Some of the problems are extremely easy, some are computationally long, and some are actual proofs. Don't be surprised if it looks too easy or too hard.
Also, do the assigned HW problems. This is just in addition to HW.

Computations

1. Consider the permutation $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 7 & 8 & 2 & 1 & 5 & 4 \end{bmatrix}$.

   (a) Write α as a product of disjoint cycles.
   <u>Answer:</u> $\alpha = (137526)(48)$
   $\boxed{\alpha = (137526)(48)}$

   (b) Find $\alpha^2 =$
   <u>Answer:</u> $\alpha^2 = (137526)(48)(137526)(48) = (172)(356)(4)(8) = (172)(356)$.
   $\boxed{\alpha^2 = (172)(356)}$

   (c) Find $\alpha^3 =$
   <u>Answer:</u> $\alpha^3 = (137526)(48)(137526)(48)(137526)(48) = (172)(356)(4)(8)(137526)(48) = (15)(23)(48)(67)$.
   $\boxed{\alpha^3 = (15)(23)(48)(67)}$

   (d) Find the order of α, i.e. $|\alpha| =$ <u>Answer:</u> $|\alpha| = |(137526)(48)| = lcm(|(137526)|, |(48)| = lcm(6, 2) = 6$
   $\boxed{|\alpha| = 6}$

2. Let $\alpha = (1325)$, $\beta = (12)(354)$. Compute:

   (a) $\alpha \cdot \beta =$

   (b) $\beta \cdot \alpha =$

   (c) $\alpha^{-1} =$

   (d) $\alpha^4 =$

   (e) $\alpha \cdot \beta \cdot \alpha^{-1} =$

3. Consider S_7, the group of permutations on $\{1, 2, 3, 4, 5, 6, 7\}$.

   (a) Find all possible disjoint cycle decompositions.
   <u>Answer:</u> (7), (6,1), (5,2), (5,1,1), (4,3), (4,2,1), (4,1,1,1), (3,3,1), (3,2,2), (3,2,1,1), (3,1,1,1,1), (2,2,2,1), (2,2,1,1,1), (2,1,1,1,1,1), (1,1,1,1,1,1,1)

   (b) For each disjoint cycle decomposition find the number of distinct permutations.
   <u>Answer:</u>
   (7) (i.e. 7-cycle) - There are $\binom{7}{6}\frac{7!}{7}$ distinct permutations.

(3,3,1) - There are $\binom{7}{3}\frac{3!}{3}\binom{4}{3}\frac{3!}{3}\frac{1}{2!}$ distinct permutations.

(3,2,2) - There are $\binom{7}{3}\frac{3!}{3}\binom{4}{2}\frac{2!}{2}\binom{2}{2}\frac{2!}{2}\frac{1}{2!}$ distinct permutations.

(2,2,2,1)- There are $\binom{7}{2}\frac{2!}{2}\binom{5}{2}\frac{2!}{2}\binom{3}{2}\frac{2!}{2}\frac{1}{3!}$ distinct permutations.

4. Consider $(Gl_2(\mathbb{R}),\cdot)$. Let $A=\begin{bmatrix}1 & -2\\ 3 & 6\end{bmatrix}$. Find the inverse of A.

   <u>Answer:</u>

   $A^{-1}=\frac{1}{6\cdot 1-(-2)\cdot 3}\begin{bmatrix}6 & 2\\ -3 & 1\end{bmatrix}=\begin{bmatrix}\frac{6}{12} & \frac{2}{12}\\ \frac{-3}{12} & \frac{1}{12}\end{bmatrix}$

   $\boxed{A^{-1}=\begin{bmatrix}\frac{6}{12} & \frac{2}{12}\\ \frac{-3}{12} & \frac{1}{12}\end{bmatrix}}$

5. Consider $(M_2(\mathbb{R}),+)$. Let $A=\begin{bmatrix}1 & -2\\ 3 & 6\end{bmatrix}$. Find the inverse of A.

   <u>Answer:</u> Since the operation in $(M_2(\mathbb{R}),+)$ is addition, the inverse of A will be $-A$.

   $\boxed{-A=\begin{bmatrix}-1 & 2\\ -3 & -6\end{bmatrix}}$

6. Consider $(Gl_2(\mathbb{Z}_7),\cdot)$. Let $A=\begin{bmatrix}[1]_7 & [1]_7\\ [3]_7 & [6]_7\end{bmatrix}$. Find the inverse of A.

