

## 1. Basic ODEs

- (a) separable:  $y' = \frac{F(x)}{G(y)}$   
 $\Rightarrow \frac{dy}{dx} = \frac{F(x)}{G(y)} \Rightarrow \int G(y)dy = \int F(x)dx$
- (b) Linear:  $y' + p(x)y = q(x)$   
 Integrating Factor:  $e^{\int p(x)dx}$   
 $\Rightarrow (e^{\int p(x)dx}y)' = e^{\int p(x)dx}q(x)$   
 $\Rightarrow e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx$
- (c)  $ay'' + by' + cy = 0$  (constant coefficient)  
 characteristic EQ:  $ar^2 + br + c = 0$   
 $\Delta > 0, y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$   
 $\Delta = 0, y = C_1 e^{rx} + C_2 x e^{rx}$   
 $\Delta < 0, r = p \pm qi : y = e^{px}[C_1 \cos(qx) + C_2 \sin(qx)]$
- (d)  $ay'' + by' + cy = f(x)$   
 $y = y_c + y_p$ ,  $y_c$  is solution to homogeneous DIFF EQ  
 $f(x)$ : a polynomial in  $x$  or single  $\sin/\cos$  function  
 $y_p = x^k$  (a polynomial of the same degree),  $k$ : # char eq's zero roots (0,1,2)  
 $f(x) = e^{ax}$  (a polynomial in  $x$ )  
 $y_p = x^k e^{ax}$  (same degree),  $k$ : # char eq's roots =  $a$  (0,1,2)  
 $f(x) = e^{ax} \cos(bx)$  (poly in  $x$ ) or  $e^{ax} \sin(bx)$  (apoly in  $x$ )  
 $y_p = x^k e^{ax}[(\text{poly in } x)\cos(bx) + (\text{poly in } x)\sin(bx)]$   
 $k$ : # char eq's root =  $a \pm bi$  (0,1)  
 (e) Second Order Cauchy-Euler Equation  
 homogeneous:  $ax^2 y'' + bxy' + cy = 0$   
 characteristic Equation:  $am^2 + (b-a)m + c = 0$   
 i. Two different real roots:  
 $y = c_1 x^{m_1} + c_2 x^{m_2}$   
 ii. Same real root:  
 $y = c_1 x^m + c_2 x^m \ln x$   
 iii. complex roots:  
 $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$   
 $y = x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$   
 nonhomogeneous:  $ax^2 y'' + bxy' + cy = f(x)$   
 $y = y_c + y_p$ , similar to the above case.

## 2. Fourier Series

Given  $F(x)$ ,  $x \in [-L, L]$  write  $F(x)$  in a series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where  $a_n, b_n$  are constants.

- (a) Orthogonality Relations  
 $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$   
 $\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0 (m \neq n), L (m = n)$   
 $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0 (m \neq n), L (m = n)$
- (b)  $a_n = \frac{1}{L} \int_{-L}^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$   
 $b_n = \frac{1}{L} \int_{-L}^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx, n \in [0, \infty], n \in \mathbb{Z}$
- (c) Convergence Statement of F.S.  
 F.S. convergence to the "periodic extension" of  $F(x)$  wherever  $F(x)$  is continuous and to the average of  $\frac{f(x^+) + f(x^-)}{2}$  at every point.

(d) F.S.S and F.C.S of  $F(x)$  on  $[0, L]$ :

Suppose  $F(x)$  is even, then  $b_n = 0$ , for all  $n$

$$\text{F.C.S} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), a_n = \frac{2}{L} \int_0^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Suppose  $F(x)$  is odd, then  $a_n = 0$ , for all  $n$

$$\text{F.S.S} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), b_n = \frac{2}{L} \int_0^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

