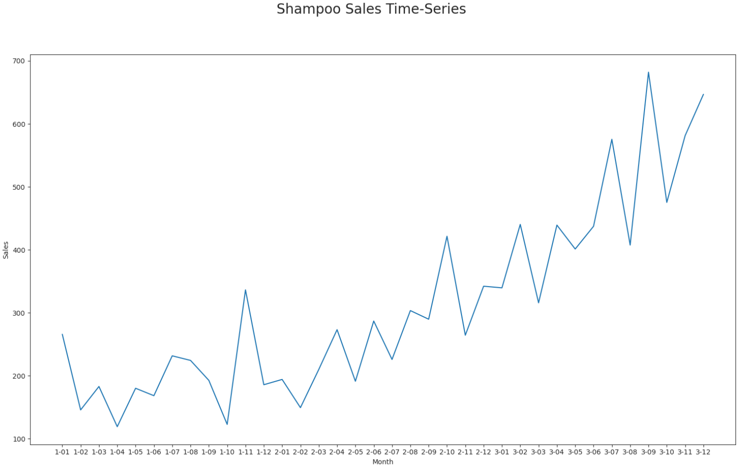
What are Time-Series?

- A time-series is a sequence of observations taken sequentially in time
- Goals of time-series modelling:
    - Understand stochastic mechanisms that give rise to the time-series
    - Predict future trajectory of time-series
- Time-series can be *univariate* or *multivariate*:

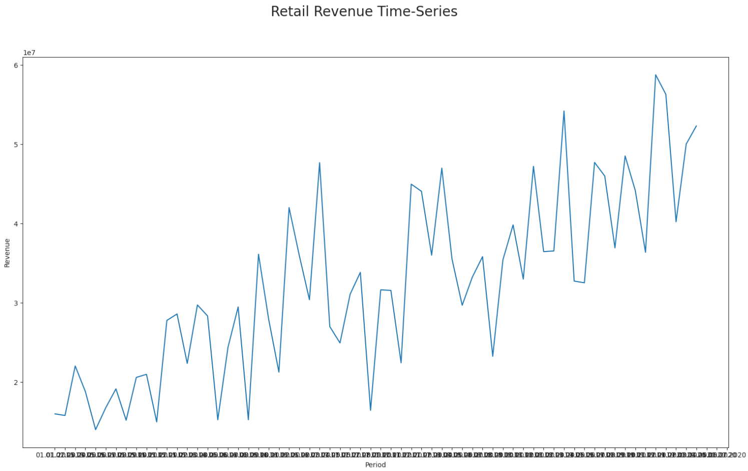
Date	Revenue
03.05.2020	845€
04.05.2020	756€
05.05.2020	545€
06.05.2020	834€
07.05.2020	957€

Date	Revenue	Weather
03.05.2020	845€	sunny
04.05.2020	756€	cloudy
05.05.2020	545€	rainy
06.05.2020	834€	rainy
07.05.2020	957€	sunny

Time-Series Examples



Time-Series Examples



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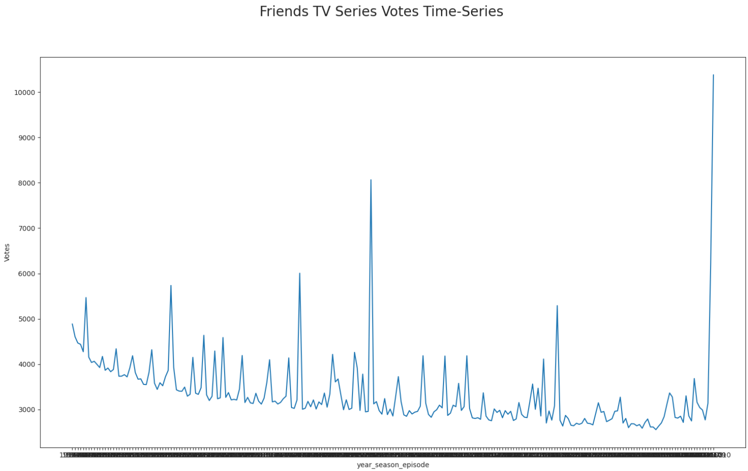
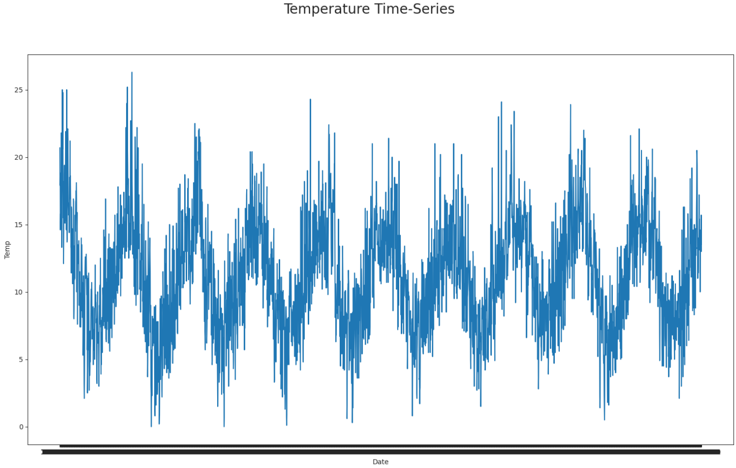


