Also, do the assigned HW problems. This is just in addition to HW.

**ALWAYS JUSTIFY YOUR ANSWER!**

## Computations

1. Describe all abelian groups of order 243 (up to isomorphism) as direct sums of cyclic groups of prime power order as $\mathbb{Z}_{p^{n_1}} \oplus \mathbb{Z}_{p^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_s}}$.
   Answer: We use $243 = 3^5$:

   - All abelian groups of order $243 = 3^5$ can be described using partitions of 5 which are: $(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)$.
   - Corresponding groups are:
     $H_1 = \mathbb{Z}_{3^5}$, $H_2 = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^1}$, $H_3 = \mathbb{Z}_{3^3} \times \mathbb{Z}_{3^1}$, $H_4 = \mathbb{Z}_{3^3} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1}$, $H_5 = \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^1}$,
     $H_6 = \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1}$, $H_7 = \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1}$.
   - Therefore all abelian groups (up to isomorphism) of order 243 are:
     $H_1 = \mathbb{Z}_{243}$, $H_2 = \mathbb{Z}_{81} \times \mathbb{Z}_3$, $H_3 = \mathbb{Z}_{27} \times \mathbb{Z}_9$, $H_4 = \mathbb{Z}_{27} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $H_5 = \mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_3$,
     $H_6 = \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $H_7 = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

2. Describe all abelian groups of order 360 (up to isomorphism) as direct sum of cyclic groups of prime power order (notice different primes).
   Answer: We use $360 = 2^3 3^2 5^1$:

   - Sylow 2-subgroup $|G_{(2)}| = 2^3 = 8$.
     All abelian groups of order $2^3$ can be described using partitions of 3 which are: $(3), (2,1), (1,1,1)$.
     Corresponding groups are $\mathbb{Z}_{2^3}$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^1}$, $\mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1}$. Therefore all abelian groups (up to isomorphism) of order 8 are:
     $H_1 = \mathbb{Z}_8$, $H_2 = \mathbb{Z}_4 \times \mathbb{Z}_2$, $H_3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
   - Sylow 3-subgroup $|G_{(3)}| = 3^2 = 9$.
     All abelian groups of order $3^2$ can be described using partitions of 2 which are: $(2), (1,1)$.
     Corresponding groups are $\mathbb{Z}_{3^2}$, $\mathbb{Z}_{3^1} \times \mathbb{Z}_{3^1}$. Therefore all abelian groups (up to isomorphism) of order 9 are:
     $K_1 = \mathbb{Z}_9$, $K_2 = \mathbb{Z}_3 \times \mathbb{Z}_3$
   - Sylow 5-subgroup $|G_{(5)}| = 5^1 = 5$.
     All abelian groups of order $5^1$ can be described using partitions of 1 which are: $(1)$.
     Corresponding groups are $\mathbb{Z}_{5^1}$. Therefore all abelian groups (up to isomorphism) of order 5 are:
     $J_1 = \mathbb{Z}_5$.
   - All possible (up to isomorphism) abelian groups of order 360, are products of one of the $H_1, H_2, H_3$ with one of $K_1, K_2$ with $J_1$:
     $H_1 \times K_1 \times J_1 = \mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
     $H_2 \times K_1 \times J_1 = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$

$H_3 \times K_1 \times J_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times Z_9 \times \mathbb{Z}_5$
$H_1 \times K_2 \times J_1 = \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
$H_2 \times K_2 \times J_1 = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
$H_3 \times K_2 \times J_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

3. Describe all abelian groups of order 360 up to isomorphism using cyclic decomposition as: $\mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_t}$ where $m_2|m_1, \quad m_3|m_2, \quad \ldots, m_t|m_{t-1}$.
<u>Answer:</u> By making the tables of powers of primes as I did in class, you can get the above groups isomorphic to the following groups:
$H_1 \times K_1 \times J_1 = \mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \cong \mathbb{Z}_{360}$
$H_2 \times K_1 \times J_1 = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \cong \mathbb{Z}_{180} \times \mathbb{Z}_2$
$H_3 \times K_1 \times J_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times Z_9 \times \mathbb{Z}_5 \cong \mathbb{Z}_{90} \times \mathbb{Z}_2 \times \mathbb{Z}_2$
$H_1 \times K_2 \times J_1 = \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{120} \times \mathbb{Z}_3$
$H_2 \times K_2 \times J_1 = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{60} \times \mathbb{Z}_6$
$H_3 \times K_2 \times J_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{30} \times \mathbb{Z}_6 \times \mathbb{Z}_2$

4. Let $G = \mathbb{Z}_{81} \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_3$.
   (a) Describe all elements of order 81.
   (b) Describe all elements of order 27.
   (c) Describe all elements of order 9.
   (d) Describe all elements of order 3.

