Also, do the assigned HW problems. This is just in addition to HW.

ALWAYS JUSTIFY YOUR ANSWER!

Computations

1. Let $G = S_6$. Let $\alpha = (12)(46)$.

   (a) Find a conjugate of α.
   Answer: Any permutation in S_6 which is a product of two disjoint 2-cycles is conjugate to α.
   $\boxed{\beta = (12)(45),\ \gamma = (26)(31) = (26)(13) = (13)(26),\ \delta = (34)(16), etc.}$

   (b) How many elements are there in the conjugacy class of α?
   Answer: $\boxed{|\{conj.class(\alpha)\}| = \binom{6}{2}\frac{2!}{2}\binom{4}{2}\frac{2!}{2}\frac{1}{2!} = 45}$

2. Let $G = \mathbb{Z}_6$. Let $x = 2$.

   (a) Find a conjugate of x.
   Answer: First notice operation is $+_6$ or $+$ for short.
   Some conjugates of $x = 2$ are: $5 + 2 + (-5) = 2$, $3 + 2 + (-3) = 2$, $1 + 2 + (-1) = 2$.
   Since $G = \mathbb{Z}_6$ is abelian, all conjugates of x are equal to x.

   (b) How many elements are there in the conjugacy class of x?
   Answer: Since $G = \mathbb{Z}_6$ is abelian, all conjugates of x are equal to x.
   $\boxed{|\{conj.class(x)\}| = |\{x\}| = 1}$

3. Let $G = \mathbb{Z}_6^\times$. Let $x = 5$.

   (a) Find a conjugate of x.
   Answer: First notice operation is $\cdot_6$ or $\cdot$ for short.
   Some conjugates of $x = 5$ are: $5 \cdot 5 \cdot \cdot 5^{-1} = 5,\ \ 3 \cdot 5 \cdot 3^{-1} = 5,\ \ 1 \cdot 5 \cdot 1^{-1}) = 5$.
   Since $G = \mathbb{Z}_6^\times$ is abelian, all conjugates of x are equal to x.

   (b) How many elements are there in the conjugacy class of x?
   $\boxed{|\{conj.class(x)\}| = |\{x\}| = 1}$

4. Let $G = \mathbb{D}_6 = \langle s, r \mid |s| = 2, |r| = 6, srs = r^5\rangle$.

   (a) Find a conjugate of s.
   Answer: First notice that the elements of G are $G = \mathbb{D}_6 = \{e, r, r^2, r^3, r^4, r^5, s, sr, r^2, sr^3, sr^4, sr^5\}$.
   Fact 1: $r^{-1} = r^5$
   Reason: $rr^5 = r^6 = e$, so by definition and uniquness of inverses it follows that the inverse of r is r^5, i.e. $r^{-1} = r^5$.

Fact 2: $rs = sr^5$
Reason: $(srs = r^5) \Rightarrow (ssrs = sr^5) \Rightarrow (s^2rs = sr^5) \Rightarrow (ers = sr^5) \Rightarrow (rs = sr^5)$
Some conjugates of s: $sss^{-1} = s$
$rsr^{-1} \overset{\text{Fact1}}{=} rsr^5 \overset{\text{Fact2}}{=} sr^5r^5 = sr^{10} = sr^4$
$r^2sr^{-2} \overset{\text{Fact1}}{=} rrsr^4 \overset{\text{Fact2}}{=} rsr^5r^4 = sr^5r^5r^4 = sr^2$
$r^3sr^{-3} \overset{\text{Fact1}}{=} rrrsr^3 \overset{\text{Fact2}}{=} rrsr^5r^3 = sr^5r^5r^5r^3 = s$
$r^4sr^{-4} = rsr^{-1} = sr^4$
$r^5sr^{-5} = r^2sr^{-2} = sr^2$
$(sr)s(sr)^{-1} = srsr^{-1}s^{-1} = ssr^4s^{-1} = r^4s = sr^5r^5r^5r^5 = sr^2$
$(sr^2)s(sr^2)^{-1} = sr^2sr^{-2}s^{-1} = ssr^2s = sr^5r^5 = sr^4$
...