Time-Series Examples



Unobserved Components Models

- Time Series = Regression Effects + Trend + Cycles + Seasons + Irregularity-Term
- Each component captures some important feature of the time-series
- Components can have own probabilistic models (but can be modeled deterministically)
- More components are possible depending on the use case
- Traditional time-series model:

$$y_t = T_t + S_t + C_t + I_t$$

T = Trend; S = Seasonal Component; C = Cyclical Component; I = Irregular Component

Unobserved Components Models

Fully specified UC Model:

$$y_t = \mu_t + \gamma_t + \psi_t + r_t + \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{j=1}^{m} \beta_j x_{jt} + \varepsilon_t$$

- y_t: time-series to be modeled
- μ_t: time varying mean (trend component)
- γ_t: seasonal component
- ψ_t: cyclical component
- r_t: autoregressive component
- $\sum_{i=1}^{p} \phi_i y_{t-i}$: momentum of time-series with respect to past timepoints
- $\sum_{j=1}^{m} \beta_j x_{jt}$: additional predictors
- ε_t: irregular component (error term)

UCM Deterministic Example

Considering a traditional UC time-series, we can construct a deterministic example:

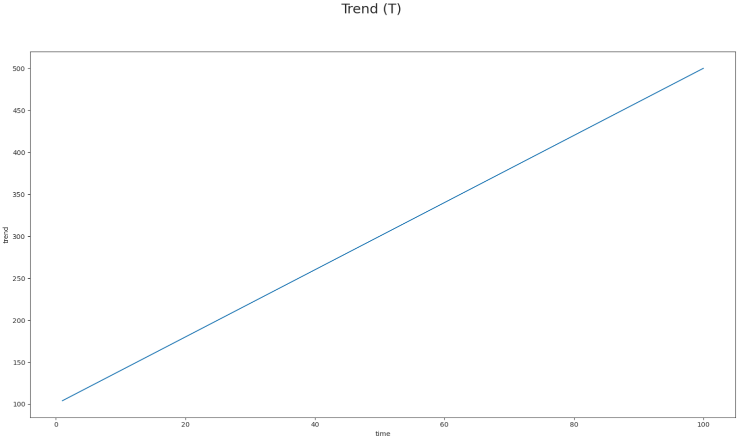
$$
\begin{aligned}
y_t &= T_t + C_t + S_t + I_t \\
&= 100 + 4t \\
&+ 50 * cos(\pi t/10) \\
&- 50jan - 25feb + 25mar - 25apr - 50may + 50jun \\
&+ 75jul + 50aug + 5sep - 25oct - 50nov + 20dec \\
&+ \varepsilon_t
\end{aligned}
$$

- jan, feb, ..., dec $\Rightarrow$ 1 if t in respective month, else 0
- $\varepsilon_t \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 0; \sigma^2 = 10$

UCM Deterministic Example - Trend

Modeling linear trend (T):

