1. Basic ODEs

   (a) seperatable: $y' = \frac{F(x)}{G(y)}$
   $\Rightarrow \frac{dy}{dx} = \frac{F(x)}{G(y)} \Rightarrow \int G(y)dy = \int F(x)dx$

   (b) Linear: $y' + p(x)y = q(x)$
   Integrating Factor: $e^{\int p(x)dx}$
   $\Rightarrow (e^{\int p(x)dx}y)' = e^{\int p(x)dx}q(x)$
   $\Rightarrow e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx$

   (c) $ay'' + by' + cy = 0$ (constant coefficient)
   characteristc EQ: $ar^2 + br + c = 0$
   $\Delta > 0, y = C_1e^{r_1x} + C_2e^{r_2x}$
   $\Delta = 0, y = C_1e^{rx} + C_2xe^{rx}$
   $\Delta < 0, r = p \pm qi : y = e^{px}[c_1cos(qx) + c_2sin(qx)]$

   (d) $ay'' + by' + cy = f(x)$
   $y = y_c + y_p$, y_c is solution to homogeneous DIFF EQ
   $f(x)$:a polynomial in x or single sin/cos function
   $y_p = x^k$(a polynomial of the same degree), k: # char eq's zero roots (0,1,2)
   $f(x) = e^{ax}$(a polynomial in x)
   $y_p = x^ke^{ax}$(same degree), k: # char eq's roots = a (0,1,2)
   $f(x) = e^{ax}cos(bx)$(poly in x) or $e^{ax}sin(bx)(apolyinx)$
   $y_p = x^ke^{ax}$[(poly in x)$cos(bx)$ + (poly in x)$sin(bx)$]
   k: # char eq's root = $a \pm bi$ (0,1)

   (e) Second Order Cauchy-Euler Equation
   homogeneous: $ax^2y'' + bxy' + cy = 0$
   characteristic Equation: $am^2 + (b-a)m + c = 0$

      i. Two different real roots:
      $y = c_1x^{m_1} + c_2x^{m_2}$

      ii. Same real root:
      $y = c_1x^m + c_2x^m lnx$

      iii. complex roots:
      $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$
      $y = x^\alpha[c_1cos(\beta lnx) + c_2sin(\beta lnx)]$

   nonhomogeneous: $ax^2y'' + bxy' + cy = f(x)$
   $y = y_c + y_p$, similar to the above case.

2. Fourier Series
   Given $F(x), x \in [-L, L]$ write $F(x)$ in a series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{L})$$

   where a_n, b_n are constants.

   (a) Orthogonality Relations
   $\int_{-L}^{L} sin(\frac{n\pi x}{L})cos(\frac{m\pi x}{L}) = 0$
   $\int_{-L}^{L} cos(\frac{n\pi x}{L})cos(\frac{m\pi x}{L}) = 0(m \neq n), L(m = n)$
   $\int_{-L}^{L} sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L}) = 0(m \neq n), L(m = n)$

   (b) $a_n = \frac{1}{L}\int_{-L}^{L} F(x)cos(\frac{n\pi x}{L})dx$
   $b_n = \frac{1}{L}\int_{-L}^{L} F(x)sin(\frac{n\pi x}{L})dx, n \in [0, \infty], n \in \mathbf{Z}$

   (c) Convergece Statement of F.S.
   F.S. convergence to the "periodic extension" of F(x) whever F(x) is continous and to the average of $\frac{f(x^+)+f(x^-)}{2}$ at every point.

   (d) F.S.S and F.C.S of $F(x)$ on $[0, L]$:
   Suppose F(x) is even, then $b_n = 0$, for all n
   F.C.S = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{L}, a_n = \frac{2}{L}\int_0^L F(x)cos(\frac{n\pi x}{L})dx$
   Suppose F(x) is odd, then $a_n = 0$, for all n
   F.S.S = $\sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{L}, b_n = \frac{2}{L}\int_0^L F(x)sin(\frac{n\pi x}{L})dx$
   F.C.S $\rightarrow$ even extension, F.S.S $\rightarrow$ odd extension