F.C.S  $\rightarrow$  even extension, F.S.S  $\rightarrow$  odd extension

## 3. Sturm-Liouville Problem

- (a) Hyperbolic sin and cos  
 $\cosh x = \frac{e^x + e^{-x}}{2}, \sinh x = \frac{e^x - e^{-x}}{2}$   
 $(\cosh x)' = \sinh x, (\cosh x)'' = \cosh x$   
 $(\sinh x)' = \cosh x, (\sinh x)'' = \sinh x$   
 $\sinh(\alpha - \beta) = \sinh \alpha \cosh \beta - \cosh \alpha \sinh \beta$   
 $\cosh(\alpha - \beta) = \cosh \alpha \cosh \beta - \sinh \alpha \sinh \beta$
- (b) Basic Examples of S-L BVP  
 $f'' + \lambda f = 0, 0 \leq x \leq L$  and Boundary conditions  
 Def: value  $\lambda$  for which the equation with the given boundary ends: has non-trivial solution is called eigen value, the corresponding solution is called eigen functions of the given S-L BVP.  
 General solutions:  
 $\lambda = 0, f(x) = \alpha x + \beta$   
 $\lambda > 0, f(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$   
 $\lambda < 0, f(x) = C_1 \cosh(ax) + C_2 \sinh(ax), a^2 = -\lambda, a > 0$   
 Impose the boundary in each case.  
 i. Boundary COND:  $f(0) = 0, f(L) = 0$   
 e-values:  $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 1, 2, 3$   
 e-functions:  $f_n \sim \sin\left(\frac{n\pi x}{L}\right)$   
 ii. Boundary COND:  $f'(0) = 0, f'(L) = 0$   
 e-values:  $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 0, 1, 2, 3$   
 e-functions:  $f_n \sim \cos\left(\frac{n\pi x}{L}\right)$   
 iii. Boundary COND:  $f(0) = 0, f'(L) = 0$   
 e-values:  $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n = 1, 2, 3$   
 e-functions:  $f_n \sim \sin\left(\frac{(2n-1)\pi}{2L}x\right)$   
 iv. Boundary COND:  $f'(0) = 0, f(L) = 0$   
 e-values:  $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n = 1, 2, 3$   
 e-functions:  $f_n \sim \cos\left(\frac{(2n-1)\pi}{2L}x\right)$
- (c) Regular S-L Problems  
 EQ:  $(pf')' + qf + \lambda \sigma f = 0, a < x < b$   
 Boundary:  $k_1 f(a) + k_2 f'(a) = 0, k_3 f(b) + k_4 f'(b) = 0$

$$g(x) \sim \sum_{n=1}^{\infty} a_n f_n(x)$$

$$a_n = \frac{\int_a^b g(x) f_n(x) \sigma(x) dx}{\int_a^b \sigma(x) f_n^2(x) dx}$$

usually  $\sigma(x) = 1$

## 4. Method of Separation: Heat and Wave Equation

- (a) Heat Equation  
 $\mu_t = k \mu_{xx}$   
 $\mu(x, t) = X(x)T(t)$  (usually  $k = 1$ )  
 Separation  $\rightarrow X(x)T'(t) = kX''(x)T(t)$   
 Let  $\frac{T'}{kT} = \frac{X''}{X} = -\lambda$   
 We get  $X'' + \lambda X = 0, T' + k\lambda T = 0$

For second EQ,  $T \sim e^{-k\lambda t}$

For first EQ, we apply S-L BVP problem

$$\mu(x, t) = \sum_{n=1}^{\infty} c_n f_n e^{-\lambda_n t}$$

Usually, we find the bound. cond. in the 4 fourier series probs.

e.g. bond cond 1:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, f_n \sim \sin\left(\frac{n\pi x}{L}\right)$$

$$\mu(x, 0) = g(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

Fouries formal solution:  $\mu(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{L^2} t}$

e.g. bond cond 2:

$$\lambda_0 = 0, X_0(x) = 1/2$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, f_n \sim \cos\left(\frac{n\pi x}{L}\right)$$

$$\mu(x, 0) = g(x) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)$$

$$g(x) = \frac{1}{2}c_0 \Rightarrow c_0 = \frac{\int_0^L \frac{1}{2}g(x)dx}{\int_0^L \left(\frac{1}{2}\right)^2 dx}$$

$$g(x) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \Rightarrow c_n = \frac{\int_0^L \cos\left(\frac{n\pi x}{L}\right)g(x)dx}{\int_0^L \left(\cos\left(\frac{n\pi x}{L}\right)\right)^2 dx}$$

$$\text{Solution: } \mu(x, t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{L^2} t}$$

If the bound. cond. not in the 4 SLBVP:

