1. Basic ODEs
   (a) seperatable: $y' = \frac{F(x)}{G(y)}$
   $\Rightarrow \frac{dy}{dx} = \frac{F(x)}{G(y)} \Rightarrow \int G(y)dy = \int F(x)dx$
   (b) Linear: $y' + p(x)y = q(x)$
   Integrating Factor: $e^{\int p(x)dx}$
   $\Rightarrow (e^{\int p(x)dx}y)' = e^{\int p(x)dx}q(x)$
   $\Rightarrow e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx$
   (c) $ay'' + by' + cy = 0$ (constant coefficient)
   characteristc EQ: $ar^2 + br + c = 0$
   $\Delta > 0, y = C_1e^{r_1x} + C_2e^{r_2x}$
   $\Delta = 0, y = C_1e^{rx} + C_2xe^{rx}$
   $\Delta < 0, r = p \pm qi : y = e^{px}[c_1cos(qx) + c_2sin(qx)]$
   (d) $ay'' + by' + cy = f(x)$
   $y = y_c + y_p$, y_c is solution to homogeneous DIFF EQ
   $f(x)$:a polynomial in x or single sin/cos function
   $y_p = x^k$(a polynomial of the same degree), k: # char eq's zero roots (0,1,2)
   $f(x) = e^{ax}$(a polynomial in x)
   $y_p = x^ke^{ax}$(same degree), k: # char eq's roots = a (0,1,2)
   $f(x) = e^{ax}cos(bx)$(poly in x) or $e^{ax}sin(bx)(apolyinx)$
   $y_p = x^ke^{ax}[$(poly in x)$cos(bx)$ + (poly in x)$sin(bx)]$
   k: # char eq's root = $a \pm bi$ (0,1)

2. Fourier Series
   Given $F(x), x \in [-L, L]$ write $F(x)$ in a series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{L})$$

   where a_n, b_n are constants.
   (a) Orthogonality Relations
   $\int_{-L}^{L} sin(\frac{n\pi x}{L})cos(\frac{m\pi x}{L}) = 0$
   $\int_{-L}^{L} cos(\frac{n\pi x}{L})cos(\frac{m\pi x}{L}) = 0(m \neq n), L(m = n)$
   $\int_{-L}^{L} sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L}) = 0(m \neq n), L(m = n)$
   (b) $a_n = \frac{1}{L}\int_{-L}^{L} F(x)cos(\frac{n\pi x}{L})dx$
   $b_n = \frac{1}{L}\int_{-L}^{L} F(x)sin(\frac{n\pi x}{L})dx, n \in [0, \infty], n \in \mathbf{Z}$
   (c) Convergece Statement of F.S.
   F.S. convergence to the "periodic extension" of F(x) whever F(x) is continous and to the average of $\frac{f(x^+)+f(x^-)}{2}$ at every point.
   (d) F.S.S and F.C.S of $F(x)$ on $[0, L]$:
   F.C.S = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{L}, a_n = \frac{2}{L}\int_0^L F(x)cos(\frac{n\pi x}{L})dx$
   F.S.S = $\sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{L}, a_n = \frac{2}{L}\int_0^L F(x)cos(\frac{n\pi x}{L})dx$
   F.C.S $\rightarrow$ even extension, F.S.S $\rightarrow$ odd extension

