Some of the problems are extremely easy, some are computationally long, and some are actual proofs. Don't be surprised if it looks too easy or too hard.
Also, do the assigned HW problems. This is just in addition to HW.

## Computations

1. Consider the permutation $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 7 & 8 & 2 & 1 & 5 & 4 \end{bmatrix}$.
   (a) Write $\alpha$ as a product of disjoint cycles.
   (b) Find $\alpha^2 =$
   (c) Find $\alpha^3 =$
   (d) Find the order of $\alpha$, i.e. $|\alpha| =$
2. Let $\alpha = (1325)$, $\beta = (12)(354)$. Compute:
   (a) $\alpha \cdot \beta =$
   (b) $\beta \cdot \alpha =$
   (c) $\alpha^{-1} =$
   (d) $\alpha^4 =$
   (e) $\alpha \cdot \beta \cdot \alpha^{-1} =$
3. Consider $S_7$, the group of permutations on $\{1, 2, 3, 4, 5, 6, 7\}$.
   (a) Find all possible disjoint cycle decompositions.
   (b) For each disjoint cycle decomposition find the number of distinct permutations.
4. Consider $(Gl_2(\mathbb{R}), \cdot)$. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix}$. Find the inverse of $A$.
5. Consider $(M_2(\mathbb{R}), +)$. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix}$. Find the inverse of $A$.
6. Consider $(Gl_2(\mathbb{Z}_7), \cdot)$. Let $A = \begin{bmatrix} [1]_7 & [1]_7 \\ [3]_7 & [6]_7 \end{bmatrix}$. Find the inverse of $A$.
7. Consider $(Gl_2(\mathbb{Z}_7), \cdot)$. Let $A = \begin{bmatrix} [1]_7 & [2]_7 \\ [3]_7 & [5]_7 \end{bmatrix}$. Find the inverse of $A$.
8. Consider $(M_2(\mathbb{Z}_7), +)$. Let $A = \begin{bmatrix} [1]_7 & [2]_7 \\ [3]_7 & [6]_7 \end{bmatrix}$. Find the inverse of $A$.

9. Find an integer $b$ so that $b \cdot 7 \equiv 1 (mod\ 10)$
10. Consider $(\mathbb{Z}_{10}, \cdot_{10})$. Find the inverse of $[7]_{10} \in \mathbb{Z}_{10}$.
11. Consider $(\mathbb{Z}_{10}, +_{10})$. Find the inverse of $[7]_{10} \in \mathbb{Z}_{10}$.
12. Find an integer $b$ so that $b \cdot 137 \equiv 1 (mod\ 532)$.
13. Consider $(\mathbb{Z}_{532}, \cdot_{532})$. Find the inverse of $[137]_{532} \in (\mathbb{Z}_{532}, \cdot_{532})$.
14. Consider the group $\mathbb{Z}_{15}^{\times}$
    (a) How many elements does $\mathbb{Z}_{15}^{\times}$ have?
    (b) Write all elements of $\mathbb{Z}_{15}^{\times}$.
    (c) Make a Cayley table for $\mathbb{Z}_{15}^{\times}$.
    (d) Find the inverse of $[2]_{15}$.
    (e) Find the inverse of $[11]_{15}$. Check your answer.
15. How many elements does $\mathbb{Z}_{200}^{\times}$ have?
16. Find 3 different subgroups of $S_3$.
17. Find all elements in $\mathbb{Z}_2 \times \mathbb{Z}_3$.
18. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.
19. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.
20. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$.
21. Prove that $\mathbb{Z}_6$ is not isomorphic to $S_3$.

**Proofs**

22. Let $(X, *)$ be a monoid. Suppose that $e$ and $e'$ are identities. Prove that $e = e'$. Make sure that you only use binary operation $*$, associative law and identity property.
23. Let $(G, *)$ be a group with identity $e$. Let $g \in G$. Prove that $g$ has a unique inverse.
24. Let $(G, \cdot)$ be a group. Suppose $a^2 = e$ for all $a \in G$. Prove that $G$ is abelian.
25. Let $G$ be a group. Prove that $(ab)^{-1} = b^{-1}a^{-1}$.
26. Let $G$ be a group. Let $e$ be the identity. Prove that $\{e\}$ is a subgroup of $G$.
27. Let $G = S_3$. Prove that $H = \{(1), (123), (132)\}$ is a subgroup of $H$.
28. Let $G$ be a group. Let $g \in G$ be an element of order 2. Prove that $a = a^{-1}$.

29. Let $G$ be a group. Let $g \in G$ be an element of order $k$. Prove that $a^{-1} = a^{k-1}$.

30. Prove that the set of permutations $\{(1), (12)(34), (13)(24), (14)(23)\}$ forms a group under multiplication of permutations.

31. Prove that the group $\{(1), (12)(34), (13)(24), (14)(23)\}$ is not cyclic.

32. Prove that the set of permutations $\{(1), (12)(34), (13)(24), (14)(23)\}$ is a subgroup of $S_4$.

33. Prove that $A = \begin{bmatrix} [5]_7 & [2]_7 \\ [3]_7 & [6]_7 \end{bmatrix}$ is in $(Gl_2(\mathbb{Z}_7), \cdot)$.

34. Let $G$ be a group. Let $Z(G) := \{a \in G \mid ag = ga \text{ for all } g \in G\}$. Prove that $Z(G)$ is a subgroup of $G$.