Do the SL-BVP manually and find e-value and e-function.

write  $\lambda_n = \left(\frac{\zeta_n}{L}\right)^2$ ,  $X_n = \sin/\cos \frac{\zeta_n}{L}$

#### (b) Wave Equation

$$\mu_{tt} = C^2 \mu_{xx}$$

$$\mu(x, t) = X(x)T(t), \text{ usually } C = 1$$

$$\mu(0, t) = 0, \mu(L, t) = 0$$

$$\text{Seperation} \rightarrow XT'' = C^2 X''T$$

$$\text{Let } \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

$$\text{We get } X'' + \lambda x = 0, T'' = -\lambda c^2 T$$

$$T_n(t) = b_{1,n} \cos \frac{n\pi ct}{L} + b_{2,n} \sin \frac{n\pi ct}{L}, X_n = \sin \frac{n\pi cx}{L} \text{ (} X_n \text{ from SL-BVP problem)}$$

When  $c = 1$ :

$$\mu(x, t) = X_n T_n = \sum_{n=1}^{\infty} (b_{1,n} \cos \frac{n\pi ct}{L} + b_{2,n} \sin \frac{n\pi ct}{L}) \sin \frac{n\pi x}{L}$$

i.  $\mu(x, 0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_{1,n} \sin \frac{n\pi x}{L}$$

$$b_{1,n} = \frac{\int_0^L f(x) \sin \frac{n\pi x}{L} dx}{\int_0^L \sin^2 \frac{n\pi x}{L} dx}$$

ii.  $\mu_t(x, 0) = g(x)$

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_{2,n} \sin \frac{n\pi x}{L}$$

$$\frac{n\pi c}{L} b_{2,n} = \frac{\int_0^L g(x) \sin \frac{n\pi x}{L} dx}{\int_0^L \sin^2 \frac{n\pi x}{L} dx}$$

#### (c) d'Alembert's solution to the Wave Equation

Given  $\mu(x, 0) = F(x), \mu_t(x, 0) = 0$  we want to get  $\mu(x, t)$

Do an odd extention on  $F(x)$ , then  $\mu(x, t) = \frac{F(x+ct) + F(x-ct)}{2}$ . Usually  $c$  is 1. Simply draw the graph of  $F(x+ct)$ (shift to left) and  $F(x-ct)$ (shift to right) and get the average.

A full period:  $\frac{2L}{c}$

#### 5. Laplace Equation

Consider the equilibrium temperature in a uniform rectangle(Heat EQ)

$$\begin{cases} \mu_{xx}(x, y) + \mu_{yy}(x, y) = 0, & 0 < x < L, & 0 < y < K \\ \mu(0, y) = f_1(y), & \mu(L, y) = f_2(y), & 0 < y < K \\ \mu(x, 0) = g_1(x), & \mu(x, K) = g_2(x), & 0 < x < L \end{cases} \quad (1)$$

Where we might have first order partial derivatives on those  $\mu$ 's on 2nd or 3rd line.

Seperate  $\rightarrow \mu = XY$ , then

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

we want to create a SL-BVP problem:

$$\text{Determine } \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \text{ or } -\lambda$$

When  $f_1(y) = f_2(y) = 0$  or  $X(0) = X(L) = 0$  we can form SL-BVP on  $Y$  and we choose  $-\lambda$ .

Similarly, when  $g_1(x) = g_2(x) = 0$  or  $Y(0) = Y(K) = 0$  we can form SL-BVP on  $X$  and we choose  $\lambda$

when we choose  $-\lambda$ : ( $X(0)$  or  $X'(0) = 0, X(L)$  or  $X'(L) = 0$ )

$$\begin{cases} X'' + \lambda X = 0, & (\text{SL-BVP}) \\ \mu(x, 0) = f_1(y), \mu(x, K) = f_2(y) \end{cases} \quad (2)$$

$\Rightarrow$  find out  $\lambda_n$  and  $X_n$  Then:

$$Y_n'' - \lambda_n Y_n = 0, 0 < y < K$$

$$Y_n = \alpha_n \sinh \frac{n\pi y}{L} + \beta_n \cosh \frac{n\pi y}{L}$$

$$\mu(x, y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi y}{L} + \beta_n \cosh \frac{n\pi y}{L}) X_n$$