   <u>Answer:</u> $A^{-1}=\frac{[1]_7}{[1]_7\cdot[6]_7-[1]_7\cdot[3]_7}\begin{bmatrix}[6]_7 & -[1]_7\\ -[3]_7 & [1]_7\end{bmatrix}=\frac{[1]_7}{[3]_7}\begin{bmatrix}[6]_7 & [6]_7\\ [4]_7 & [1]_7\end{bmatrix}$

   $\frac{[1]_7}{[3]_7}=[5]_7$ since $[5]_7\cdot[3]_7=[1]_7$

   Therefore: $\frac{[1]_7}{[3]_7}\begin{bmatrix}[6]_7 & [6]_7\\ [4]_7 & [1]_7\end{bmatrix}=[5]_7\begin{bmatrix}[6]_7 & [6]_7\\ [4]_7 & [1]_7\end{bmatrix}=\begin{bmatrix}[30]_7 & [30]_7\\ [20]_7 & [5]_7\end{bmatrix}=\begin{bmatrix}[2]_7 & [2]_7\\ [6]_7 & [5]_7\end{bmatrix}$.

   $\boxed{A^{-1}=\begin{bmatrix}[2]_7 & [2]_7\\ [6]_7 & [5]_7\end{bmatrix}}$

   Remember to check your answer, i.e.

   check $AA^{-1}=I_2=\begin{bmatrix}[1]_7 & [0]_7\\ [0]_7 & [1]_7\end{bmatrix}$ and $A^{-1}A=I_2\begin{bmatrix}[1]_7 & [0]_7\\ [0]_7 & [1]_7\end{bmatrix}$.

7. Consider $(Gl_2(\mathbb{Z}_7),\cdot)$. Let $A=\begin{bmatrix}[1]_7 & [2]_7\\ [3]_7 & [5]_7\end{bmatrix}$. Find the inverse of A.

8. Consider $(M_2(\mathbb{Z}_7),+)$. Let $A=\begin{bmatrix}[1]_7 & [2]_7\\ [3]_7 & [6]_7\end{bmatrix}$. Find the inverse of A.

Answer: Since the operation in $(M_2(\mathbb{Z}_7), +)$ is addition, the inverse of A will be $-A$.
$-A = \begin{bmatrix} [-1]_7 & [-2]_7 \\ [-3]_7 & [-6]_7 \end{bmatrix} = \begin{bmatrix} [6]_7 & [5]_7 \\ [4]_7 & [1]_7 \end{bmatrix}$
$\boxed{-A = \begin{bmatrix} [6]_7 & [5]_7 \\ [4]_7 & [1]_7 \end{bmatrix}}$

Remember to check your answer, i.e.
check $A + (-A) = 0_2 = \begin{bmatrix} [0]_7 & [0]_7 \\ [0]_7 & [0]_7 \end{bmatrix}$ and $(-A) + A = 0_2 \begin{bmatrix} [0]_7 & [0]_7 \\ [0]_7 & [0]_7 \end{bmatrix}$.

9. Find an integer b so that $b \cdot 7 \equiv 1(mod\ 10)$
Answer: $b = 3$ since $3 \cdot 7 = 21 = 2 \cdot 10 + 1 \equiv (2 \cdot 0 + 1)(mod\ 10) \equiv 1\ (mod\ 10)$.

10. Consider $(\mathbb{Z}_{10}, \cdot_{10})$. Find the inverse of $[7]_{10} \in \mathbb{Z}_{10}$.
Answer: The inverse of $[7]_{10}$ is $[3]_{10}$ since
$[3]_{10} \cdot [7]_{10} = [21]_{10} = 2 \cdot [10]_{10} + [1]_{10} = 2 \cdot [0]_{10} + [1]_{10} = [1]_{10}$.