5. Let $G = \mathbb{Z}_8 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_5$.
<u>Answer:</u> We will use the following facts:
All elements in $G$ are of the form $(a, b, c)$ with $a \in \mathbb{Z}_8, \; b \in \mathbb{Z}_{27}, \; c \in \mathbb{Z}_5$.
The orders are given by: $|(a, b, c)| = lcm(|a|, |b|, |c|)$.
   (a) Describe all elements of order 8.
   <u>Answer:</u> $|(a,b,c)| = lcm(|a|,|b|,|c|) = 8 \implies \quad |a| = 8, \; |b| = 1, \; |c| = 1 \implies$
   $a \in \{1,3,5,7\}, \quad b = 0, \quad c = 0 \; \boxed{\{(1,0,0), \; (3,0,0), \; (5,0,0), \; (7,0,0)\}}$
   (b) Describe all elements of order 24.
   <u>Answer:</u> $|(a,b,c)| = lcm(|a|,|b|,|c|) = 24 \implies \quad |a| = 8, \; |b| = 3, \; |c| = 1 \implies$
   $a \in \{1,3,5,7\}, \quad b \in \{9,18\}, \quad c = 0$
   $\boxed{\{(1,9,0), \; (3,9,0), \; (5,9,0), \; (7,9,0)\}, \{(1,18,0), \; (3,18,0), \; (5,18,0), \; (7,18,0)\}}$
   (c) Describe all elements of order 5.
   <u>Answer:</u> $|(a,b,c)| = lcm(|a|,|b|,|c|) = 5 \implies \quad |a| = 1, \; |b| = 1, \; |c| = 5 \implies$
   $a = 0, \quad b = 0, \quad c \in \{1,2,3,4\} \; \boxed{\{(0,0,1), \; (0,0,2), \; (0,0,3), \; (0,0,4)\}}$
   (d) Describe all elements of order 120.
   <u>Answer:</u> $|(a,b,c)| = lcm(|a|,|b|,|c|) = 120 \implies \quad |a| = 8, \; |b| = 3, \; |c| = 5 \implies$
   $a \in \{1,3,5,7\}, \quad b \in \{9,18\}, \quad c \in \{1,2,3,4\}$

$\boxed{\{(1,9,1),\ (3,9,1),\ (5,9,1),\ (7,9,1)\}, \{(1,18,1),\ (3,18,1),\ (5,18,1),\ (7,18,1)\}}$
$\boxed{\{(1,9,2),\ (3,9,2),\ (5,9,2),\ (7,9,2)\}, \{(1,18,2),\ (3,18,2),\ (5,18,2),\ (7,18,2)\}}$
$\boxed{\{(1,9,3),\ (3,9,3),\ (5,9,3),\ (7,9,3)\}, \{(1,18,3),\ (3,18,3),\ (5,18,3),\ (7,18,3)\}}$
$\boxed{\{(1,9,4),\ (3,9,4),\ (5,9,4),\ (7,9,4)\}, \{(1,18,4),\ (3,18,4),\ (5,18,4),\ (7,18,4)\}}$

(e) Describe all elements of order 1.
Answer: $|(a,b,c)| = lcm(|a|,|b|,|c|) = 1 \implies \quad |a| = 1,\ |b| = 1,\ |c| = 1 \implies$
$a = 0, \quad b = 0, \quad c = 0$ $\boxed{\{(0,0,0)\}}$

6. Let $G = \mathbb{D}_6 = \langle s, r \mid |s| = 2, |r| = 6, srs = r^5 = r^{-1}\rangle$.
Answer: We will use the following description of elements of $G$:
$G = \mathbb{D}_6 = \{e, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}$.