(b) How many elements are there in the conjugacy class of s?
Answer: $\boxed{|conj.class(s)| = |\{s, sr^2, sr^4\}| = 3}$

5. Let $G = \mathbb{D}_6 = \langle s, r \mid |s| = 2, |r| = 6, srs = r^5 \rangle$.

   (a) Find a conjugate of r.

   (b) How many elements are there in the conjugacy class of r?

6. Let $G = \mathbb{D}_6 = \langle s, r \mid |s| = 2, |r| = 6, srs = r^5 \rangle$.

   (a) Find a conjugate of sr.

   (b) How many elements are there in the conjugacy class of sr?

7. Let $G = M_2(\mathbb{R})$ be the group of 2×2 matrices with entries in $\mathbb{R}$.
   Find the conjugate of matrix $X = \begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix}$ by $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$.

8. Let $G = Gl_2(\mathbb{R})$ be the group of 2×2 (multiplicatively) invertible matrices with entries in $\mathbb{R}$.
   Find the conjugate of matrix $X = \begin{bmatrix} 1 & -2 \\ 3 & 6 \end{bmatrix}$ by $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$.

9. Let $G = S_4$.

   (a) Find the subgroup $H = \langle (1342) \rangle$.

   (b) Find the conjugate of H by $\beta = (14)$.

10. Let $G = S_4$.

    (a) Find the subgroup A_4 of even permutations.

    (b) Find the conjugate of A_4 by $\beta = (14)$ in G.

11. Consider the group $\mathbb{Z}_{15}$.

(a) Find the subgroup $H = \langle 3 \rangle$

(b) Find the conjugate of H by 4.

12. Consider the group $\mathbb{Z}_{15}^{\times}$.

(a) Find the subgroup $H = \langle 2 \rangle$

(b) Find the conjugate of H by 4.

13. Consider the group $G = \mathbb{D}_6 = \langle s, r \mid |s| = 2, |r| = 6, srs = r^5 \rangle$. Answer: First notice that the elements of G are $G = \mathbb{D}_6 = \{e, r, r^2, r^3, r^4, r^5, s, sr, r^2, sr^3, sr^4, sr^5\}$.

Fact 1: $r^{-1} = r^5$

Reason: $rr^5 = r^6 = e$, so by definition and uniquness of inverses it follows that the inverse of r is r^5, i.e. $r^{-1} = r^5$.

Fact 2: $rs = sr^5$

Reason: $(srs = r^5) \Rightarrow (ssrs = sr^5) \Rightarrow (s^2rs = sr^5) \Rightarrow (ers = sr^5) \Rightarrow (rs = sr^5)$

(a) Find the subgroup $H = \langle s \rangle$

Answer: $H = \langle s \rangle = \{s, s^2 = e\}$, $\boxed{H = \langle s \rangle = \{s, e\}}$

(b) Find the conjugate of H by the identity e

Answer: $eHe^{-1} = eHe = \{ese, eee\} = \{s, e\} = H$, $\boxed{eHe^{-1} = \{s, e\}}$

(c) Find the conjugate of H by s

Answer: $sHs^{-1} = sHs = \{sss, ses\} = \{s, e\} = H$, $\boxed{sHs^{-1} = \{s, e\}}$

(d) Find the conjugate of H by r

Answer: $rHr^{-1} = rHr^5 = \{rsr^5, rer^5\} = \{sr^4, e\} \neq H$, $\boxed{rHr^{-1} = \{sr^4, e\}}$

Use: $rsr^{-1} \overset{\text{Fact1}}{=} rsr^5 \overset{\text{Fact2}}{=} sr^5r^5 = sr^{10} = sr^4$