For simplicity, we transform to:

$$(a) \mu(x, 0) = g_1(x), \mu(x, K) = g_2(x)$$

$$Y_n = \alpha_n \sinh \frac{n\pi(y-K)}{L} + \beta_n \sinh \frac{n\pi y}{L}$$

$$(b) \mu_y(x, 0) = g_1(x), \mu_y(x, K) = g_2(x)$$

$$Y_n = \alpha_n \cosh \frac{n\pi(y-K)}{L} + \beta_n \cosh \frac{n\pi y}{L}$$

$$(c) \mu(x, 0) = g_1(x), \mu_y(x, K) = g_2(x)$$

$$Y_n = \alpha_n \cosh \frac{n\pi(y-K)}{L} + \beta_n \cosh \frac{n\pi y}{L}$$

$$(d) \mu_y(x, 0) = g_1(x), \mu(x, K) = g_2(x)$$

$$Y_n = \alpha_n \cosh \frac{n\pi y}{L} + \beta_n \sinh \frac{n\pi(y-K)}{L}$$

Then we get  $\mu(x, 0), \mu(x, K)$  (as long as they fit the other 2 equations) and try to figure out  $\alpha_n, \beta_n$  using match or SL-BVP.

**Note** : in bond cond 2,  $n$  starts from 0. calc case 0 seperately:  $\lambda_0 = 0, X_0 = 1 \Rightarrow Y_0'' - 0Y_0 = 0 \Rightarrow Y_0 = ay + b$

#### 6. Method of Eigenfunction Expansion

Consider

$$PDE : \mu_t = k \mu_{xx}(x, t) + q(x, t), 0 < x < L, t > 0$$

$$BCs : \mu(0, t) = 0, \mu(L, t) = 0, t > 0$$

$$IC : \mu(x, 0) = f(x), 0 < x < L$$

(a) BCs  $\Rightarrow$  Eigenfunction:  $X_n(x), \lambda_n$

$$\Rightarrow \mu(x, t) = \sum_{n=1}^{\infty} C_n(t) X_n(x)$$

$$\text{Write } q(x, t) = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

(b) PDE:  $X_n'' + \lambda_n X_n = 0$

$$\Rightarrow \sum_{n=1}^{\infty} C_n'(t) X_n(x) = \sum_{n=1}^{\infty} C_n(t) X_n''(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} C_n'(t) X_n(x) = - \sum_{n=1}^{\infty} C_n(t) \lambda_n X_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C_n'(t) + C_n(t) \lambda_n] X_n(x) = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C'_n(t) + C_n(t)\lambda_n] = \sum_{n=1}^{\infty} q_n(t)$$

Form may vary ( $\mu$  is differentiated in other ways)

(c) IC  $\Rightarrow \mu(x, 0) = f(x) = \sum_{n=1}^{\infty} C_n(0)X_n(x)$  (SL-BVP)

either match up or  $C_n(0) = \frac{\int_0^L f(x)X_n(x)dx}{\int_0^L X_n^2(x)dx}$

(d) Then we solve functions  $C'_n(t) + C_n(t)\lambda_n = q_n(t)$  for each  $n$  with initial condition  $C_n(0)$ 's and  $q_n(0)$ 's

## 7. Laplace Transform

Def:  $\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$

$$Def: H_a(t) = H(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases} \quad (3)$$

Def:  $erf(x) = \frac{2}{\pi} \int_0^x e^{-u^2} du$ ,  $erfc(x) = 1 - erf(x)$

$f(t) = \mathcal{L}^{-1}F(t)$	$F(t) = \mathcal{L}f(s)$
$f^{(n)}(t)$ (nth derivative)	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$
$H(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$(f * g)(t)$	$F(s)G(s)$
1	$\frac{1}{s} \ (s > 0)$
$t^n$ (n is positive integer)	$\frac{n!}{s^{n+1}} \ (s > 0)$
$e^{at}$	$\frac{1}{s-a} \ (s > a)$
$\sin(at)$	$\frac{a}{s^2+a^2} \ (s > 0)$
$\cos(at)$	$\frac{s}{s^2+a^2} \ (s > 0)$
$\sinh(at)$	$\frac{a}{s^2-a^2} \ (s >  a )$
$\cosh(at)$	$\frac{s}{s^2-a^2} \ (s >  a )$
$\delta(t-a) \ (a \geq 0)$	$e^{-as}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin bt$	$\frac{b}{((s-a)^2+b^2)}$
$e^{at} \cos bt$	$\frac{s-a}{((s-a)^2+b^2)}$
$erfc(\frac{a}{2\sqrt{t}})$	$\frac{1}{s} e^{-a\sqrt{s}}$

