Data-driven systems theory, signal processing, and control

Ivan Markovsky

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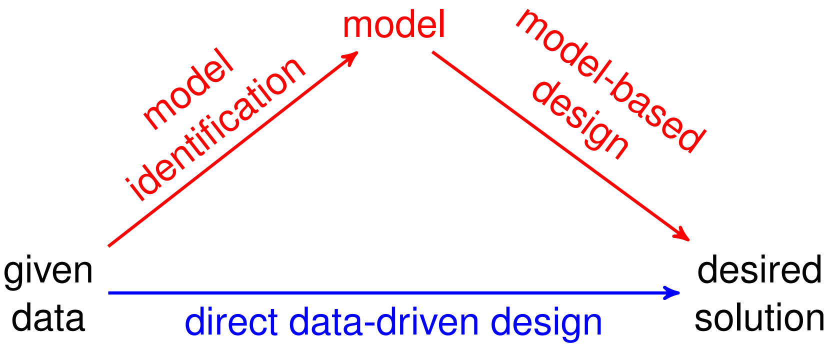
today:	object	linear time-invariant systems
	problem	given u, predict, filter, control
	approach	behavioral systems theory

We are aiming at direct data-driven methods for analysis and design of dynamical systems



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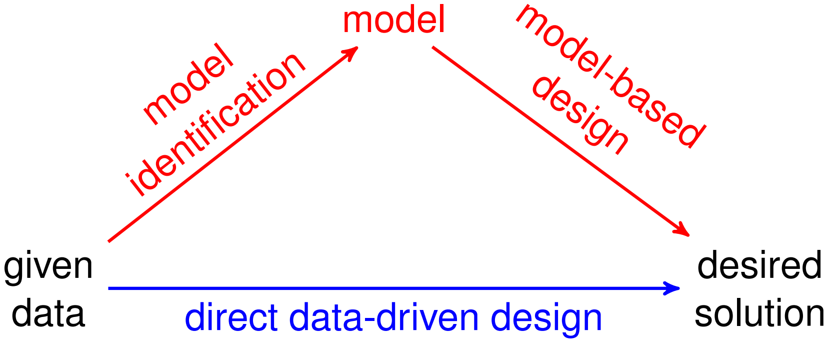

the classical approach is “indirect data-driven”

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but don't use *parametric representations*

they are *non-parametric* using directly the data

Outline

Example: Free fall prediction

Linear time-invariant systems

Data-driven representation

Dealing with noise

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- data-driven approach: data $w_{\mathrm{d}}^1, \ldots, w_{\mathrm{d}}^N$ + ini. cond. $\mapsto$ w

Modeling from first principles yields affine time-invariant dynamical system

second law of Newton + the law of gravity

$$m\ddot{w} = m \begin{bmatrix} 0 \\ -9.81 \end{bmatrix} + f, \quad w(0) = w_{\text{ini}} \text{ and } \dot{w}(0) = v_{\text{ini}}$$

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1st order equation

$$\dot{x} = Ax, \quad w = Cx, \quad x(0) = x_{\text{ini}}$$

- state $x := (w_1, \dot{w}_1, w_2, \dot{w}_2, -9.81)$
- initial state $x_{\text{ini}} := (w_{\text{ini},1}, v_{\text{ini},1}, w_{\text{ini},2}, v_{\text{ini},2}, -9.81)$
- A, C — model parameters (depend on m and γ)

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algorithm for data-driven prediction:

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2. define $w := \begin{bmatrix} w_d^1 & \cdots & w_d^N \end{bmatrix} g$

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- bypasses the knowledge of the physical laws
- and prior knowledge or estimation of model parameters
- no hyper-parameters to tune

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$w(t) \in \mathbb{R}^q$ is the value of w at time $t \in \mathscr{T}$

Signals are classified according to # of variables q and type of time axis $\mathscr{T}$

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$(\mathbb{R}^q)^{\mathbb{R}} \mapsto (\mathbb{R}^q)^{\mathbb{Z}}$ — sampling / time-discretization

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$w|_{[t_1,t_2]}$ and $w|_T$ — restriction to interval

The classical view of dynamical system is a “signal processor”: an input/output map

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intuition: the input *causes* the output

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$\mathscr{B}|_T$ — restriction of $\mathscr{B}$ to the interval $1, \ldots, T$

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- noise filtering, prediction, control, ...

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representations

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$$= \Big\{ w \in (\mathbb{R}^2)^{\mathbb{R}_+} \mid \text{there is } x \in (\mathbb{R}^5)^{\mathbb{R}_+}\text{, such that}$$

$$\dot{x} = Ax,\ w = Cx,\ x_5(0) = -9.81 \Big\}$$

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$$\mathscr{B} \in \mathscr{L}^q \quad \Longrightarrow \quad \dim \mathscr{B}|_L = \mathbf{m}(\mathscr{B})L + \mathbf{n}(\mathscr{B}), \text{ for all } L \geq \boldsymbol{\ell}(\mathscr{B})$$

Kernel representation $\mathscr{B} = \ker R(\sigma)$ is ℓ-th order vector difference equation

$$\left\{ w \mid R_0 w(t) + R_1 w(t+1) + \cdots + R_\ell w(t+\ell) = 0, \text{ for all } t \in \mathscr{T} \right\}$$

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the parameter is a polynomial matrix $R(z) \in \mathbb{R}^{g \times q}[z]$

Input/state/output representation is 1-st order vector difference equation

$$\left\{ w = \Pi \begin{bmatrix} u \\ y \end{bmatrix} \;\middle|\; \text{there is } x \in (\mathbb{R}^n)^{\mathbb{N}}, \text{ such that } \sigma x = Ax + Bu,\; y = Cx + Du \right\} \quad \text{(I/S/O)}$$

x	—	state	,	n	$:=$	$\dim x$	—	order
u	—	input	,	m	$:=$	$\dim u$	—	# of inputs
y	—	output	,	p	$:=$	$\dim y$	—	# of outputs

the parameters are:

- permutation matrix $\Pi \in \mathbb{R}^{q \times q}$ and
- matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$

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- complexity: (# of inputs, lag, order)
- $\mathscr{B} = \ker R(\sigma)$ kernel representation

Summary: linear time-invariant systems

$w \in (\mathbb{R}^q)^{\mathscr{T}}$ signals are functions of time

$\mathscr{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ systems are sets of signals

$\mathscr{B}$ can be represented by different equations

$\mathscr{L}^q$ LTI model class: shift-invariant subspaces

- complexity: (# of inputs, lag, order)
- $\mathscr{B} = \ker R(\sigma)$ kernel representation
- input/state/output representation

Outline

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The finite-horizon behavior $\mathscr{B}|_L$ is used for both analysis and computations

restriction of w to finite interval $[1, L]$

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if $\mathscr{B}$ is linear, $\mathscr{B}|_L$ is a linear subspace of $(\mathbb{R}^q)^L$

$\mathscr{B}|_L$ can be obtained experimentally by collecting "informative" data

collect $N \geq qL$ random trajectories

$$w_{\mathrm{d}}^1, \ldots, w_{\mathrm{d}}^N \in \mathscr{B}|_L$$

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with probability one equality holds

Discrete-time LTI systems over finite horizon can be studied using linear algebra only

$$\underbrace{\begin{bmatrix} w_\mathrm{d}^1 & \cdots & w_\mathrm{d}^N \end{bmatrix}}_{W} \in \mathbb{R}^{qL\times N}$$ — "trajectory matrix"

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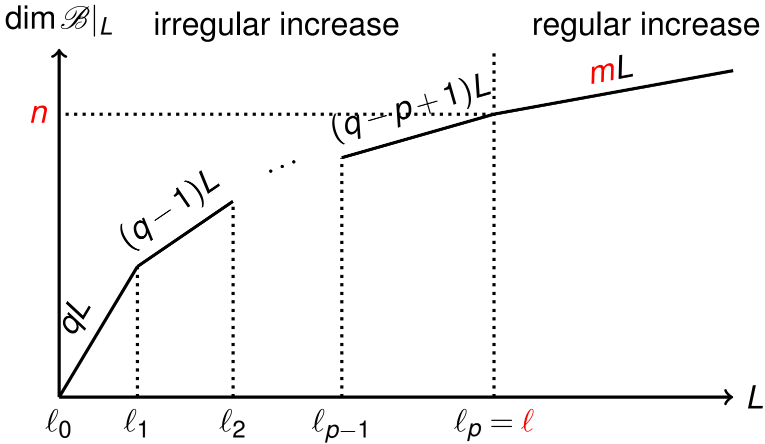
now, we can do explorations, in particular check

$$\dim \mathscr{B}|_L = \mathbf{m}(\mathscr{B})L + \mathbf{n}(\mathscr{B}) \geq \operatorname{rank} W, \quad \text{for } L \geq \boldsymbol{\ell}(\mathscr{B})$$

$\dim \mathscr{B}|_L$ is a piecewise affine function of L



$$\dim \mathscr{B}|_L = mL + n, \quad \text{for all } L \geq \ell$$

Data-driven representation (infinite horizon)

data: exact infinite trajectory w_d of $\mathscr{B} \in \mathscr{L}$

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identifiability condition: $\quad \mathscr{B} = \widehat{\mathscr{B}}$

Consecutive application of σ on finite w_d results in Hankel matrix with missing values

$$
\begin{array}{cccc}
\sigma^0 w_d & \sigma^1 w_d & \cdots & \sigma^{T_d - 1} w_d \\
\hline
w_d(1) & w_d(2) & \cdots & w_d(T_d) \\
w_d(2) & \vdots & \iddots & ? \\
\vdots & w_d(T_d) & \iddots & \vdots \\
w_d(T_d) & ? & \cdots & ?
\end{array}
$$

for $w_d = \big(w_d(1), \dots, w_d(T_d)\big)$ and $1 \leq L \leq T_d$

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for $w_\mathrm{d} = \big(w_\mathrm{d}(1), \dots, w_\mathrm{d}(T_\mathrm{d})\big)$ and $1 \leq L \leq T_\mathrm{d}$

$$
\mathscr{H}_L(w_\mathrm{d}) := \begin{bmatrix} (\sigma^0 w_\mathrm{d})|_L & (\sigma^1 w_\mathrm{d})|_L & \cdots & (\sigma^{T_\mathrm{d}-L} w_\mathrm{d})|_L \end{bmatrix}
$$

Data-driven representation (finite horizon)

the finite horizon data-driven representation

$$\mathscr{B}|_L = \widehat{\mathscr{B}}|_L := \text{image}\, \mathscr{H}_L(w_\text{d}) \qquad \text{(DD-REPR)}$$

holds if and only if

$$\text{rank}\, \mathscr{H}_L(w_\text{d}) = L\mathbf{m}(\mathscr{B}) + \mathbf{n}(\mathscr{B}) \qquad \text{(GPE)}$$

GPE — generalized persistency of excitation

Identifiability condition verifiable from $w_{\mathrm{d}} \in \mathscr{B}|_{T_{\mathrm{d}}}$ and (m, ℓ, n)

fact: $\mathscr{B} = \mathscr{B}' \iff \mathscr{B}|_{\ell+1} = \mathscr{B}'|_{\ell+1}$

