e-function expansion 7.1.2

#1 $\begin{cases} u_t = u_{xx} + 2 + \cos 2\pi x \\ u_x(0,t) = 0 \quad u_x(1,t) = 0 \\ u(x,0) = 2\cos\pi x - \cos 2\pi x \end{cases}$

soln: e-function assumption $u(x,t) = \sum_{n=0}^{\infty} c_n(t) \cos n\pi x$

after substitution: $c_n' + n^2\pi^2 c_n =$ n^{th} coeff. of $q(x,t) = 2 + \cos 2\pi x$

$c_n(0)$: n^{th} coeff of $f(x) = 2\cos\pi x - \cos 2\pi x$

$q_n(t) =$ n^{th} coeff of $q(x,t) \Longrightarrow$

$$2 + \cos 2\pi x = \sum_{n=0}^{\infty} q_n(t) \cos n\pi x$$

match up $q_0(t) = 2$ $\quad q_2(t) = 1$ all others are 0

$c_n(0) =$ n^{th} coeff of $f(x) \Longrightarrow$

$$2\cos\pi x - \cos 2\pi x = \sum_{n=1}^{\infty} c_n(0) \cos n\pi x$$

match up $n=1$ $\quad c_1(0) = 2$ $\quad n=2$ $\quad c_2(0) = -1$

all others are 0

O.D.E.'s			
	$c_0' = 2$ $c_0(0) = 0$	$c_1' + \pi^2 c_1 = 0$ $c_1(0) = 2$	$c_2' + 4\pi^2 c_2 = 1$ $c_2(0) = -1$

all other $c_n \equiv 0$

solving: $c_0 = 2t$; $c_1 = 2e^{-\pi^2 t}$; $c_2 = \frac{1}{4\pi^2} - \frac{4\pi^2+1}{4\pi^2} e^{-4\pi^2 t}$

Assembling

$$u(x,t) = 2t + 2e^{-\pi^2 t}\cos\pi x + \left[\frac{1}{4\pi^2} - \frac{4\pi^2+1}{4\pi^2} e^{-4\pi^2 t}\right] \cos 2\pi x$$

Given

7.6.2 #7 $\quad u(x,0) = 1 - 3\cos 2x \qquad q(x,t) = -2xt$

e-function assumption $\quad u(x,t) = \sum_{n=0}^{\infty} c_n(t) \cos n\pi x$

after substitution $\quad c_n' + n^2\pi^2 c_n = n^{th}$ coeff of $q = -2xt = q_n(t)$

$c_n(0) = n^{th}$ coeff of $u(x,0) = 1 - 3\cos 2x = c_n(0)$

for q_n $\qquad q_n(t) = \dfrac{\int_0^1 -2xt \cos n\pi x \, dx}{\int_0^1 \cos^2 n\pi x \, dx}$

for $c_n(0)$ $\qquad c_n(0) = \begin{cases} 1 & \text{when } n = 0 \\ -3 & \text{" } n = 2 \\ 0 & \text{otherwise} \end{cases}$

after integration $\quad q_0(t) = -t \qquad q_n(t) = \dfrac{4}{n^2\pi^2} t (1 - \cos n\pi)$

equations

$n = 0$	$n = 2$	otherwise
$c_0' = -t$	$c_2' + 4\pi^2 c_n = 0$ (since $\cos 2\pi = 1$)	$c_n' + n^2\pi^2 c_n = \frac{4t}{n^2\pi^2}(1 - \cos n\pi)$
$c_0(0) = 1$	$c_2(0) = -3$	$c_n(0) = 0$

solutions $\quad c_0 = -\dfrac{t^2}{2} + 1 \qquad c_2 = -3e^{-4\pi^2 t}$

$n \neq 0, 2 \qquad c_n = \dfrac{4(1 - \cos n\pi)}{n^6\pi^6} \left(n^2\pi^2 t - 1 + e^{-n^2\pi^2 t} \right)$ see other side