3. Sturm-Liouville Problem

   (a) Hyperbolic sin and cos
   $coshx = \frac{e^x+e^{-x}}{2}$, $sinhx = \frac{e^x-e^{-x}}{2}$
   $(coshax)' == asinhax$, $(coshax)'' = a^2coshax$
   $(sinhax)' == acoshax$, $(sinhax)'' = a^2sinhax$
   $sinh(\alpha - \beta) = sinh\alpha cosh\beta - cosh\alpha sinh\beta$
   $cosh(\alpha - \beta) = cosh\alpha cosh\beta - sinh\alpha sinh\beta$

   (b) Basic Examples of S-L BVP
   $f'' + \lambda f = 0, 0 \leq x \leq L$ and Boundary conditions
   Def: value λ for which the equation with the given boundary ends: has non-trival solution is called eigen value, the corresponding solution is called eigeon functions of the given S-L BVP.
   General solutions:
   $\lambda = 0, f(x) = \alpha x + \beta$
   $\lambda > 0, f(x) = C_1cos(\sqrt{\lambda}x) + C_2sin(\sqrt{\lambda}x)$
   $\lambda < 0, f(x) = C_1cosh(ax) + C_2sinh(ax), a^2 = -\lambda, a > 0$
   Impose the boundary in each case.

      i. Boundary COND: $f(0) = 0, f(L) = 0$
      e-values: $\lambda_n = \frac{n^2\pi^2}{L^2}, n = 1, 2, 3$
      e-functions: $f_n \sim sin(\frac{n\pi x}{L})$

      ii. Boundary COND: $f'(0) = 0, f'(L) = 0$
      e-values: $\lambda_n = \frac{n^2\pi^2}{L^2}, n = 0, 1, 2, 3$
      e-functions: $f_n \sim cos(\frac{n\pi x}{L})$

      iii. Boundary COND: $f(0) = 0, f'(L) = 0$
      e-values: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2, n = 1, 2, 3$
      e-functions: $f_n \sim sin(\frac{(2n-1)\pi}{2L}x)$

      iv. Boundary COND: $f'(0) = 0, f(L) = 0$
      e-values: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2, n = 1, 2, 3$
      e-functions: $f_n \sim cos(\frac{(2n-1)\pi}{2L}x)$

   (c) Regular S-L Problems
   EQ: $(pf')' + qf + \lambda\sigma f = 0, a < x < b$
   Boundary: $k_1f(a) + k_2f'(a) = 0, k_3f(b) + k_4f'(b) = 0$

$$g(x) \sim \sum_{n=1}^{\infty} a_n f_n(x)$$

$$a_n = \frac{\int_a^b g(x)f_n(x)\sigma(x)dx}{\int_a^b \sigma(x)f_n^2(x)dx}$$

   ususlly $\sigma(x) = 1$

4. Method of Separationg: Heat and Wave Equation

   (a) Heat Equation
   $\mu_t = k\mu_{xx}$
   $\mu(x, t) = X(x)T(t)$ (usually k = 1)
   Seperation$\rightarrow$ $X(x)T'(t) = kX''(x)T(t)$
   Let $\frac{T'}{kT} = \frac{kX''}{x} = -\lambda$
   We get $X'' + \lambda x = 0$, $T' + k\lambda T = 0$

For second EQ, T $\sim e^{-k\lambda t}$
For first EQ, we apply S-L BVP problem
$\mu(x,t) = \sum_{n=1}^{\infty} c_n f_n e^{-\lambda_n t}$
Usually, we find the bound. cond. in the 4 fourier series probs.
e.g. bond cond 1:
$\lambda_n = (\frac{n\pi}{L})^2$, $f_n \sim sin(\frac{n\pi x}{L})$
$\mu(x,0) = g(x) = \sum_{n=1}^{\infty} c_n sin(\frac{n\pi x}{L})$
Fouries formal solution: $\mu(x,t) = \sum_{n=1}^{\infty} c_n sin(\frac{n\pi x}{L}) e^{-\frac{n^2\pi^2}{L^2}t}$
e.g. bond cond 2:
$\lambda_0 = 0$, $X_0(x) = 1/2$
$\lambda_n = (\frac{n\pi}{L})^2$, $f_n \sim cos(\frac{n\pi x}{L})$
$\mu(x,0) = g(x) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n cos(\frac{n\pi x}{L})$
$g(x) = \frac{1}{2}c_0 \Rightarrow c_0 = \frac{\int_0^L \frac{1}{2}g(x)dx}{\int_0^L (\frac{1}{2})^2 dx}$
$g(x) = \sum_{n=1}^{\infty} c_n cos(\frac{n\pi x}{L}) \Rightarrow c_n = \frac{\int_0^L cos(\frac{n\pi x}{L})g(x)dx}{\int_0^L (cos(\frac{n\pi x}{L})^2 dx}$
Solution: $\mu(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n cos(\frac{n\pi x}{L}) e^{-\frac{n^2\pi^2}{L^2}t}$
If the bound. cond. not in the 4 SLBVP:
Do the SL-BVP manually and find e-value and e-function.
write $\lambda_n = (\frac{\zeta_n}{L})$, $X_n = sin/cos\frac{\zeta_n}{L}$