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$\mathscr{B}$ is identifiable from $w_{\mathrm{d}} \in \mathscr{B}|_{T_{\mathrm{d}}}$ if and only if

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$w_\mathrm{d} \mapsto \mathscr{B}$ — system identification

Generic data-driven problem: trajectory interpolation/approximation

given:	"data trajectory"	$w_{\mathrm{d}} \in \mathscr{B}\|_{T_{\mathrm{d}}}$
	and elements	$w\|_{I_{\text{given}}}$
	of a trajectory	$w \in \mathscr{B}\|_{T}$

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$$\widehat{w} = \mathscr{H}_T(w_{\mathrm{d}})\big(\mathscr{H}_T(w_{\mathrm{d}})|_{I_{\text{given}}}\big)^{+} w|_{I_{\text{given}}} \qquad \text{(SOL)}$$

Special cases

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- given data: initial condition and input
- to-be-found: output (exact interpolation)

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tracking control

- given data: to-be-tracked trajectory
- to-be-found: ℓ_2-optimal approximation

Generalizations

multiple data trajectories $w_d^1, \dots, w_d^N$

$$\widehat{\mathscr{B}}|_L = \text{image}\, \underbrace{\begin{bmatrix} \mathscr{H}_L(w_d^1) & \cdots & \mathscr{H}_L(w_d^N) \end{bmatrix}}_{\text{mosaic-Hankel matrix}}$$

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$\rightsquigarrow$ Hankel structured low-rank approximation/completion
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nonlinear systems

results for special classes of nonlinear systems:
Volterra, Wiener-Hammerstein, bilinear, ...

Summary: data-driven representation

assuming $\quad \operatorname{rank} \mathscr{H}_L(w_{\mathrm{d}}) = \mathbf{m}(\mathscr{B})L + \mathbf{n}(\mathscr{B})$

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replaces parametric representations

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w_d exact and satisfying (GPE)

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- heuristics: convex relaxations of the ML estimator

The maximum-likelihood estimation problem in the errors-in-variables setup is nonconvex

errors-in-variables setup: $\quad w_{\mathrm{d}} = \overline{w}_{\mathrm{d}} + \widetilde{w}_{\mathrm{d}}$

- $\overline{w}_{\mathrm{d}}$ — true data, $\overline{w}_{\mathrm{d}} \in \mathscr{B}|_{T_{\mathrm{d}}}$, $\mathscr{B} \in \mathscr{L}_c^q$
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ML problem: given w_d, c, and $w|_{I_\mathrm{given}}$

$$\begin{aligned} \underset{g}{\text{minimize}} \quad & \|w|_{I_\mathrm{given}} - \mathscr{H}_T(\widehat{w}_\mathrm{d}^*)|_{I_\mathrm{given}} g\| \\ \text{subject to} \quad & \widehat{w}_\mathrm{d}^* = \arg\min_{\widehat{w}_\mathrm{d}, \widehat{\mathscr{B}}} \quad \|w_\mathrm{d} - \widehat{w}_\mathrm{d}\| \\ & \qquad \text{subject to} \quad \widehat{w}_\mathrm{d} \in \widehat{\mathscr{B}}|_{T_\mathrm{d}} \text{ and } \widehat{\mathscr{B}} \in \mathscr{L}_c^q \end{aligned}$$

The ML estimation problem is equivalent to Hankel structured low-rank approximation

$$\begin{aligned} \underset{g}{\text{minimize}} \quad & \| w|_{I_{\text{given}}} - \mathscr{H}_T(\widehat{w}_{\text{d}}^*)|_{I_{\text{given}}} g \| \\ \text{subject to} \quad & \widehat{w}_{\text{d}}^* = \arg\min_{\widehat{w}_{\text{d}}, \widehat{\mathscr{B}}} \quad \| w_{\text{d}} - \widehat{w}_{\text{d}} \| \\ & \qquad \text{subject to} \quad \widehat{w}_{\text{d}} \in \widehat{\mathscr{B}}|_{T_{\text{d}}} \text{ and } \widehat{\mathscr{B}} \in \mathscr{L}_c^q \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \underset{g}{\text{minimize}} \quad & \| w|_{I_{\text{given}}} - \mathscr{H}_T(\widehat{w}_{\text{d}}^*)|_{I_{\text{given}}} g \| \\ \text{subject to} \quad & \widehat{w}_{\text{d}}^* = \arg\min_{\widehat{w}_{\text{d}}} \quad \| w_{\text{d}} - \widehat{w}_{\text{d}} \| \\ & \qquad \text{subject to} \quad \operatorname{rank} \mathscr{H}_{\ell+1}(\widehat{w}_{\text{d}}) \leq (\ell+1)m + n \end{aligned}$$

Solution methods

local optimization

- choose a parametric representation of $\widehat{\mathscr{B}}(\theta)$
- optimize over $\widehat{w}$, $\widehat{w_{\mathrm{d}}}$, and θ
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convex relaxation based on ℓ_1-norm (LASSO)

$$\text{minimize} \quad \text{over } g \quad \| w|_{I_{\text{given}}} - \mathscr{H}_T(w_{\mathrm{d}})|_{I_{\text{given}}} g \| + \lambda \| g \|_1$$

Empirical validation on real-life datasets

	data set name	T_{d}	m	p
1	Air passengers data	144	0	1
2	Distillation column	90	5	3
3	pH process	2001	2	1
4	Hair dryer	1000	1	1
5	Heat flow density	1680	2	1
6	Heating system	801	1	1