Notice when $n=0$ and $n=2$ this.

assembling:
$$u(x,t) = -\frac{t^2}{2} + 1 - 3e^{-4\pi^2 t} \cos 2\pi x + \sum_{n=1}^{\infty} \frac{4(1 - \cos n\pi)}{n^6\pi^6} \left(n^2\pi^2 t - 1 + e^{-n^2\pi^2 t} \right) \cos n\pi x$$

For $\quad c_n' + n^2\pi^2 c_n = \frac{4}{n^2\pi^2}(1-\cos n\pi)\, t \qquad c_n(0)=0$

I.F.: $\quad e^{n^2\pi^2 t} \quad \Rightarrow$

$$\left(e^{n^2\pi^2 t} c_n\right)' = \frac{4}{n^2\pi^2}(1-\cos n\pi)\; t e^{n^2\pi^2 t}$$

parts.

$$\int t e^{n^2\pi^2 t}\,dt = \frac{t}{n^2\pi^2} e^{n^2\pi^2 t} - \frac{1}{n^4\pi^4} e^{n^2\pi^2 t}$$

$$= \frac{n^2\pi^2 t - 1}{n^4\pi^4} e^{n^2\pi^2 t}$$

$$e^{n^2\pi^2 t} c_n = \frac{4(1-\cos n\pi)}{n^2\pi^2}\left(\frac{n^2\pi^2 t - 1}{n^6\pi^6}\right) e^{n^2\pi^2 t} + k$$

$$c_n = \frac{4(1-\cos n\pi)(n^2\pi^2 t - 1)}{n^6\pi^6} + k e^{-n^2\pi^2 t}$$

$$c_n(0) = 0 \Rightarrow k = \frac{4(1-\cos n\pi)}{n^6\pi^6}$$

$$c_n = \frac{4(1-\cos n\pi)}{n^6\pi^6}\left(n^2\pi^2 t - 1 + e^{-n^2\pi^2 t}\right)$$

heat problems

7.1.3 #1

$u_t = u_{xx} + \sin\left(\frac{3}{2}\pi x\right) - 2\sin\left(\frac{5}{2}\pi x\right) \qquad 0<x<1 \qquad t>0$

$u(0,t)=0 \qquad u_x(1,t)=0$

$u(x,0) = \sin\frac{3}{2}\pi x = f(x)$

e-function assumption $\quad u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin\frac{(2n-1)\pi}{2}x$

substitution $\quad \sum c_n' \sin\frac{(2n-1)\pi}{2}x = \sum -\frac{(2n-1)^2\pi^2}{4} \cdot c_n \sin\frac{(2n-1)\pi}{2}x + q(x,t)$

comparing: $\quad c_n' + \frac{(2n-1)^2\pi^2}{4}c_n = q_n \qquad q_n$ is the n^{th} coeff of $q(x,t)$

get o.d.e $\quad c_n(0) = n^{th}$ coeff of $f(x)$.

on $q_n(t)$ $\quad \sin\frac{3}{2}\pi x - 2\sin\frac{5}{2}\pi x = \sum_{n=1} q_n(t) \sin\frac{(2n-1)\pi}{2}x$

matching up $\quad n=2 \Rightarrow q_2(t)=1$

$n=3 \Rightarrow q_3(t)=-2 \qquad$ all others are 0

on $c_n(0)$ $\quad f(x) = u(x,0) = \sin\frac{3}{2}\pi x = \sum_{n=1}^{\infty} c_n(0)\sin\frac{(2n-1)\pi}{2}x$

