Skew bracoids, Semi-braces and the Yang-Baxter Equation

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Outline

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The skew brace

Definition (Guarnieri and Vendramin, 2017)

A *skew brace* is a triple $(G, +, \cdot)$, where $(G, +)$ and $(G, \cdot)$ are groups and for all $g, h, f \in G$

$$g \cdot (h + f) = (g \cdot h) - g + (g \cdot f). \tag{1}$$

- We refer to
  - (1) as the *skew brace relation*;
  - $(G, \cdot)$ as the *multiplicative* group;
  - $(G, +)$ as the *additive* group, though we do not assume $+$ is abelian (and do not always write it additively).
- The additive and multiplicative identities coincide.
- If $(G, \cdot)$ is a group then $(G, \cdot, \cdot)$ and $(G, \cdot^{opp}, \cdot)$ are skew braces.

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Definition/Proposition

The map $\gamma : (G, \cdot) \to \mathrm{Perm}(G, +)$, sending g to γ_g, given by

$$\gamma_g(f) = -g + (g \cdot f),$$

for $g, f \in G$, is in fact a homomorphism, with image in $\mathrm{Aut}(G, +)$. We call this map the γ-*function* of the skew brace.

Applications

Yang-Baxter Equation

Skew braces are associated with *solutions to the set-theoretic Yang-Baxter equation*. A solution is a set G together with a map $\boldsymbol{r}(x, y) = (\lambda_x(y), \rho_y(x))$ from $G \times G$ to $G \times G$ satisfying

$$(\boldsymbol{r} \times 1)(1 \times \boldsymbol{r})(\boldsymbol{r} \times 1) = (1 \times \boldsymbol{r})(\boldsymbol{r} \times 1)(1 \times \boldsymbol{r}).$$

The $\boldsymbol{r}$'s coming from skew braces are bijective and so are each λ_g, ρ_h so the solutions are *non-degenerate*.

Hopf-Galois Theory

Skew braces are connected to Hopf-Galois structures on finite Galois extensions of fields. The multiplicative group $(G, \cdot)$ is the Galois group and $(G, +)$ occurs as the *type* of the structure.

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The semi-brace

Definition (Catino, Colazzo, and Stefanelli, 2017)

A *semi-brace* is a triple $(G, +, \cdot)$ with $(G, \cdot)$ a group and $(G, +)$ a left-cancellative semi-group satisfying

$$g \cdot (h + h') = g \cdot h + g \cdot (g^{-1} + h')$$

for all $g, h, h' \in G$.

Semi-braces give rise to solutions, these are left non-degenerate but unlike those from skew braces not necessarily right non-degenerate.

Examples

Examples

- Skew braces are semi-braces.
- If $(G, \cdot)$ is a group then $(G, +, \cdot)$ is a semi-brace with $g + h = h$ for all $g, h \in G$, this is called the *trivial semi-brace on* G.
- For $n \in \mathbb{N}$, take $(G, \cdot) = \langle r, s \mid r^n = s^2 = e, srs^{-1} = r^{-1} \rangle \cong D_{2n}$ and define $r^i s^j + r^k s^\ell := r^{i+k} s^\ell$ then $(G, +, \cdot)$ is a semi-brace.

Some Facts

Proposition

Let $(G, +, \cdot)$ be a semi-brace, write E for the set of additive idempotents. Then

- the multiplicative identity $e \in E$;
- E is a subgroup of G under $\cdot$ and $(E, +, \cdot)$ is also a semi-brace;
- $G + e$ is a group under both $+$ and $\cdot$, and $(G + e, +, \cdot)$ is a skew brace;
- $(G, \cdot)$ admits an exact factorisation by $G + e$ and E.

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The skew bracoid

Definition (M-L and Truman, 2024)

A *skew bracoid* is a quintuple $(G, \cdot, N, +, \odot)$ (or a double (G, N)) with $(G, \cdot)$ and $(N, +)$ groups and $\odot$ a transitive action of G on N satisfying

$$g \odot (\eta + \mu) = (g \odot \eta) - (g \odot e_N) + (g \odot \mu) \tag{2}$$

for all $g \in G$ and all $\eta, \mu \in N$.

Skew bracoids are associated with Hopf-Galois structures on separable extensions of fields.

Examples

Examples

- A skew brace $(G, +, \cdot)$ can be thought of as a skew bracoid $(G, \cdot, G, +, \cdot)$. If a skew bracoid (G, N) has $\mathrm{Stab}_G(e_N) = \{e_G\}$ then the evaluation map defines a bijection between G and N that we can use to transfer the operation from one group to the other and produce a bone fide skew brace. We say that such a skew bracoid is *essentially a skew brace.*
- For any group G, $(G, \{e\})$ is a skew bracoid.
- Let $d, n \in \mathbb{N}$ such that $d|n$. Take $G = \langle r, s\rangle \cong D_{2n}$ and $N = \langle \eta \rangle \cong C_d$. Then we get a skew bracoid (G, N) using the action $\odot$ given by
$$r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}.$$

The γ-function

Definition/Proposition

Given a skew bracoid $(G, \cdot, N, +, \odot)$, we define the map $\gamma : g \mapsto \gamma_g$, by

$$\gamma_g(\eta) = -(g \odot e_N) + (g \odot \eta),$$

for $g \in G, \eta \in N$. Then γ is a homomorphism with image in $\mathrm{Aut}(N, +)$.

Examples

- In $(G, \{e\})$, $\gamma_g = id$ for all $g \in G$.
- In (D_{2n}, C_d), recall $r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}$, we have

$$\gamma_{r^i s^j}(\eta^k) = (r^i s^j \odot e_N)^{-1}(r^i s^j \odot \eta^k) = \eta^{-i}\eta^{i+(-1)^j k} = \eta^{(-1)^j k}.$$

A family of skew bracoids

Definition

A skew bracoid (G, N) is said to *contain a brace* if $S = \mathrm{Stab}_G(e_N)$ has a complement H in G, so that G has the exact factorisation HS.