G. Box, and G. Jenkins. Time Series Analysis: Forecasting and Control, Holden-Day, 1976

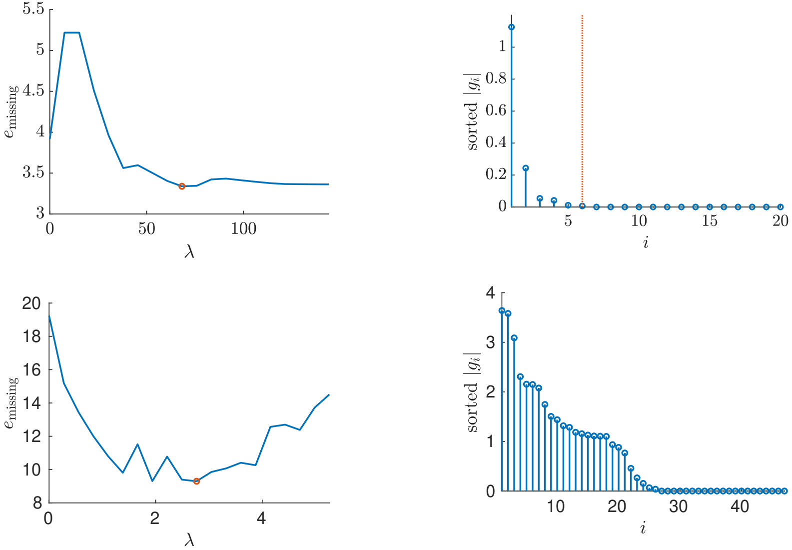
B. De Moor, et al.*DAISY: A database for identification of systems. Journal A, 38:4–5, 1997*

ℓ_1-norm regularization with optimized λ achieves the best performance

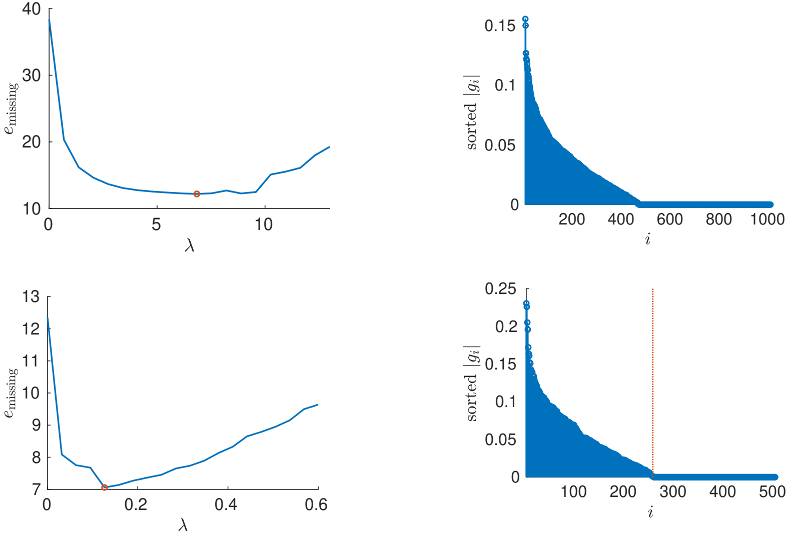
$$e_{\text{missing}} := \frac{\| w|_{I_{\text{missing}}} - \widehat{w}|_{I_{\text{missing}}} \|}{\| w|_{I_{\text{missing}}} \|} \, 100\%$$

	data set name	naive	ML	LASSO
1	Air passengers data	3.9	fail	3.3
2	Distillation column	19.24	17.44	9.30
3	pH process	38.38	85.71	12.19
4	Hair dryer	12.35	8.96	7.06
5	Heat flow density	7.16	44.10	3.98
6	Heating system	0.92	1.35	0.36

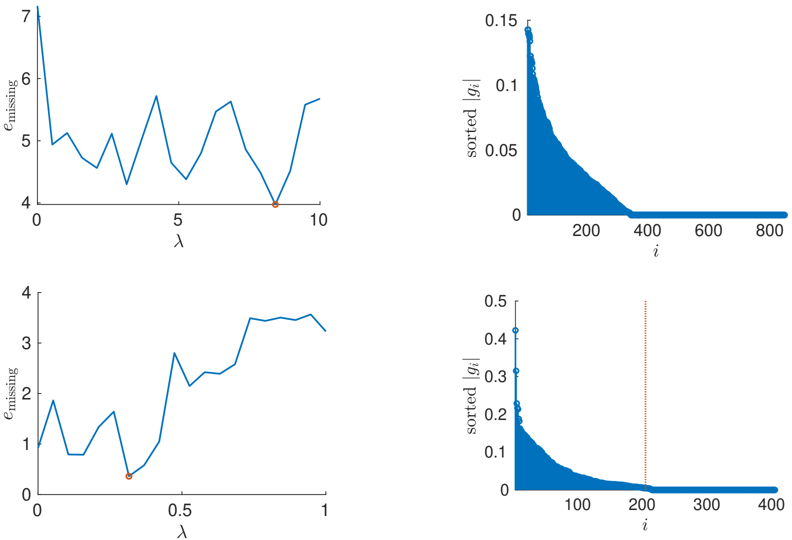
Tuning of λ and sparsity of g (datasets 1, 2)



Tuning of λ and sparsity of g (datasets 3, 4)



Tuning of λ and sparsity of g (datasets 5, 6)



Summary: convex relaxations

w_{d} exact $\rightsquigarrow$ systems theory

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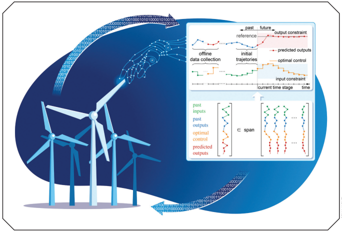
empirical validation

- the naive approach works (surprisingly) well
- parametric local optimization is not robust
- ℓ_1-norm regularization gives the best results