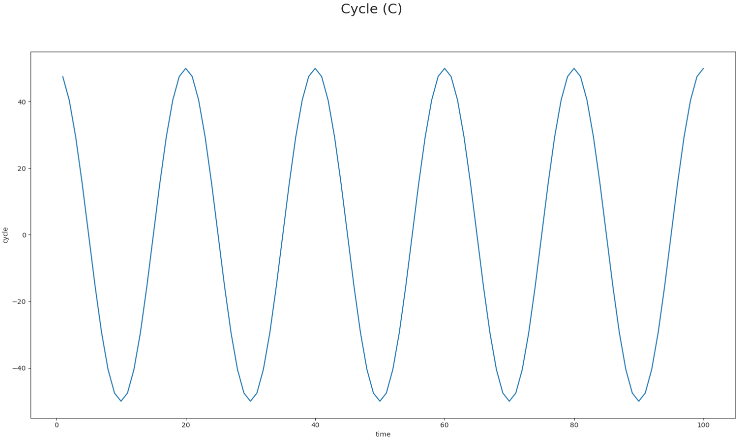
$$T_t = 100 + 4t$$



UCM Deterministic Example - Cyclical Component

Modeling cyclicality (C):

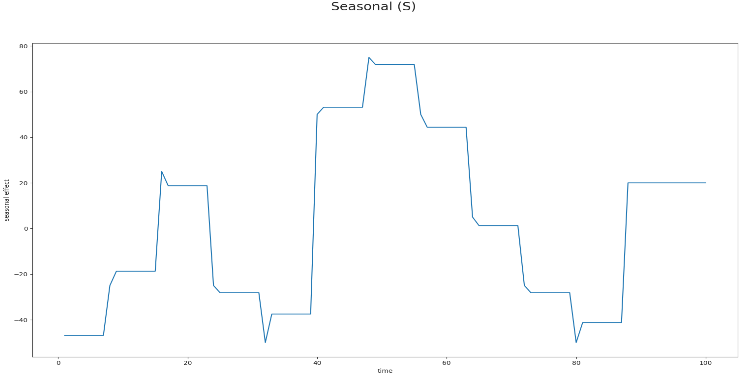
$$C_t = 50 * cos(\pi t/10)$$



UCM Deterministic Example - Seasonal Component

Modeling seasonality (S):

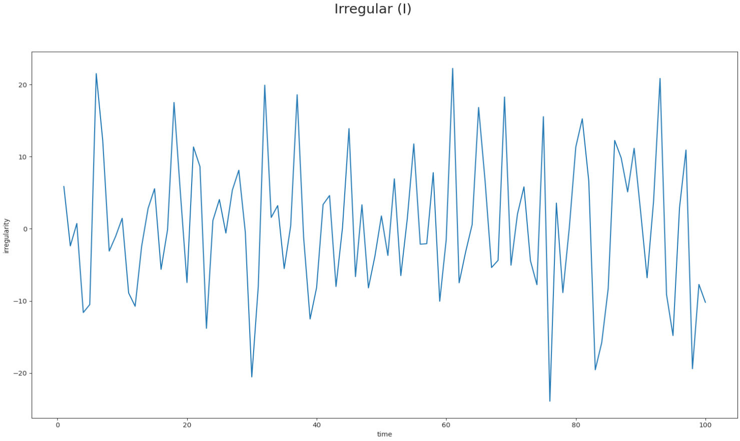
$$S_t = -50jan - 25feb + 25mar - 25apr - 50may + 50jun \\ + 75jul + 50aug + 5sep - 25oct - 50nov + 20dec$$



UCM Deterministic Example - Irregular Component

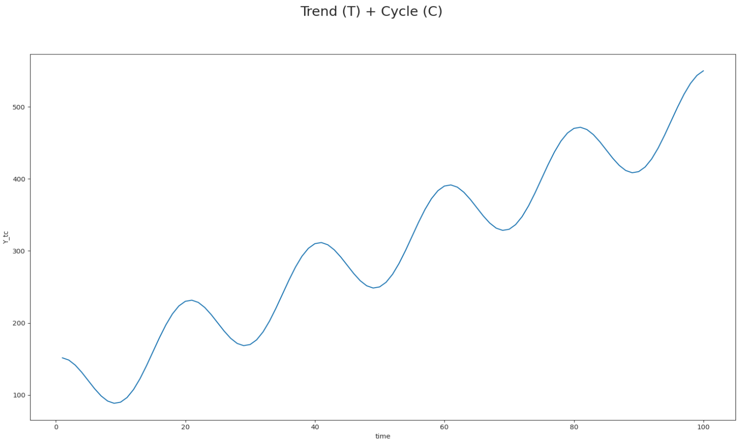
Modeling irregularity (I):

