Some of the problems are extremely easy, some are computationally long, and some are actual proofs. Don't be surprised if it looks too easy or too hard. Some problems are from Practice Quiz2. Also, do the assigned HW problems. This is just in addition to HW.

ALWAYS JUSTIFY YOUR ANSWER!

Computations

1. Consider the group $G = \langle a \rangle$, where $|a| = 45$.

   (a) Find all divisors of 45.
   Answer: Divisors of $45 = 3 \cdot 3 \cdot 5$ are $\{1, 3, 5, 9, 15, 45\}$
   $\boxed{1, 3, 5, 9, 15, 45}$

   (b) Describe all subgroups of $G = \langle a \rangle$.
   Answer: For each divisor d of 45, there is exactly one subgroup of order d, which can be generated by $a^{45/d} \in G = \langle a \rangle$.
   – $d = 1$, $a^{45/1} = a^{45} = a^0 = e \in G = \langle e \rangle$. So $\boxed{H_1 = \langle e \rangle = \{e\}}$. $\boxed{|H_1| = 1}$
   – $d = 3$, $a^{45/3} = a^{15}$. So $\boxed{H_2 = \left\langle a^{15} \right\rangle = \{a^{15}, a^{30}, a^{45} = e\}}$. $\boxed{|H_2| = 3}$
   – $d = 5$, $a^{45/5} = a^9$. So $\boxed{H_3 = \left\langle a^9 \right\rangle = \{a^9, a^{18}, a^{27}, a^{36}, a^{45} = e\}}$. $\boxed{|H_3| = 5}$
   – $d = 9$, $a^{45/9} = a^5$. So $\boxed{H_4 = \left\langle a^5 \right\rangle = \{a^5, a^{10}, a^{15}, a^{20}, a^{25}, a^{30}, a^{35}, a^{40}, a^{45} = e\}}$. $\boxed{|H_4| = 9}$
   – $d = 15$, $a^{45/15} = a^3$. So $\boxed{H_5 = \left\langle a^3 \right\rangle = \{a^3, a^6, a^9, \ldots, a^{39}, a^{42}, a^{45} = e\}}$. $\boxed{|H_5| = 15}$
   – $d = 45$, $a^{45/45} = a^1 = a$. So $\boxed{H_6 = \langle a \rangle = \{a, a^2, a^3, \ldots, a^{43}, a^{44}, a^{45} = e\}}$. $\boxed{|H_6| = 45}$

   (c) Find the subgroup of G generated by a^5.
   Answer: From the above computations:
   $\boxed{H_4 = \left\langle a^5 \right\rangle = \{a^5, a^{10}, a^{15}, a^{20}, a^{25}, a^{30}, a^{35}, a^{40}, a^{45} = e\}}$. $\boxed{|H_4| = 9}$

   (d) Find the subgroup $\langle a^{10} \rangle$ of $G = \langle a \rangle$.
   Answer: From class: $\langle a^{10} \rangle = \left\langle a^{gcd(10,45)} \right\rangle = \langle a^5 \rangle$. Therefore:
   $\boxed{\left\langle a^{10} \right\rangle = \left\langle a^5 \right\rangle = H_4 = \{a^5, a^{10}, a^{15}, a^{20}, a^{25}, a^{30}, a^{35}, a^{40}, a^{45} = e\}}$

   (e) Find the subgroup $\langle 25 \rangle$ of G.
   Answer: From class: $\langle a^{25} \rangle = \left\langle a^{gcd(25,45)} \right\rangle = \langle a^5 \rangle$. Therefore:
   $\boxed{\left\langle a^{25} \right\rangle = \left\langle a^5 \right\rangle = H_4 = \{a^5, a^{10}, a^{15}, a^{20}, a^{25}, a^{30}, a^{35}, a^{40}, a^{45} = e\}}$

   (f) Find the order of a^{30} in G.
   Answer: From class: $|a^{30}| = |a^{gcd(30,45)}| = |a^{15}| = 45/15 = 3$.
   $\boxed{|a^{30}| = 3}$

