1. Primitive Pythagorean Triple
   (a) Definition
   a triple of numbers(a,b,c) such that a,b,c have no common factors and $a^2+b^2=c^2$
   (b) PPT can be expressed as $a=st, b=\frac{s^2-t^2}{2}, c=\frac{s^2+t^2}{2}, s>t\geq 1$, gcd(s,t) = 1, s and t are odd.
   Thm1: a and b cannot both be odd

   Proof. suppose they are all odd
   a = 2k+1, b = 2p+1, c = 2z
   $a^2+b^2=4(k^2+k+p^2+p)+2, c^2=4z^2$
   $a^2+b^2\equiv 2 \pmod 4$, $4|c^2$, contradits.
   So a and b cannot both be odd. □

   Thm2: Suppose a ← odd b ← even c ← odd, $a^2=c^2-b^2$, a,b,c are coprime. we want to prove (c-b)(c+b) are coprime.

   Proof. Then $a^2=(c-b)(c+b)$
   Suppose they are not coprime(prove by contradict)
   Then There exist a prime p s.t. $p|c-b$ or $p|c+b$
   $p|(c+b)+(c-b)\Rightarrow p|2c$, $p|(c+b)-(c-b)\Rightarrow p|2b$
   Since $p|(c-b)(c+b)\Rightarrow p|a^2\Rightarrow p|a\Rightarrow$ p is odd. Then $p|c$ & $p|b$, contradicts with c, b are coprime.
   $a^2=(c-b)(c+b)$ and (c-b), (c+b) are coprime. □

   Thm3: x,y are coprime, $a^2=xy\Rightarrow$ x, y are perfect squares. Can be proved by Fundmental Theorem of Arithmetic.
   We express $c-b=s^2$, $c+b=t^2$, then $a=st, b=\frac{s^2-t^2}{2}, c=\frac{s^2+t^2}{2}$

2. Fermat's Last Theorem
   If $n\in\mathbf{N}$, $n\geq 3$, $x^n+y^n=z^n$ has no natural number solutions

3. Euclidean algorithm
   Suppose $A,B\in\mathbf{N}$ There exist unique Q and R such that $Q\in\mathbf{N}, R\in\mathbf{N}, A=QB+R$. Then gcd(A,B) = gcd(B,R)
   Proof of Correctness:

   Proof. Let $d=gcd(A,B), d_0=gcd(B,R)$
   On one hand
   $\Rightarrow d|A, d|B\Rightarrow d|R$
   $d|B, d|R\Rightarrow d\leq d_0$
   On the other hand
   $d_0|B, d_0|R\Rightarrow d_0|A\Rightarrow d_0\leq d$
   In all: $d=d_0$ i.e. gcd(A,B) = gcd(B,R) □

   Thm $LCM(a,b)gcd(a,b)=ab$

   Proof. let d = gcd(a,b)
   we need to find LCM(a,b) by find the smallest LCM(a,b) = ja = kb.
   Then $j\frac{a}{d}=k\frac{b}{d}$
   $\Rightarrow \frac{a}{d}|\frac{b}{d}k$
   since gcd($\frac{a}{d},\frac{b}{d}$)=1
   $\Rightarrow \frac{a}{d}|k$
   smallest k = $\frac{a}{d}$
   Same process, we get j = $\frac{b}{d}$
   Then LCM = ja = $\frac{ab}{d}\Rightarrow LCM(a,b)\cdot gcd(a,b)=ab$ □

4. Linear Equations
   we can use Euclidean algorithm to get a Linear Equation: $ax_0+by_0=gcd(a,b)$. Thus, we can find a solution to $ax+by=n$ iff $gcd(a,b)|n$
   Thm1: gcd(m,n) = 1, $m|nc\Rightarrow m|c$

   Proof. $\exists x_0,y_0$ s.t. $mx_0+ny_0=1$
   Then $mx_0c+ny_0c=c$
   $m|nc\Rightarrow m|(mx_0c+ny_0c)\Rightarrow m|c$ □