$$I_t = \varepsilon_t \sim \mathcal{N}(\mu, \sigma^2); \mu = 0; \sigma^2 = 10$$



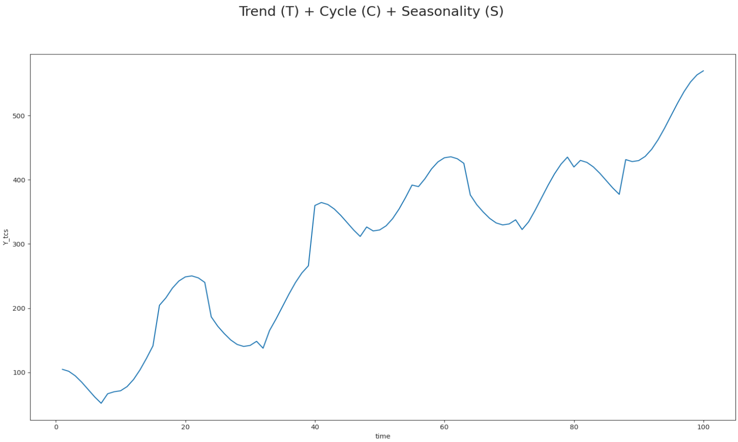
UCM Deterministic Example: T + C

$$y_t = T_t + C_t$$



UCM Deterministic Example: T + C + S

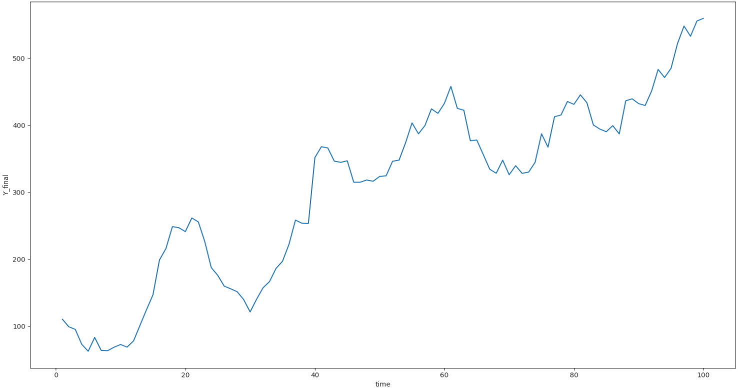
$$y_t = T_t + C_t + S_t$$



UCM Deterministic Example: T + C + S + I

$$y_t = T_t + C_t + S_t + I_t$$

Trend (T) + Cycle (C) + Seasonality (S) + Irregularity (I)



Local Linear Trend (LLT)

- Trend can also be modeled stochastically as a local linear trend (LLT)

$$
\begin{aligned}
\mu_t &= \mu_{t-1} + \eta_{t-1} + \nu_t && \text{level equation} \\
\eta_t &= \eta_{t-1} + \xi_t && \text{slope equation}
\end{aligned}
$$

- disturbances in the level equation modeled as $\nu_t \sim \mathcal{N}(0,\, \sigma_\nu^2)$
- disturbances in the slope equation modeled as $\xi_t \sim \mathcal{N}(0,\, \sigma_\xi^2)$
- μ_t and η_t can be initialized as unknown or random vectors

LLT in vector recursion form:

$$
\begin{pmatrix} \mu_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ \eta_{t-1} \end{pmatrix} + \begin{pmatrix} \nu_t \\ \xi_t \end{pmatrix}
$$

LLT - Special Cases

- Special cases of LLT can be constructed via elimination of specific parts
- Deterministic examples are special cases of LLT

model	*description*
$\mu_t = \mu_{t-1} + \nu_t$	initial slope and its disturbance $= 0$
$\mu_t = \mu_{t-1} + \eta_1 + \nu_t$	slope disturbance $= 0$
$\mu_t = \mu_{t-1} + \eta_{t-1};\ \eta_t = \eta_{t-1} + \xi_t$	level disturbance $= 0$
$\mu_t = \mu_1 + t\eta_1$	deterministic time trend
$\mu = \mu_1$	degenerate time invariant