35. Let $\phi : G \to G'$ be an isomorphism. Prove that $G$ is abelian if and only if $G'$ is abelian.

36. Let $\phi : G \to G'$ be an isomorphism. Let $g \in G$. Prove that $|\phi(g)| = |g|$.

37. Let $\phi : G \to G'$ be an isomorphism. Prove that $G$ is cyclic if and only if $G'$ is cyclic.

## True -False - Sometimes

38. True -False - Sometimes

    T F S - Let $G$ be a group. Then $G$ has an identity element.

    T F S - Let $M$ be a monoid. Then $M$ has an identity element.

    T F S - Let $S$ be a semigroup. Then $S$ has an identity element.

    T F S - Every group is a monoid.

    T F S - Every monoid is a group.

    T F S - Let $M$ be a monoid. Then $M$ is a group.

    T F S - $(\mathbb{Z}, +)$ is an abelian group.

    T F S - $(\mathbb{Z}_n, +_n)$ is an abelian group.

    T F S - $(\mathbb{Z}, \cdot)$ is an abelian group.

    T F S - $(\mathbb{Z}_n, \cdot_n)$ is an abelian group.

    T F S - $(\mathbb{Z}_n^\times, \cdot_n)$ is an abelian group.

    T F S - $(\mathbb{Z}_8, \cdot_8)$ is a monoid.

    T F S - Identity element in $(\mathbb{Z}, +)$ is 1.

    T F S - Identity element in $(\mathbb{Z}, \cdot)$ is 1.

    T F S - $(2\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$

    T F S - $[6]_{10}$ has an inverse in $(\mathbb{Z}_{10}, +_{10})$

T F S - $[6]_{10}$ has an inverse in $(\mathbb{Z}_{10}, \cdot_{10})$

T F S - Let $f : X \to Y$ and $g : Y \to Z$ be two bijective functions. Then $g \cdot f$ is bijective.

T F S - Let $f : X \to Y$ be a bijective function. Then $f$ has an inverse.

T F S - All subgroups of $S_3$ are cyclic.

T F S - All proper subgroups of $S_3$ are cyclic.

T F S - All subgroups of $S_4$ are cyclic.

T F S - All proper subgroups of $S_4$ are cyclic.

T F S - $Gl_2(\mathbb{R})$ is a subgroup of the group $M_2(\mathbb{R})$.

T F S - Let $G$ be a group of order $|G| = 5$. Let $g \in G$. Then $|g| = 4$.

T F S - Let $G$ be a group of order $|G| = 15$. Let $g \in G$. Then $|g| = 15$.

T F S - Let $G$ be a group of order $|G| = 15$. Let $g \in G$. Then $|g| = 5$.

T F S - Let $G$ be a group of order $|G| = 15$. Let $g \in G$. Then $|g| = 10$.

T F S - Let $G$ be a group of order $|G| = 15$. Let $H < G$ be a subgroup. Then $|H| = 10$.

T F S - Let $G$ be a group of order $|G| = 15$. Let $H < G$ be a subgroup. Then $|H| = 1$.

T F S - Let $\phi : G \to G'$ be an isomorphism. Then $\phi(e) = e$.

## Examples

39. Give an example of a semigroup which is not a monoid.

40. Give an example of a semigroup which is a monoid.

41. Give an example of a group which is not abelian.

42. Give an example of a semigroup which is a monoid.

43. Give an example of a monoid and an element which has an inverse and an element which does not have an inverse.

44. Consider $(M_2(\mathbb{R}), \cdot)$. Give an example of a matrix which has an inverse and find the inverse.

45. Consider $(M_2(\mathbb{R}), +)$. Give an example of a matrix which has an inverse and find the inverse.

46. Consider $(Gl_2(\mathbb{R}), \cdot)$. Give an example of a matrix which has an inverse and find the inverse.

47. Give an example of a matrix in $(Sl_2(\mathbb{R}), \cdot)$.

48. Give an example of a matrix in $(Gl_2(\mathbb{Z}_5), \cdot)$.

49. Give an example of a matrix in $(Gl_2(\mathbb{Z}_5), \cdot)$ which is not in $(Sl_2(\mathbb{Z}_5), \cdot)$.

50. Give an example of a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ and its inverse.
51. Give an example of a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ which has order 5.
52. Give an example of a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ which has order 12.
53. Give an example of a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ which fixes 3, 4, 5 and 6.
54. Give an example of a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ which does not fix any element of $\{1, 2, 3, 4, 5, 6\}$.
55. Give an example of a function $f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ which is not a permutation.
56. Give an example of a group and a subgroup.
57. Give an example of a group and a subset which is not a subgroup.
58. Give an example of $f : \{1, 2, 3, 4, 5, 6\} \to \mathbb{R}$ which is not a function.
59. Give an example of $f : \mathbb{R} \to \mathbb{R}$ which is not a function.
60. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is not surjective.
61. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is surjective.
62. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is not injective.
63. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is injective.
64. Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is not an isomorphism of groups.
65. Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is an isomorphism of groups.
66. Give an example of a function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ which is not an isomorphism of groups.
67. Give an example of a function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ which is an isomorphism of groups.
68. Give an example of a function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ which is an isomorphism of groups but $f(1) \neq 1$.