Show all work. **Math 3175 Final Exam - M** Fall 2018

1) (8 points) Suppose we have a set G and function $*: G \times G \to G$ (a binary operation on G), where the value of the function on a pair (g_1, g_2) is denoted by $g_1 * g_2$.

G and $*$ must satisfy what conditions for $(G, *)$ to be a group?

2) Suppose that G and L are groups (where we use multiplicative notation for both), and that $\phi : G \to L$ is a function.

a) (4 points) Define what is means for ϕ to be a homomorphism.

b) (6 points) Suppose that ϕ is a homomorphism. Prove that the image, $\operatorname{im} \phi$, of ϕ is a subgroup of L.

3) (8 points) Describe all of the homomorphisms from $\mathbb{Z}_{18}$ to $\mathbb{Z}_{12}$ (where we use addition for both groups).

4)

a) (4 points) List the elements of D_4 where D_4 is considered as a subgroup of S_4. Give each element as a product of disjoint cycles.

b) (4 points) Which of the elements from part (a) are conjugate to each other in S_4? How do you know? (Note that here you are allowed to conjugate by elements of S_4.)

c) (4 points) What is the center of D_4? List the elements.

5) Let G be a group of order 21. Suppose H is a subgroup of G.

a) (3 points) What are the possible orders of H?

b) (4 points) Must a group of order 21 have subgroups of all of the orders you gave in part (a)? Justify your answer.

c) (5 points) Show that G must have a proper non-trivial **normal** subgroup.

6) (7 points) Suppose that $\phi : G \to L$ is a group homomorphism, and that G is a simple group. Prove that either the image of ϕ is the trivial subgroup of L or that the image of ϕ is isomorphic to G.

7) (6 points) What is the order of $([5]_6, (3\ 7\ 10\ 1))$ in $\mathbb{Z}_6 \times S_{12}$? (Here, as usual, we are using modular addition in $\mathbb{Z}_6$ and multiplication/composition in S_{12}.) Show your work.

8) Consider $\sigma := (2\ 7\ 5\ 3)(3\ 7\ 1)(5\ 8) \in S_8$.

a) (4 points) Write σ as the product of disjoint cycles.

b) (3 points) What is the order of σ? Justify your answer.

c) (2 points) Is σ even or odd? Justify your answer.

9) (6 points) Let G be a group, and let $a \in G$. Define a function $\phi_a : G \to G$ by $\phi_a(x) = axa^{-1}$ for all $x \in G$. Prove that ϕ_a is an isomorphism.

10) (7 points) A group G of order 121 acts on the symmetric group S_7. Prove that there must be at least one fixed point of the action.

11) (7 points) Recall that the center of A_4 is trivial. Now, suppose that A_4 acts on itself by conjugation. Given that one of the conjugacy classes contains precisely 4 elements and that no conjugacy class contains precisely 2 elements, how many conjugacy classes are there and what are their orders? (Hint: You want to do this without determining the actual conjugacy classes.)

12) (8 points) Suppose that G is an abelian group of order 200. Give a list of representatives of the all of the possible isomorphism classes G, where no two groups in your list are isomorphic to each other. In other words, G must be isomorphic to one and one only of what finite list of groups?