Cycles

- Stochastic cycle can be modeled using recursion
- For $t = 1,2,3,...,$ and $0 < \omega < \pi$;
  $\psi_t = \gamma \cos(\omega t - \phi)$, $\gamma = \sqrt{a^2 + b^2}$, $\phi = \arctan(b/a)$; and
  $0 \leq \rho \leq 1$; $\nu_t \sim \mathcal{N}(0,\, \sigma_\nu^2)$ and $\nu_t^* \sim \mathcal{N}(0,\, \sigma_\nu^2)$

$$\begin{pmatrix} \psi_t \\ \psi_t^* \end{pmatrix} = \rho \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix} \begin{pmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{pmatrix} + \begin{pmatrix} \nu_t \\ \nu_t^* \end{pmatrix}$$

- resulting sequence ψ_t is pseudo-cyclical with time-varying amplitude, phase, and period

Seasons

- Seasons are corrections to general trend of the time-series
- Seasonal effects sum to 0 over one full season cycle
- Two popular representations of seasonal patterns:

Stochastic Dummy Type	*Stochastic Trigonometric Type*
$\sum_{i=0}^{s-1} \gamma_{t-i} = \nu_t, \nu_t \sim \mathcal{N}(0, \sigma_\nu^2)$	$\gamma_t = \sum_{j=1}^{[s/2]} \psi_{j,t}; \psi_{j,t}$ has period s/j
list of s numbers that sum to 0	sum of $s/2$ deterministic cycles (harmonics) $\rightarrow s, s/2, s/3, ...$
variance σ^2 controls disturbance	all cycles have common disturbance variance σ^2
if $\sigma^2 = 0$, then deterministic seasonal	if all $\sigma^2 = 0$ then deterministic seasonal

UCM as State-Space Models

- UCM models can be written in state-space form
- Y_t evolves over time in Markovian fashion
- state space formulation enables the use of the Kalman filter (KF) for computation of predictions:

$$
\begin{aligned}
Y_t &= Z_t\alpha_t + D_t + \varepsilon_t && \text{observation equation} \\
\alpha_{t+1} &= T_t\alpha_t + C_t + R_t\eta_t && \text{state transition equation}
\end{aligned}
$$

Z_t is an $N \times m$ design matrix for α_t; D_t is an $N \times 1$ vector
ε_t is an $N \times 1$ error vector with $\varepsilon_t \sim \mathcal{N}(0, H_t)$

T_t is an $m \times m$ transition matrix; C_t is an $m \times 1$ vector
R_t is a $m \times g$ matrix; η_t is a $g \times 1$ error vector with $\eta_t \sim \mathcal{N}(0, Q_t)$

system matrices: $Z_t, D_t, H_t, T_t, C_t, R_t$, and Q_t

initial state $\rightarrow \alpha_0 \sim \mathcal{N}(a_0, P_0)$

UCM as State-Space Models - Simple Example

Example: Autoregressive model (AR) with lag of two (AR(2))

$$y_t = \alpha + \phi_t y_{t-1} + \phi_t y_{t-2} + \eta_t; \eta_t \sim \mathcal{N}(0, \sigma^2)$$

One possible state space transition equation of AR(2):

We define $\alpha_t = (y_t, y_{t-1})'$

$$\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \eta_t$$

Observation equation of AR(2):

$$y_t = (1, 0)\alpha_t$$

$\Rightarrow$ system matrices: $T = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}; R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; C = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}; Q = \sigma^2$

$$Z_t = (1, 0); D_t = 0; \varepsilon_t = 0; H_t = 0$$

Kalman Filter

Kalman Smoother

Kalman Filter/Smoother Algorithm

Spike-and-Slab Prior

BSTM: State-Space Definition

BSTM: Plate Notation

BSTM: Graphical Model

BSTM: Inference

BSTM: Impact Evaluation

Causal Impact in BSTM

References

- *Beamer Paket*
  http://latex-beamer.sourceforge.net/
- *User's Guide to the Beamer*
- *DANTE e.V.* http://www.dante.de