

- **Bijection:**

Function $f : X \rightarrow Y$ is bijection if f is both surjection(on to) and injection (one to one)

Proposition:

1. $f : X \rightarrow Y$ is bijection \Leftrightarrow
 $\exists g : Y \rightarrow X$ s.t. $g \circ f = id_x, f \circ g = id_y$ ($id_x \rightarrow$ identity)
2. Composition Properties:
 - Composition of two injective functions is injective.
 - Composition of two surjective functions is surjective.
 - Composition of two bijective functions is bijective.

- **Permutation:**

Permutation on set X is a bijection $f : X \rightarrow X$

If $X = \{1, 2, \dots, n\}$ then, $S_n := \{\text{all permutation on } X\}$ **Proposition:**

1. if $f : X \rightarrow X$ is a permutation then $\exists f^{-1} : X \rightarrow X$ which is also permutation.
2. composition of two permutation is again a permutation.

- **Group 5 Rules:**

1. Closed under binary operation
2. associative: $(ab)c = a(bc)$
3. identity: $\exists e \in G, ea = ae = a \forall a \in G$
4. inverse: $\forall a \in G, \exists ! a^{-1} \text{ s.t. } a^{-1}a = aa^{-1} = e$
5. commutative $a, b \in G, ab = ba$.

1,2: semigroup

1,2,3: monoid

1,2,3,4: group

1,2,3,4,5: Abelian group

- **Equivalence Relation:**

Operation in Group G is equivalence if

1. Reflective: $g \sim g, \forall g \in G$
2. Symmetry: $g \sim g' \Rightarrow g' \sim g, \forall g, g' \in G$
3. transitive: $x \sim y, y \sim z \Rightarrow x \sim z \forall x, y, z$

- **Subgroup:** H is a subgroup of G if

- $H \subseteq G$
- H is a group

CHECK a SUBGROUP:

- $H \subseteq G$ (subset)
- $e \in H$ (non empty)
- $\forall a, b \in H, ab \in H$ (closed)

$$- \forall a \in H, a^{-1} \in H$$

Proper subgroup: subgroup H that is not $H = G$

- **Order:**

Order of a group: $|G| = \#$ of elements in the group

Order of an element: $g \in G, |g| = \text{smallest positive integer } n, \text{ s.t. } x^n = e$

Propositions:

$$- \text{ If } H \text{ is a subgroup of } G \text{ then } |H| \mid |G|. \text{ If } x \in G, \text{ then } |x| \mid |G|$$

- $\langle x \rangle := \{ x^n \mid n \in \mathbb{Z} \}$

- **Conjugate:** $x, g \in G$, conjugate of x by g : gxg^{-1}

Conjugate class of $x := \{gxg^{-1} \mid \forall g \in G\}$

- **ISOMORPHISMS of GROUP:** a function $f : G \rightarrow G'$ is called isomorphism if:

1. $f(xy) = f(x)f(y)$

2. f is one to one (injective)

3. f is onto (surjective)

- **Cyclic:** $\exists a \in G, \text{ s.t. } \langle a \rangle = G$

- **Center of Group:**

Center of a Group $G : Z(G) := \{Z \in G \mid gz = zg, \forall g \in G\}$

Proposition:

1. $Z(G)$ is a subgroup of G .

2. If G is abelian, then $Z(G) = G$

- **External direct product of Groups:**

Group G, H , Define $G \times H := \{(x, y) \mid x \in G, y \in H\}$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$$

Proposition:

1. $e_{G \times H} = (e_G, e_H)$

2. $(x, y)^{-1} = (x^{-1}, y^{-1})$

3. $|(x, y)| = \text{LCM}(|x|, |y|)$

- **Internal product of groups:**

Group G has subgroup H, K . Define $HK := \{xy \mid x \in H, y \in K\}$

NOTE: HK is not always a subgroup.

Proposition:

1. H, K are subgroup of G .

Suppose $x^{-1}yx \in K, \forall x \in H, y \in K$ Then HK is a subgroup of G .

Corollary: H, K are subgroup of abelian group G , then HK is a subgroup of G .

- **Group Homomorphisms:**

$f : G \rightarrow G'$ if $f(xy) = f(x)f(y) \forall x, y \in G$

Compared to isomorphism, we don't need bijection.

- **Kernal and Image:**

$f : G \rightarrow G'$, Define:

$\text{Kerf} := \{g \in G \mid f(g) = e_{G'}\}$

$\text{Imf} := \{y \in G' \mid \exists x \in G, s.t. f(x) = y\} \equiv \{f(x) \mid x \in G\}$

Lemma:

$f : G \rightarrow G'$ be a group homomorphism.

1. $f(e_G) = e_{G'}$
2. $f(a^n) = (f(a))^n, \forall n > 0, n \in \mathbb{Z}$
3. $f(a^{-1}) = (f(a))^{-1}$
4. From 2,3 we can conclude: $f(a^n) = (f(a))^n$

Proposition:

1. $f : G \rightarrow G'$ be group homomorphism:
 - kerf is a subgroup of G
 - Imf is a subgroup of G'
2. If $G = \langle a \rangle$ i.e. G is a cyclic group. Then, it is enough to define homomorphism $f : G \rightarrow G'$ on a and extend to all a^n .
3. $f : G \rightarrow G'$ be a group homomorphism, then $|f(a)| \mid |a|$
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