(a) Find the possible numbers $n_p$ of Sylow $p$-subgroups for each prime $p||G|$.
Answer: The order of $G$ is $|G| = 2 \cdot 6 = 2^2 \cdot 3$. So the primes are $p = 2$ and $p = 3$.
From Sylow theorems: (1) $n_p||G|$ and (2) $n_p \equiv 1(mod\ p)$.
$p = 2$
(1) $n_p||G| \implies n_2|12 \implies n_2 \in \{1,2,3,4,6,12\}$
(2) $n_p \equiv 1(mod\ p) \implies n_2 \equiv 1(mod\ 2) \implies n_2 \in \{1,3,5,7,9,11\}$.
$\therefore n_2 \in \{1,3\}$
$p = 3$
(1) $n_p||G| \implies n_3|12 \implies n_3 \in \{1,2,3,4,6,12\}$
(2) $n_p \equiv 1(mod\ p) \implies n_3 \equiv 1(mod\ 3) \implies n_3 \in \{1,4,7,10,\}$.
$\therefore n_3 \in \{1,4\}$.

(b) For each prime $p||G|$ describe all Sylow $p$-subgroups?
Answer:
Orders of elements of $G = D_6$:
$o(e) = 1, o(r) = 6, o(r^2) = 3, o(r^3) = 2, o(r^4) = 3, o(r^5) = 6,$
$o(s) = 2, o(sr) = 2, o(sr^2) = 2, o(sr^3) = 2, o(sr^4) = 2, o(sr^5) = 2$
$p = 2$
Since $|G| = 2^2 \cdot 3$, the order of Sylow 2-subgroups is equal to $2^2$, i.e. $|P_{(2)}| = 4$.
Orders of elements in Sylow 2-subgroups must be powers of 2 and divide 4.
Therefore: 1,2 or 4.
$P_{(2),1} = \langle r^3, s\rangle = \{e, r^3, s, sr^3\}$(you have to check that it is a subgroup)
$P_{(2),2} = \langle r^3, sr\rangle = \{e, r^3, sr, sr^4\}$ (you have to check that it is a subgroup)
$P_{(2),3} = \langle r^3, sr^2\rangle = \{e, r^3, sr^2, sr^5\}$ (you have to check that it is a subgroup)
$p = 3$
Since $|G| = 2^2 \cdot 3$, the order of any Sylow 3-subgroup is equal to 3, i.e. $|P_{(3)}| = 3$.
Orders of elements in Sylow 3-subgroups must be powers of 3 and divide 3.
Therefore: 1 or 3.
$P_{(3)} = \langle r^2\rangle = \{e, r^2, r^4\}$(you have to check that it is a subgroup)
There are no more elements of order 3, therefore this is the only Sylow 3-subgroup.

(c) For each prime $p||G|$ show explicitly how all Sylow $p$-subgroups are conjugate?
$p = 2$
The three Sylow 2-subgroups $\{P_{(2),1}, P_{(2),2}, P_{(2),3}\}$ are conjugate:
$P_{(2),1} = \{e, r^3, s, sr^3\}$
$rP_{(2),1}r^{-1} = \{rer^{-1}, rr^3r^{-1}, rsr^{-1}, rsr^3r^{-1}\} = \{e, r^3, sr^5r^{-1}, sr^5r^3r^{-1}\}$
$rP_{(2),1}r^{-1} = \{e, r^3, sr^4, sr\} = P_{(2),2}$
$r^2P_{(2),1}r^{-2} = \{r^2er^{-2}, r^2r^3r^{-2}, r^2sr^{-2}, r^2sr^3r^{-2}\} = \{e, r^3, sr^5r^5r^{-2}, sr^5r^5r^3r^{-2}\}$
$r^2P_{(2),1}r^{-2} = \{e, r^3, sr^2, sr^5\} = P_{(2),3}$
$\{P_{(2),1}, P_{(2),2}, P_{(2),3}\} = \{P_{(2),1}, rP_{(2),1}r^{-1}, r^2P_{(2),1}r^{-2}\}$
$p = 3$
There is only one Sylow 3-subgroup $\{P_{(3)}\}$, so $gP_{(3)}g^{-1} = P_{(3)}$ for all $g \in G$.

7. Let $G = \mathbb{D}_5 = \langle s, r \mid |s| = 2, |r| = 5, srs = r^4 = r^{-1}\rangle$.
Answer: We will use the following description of elements of $G$:
$G = \mathbb{D}_5 = \{e, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\}$.

