1. Basic ODEs

- (a) separatable: $y' = \frac{F(x)}{G(y)}$ $\Rightarrow \frac{dy}{dx} = \frac{F(x)}{G(y)} \Rightarrow \int G(y) dy = \int F(x) dx$
- (b) Linear: y' + p(x)y = q(x)Integrating Factor: $e^{\int p(x)dx}$ $\Rightarrow (e^{\int p(x)dx}y)' = e^{\int p(x)dx}q(x)$ $\Rightarrow e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx$
- (c) ay'' + by' + cy = 0 (constant coefficient) characteristc EQ: $ar^2 + br + c = 0$ $\Delta > 0$, $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ $\Delta = 0, y = C_1 e^{rx} + C_2 x e^{rx}$ $\Delta < 0, r = p \pm qi : y = e^{px} [c_1 cos(qx) + c_2 sin(qx)]$

2. Fourier Series

Given F(x), $x \in [-L, L]$ write F(x) in a series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{L})$$

where a_n , b_n are constants.

(a) Orthogonality Relations

$$\int_{-L}^{L} sin(\frac{n\pi x}{L})cos(\frac{m\pi x}{L}) = 0$$

$$\int_{-L}^{L} cos(\frac{n\pi x}{L})cos(\frac{m\pi x}{L}) = 0 (m \neq n), L(m = n)$$

$$\int_{-L}^{L} sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L}) = 0 (m \neq n), L(m = n)$$

- (b) $a_n = \frac{1}{L} \int_{-L}^{L} F(x) cos(\frac{n\pi x}{L}) dx$ $b_n = \frac{1}{\tau} \int_{-\tau}^{L} F(x) \sin(\frac{n\pi x}{\tau}) dx, n \in [0, \infty], n \in \mathbf{Z}$
- (c) Convergece Statement of F.S. F.S. convergence to the "periodic extension" of F(x)whever F(x) is continous and to the average of $\frac{f(x^+)+f(x^-)}{2}$ at every point.
- (d) F.S.S and F.C.S of F(x) on [0,L]: F.C.S = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$, $a_n = \frac{2}{L} \int_0^L F(x) \cos \left(\frac{n\pi x}{L}\right) dx$ F.S.S = $\sum_{n=1}^{\infty} b_n sin \frac{n\pi x}{L}$, $a_n = \frac{2}{L} \int_0^L F(x) cos(\frac{n\pi x}{L}) dx$ F.C.S \rightarrow even extension, F.S.S \rightarrow odd extension

3. Sturm-Liouville Problem

(a) Basic Examples of S-L BVP

 $f'' + \lambda f = 0, 0 \le x \le L$ and Boundary conditions Def: value λ for which the equation with the given boundary ends: has non-trival solution is called eigen value, the corresponding solution is called eigeon functions of the given S-L BVP.

First, general solutions:

Then, impose the boundary in each case.
$$\lambda = 0, f(x) = \alpha x + \beta$$

$$\lambda > 0, f(x) = C_1 cos(\sqrt{\lambda}x) + C_2 sin(\sqrt{\lambda}x)$$

$$\lambda < 0, f(x) = C_1 cosh(ax) + C_2 sinh(ax), a^2 = -\lambda, a > 0$$
Then, impose the boundary in each case.

i. Boundary COND: f(0) = 0, f(L) = 0e-values: $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 1, 2, 3$ e-functions: $f_n \sim sin(\frac{n\pi x}{L})$

- ii. Boundary COND: f'(0) = 0, f'(L) = 0e-values: $\lambda_n = \frac{n^2 \pi^2}{L^2}$, n = 0, 1, 2, 3e-functions: $f_n \sim cos(\frac{n\pi x}{I})$
- iii. Boundary COND: f(0) = 0, f'(L) = 0e-values: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2, n = 1, 2, 3$ e-functions: $f_n \sim sin(\frac{(2n-1)\pi}{2L}x)$
- iv. Boundary COND: f'(0) = 0, f(L) = 0e-values: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2, n = 1, 2, 3$ e-functions: $f_n \sim cos(\frac{(2n-1)\pi}{2L}x)$
- (b) Regular S-L Problems

EQ:
$$(pf')' + qf + \lambda \sigma f = 0, a < x < b$$

Boundary: $k_1 f(a) + k_2 f'(a) = 0, k_3 f(b) + k_4 f'(b) = 0$

Boundary:
$$k_1 f(a) + k_2 f'(a) = 0, k_3 f(b) + k_4 f'(b) = 0$$

$$g(x) \sim \sum_{n=1}^{\infty} a_n f_n(x)$$

$$a_n = \frac{\int_a^b g(x) f_n(x) \sigma(x) dx}{\int_a^b \sigma(x) f_n^2(x) dx}$$