Such an H acts regularly on N as $N = G \odot e_N = HS \odot e_N = H \odot e_N$ and $\mathrm{Stab}_H(e_N) = \{e_G\}$. Then we have

- (H, N) essentially a skew brace,
- $(S, \{e\})$ a skew bracoid, and
- $G = HS$.

A sense of strength

Example

For $(G, N) \cong (D_{2n}, C_n)$ where $r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}$, $S = \mathrm{Stab}_G(e_N) = \langle s \rangle$ which has $R = \langle r \rangle$ as a complement and if n is even $H = \langle r^2, rs \rangle$ as an additional complement.

Non-example

In the family of $(G, N) \cong (D_{2n}, C_d)$ if n is odd and $(d, \frac{n}{d}) > 1$ then $S = \langle r^d, s \rangle$ does not have a complement in G and the skew bracoid does not contain a brace.

Outline

The connection

Theorem (Colazzo, Koch, M-L, and Truman, 2024)

Let $(G, \cdot)$ be a group with exact factorisation HS and identity e. There is a bijection between

1. binary operations $+$ and transitive actions $\odot$ of G on H such that $(G, \cdot, H, +, \odot)$ is a skew bracoid with $S = \mathrm{Stab}_G(e)$,
2. and binary operations $\hat{+}$ for which $(G, \hat{+}, \cdot)$ is a semi-brace with S the set of additive idempotents and $G \hat{+} e = H$.

Key Ideas.

From (1) to (2) define $g_1 \hat{+} g_2 = g_2 \gamma_{g_2^{-1}}(g_1 \odot e)$ for $g_1, g_2 \in G$.
From (2) to (1) define $h_1 + h_2 = h_2 \hat{+} h_1$ for $h_1, h_2 \in H$ and $g \odot h = g \cdot h + e$ for $g \in G$ and $h \in H$. $\square$

Examples

- Starting with the skew bracoid $(G, \{e\})$ we define

$$g_1 \hat{+} g_2 := g_2 \gamma_{g_2^{-1}}(g_1 \odot e) = g_2,$$

to recover the trivial semi-brace on G.

- Consider the semi-brace $(G, +, \cdot)$ where $(G, \cdot) = \langle r, s \rangle \cong D_{2n}$ and $r^i s^j + r^k s^\ell := r^{i+k} s^\ell$, here $G + e = \langle r \rangle$. If we define

$$r^i s^j \odot r^k := r^i s^j \cdot r^k + e = r^{i+(-1)^j k} s^j + e = r^{i+(-1)^j k},$$

we get our skew bracoid (D_{2n}, C_n) with η relabelled as r.

Outline

Solutions from skew bracoids

Let (G, H) be a skew bracoid that contains a brace (H, H). With this we can define

$$\lambda_{g_1}(g_2) = \gamma_{g_1}(g_2 \odot e)$$

and then

$$\rho_{g_2}(g_1) = \lambda_{g_1}(g_2)^{-1} g_1 g_2,$$

for all $g_1, g_2 \in G$.

Then G with $\boldsymbol{r}(x, y) = (\lambda_x(y), \rho_y(x))$ forms a (right non-degenerate but possibly left degenerate) solution.

Example

Example

In our (D_{2n}, C_n) example, recall we may present this as (G, R) with $R = \langle r \rangle$ so we get

$$\begin{aligned}
\lambda_{r^i s^j}(r^k s^\ell) &= \gamma_{r^i s^j}(r^k s^\ell \odot e) \\
&= \gamma_{r^i s^j}(r^k) \\
&= r^{(-1)^j k}, \\
\rho_{r^k s^\ell}(r^i s^j) &= r^{-(-1)^j k} r^i s^j r^k s^\ell \\
&= r^{-(-1)^j k + i + (-1)^j k} s^{j+\ell} \\
&= r^i s^{j+\ell}.
\end{aligned}$$

Hence G with $\boldsymbol{r}(r^i s^j, r^k s^\ell) = (r^{(-1)^j k}, r^i s^{j+\ell})$ is a solution.

Or alternatively

Example

If we take n to be even and use the complement $H = \langle r^2, rs \rangle$ to S in $G \cong D_{2n}$.

We find that G with $\boldsymbol{r}(r^i s^j, r^k s^\ell) = (r^{(-1)^j k} s^k, r^{(-1)^k i} s^{j+k+\ell})$ is a also solution.

Thank you for your attention!

References I

L. Guarnieri and L. Vendramin. Skew braces and the Yang–Baxter equation. *Mathematics of Computation*, 86(307):2519–2534, 2017. ISSN 1088-6842. doi: 10.1090/mcom/3161. URL http://dx.doi.org/10.1090/mcom/3161.

Francesco Catino, Ilaria Colazzo, and Paola Stefanelli. Semi-braces and the Yang–Baxter equation. *Journal of Algebra*, 483:163–187, 2017. ISSN 0021-8693. doi: https://doi.org/10.1016/j.jalgebra.2017.03.035. URL https://www.sciencedirect.com/science/article/pii/S0021869317302284.

I M-L and Paul J. Truman. Skew bracoids. *Journal of Algebra*, 638:751–787, 2024. ISSN 0021-8693. doi: https://doi.org/10.1016/j.jalgebra.2023.10.005. URL https://www.sciencedirect.com/science/article/pii/S0021869323005136.

Ilaria Colazzo, Alan Koch, I M-L, and Paul J. Truman. Skew bracoids and the Yang-Baxter equation, 2024.