   Thm2: suppose p is prime, $p|ab\Rightarrow p|a$ or $p|b$

   Proof. if $p|a$, this is true. if $p\nmid a$, then gcd(p,a) = 1 since p is prime.
   Then $p|ab\Rightarrow p|b$ □

   Thm3: all solutions to $ax+by=gcd(a,b)$
   we can find $ax_0+by_0=gcd(a,b)$ by Euclidean algorithm.
   Then we have $ax+by=ax_0+by_0=gcd(a,b)$
   Then $a(x_0-x)+b(y_0-y)=0\Rightarrow a(x_0-x)=b(y-y_0)$
   Divides both sides by gcd(a,b)
   $\frac{a(x_0-x)}{gcd(a,b)}=\frac{b(y-y_0)}{gcd(a,b)}$
   $\Rightarrow \frac{a}{gcd(a,b)}|\frac{b}{gcd(a,b)}\cdot(y-y_0)$
   since $\frac{a}{gcd(a,b)},\frac{b}{gcd(a,b)}$ are co-prime $\Rightarrow \frac{a}{gcd(a,b)}|y-y_0$
   $y=y_0+k\frac{a}{gcd(a,b)}$
   similarly, $x=x_0-k\frac{b}{gcd(a,b)}$, where k are the same.

5. Fundmental Theorem of Arithmetic
   For all $n\in\mathbf{N}$ where $n\geq 2$, n factors as a product of prime numbers, and does so in a unique way.

6. Chapter 8 Congruences
   Properties:
   · $a\equiv a$ (mod m)
   · $a\equiv b$ (mod m) $\Rightarrow b\equiv a$ (mod m)
   · $a\equiv b$ (mod m), $b\equiv c$ (mod m) $\Rightarrow a\equiv c$ (mod m)
   · if $a\equiv b$ (mod m), $c\equiv d$ (mod m) $\Rightarrow$
   $a+c\equiv b+d$ (mod m)
   $a-c\equiv b-d$ (mod m)
   $ac\equiv bd$ (mod m)

   Proof. $m|a-b, m|c-d\Rightarrow m|ac-bc, m|bc-bd\Rightarrow$
   $m|ac-bc+bc-bd\Rightarrow m|ac-bd\Rightarrow ac\equiv bd$(mod m) □

   · gcd(m,c) = 1, $ca\equiv cb$ (mod m)$\Rightarrow a\equiv b$ (mod m)

   Proof. $m|ca-cb\Rightarrow m|c(a-d)$ since gcd(m,c) = 1, they are coprime$\Rightarrow m|(a-b)\Rightarrow a\equiv b$(mod m) □

7. Fermat's Little Theorem
   If p is prime, and $p\nmid a$ then $a^{p-1}\equiv 1$ (mod p)

8. Euler's phi function:
   $\phi:\mathbf{N}\to\mathbf{N}, \phi=\#\{a|1\leq a\leq m, gcd(a,m)=1\}$
   Properties:
   (a) For prime p: $\phi(p)=p-1$
   (b) If gcd(m,n) = 1, $\phi(mn)=\phi(m)\cdot\phi(n)$

(c) For prime p: $\phi(p^k) = p^k - p^{k-1}$

(d) For number $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$
$\phi(n) = (p_1^{\alpha_1} - p_1^{\alpha_1 - 1})(p_2^{\alpha_2} - p_2^{\alpha_2 - 1}) \dots (p_k^{\alpha_k} - p_k^{\alpha_k - 1})$
$= n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \dots (1 - \frac{1}{p_k})$