(b) Wave Equation
$\mu_{tt} = C^2\mu_{xx}$
$\mu(x,t) = X(x)T(t)$, usually $C = 1$
$\mu(0,t) = 0$, $\mu(L,t) = 0$
Seperation $\rightarrow XT'' = C^2X''T$
Let $\frac{T''}{c^2T} = \frac{X''}{x} = -\lambda$
We get $X'' + \lambda x = 0$, $T'' = -\lambda c^2 T$
$T_n(t) = b_{1,n} cos\frac{n\pi ct}{L} + b_{2,n} sin\frac{n\pi ct}{L}$, $X_n = sin\frac{n\pi cx}{L}$ (X_n from SL-BVP problem)
When c = 1:
$\mu(x,t) = X_nT_n = \sum_{n=1}^{\infty}(b_{1,n}cos\frac{n\pi ct}{L} + b_{2,n}sin\frac{n\pi ct}{L})sin\frac{n\pi x}{L}$

i. $\mu(x,0) = f(x)$
$f(x) = \sum_{n=1}^{\infty} b_{1,n} sin\frac{n\pi x}{L}$
$b_{1,n} = \frac{\int_0^L f(x) sin\frac{n\pi x}{L}dx}{\int_0^L sin^2\frac{n\pi x}{L}dx}$

ii. $\mu_t(x,0) = g(x)$
$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_{2,n} sin\frac{n\pi x}{L}$
$\frac{n\pi c}{L} b_{2,n} = \frac{\int_0^L g(x) sin\frac{n\pi x}{L}dx}{\int_0^L sin^2\frac{n\pi x}{L}dx}$

(c) d'Alembert's solution to the Wave Equation
Given $\mu(x,0) = F(x)$,$\mu_t(x,0) = 0$ we want to get $\mu(x,t)$
Do an odd extention on $F(x)$, then $\mu(x,t) = \frac{F(x+ct)+F(x-ct)}{2}$. Usually c is 1. Simply draw the graph of $F(x+ct)$(shift to left) and $F(x-ct)$(shift to right) and get the average.
A full period: $\frac{2L}{c}$

5. Laplace Equation
Consider the equilibrium temperature in a uniform rectangle(Heat EQ)

$$\begin{cases} \mu_{xx}(x,y) + \mu_{yy}(x,y) = 0, \ 0 < x < L, \ 0 < y < K \\ \mu(0,y) = f_1(y), \ \mu(L,y) = f_2(y), \ 0 < y < K \\ \mu(x,0) = g_1(x), \ \mu(x,K) = g_2(x), \ 0 < x < L \end{cases} \tag{1}$$

Where we might have first order partial derivatives on those μ's on 2nd or 3rd line.
Seperate $\rightarrow \mu = XY$, then
$X''Y + XY'' = 0$
$\frac{X''}{X} = -\frac{Y''}{Y}$
we want to create a SL-BVP problem:
Determine $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$ or $-\lambda$
When $f_1(y) = f_2(y) = 0$ or $X(0) = X(L) = 0$ we can form SL-BVP on Y and we choose $-\lambda$.
Similarly, when $g_1(x) = g_2(x) = 0$ or $Y(0) = Y(K) = 0$ we can form SL-BVP on X and we choose λ
when we choose $-\lambda$: $(X(0)\ or\ X'(0) = 0, X(L)\ or\ X'(L) = 0)$

$$\begin{cases} X'' + \lambda X = 0, \ \ (SL-BVP) \\ \mu(x,0) = f_1(y), \mu(x,K) = f_2(y) \end{cases} \tag{2}$$