(a) Find the possible numbers $n_p$ of Sylow $p$-subgroups for each prime $p||G|$.
Answer: The order of $G$ is $|G| = 2 \cdot 5$. So the primes are $p = 2$ and $p = 5$.
From Sylow theorems: (1) $n_p||G|$ and (2) $n_p \equiv 1(mod\ p)$.
$p = 2$
(1) $n_p||G| \implies n_2|10 \implies n_2 \in \{1, 2, 5, 10\}$
(2) $n_p \equiv 1(mod\ p) \implies n_2 \equiv 1(mod\ 2) \implies n_2 \in \{1, 3, 5, 7, 9\}$.
$\therefore n_2 \in \{1, 5\}$
$p = 5$
(1) $n_p||G| \implies n_5|10 \implies n_5 \in \{1, 2, 5, 10\}$
(2) $n_p \equiv 1(mod\ p) \implies n_5 \equiv 1(mod\ 5) \implies n_5 \in \{1, 6\}$.
$\therefore n_5 \in \{1\}$, i.e. $n_5 = 1$.

(b) For each prime $p||G|$ describe all Sylow $p$-subgroups?
Answer:
$p = 2$
Since $|G| = 2 \cdot 5$, the order of Sylow 2-subgroups is equal to 2, i.e. $|P_{(2)}| = 2$.
$P_{(2),1} = \langle s\rangle = \{e, s\}$
$P_{(2),2} = \langle sr\rangle = \{e, sr\}$ (you have to check that $(sr)(sr) = e$
$P_{(2),3} = \langle sr^2\rangle = \{e, sr^2\}$ (you have to check that $(sr^2)(sr^2) = e$
$P_{(2),4} = \langle sr^3\rangle = \{e, sr^3\}$ (you have to check that $(sr^3)(sr^3) = e$
$P_{(2),5} = \langle sr^4\rangle = \{e, sr^4\}$ (you have to check that $(sr^4)(sr^4) = e$
Notice that this agrees nicely with $n_2 = 5$, so there are 5 Sylow 2-subgroups.
$p = 5$
Since $|G| = 2 \cdot 5$, the order of Sylow 5-subgroups is equal to 5, i.e. $|P_{(5)}| = 5$.
$P_{(5)} = \langle r\rangle = \{e, r, r^2, r^3, r^4\}$
This is the only Sylow 5-subgroup which agrees with $n_5 = 1$.

(c) For each prime $p||G|$ show explicitly how all Sylow $p$-subgroups are conjugate?
$p = 2$

It will be used $rs = sr^4$ which follows from $srs = r^4$.
Consider $P_{(2),1} = \langle s \rangle = \{e, s\}$. For simplicity denote it by $P$.
$rPr^{-1} = r\langle s \rangle r^{-1} = r\{e, s\}r^{-1} = \{rer^{-1}, rsr^{-1}\} = \{e, sr^3\} = P_{(2),4}$
$r^2P(r^2)^{-1} = r^2\langle s \rangle r^{-2} = r^2\{e, s\}r^{-2} = \{r^2er^{-2}, r^2sr^{-2}\} = \{e, sr\} = P_{(2),2}$
$r^3P(r^3)^{-1} = r^3\langle s \rangle r^{-3} = r^3\{e, s\}r^{-3} = \{r^3er^{-3}, r^3sr^{-3}\} = \{e, sr^4\} = P_{(2),5}$
$r^4P(r^4)^{-1} = r^4\langle s \rangle r^{-4} = r^4\{e, s\}r^{-4} = \{r^4er^{-4}, r^4sr^{-4}\} = \{e, sr^2\} = P_{(2),3}$
$\underline{p = 5}$
$P_{(5)} = \langle r \rangle = \{e, r, r^2, r^3, r^4\}$ is the only Sylow 5-subgroup.
Every conjugate of Sylow 5-subgroup is again Sylow 5-subgroup by problem #15 (notice I added a few problems).
So all conjugates of $P_{(5)}$ are equal to $P_{(5)}$, i.e. $gP_{(5)}g^{-1} = P_{(5)}$ for all $g \in G$.

8. Let $G = S_4$.
   (a) Find the possible numbers $n_p$ of Sylow $p$-subgroups for each prime $p||G|$.
   (b) For each prime $p||G|$ describe all Sylow $p$-subgroups?
   (c) For each prime $p||G|$ show explicitly how all Sylow $p$-subgroups are conjugate?

9. Let $G = \mathbb{Z}_{15}^{\times}$.
   (a) Find the possible numbers $n_p$ of Sylow $p$-subgroups for each prime $p||G|$.
   (b) For each prime $p||G|$ describe all Sylow $p$-subgroups?
   (c) For each prime $p||G|$ show explicitly how all Sylow $p$-subgroups are conjugate?