9. Euler's Phi Formula:
If gcd(a,m) = 1, $a^{\phi(m)} \equiv 1$(mod m)

Proof. Suppose gcd(a,m) = 1. $b_n, 1 \le n \le \phi(m)$ represents all numbers that are co-prime to m.
Consider A = $ab_1, ab_2, ab_3, \dots, ab_{\phi(m)}$ (mod m) and B = $b_1, b_2, b_3, \dots, b_{\phi(m)}$ (mod m). They have the same number of elements. If all elements in A are congruent to different number mod m, two set are the same.
We prove by contradiction, suppose $ab_i \equiv ab_j$ (mod m)$\Rightarrow m | a(b_i - b_j) \Rightarrow m | (b_i - b_j) \Rightarrow b_i \equiv b_j$ contradicts!
Then $b_1 b_2 \dots b_{\phi(m)} \equiv ab_1 ab_2 \dots ab_{\phi(m)}$(mod m)
$\Rightarrow \prod_{i=1}^{\phi(m)} b_i \equiv a^{\phi(m)} \prod_{i=1}^{\phi(m)} b_i$ (mod m)
since $b_i's$ are coprime to m, $\prod_{i=1}^{\phi(m)} b_i$ are coprime to m.
$\Rightarrow 1 \equiv a^{\phi(m)}$ (mod m) □

10. Chinese Remainder Theorem
If gcd(m,n) = 1, let $b, c \in \mathbf{Z}$. Then there exist a solution to the simultaneous congruence:

$$\begin{cases} x \equiv b \ \ mod \ m \\ x \equiv c \ \ mod \ m \end{cases} \tag{1}$$

and such a solution is unique modulo mn

Proof. Existence:
$m | x - b, n | x - c \Rightarrow m\alpha = x - b, m\beta = x - c$
$\Rightarrow m\alpha - n\beta = c - b$ since m,n are coprime
$\Rightarrow$ we can find solution α_0, β_0 such that $m\alpha_0 - n\beta_0 = 1$
$\Rightarrow m\alpha_0(c - b) - n\beta_0(c - b) = c - b$
We get $x = m\alpha_0(c - b) + b$
Uniqueness:
suppose that there are two solution x_0, x_1
$\Rightarrow x_0 \equiv x_1 \equiv b$ (mod m) $x_0 \equiv x_1 \equiv c$ (mod n)
$\Rightarrow m | (x_1 - x_0), n | (x_1 - x_0) \Rightarrow mn | x_1 - x_0$
$\Rightarrow x_1 \equiv x_0$ (mod mn) □

11. Solving congruences functions

(a) $x^2 \equiv k^2$ (mod p), p is prime
$p | x^2 - k^2 \Rightarrow p | (x - k)(x + k)$
$\Rightarrow x \equiv k$ (mod m) or $x \equiv -k$(mod m)

(b) $a^k \equiv 1$ (mod m), gcd(m,a) = 1
Use Euler's Phi Formula to decrease k.

(c) $ax \equiv c$ (mod m), $gcd(a, m) | c$
There is no solution if $gcd(a, m) \nmid c$
$m | ax - c \Rightarrow ym = ax - c \Rightarrow c = ax - ym$
Find an x_0 suits the function by Euclidean Algorithm
Then $ax_0 \equiv c$(mod m). We want to find all x.
Then $ax_0 = ax$ (mod m)
$m | a(x - x_0) \Rightarrow \frac{m}{gcd(m,a)} | \frac{a}{gcd(m,a)}(x - x_0)$
$gcd(\frac{m}{gcd(m,a)}, \frac{a}{gcd(m,a)}) = 1 \Rightarrow \frac{m}{gcd(m,a)} | x - x_0$
$\Rightarrow x = x_0 + k\frac{m}{gcd(m,a)}$

(d) $x \equiv b$ (mod m), $x \equiv c$ (mod m), gcd(m,n) = 1
Use Chinese Remainder Theorm's proof

12. Dirichlet's Theorem
If gcd(a,b) = 1, there are infinite number of prime of the form $an + b$.
In book we have primes 3 (mod 4) Theorem, which is a special case of Dirichlet's Theorem.
primes 3 (mod 4) Theorem:
There are infinite number of primes of the form $4n + 3$