$\Rightarrow$ find out λ_n and X_n Then:

$$Y_n'' - \lambda_n Y_n = 0, 0 < y < K$$

$$Y_n = \alpha_n sinh\frac{n\pi y}{L} + \beta_n cosh\frac{n\pi y}{L}$$

$$\mu(x,y) = \sum_{n=1}^{\infty}(\alpha_n sinh\frac{n\pi y}{L} + \beta_n cosh\frac{n\pi y}{L})X_n$$

For simplicity, we transform to:

(a) $\mu(x,0) = g_1(x)$, $\mu(x,K) = g_2(x)$
$Y_n = \alpha_n sinh\frac{n\pi(y-K)}{L} + \beta_n sinh\frac{n\pi y}{L}$
(b) $\mu_y(x,0) = g_1(x)$, $\mu_y(x,K) = g_2(x)$
$Y_n = \alpha_n cosh\frac{n\pi(y-K)}{L} + \beta_n cosh\frac{n\pi y}{L}$
(c) $\mu(x,0) = g_1(x)$, $\mu_y(x,K) = g_2(x)$
$Y_n = \alpha_n cosh\frac{n\pi(y-K)}{L} + \beta_n sinh\frac{n\pi y}{L}$
(d) $\mu_y(x,0) = g_1(x)$, $\mu(x,K) = g_2(x)$
$Y_n = \alpha_n cosh\frac{n\pi y}{L} + \beta_n sinh\frac{n\pi(y-K)}{L}$

Then we get $\mu(x,0)$, $\mu(x,K)$ (as long as they fit the other 2 equations) and try to figure out α_n, β_n using match or SL-BVP.
Note : in bond cond 2, n starts from 0. calc case 0 seperately: $\lambda_0 = 0$, $X_0 = 1 \Rightarrow Y_0'' - 0Y_0 = 0 \Rightarrow Y_0 = ay + b$

6. Method of Eigenfunction Expansion
Consider
$PDE: \mu_t = k\mu_{xx}(x,t) + q(x,t), 0 < x < L, t > 0$
$BCs: \mu(0,t) = 0, \mu(L,t) = 0, t > 0$
$IC: \mu(x,0) = f(x), 0 < x < L$

(a) BCs $\Rightarrow$ Eigenfunction: $X_n(x), \lambda_n$
$\Rightarrow \mu(x,t) = \sum_{n=1}^{\infty} C_n(t)X_n(x)$
Write $q(x,t) = \sum_{n=1}^{\infty} q_n(t)X_n(x)$
(b) PDE: $X_n'' + \lambda_n X_n = 0$

$$\Rightarrow \sum_{n=1}^{\infty} C_n'(t)X_n(x) = \sum_{n=1}^{\infty} C_n(t)X_n''(x) + \sum_{n=1}^{\infty} q_n(t)X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} C_n'(t)X_n(x) = -\sum_{n=1}^{\infty} C_n(t)\lambda_n X_n(x) + \sum_{n=1}^{\infty} q_n(t)X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty}[C_n'(t) + C_n(t)\lambda_n]X_n(x) = \sum_{n=1}^{\infty} q_n(t)X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty}[C_n'(t)+C_n(t)\lambda_n] = \sum_{n=1}^{\infty} q_n(t)$$

Form may vary (μ is differentiated in other ways)

(c) IC $\Rightarrow \mu(x,0) = f(x) = \sum_{n=1}^{\infty} C_n(0)X_n(x)$ (SL-BVP)
either match up or $C_n(0) = \frac{\int_0^L f(x)X_n(x)dx}{\int_0^L X_n^2(x)dx}$

(d) Then we solve functions $C_n'(t)+C_n(t)\lambda_n = q_n(t)$ for each n with initial condition $C_n(0)'s$ and $q_n(0)'s$

7. Laplace Transform
Def: $\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$

$$Def: H_a(t) = H(t-a) = \begin{cases} 1, \ t \geq a \\ 0, \ t < a \end{cases} \tag{3}$$