11. Consider $(\mathbb{Z}_{10}, +_{10})$. Find the inverse of $[7]_{10} \in \mathbb{Z}_{10}$.

12. Find an integer b so that $b \cdot 137 \equiv 1(mod\ 532)$.
Answer: Use Euclidean algorithm.

13. Consider $(\mathbb{Z}_{532}, \cdot_{532})$. Find the inverse of $[137]_{532} \in (\mathbb{Z}_{532}, \cdot_{532})$.

14. Consider the group $\mathbb{Z}_{15}^{\times}$

(a) How many elements does $\mathbb{Z}_{15}^{\times}$ have?
Answer: $|\mathbb{Z}_{15}^{\times}| = \varphi(15) = \varphi(3)\varphi(5) = (3-1)(5-1) = 2 \cdot 4 = 8$.
$\boxed{|\mathbb{Z}_{15}^{\times}| = 8}$

(b) Write all elements of $\mathbb{Z}_{15}^{\times}$.
Answer: $\mathbb{Z}_{15}^{\times} = \{$all integers between 1 and 15 which are relatively prime to 15$\} =$
$\{i \in \mathbb{Z} \mid 1 \le i \le 15, gcd(i, 15) = 1\} = \{1, 2, 4, 7, 8, 11, 13, 14\}$
$\boxed{elements\ (\mathbb{Z}_{15}^{\times}) = \{1, 2, 4, 7, 8, 11, 13, 14\}}$

(c) Make a Cayley table for $\mathbb{Z}_{15}^{\times}$.
Answer:

(d) Find the inverse of $[2]_{15}$, or Find the inverse of $2 \in \mathbb{Z}_{15}^{\times}$
Answer: $2^{-1} = 8$ in $\mathbb{Z}_{15}^{\times}$ since $2 \cdot 8 = 16 = 15 + 1 \equiv 1(mod\ 15)$.
$\boxed{2^{-1} = 8 \in \mathbb{Z}_{15}^{\times}}$

(e) Find the inverse of $[11]_{15}$. Check your answer.

15. How many elements does $\mathbb{Z}_{200}^{\times}$ have?
Answer: $|\mathbb{Z}_{200}^{\times}| = \varphi(200) = \varphi(2^3)\varphi(5^2) = (2-1)2^{3-1}(5-1)5^{2-1} = 4 \cdot 4 \cdot 5 = 80.$
$\boxed{|\mathbb{Z}_{200}^{\times}| = 80}$

16. Find 3 different subgroups of S_3.
Answer: $S_3 = \{(1), (123), (132), (12), (13), (23)\}$
$\boxed{Subgroups\ of\ S_3 :}$
$\boxed{H_1 = S_3, H_2 = \{(1)\}, H_3 = \{(1), (123), (132)\}, H_4 = \{(1), (12)\}, H_5 = \{(1), (13)\}, H_6 = \{(1), (23)\}}$

17. Find all elements in $\mathbb{Z}_2 \times \mathbb{Z}_3$.
Answer: $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(x, y) \mid x \in \mathbb{Z}_2,\ y \in \mathbb{Z}_3\}$
$\boxed{elements\ in\ \mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}}$

18. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.
Answer:
- (*) From class: A group of order n is cyclic if and only if it has an element of order n.
- Order of $\mathbb{Z}_2 \times \mathbb{Z}_3$ is $|\mathbb{Z}_2 \times \mathbb{Z}_3| = |\mathbb{Z}_2| \cdot |\mathbb{Z}_3| = 2 \cdot 3 = 6$.
- Order of element (x, y) is $|(x, y)| = lcm(|x|, |y|)$.
- Order of $(1, 1)$ is $|(1, 1)| = lcm(|1|, |1|) = lcm(2, 3) = 6$.
- Therefore $\mathbb{Z}_2 \times \mathbb{Z}_3$ has an element of order 6, hence
- $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic by (*).

19. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.
Answer:
- By Problem 18. the group $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic of order 6.
- Theorem (**) from class: Every cyclic group of order n is isomorphic to $\mathbb{Z}_n$.
- By Theorem (**) it follows that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$, i.e.
- $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$.

20. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$.
Answer:
- Elements in $\mathbb{Z}_2$ have orders either 1 or 2.
- Elements in $\mathbb{Z}_2 \times \mathbb{Z}_2$ have orders either 1 or 2, since $(x, y)| = lcm(|x|, |y|)$.
- $lcm(1, 1) = 1$, $lcm(1, 2) = 2$, $lcm(2, 1) = 2$, $lcm(2, 2) = 2$
- $\mathbb{Z}_4$ has an element of order 4.
- Isomorphic groups must have the same number of the elements of each of the orders.
- Therefore $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$, i.e.
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \ncong \mathbb{Z}_4$.