(g) Find an element of order 9 in G.
Answer: From class: An element of order 9 is $a^{45/9} = a^5$.
$\boxed{a^5}$

(h) Find all element of order 9 in G.
Answer:
- Integers relatively prime to 9, between 1 and 9 are $\{1, 2, 4, 5, 7, 8\}$
- Elements of order 9 are $\{a^{1\cdot 5}, a^{2\cdot 5}, a^{4\cdot 5}, a^{5\cdot 5}, a^{7\cdot 5}, a^{8\cdot 5}\}$

$\boxed{a^5, a^{10}, a^{20}, a^{25}, a^{35}, a^{40}}$

(i) Find a generator of the subgroup of order 9 in G.
Answer: This is the same as the question: Find an element of order 9 in G.
$\boxed{a^5}$

(j) Find all generators of the subgroup of order 9 in G.
Answer: This is the same as the question: Find all element of order 9 in G.
$\boxed{a^5, a^{10}, a^{20}, a^{25}, a^{35}, a^{40}}$

(k) Find all generators of G.
Answer: This is the same as the question: Find all element of order 45 in G.
- Find all numbers relatively prime to 45.
- There are $\varphi(45)$ such elements.
- $\varphi(45) = \varphi(3^2 5)\varphi(3^2)\varphi(5) = (3-1)3^{(2-1)}(5-1) = 2 \cdot 3 \cdot 4 = 24$
- $\{1, 2, 4, 7, 8, 11, 13, 14, 16, 17, 19, 22, 23, 26, 28, 29, 31, 32, 34, 37, 38, 41, 43, 44\}$

$\boxed{a, a^2, a^4, a^7, a^8, a^{11}, a^{13}, a^{14}, a^{16}, a^{17}, a^{19}, a^{22}, a^{23}, a^{26}, a^{28}, a^{29}, a^{31}, a^{32}, a^{34}, a^{37}, a^{38}, a^{41}, a^{43}, a^{44}}$

(l) Describe the diagram of subgroups.

2. Consider the group $\mathbb{Z}_{45}$.

(a) Find all divisors of 45.
Answer: Divisors of $45 = 3 \cdot 3 \cdot 5$ are $\{1, 3, 5, 9, 15, 45\}$
$\boxed{1, 3, 5, 9, 15, 45}$

(b) Describe all subgroups of $\mathbb{Z}_{45}$.
Answer: For each divisor d of 45, there is exactly one subgroup of order d, which can be generated by $x = 45/d \in \mathbb{Z}_{45}$.
– $d = 1$, $x = 45/1 = 45 = 0 \in \mathbb{Z}_{45}$ So $\boxed{H_1 = \langle 0 \rangle = \{0\}}$. $\boxed{|H_1| = 1}$
– $d = 3$, $x = 45/3 = 15$. So $\boxed{H_2 = \langle 15 \rangle = \{15, 30, 45 = 0\}}$. $\boxed{|H_2| = 3}$
– $d = 5$, $x = 45/5 = 9$. So $\boxed{H_3 = \langle 9 \rangle = \{9, 18, 27, 36, 45 = 0\}}$. $\boxed{|H_3| = 5}$
– $d = 9$, $x = 45/9 = 5$. So $\boxed{H_4 = \langle 5 \rangle = \{5, 10, 15, 20, 25, 30, 35, 40, 45 = 0\}}$. $\boxed{|H_4| = 9}$

– $d = 15$, $x = 45/15 = 3$. So $\boxed{H_5 = \langle 3 \rangle = \{3, 6, 9, \ldots, 39, 42, 45 = 0\}}$. $\boxed{|H_5| = 15}$
– $d = 45$, $45/45 = 1$. So $\boxed{H_6 = \langle 1 \rangle = \{1, 2, 3, \ldots, 43, 44, 45 = 0\}}$. $\boxed{|H_6| = 45}$