matching up $\quad n=2 \Rightarrow c_2(0)=1$

o.d.e.s

$n=2$: $\quad c_2' + \frac{9\pi^2}{4}c_2 = 1, \quad c_2(0)=1$

$n=3$: $\quad c_3' + \frac{25}{4}\pi^2 c_3 = -2, \quad c_3(0) = 0$

all others are 0

after solving the o.des. for c_n

$$u(x,t) = \left[\left(\frac{9\pi^2-4}{9\pi^2}\right)e^{-\frac{9\pi^2}{4}t} + \frac{4}{9\pi^2}\right]\sin\frac{3\pi x}{2} + \left[\frac{8}{25\pi^2}e^{-\frac{25\pi^2}{4}t} - \frac{8}{25\pi^2}\right]\sin\frac{5\pi x}{2}$$

heat eqn with

7.1.3 $\quad u_x(0,t) = 0 \qquad u(1,t) = 0$

$q(x,t) = t\cos\frac{\pi x}{2} \qquad u(x,0) = \cos\frac{\pi x}{2} + 2\cos\frac{5\pi x}{2}$

Soln

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\cos\frac{(2n-1)\pi x}{2}$$

$$\Rightarrow \quad c_n'' + \frac{(2n-1)^2\pi^2}{4}c_n = q_n = n^{th} \text{ coeff of } q(x,t)$$

$$c_n(0) = n^{th} \text{ coeff of } u(x,0)$$

For q_n

$$t\cos\frac{\pi x}{2} = \sum_{n=1}^{\infty} q_n(t)\cos\frac{(2n-1)\pi x}{2} \Rightarrow \begin{cases} n=1 \\ q_1(t) = t \\ \text{all others are } 0 \end{cases}$$

for $c_n(0)$

$$\cos\frac{\pi x}{2} + 2\cos\frac{5\pi x}{2} = \sum_{n=1}^{\infty} c_n(0)\cos\frac{(2n-1)\pi x}{2} \Rightarrow \begin{cases} c_1(0) = 1 \\ c_3(0) = 2 \\ \text{all others are } 0 \end{cases}$$

$n=1$	$n=3$	others
$c_1' + \frac{\pi^2}{4}c_1 = t$	$c_3' + \frac{25\pi^2}{4}c_3 = 0$	$c_n' + \frac{(2n-1)^2\pi^2}{4}c_n = 0$
$c_1(0) = 1$	$c_3(0) = 2$	$c_n(0) = 0$

after solving the equations.

solns: $\quad n=1 \quad c_1 = \frac{4t}{\pi^2} - \frac{16}{\pi^4} + \left(1 + \frac{16}{\pi^4}\right)e^{-\frac{\pi^2}{4}t}$

$n=3 \quad c_3 = 2e^{-\frac{25\pi^2}{4}t}$

$n \neq 1, 3 \quad c_n \equiv 0$

$$u(x,t) = \left(\frac{4t}{\pi^2} - \frac{16}{\pi^4} + \left(1 + \frac{16}{\pi^4}\right)e^{-\frac{\pi^2}{4}t}\right)\cos\frac{\pi x}{2} + 2e^{-\frac{25\pi^2}{4}t}\cos\frac{5\pi x}{2}$$

7.2.1 WAVE PROBLEM 35

#5 $u_{tt} = u_{xx} + (t+1)\sin 2\pi x \qquad 0<x<1$

$u(0,t) = u(1,t) = 0 \;; \quad u(x,0) = 0 \qquad u_t(x,0) = 2x-1$

SOLN:

e-function assumption $\quad u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin n\pi x$

after substitution $\quad c_n'' + n^2\pi^2 c_n =$ n-th coeff of $f(x,t) = (t+1)\sin 2\pi x$

