Almost Classical Skew Bracoids

Isabel Martin-Lyons

with the supervision of Paul Truman

Keele University, UK

Groups, Rings and the Yang-Baxter equation

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The Objects at Play

Definition

A *skew (left) brace* is a triple $(G, \star, \circ)$, where $(G, \star)$ and $(G, \circ)$ are groups and for all $g, h, f \in G$

$$g \circ (h \star f) = (g \circ h) \star g^{-1} \star (g \circ f).$$

Definition

A *skew (left) bracoid* is a 5-tuple $(G, \circ, N, \star, \odot)$, where $(G, \circ)$ and $(N, \star)$ are groups and $\odot$ is a transitive action of G on N for which

$$g \odot (\eta \star \mu) = (g \odot \eta) \star (g \odot e_N)^{-1} \star (g \odot \mu),$$

for all $g \in G$ and $\eta, \mu \in N$.

Housekeeping

- We will assume everything is finite.
- We will frequently write (G, N) for $(G, \circ, N, \star, \odot)$ (and then flagrantly use $\star$ to mean the operation in N and $\odot$ the action of G on N).

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On the other hand, suppose we have a skew bracoid $(G, \circ, N, \star, \odot)$ with $|G| = |N|$. We then have a bijection $g \mapsto g \odot e_N$, which we can use to transfer the operation from one group onto the other, to produce a skew brace.

In this situation we will say (G, N) is *essentially a skew brace*.

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- Then $H \cap S = \{e_N\}$, $|G| = |H||S|$, so H is a complement to S in G.

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Remark

Let $(G, \circ, G, \star, \odot)$ be a skew brace thought of as a skew brace and $S = \mathrm{Stab}_G(e_N)$. We say (G, G) is *almost classical* if S has a normal complement H in G for which (H, G) is essentially a trivial skew brace.

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The λ-functions

Let (G, N) be an almost classical skew bracoid with H such that $G \cong H \rtimes S$ and (H, N) is essentially a trivial skew brace.

Let $hx \in G$, with $h \in H$ and $x \in S$, and $n = h_n \odot e_N \in N$ then,

$$\begin{aligned}\lambda_{hx}(n) &= (hx \odot e_N)^{-1} \star (hx \odot n) \\ &= (h \odot e_N)^{-1} \star (hxh_n \odot e_N) \\ &= (h^{-1} \odot e_N) \star (hxh_n x^{-1} \odot e_N) \\ &= xh_n x^{-1} \odot e_N.\end{aligned}$$

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So we conjugate by the S part and the H part acts trivially.

To the holomorph!

We have a correspondence between (equivalence classes of) skew bracoids (G, N) and transitive subgroups of $\mathrm{Hol}(N) = N \rtimes \mathrm{Aut}(N)$. We use the map $\Lambda : G \to \mathrm{Hol}(N)$ given by $g \mapsto (g \odot e_N, \lambda_g)$ for the forward direction.

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Example

Let (G, N) be an almost classical skew bracoid and H be a normal complement to $S = \mathrm{Stab}_G(e_N)$ in G with (H, N) essentially a trivial skew brace.

Then,

$$\begin{aligned}\Lambda(H) &= \{(h \odot e_N, \lambda_h) \mid h \in H\} \\ &= \{(h \odot e_N, id) \mid h \in H\} \\ &= (N, id),\end{aligned}$$

Example (continued)

Also,

$$\begin{aligned}\Lambda(S) &= \{(x \odot e_N, \lambda_x) \mid x \in S\} \\ &= \{(e_N, \lambda_x) \mid x \in S\} \\ &= \Lambda(G) \cap (e_N, \mathrm{Aut}(N))\end{aligned}$$

since any $g \mapsto (e_N, \lambda_g)$ must in particular have $g \odot e_N = e_N$.

So $\Lambda(G) = N \rtimes \Lambda(S)$.

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Conversely, let $A \leq \mathrm{Aut}(N)$. Packaging up $N \rtimes A \subseteq \mathrm{Hol}(N)$ with N, we get an almost classical skew bracoid $(N \rtimes A, N)$.

Left Ideals

Proposition

If (G, N) is an almost classical skew bracoid and $S = \mathrm{Stab}_G(e_N)$, then $G' \odot e_N$ is a left ideal for all G' with $S \leq G' \leq G$.

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Proof.

Let G' be a subgroup of G containing S. Since $S \subseteq G'$ if $hx \in G'$ then $h = hxx^{-1} \in G'$. Then for $hx \in G$ and $h_1'x_1', h_2'x_2' \in G'$

- $(h_1'x_1' \odot e_N)(h_2'x_2' \odot e_N) = (h_1' \odot e_N)(h_2' \odot e_N) = h_1'h_2' \odot e_N \in G' \odot e_N$. Hence $G' \odot e_N$ is a subgroup of N.

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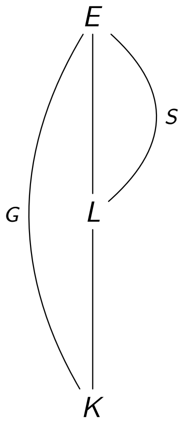
- $\lambda_{hx}(h_1'x_1' \odot e_N) = \lambda_{hx}(h_1' \odot e_N) = xh_1'x^{-1} \odot e_N \in G' \odot e_N$. Hence $G' \odot e_N$ is closed under the $\lambda-$functions of G.

□

To a relevant extension of fields!

Let L/K be a separable extension of fields with Galois closure E, and write $G = \mathrm{Gal}(E/K)$ and $S = \mathrm{Gal}(E/L)$.



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- Suppose $(G, G/S)$ is almost classical skew bracoid thanks to H. We take $\mathcal{R}_\star(G/S) \subseteq \mathrm{Perm}(G/S)$

$$\mathcal{R}_\star(hS)[h'S] = h'S \star (hS)^{-1} = h'h^{-1}S.$$

$$\begin{tikzcd}
& E \arrow[d, dash] \arrow[dd, dash, bend right=60, "G"'] \arrow[d, dash, bend left=60, "S"] \\
& L \arrow[d, dash] \\
& K
\end{tikzcd}$$

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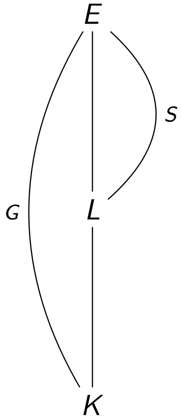

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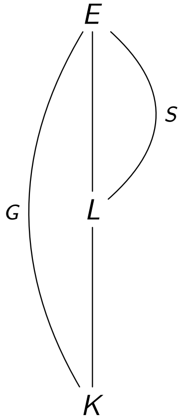
- $\mathcal{L}(H)^{opp} := \mathrm{Cent}_{\mathrm{Perm}(G/S)}(\mathcal{L}(H))$.



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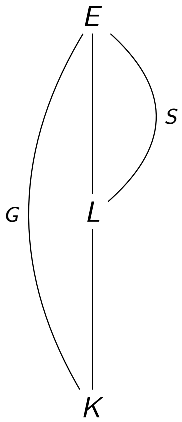

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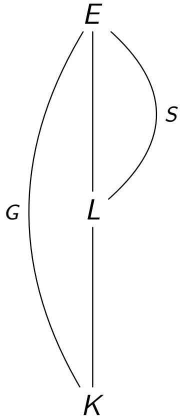
The Hopf-Galois structures corresponding to such $\mathcal{L}(H)^{opp}$ are precisely the almost classical HGS on L/K.



Thank you for your attention!