

DS4400 Notes

DS4400 Notes 01/24

1. Convex functions:

A function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex iff $\forall \theta_1, \theta_2 \in \mathbb{R}^d$ and $\forall \alpha \in [0, 1]$ we have $f(\alpha\theta_1 + (1-\alpha)\theta_2) \leq \alpha f(\theta_1) + (1-\alpha)f(\theta_2)$

In the special case ($d = 1$) $f : \mathbb{R} \rightarrow \mathbb{R}$, f is convex iff $\forall \theta, f''(\theta) \geq 0$

When the function is convex, **local min** \equiv **global min**. When the system is not convex, we might find only a **local min** but not a **global min**

2. Dealing with non convex function:

In gradient descent:

(a) use larger ρ in the beginning and gradually decrease ρ with iteration.

(b) Run SGD/GD with multiple random initializations $\theta_1^{(0)}, \theta_2^{(0)} \dots$ and keep the best solution.

3. $\arg \min_{\theta} \sum_{i=1}^N (y_i - \theta^T x_i)^2 \triangleq J(\theta)$

In linear regression, $J(\theta)$ is convex.

4. Robustness of Regression to outliers:

(a) Run outlier detection algorithm, remove detected outliers, then run Linear Regression on remaining points.

(b) Robust Regression cost function.

$$\arg \min_{\theta} \sum_{i=1}^N e_i^2, \quad e_i \triangleq y_i - \theta^T x_i$$

e^2 is extremely unhappy with large errors.

we might use $|e|$ to replace the function. This might be more tolerance. Then,

$$\arg \min_{\theta} \sum_{i=1}^N |y_i - \theta^T x_i|$$

5. **Exercise:** $D = \{(x_1, y_1 = 100) \dots (x_1, y_1 = 0), (x_{11}, y_{11} = 0), (x_{12}, y_{12} = 0)\}$

$$e^2: 10(\theta - 100)^2 + 2\theta^2 \rightarrow$$

$$\frac{\partial}{\partial \theta} = 20(\theta - 100) + 4\theta = 0 \rightarrow$$

$$\theta = 83.3$$

$$|e| : \min_{\theta} \sum_{i=1}^{12} |\theta - y_i| = 10|\theta - 100| + 2\theta$$

$$(\theta \leq 100) = \min_{\theta} 10(100 - \theta) + 2\theta$$

$$= 1000 - 8\theta \rightarrow \theta = 100$$

$$(\theta \geq 100) = \min_{\theta} 10(\theta - 100) + 2\theta$$

$$= 12\theta - 1000 \rightarrow \theta = 100$$

6. How to solve l1-norms cost functions?

- (a) No closed form
- (b) we need to be careful with gradient descent
- (c) We need to use convex programming toolboxes (convex optimizations)

7. Huber loss funct

$$l_{\delta}(e) = \begin{cases} \frac{1}{2}e^2 & |e| \leq \delta \\ \delta|e| - \frac{\delta^2}{2} & |e| \geq \delta \end{cases}$$

$$\frac{\partial l_{\delta}(e)}{\partial e} = \begin{cases} e & -\delta \leq e \leq \delta \\ \delta & e > \delta \\ -\delta & e < -\delta \end{cases}$$

in huber loss function, we don't have closed form solution but we can run gradient descent now.

8. **Definition:** Overfitting:

Learning a system from training data that does very well on training data itself (e.g, very low regression error on training data), but performs poorly on test data.

9. **Definition:** Overfitting in Linear Regression

$$\Phi^T \Phi \theta = \Phi^T Y$$

$$\Rightarrow \theta^* = (\Phi^T \Phi)^{-1} \Phi^T Y$$

$$\text{rank}(\Phi^T \Phi) \leq \min\{\text{rk}(\Phi^T), \text{rk}(\Phi)\} = \text{rk}(\Phi) \leq \min\{N, d\}$$

$\Phi^T \Phi$ is $d \times d$ matrix, then rank is $\leq d$.