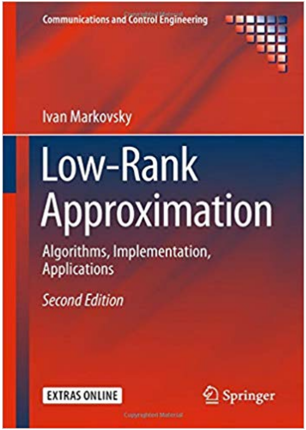
References

Data-Driven Control Based on the Behavioral Approach

FROM THEORY TO APPLICATIONS IN POWER SYSTEMS



IVAN MARKOVSKY, LINBIN HUANG, and FLORIAN DÖRFLER



Outline

Case study: Dynamic measurement

Nonparametric frequency response estimation

Generalization for nonlinear systems

A textbook problem

D. G. Luenberger, Introduction to Dynamical Systems: Theory, Models and Applications. John Wiley, 1979.

"A thermometer reading $21^\circ C$*, which has been inside a house for a long time, is taken outside. After one minute the thermometer reads* $15^\circ C$*; after two minutes it reads* $11^\circ C$*. What is the outside temperature?"*

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According to Newton's law of cooling, an object of higher temperature than its environment cools at a rate that is proportional to the difference in temperature.

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textbook problem:

- 1st order dynamics
- 3 noise-free samples
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- recursive computation

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implementation and practical validation

Thermometer: first order dynamical system

environmental temperature $\bar{u}$ $\xrightarrow{\text{heat transfer}}$ thermometer's reading y

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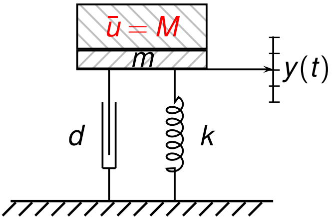
$$\begin{array}{c} \text{environmental} \\ \text{temperature } \bar{u} \end{array} \xrightarrow{\text{heat transfer}} \begin{array}{c} \text{thermometer's} \\ \text{reading } y \end{array}$$

measurement process: Newton's law of cooling

$$y = a\left(\bar{u} - y\right)$$

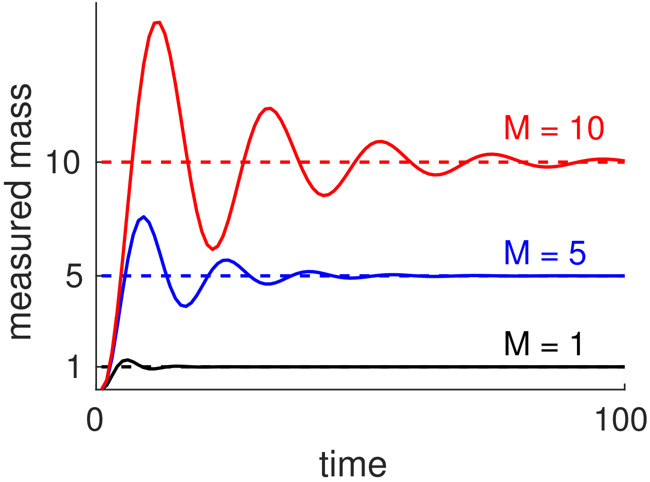
heat transfer coefficient $a > 0$

Scale: second order dynamical system



$$(M+m)\frac{\mathrm{d}}{\mathrm{d}\,t}y + dy + ky = g\bar{u}$$

The measurement process dynamics depends on the to-be-measured mass



Dynamic measurement: take into account the dynamical properties of the sensor

$$\begin{array}{c}\text{to-be-measured}\\ \text{variable } u\end{array} \xrightarrow{\text{measurement process}} \begin{array}{c}\text{measured}\\ \text{variable } y\end{array}$$

assumption 1: measured variable is constant $u(t) = \bar{u}$

assumption 2: the sensor is stable LTI system

assumption 3: sensor's DC-gain $= 1$ (calibrated sensor)

The data is generated from LTI system with output noise and constant input

$$\underbrace{y_{\mathrm{d}}}_{\substack{\text{measured}\\ \text{data}}} = \underbrace{y}_{\substack{\text{true}\\ \text{value}}} + \underbrace{e}_{\substack{\text{measurement}\\ \text{noise}}}$$

$$\underbrace{y}_{\substack{\text{true}\\ \text{value}}} = \underbrace{\bar{u}}_{\substack{\text{steady-state}\\ \text{value}}} + \underbrace{y_0}_{\substack{\text{transient}\\ \text{response}}}$$

assumption 4: e is a zero mean, white, Gaussian noise using a state space representation of the sensor

$$
\begin{aligned}
x(t+1) &= Ax(t), \qquad x(0) = x_0 \\
y_0(t) &= cx(t)
\end{aligned}
$$

we obtain

$$
\underbrace{\begin{bmatrix} y_{\mathrm{d}}(1) \\ y_{\mathrm{d}}(2) \\ \vdots \\ y_{\mathrm{d}}(T) \end{bmatrix}}_{y_{\mathrm{d}}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{1}_T} \bar{u} + \underbrace{\begin{bmatrix} c \\ cA \\ \vdots \\ cA^{T-1} \end{bmatrix}}_{\mathscr{O}_T} x_0 + \underbrace{\begin{bmatrix} e(1) \\ e(2) \\ \vdots \\ e(T) \end{bmatrix}}_{e}
$$