$c_n(0) = 0 \quad =$ n-th coefficient of $f(x) = 0$

$c_n'(0) =$ n-th coeff of $g(x) = 2x-1$

for q_n: $\quad (t+1)\sin 2\pi x = \sum_{n=1}^{\infty} q_n(t)\sin n\pi x \implies \begin{cases} q_2(t) = t+1 \\ \text{all other } c_n \equiv 0 \end{cases}$

for $c_n(0)$: $\quad c_n(0) = 0$

for $c_n'(0)$: $\quad 2x-1 = \sum_{n=1}^{\infty} c_n'(0)\sin n\pi x \implies$

$$c_n'(0) = \frac{\int_0^1 (2x-1)\sin n\pi x\,dx}{1/2} \qquad u = 2x-1 \quad dv = \sin n\pi x$$
$$du = 2 \quad v = -\frac{1}{n\pi}\cos n\pi x$$

num. $\displaystyle -\frac{(2x-1)}{n\pi}\cos n\pi x\Big|_0^1 + \frac{2}{n\pi}\cancel{\int_0^1 \cos n\pi x\,dx}^{\,0} =$

$\displaystyle = -\frac{1}{n\pi}\cos n\pi \quad \frac{1}{n\pi} = \frac{1}{n\pi}(1-\cos n\pi)$

$$c_n'(0) = \frac{-2}{n\pi}(1+\cos n\pi)$$

7.2.1 continued (completed in class)

for $n=2$ $\quad c_n'' + n^2\pi^2 c_n = t+1 \qquad c_n(0)=0 \qquad c_n'(0) = -\frac{2}{n\pi}(1+\cos n\pi)$

for $n \neq 2$ $\quad c_n'' + n^2\pi^2 c_n = 0 \qquad c_n(0)=0 \qquad c_n'(0) = -\frac{2}{n\pi}(1+\cos n\pi)$

$n=2$

$$\left.\begin{array}{l}(c_2)_c = k_1 \cos n\pi t + k_2 \sin n\pi t \\ (c_2)_p = \dfrac{t+1}{n^2\pi^2}\end{array}\right\} \Rightarrow c_2 = k_1 \cos n\pi t + k_2 \sin n\pi t + \frac{t+1}{n^2\pi^2}$$

$$c_2' = -n\pi k_1 \sin n\pi t + k_2 \cdot n\pi \cos n\pi t + \frac{1}{n^2\pi^2}$$

$$c_n(0)=0 \Rightarrow k_1 + \frac{1}{n^2\pi^2} = 0 \Rightarrow k_1 = -\frac{1}{n^2\pi^2}$$

$$c_n'(0) = -\frac{2}{n\pi}(1+\cos n\pi) \Rightarrow n\pi k_2 + \frac{1}{n^2\pi^2} = -\frac{2}{n\pi}(1+\cos n\pi)$$

$$k_2 = -\frac{1}{n^3\pi^3} - \frac{2}{n^2\pi^2}(1+\cos n\pi)$$

$$c_n = -\frac{1}{n^2\pi^2}\cos n\pi t - \left(\frac{1}{n^3\pi^3} + \frac{2}{n^2\pi^2}(1+\cos n\pi)\right)\sin n\pi t + \frac{t+1}{n^2\pi^2}$$

$n \neq 2$

$$c_n = k_1 \cos n\pi t + k_2 \sin n\pi t \qquad c_n' = -n\pi k_1 \sin n\pi t + n\pi k_2 \cos n\pi t$$

$$c_n(0)=0 \Rightarrow k_1 = 0 \qquad c_n'(0) = -\frac{2}{n\pi}(1+\cos n\pi) \Rightarrow k_2 = \frac{-2}{n^2\pi^2}(1+\cos n\pi)$$

$$c_n = \frac{-2}{n^2\pi^2}(1+\cos n\pi)\sin n\pi t$$

Notice that the term we computed when $n=2$ agrees with one of the terms in c_n for $n=2$. So we can include the latter in the sum.

Assembling

$$u(x,t) = \left(-\frac{1}{4\pi^2}\cos 2\pi t - \frac{1}{8\pi^3}\sin 2\pi t + \frac{t+1}{4\pi^2}\right)\sin 2\pi x + \sum_{n=1}^{\infty} \frac{-2}{n^2\pi^2}(1+\cos n\pi)\sin n\pi t \sin n\pi x$$