Therefore, when $N < d$ it is not invertible which means we have multiple solutions and results in overfitting.

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1. **Definition:** Overfitting

Refers to situation where the learned model does well on training data and poorly on testing data.

As d (dimension of system) increases, then training error goes down (can be exactly ZERO for sufficiently large d)

2. In Linear regression:

$$\min \sum_{i=1}^n (\theta^T \phi(x_i) - y_i)^2$$

set the derivative to 0 and we find

$$\Phi^T \Phi \theta = \Phi^T Y$$

$$\text{Then } \theta^* = (\Phi^T \Phi)^{-1} \Phi^T Y$$

When is it the case that $\Phi^T \Phi$ is not invertible?

Since $\Phi^T \Phi \in \mathbb{R}^{N \times d}$

$$rk(\Phi^T \Phi) \leq rk(\Phi) \leq \min\{N, d\}$$

$\Phi^T \Phi \in \mathbb{R}^{d \times d}$ is invertible when $rk(\Phi^T \Phi) = d$. Therefore, when $N < d$, $rk(\Phi^T \Phi) = N$, $\Phi^T \Phi$ is not invertible. There will be infinitely many solutions for θ .

Generally, need sufficient # samples

3. Test overfitting.

If $\Phi^T \Phi$ is not invertible,

$$\exists v \neq 0, \Phi^T \Phi v = 0$$

$\Rightarrow \theta^* + \alpha v$ is also a solution for any $\alpha \in \mathbb{R}$

$$\Phi^T \Phi(\theta^* + \alpha v) = \Phi^T \Phi \theta^* + \Phi^T \Phi(\alpha v)$$

$$= \Phi^T \Phi \theta^* + \alpha \Phi^T \Phi v$$

$$= \Phi^T \Phi \theta^* = \Phi^T Y$$

We can find large α so that θ^* have extremely large entries.

Generally, if the entries are very large (abs) we might have overfitting

4. Treat overfitting

We want to change regression optimization to prevent θ from very large terms.

then we change the cost function:

$$\min_{\theta} \sum_{i=1}^N (\theta^T \phi(x_i) - y_i)^2 + \lambda \sum_{j=1}^d \theta_j^2$$

λ : regularization parameter (> 0)

$\sum_{j=1}^d \theta_j^2$: regularizer.

$\lambda \rightarrow 0$: back to overfitting

$\lambda \rightarrow \infty$: $\theta^* = 0$, underfitting

(a) closed-form

$$\frac{\partial J}{\partial \theta}$$

$$= 2\Phi^T(\Phi\theta - Y) + \lambda \frac{\partial \sum_{j=1}^d \theta_j^2}{\partial \theta}$$

$$= 2\Phi^T(\Phi\theta - Y) + 2\lambda\theta$$

Let it be zero:

$$\Phi^T \Phi \theta + \lambda \theta = \Phi^T Y$$

$$(\Phi^T \Phi + \lambda I_d) \theta = \Phi^T Y$$

$$\text{Then } \theta^* = (\Phi^T \Phi + \lambda I_d)^{-1} \Phi^T Y$$

(b) Gradient descent

Find initial $\theta^{(0)}$

$$\theta^t = \theta^{(t-1)} - \rho \frac{\partial J}{\partial \theta} |_{\theta^{(t-1)}}$$

$$= \theta^{(t-1)} - 2\Phi^T(\Phi\theta^{(t-1)} - Y) + 2\lambda\theta^{(t-1)}$$

5. Hyperparameter Tuning

GD: set learning rate ρ

Robust Reg: Huber loss δ

overfitting and regularization: λ

ρ, δ, λ = hyperparameters

How to pick hyperparameters?

BAD APPROACH 1:

(a) pick some set of possible $\lambda_i \in \{\lambda_1, \lambda_2, \dots\}$

Run regression with λ_i and find θ_i^*

Measure regression error:

$$\epsilon_{tr}(\lambda) = \sum_{i=1}^N ((\theta^*(\lambda))^T x_i - y_i)^2$$

To sum: just find λ for which $\epsilon_{tr}(\lambda)$ is minimum

This approach is setting λ back to 0

Test data needed!!!