21. Prove that $\mathbb{Z}_6$ is not isomorphic to S_3.
Answer:
- Theorem (*) from class: If G is isomorphic to G' then G is abelian if and only if G' is abelian.
- $\mathbb{Z}_6$ is abelian group.
- S_3 is not abelian since $(12)(23) = (123)$ but $(23)(12) = (132)$ and so $(12)(23) \neq (23)(12)$.
- Therefore by the above theorem $\mathbb{Z}_6$ is not isomorphic to S_3, i.e.
- $\mathbb{Z}_6 \ncong S_3$

Proofs

22. Let $(X, *)$ be a monoid. Suppose that e and e' are identities. Prove that $e = e'$. Make sure that you only use binary operation $*$, associative law and identity property.
Answer: Done in class.

23. Let $(G, *)$ be a group with identity e. Let $g \in G$. Prove that g has a unique inverse.
Answer: Done in class.

24. Let $(G, \cdot)$ be a group. Suppose $a^2 = e$ for all $a \in G$. Prove that G is abelian.
Answer/Proof:
- Want to show that $xy = yx$ for all elements $x, y \in G$.
- Let $x, y \in G$.
- Then $xy \in G$ since G is a group and therefore closed under operation.
- $(xy)^2 = e$ by the assumption that $a^2 = e$ for every element of G (so it is true for xy).
- $(xy)^2 = xyxy$ and therefore
- $xyxy = e$. Multiply this on the left by x and get:
- $xxyxy = xe$. Now use that $xx = x^2 = e$ by assumption, and get:
- $eyxy = xe$. Now use that e is identity and hence $eyxy = yxy$ and $xe = x$. Therefore:
- $yxy = x$. Multiply this on the left by y and get:
- $yyxy = yx$. Now use that $yy = y^2 = e$ by assumption, and get:
- $exy = yx$. Now use that e is identity and hence $exy = xy$. Therefore:
- $xy = yx$ for all $x, y \in G$.
- Therefore G is abelian.

25. Let G be a group. Prove that $(ab)^{-1} = b^{-1}a^{-1}$.
Answer/Proof:
- By definition of inverse in a group (property (4) of the definition) and by the proposition that the inverse of each element is unique, the inverse of ab is the unique element which when multiplied by ab gives the identity e.
- $(ab)(b^{-1}a^{-1}) =$ use associative law and get:
- $((ab)b^{-1})a^{-1} =$ use again associative law and get:
- $(a(bb^{-1}))a^{-1} =$ now use inverse property for b and b^{-1} and get:
- $(ae)a^{-1} =$ now use identity property of e and get:
- $aa^{-1} = e$. Here the last step comes from inverse property of a and a^{-1}.
- Therefore $(ab)(b^{-1}a^{-1}) = e$ (transitive property of "=")
- Similarly $(b^{-1}a^{-1})(ab) = e$

- By the definition and uniqueness of inverses, it follows that $b^{-1}a^{-1}$ is the inverse of ab, i.e.
- $\boxed{(ab)^{-1} = b^{-1}a^{-1}}$.

26. Let G be a group. Let e be the identity in G. Prove that $\{e\}$ is a subgroup of G.
Answer/Proof:
Recall that in order to check that H is a subgroup of a group G, you have to check:
(0) H is a subset of G
(00) H is nonempty
(1) H is closed under operation
(2) H is closed under inverses
- (0) Since $e \in G$ it follows that $\{e\} \subset G$, i.e. $\{e\}$ is a subset of G.
- (00) Since $e \in \{e\}$ it follows that $\{e\} \neq \emptyset$.
- (1) Since $ee = e \in \{e\}$ it follows that $\{e\}$ is closed under operation.
- (2) Since $ee = e$ it follows that $e^{-1} = e$. So $e^{-1} \in \{e\}$. Hence $\{e\}$ is closed under inverses.
Therefore $\{e\}$ is a subgroup of G.

