A SL BVP

Given the S-L BVP $\begin{cases} f'' + \lambda f = 0, & f = f(x) \quad 0 < x < 1 \\ f(0) = 0 & 3f(1) - 2f'(1) = 0 \end{cases}$

Find e-vals and e-functions

SOLN:

$\lambda = 0$ $\quad f(x) = \alpha x + \beta \quad f'(x) = \alpha$

$f(0) = 0 \Rightarrow \beta = 0 \Rightarrow f(x) = \alpha x \quad f'(x) = \alpha$

$3f(1) - 2f'(1) = 0 \Rightarrow 3\alpha - 2\alpha = 0 \Rightarrow \alpha = 0$ $\quad \beta = \alpha = 0 \Rightarrow \lambda = 0$ not an e-val.

$\lambda > 0$ $\quad f(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x \Rightarrow f'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$

$f(0) = 0 \Rightarrow c_1 = 0 \Rightarrow f(x) = c_2 \sin\sqrt{\lambda}\, x \quad f'(x) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$

$3f(1) - 2f'(1) = 0 \Rightarrow 3c_2\sin\sqrt{\lambda} - 2c_2\sqrt{\lambda}\cos\sqrt{\lambda} = 0$

$\tan\sqrt{\lambda} = \frac{2}{3}\sqrt{\lambda}$ eqn for e-vals

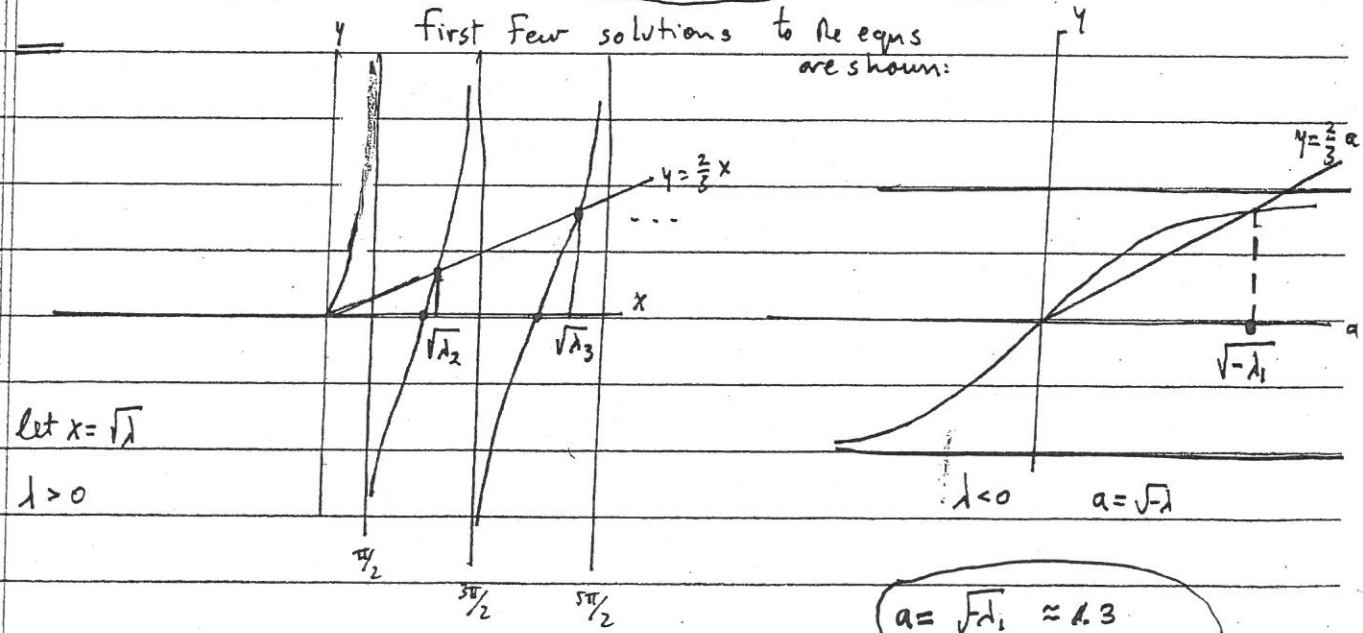
$\lambda < 0$ let $\lambda = -a^2$, $a > 0$ $\quad f(x) = c_1\cosh ax + c_2 \sinh ax \quad f'(x) = ac_1\sinh ax + ac_2\cosh ax$

$f(0) = 0 \Rightarrow c_1 = 0 \Rightarrow f(x) = c_2 \sinh ax$

$f'(x) = ac_2\cosh ax$

$3f(1) - 2f'(1) = 0 \Rightarrow 3c_2\sinh a - 2ac_2\cosh a = 0$

eqn for e-vals: $\tanh a = \frac{2}{3}a$ $\quad a = \sqrt{-\lambda}$



$\sqrt{\lambda_2} \approx 4.4 \quad \sqrt{\lambda_3} \approx 7.7, \cdots, \sqrt{\lambda_n} \approx \frac{(2n-1)\pi}{2}$ n large

$f_n \sim \sin\sqrt{\lambda_n}\, x$ for $n > 1$

$a = \sqrt{-\lambda_1} \approx 1.3$

$\lambda_1 \approx -1.69$

$f_1 \sim \sinh 1.3x$