(a) We need to Train λ_i on **training set** to minimize the cost function

$$2\Phi^T(\Phi\theta - Y) + 2\lambda\theta$$

to find θ_i^*

(b) Measure regression error on the **hold-out set** D^{ho}

$$\epsilon_{tr} = \sum_{x_i, y_i \in D^{ho}} (y_i - (\theta^*(\lambda))^T x_i)^2$$

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1. Hyperparameter Tuning:

$$\min_{\theta} \sum_{i=1}^N (\theta^T \phi(x_i) - y_i)^2 + \lambda \sum_{j=1}^d \theta_j^2$$

- For $\lambda \in \{\lambda_1, \lambda_2, \dots, \lambda_p\}$
 - Train using D^{tr} with $\lambda \rightarrow \theta^*(\lambda)$
 - Measure validation error

$$\epsilon^{tr}(\lambda) = \sum_{x_i, y_i \in D^{ho}} (y_i - (\theta^*(\lambda))^T x_i)^2$$

- select λ which minimizes

$$\epsilon^{ho}(\lambda) \rightarrow \lambda^* = \min_{\{\lambda_1, \lambda_2, \dots, \lambda_p\}} \epsilon^{ho}(\lambda)$$

2. Problems:

- Take much longer time since we are training the models multiple times
- Each training is using a subset of the data set, then each training is amplifying the problem of overfitting.

3. K-fold cross validation

divide Data set to k equally large sets $\{D_1, D_2, \dots, D_k\} \in D$

- For $\lambda \in \{\lambda_1, \lambda_2, \dots, \lambda_p\}$
 - For $i = 1, 2, \dots, k$
 - * train on $\bigcup_{j \neq i} D^j$ and get $\theta_i^*(\lambda)$
 - * compute validate error on $D^i \rightarrow \epsilon_i^{ho}(\lambda)$
 - compute average of $\{\epsilon_i^{ho}(\lambda)\}$: $\epsilon^{ho} = \frac{1}{k} \sum_{i=1}^k \epsilon_i^{ho}(\lambda)$
- select $\lambda^* = \min_{\{\lambda_1, \lambda_2, \dots, \lambda_p\}} \epsilon^{ho}(\lambda)$

Once we find the best λ , train the model on the whole set.

4. PROBABILITY REVIEW

- Random Variable: a variable that takes values corresponding to outcome of a random phenomenon.
- Discrete r.v.: discrete values
- continuous r.v. continuous range of values
- Condition: $P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$

$$P(X, Y) = P(X|Y)P(Y)$$

$$P(X, Y) = P(Y|X)P(X)$$

Chain rule:

$$P(X_1, X_2, \dots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots P(X_N|X_1, X_2 \dots X_{N-1})$$

- Marginalization

$$p(x, y) \text{ known}$$
$$p(x) = \sum_y p(x, Y = y) = \int p(x, y) dy$$

- Bayes Rule:

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)} = \frac{P(X|Y)P(Y)}{P(X)}$$

- Independence:

r.v. are independent ($X \perp\!\!\!\perp Y$) iff

$$P(X|Y) = p(X), P(Y|X) = p(Y)$$

or $P(x, y) = P(x)p(y)$

- conditional independence example: X = height of person, Y = vocabulary, X is not independent of Y since babies may have less vocabulary and with lower heights. However, X = height, Y = vocab, Z = age. Then $(X \perp\!\!\!\perp Y) \mid Z$

$$P(X, Y|Z) = P(X|Z)P(Y|Z)$$

$$\Rightarrow P(X|Y, Z) = P(X|Z)$$

- Expectation:
 $E(X) = \sum xp(x)$ or $\int xp(x)dx$
 $E(f(X)) = \sum f(x)p(x)$ or $\int f(x)p(x)dx$
 Given $X \perp\!\!\!\perp Y$, $E[XY] = E[X]E[Y]$
 hint: $(E[XY] = E[f(x, y)])$
- IID r.v: independent and identically distributed
 $p(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = p(X_1 = x_1)p(X_2 = x_2) \dots p(X_n = x_n)$ and each experiment is identical.
 $P(X_1 = \theta) = P(X_2 = \theta) = \dots = P(X_n = \theta)$