10. Let $G = \mathbb{Z}_{36}^{\times}$.
    Answer: First notice that the elements of $G = \mathbb{Z}_{36}^{\times}$ are the integers between 1 and 35 which are relatively prime to 36, i.e. $G = \mathbb{Z}_{36}^{\times} = \{i \mid 1 \leq i \leq 35,\ gcd(i, 36) = 1\} = \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35\}$.
    Check, there should be $\varphi(36) = \varphi(2^2 3^2) = \varphi(2^2)\varphi(3^2) = (2-1)2(3-1)3 = 12$ elements.
    (a) Find the possible numbers $n_p$ of Sylow $p$-subgroups for each prime $p||G|$.
    $|G| = 12 = 2^2 3$. So, one should consider primes $p = 2$ and $p = 3$.
    - One can compute $n_2$ and $n_3$ using Sylow theorems: (1) $n_p||G|$ and (2) $n_p \equiv 1(mod\ p)$.
    - Another way: since $G$ is abelian, all subgroups are normal.
    - One more Sylow theorem - For each prime $p$, all Sylow p-subgroups are conjugate.
    - From the last two statements it follows that for each prime $p = 2$ and $p = 3$ there is exactly one Sylow p-subgroup.
    - $n_2 = 1$, $n_3 = 1$.
    (b) For each prime $p||G|$ describe all Sylow $p$-subgroups?
    Answer: In order to find Sylow subgroups, we need to compute orders of elements:
    If $a \in G$ then $|a|||G|$. Therefore, possible orders of elements in $G = \mathbb{Z}_{36}^{\times}$ must divide 12.
    Possible orders: 1,2,3,4,6,12.
    Sylow 2-subgroup can have only elements of order 1,2,4. $|P_2| = 4$

Sylow 3-subgroup can have only elements of order 1,3. $|P_3| = 3$
$(5, 5^2 = 25, 5^3 = 17, 5^4 = 13, 5^5 = 29, 5^6 = 1)$ implies $|5| = 6$, $|5^2| = 3$, $|5^3| = 2$
Sylow 3-subgroup $P_3 = \langle 25 \rangle = \{13, 25, 1\}$ since $25^3 = 1$
Sylow 2-subgroup $P_2 = \{17, 19, 35, 1\}$

(c) For each prime $p||G|$ show explicitly how all Sylow $p$-subgroups are conjugate?
Answer:
There is only one Sylow 2-subgroup $P_2$. Therefore all conjugates of $P_2$ are equal to $P_2$, i.e. $gP_2g^{-1} = P_2$ for all $g \in G = \mathbb{Z}_{36}^{\times}$.
There is only one Sylow 3-subgroup $P_3$. Therefore all conjugates of $P_3$ are equal to $P_3$, i.e. $gP_3g^{-1} = P_3$ for all $g \in G = \mathbb{Z}_{36}^{\times}$.

## Theoretic Questions

11. Write the definition of *p-subgroup*.

12. Write the definition of *Sylow p-subgroup*.

## Proofs

13. Let $H$ be a subgroup of a group $G$. Let $g \in G$. Prove that $gHg^{-1}$ is a subgroup of $G$.
Proof:

- Claim 1: $aHa^{-1} \subset G$, i.e. is a subset of $G$.
Proof of Claim 1: If $x \in aHa^{-1}$, then $x = aha^{-1}$ for some $h \in H$.
Since $H \subset G$, then $h \in G$. Since $G$ is a group it is closed under inverses and multiplications. Therefore $aha^{-1} \in G$.Therefore $x \in G$. Therefore $aHa^{-1} \subset G$.
- Claim 2: $aHa^{-1} \neq \emptyset$, i.e. is a nonempty set.
Proof of Claim 2: Since $H$ is a subgroup of $G$, the identity of $G$ is in $H$, i.e. $e \in H$. Therefore $aea^{-1} \in aHa^{-1}$. So $e = aea^{-1}$ using inverse and identity properties in a group. Therefore $e \in aHa^{-1}$. Therefore $aHa^{-1} \neq \emptyset$.
- Claim 3: If $x, y \in aHa^{-1}$, then $xy \in aHa^{-1}$.
Proof of Claim 3: Let $x, y \in aHa^{-1}$.
Then, there exist $h_1, h_2 \in H$ such that $x = ah_1a^{-1}$ and $y = ah_2a^{-1}$, by defn of $aHa^{-1}$.
Therefore $xy = (ah_1a^{-1})(ah_2a^{-1}) = ah_1eh_2a^{-1} = ah_1h_2a^{-1} = aha^{-1}$ where $h = h_1h_2$.
Then $h \in H$ ($H$ is closed under operation since it is a subgroup).
Therefore $xy \in aHa^{-1}$.
- Claim 4: If $x \in aHa^{-1}$, then $x^{-1} \in aHa^{-1}$.
Proof of Claim 4: Let $x \in aHa^{-1}$.
Then there exist an $h \in H$ such that $x = aha^{-1}$.
$x^{-1} = (aha^{-1})^{-1} = (a^{-1})^{-1}h^{-1}a^{-1} = ah^{-1}a^{-1}$.
Since $H$ is a subgroup, it is closed under inverses. Therefore $h^{-1} \in H$. So $x^{-1} \in aHa^{-1}$.