(e) Find the conjugate of H by r^2

Answer: $r^2Hr^{-2} = r^2Hr^4 = \{r^2sr^4, r^2er^4\} = \{sr^2, e\} \neq H$, $\boxed{r^2Hr^{-2} = \{sr^2, e\}}$

(f) Find the conjugate of H by r^3

Answer: $r^3Hr^{-3} = r^3Hr^3 = \{r^3sr^3, r^3er^3\} = \{s, e\} = H$, $\boxed{r^3Hr^{-3} = \{s, e\}}$

(g) Find the conjugate of H by r^4

Answer: $r^4Hr^{-4} = r^4Hr^2 = \{r^4sr^2, r^2er^4\} = \{sr^4, e\} \neq H$, $\boxed{r^4Hr^{-4} = \{sr^4, e\}}$

(h) Find the conjugate of H by r^5

Answer: $r^5Hr^{-5} = r^5Hr^1 = \{r^5sr^1, r^5er^1\} = \{sr^2, e\} \neq H$, $\boxed{r^5Hr^{-5} = \{sr^2, e\}}$

(i) Find the conjugate of H by sr

Answer: $\boxed{(sr)H(sr)^{-1} = \{sr^2, e\}}$

(j) Find the conjugate of H by sr^2

Answer: $\boxed{(sr^2)H(sr^2)^{-1} = \{sr^4, e\}}$

(k) Find the conjugate of H by sr^3
Answer: $\boxed{(sr^3)H(sr^3)^{-1} = \{sr^?, e\}}$

(l) Find the conjugate of H by sr^4
Answer: $\boxed{(sr^4)H(sr^4)^{-1} = \{sr^?, e\}}$

(m) Find the conjugate of H by sr^5
Answer: $\boxed{(sr^5)H(sr^5)^{-1} = \{sr^?, e\}}$

(n) Find the conjugacy class of H
Answer: Conjugacy class of H is $\boxed{\{H, \{sr^2, e\}, \{sr^4, e\}\}}$

Theoretic Questions

14. Write the definition of *Action of group on a set*.

15. Write the definition of *Orbit of a point*.

16. Write the definition of *Stabilizer*.

17. Write the definition of *Conjugacy class of an element*.

18. Write the definition of *Conjugacy class of a subgroup*.

19. Write the definition of gHg^{-1}.

Proofs

20. Let $H < G$ be a subgroup of G. Let $g \in G$. Prove that gHg^{-1} is a subgroup of G.

21. Let $H < G$ be a subgroup of G. Let $h \in H$. Prove that $hHh^{-1} = H$.

22. Let H be a subgroup of an abelian group G. Prove that H is normal subgroup of G.

23. Prove that the center of a group is normal subgroup, i.e. $Z(G)$ is normal subgroup in G.

24. Let H be a subgroup of a group G. Assume that H is contained in $Z(G)$, the center of G. Prove that H is a normal subgroup of G.

25. Let G be a group with $|G| = 25$ acting on a set X. What are the possible sizes of orbits?
Answer: Let group G acts on set X. Let $x \in X$. Then:
Let $|orbit(x)| = |o(x)|$. Then $|o(x)| \mid |G|$. Therefore $|o(x)| = 1, 5, 25$.

26. Let G be a group with $|G| = 25$ acting on a set X with $|X| = 91$. Prove that there must be a fixed point. Answer: Let group G acts on set X. Let $x \in X$. Then:
(1) Let $|orbit(x)| = |o(x)|$. Then $|o(x)| \mid |G|$. Therefore $|o(x)| = 1, 5, 25$.
(2) $|X| = \Sigma|o(x)|$ where sum is over all distinct orbits.
From (2) it follows that $91 = a \cdot 1 + b \cdot 5 + c \cdot 25$, where
a = # of orbits of size 1,
b = # of orbits of size 5,
c = # of orbits of size 25.
If $a = 0$ then $91 = b \cdot 5 + c \cdot 25 = 5(b + c \cdot 5)$. This gives a contradiction since 5 does not divide 91, but 5 divides right side. Therefore $a \neq 0$. Therefore there exist at least one orbit with $|o(x)| = 1$.
From proposition in class: $(|o(x)| = 1) \iff (o(x) = \{x\}) \iff (x \text{ is a fixed point})$
Therefore there is at least one fixed point.
Another solution: Try to write 91 as a sum of 25, 5 and 1 and you will see that in every possibility you have to use 1, which will imply that there is always and orbit of size 1. And then: Therefore there exist at least one orbit with $|o(x)| = 1$.
From proposition in class: $(|o(x)| = 1) \iff (o(x) = \{x\}) \iff (x \text{ is a fixed point})$
Therefore there is at least one fixed point.