(c) Find the subgroup of $\mathbb{Z}_{45}$ generated by 5.
Answer: From the above computations:
$\boxed{H_4 = \langle 5 \rangle = \{5, 10, 15, 20, 25, 30, 35, 40, 45 = 0\}}$

(d) Find the subgroup $\langle 10 \rangle$ of $\mathbb{Z}_{45}$. Answer: From class: $\langle 10 \rangle = \langle gcd(10, 45) \rangle = \langle 5 \rangle$. Therefore:
$\boxed{\langle 10 \rangle = \langle 5 \rangle = H_4 = \{5, 10, 15, 20, 25, 30, 35, 40, 45 = 0\}}$

(e) Find the subgroup $\langle 25 \rangle$ of $\mathbb{Z}_{45}$.
(f) Find the order of 30 in $\mathbb{Z}_{45}$.
(g) Find an element of order 9 in $\mathbb{Z}_{45}$.
(h) Find all element of order 9 in $\mathbb{Z}_{45}$.
(i) Find a generator of the subgroup of order 9 in $\mathbb{Z}_{45}$.
(j) Find all generators of the subgroup of order 9 in $\mathbb{Z}_{45}$.
(k) Find all generators of $\mathbb{Z}_{45}$.
Answer: This is the same as the question: Find all element of order 45 in $\mathbb{Z}_{45}$.
- Find all numbers relatively prime to 45.
- There are $\varphi(45)$ such elements.
- $\varphi(45) = \varphi(3^2 5) = \varphi(3^2)\varphi(5) = (3-1)3^{(2-1)}(5-1) = 2 \cdot 3 \cdot 4 = 24$
$\boxed{\{1, 2, 4, 7, 8, 11, 13, 14, 16, 17, 19, 22, 23, 26, 28, 29, 31, 32, 34, 37, 38, 41, 43, 44\}}$

(l) Describe the diagram of subgroups.

3. Let $\alpha = (1325)$. Find the subgroup of S_5 generated by α.
Answer: $\langle \alpha \rangle = \langle (1325) \rangle = \{(1325), (1325)^2 = (12)(35), (1325)^3 = (1523), (1325)^4 = (1)\}$
$\boxed{\{(1325), (12)(35), (1523), (1)\}}$

4. Let $\alpha = (1325)$. Write α as a product of transpositions in 3 different ways.
Answer: $\alpha = (1325) = (15)(12)(13)$ or $\alpha = (1325) = (3251) = (31)(35)(32)$ or
$\alpha = (1325) = (31)(35)(32)(14)(14) = (31)(35)(14)(32)(14) = (24)(31)(24)(35)(14)(32)(14) = (24)(31)(35)(24)(14)(32)(14)$
$\boxed{\{\alpha = (15)(12)(13) = (31)(35)(14)(32)(14) = (24)(31)(24)(35)(14)(32)(14) = (24)(31)(35)(24)(14)(32)(14)\}}$

5. Consider the group $\mathbb{Z}_8^\times$. What is the order of $\mathbb{Z}_8^\times$.
Answer: $|\mathbb{Z}_8^\times| = \varphi(8) = \varphi(2^3) = (2-1)2^{(3-1)} = 4$

