

1. Basic ODEs

- (a) separable: $y' = \frac{F(x)}{G(y)}$
 $\Rightarrow \frac{dy}{dx} = \frac{F(x)}{G(y)} \Rightarrow \int G(y)dy = \int F(x)dx$
- (b) Linear: $y' + p(x)y = q(x)$
 Integrating Factor: $e^{\int p(x)dx}$
 $\Rightarrow (e^{\int p(x)dx}y)' = e^{\int p(x)dx}q(x)$
 $\Rightarrow e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx$
- (c) $ay'' + by' + cy = 0$ (constant coefficient)
 characteristic EQ: $ar^2 + br + c = 0$
 $\Delta > 0, y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
 $\Delta = 0, y = C_1 e^{rx} + C_2 x e^{rx}$
 $\Delta < 0, r = p \pm qi : y = e^{ax}[c_1 \cos(bx) + c_2 \sin(bx)]$

2. Fourier Series

Given $F(x), x \in [-L, L]$ write $F(x)$ in a series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where a_n, b_n are constants.

- (a) Orthogonality Relations
 $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$
 $\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0 (m \neq n), L (m = n)$
 $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0 (m \neq n), L (m = n)$
- (b) $a_n = \frac{1}{L} \int_{-L}^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$
 $b_n = \frac{1}{L} \int_{-L}^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx, n \in [0, \infty], n \in \mathbb{Z}$
- (c) Convergence Statement of F.S.
 F.S. convergence to the "periodic extension" of $F(x)$ wherever $F(x)$ is continuous and to the average of $\frac{f(x^+) + f(x^-)}{2}$ at every point.
- (d) F.S.S and F.C.S of $F(x)$ on $[0, L]$:
 F.C.S = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), a_n = \frac{2}{L} \int_0^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$
 F.S.S = $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), a_n = \frac{2}{L} \int_0^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$
 F.C.S \rightarrow even extension, F.S.S \rightarrow odd extension

3. Sturm-Liouville Problem

- (a) Basic Examples of S-L BVP
 $f'' + \lambda f = 0, 0 \leq x \leq L$ and Boundary conditions
 Def: value λ for which the equation with the given boundary ends: has non-trivial solution is called eigen value, the corresponding solution is called eigen functions of the given S-L BVP.
 First, general solutions:
 $\lambda = 0, f(x) = \alpha x + \beta$
 $\lambda > 0, f(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$
 $\lambda < 0, f(x) = C_1 \cosh(ax) + C_2 \sinh(ax), a^2 = -\lambda, a > 0$
 Then, impose the boundary in each case.
- i. Boundary COND: $f(0) = 0, f(L) = 0$
 e-values: $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 1, 2, 3$
 e-functions: $f_n \sim \sin\left(\frac{n\pi x}{L}\right)$

ii. Boundary COND: $f'(0) = 0, f'(L) = 0$

e-values: $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 0, 1, 2, 3$

e-functions: $f_n \sim \cos\left(\frac{n\pi x}{L}\right)$

iii. Boundary COND: $f(0) = 0, f'(L) = 0$

e-values: $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n = 1, 2, 3$

e-functions: $f_n \sim \sin\left(\frac{(2n-1)\pi}{2L}x\right)$

iv. Boundary COND: $f'(0) = 0, f(L) = 0$

e-values: $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n = 1, 2, 3$

e-functions: $f_n \sim \cos\left(\frac{(2n-1)\pi}{2L}x\right)$

(b) Regular S-L Problems

EQ: $(pf')' + qf + \lambda \sigma f = 0, a < x < b$

Boundary: $k_1 f(a) + k_2 f'(a) = 0, k_3 f(b) + k_4 f'(b) = 0$

$$g(x) \sim \sum_{n=1}^{\infty} a_n f_n(x)$$

$$a_n = \frac{\int_a^b g(x) f_n(x) \sigma(x) dx}{\int_a^b \sigma(x) f_n^2(x) dx}$$

4. Heat and Wave Equation

(a) Heat Equation

$\mu_t = k \mu_{xx}, \mu(x, t) = X(x)T(t)$ (usually $k = 1$)

$\rightarrow X(x)T'(t) = kX''(x)T(t)$

Let $\frac{T'}{kT} = \frac{kX''}{X} = -\lambda$

We get $X'' + \lambda x = 0, T' + k\lambda T = 0$

For second EQ, $T \sim e^{-k\lambda t}$

For first EQ, we apply S-L BVP problem

$\mu(x, t) = \sum_{n=1}^{\infty} c_n f_n e^{-\lambda_n t}$

Usually, we find the bound. cond. in the 4 fourier series probs.

e.g. bond cond 1:

$\lambda_n = \left(\frac{n\pi}{L}\right)^2, f_n \sim \sin\left(\frac{n\pi x}{L}\right)$

$\mu(x, 0) = g(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

Fouries formal solution: $\mu(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{L^2} t}$

note : in bond cond 2, n starts from 0. calculate case 0 separately.

(b) Wave Equation

$\mu_{tt} = C^2 \mu_{xx}, \mu(x, t) = X(x)T(t)$

$XT'' = C^2 X''T$

Let $\frac{T''}{C^2 T} = \frac{X''}{X} = -\lambda$

We get $X'' + \lambda x = 0, T'' = -\lambda C^2 T$

$\mu(x, t) = \sum_{n=1}^{\infty} (b_{1,n} \cos\left(\frac{n\pi ct}{L}\right) + b_{2,n} \sin\left(\frac{n\pi ct}{L}\right)) \sin\left(\frac{n\pi x}{L}\right)$

i. $\mu(x, 0) = f(x)$

$f(x) = \sum_{n=1}^{\infty} b_{1,n} \sin\left(\frac{n\pi x}{L}\right)$

$b_{1,n} = \frac{\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx}$

ii. $\mu_t(x, 0) = g(x)$

$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_{2,n} \sin\left(\frac{n\pi x}{L}\right)$

$\frac{n\pi c}{L} b_{2,n} = \frac{\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx}$