Example:

$$\mu_{tt}(x, t) = \mu_{xx}(x, t) + 1, \mu(x, 0) = \mu_t(x, 0) = -1, \mu(0, t) = t$$

$$\mathcal{L}(\mu_{tt}) = s^2 U - s\mu(x, 0) - \mu_t(x, 0) = s^2 U + 1$$

$$\mathcal{L}(\mu_{xx} + 1) = U_{xx} + \frac{1}{s}$$

$$\Rightarrow U'' - s^2 U = 1 - \frac{1}{s} \quad (x \text{ is variable, } s \text{ is scalar})$$

$$U_c = k_1 e^{sx} + k_2 e^{-sx}, k_1 = 0$$

$$U_p = \frac{1-\frac{1}{s}}{-s^2} = \frac{1}{s^3} - \frac{1}{s^2}$$

$$\mathcal{L}\mu(0, t) = \mathcal{L}t = \frac{1}{s} = U(0, s)$$

$$\Rightarrow k_2 + \frac{1}{s^3} - \frac{1}{s^2} = \frac{1}{s^2}$$

$$\Rightarrow U = (\frac{2}{s^2} - \frac{1}{s^3})e^{-sx} - \frac{1}{s^2} + \frac{1}{s^3}$$

$$\mu = \mathcal{L}^{-1}(U) = H(t-x)[2(t-x) - \frac{1}{2}(t-x)^2] - t + \frac{1}{2}t^2$$

(a) wave eq

$$\mu_{tt} = c^2 \mu_{xx}, x > 0, t > 0$$

$$\mu(0, t) = f(t), t > 0$$

$$\mu(x, 0) = 0, \mu_t(x, 0) = 0, x > 0$$

$$\mathcal{L}\mu(x, s) = U(x, s), \mathcal{L}f(s) = F(s)$$

$$\Rightarrow s^2 U(x, s) = c^2 U''(x, s) \Rightarrow U''(x, s) - (\frac{s}{c})^2 U(x, s) = 0$$

$$\text{General Solution: } U(x, s) = C_1(s)e^{\frac{s}{c}x} + C_2(s)e^{-\frac{s}{c}x}$$

$$C_1(s) = 0, \text{ then } U(x, s) = F(s)e^{-\frac{s}{c}x} = F(s)e^{-\frac{x}{c}s}$$

$$u(x, t) = H(t-x/c)f(t-x/c)$$

(b) Heat equation

$$\mu_t(x, t) = \mu_{tt}(x, t)$$

$$\mu_x(0, t) - \mu(0, t) = 0$$

$$\mu(x, 0) = \mu_0 = \text{const}$$

$$\Rightarrow U''(x, s) - sU(x, s) + \mu_0 = 0$$

$$U'(0, s) - U(0, s) = 0$$

$$\text{General Solut: } U(x, s) = C_1(s)e^{\sqrt{s}x} + C_2(s)e^{-\sqrt{s}x} + \frac{1}{s}\mu_0$$

$$C_1(s) = 0, C_2(s) = -\frac{\mu_0}{s(\sqrt{s}+1)}$$

$$\Rightarrow U(x, s) = \mu_0[-\frac{1}{s(\sqrt{s}+1)}e^{-\sqrt{s}x} + \frac{1}{s}]$$

$$\mu(x, t) = \mu_0[1 - erfc(\frac{x}{2\sqrt{t}}) - erfc(\sqrt{t} + \frac{x}{2\sqrt{t}})e^{x+t}]$$

## 8. method of characteristics

$$\mu_t + A\mu_x = B$$

$$A = \frac{dx}{dt}, B = \frac{du}{dt}$$