$\boxed{|\mathbb{Z}_8^\times| = 4}$

6. Consider the group $\mathbb{Z}_{15}^\times$. Decide if this group is cyclic and prove your statement.
   Answer:
   – $|\mathbb{Z}_{15}^\times| = \varphi(15) = \varphi(3 \cdot 5) = \varphi(3)\varphi(5) = (3-1)(5-1) = 8$
   – Elements of $\mathbb{Z}_{15}^\times$ are integers relatively prime to 15 between 1 and 15.
   – $\mathbb{Z}_{15}^\times = \{1, 2, 4, 7, 8, 11, 13, 14\}$
   – Find orders of each of the elements:
   $\langle 1 \rangle = \{1\}$, so $|1| = 1$
   $\langle 2 \rangle = \{2, 4, 8, 1\}$, so $|2| = 4$.
   $\langle 4 \rangle = \{4, 1\}$, so $|4| = 2$.
   $\langle 7 \rangle = \{7, 7^2 = 4, 7^3 = 13, 7^4 = 1\}$, so $|7| = 4$.
   $\langle 8 \rangle = \{8, 8^2 = 4, 8^3 = 2, 8^4 = 1\}$, so $|8| = 4$.
   $\langle 11 \rangle = \{11, 11^2 = 1\}$, so $|11| = 2$.
   $\langle 13 \rangle = \{13, 13^2 = 4, 13^3 = 7, 13^4 = 1\}$, so $|13| =$.
   $\langle 11 \rangle = \{14, 14^2 = 1\}$, so $|14| = 2$.
   – So there is no element of order 8, hence there is no element that generates the group $\mathbb{Z}_{15}^\times$.
   $\boxed{\mathbb{Z}_{15}^\times \textit{ is not cyclic}}$

7. Find all elements in $\mathbb{Z}_2 \times \mathbb{Z}_3$.

8. What are the possible orders of elements in $\mathbb{Z}_8 \times \mathbb{Z}_6$.

9. What are the possible orders of elements in $\mathbb{Z}_4 \times \mathbb{Z}_3$.

10. Prove that $\mathbb{Z}_4 \times \mathbb{Z}_3$ is cyclic.

11. Prove that $\mathbb{Z}_4 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_{12}$.

12. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$.

13. Prove that $\mathbb{Z}_6$ is not isomorphic to S_3.

14. Prove that the group $\mathbb{Z}_8 \times \mathbb{Z}_{81}$ is cyclic.

Theoretic Questions

15. Write the definition of *Group*.

16. Write the definition of *Cyclic Group*.

17. Write the definition of *Subgroup*.

18. Let G be a group.

(a) Let $b \in G$. What does it mean to say that b has order n in G (where n is some positive integer)?

(b) Let $b \in G$. What does it mean to say that $\langle b \rangle$ has order n in G (where n is some positive integer)?

(c) Let H be a subset of G. What do you have to check in order to prove that H is a subgroup of G.

(d) Let H be a set and let G be a group. What do you have to check in order to prove that H is a subgroup of G.

(e) Suppose $a, b \in G$. How can you prove that b is inverse of a in G?

19. Let G be a group of order n. How can you prove that G is cyclic?

Proofs

20. Let $(X, *)$ be a monoid. Suppose that e and e' are identities. Prove that $e = e'$. Make sure that you only use binary operation $*$, associative law and identity property.

21. Let $(G, *)$ be a group with identity e. Let $g \in G$. Prove that g has a unique inverse.

22. Let $(G, \cdot)$ be a group. Suppose $a^2 = e$ for all $a \in G$. Prove that G is abelian.

23. Let G be a group. Prove that $(ab)^{-1} = b^{-1}a^{-1}$.

24. Let $G = S_3$. Prove that $H = \{(1), (12), (132)\}$ is not a subgroup of G.

25. Let G be a group. Let $g \in G$ be an element of order 2. Prove that $a = a^{-1}$.

26. Let G be a group. Let $g \in G$ be an element of order k. Prove that $a^{-1} = a^{k-1}$.

27. Prove that the group $\{(1), (12)(34), (13)(24), (14)(23)\}$ is not cyclic.

28. Prove that the set of permutations $\{(1), (12)(34), (13)(24), (14)(23)\}$ is a subgroup of S_4.

29. Prove that $A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$ is in $(Gl_2(\mathbb{Z}_7), \cdot)$.

30. Let $\phi : G \to G'$ be an isomorphism. Prove that G is abelian if and only if G' is abelian.

31. Let $\phi : G \to G'$ be an isomorphism. Let $g \in G$. Prove that $|\phi(g)| = |g|$.

32. Let $\phi : G \to G'$ be an isomorphism. Prove that G is cyclic if and only if G' is cyclic.

33. Let $\phi : G \to G'$ be a homomorphism. Let $g \in G$. Prove that $|\phi(g)|$ divides $|g|$.

34. Let $t \in S_n$ be a transposition. Prove that $t^{-1} = t$.

35. Let α be an even permutation. Prove that α^{-1} is an even permutation.

36. Let γ be an even permutation in S_n. Let $\beta \in S_n$ be any permutation. Prove that $\beta\gamma\beta^{-1}$ is an even permutation.