Maximum-likelihood model-based estimator

solve approximately

$$\begin{bmatrix} \mathbf{1}_T & \mathscr{O}_T \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \widehat{x}_0 \end{bmatrix} \approx y_\mathrm{d}$$

standard least-squares problem

$$\begin{aligned} &\text{minimize} \quad \text{over } \widehat{y},\, \widehat{u},\, \widehat{x}_0 \quad \|y_\mathrm{d} - \widehat{y}\| \\ &\text{subject to} \quad \begin{bmatrix} \mathbf{1}_T & \mathscr{O}_T \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \widehat{x}_0 \end{bmatrix} = \widehat{y} \end{aligned}$$

recursive implementation $\rightsquigarrow$ Kalman filter

Subspace model-free method

goal: avoid using the model parameters (A, C, $\mathscr{O}_T$)

in the noise-free case, due to the LTI assumption,

$$\Delta y(t) := y(t) - y(t-1) = y_0(t) - y_0(t-1)$$

satisfies the same dynamics as y_0, *i.e.*,

$$\begin{aligned} x(t+1) &= Ax(t), \qquad x(0) = \Delta x \\ \Delta y(t) &= cx(t) \end{aligned}$$

Hankel matrix—construction of multiple "short" trajectories from one "long" trajectory

$$\mathscr{H}(\Delta y) := \begin{bmatrix} \Delta y(1) & \Delta y(2) & \cdots & \Delta y(\mathrm{n}) \\ \Delta y(2) & \Delta y(3) & \cdots & \Delta y(\mathrm{n}+1) \\ \Delta y(3) & \Delta y(4) & \cdots & \Delta y(\mathrm{n}+2) \\ \vdots & \vdots & & \vdots \\ \Delta y(T-\mathrm{n}) & \Delta y(T-\mathrm{n}) & \cdots & \Delta y(T-1) \end{bmatrix}$$

fact: if $\operatorname{rank} \mathscr{H}(\Delta y) = \mathrm{n}$, then

$$\operatorname{image} \mathscr{O}_{T-\mathrm{n}} = \operatorname{image} \mathscr{H}(\Delta y)$$

model-based equation

$$\begin{bmatrix} \mathbf{1}_T & \mathscr{O}_T \end{bmatrix} \begin{bmatrix} \bar{u} \\ \widehat{x}_0 \end{bmatrix} = y$$

data-driven equation

$$\begin{bmatrix} \mathbf{1}_{T-\mathtt{n}} & \mathscr{H}(\Delta y) \end{bmatrix} \begin{bmatrix} \bar{u} \\ \ell \end{bmatrix} = y|_{T-\mathtt{n}} \qquad (*)$$

subspace method

solve $(*)$ by (recursive) least squares

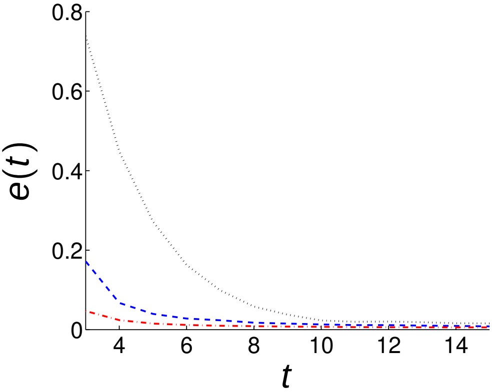
Empirical validation

dashed	—	true parameter value $\bar{u}$
solid	—	true output trajectory y_0
dotted	—	naive estimate $\widehat{u} = G^+ y$
dashed	—	model-based Kalman filter
bashed-dotted	—	data-driven method

estimation error: $e := \frac{1}{N} \sum_{i=1}^{N} \|\bar{u} - \widehat{u}^{(i)}\|$

(for $N = 100$ Monte-Carlo repetitions)

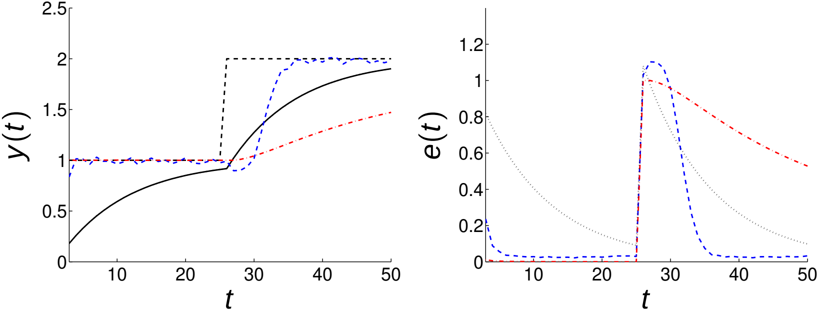
Simulated data of dynamic cooling process



$e(t) \to 0$ as $t \to \infty$ at different rates

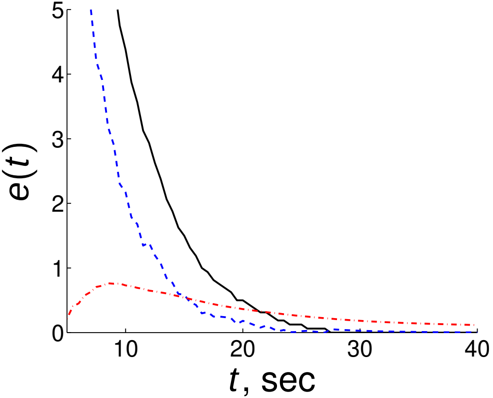
best is the Kalman filter (maximum likelihood estimator)

Simulation with time-varying parameter



Proof of concept prototype

Results in real-life experiment



Summary

dynamic measurement

steady-state value prediction

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the subspace method is applicable for

- high order dynamics
- noisy vector observations
- online computation

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future work / open problems

- numerical efficiency
- real-time uncertainty quantification
- generalization to nonlinear systems