3. Sturm-Liouville Problem
   (a) Basic Examples of S-L BVP
   $f'' + \lambda f = 0, 0 \leq x \leq L$ and Boundary conditions
   Def: value λ for which the equation with the given boundary ends: has non-trival solution is called eigen value, the corresponding solution is called eigeon functions of the given S-L BVP.
   First, general solutions:
   $\lambda = 0, f(x) = \alpha x + \beta$
   $\lambda > 0, f(x) = C_1cos(\sqrt{\lambda}x) + C_2sin(\sqrt{\lambda}x)$
   $\lambda < 0, f(x) = C_1cosh(ax) + C_2sinh(ax), a^2 = -\lambda, a > 0$
   Then, impose the boundary in each case.
      i. Boundary COND: $f(0) = 0, f(L) = 0$
      e-values: $\lambda_n = \frac{n^2\pi^2}{L^2}, n = 1, 2, 3$
      e-functions: $f_n \sim sin(\frac{n\pi x}{L})$
      ii. Boundary COND: $f'(0) = 0, f'(L) = 0$
      e-values: $\lambda_n = \frac{n^2\pi^2}{L^2}, n = 0, 1, 2, 3$
      e-functions: $f_n \sim cos(\frac{n\pi x}{L})$
      iii. Boundary COND: $f(0) = 0, f'(L) = 0$
      e-values: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2, n = 1, 2, 3$
      e-functions: $f_n \sim sin(\frac{(2n-1)\pi}{2L}x)$
      iv. Boundary COND: $f'(0) = 0, f(L) = 0$
      e-values: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2, n = 1, 2, 3$
      e-functions: $f_n \sim cos(\frac{(2n-1)\pi}{2L}x)$
   (b) Regular S-L Problems
   EQ: $(pf')' + qf + \lambda\sigma f = 0, a < x < b$
   Boundary: $k_1f(a) + k_2f'(a) = 0, k_3f(b) + k_4f'(b) = 0$

$$g(x) \sim \sum_{n=1}^{\infty} a_n f_n(x)$$

$$a_n = \frac{\int_a^b g(x)f_n(x)\sigma(x)dx}{\int_a^b \sigma(x)f_n^2(x)dx}$$

4. Method of Separationg: Heat and Wave Equation
   (a) Heat Equation
   $\mu_t = k\mu_{xx}$, $\mu(x,t) = X(x)T(t)$ (usually k = 1)
   Seperation$\rightarrow X(x)T'(t) = kX''(x)T(t)$
   Let $\frac{T'}{kT} = \frac{kX''}{x} = -\lambda$
   We get $X'' + \lambda x = 0$, $T' + k\lambda T = 0$
   For second EQ, T $\sim e^{-k\lambda t}$
   For first EQ, we apply S-L BVP problem
   $\mu(x,t) = \sum_{n=1}^{\infty} c_nf_ne^{-\lambda_nt}$
   Usually, we find the bound. cond. in the 4 fourier series probs.
   e.g. bond cond 1:
   $\lambda_n = (\frac{n\pi}{L})^2$, $f_n \sim sin(\frac{n\pi x}{L})$
   $\mu(x,0) = g(x) = \sum_{n=1}^{\infty} c_nsin(\frac{n\pi x}{L})$
   Fouries formal solution: $\mu(x,t) = \sum_{n=1}^{\infty} c_nsin(\frac{n\pi x}{L})e^{-\frac{n^2\pi^2}{L^2}t}$
   note : in bond cond 2, n starts from 0. calculate case 0 seperately.
   (b) Wave Equation
   $\mu_{tt} = C^2\mu_{xx}, \mu(x,t) = X(x)T(t)$, usually $C = 1$
   $\mu(0,t) = 0$, $\mu(L,t) = 0$
   Seperation $\rightarrow XT'' = C^2X''T$
   Let $\frac{T''}{c^2T} = \frac{X''}{x} = -\lambda$
   We get $X'' + \lambda x = 0$, $T'' = -\lambda c^2T$
   $\mu(x,t) = \sum_{n=1}^{\infty}(b_{1,n}cos\frac{n\pi ct}{L} + b_{2,n}sin\frac{n\pi ct}{L})sin\frac{n\pi x}{L}$
      i. $\mu(x,0) = f(x)$
      $f(x) = \sum_{n=1}^{\infty} b_{1,n}sin\frac{n\pi x}{L}$
      $b_{1,n} = \frac{\int_0^L f(x)sin\frac{n\pi x}{L}dx}{\int_0^L sin^2\frac{n\pi x}{L}}$

ii. $\mu_t(x,0) = g(x)$
$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_{2,n} sin \frac{n\pi x}{L}$
$\frac{n\pi c}{L} b_{2,n} = \frac{\int_0^L g(x) sin \frac{n\pi x}{L} dx}{\int_0^L sin^2 \frac{n\pi x}{L}}$

(c) d'Alembert's solution to the Wave Equation
Given $\mu(x,0) = F(x)$,$\mu_t(x,0) = 0$ we want to get $\mu(x,t)$
Do an odd extention on $F(x)$, then $\mu(x,t) = \frac{F(x+ct)+F(x-ct)}{2}$. Usually c is 1. Simply draw the graph of $F(x+ct)$(shift to left) and $F(x-ct)$(shift to right) and get the average.