DS4400 Notes 02/04

Maximum Likelihood Estimation

1. Some distributions:

- Gaussian Dist.

$$P(X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Laplace Dist.

$$P(X = x) = \frac{1}{2\lambda} e^{-\frac{|x-\mu|}{\lambda}}$$

- Bernoulli Dist.

$$P_\theta(x = 1) = \theta, P_\theta(x = 0) = 1 - \theta$$

2. Goal: Learn parameters of probability models. (fix the prob. model class)

In ML, we learn these parameters (θ) using training data, D .

We want to measure $P_\theta(D)$

MLE: $\theta^* = \arg \max_{\theta} P_\theta(D)$ Under such θ^* , the probability of observing the given dataset is maximum.

3. **Exercise:** Flipping a coin.

This is a Binomial Dist. (n time Bernoulli Trials)

model: $p(X = x) = \theta^x(1 - \theta)^{1-x}, x = 0, 1$

$$P_\theta(D) = P_\theta(X_1 = x_1, X_2 = x_2 \dots)$$

Assuming that tossing coins are iids:

$$P_\theta(D) = P_\theta(x_1)P_\theta(x_2) \dots$$

$$= \theta^{\sum x_i} (1 - \theta)^{\sum (1-x_i)}$$

Then likelihood function:

$$L(\theta) = P_{\theta}(D)$$

Take the logarithm of both sides (simplify product to sum)

$$\theta^* = \arg \max_{\theta} \log L(\theta)$$

REASON:

1. log is monotonically increasing
2. simplify the powers to scale, the product to sum.
3. Increase the dynamic range (working with small numbers is not accurate on computers and memory consuming)

$$\frac{\partial \log L(\theta)}{\partial \theta} = \sum x_i \frac{1}{\theta} + (N - \sum x_i) \frac{-1}{1 - \theta}$$

Let the derivative equals to 0.

$$\begin{aligned} \frac{1}{\theta} \sum x_i &= \frac{1}{\theta - 1} (N - \sum x_i) \\ \theta &= \frac{\sum x_i}{N} \end{aligned}$$

4. **Exercise:** People's height

Use model: normal distribution.

$$L(\theta) = P_{\sigma, \mu}(D) = \prod_{i=1}^N P_e(x_i)$$

$$\log L(\theta) = -\frac{N}{2} \log(2\pi\sigma^2) - \frac{\sum_{i=1}^N (x_i - \mu)^2}{2\sigma^2} = -\frac{N}{2} \log(2\pi) - \frac{N}{2} \log(\sigma^2) - \frac{\sum_{i=1}^N (x_i - \mu)^2}{2\sigma^2}$$

$$\frac{\partial \log L(\theta)}{\partial \mu} = - \sum_{i=1}^N (\mu - x_i) / \sigma^2$$

$$\Rightarrow \hat{\mu} = \frac{\sum_{i=1}^N x_i}{N}$$

$$\frac{\partial \log L(\theta)}{\partial \sigma^2} = -\frac{N}{2} \frac{1}{\sigma^2} - \frac{\sum_{i=1}^N (x_i - \mu)^2}{2} \frac{-1}{\sigma^4}$$

$$= \frac{-N}{2\sigma^2} + \frac{\sum_{i=1}^N (x_i - \mu)^2}{2\sigma^4}$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \hat{\mu})^2$$

Classification:

Binary Classification:input = Email \rightarrow output = 'spam' vs 'non-spam'**Multiclass Classification:**input = Image \rightarrow output = 'car', 'bike', 'stop-sign', ...• **Classification Setup:**