- Conclusion: $aHa^{-1}$ is a subgroup in $G$. This follows by the Theorem on subgroups: A nonempty subset of a group is a subgroup if it is closed under group operation and inverses.

14. Let $H$ be a subgroup of a group $G$. Let $g \in G$. Prove that the conjugate subgroup $gHg^{-1}$ is isomorphic to $H$. Proof:

- Define $f : H \to gHg^{-1}$ by $f(x) := gxg^{-1}$.
- Claim 1: $f$ is a group homomorphism.
Proof of Claim 1: Have to show $f(xy) = f(x)f(y)$ for all $x, y \in H$.
$f(xy) \overset{\text{(def. of f)}}{=} g(xy)g^{-1} \overset{\text{(identity e)}}{=} gxeyg^{-1} \overset{\text{(inverse)}}{=} gx(g^{-1}g)yg^{-1} \overset{\text{(associative)}}{=} (gxg^{-1})(gyg^{-1}) \overset{\text{(def. of f)}}{=}$ $f(x)f(y)$ Therefore $f(xy) = f(x)f(y)$ for all $x, y \in H$. Therefore $f$ is a group homomorphism.
- Claim 2: $f$ is injective (one-to-one).
Proof of Claim 2: Have to show: If $f(x_1) = f(x_2)$ then $x_1 = x_2$.
Suppose $f(x_1) = f(x_2)$. Then $gx_1g^{-1} = gx_2g^{-1}$.
Use cancellation law on the left for $g$ and get $x_1g^{-1} = x_2g^{-1}$
Use cancellation law on the right side for $g^{-1}$ and get $x_1 = x_2$
Therefore $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Therefore $f$ is injective by definition.
- Claim 3: $f$ is surjective (onto).
Proof of Claim 3: Have to show: given $y \in gHg^{-1}$ there is $x \in H$ such that $f(x) = y$.
$y \in gHg^{-1}$ implies there is an $x \in H$ such that $y = gxg^{-1}$. Therefore $f(x) = gx.g^{-1} = y$.
Therefore $f$ is onto, i.e. surjective.
- Therefore $f : H \to gHg^{-1}$ is a group homomorphism which is bijection, hence it is an isomorphism by definition of isomorphism.

15. Let $P$ be a Sylow p-subgroup of a finite group $G$. Let $g \in G$. Prove that the conjugate subgroup $gPg^{-1}$ is a Sylow p-subgroup of $G$.
Proof:

- Let $|G| = p^k m$, with $p$ prime, $k \geq 1$, $gcd(p, m) = 1$.
- Sylow $p$-subgroups are subgroups $P$ such that $|P| = p^k$.
- Let $P$ be a Sylow $p$-subgroup of $G$. Let $gPg^{-1}$ be a conjugate of $P$.
- Then $gPg^{-1} \cong P$ by Problem #14. Therefore
- $|gPg^{-1}| = |P| = p^k$. Therefore $gPg^{-1}$ is Sylow $p$-subgroup.

16. Let $H$ be a subgroup of a group $G$. Let $conj.cl(H)$ be the conjugacy class of $H$. Prove that $H$ is a normal subgroup of $G$ if and only if $|conj.cl(H)| = 1$.
Proof:

- Recall: A subgroup $H$ of $G$ is normal subgroup of $G$ if $gHg^{-1} = H$ for all $g \in G$.
- Recal definition: $conj.cl(H) = \{gHg^{-1} \mid g \in G\}$
- Proof of ($\Rightarrow$) Suppose $H$ is normal subgroup in $G$. Then $gHg^{-1} = H$ for all $g \in G$. Therefore $conj.cl(H) = \{H\}$. Therefore $|conj.cl(H)| = 1$.
- Proof of ($\Leftarrow$) Suppose $|conj.cl(H)| = 1$. Then $conj.cl(H) = \{H\}$ since $H \in conj.cl(H)$. Therefore $gHg^{-1} = H$ for all $g \in G$. Therefore $H$ is normal subgroup in $G$.