5. Laplace Equation
Consider the equilibrium temperature in a uniform rectangle(Heat EQ)

$$\begin{cases} \mu_{xx}(x,y) + \mu_y y(x,y) = 0, \ 0 < x < L, \ 0 < y < K \\ \mu(0,y) = f_1(y), \ \mu(L,y) = f_2(y), \ 0 < y < K \\ \mu(x,0) = g_1(x), \ \mu(x,K) = g_2(x), \ 0 < x < L \end{cases} \tag{1}$$

Where we might have first order partial derivatives on those $\mu's$ on 2nd or 3rd line.
Seperate $\rightarrow \mu = XY$, then
$X''Y + XY'' = 0$
$\frac{X''}{X} = -\frac{Y''}{Y}$
we want to create a SL-BVP problem:
Determine $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$ or $-\lambda$
When $f_1(y) = f_2(y) = 0$ we can form SL-BVP on Y and we choose $-\lambda$. Similarly, when $g_1(x) = g_2(x) = 0$ we can form SL-BVP on X and we choose λ
Case1: $-\lambda$:$(X(0) \ or \ X'(0) = 0, X(L) \ or \ X'(L) = 0)$

$$\begin{cases} X'' + \lambda X = 0, \ \ (SL-BVP) \\ \mu(x,0) = f_1(y), \mu(x,K) = f_2(y) \end{cases} \tag{2}$$

Get λ_n and X_n from SL-BVP.

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n sinh \frac{n\pi y}{L} + \beta_n cosh \frac{n\pi y}{L}) X_n$$

For simplicity, we transform to:

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n sinh \frac{n\pi y}{L} + \beta_n sinh \frac{n\pi (y-K)}{L}) X_n$$

Then we get $\mu(x,0)$, $\mu(x,K)$ (as long as they fit the other 2 equations) and try to figure out α_n, β_n using match or SL-BVP. **Note** : in bond cond 2, n starts from 0. calc case 0 seperately.
Case2: λ:$(Y(0) \ or \ Y'(0) = 0, Y(K) \ or \ Y'(K) = 0)$

$$\begin{cases} Y'' + \lambda X = 0, \ \ (SL-BVP) \\ \mu(0,y) = g_1(y), \mu(L,y) = g_2(x) \end{cases} \tag{3}$$

Get λ_n and Y_n from SL-BVP.

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n sinh \frac{n\pi x}{K} + \beta_n cosh \frac{n\pi x}{K}) Y_n$$

For simplicity, we transform to:

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n sinh \frac{n\pi x}{K} + \beta_n cosh \frac{n\pi (x-L)}{K}) Y_n$$

Then we get $\mu(0,y)$, $\mu(L,y)$ (as long as they fit the other 2 equations) and try to figure out α_n, β_n using match or SL-BVP. **Note** : in bond cond 2, n starts from 0. calc case 0 seperately.

6. Method of Eigenfunction Expansion
Consider
$PDE: \mu_t = k\mu_{xx}(x,t) + q(x,t), 0 < x < L, t > 0$
$BCs: \mu(0,t) = 0, \mu(L,t) = 0, t > 0$
$IC: \mu(x,0) = f(x), 0 < x < L$

(a) BCs $\Rightarrow$ Eigenfunction: $X_n(x), \lambda_n$
$\Rightarrow \mu(x,t) = \sum_{n=1}^{\infty} C_n(t) X_n(x)$
Write $q(x,t) = \sum_{n=1}^{\infty} q_n(t) X_n(x)$

