MATH 3175 Notes

Xin Guan

1. 01.22

   (a) Definition: 7.1
   Function $f : X \to Y$ is bijection if f is both surjection(on to) and injection (one to one)

   (b) Theorem: 7.2
   $f : X \to Y$ is bijection $\Leftrightarrow$
   $\exists g : Y \to X$ s.t. $g \circ f = id_x, f \circ g = id_y$ (id_x means identity)
   Such g is called the inverse of f. Denoted by f^{-1}

   (c) Recall:
   ◦ Composition of two injective functions is injective.
   ◦ Composition of two surjective functions is surjective.
   ◦ Composition of two bijective functions is bijective.

   (d) Definition: 7.4 Permutation:
   Permutation on set X is a bijection $f : X \to X$

   (e) prop 7.5
      i. if $f : X \to X$ is a permutation then $\exists f^{-1} : X \to X$ which is also permutation.
      ii. composition of two permutation is again a permutation.

   (f) Definition: 7.6
   if $X = \{1, 2, \ldots, n\}$ then, $S_n := \{\text{all permutation on } X\}$

   (g) EX 7.7
   $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$
   $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$
   Find $\alpha\beta$ (composition of α and β), α^{-1}
   Solution:
   $(\alpha\beta)(1) = \alpha(\beta(1)) = \alpha(5) = 5$
   $(\alpha\beta)(2) = \alpha(\beta(2)) = \alpha(1) = 3$
   $(\alpha\beta)(3) = \alpha(\beta(3)) = \alpha(4) = 2$
   $(\alpha\beta)(4) = \alpha(\beta(4)) = \alpha(5) = 1$
   $(\alpha\beta)(5) = \alpha(\beta(5)) = \alpha(2) = 4$
   Then, $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$

$\alpha^{-1} = \begin{pmatrix} 3 & 4 & 1 & 2 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$
rearrange:
$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$

(h) Homework: 2.1 9(b)
$g : \mathbf{Z}_8 \Rightarrow \mathbf{Z}_1 2, g([x]_8) = [6x]_1 2$ show that g is well defined.
Solution:

Proof. Suppose $[x]_8 = [x']_8$, WTS $g([x]_8) = g([x']_8)$
Let $[x]_8 = [x']_8$
$\Rightarrow x \equiv x' (mod\ 8)$
$\Rightarrow 8|(x - x')$
$\Rightarrow x - x' = 8 * q$ for some $q \in \mathbf{Z}$
$x = 8 \cdot q + x'$
By definition of g, $g([x]_8) = [6x]_{12}$
Then, $g([x]_8) = [6(8q + x')]_{12} = [48q + 6x']_{12}, g([x']_8) = [6x']_{12}$
WTS $[48q + 6x']_{12} = [6x']_{12}$
Enough to show: $12|(48q + 6x' - 6x')$
Since $48q + 6x' - 6x' = 48q = 12 \cdot 4 \cdot q$
$\Rightarrow 12|12 \cdot 4 \cdot q$
$\Rightarrow 12|(48q + 6x' - 6x')$
$\Rightarrow g([x]_8) = g([x']_8)$ □

2. 01.23

(a) Recall:
DEF: Permutation on set X is a bijection $f : X \to X$
NOTE: S_x = {permutation on X}, S_n = {permutation on {1,2,3, ... n}}
PROPERTIES:
composition of permutation is again a permutation.
identity map: $id : X \to X(id(x) = x)$ is a permutation.
each permutation f there is an inverse f^{-1} such that $f \circ f^{-1} = id, f^{-1} \circ f = id$.

(b) Definition: 8.1 Disjoint cycle decomposition
Suppose $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 8 & 1 & 2 & 6 & 7 & 5 \end{pmatrix}$
= (1 3 8 5 2 4)(6)(7) or (1 3 8 5 2 4) (in cycle notation)

(c) Definition: 8.2
2-cycle $\to$ (i j) $i \neq j$
3-cycle $\to$ (i j k) i,j,k distinct
r-cycle $\to (i_1, i_2, \ldots, i_r)$, $i_1, i_2, \ldots, i_r$ distinct

(d) Example: 8.3
$\alpha = (142), \beta = (13)$ $\alpha \to$ 3-cycle, $\beta \to$ 2-cycle.

(e) identity permutation in S_n

i. (1)(2)...(n)

ii. fixes $\forall i$

iii. 1-cycle (i) fixes i

iv. ofthen we donot note 1-cycle: $\alpha = (142) = (142)(3)$

v. id = (1) = (1)(2)...(n)

(f) Example: 8.5 nultiplication of permutation
$\alpha = (142), \beta = (13), \in S_4$
compute - write as a product of disjoint cycles (same as Example: 7.7 with new notation)
$\alpha\beta = (142)(13) = (1342)$
HOWTO: $\beta : 1 \longrightarrow 3$, then $\alpha 3 \longrightarrow 3$, then (13) now.
$\beta 3 \longrightarrow 1$, then $\alpha 1 \longrightarrow 4$, then (134) now.
$\beta 4 \longrightarrow 4$, then $\alpha 4 \longrightarrow 2$, then (1342).
Similarly: $\beta\alpha = (13)(142) = (1423)$

(g) Remark: 8.6 Ingeneral $\alpha\beta \neq \beta\alpha$
if α, β are disjoint then $\alpha\beta = \beta\alpha$

3. Definition: 8.7

   Order of permutation α is the smallest positive integer n such that $\alpha^n = (1)$ where $\alpha^n = \alpha\alpha\ldots\alpha$ (there are n α's)

4. Example: 8.8

   $\alpha = (142)$

   $\alpha^2 = \alpha\alpha = (142)(142) = (124)$

   $\alpha^3 = \alpha\alpha\alpha = (142)(142)(142) = (142)(124) = (1)(2)(4) = (1)$

   Then $|\alpha| = 3$. Order of α is 3.

   $\beta = (13)$
   $\beta^2 = (13)(13) = (1)$
   Then $|\beta| = 2$

5. Prop: 8.10 Order of an r-cycle is r

6. Example: 8.11 $\alpha = (143)(25)$
   $|\alpha| = LCM(|(143)|, |(25)|) = LCM(3, 2) = 6$

7. Prop: 8.12 Let α, β be two disgoint permutation. Then $|\alpha\beta| = LCM(|\alpha|, |\beta|)$

8. Possible Disjoint Cycles

Partition of 6	Disjoint cycles	Example	Order	How many different permutation
6	6 cycle	(132654)	6	$\frac{6!}{6} = 5!$
5 + 1	5 cycle, 1 cycle	(13465)(2)	5	$\binom{6}{5}\frac{5!}{5}\frac{1!}{1} = \binom{6}{5} \cdot 4!$
4 + 2	4 cycle, 2 cycle	(1354)(26)	4	$\binom{6}{4}\binom{2}{2}\frac{4!}{4}\frac{2!}{2}$

NOTE: We need to divide by the order since (123) = (231) = (312). We need to eliminate repeative terms.