DS4400 HW4

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1. **SVM**. Consider a supervised learning problem in which the training examples are points in 2-dimensional space. The positive examples (samples in class 1) are $(1,1)$ and $(-1,-1)$. The negative examples (samples in class 0) are $(1,-1)$ and $(-1,1)$. Are the positive examples linearly separable from the negative examples in the original space? If so, give the coefficients of ω.

   (a) For the example above, consider the feature transformation $\phi(x) = [1, x_1, x_2, x_1x_2]$, where x_1 and x_2 are, respectively, the first and second coordinates of a generic example x. Can we find a hyperplance $\omega^T\phi(x)$ in this feature space that can separate the data from positive and negative class. If so, give the coefficients of ω (You should be able to do this by inspection, without significant computation).

   Solution:

   We find that when x_1, x_2 have the same sign, they belong to $+$. If x_1, x_2 have different sign, they belong to $-$ Therefore, we can use x_1x_2 to decide its category. Then we can pick $\omega = [0,0,0,1]$.

   (b) What is the kernel corresponding to the feature map $\phi(\cdot)$ in the last part. In other words provde the kernel function $K(x,z) = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

   Solution:

   $K(x,z) = [1, x_1, x_2, x_1x_2]^T \cdot [1, z_1, z_2, z_1z_2] = 1 + x_1z_1 + x_2z_2 + x_1x_2z_1z_2 = 1 + xz + \frac{(xz)^2 - \|xz\|_2^2}{2}$

2. **Neural Network** Consider a neural net for a binary classification which has one hidden layer as shown in the figure below. We use a linear activation function $a(z) = cz$ at hidden units and a sigmoid activation function $a(z) = 1/(1+e^{-z})$ at the output unit to learn the function for $P(y=1|x,w)$ where $x = (x_1, x_2)$ and $w = (w_1, w_2, \ldots, w_9)$.

   (a) What is the output $P(y=1|x,w)$ from the above neural net? Express it in terms of x_i,c and weights w_i. What is the final classification boundary?

   Solution:

   For the first neuron of the hidden level: $A = a_1(w_1 + x_1w_3 + x_2w_5) = c(w_1 + x_1w_3 + x_2w_5)$. For the second neuron of the hidden level: $B = a_1(w_2 + x_1w_4 + x_2w_6) = c(w_2 + x_1w_4 + x_2w_6)$. Then the output neuron is $a_2(w_7 + w_8A + w_9B) = \frac{1}{(1+e^{-(w_7+w_8(c(w_1+x_1w_3+x_2w_5))+w_9(c(w_2+x_1w_4+x_2w_6)))})}$

   The boundary: let $w_7 + w_8A + w_9B = 0$. Then we have

   $$w_7 + w_8(c(w_1 + x_1w_3 + x_2w_5)) + w_9(c(w_2 + x_1w_4 + x_2w_6)) = 0$$

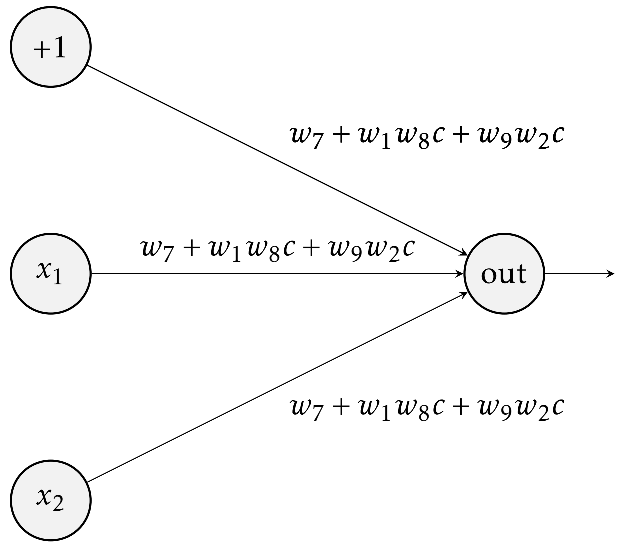
   Or we formulate it as:

   $$(w_8w_3 + w_9w_4)cx_1 + (w_8w_5 + w_9w_6)cx_2 + w_7 + w_1w_8c + w_9w_2c = 0$$

   Therefore, $y = \begin{cases} 1 & (w_8w_3 + w_9w_4)cx_1 + (w_8w_5 + w_9w_6)cx_2 + w_7 + w_1w_8c + w_9w_2c > 0 \\ 0 & (w_8w_3 + w_9w_4)cx_1 + (w_8w_5 + w_9w_6)cx_2 + w_7 + w_1w_8c + w_9w_2c < 0 \end{cases}$

(b) Draw a neural net with no hidden layer which is equivalent to the given neural net, and write weights $\tilde{w}$ of this new neural net in terms of c and w_i.

Solution:



The out node is using a sigmoid function.

(c) Is it true that any multi-layered neural net with linear activation functions at hidden layers can be represented as a neural net without any hidden layer? Explain your answer.

Solution:

It is true. linear activation functions is applying linear transformations on the inputs. Therefore, no mather how many layers it have, it is still a linear combination of the inputs. i.e. we can finally formulate the output to be something like $(\dots)x_1 + (\dots)x_2 + C$, where C is some constant. Therefore, we can use a neural network with no hidden layer to directly map to the output layer.