27. Let $G = S_3$. Prove that $H = \{(1), (123), (132)\}$ is a subgroup of H.
Answer/Proof:
Recall that in order to check that H is a subgroup of a group G, you have to check:
(0) H is a subset of G
(00) H is nonempty
(1) H is closed under operation
(2) H is closed under inverses
- (0) Since all elements of H are permutations in S_3, it follows that H is a subset of G.
- (00) Since $(1) \in H$ it follows that $H \neq \emptyset$
- (1) and (2) can be checked using Cayley table:

H	(1)	(123)	(132)
(1)	(1)	(123)	(132)
(123)	(123)	(132)	(1)
(132)	(132)	(1)	(123)

- (1) Holds since all the entries in the Cayley table are among the elements of H.
- (2) Each row has identity $e = (1)$. Therefore $(1)^{-1} = (1)$, $(123)^{-1} = (132)$, $(132)^{-1} = (123)$.

28. Let G be a group. Let $g \in G$ be an element of order 2. Prove that $a = a^{-1}$.
Answer/Proof:
- By definition of inverse in a group (property (4) of the definition) and by the proposition that the inverse of each element is unique, the inverse of a is the unique element which when multiplied by a gives the identity e.
- $a^2 = e$ by assumption that order of a is 2.
- $aa = a^2 = e$ implies that a is that unique element that multiplies a to get e. Therefore: - $\boxed{a^{-1} = a}$

29. Let G be a group. Let $g \in G$ be an element of order k. Prove that $a^{-1} = a^{k-1}$.
Answer:

30. Prove that the set of permutations $\{(1), (12)(34), (13)(24), (14)(23)\}$ forms a group under multiplication of permutations.
Answer:

31. Prove that the group $\{(1), (12)(34), (13)(24), (14)(23)\}$ is not cyclic.
Answer:

32. Prove that the set of permutations $\{(1), (12)(34), (13)(24), (14)(23)\}$ is a subgroup of S_4.
Answer:

33. Prove that $A = \begin{bmatrix} [5]_7 & [2]_7 \\ [3]_7 & [6]_7 \end{bmatrix}$ is in $(Gl_2(\mathbb{Z}_7), \cdot)$.
Answer/Proof:
- $(Gl_2(\mathbb{Z}_7), \cdot)$ consists of 2x2 matrices with entries in $\mathbb{Z}_7$ which have determinant $\neq 0$.
- The matrix A is 2x2 with entries in $\mathbb{Z}_7$.
- $detA = [5]_7[6]_7 - [2]_7[6]_7 = [24]_7 = [3]_7 \neq [0]_7$. Therefore:
- $\boxed{A \in (Gl_2(\mathbb{Z}_7), \cdot)}$

34. Let G be a group. Let $Z(G) := \{a \in G \mid ag = ga \text{ for all } g \in G\}$. Prove that $Z(G)$ is a subgroup of G.
Answer:

35. Let $\phi : G \to G'$ be an isomorphism. Prove that G is abelian if and only if G' is abelian.
Answer/Proof:
Done in class.

36. Let $\phi : G \to G'$ be an isomorphism. Let $g \in G$. Prove that $|\phi(g)| = |g|$.
Answer/Proof: First assume that the order of g is finite.
- If $|g| = 0$ then $g = e_G$ and $\phi(g) = \phi(e_G) = e_{G'}$. Hence $|\phi(g)| = 0$. So $|\phi(g)| = |g|$.
- Let $|g| = n > 0$. Then n is the smallest positive integer, so that $g^n = e_G$.
- Let $|\phi(g)| = m$
- $\phi(g^n) = (\phi(g))^n$ since ϕ is a isomorphism (follows from $\phi(ab) = \phi(a)\phi(b)$. Therefore:
- $(\phi(g))^n = \phi(g^n) = \phi(e_G) = e_{G'}$ as shown in class (also using $\phi(ab) = \phi(a)\phi(b)$).
- Since $|\phi(g)| = m$ it follows that $m|n$. (Proved in class: If $|x| = m$ and $x^n = e$ then $m|n$.)
- Then $\phi(g^m) = (\phi(g))^m = e_{G'}$ and $\phi(e_G) = e_{g'}$.
- It follows that $g^m = e_G$ since ϕ is injective (one to one).
- Therefore $m = n$ since n was the smallest positive integer such that $g^n = e_G$. So:
- $\boxed{|\phi(g)| = |g|}$

37. Let $\phi : G \to G'$ be an isomorphism. Prove that G is cyclic if and only if G' is cyclic.
Answer:

True -False - Sometimes

38. True -False - Sometimes

$\boxed{T}$ F S - Let G be a group. Then G has an identity element.

$\boxed{T}$ F S - Let M be a monoid. Then M has an identity element.

