Also, do the assigned HW problems. This is just in addition to HW.

ALWAYS JUSTIFY YOUR ANSWER!

Computations

1. Describe all abelian groups of order 243 (up to isomorphism) as direct sums of cyclic groups of prime power order as $\mathbb{Z}_{p^{n_1}} \oplus \mathbb{Z}_{p^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{n_s}}$.
2. Describe all abelian groups of order 360 (up to isomorphism) as direct sum of cyclic groups of prime power order (notice different primes).
3. Describe all abelian groups of order 360 up to isomorphism using cyclic decomposition as: $\mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_t}$ where $m_2|m_1, \quad m_3|m_2, \quad \ldots\ , m_t|m_{t-1}$.
4. Let $G = \mathbb{Z}_{81} \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_3$.
   (a) Describe all elements of order 81.
   (b) Describe all elements of order 27.
   (c) Describe all elements of order 9.
   (d) Describe all elements of order 3.
5. Let $G = \mathbb{Z}_8 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_5$.
   (a) Describe all elements of order 8.
   (b) Describe all elements of order 24.
   (c) Describe all elements of order 5.
   (d) Describe all elements of order 120.
   (e) Describe all elements of order 1.
6. Let $G = \mathbb{D}_6 = \langle s, r \mid |s| = 2, |r| = 6, srs = r^5 = r^{-1} \rangle$.
   (a) Find the possible numbers n_p of Sylow p-subgroups for each prime $p||G|$.
   (b) For each prime $p||G|$ describe all Sylow p-subgroups?
   (c) For each prime $p||G|$ show explicitly how all Sylow p-subgroups are conjugate?
7. Let $G = \mathbb{D}_5 = \langle s, r \mid |s| = 2, |r| = 5, srs = r^4 = r^{-1} \rangle$.
   (a) Find the possible numbers n_p of Sylow p-subgroups for each prime $p||G|$.
   (b) For each prime $p||G|$ describe all Sylow p-subgroups?
   (c) For each prime $p||G|$ show explicitly how all Sylow p-subgroups are conjugate?
8. Let $G = S_4$.

(a) Find the possible numbers n_p of Sylow p-subgroups for each prime $p||G|$.
(b) For each prime $p||G|$ describe all Sylow p-subgroups?
(c) For each prime $p||G|$ show explicitly how all Sylow p-subgroups are conjugate?

9. Let $G = \mathbb{Z}_{15}^{\times}$.
   (a) Find the possible numbers n_p of Sylow p-subgroups for each prime $p||G|$.
   (b) For each prime $p||G|$ describe all Sylow p-subgroups?
   (c) For each prime $p||G|$ show explicitly how all Sylow p-subgroups are conjugate?

10. Let $G = \mathbb{Z}_{36}^{\times}$.
    (a) Find the possible numbers n_p of Sylow p-subgroups for each prime $p||G|$.
    (b) For each prime $p||G|$ describe all Sylow p-subgroups?
    (c) For each prime $p||G|$ show explicitly how all Sylow p-subgroups are conjugate?

Theoretic Questions

11. Write the definition of *p-subgroup.*
12. Write the definition of *Sylow p-subgroup.*

Proofs

13. Let G be a group of order $|G| = 33$. Prove that G has a normal subgroup.
14. Let G be a group of order $|G| = 21$. Prove that G has a normal subgroup.
15. Let G be a group of order $|G| = 2p$ where $p \neq 2$ is prime. Prove that G is not simple.
16. Let G be a group of order $|G| = 56$. Prove that G is not simple.
17. Let G be a group of order $|G| = 125$. Prove that the center $Z(G)$ of G has at least 5 elements.
18. Let G be a group of order $|G| = 125$. Prove that G is not simple.
19. Prove that abelian group of order 55 must be cyclic.
20. Prove that $\mathbb{Z}_8 \times \mathbb{Z}_5 \cong \mathbb{Z}_{40}$.
21. Prove that every group of order 5 is cyclic.
22. Prove that every group of prime order is cyclic.
23. Prove that every group of prime order p is isomorphic to $\mathbb{Z}_p$.

24. Prove that every group of order 4 is either cyclic or isomorphic to Klein Four Group.

25. Prove that every group of order 4 is isomorphic either to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

True -False - Sometimes

26. True -False - Sometimes

    T F S - Let P_p be a Sylow p-subgroup of G. Then P_p is normal subgroup.

    T F S - Let $G = (\mathbb{Z}_n, +_n)$, let P_p be a Sylow p-subgroup of G. Then P_p is normal subgroup.

    T F S - Let G be a group with $|G| = 150$. Let P_2 be a Sylow 2-subgroup of G. Then $|P_2| = 2$.

    T F S - Let G be a group with $|G| = 150$. Let P_5 be a Sylow 5-subgroup of G. Then $|P_5| = 5$.

    T F S -Let G be a group with $|G| = 150$. Let P_5 be a Sylow 5-subgroup of G. Then $|P_5| = 25$.

    T F S - $\mathbb{Z}_4 \times \mathbb{Z}_5 \cong \mathbb{Z}_{20}$.

    T F S - $\mathbb{Z}_4 \times \mathbb{Z}_2 \cong \mathbb{Z}_8$.

    T F S - $\mathbb{Z}_4 \times \mathbb{Z}_4 \cong \mathbb{Z}_4$.

    T F S - $\mathbb{Z}_4 \times \mathbb{Z}_4 \cong \mathbb{Z}_{16}$.

    T F S - $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_4$.

    T F S - $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong K$, the Klein Four Group.

Examples

27. Give an example of a group G and a p-subgroup of G which is not Sylow p-subgroup of G.

28. Give an example of a group G and a p-subgroup of G which is Sylow p-subgroup of G.

29. Give an example of a group G and a subgroup of G which is not a p-subgroup of G.

30. Consider the Klein Four Group:

$$K = \{e, a, b, c \mid a^2 = b^2 = c^2 = e, ab = ba = c, bc = cb = a, ac = ca = b\}.$$

    (a) Make the Cayley table for K.
    (b) Make the Cayley table for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
    (c) Write an explicit isomorphism $f : K \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$.