Given a training dataset $D = \{(x_1, y_1) \cdots (x_n, y_n)\}$ where $x_i \in \mathbb{R}^d$ is input feature vector and $y_i \in \{0, 1, 2, \dots, L-1\}$, Find a mapping $g : \mathbb{R}^d \rightarrow \{0, 1, 2, \dots, L-1\}$ s.t. $g_w(x_i) = y_i$ for many i 's $\in \{1, 2, \dots, N\}$

• **Assumption:**

Assume that there is a hyperplane $w^T \phi(x) = 0$ that separates data into two classes.
Then set: $w^T \phi(x) > 0 \rightarrow y = 1$, set: $w^T \phi(x) < 0 \rightarrow y = 0$

• **Logistic Regression Model:**

$$P_w(y = 1|x) \propto e^{w^T \phi(x)/2}$$

$$P_w(y = 0|x) \propto e^{-w^T \phi(x)/2}$$

Determine Z: $P_w(y = 1|x) + P_w(y = 0|x) = 1$:

$$\frac{1}{z} e^{w^T \phi(x)/2} + \frac{1}{z} e^{-w^T \phi(x)/2} = 1$$

$$\rightarrow z = e^{w^T \phi(x)/2} + e^{-w^T \phi(x)/2}$$

$$\rightarrow P_w(y = 1|x) = \frac{1}{z} e^{w^T \phi(x)/2} = \frac{1}{1 + e^{-w^T \phi(x)}}$$

$$\text{Model: } P_w(y = 1|x) = \frac{1}{1 + e^{-w^T \phi(x)}}$$

$$\text{Model: } P_w(y = 0|x) = 1 - \frac{1}{1 + e^{-w^T \phi(x)}}$$

• **sigmoid/logistic function:**

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

• **Logistic regression:**

$$P_w(y = 1|x) = \frac{1}{1 + e^{-w^T \phi(x)}} = \sigma(w^T \phi(x))$$

• **Training: Learn w^* given training data D**

$$\text{• Testing: } P_w(y^n = 1|x^n) = \frac{1}{1 + e^{-w^{*T} \phi(x^n)}}$$

• **Assign $P > 0.5 \rightarrow \text{class1}$, $P \leq 0.5 \rightarrow \text{class0}$** • **Training via MLE:**

$$\max_w P_w(D) = \max_w P_w(y_1|x_1) \cdots P_w(y_N|x_N)$$

$$= \max_w \prod_{i=1}^N P_w(y_i|x_i)$$

We can write :

$$P_w(y_i|x_i) = P_w(y_i = 1|x_i)^{y_i} P_w(y_i = 0|x_i)^{1-y_i}$$

Apply natural log:

$$\max_w \log P_w(D) = \max_w \sum_{i=1}^N \log \left[\left(\frac{1}{1 + e^{-w^T \phi(x_i)}} \right)^{y_i} + \left(\frac{1}{1 + e^{w^T \phi(x_i)}} \right)^{1-y_i} \right]$$

$$= \max_w \sum_{i=1}^N (y_i) \log \left(\frac{1}{1 + e^{-w^T \phi(x_i)}} \right) + (1 - y_i) \log \left(\frac{1}{1 + e^{w^T \phi(x_i)}} \right)$$

$$\begin{aligned}
&= \max_w \sum_{i=1}^N (y_i) \left[\log\left(\frac{1}{1+e^{-w^T \phi(x_i)}}\right) - \log\left(\frac{1}{1+e^{w^T \phi(x_i)}}\right) \right] + \log \frac{1}{1+e^{w^T \phi(x_i)}} \\
&= \max_w \sum_{i=1}^N (y_i) \left[\log\left(\frac{1+e^{w^T \phi(x_i)}}{1+e^{-w^T \phi(x_i)}}\right) \right] + \log \frac{1}{1+e^{w^T \phi(x_i)}} \\
&= \max_w \sum_{i=1}^N (y_i) \left[\log(e^{w^T \phi(x_i)}) \right] + \log \frac{1}{1+e^{w^T \phi(x_i)}} \\
&= \max_w \sum_{i=1}^N (y_i w^T \phi(x_i)) - \log(1 + e^{w^T \phi(x_i)}) \\
&\equiv \min_w \sum_{i=1}^N -y_i w^T \phi(x_i) + \log(1 + e^{w^T \phi(x_i)})
\end{aligned}$$