17. Let $G$ be a group. Let $G$ act on $X = \{subgroups\ of\ G\}$ by conjugation. Let $H$ be a subgroup of $G$. Prove that $H$ is a normal subgroup of $G$ if and only if $o(H)$, the orbit of $H$, has only one point.
Proof:

- Use the fact that the orbit of a subgroup $H$ under conjugation action is the same as the conjugacy class of $H$.
- $o(H) = conj.cl(H)$
- Apply Problem #16.

18. Let $n_p = \#\{\text{Sylow } p\text{-subgroups of } G\}$. Suppose $n_p = 1$. Let $P$ be a $p$-Sylow subgroup of $G$. prove that $P$ is a normal subgroup.
Proof:

- Suppose $n_p = 1$. Then
- $\#\{\text{Sylow p-subgroups}\} = 1$.
- Let $P$ be a Sylow p-subgroup. Then all conjugates of $P$ are also Sylow p-subgroups by problem #15.
- $\{gPg^{-1} \mid g \in G\} = \{\text{conjugates of } P\} \subseteq \{\text{ Sylow p-subgroups}\} = \{P\}$ since there is only one Sylow p-subgroup.
- $gPg^{-1} = P$ for all $g \in G$. Therefore $P$ is normal subgroup in $G$.

19. Let $G$ be a group. Suppose $|G| = p^k m$ where $p$ is prime, $k \geq 1$, $m > 1$, $gcd(p, m) = 1$. Suppose $n_p = 1$. Prove that $G$ has a normal subgroup.
Proof:

- Suppose $n_p = 1$. Therefore
- $\#\{\text{Sylow p-subgroups}\} = 1$.
- Let $P$ be a Sylow p-subgroup. Then all conjugates of $P$ are also Sylow p-subgroups by problem #15.
- $\{gPg^{-1} \mid g \in G\} = \{\text{conjugates of } P\} \subseteq \{\text{ Sylow p-subgroups}\} = \{P\}$ since there is only one Sylow p-subgroup.
- $gPg^{-1} = P$ for all $g \in G$. Therefore $P$ is normal subgroup in $G$.

20. Let $G$ be a group of order $|G| = 33$. Prove that $G$ has a normal subgroup.
    Proof:

21. Let $G$ be a group of order $|G| = 21$. Prove that $G$ has a normal subgroup.

    - $|G| = 21 = 3 \cdot 7$. So we consider primes $p = 3$ and $p = 7$.
    - $p = 3$. Then:
      (1) $n_p \| G| \implies n_3 | 21 \implies n_3 \in \{1, 3, 7, 21\}$
      (2) $n_p \equiv 1 (mod\ p) \implies n_3 \equiv 1 (mod\ 3) \implies n_3 \in \{1, 7, 10, 13\}$.
      Therefore $n_3$ might be 1 or 7.
    - $p = 7$. Then:
      (1) $n_p \| G| \implies n_7 | 21 \implies n_7 \in \{1, 3, 7, 21\}$
      (2) $n_p \equiv 1 (mod\ p) \implies n_7 \equiv 1 (mod\ 7) \implies n_7 \in \{1, 8\}$.
      Therefore $n_7 = 1$.
    - Since $n_7 = 1$ it follows by Problem #18 that Sylow 7-subgroup is normal subgroup in $G$.
    - Notice that we could not conclude that $n_3 = 1$ and anything about Sylow 3-subgroup, but we could conclude that Sylow 7-subgroup is normal in $G$.

