5.1.2. Heat Eqn using the 2nd basic SL BVP.

#2 $u_t = u_{xx} \qquad 0 \le x \le 1 \qquad t > 0$

$u_x(0,t) = 0 \qquad u_x(1,t) = 0$

$u(x,0) = f(x) = \cos 2\pi x - 3\cos 3\pi x$

Soln: The formal solution to this problem is

$$u(x,t) = \sum_{n=0}^{\infty} c_n \cos n\pi x \, e^{-n^2\pi^2 t}$$

$u(x,0) = \cos 2\pi x - 3\cos 3\pi x \implies$ match up $\implies$

$\cos 2\pi x - 3\cos 3\pi x = \sum_{n=0}^{\infty} c_n \cos n\pi x \implies c_2 = 1 \qquad c_3 = -3$

$$u(x,t) = \cos 2\pi x \, e^{-4\pi^2 t} - 3\cos 3\pi x \, e^{-9\pi^2 t}$$

#5 $u(x,t) = \sum_{n=0}^{\infty} c_n \cos n\pi x \, e^{-n^2\pi^2 t}$

$u(x,0) = f(x) = \begin{cases} -2 & 0 \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} < x \le 1 \end{cases} \implies c_n = \dfrac{\int_0^1 f(x)\cos n\pi x \, dx}{\int_0^1 \cos^2 n\pi x \, dx}$

$n = 0 \qquad c_0 = \dfrac{\int_0^1 f(x)\,dx}{\int_0^1 dx} = \dfrac{-1}{1} = -1$

$n \ge 1 \qquad c_n = \dfrac{\int_0^{1/2} -2\cos n\pi x \, dx}{\int_0^1 \cos^2 n\pi x \, dx}$

num: $-\frac{2}{n\pi} \sin n\pi x \Big|_0^{1/2} = -\frac{2}{n\pi} \sin n\pi/2$

denom: $\frac{1}{2}$

$$u(x,t) = -1 - \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\frac{n\pi}{2} \cos n\pi x \, e^{-n^2\pi^2 t}$$

# Heat Eqn with SL BVP of type (3) and (4)

S.1.3 #3 $\quad u_t = u_{xx} \qquad 0 \le x \le 1 \qquad t > 0$

$u(x,0) = f(x) = 2 + x$

$u(0,t) = u_x(1,t) = 0$

Soln: Formal Soln is $u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{(2n-1)\pi}{2} x \, e^{-\frac{(2n-1)^2\pi^2}{4} t}$

(The e-functions of the translated bdy conditions are those of SL BVP (3))

$$c_n = \frac{\int_0^1 f(x) \sin \frac{(2n-1)\pi}{2} x \, dx}{\int_0^1 \sin^2 \frac{(2n-1)\pi}{2} x \, dx}$$

$u = 2 + x \qquad dv = \sin \frac{(2n-1)\pi}{2} x$

$du = dx \qquad v = -\frac{2}{(2n-1)\pi} \cos \frac{(2n-1)\pi}{2} x$

num: $\int_0^1 (2+x) \sin \frac{(2n-1)\pi}{2} x \, dx = \frac{-2}{(2n-1)\pi} (2+x) \cos \frac{(2n-1)\pi}{2} x \Big|_0^1 + \int_0^1 \frac{2}{(2n-1)\pi} \cos \frac{(2n-1)\pi}{2} x \, dx$

$= \frac{4}{(2n-1)\pi} + \frac{4}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi}{2} x \Big|_0^1 = \frac{4}{(2n-1)\pi} + \frac{4}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi}{2}$

denom $= 1/2 \Rightarrow$ $\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[ \frac{8}{(2n-1)\pi} + \frac{8}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi}{2} \right] \sin \frac{(2n-1)\pi}{2} x \, e^{-\frac{(2n-1)^2\pi^2}{4} t}}$

5) $u_t = u_{xx} \quad 0 \le x \le 1 \qquad t > 0$

$u_x(0,t) = u(0,t) = 0 \qquad u(x,0) = 2 \cos \frac{5\pi}{2} x$

Soln Translated bdy conds are those of SL BVP

Formal soln is $u(x,t) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi}{2} x \, e$

$u(x,0) = 2 \cos \frac{5\pi}{2} x \implies 2\cos(5\pi x/2) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi}{2} x$

match up $\quad c_3 = 2 \quad$ all other $c$'s $= 0$

$u(x,t) = 2 \cos \frac{5\pi x}{2} \, e^{-\frac{25\pi^2}{4} t}$