Derivative:
 $\frac{\partial J}{\partial w}$

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1. REVIEW: logistic model: $P_w(y = 1|x) = \frac{1}{1+e^{-w^T \phi(x)}}$

MLE to learn w:

$$\begin{aligned}
\ell(x) &= \log P_w(D) = \log \prod_{i=1}^N P_w(y_i|x_i) \\
&= \sum_{i=1}^N [y_i \phi(x_i)^T w - \log(1 + e^{w^T \phi(x_i)})] \text{ maximizing } \ell(w) \equiv \text{minimizing } -\ell(w). \text{ Then } \min_w -\ell(w) \\
&= \min_w \sum_{i=1}^N -y_i w^T \phi(x_i) + \log(1 + e^{w^T \phi(x_i)}) \\
&= \min_w -y_i \phi(x_i)^T w + \log(1 + e^{w^T \phi(x_i)})
\end{aligned}$$

derivative:

$$\begin{aligned}
\frac{\partial -\ell(w)}{\partial w} &= \sum_i -y_i \phi(x_i) + \frac{1}{1+e^{w^T \phi(x_i)}} (e^{w^T \phi(x_i)}) (\phi(x_i)) \\
&= \sum_i -y_i \phi(x_i) + \frac{e^{w^T \phi(x_i)} \phi(x_i)}{1+e^{w^T \phi(x_i)}} \\
&= \sum_i -y_i \phi(x_i) + \frac{\phi(x_i)}{1+e^{-w^T \phi(x_i)}} \\
&= \sum_i (-y_i + \frac{1}{1+e^{-w^T \phi(x_i)}}) \phi(x_i)
\end{aligned}$$

No closed form solution for = 0.

2. GD of logistic regression

- Initialize w^0
- For $t = 1, 2, \dots$ (until converge)
 - $w^t = w^{t-1} - \rho \frac{\partial J}{\partial w}|_{t-1}$
 - $w^t = w^{t-1} - \rho \sum (-y_i + \frac{1}{1+e^{-w^{t-1} \phi(x_i)}}) \phi(x_i)|_{t-1}$

3. overfitting.

overfitting: do well on training but poorly on testing.

Symptom: w with large entries.

4. Regularized logistic regression:

$$\min_w J(w) = -\ell(w) + \frac{\lambda}{2} \|w\|_2^2$$

$$\text{GD: } \frac{\partial J + \frac{\lambda}{2} \|w\|_2^2}{\partial w} = \frac{\partial J}{\partial w} + \lambda w$$

$$= \sum_i (-y_i + \frac{1}{1+e^{-w^T \phi(x_i)}}) \phi(x_i) + \lambda w^{t-1}$$

5. clustering more than 2

one of the methods: Just creating n models for n type of data. each model is a i vs rest.

For each model we have:

$$P(y = i|x) = \sigma(w_i^T x)$$

Then see which have the max probability.

$$y^{test} = \arg \max_{i \in \{0,1,\dots,N\}} \sigma(w_i^T \phi(x^{test}))$$

6. MAXIMUM a Posteriori (MAP) Estimation:

Incorporating with prior knowledge with parameters. When the data is not enough and we have some prior knowledge, we do MAP.