22. Let $G$ be a group of order $|G| = 2p$ where $p \neq 2$ is prime. Prove that $G$ is not simple.
    Proof:

    - Recall, group $G$ is simple if it does not have any proper normal subgroups $H$, i.e. if it does not have normal subgroup $H$, so that $\{e\} \subsetneq H \subsetneq G$.
    - $p$ is prime. Then:
      (1) $n_p \| G| \implies n_p | 2p \implies n_p \in \{1, 2, p, 2p\}$
      (2) $n_p \equiv 1 (mod\ p) \implies n_p \in \{1, p+1\}$.
      Notice since $p$ is prime and $p \neq 2$ it follows that $p + 1 \neq 2$, $p + 1 \neq p$, $p + 1 \neq 2p$
      (If $p + 1 = 2p$ then $1 = p$ which gives a contradiction that $p$ is prime.)
    - Conclusion: $n_p = 1$.
    - The Sylow $p$-subgroup $P$ is normal in $G$ by Problem #18.
    - $1 < p < 2p$ implies $|\{e\}| < |P| < |G|$ which implies $\{e\} \subsetneq P \subsetneq G$
    - The Sylow subgroup $P$ is proper normal subgroup of $G$. Therefore $G$ is not simple.

23. Let $G$ be a group of order $|G| = 56$. Prove that $G$ is not simple.

24. Let $G$ be a group of order $|G| = 125$. Prove that the center $Z(G)$ of $G$ has at least 5 elements.
    Proof: Use the same proof as done in class for $|G| = p^k$ since $|G| = 5^3$ and prove that $|Z(G)| \geq 5$ and therefore $Z(G)$ has at least 5 elements.

25. Let $G$ be a group of order $|G| = 125$. Prove that $G$ is not simple.
    Proof:

- The center $Z(G)$ of $G$ is a subgroup (proved several weeks ago).
- The center $Z(G)$ of $G$ is a normal subgroup (proved several weeks ago).
- The center $Z(G)$ of $G$ has at least 5 elements.
- The center $Z(G)$ of $G$ is a proper normal subgroup.
- The group $G$ is not simple.

26. Prove that abelian group of order 55 must be cyclic.

27. Prove that $\mathbb{Z}_8 \times \mathbb{Z}_5 \cong \mathbb{Z}_{40}$.

28. Prove that every group of order 5 is cyclic.

29. Prove that every group of prime order is cyclic.

30. Prove that every group of prime order $p$ is isomorphic to $\mathbb{Z}_p$.

31. Prove that every group of order 4 is either cyclic or isomorphic to Klein Four Group.
    Proof: Done in class.

32. Prove that every group of order 4 is isomorphic either to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

## True -False - Sometimes

33. True -False - Sometimes

    T F $\boxed{S}$ Let $P_p$ be a Sylow $p$-subgroup of $G$. Then $P_p$ is normal subgroup.

    $\boxed{T}$ F S Let $G = (\mathbb{Z}_n, +_n)$, let $P_p$ be a Sylow $p$-subgroup of $G$. Then $P_p$ is normal subgroup.

    $\boxed{T}$ F S Let $G$ be a group with $|G| = 150$. Let $P_2$ be a Sylow 2-subgroup of $G$. Then $|P_2| = 2$.

    T $\boxed{F}$ S Let $G$ be a group with $|G| = 150$. Let $P_5$ be a Sylow 5-subgroup of $G$. Then $|P_5| = 5$.

    $\boxed{T}$ F S Let $G$ be a group, $|G| = 150$. Let $P_5$ be a Sylow 5-subgroup of $G$. Then $|P_5| = 25$.

    $\boxed{T}$ F S $\mathbb{Z}_4 \times \mathbb{Z}_5 \cong \mathbb{Z}_{20}$.

    T $\boxed{F}$ S $\mathbb{Z}_4 \times \mathbb{Z}_2 \cong \mathbb{Z}_8$.

    T $\boxed{F}$ S $\mathbb{Z}_4 \times \mathbb{Z}_4 \cong \mathbb{Z}_4$.

    T $\boxed{F}$ S $\mathbb{Z}_4 \times \mathbb{Z}_4 \cong \mathbb{Z}_{16}$.

    T $\boxed{F}$ S $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_4$.

    $\boxed{T}$ F S $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong K$, the Klein Four Group.

## Examples

34. Give an example of a group $G$ and a $p$-subgroup of $G$ which is not Sylow $p$-subgroup of $G$.

35. Give an example of a group $G$ and a $p$-subgroup of $G$ which is Sylow $p$-subgroup of $G$.

36. Give an example of a group $G$ and a subgroup of $G$ which is not a $p$-subgroup of $G$.

37. Consider the Klein Four Group:

$$K = \{e, a, b, c \mid a^2 = b^2 = c^2 = e, ab = ba = c, bc = cb = a, ac = ca = b\}.$$

(a) Make the Cayley table for $K$.

(b) Make the Cayley table for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(c) Write an explicit isomorphism $f : K \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$.