(b) PDE

$$\Rightarrow \sum_{n=1}^{\infty} C_n'(t) X_n(x) = \sum_{n=1}^{\infty} C_n(t) X_n''(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} C_n'(t) X_n(x) = -\sum_{n=1}^{\infty} C_n(t) \lambda_n X_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C_n'(t) + C_n(t)\lambda_n] X_n(x) = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C_n'(t) + C_n(t)\lambda_n] = \sum_{n=1}^{\infty} q_n(t)$$

Form may vary (μ is differentiated in other ways)

(c) IC $\Rightarrow \mu(x,0) = f(x) = \sum_{n=1}^{\infty} C_n(0) X_n(x)$ (SL-BVP)
either match up or $C_n(0) = \frac{\int_0^L f(x) X_n(x) dx}{\int_0^L X_n^2(x) dx}$

(d) Then we solve functions $C_n'(t) + C_n(t)\lambda_n = q_n(t)$ for each n with initial condition $C_n(0)'s$ and $q_n(0)'s$

7. Laplace Transform
Def: $\mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$

$$Def: H_a(t) = H(t-a) = \begin{cases} 1, \ t \geq a \\ 0, \ t < a \end{cases} \tag{4}$$

Def: $erf(x) = \frac{2}{\pi} \int_0^x e^{-u^2} du$, $erfc(x) = 1 - erf(x)$

$f(t) = \mathcal{L}^{-1} F(t)$	$F(t) = \mathcal{L} f(s)$
$f^{(n)}(t)$ (nth derivative)	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
$H(t-a) f(t-a)$	$e^{-as} F(s)$
$e^{at} f(t)$	$F(s-a)$
$(f * g)(t)$	$F(s)G(s)$
1	$\frac{1}{s}$ $(s > 0)$
t^n (n is positive integer)	$\frac{n!}{s^{n+1}}$ $(s > 0)$
e^{at}	$\frac{1}{s-a}$ $(s > a)$
$sin(at)$	$\frac{a}{s^2+a^2}$ $(s > 0)$
$cos(at)$	$\frac{s}{s^2+a^2}$ $(s > 0)$
$sinh(at)$	$\frac{a}{s^2-a^2}$ $(s > \lvert a \rvert)$
$cosh(at)$	$\frac{s}{s^2-a^2}$ $(s > \lvert a \rvert)$
$\delta(t-a)$ $(a \geq 0)$	e^{-as}
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} sinbt$	$\frac{b}{((s-a)^2+b^2)}$
$e^a tcosbt$	$\frac{s-a}{((s-a)^2+b^2)}$
$erfc(\frac{a}{2\sqrt{t}})$	$\frac{1}{s} e^{-a\sqrt{s}}$

Example:

$\mu_{tt}(x,t) = \mu_{xx}(x,t) + 1$, $\mu(x,0) = \mu_t(x,0) = -1$, $\mu(0,t) = t$

$\mathcal{L}\mu_{tt} = s^2U - s\mu(x,0) - \mu_t(x,0) = s^2U + 1$

$\mathcal{L}\mu_{xx} + 1 = U_{xx} + \frac{1}{s}$

$\Rightarrow U'' - s^2U = 1 - \frac{1}{s}$ (x is variable, s is scalar)

$U_c = k_1e^{sx} + k_2e^{-sx}$, $k_1 = 0$

$U_p = \frac{1-\frac{1}{s}}{-s^2} = \frac{1}{s^3} - \frac{1}{s^2}$

$\mathcal{L}\mu(0,t) = \mathcal{L}t = \frac{1}{s} = U(0,s)$

$\Rightarrow k_2 + \frac{1}{s^3} - \frac{1}{s^2} = \frac{1}{s^2}$

$\Rightarrow U = (\frac{2}{s^2} - \frac{1}{s^3})e^{-sx} - \frac{1}{s^2} + \frac{1}{s^3}$

$\mu = \mathcal{L}^{-1}(U) = H(t-x)[2(t-x) - \frac{1}{2}(t-x)^2] - t + \frac{1}{2}t^2$