T F $\boxed{S}$ - Let S be a semigroup. Then S has an identity element.

$\boxed{T}$ F S - Every group is a monoid.

T $\boxed{F}$ S - Every monoid is a group.

T F $\boxed{S}$ - Let M be a monoid. Then M is a group.

$\boxed{T}$ F S - $(\mathbb{Z}, +)$ is an abelian group.

$\boxed{T}$ F S - $(\mathbb{Z}_n, +_n)$ is an abelian group.

T $\boxed{F}$ S - $(\mathbb{Z}, \cdot)$ is an abelian group.

T $\boxed{F}$ S - $(\mathbb{Z}_n, \cdot_n)$ is an abelian group.

$\boxed{T}$ F S - $(\mathbb{Z}_n^\times, \cdot_n)$ is an abelian group.

$\boxed{T}$ F S - $(\mathbb{Z}_8, \cdot_8)$ is a monoid.

T $\boxed{F}$ S - Identity element in $(\mathbb{Z}, +)$ is 1.

$\boxed{T}$ F S - Identity element in $(\mathbb{Z}, \cdot)$ is 1.

$\boxed{T}$ F S - $(2\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$

$\boxed{T}$ F S - $[6]_{10}$ has an inverse in $(\mathbb{Z}_{10}, +_{10})$

T $\boxed{F}$ S - $[6]_{10}$ has an inverse in $(\mathbb{Z}_{10}, \cdot_{10})$

$\boxed{T}$ F S - Let $f : X \to Y$ and $g : Y \to Z$ be two bijective functions. Then $g \cdot f$ is bijective.

$\boxed{T}$ F S - Let $f : X \to Y$ be a bijective function. Then f has an inverse.

T $\boxed{F}$ S - All subgroups of S_3 are cyclic.

$\boxed{T}$ F S - All proper subgroups of S_3 are cyclic.

T $\boxed{F}$ S - All subgroups of S_4 are cyclic.

T $\boxed{F}$ S - All proper subgroups of S_4 are cyclic.

T $\boxed{F}$ S - $Gl_2(\mathbb{R})$ is a subgroup of the group $M_2(\mathbb{R})$.

T $\boxed{F}$ S - Let G be a group of order $|G| = 5$. Let $g \in G$. Then $|g| = 4$.

T F $\boxed{S}$ - Let G be a group of order $|G| = 15$. Let $g \in G$. Then $|g| = 15$.

T F $\boxed{S}$ - Let G be a group of order $|G| = 15$. Let $g \in G$. Then $|g| = 5$.

T $\boxed{F}$ S - Let G be a group of order $|G| = 15$. Let $g \in G$. Then $|g| = 10$.

T $\boxed{F}$ S - Let G be a group of order $|G| = 15$. Let $H < G$ be a subgroup. Then $|H| = 10$.

T F $\boxed{S}$ - Let G be a group of order $|G| = 15$. Let $H < G$ be a subgroup. Then $|H| = 1$.

$\boxed{T}$ F S - Let $\phi : G \to G'$ be an isomorphism. Then $\phi(e) = e$.

Examples

Justify your answer, i.e. explain/prove that your example is an example of the requested concept!!!
(These solutions are not as detailed as they should be.)

39. Give an example of a semigroup which is not a monoid.
Answer:
- $(2\mathbb{Z}, \cdot)$. These are all even integers.
- Product of any two even integers is again an even integer.
- $(2\mathbb{Z}, \cdot)$ is closed under multiplication.
- Multiplication of integers is associative.Therefore:
- $(2\mathbb{Z}, \cdot)$ is a semigroup.
- $(2\mathbb{Z}, \cdot)$ is not monoid since it does not have identity for multiplication since $1 \notin (2\mathbb{Z}, \cdot)$.

40. Give an example of a semigroup which is a monoid.
Answer:

41. Give an example of a group which is not abelian.
Answer:

42. Give an example of a semigroup which is a monoid.
Answer:

43. Give an example of a monoid and an element which has an inverse and an element which does not have an inverse.
Answer:

44. Consider $(M_2(\mathbb{R}), \cdot)$. Give an example of a matrix which has an inverse and find the inverse.
Answer:
$A = \begin{bmatrix} 1 & 0 \\ 3 & 6 \end{bmatrix}$. Notice, operation is multiplication. Find multiplicative inverse.