Outline

Case study: Dynamic measurement

Nonparametric frequency response estimation

Generalization for nonlinear systems

Problem formulation

given: "data" trajectory $(u_\mathrm{d}, y_\mathrm{d}) \in \mathscr{B}|_{T_\mathrm{d}}$ and $z \in \mathbb{C}$

find: $H(z)$, where H is the transfer function of $\mathscr{B}$

Direct data-driven solution

we are interested in trajectory

$$w = \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \exp_z \\ \widehat{H} \exp_z \end{bmatrix} \in \mathscr{B}, \quad \text{where } \exp_z(t) := z^t$$

using the data-driven representation, we have

$$\begin{bmatrix} \mathscr{H}_L(u_\mathrm{d}) \\ \mathscr{H}_L(y_\mathrm{d}) \end{bmatrix} g = \begin{bmatrix} \mathbf{z} \\ \widehat{H}\mathbf{z} \end{bmatrix}, \quad \text{where } \mathbf{z} := \begin{bmatrix} z^1 \\ \vdots \\ z^L \end{bmatrix}$$

which leads to the system

$$\begin{bmatrix} 0 & \mathscr{H}_L(u_\mathrm{d}) \\ -\mathbf{z} & \mathscr{H}_L(y_\mathrm{d}) \end{bmatrix} \begin{bmatrix} \widehat{H} \\ g \end{bmatrix} = \begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix} \qquad \text{(SYS)}$$

Solution method: solve (SYS) for $\widehat{H}$

under (GPE) with $L \geq \ell + 1$, $\widehat{H} = H(z)$

without prior knowledge of ℓ

$$L = L_{\max} := \lfloor (T_{\rm d} + 1)/3 \rfloor$$

trivial generalization to

- multivariable systems
- multiple data trajectories $\{ w_{\rm d}^1, \ldots, w_{\rm d}^N \}$
- evaluation of $H(z)$ at multiple points in $\{ z_1, \ldots, z_K \} \in \mathbb{C}^K$

Comparison with classical nonparametric frequency response estimation methods

ignored initial/terminal conditions $\rightsquigarrow$ *leakage*

DFT grid $\rightsquigarrow$ limited *frequency resolution*

improvements by windowing and interpolation

- the leakage is not eliminated
- the methods involve *hyper-parameters*

Generalization of (SYS) to noisy data

preprocessing: rank-$mL+n$ approx. of $\mathscr{H}_L(w_\mathrm{d})$

- hyper-parameters $L \geq \ell + 1$ and n
- if the approximation preserves the Hankel structure, the method is maximum-likelihood in the EIV setting

regularization with $\|g\|_1$

- hyper-parameter: the 1-norm regularization parameter

regularization with the nuclear norm of $\mathscr{H}_L(\widehat{w}_\mathrm{d})$

- hyper-parameters: L and the regularization parameter

Matlab implementation

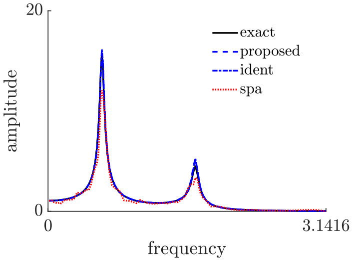
```
function Hh = dd_frest(ud, yd, z, n)
L = n + 1; t = (1:L)';
m = size(ud, 2); p = size(yd, 2);

%% preprocessing by low-rank approximation
H = [moshank(ud, L); moshank(yd, L)];
[U, ~, ~] = svd(H); P = U(:, 1:m * L + n);

%% form and solve the system of equations
for k = 1:length(z)
  A  = [[zeros(m*L, p); -kron(z(k).^t, eye(p))] P];
  hg = A \ [kron(z(k).^t, eye(m)); zeros(p*L, m)];
  Hh(:, :, k) = hg(1:p, :);
end
```

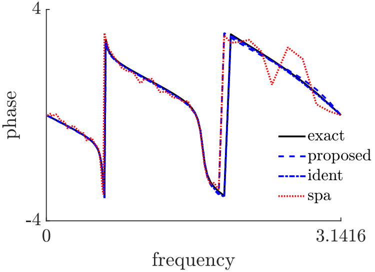
- effectively 5 lines of code
- MIMO case, multiple evaluation points
- $L = n + 1$ in order to have a single hyper-parameter

Example: EIV setup with 4th order system

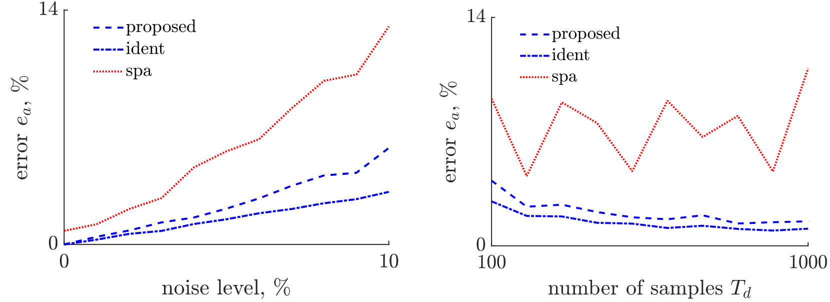
`dd_frest` is compared with

- `ident` — parametric maximum-likelihood estimator
- `spa` — nonparameteric estimator with Welch filter





Monte-Carlo simulation over different noise levels and number of samples



$$e_a := 100\% \cdot |(|\overline{H}_z| - |\widehat{H}_z|)| \, / \, |\overline{H}_z|$$

Outline

Case study: Dynamic measurement

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Generalization for nonlinear systems