37. Let $f : G \to G'$ be a group homomorphism. Prove that $ker(f)$ is a subgroup of G.

38. Let $f : G \to G'$ be a group homomorphism. Prove that $Im(f)$ is a subgroup of G'.

True -False - Sometimes

39. True -False - Sometimes

    T F S - $(\mathbb{Z}_n, +_n)$ is an abelian group.

    T F S - $(\mathbb{Z}, \cdot)$ is an abelian group.

    T F S - $(\mathbb{Z}_n, \cdot_n)$ is an abelian group.

    T F S - $(\mathbb{Z}_n^\times, \cdot_n)$ is an abelian group.

    T F S - $(\mathbb{Z}_8, \cdot_8)$ is a monoid.

    T F S - $(\mathbb{Z}_8^\times, \cdot_8)$ is a group.

    T F S - $(\mathbb{Z}_8^\times, +_8)$ is a semigroup.

    T F S - $(\mathbb{Z}_8^\times, +_8)$ is a group.

    T F S - $\mathbb{Z}_8^\times$ has 8 elements.

    T F S - $\mathbb{Z}_8^\times$ has 4 elements.

    T F S - Identity element in $(\mathbb{Z}, +)$ is 1.

    T F S - $(2\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$

    T F S - $\langle 6 \rangle$ is a cyclic subgroup of $(\mathbb{Z}_{10}, +_{10})$

    T F S - Let H be a subgroup of S_3. Then H is cyclic.

    T F S - All proper subgroups of S_4 are cyclic.

    T F S - Let G be a cyclic group of order $|G| = 5$. Let $g \in G$. Then $|g| = 4$.

    T F S - Let G be a cyclic group of order $|G| = 5$. Let $g \in G$. Then $|g| = 1$.

    T F S - Let G be a cyclic group of order $|G| = 5$. Let $g \in G$. Then $|g| = 5$.

    T F S - Let G be a cyclic group of order $|G| = 15$. Let $g \in G$. Then $|g| = 15$.

    T F S - Let G be a cyclic group of order $|G| = 15$. Let $g \in G$. Then $|g| = 5$.

    T F S - Let G be a cyclic group of order $|G| = 15$. Let $g \in G$. Then $|g| = 10$.

    T F S - Let G be a group. Let $g \in G$. Then $\langle g \rangle$ is cyclic group.

    T F S - Let $f : G \to G'$ be a group homomorphism. Then $f(e_G) = e_{G'}$.

    T F S - Let $\alpha \in S_4$. Then α is an even permutation.

    T F S - Product of two odd permutations is odd permutation.

T F S - Product of two odd permutations is even permutation.

Examples

40. Give an example of a group and a subgroup which is not cyclic. Prove your statement.
41. Give an example of a non cyclic group and a subgroup which is not cyclic. Prove your statement.
42. Give an example of a non cyclic group and a subgroup which is cyclic. Prove your statement.
43. Give an example of a group and a subset which is not a subgroup. Prove your statement.
44. Give an example of an even permutation. Prove your statement.
45. Give an example of an odd permutation. Prove your statement.
46. Give an example of a group homomorphism which is not surjective.
47. Give an example of a group homomorphism which is not injective.
48. Give an example of a group homomorphism $f : G \to G'$ such that $ker(f) = \{e_G\}$.
49. Give an example of a group homomorphism $f : G \to G'$ such that $ker(f) \neq \{e_G\}$.
50. Give an example of a group homomorphism $f : G \to G'$ such that $Im(f) = G'$.
51. Give an example of a group homomorphism $f : G \to G'$ such that $Im(f) \neq G'$.