$A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{-3}{6} & \frac{1}{6} \end{bmatrix}$

45. Consider $(M_2(\mathbb{R}), +)$. Give an example of a matrix which has an inverse and find the inverse.
Answer:
$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$. Additive inverse is $-A = \begin{bmatrix} -1 & -1 \\ -3 & -3 \end{bmatrix}$.

46. Consider $(Gl_2(\mathbb{R}), \cdot)$. Give an example of a matrix which has an inverse and find the inverse.
Answer:
$A = \begin{bmatrix} 1 & 0 \\ 3 & 6 \end{bmatrix}$. Multiplicative inverse is $A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{-3}{6} & \frac{1}{6} \end{bmatrix}$.

47. Give an example of a matrix in $(Sl_2(\mathbb{R}), \cdot)$.
Answer:
$A = \begin{bmatrix} 1 & -1 \\ -5 & 6 \end{bmatrix}$. Check: $detA = 1 \cdot 6 - (-1) \cdot (-5) = 1$.

48. Give an example of a matrix in $(Gl_2(\mathbb{Z}_5), \cdot)$.
Answer:
$A = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}$. Check: $detA = 3 \cdot 4 = 12 \equiv 2 \ (mod\ 5)$.

49. Give an example of a matrix in $(Gl_2(\mathbb{Z}_5), \cdot)$ which is not in $(Sl_2(\mathbb{Z}_5), \cdot)$.
Answer:

50. Give an example of a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ and its inverse.
Answer:

51. Give an example of a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ which has order 5.
Answer:

52. Give an example of a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ which has order 12.
Answer:
$\alpha = (1467)(235)$ Then $|\alpha| = |(1467)(235)| = lcm(4, 3) = 12$.

53. Give an example of a permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ which fixes 3, 4, 5 and 6.
Answer:
$\alpha = (127)$ or $\beta = (17)$

54. Give an example of a permutation of $\{1,2,3,4,5,6,7\}$ which does not fix any element of $\{1,2,3,4,5,6\}$.
Answer:
$\alpha = (1345)(26)$ or $\beta = (13)(45)(26)$ or $\gamma = (1,2,3,4,5,6,7)$

55. Give an example of a function $f : \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$ which is not a permutation.
Answer:
$f(1) = 1,\ f(2) = 1,\ f(3) = 1,\ f(4) = 6,\ f(5) = 1,\ f(6) = 1.$

56. Give an example of a group and a subgroup.
Answer:

57. Give an example of a group and a subset which is not a subgroup.
Answer:
$G = S_4,\ H = \{(1),(12),(23)\}$ or
$G = S_4,\ H = \{(12),(23),(123)\}$

58. Give an example of $f : \{1,2,3,4,5,6\} \to \mathbb{R}$ which is not a function.
Answer:
$f(1) = 1,\ f(1) = 2,\ f(3) = 1,\ f(4) = 6,\ f(5) = 1,\ f(6) = 1.$ or
$g(1) = 1,\ g(2) = 2,\ g(3) = 1,\ g(4) = 6,\ g(5) = 1.$

59. Give an example of $f : \mathbb{R} \to \mathbb{R}$ which is not a function.
Answer:
$f(x) = \pm\sqrt{x}$

60. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is not surjective.
Answer:

61. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is surjective.
Answer:

62. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is not injective.
Answer:

63. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is injective.
    Answer:

64. Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is not an isomorphism of groups.
    Answer:
    $f(x) = x + 3$ or $g(x) = x^2$ or $h(x) = 5$

65. Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is an isomorphism of groups.
    Answer:
    $f(x) = x$ or $g(x) = 2x$ or $h(x) = 5x$

66. Give an example of a function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ which is not an isomorphism of groups.
    Answer:
    $f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 0$

67. Give an example of a function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ which is an isomorphism of groups.
    Answer:
    $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 4$ or
    $f(0) = 0$, $f(1) = 2$, $f(2) = 4$, $f(3) = 1$, $f(4) = 3$ or
    $f(0) = 0$, $f(1) = 3$, $f(2) = 1$, $f(3) = 4$, $f(4) = 2$

68. Give an example of a function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ which is an isomorphism of groups but $f(1) \neq 1$.
    Answer:
    $f(0) = 0$, $f(1) = 3$, $f(2) = 1$, $f(3) = 4$, $f(4) = 2$