- 4. Method of Separationg: Heat and Wave Equation
 - (a) Heat Equation

 $\mu_t = k\mu_{xx}$, $\mu(x, t) = X(x)T(t)$ (usually k = 1) Separation $\rightarrow X(x)T'(t) = kX''(x)T(t)$

Let
$$\frac{T'}{kT} = \frac{kX''}{kT} = -\lambda$$

Let $\frac{T'}{kT} = \frac{kX''}{x} = -\lambda$ We get $X'' + \lambda x = 0$, $T' + k\lambda T = 0$ For second EQ, $T \sim e^{-k\lambda t}$

For first EQ, we apply S-L BVP problem

$$\mu(x,t) = \sum_{n=1}^{\infty} c_n f_n e^{-\lambda_n t}$$

 $\mu(x,t) = \sum_{n=1}^{\infty} c_n f_n e^{-\lambda_n t}$ Usually, we find the bound. cond. in the 4 fourier series probs.

e.g. bond cond 1:

$$\begin{array}{l} \lambda_n = (\frac{n\pi}{L})^2, \, f_n \sim sin(\frac{n\pi x}{L}) \\ \mu(x,0) = g(x) = \sum_{n=1}^{\infty} c_n sin(\frac{n\pi x}{L}) \end{array}$$

Fouries formal solution: $\mu(x,t) = \sum_{n=1}^{\infty} c_n sin(\frac{n\pi x}{L}) e^{-\frac{n^2 \pi^2}{L^2} t}$ note: in bond cond 2, n starts from 0. calculate case 0 seperately.

(b) Wave Equation

1

$$\mu_{tt} = C^2 \mu_{xx}, \mu(x,t) = X(x)T(t), \text{ usually } C = 1$$

 $\mu(0,t) = 0, \mu(L,t) = 0$

Separation
$$\rightarrow XT'' = C^2X''T$$

Let
$$\frac{T''}{2\pi} = \frac{X''}{\pi} = -\lambda$$

Separation
$$\rightarrow XT'' = C^2X''T$$

Let $\frac{T''}{c^2T} = \frac{X''}{x} = -\lambda$
We get $X'' + \lambda x = 0$, $T'' = -\lambda c^2T$

$$\mu(x,t) = \sum_{n=1}^{\infty} (b_{1,n} \cos \frac{n\pi ct}{L} + b_{2,n} \sin \frac{n\pi ct}{L}) \sin \frac{n\pi x}{L}$$

i.
$$\mu(x,0) = f(x)$$

 $f(x) = \sum_{n=1}^{\infty} b_{1,n} \sin \frac{n\pi x}{L}$
 $b_{1,n} = \frac{\int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx}{\int_{0}^{L} \sin^{2} \frac{n\pi x}{L}}$

ii.
$$\mu_t(x,0) = g(x)$$

 $g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_{2,n} \sin \frac{n\pi x}{L}$
 $\frac{n\pi c}{L} b_{2,n} = \frac{\int_0^L g(x) \sin \frac{n\pi x}{L} dx}{\int_0^L \sin \frac{n\pi x}{L}}$

(c) d'Alembert's solution to the Wave Equation Given $\mu(x,0) = F(x), \mu_t(x,0) = 0$ we want to get $\mu(x,t)$ Do an odd extention on F(x), then $\mu(x,t) =$ $\frac{F(x+ct)+F(x-ct)}{2}$. Usually c is 1. Simply draw the graph of F(x+ct)(shift to left) and F(x-ct)(shift to right) and get the average.

5. Laplace Equation

Consider the equilibrium temperature in a uniform rectangle(Heat EQ)

$$\begin{cases} \mu_{xx}(x,y) + \mu_{y}y(x,y) = 0, \ 0 < x < L, \ 0 < y < K \\ \mu(0,y) = f_{1}(y), \ \mu(L,y) = f_{2}(y), \ 0 < y < K \\ \mu(x,0) = g_{1}(x), \ \mu(x,K) = g_{2}(x), \ 0 < x < L \end{cases}$$
 (1)

Where we might have first order partial derivatives on those μ 's on 2nd or 3rd line.

Separate $\rightarrow \mu = XY$, then

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

we want to create a SL-BVP problem:

Determine $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$ or $-\lambda$

When $f_1(y) = f_2(y) = 0$ we can form SL-BVP on Y and we choose $-\lambda$. Similarly, when $g_1(x) = g_2(x) = 0$ we can form SL-BVP on X and we choose λ

Case1: $-\lambda$:(X(0) or X'(0) = 0, X(L) or X'(L) = 0)

$$\begin{cases} X'' + \lambda X = 0, & (SL - BVP) \\ \mu(x, 0) = f_1(y), \mu(x, K) = f_2(y) \end{cases}$$
 (2)

Get λ_n and X_n from SL-BVP.