Proof. Prove by Contradiction
Suppose there are finite number of P $\{3, p_1, p_2, p_3 ... p_n\}$in 4n+3 form.
Consider $A = 4p_1 p_2 \dots p_n + 3$
Since A can be factored to product of primes, $A = q_1 q_2 q_3 \cdots q_k$
$q_i \equiv 1, 3$(mod 4) since primes are odd. At least one of the $q_i \equiv 3$ (mod 4)
If $q_i = 3$, $3 | q_1 q_2 \dots q_k \Rightarrow 3 | 4p_1 p_2 \dots p_n + 3 \Rightarrow 3 | 4p_1 p_2 \dots p_n$
Since $3 \nmid 4$, we have $3 | p_1 p_2 \dots p_n$ which contradicts.($p_i's$ are primes bigger than 3)
If q_i is one of $\{p_1, p_2 \dots p_n\}$
$q_i | A \Rightarrow q_i | 4p_1 p_2 \dots p_n + 3$
Since $q_i | p_1 p_2 \dots p_n$, $q_i | 4p_1 p_2 \dots p_n$
Then $q_i | 3$. However, q_i is prime and should not divide 3, contradicts.
Therefore, q_i is a new prime of the form $4n + 3$.
Therefore, there are infinite number of primes of the form $4n + 3$ □

13. Prime Number Theorem:

$$\lim_{n \to \infty} \frac{\pi(n)}{\frac{n}{lnn}} = 1$$

where $pi(n) :=$ # of prime numbers $\le$ n

14. Mersenne Primes
Def: primes have the form of $a^n - 1$
Look at geometic series: $S = 1 + x + x^2 + \cdots + x^{n-1}, x > 1, x \in \mathbf{N}$
$S = \frac{x^n - 1}{x - 1}$. So when $x > 2$, $x^n - 1$ is composite.When n is composite, write n as pq. Thus, $x^p - 1 | x^{p^q} - 1 = x^n$, i.e x^n is composite.
Thus, Mersenne Primes have the form $2^p - 1$

15. Perfect Numbers
Def: a number equal to the sum of its proper divisors.
$\sigma(n) = \sum_{d \ge 1, d | n} d$ (sum of all the divisors)
$\hat{\sigma}(n) = \sigma(n) - n$, n is perfect iff $\sigma(n) = 2n$
Property of $\sigma(n)$:

(a) If P is prime, $\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1}$
(b) $gcd(m, n) = 1$, $\sigma(mn) = \sigma(m)\sigma(n)$

All even perfect number have the form:$n = 2^{p-1}(2^p - 1)$

Proof. n is even, write n as $2^a m$, $a \ge 1$, m is odd.
gcd(m, 2^a) = 1 $\rightarrow \sigma(2^a m) = \sigma(2^a)\sigma(m) = (2^{a+1} - 1)\sigma(m)$
since n is perfect number, $\sigma(n) = 2n = 2^{a+1} m$
$\rightarrow 2^{a+1} m = (2^{a+1} - 1)\sigma(m)$
$\rightarrow 2^{a+1} m = (2^{a+1} - 1)(\hat{\sigma}(m) + m)$
$\rightarrow 0 = 2^{a+1}\hat{\sigma}(m) - \hat{\sigma}(m) - m$

$\rightarrow m = (2^{a+1} - 1)\hat{\sigma}(m)$
since $a > 1$, $\hat{\sigma}(m)|m, m > \hat{\sigma}(m)$
so $\hat{\sigma}(m)$ is a proper divisor of m and $\hat{\sigma}(m) = \sum d$
$\rightarrow \hat{\sigma}(m) = 1$: m has only one divsor that is smaller than itself: 1$\rightarrow$ m is a prime.
$m = 2^{a+1} - 1$, m is a Mersenne Prime. m has the form of $2^p - 1$ and $n = 2^{p-1}(2^p - 1)$ $\square$