Characteristics – 2 worked problems

ex 1 $u_t - 3u_x = 1 - x$ $\qquad u(x,0) = x^2 + 1$

soln: $\frac{dx}{dt} = -3$

$x = -3t + x_0 \longrightarrow$ $\frac{du}{dt} = 1 - x$

$x_0 = x + 3t$ $\qquad \frac{du}{dt} = 1 + 3t - x_0$

$u = t + \frac{3}{2}t^2 - x_0 t + K$

$\left.\begin{aligned} u(x_0,0) &= K \\ u(x_0,0) &= x_0^2 + 1 \end{aligned}\right\} \Rightarrow K = x_0^2 + 1$

$$u = t + \frac{3}{2}t^2 - (x+3t)t + (x+3t)^2 + 1$$

simplifying $u = t + \frac{3}{2}t^2 - xt - 3t^2 + x^2 + 6xt + 9t^2 + 1$

$$u = t + \frac{15}{2}t^2 + 5xt + x^2 + 1$$

ex 2 $u_t + u^3 u_x = 0$ $\qquad u(x,0) = x^{1/3}$

soln $\frac{dx}{dt} = u^3$ $\qquad \frac{du}{dt} = 0$

$u = C$

$\left.\begin{aligned} u(x_0,0) &= C \\ u(x_0,0) &= x_0^{1/3} \end{aligned}\right\} \Rightarrow C = x_0^{1/3}$

$\leftarrow \quad u = x_0^{1/3}$

$\frac{dx}{dt} = x_0$

$x = x_0 t + x_0$

$x = x_0(t+1)$ $\longrightarrow$ $u = \left(\frac{x}{t+1}\right)^{1/3}$

$x_0 = \frac{x}{t+1}$

12.2 #1

$u_t - (2u+1)u_x = 3$ $\qquad u(x,0) = F(x) = 1-x$

sol

$\frac{dx}{dt} = -2u-1$ $\qquad \frac{du}{dt} = 3$

$u = 3t + C$

$\begin{cases} u(x_0,0) = C \\ u(x_0,0) = 1-x_0 \end{cases} \Rightarrow \quad C = 1-x_0$

$\frac{dx}{dt} = -6t - 3 + 2x_0 \quad \leftarrow \quad u = 3t + 1 - x_0$

$x = -3t^2 - 3t + 2x_0 t + K$

$x(0) = x_0 \Rightarrow \quad K = x_0$

$x = -3t^2 - 3t + 2x_0 t + x_0$

$x = -3t^2 - 3t + (2t+1)x_0$

$x_0 = \frac{x + 3t^2 + 3t}{2t+1} \longrightarrow u = 3t + 1 - \frac{x + 3t^2 + 3t}{2t+1}$

$u = \frac{6t^2 + 3t + 2t + 1 - x - 3t^2 - 3t}{2t+1}$

$u = \frac{3t^2 + 2t + 1}{2t+1}$

Characteristics

12.2 #3

$u_t + (u+t)u_x = 1$ $\qquad u(x,0) = f(x) = x+1$

Soln

$\frac{dx}{dt} = u+t$ $\qquad \frac{du}{dt} = 1$

$u = t + C$

$\begin{cases} u(x_0,0) = C \\ u(x_0,0) = x_0 + 1 \end{cases} \Rightarrow \quad C = x_0 + 1$

$\leftarrow$ $u = t + x_0 + 1$

$\frac{dx}{dt} = 2t + x_0 + 1$

$x = t^2 + x_0 t + t + K$

$x(0) = x_0 \Rightarrow x_0 = K$

$x = t^2 + x_0 t + t + x_0$

$x_0 = \frac{x - t^2 - t}{t+1}$ $\longrightarrow$ $u = t + \frac{x - t^2 - t}{t+1} + 1$

$u = \frac{t^2 + t + x - t^2 - t + t + 1}{t+1}$

Ans. $u = \frac{x+t+1}{t+1}$

#5 $u_t + u_x = 2 + u^2$ $\quad u(x,0) = f(x) = -e^{-x}$

Sol

$\frac{dx}{dt} = 1$ $\quad \frac{du}{dt} = 2 + u^2$

$x = t + C$ $\quad \int \frac{du}{u^2} = \int 2 + dt$

$x(0) = x_0 \Rightarrow C = x_0$ $\quad -\frac{1}{u} = t^2 + k$

$x = t + x_0$ — $\quad u = \frac{-1}{t^2 + k}$

$\begin{cases} u(x_0,0) = -\frac{1}{k} \Rightarrow k = e^{x_0} \\ u(x_0,0) = -e^{-x_0} \end{cases}$

$x_0 = x - t$ $\quad u = \frac{-1}{t^2 + e^{x_0}}$

$u = \frac{-1}{t^2 + e^{x-t}}$