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi y}{L} + \beta_n \cosh \frac{n\pi y}{L}) X_n$$

For simplicity, we transform to:

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi y}{L} + \beta_n \sinh \frac{n\pi (y-K)}{L}) X_n$$

Then we get $\mu(x,0)$, $\mu(x,K)$ (as long as they fit the other 2 equations) and try to figure out α_n , β_n using match or SL-BVP. **Note**: in bond cond 2, n starts from 0. calc case 0 seperately.

Case2: λ :(Y(0) or Y'(0) = 0, Y(K) or Y'(K) = 0)

$$\begin{cases} Y'' + \lambda X = 0, & (SL - BVP) \\ \mu(0, y) = g_1(y), \mu(L, y) = g_2(x) \end{cases}$$
 (3)

Get λ_n and Y_n from SL-BVP.

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi x}{K} + \beta_n \cosh \frac{n\pi x}{K}) Y_n$$

For simplicity, we transform to:

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi x}{K} + \beta_n \cosh \frac{n\pi (x-L)}{K}) Y_n$$

Then we get $\mu(0,y)$, $\mu(L,y)$ (as long as they fit the other 2 equations) and try to figure out α_n , β_n using match or SL-BVP. **Note**: in bond cond 2, n starts from 0. calc case 0 seperately.

6. Method of Eigenfunction Expansion Consider

$$PDE: \mu_t = k\mu_{xx}(x, t) + q(x, t), 0 < x < L, t > 0$$

$$BCs: \mu(0,t) = 0, \mu(L,t) = 0, t > 0$$

$$IC : \mu(x, 0) = f(x), 0 < x < L$$

(a) BCs
$$\Rightarrow$$
 Eigenfunction: $X_n(x)$, λ_n
 $\Rightarrow \mu(x,t) = \sum_{n=1}^{\infty} C_n(t) X_n(x)$
Write $q(x,t) = \sum_{n=1}^{\infty} q_n(t) X_n(x)$

(b) PDE

$$\Rightarrow \sum_{n=1}^{\infty} C'_n(t)X_n(x) = \sum_{n=1}^{\infty} C_n(t)X''_n(x) + \sum_{n=1}^{\infty} q_n(t)X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} C'_n(t)X_n(x) = -\sum_{n=1}^{\infty} C_n(t)\lambda_n X_n(x) + \sum_{n=1}^{\infty} q_n(t)X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C'_n(t) + C_n(t)\lambda_n]X_n(x) = \sum_{n=1}^{\infty} q_n(t)X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C'_n(t) + C_n(t)\lambda_n] = \sum_{n=1}^{\infty} q_n(t)$$

Form may vary (μ is differentiated in other ways)

- (c) IC $\Rightarrow \mu(x,0) = f(x) = \sum_{n=1}^{\infty} C_n(0) X_n(x)$ (SL-BVP) either match up or $C_n(0) = \frac{\int_0^L f(x) X_n(x) dx}{\int_0^L X_n^2(x) dx}$
- (d) Then we solve functions $C'_n(t) + C_n(t)\lambda_n = q_n(t)$ for each n with initial condition $C_n(0)$'s and $q_n(0)$'s
- 7. Laplace Transform

Def:
$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$Def: H_a(t) = H(t - a) = \begin{cases} 1, & t \ge a \\ 0, & t < a \end{cases}$$
 (4)

$f(t) = \mathcal{L}^{-1}F(t)$	$F(t) = \mathcal{L}f(s)$
$f^{(n)}(t)$ (nth derivative)	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$
H(t-a)f(t-a)	$e^{-as}F(s)$
$e^{at}f(t)$	F(s-a)
(f * g)(t)	F(s)G(s)
1	$\frac{1}{s} (s > 0)$
t^n (n is positive integer)	$\frac{n!}{s^{n+1}}$ (s > 0)
$e^a t$	$\frac{3}{s-a}$ $(s>a)$
sin(at)	$\frac{\ddot{a}}{s^2+a^2}$ (s > 0)
cos(at)	$\frac{s^2+a^2}{s^2+a^2}$ (s > 0)
sinh(at)	$\frac{s}{s^2-a^2}$ $(s> a)$
cosh(at)	$\frac{s-a}{s^2-a^2}$ $(s> a)$
$\delta(t-a) \ (a \ge 0)$	e^{-as}
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
e ^{at} sinbt	` h
	$\frac{\overline{((s-a)^2+b^2)}}{\frac{s-a}{s}}$
e ^a tcosbt	$\frac{\sqrt{(s-a)^2+b^2}}{\sqrt{(s-a)^2+b^2}}$