Problem done in class

Given the SL-BVP $\begin{cases} f'' + \lambda f = 0, & f = f(x) \quad 0 \le x \le 1 \\ f'(0) = 0 & f(1) - f'(1) = 0. \end{cases}$

a) Determine all e-values of the problem graphically.

b) Determine all e-functions of the problem.

c) No estimates required but show the 1st three e-values on your graphs.

Soln

$\lambda = 0 \implies f = ax + b \qquad f' = a$

$f'(0) = 0 \implies \boxed{a = 0} \implies f = b \qquad f' = 0$

$f(1) - f'(1) = 0 \implies \boxed{b = 0}$ So $\lambda = 0$ is <u>not</u> an e-value

$\lambda > 0 \implies f = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x \qquad f' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$

$f'(0) = 0 \implies c_2 = 0 \implies \boxed{f = c_1 \cos\sqrt{\lambda}\, x} \qquad f' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x$

$f(1) - f'(1) = 0 \implies \cos\sqrt{\lambda} + \sqrt{\lambda}\sin\sqrt{\lambda} = 0 \implies$
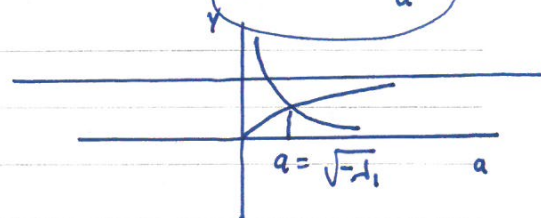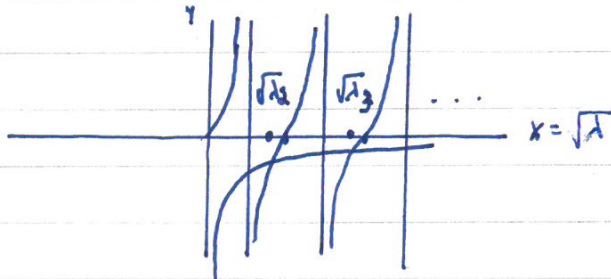
$$\boxed{\tan\sqrt{\lambda} = -\frac{1}{\sqrt{\lambda}}}$$

$\lambda < 0 \qquad \lambda = -a^2 \qquad a > 0$

$\qquad f = c_1 \cosh ax + c_2 \sinh ax \qquad f' = c_1 a \sinh ax + c_2 a \cosh ax$

$f'(0) = 0 \implies c_2 = 0 \implies \boxed{f = c_1 \cosh ax} \qquad f' = c_1 a \sinh ax$

$f(1) - f'(1) = 0 \implies \cosh a - a \sinh a = 0 \implies \boxed{\tanh a = \frac{1}{a}}$

The positive e-values are the solutions $\lambda_2, \lambda_3 \dots$ to $\tan\sqrt{\lambda_n} = -\frac{1}{\sqrt{\lambda_n}}$

The e-functions $\boxed{f_n \sim \cos\sqrt{\lambda_n}\, x} \qquad n = 2, 3, -$

There is one negative e-value $\boxed{\lambda_1 = -a^2}$ where a solve $\boxed{\tanh a = \frac{1}{a}}$

The corresponding e-function is $\boxed{f_1 \sim \cosh ax}$

(estimates $\lambda_1 \approx -1.44 \qquad \lambda_2 \approx 7.8 \qquad \lambda_3 \approx 37.2$) $\qquad \sqrt{\lambda_n} \approx (n-1)\pi \quad n$ large