Kernel representation

LTI systems

$$\mathscr{B} = \ker R(\sigma) := \{\, w \mid R(\sigma)w = 0 \,\}$$
$$= \{\, w \mid R_0 w + R_1 \sigma w + \cdots + R_\ell \sigma^\ell w = 0 \,\}$$

nonlinear time-invariant system

$$\mathscr{B} = \Big\{\, w \mid R(\underbrace{w, \sigma w, \ldots, \sigma^\ell w}_{x}) = 0 \,\Big\}$$

linearly parameterized R

$$R(x) = \sum \theta_i \phi_i(x) = \theta^\top \phi(x),$$

ϕ — model structure

θ — parameter vector

Polynomial SISO NARX system

$$\mathscr{B}(\theta) = \left\{ w = \begin{bmatrix} u \\ y \end{bmatrix} \mid y = f\big(u, \sigma w, \ldots, \sigma^{\ell} w\big) \right\}$$

split f into 1st order (linear) and other (nonlinear) terms

$$f(x) = \theta_{\text{li}}^{\top} x + \theta_{\text{nl}}^{\top} \phi_{\text{nl}}(x)$$

ϕ_{nl} — vector of monomials

Special cases

Hammerstein

$$\phi_{\mathrm{nl}}(x) = \begin{bmatrix}\phi_u(u) & \phi_u(\sigma u) & \cdots & \phi_u(\sigma^\ell u)\end{bmatrix}^\top$$

FIR Volterra

$$\phi_{\mathrm{nl}}(x) = \phi_{\mathrm{nl}}(x_u), \quad \text{where } x_u := \mathrm{vec}(u, \sigma u, \ldots, \sigma^\ell u).$$

bilinear

$$\phi_{\mathrm{nl}}(x) = x_u \otimes x_y, \quad \text{where } x_y := \mathrm{vec}(y, \sigma y, \ldots, \sigma^{\ell-1} y)$$

generalized bilinear

$$\phi_{\mathrm{nl}}(x) = \phi_{u,\mathrm{nl}}(x_u) \otimes x_y$$

LTI embedding of polynomial NARX system

$$\mathscr{B}_{\text{ext}}(\theta) := \left\{ w_{\text{ext}} = \begin{bmatrix} u \\ u_{\text{nl}} \\ y \end{bmatrix} \;\middle|\; \sigma^{\ell} y = \theta_{\text{li}}^{\top} x + \theta_{\text{nl}}^{\top} u_{\text{nl}} \right\}$$

define: $\quad \Pi_w w_{\text{ext}} := w \quad$ and $\quad \Pi_{u_{\text{nl}}} w_{\text{ext}} := u_{\text{nl}}$

fact: $\quad \mathscr{B}(\theta) \subseteq \Pi_w \mathscr{B}_{\text{ext}}(\theta)$, moreover

$$\mathscr{B}(\theta) = \Pi_w \left\{ w_{\text{ext}} \in \mathscr{B}_{\text{ext}}(\theta) \mid \Pi_{u_{\text{nl}}} w_{\text{ext}} = \phi_{\text{nl}}(x) \right\}$$

FIR Volterra data-driven simulation

given

data $w_{\rm d} = (u_{\rm d}, y_{\rm d})$ of lag-ℓ FIR Volterra system $\mathscr{B}$
$\phi_{\rm nl}$ — system's model structure

assume ID conditions for $\mathscr{B}_{\rm ext}$ hold

then, $\mathscr{B}|_L = \text{image}\, M$, where

$$M(w_{\rm ini}, u) := \mathscr{H}_L(\sigma^\ell y_{\rm d}) \underbrace{\begin{bmatrix} \mathscr{H}_\ell(w_{\rm d}) \\ \mathscr{H}_L(\sigma^\ell u_{\rm d}) \\ \hline \mathscr{H}_\ell\big(\phi_{\rm nl}(x_{u_{\rm d}})\big) \\ \mathscr{H}_L\big(\sigma^\ell \phi_{\rm nl}(x_{u_{\rm d}})\big) \end{bmatrix}^\dagger \begin{bmatrix} w_{\rm ini} \\ u \\ \hline \phi_{\rm nl}(x_{u_{\rm ini}}) \\ \phi_{\rm nl}(x_u) \end{bmatrix}}_{g}$$

proof

$$\left[\begin{array}{c} \mathscr{H}_\ell(w_\mathrm{d}) \\ \mathscr{H}_L(\sigma^\ell u_\mathrm{d}) \\ \hline \mathscr{H}_\ell\big(\phi_\mathrm{nl}(x_{u_\mathrm{d}})\big) \\ \mathscr{H}_L\big(\sigma^\ell \phi_\mathrm{nl}(x_{u_\mathrm{d}})\big) \\ \hline \color{red}{\mathscr{H}_L(\sigma^\ell y_\mathrm{d})} \end{array}\right] g = \left[\begin{array}{c} w_\mathrm{ini} \\ u \\ \hline \phi_\mathrm{nl}(x_{u_\mathrm{ini}}) \\ \phi_\mathrm{nl}(x_u) \\ \hline \color{red}{y} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ b \end{array}}\right\} \text{B1} \\ \left.\vphantom{\begin{array}{c} a \\ b \end{array}}\right\} \text{B2} \\ \left.\vphantom{a}\right\} \text{B3} \end{array}$$

B1 constraint on g, such that $w_\mathrm{ini} \wedge \big(u, \mathscr{H}_L(\sigma^\ell y_\mathrm{d})g\big) \in \mathscr{B}_\mathrm{ext}$
B2 constraint $u_\mathrm{nl} = \phi_\mathrm{nl}(x) \iff \mathscr{B}_\mathrm{ext} = \mathscr{B}(\theta)$
B3 defines the to-be-computed output y

generalized bilinear models

also tractable because B2: $u_\mathrm{nl} = \phi_\mathrm{nl}(x)$ is still linear in y