Def: $erf(x) = \frac{2}{\pi}\int_0^x e^{-u^2}du$, $erfc(x) = 1 - erf(x)$

$f(t) = \mathcal{L}^{-1}F(t)$	$F(t) = \mathcal{L}f(s)$
$f^{(n)}(t)$ (nth derivative)	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$H(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$(f * g)(t)$	$F(s)G(s)$
1	$\frac{1}{s}$ $(s>0)$
t^n (n is positive integer)	$\frac{n!}{s^{n+1}}$ $(s>0)$
e^{at}	$\frac{1}{s-a}$ $(s>a)$
$sin(at)$	$\frac{a}{s^2+a^2}$ $(s>0)$
$cos(at)$	$\frac{s}{s^2+a^2}$ $(s>0)$
$sinh(at)$	$\frac{a}{s^2-a^2}$ $(s>\lvert a\rvert)$
$cosh(at)$	$\frac{s}{s^2-a^2}$ $(s>\lvert a\rvert)$
$\delta(t-a)$ $(a \geq 0)$	e^{-as}
t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}sinbt$	$\frac{b}{((s-a)^2+b^2)}$
$e^atcosbt$	$\frac{s-a}{((s-a)^2+b^2)}$
$erfc(\frac{a}{2\sqrt{t}})$	$\frac{1}{s}e^{-a\sqrt{s}}$

Example:
$\mu_{tt}(x,t) = \mu_{xx}(x,t)+1$, $\mu(x,0) = \mu_t(x,0) = -1, \mu(0,t) = t$
$\mathcal{L}(\mu_{tt}) = s^2U - s\mu(x,0) - \mu_t(x,0) = s^2U + 1$
$\mathcal{L}(\mu_{xx}+1) = U_{xx} + \frac{1}{s}$
$\Rightarrow U'' - s^2U = 1 - \frac{1}{s}$ (x is variable, s is scalar)
$U_c = k_1e^{sx} + k_2e^{-sx}$, $k_1 = 0$
$U_p = \frac{1-\frac{1}{s}}{-s^2} = \frac{1}{s^3} - \frac{1}{s^2}$
$\mathcal{L}\mu(0,t) = \mathcal{L}t = \frac{1}{s} = U(0,s)$
$\Rightarrow k_2 + \frac{1}{s^3} - \frac{1}{s^2} = \frac{1}{s^2}$
$\Rightarrow U = (\frac{2}{s^2} - \frac{1}{s^3})e^{-sx} - \frac{1}{s^2} + \frac{1}{s^3}$
$\mu = \mathcal{L}^{-1}(U) = H(t-x)[2(t-x) - \frac{1}{2}(t-x)^2] - t + \frac{1}{2}t^2$

(a) wave eq
$\mu_{tt} = c^2\mu_{xx}, x>0, t>0$
$\mu(0,t) = f(t), t>0$
$\mu(x,0) = 0, \mu_t(x,0) = 0, x>0$
$\mathcal{L}\mu(x,s) = U(x,s), \mathcal{L}f(s) = F(s)$
$\Rightarrow s^2U(x,s) = c^2U''(x,s) \Rightarrow U''(x,s) - (\frac{s}{c})^2U(x,s) = 0$
General Solution:$U(x,s) = C_1(s)e^{\frac{s}{c}x} + C_2(s)e^{-\frac{s}{c}x}$
$C_1(s) = 0$, then $U(x,s) = F(s)e^{-\frac{s}{c}x} = F(s)e^{-\frac{x}{c}s}$
$u(x,t) = H(t-x/c)f(t-x/c)$

(b) Heat equation
$\mu_t(x,t) = \mu_{tt}(x,t)$
$\mu_x(0,t) - \mu(0,t) = 0$
$\mu(x,0) = \mu_0 = const$
$\Rightarrow U''(x,s) - sU(x,s) + \mu_0 = 0$
$U'(0,s) - U(0,s) = 0$
General Solut: $U(x,s) = C_1(s)e^{\sqrt{s}x} + C_2(s)e^{-\sqrt{s}x} + \frac{1}{s}\mu_0$
$C_1(s) = 0$, $C_2(s) = -\frac{\mu_0}{s(\sqrt{s}+1)}$
$\Rightarrow U(x,s) = \mu_0[-\frac{1}{s(\sqrt{s}+1)}e^{-\sqrt{s}x} + \frac{1}{s}]$
$\mu(x,t) = \mu_0[1 - erfc(\frac{x}{2\sqrt{t}}) - erfc(\sqrt{t} + \frac{x}{2\sqrt{t}})e^{x+t}]$

8. method of characteristics
$\mu_t + A\mu_x = B$
$A = \frac{dx}{dt}$, $B = \frac{du}{dt}$