27. Let G be a group with $|G| = 15$ acting on a set X with $|X| = 9$. Prove that there must be at least 3 orbits.

28. Suppose x is conjugate to y and y is conjugate to z. Prove that x is conjugate to z.

29. Prove that a factor group of a cyclic group is cyclic.

30. Prove that any subgroup of an Abelian group is normal subgroup.

True -False - Sometimes

31. True -False - Sometimes

T F $\boxed{S}$ - Let H be a subgroup of G. Then H is normal subgroup.

$\boxed{T}$ F S - Let $G = (\mathbb{Z}_n, +_n)$, let H be a subgroup of G. Then H is normal subgroup.

$\boxed{T}$ F S - Let H be a subgroup of G. Then $|H| = |gHg^{-1}|$ for all $g \in G$.

T F $\boxed{S}$ - Let H be a subgroup of G. Then $H = gHg^{-1}$ for all $g \in G$.

$\boxed{T}$ F S - Let H be a subgroup of an abelian group G. Then $H = gHg^{-1}$ for all $g \in G$.

T F $\boxed{S}$ - Let G be a group of order $|G| = 5$ acting on a set X with $|X| = 10$. Then there is a fixed point.

$\boxed{T}$ F S - Let G be a group of order $|G| = 5$. Let $H < G$. Then $|H| = 1$ or $|H| = 5$.

T F $\boxed{S}$ - Let G be a group of order $|G| = 10$. Let $C(x)$ be a conjugacy class. Then $|C(x)| = 2$.

T $\boxed{F}$ S - Let G be a cyclic group of order $|G| = 15$. Let $C(x)$ be a conjugacy class. Then $|C(x)| = 2$.

T $\boxed{F}$ S - Let G be a group of order $|G| = 15$. Let $C(x)$ be a conjugacy class. Then $|C(x)| = 2$.

$\boxed{T}$ F S - Let G be a cyclic group of order $|G| = 15$. Let $C(x)$ be a conjugacy class. Then $|C(x)| = 1$.

T F $\boxed{S}$ - Let G be a group of order $|G| = 150$. Let $C(x)$ be a conjugacy class. Then $|C(x)| = 1$.

T F $\boxed{S}$ - Let G be a group. Let $g \in G$. Then $\langle g \rangle$ is normal subgroup.

T F $\boxed{S}$ - Let $G = D_5$. Let $g \in G$. Then $\langle g \rangle$ is normal subgroup.

T F $\boxed{S}$ - Let $G = S_5$. Let $g \in G$. Then $\langle g \rangle$ is normal subgroup.

$\boxed{T}$ F S - Let $G = \mathbb{Z}_5$. Let $g \in G$. Then $\langle g \rangle$ is normal subgroup.

Examples

32. Give an example of a group and an element x such that it's conjugacy class $C(x)$ has exactly one element. Prove your statement.

33. Give an example of a group and an element x such that it's conjugacy class $C(x)$ has more then one element. Prove your statement.

34. Give an example of a group G and a subgroup H which is not isomorphic to all of its conjugates.

35. Give an example of a group G and a subgroup H which is isomorphic to all of its conjugates.

36. Describe all conjugacy classes in D_5.

37. Describe all conjugacy classes in D_6.