MAP setting:

- we start with a "prior" model on parameters of systems $\rightarrow P_{prior}(\theta)$
- we observe a dataset $D \rightarrow P_\theta(D)$
- Given D , how the prior knowledge on θ changes $\rightarrow P(\theta|D)$

$$\text{MAP: } \max_{\theta} P(\theta|D) \rightarrow \hat{\theta}_{MAP}$$

DS4400 Notes 02/14

Maximum A Posteriori (MAP) Estimation:

Incorporate prior knowledge into system(parameter) learning

1. **Exercise:** bernoulli experiment: $\theta = P(x = 1) \rightarrow \hat{\theta}_{MLE}$ can be learned from training set $X_1 = x_1 \dots$

Tossing a coin : $\theta = P(X = 1) = P('H'), D = H, H \rightarrow \hat{\theta}_{MLE} = \frac{2}{2} = 1$

2. MAP setting:

- Put a prior distribution on θ (that encodes prior knowledge / domain expertise)
- Observe a data set $D = \{X_1 = x_1, \dots, x_N = N\} \rightarrow P(D|\theta) = P_\theta(D)$
- How much our knowledge about θ changes after seeing data, D : $P(\theta|D) \rightarrow$ posterior dist.

$$\text{MAP: } \max_{\theta} P(\theta|D) \equiv \frac{P(D|\theta)P(\theta)}{P(D)} = \max_{\theta} P(D|\theta)P(\theta) = \max_{\theta} L(\theta)P(\theta)$$

3. **Exercise:**

$$X_1 = x_1, X_2 = x_2, \dots, X_N = x_N, x_i \in \{0, 1\}$$

$$P(x_i = 1) = \theta, p(x_i = 0) = 1 - \theta$$

$$\text{Using Beta distribution: } P_{\alpha,\beta}(\theta) \propto \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

$$\text{MAP: } \max_{\theta} P(\theta|D) = \max P(x_1, x_2, \dots, x_N | \theta) P_{\alpha,\beta}(\theta)$$

$$= \prod_{i=1}^N P(X_i = x_i) P_{\alpha,\beta}(\theta)$$

$$= \theta^{\sum x_i} (1 - \theta)^{N - \sum x_i} \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

$$= \theta^{\sum x_i + \alpha - 1} (1 - \theta)^{N - \sum x_i + \beta - 1}$$

$$\text{Let } \alpha' = \sum x_i + \alpha - 1, \beta' = N - \sum x_i + \beta - 1$$

$$\text{Then } P(\theta|D) \propto P_{\alpha',\beta'}(\theta)$$

$$\max_{\theta} P(\theta|D) = \max_{\theta} P_{\alpha',\beta'}(\theta) = \frac{\alpha'-1}{\alpha'+\beta'-2}$$

$$\hat{\theta}_{MAP} = \frac{\alpha'-1}{\alpha'+\beta'-2} = \frac{\sum x_i + \alpha - 1}{N + \alpha + \beta - 2}$$

$$= \frac{N}{N + \alpha + \beta - 2} \frac{\sum x_i}{N} + \frac{\alpha + \beta - 2}{N + \alpha + \beta - 2} \frac{\alpha - 1}{\alpha + \beta - 2}$$

$$= \eta \hat{\theta}_{MLE} + \eta \hat{\theta}_{prior mode}$$

4. Conclusion:

$$N \rightarrow \infty \Rightarrow \hat{\theta}_{MAP} \sim \hat{\theta}_{MLE}$$

$$N \rightarrow 0 \Rightarrow \hat{\theta}_{MAP} \sim \hat{\theta}_{prior\ mode}$$

having prior mode is just adding fake observations generated by prior mode. The bigger the $1 - \eta$, the more fake observations we add to dataset

5. MAP on logistic regression:

$\rightarrow \min_{\omega} J(\omega)$. We might have θ of very large terms.

$$\text{Then choose } p(\omega) \propto e^{-\omega^T \omega / 2\sigma^2} = e^{-\|\omega\|_2^2 / 2\sigma^2}$$

$$\max_{\omega} P(\omega|D) \propto \max_{\omega} P(D|\omega)P(\omega)$$

$$\text{Then } \max_{\omega} \log(P(\omega|D)) \propto \max_{\omega} \log P(D|\omega) + \log P(\omega)$$

$$= \max_{\omega} J(\omega) - \frac{1}{2\sigma^2} \|\omega\|_2^2$$

$$\equiv \min_{\omega} -J(\omega) + \lambda \|\omega\|_2^2$$