

## 1. Basic ODEs

- (a) separable:  $y' = \frac{F(x)}{G(y)}$   
 $\Rightarrow \frac{dy}{dx} = \frac{F(x)}{G(y)} \Rightarrow \int G(y)dy = \int F(x)dx$
- (b) Linear:  $y' + p(x)y = q(x)$   
 Integrating Factor:  $e^{\int p(x)dx}$   
 $\Rightarrow (e^{\int p(x)dx}y)' = e^{\int p(x)dx}q(x)$   
 $\Rightarrow e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx$
- (c)  $ay'' + by' + cy = 0$  (constant coefficient)  
 characteristic EQ:  $ar^2 + br + c = 0$   
 $\Delta > 0, y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$   
 $\Delta = 0, y = C_1 e^{rx} + C_2 x e^{rx}$   
 $\Delta < 0, r = p \pm qi : y = e^{px}[c_1 \cos(qx) + c_2 \sin(qx)]$
- (d)  $ay'' + by' + cy = f(x)$   
 $y = y_c + y_p$ ,  $y_c$  is solution to homogeneous DIFF EQ  
 $f(x)$ : a polynomial in  $x$   
 $y_p = x^k$  (a polynomial of the same degree),  $k$ : # char eq's zero roots (0,1,2)  
 $f(x) = e^{ax}$  (a polynomial in  $x$ )  
 $y_p = x^k e^{ax}$  (same degree),  $k$ : # char eq's roots =  $a$  (0,1,2)  
 $f(x) = e^{ax} \cos(bx)$  (poly in  $x$ ) or  $e^{ax} \sin(bx)$  (apoly in  $x$ )  
 $y_p = x^k e^{ax}[(\text{poly in } x)\cos(bx) + (\text{poly in } x)\sin(bx)]$   
 $k$ : # char eq's root =  $a \pm bi$  (0,1)

## 2. Fourier Series

Given  $F(x), x \in [-L, L]$  write  $F(x)$  in a series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where  $a_n, b_n$  are constants.

- (a) Orthogonality Relations  
 $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$   
 $\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0 (m \neq n), L (m = n)$   
 $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0 (m \neq n), L (m = n)$
- (b)  $a_n = \frac{1}{L} \int_{-L}^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$   
 $b_n = \frac{1}{L} \int_{-L}^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx, n \in [0, \infty], n \in \mathbb{Z}$
- (c) Convergence Statement of F.S.  
 F.S. convergence to the "periodic extension" of  $F(x)$   
 whenever  $F(x)$  is continuous and to the average of  $\frac{f(x^+) + f(x^-)}{2}$  at every point.
- (d) F.S.S and F.C.S of  $F(x)$  on  $[0, L]$ :  
 F.C.S =  $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), a_n = \frac{2}{L} \int_0^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$   
 F.S.S =  $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), a_n = \frac{2}{L} \int_0^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$   
 F.C.S  $\rightarrow$  even extension, F.S.S  $\rightarrow$  odd extension

## 3. Sturm-Liouville Problem

- (a) Basic Examples of S-L BVP  
 $f'' + \lambda f = 0, 0 \leq x \leq L$  and Boundary conditions  
 Def: value  $\lambda$  for which the equation with the given boundary ends: has non-trivial solution is called eigen value, the corresponding solution is called eigen functions of the given S-L BVP.

First, general solutions:

$$\lambda = 0, f(x) = \alpha x + \beta$$

$$\lambda > 0, f(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

$$\lambda < 0, f(x) = C_1 \cosh(ax) + C_2 \sinh(ax), a^2 = -\lambda, a > 0$$

Then, impose the boundary in each case.

- i. Boundary COND:  $f(0) = 0, f(L) = 0$   
 e-values:  $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 1, 2, 3$   
 e-functions:  $f_n \sim \sin\left(\frac{n\pi x}{L}\right)$
- ii. Boundary COND:  $f'(0) = 0, f'(L) = 0$   
 e-values:  $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 0, 1, 2, 3$   
 e-functions:  $f_n \sim \cos\left(\frac{n\pi x}{L}\right)$
- iii. Boundary COND:  $f(0) = 0, f'(L) = 0$   
 e-values:  $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n = 1, 2, 3$   
 e-functions:  $f_n \sim \sin\left(\frac{(2n-1)\pi x}{2L}\right)$
- iv. Boundary COND:  $f'(0) = 0, f(L) = 0$   
 e-values:  $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n = 1, 2, 3$   
 e-functions:  $f_n \sim \cos\left(\frac{(2n-1)\pi x}{2L}\right)$
- (b) Regular S-L Problems  
 EQ:  $(pf')' + qf + \lambda \sigma f = 0, a < x < b$   
 Boundary:  $k_1 f(a) + k_2 f'(a) = 0, k_3 f(b) + k_4 f'(b) = 0$

$$g(x) \sim \sum_{n=1}^{\infty} a_n f_n(x)$$

$$a_n = \frac{\int_a^b g(x) f_n(x) \sigma(x) dx}{\int_a^b \sigma(x) f_n^2(x) dx}$$

## 4. Method of Separation: Heat and Wave Equation

### (a) Heat Equation

$$\mu_t = k \mu_{xx}, \mu(x, t) = X(x)T(t) \text{ (usually } k = 1)$$

$$\text{Separation} \rightarrow X(x)T'(t) = kX''(x)T(t)$$

$$\text{Let } \frac{T'}{kT} = \frac{kX''}{X} = -\lambda$$

$$\text{We get } X'' + \lambda x = 0, T' + k\lambda T = 0$$

$$\text{For second EQ, } T \sim e^{-k\lambda t}$$

For first EQ, we apply S-L BVP problem

$$\mu(x, t) = \sum_{n=1}^{\infty} c_n f_n e^{-\lambda_n t}$$

Usually, we find the bound. cond. in the 4 fourier series probs.

e.g. bond cond 1:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, f_n \sim \sin\left(\frac{n\pi x}{L}\right)$$

$$\mu(x, 0) = g(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{Fouries formal solution: } \mu(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{L^2} t}$$

**note**: in bond cond 2,  $n$  starts from 0. calculate case 0 separately.

### (b) Wave Equation

$$\mu_{tt} = C^2 \mu_{xx}, \mu(x, t) = X(x)T(t), \text{ usually } C = 1$$

$$\mu(0, t) = 0, \mu(L, t) = 0$$

$$\text{Separation} \rightarrow XT'' = C^2 X''T$$

$$\text{Let } \frac{T''}{C^2 T} = \frac{X''}{X} = -\lambda$$

$$\text{We get } X'' + \lambda x = 0, T'' = -\lambda C^2 T$$

$$\mu(x, t) = \sum_{n=1}^{\infty} (b_{1,n} \cos\left(\frac{n\pi ct}{L}\right) + b_{2,n} \sin\left(\frac{n\pi ct}{L}\right)) \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{i. } \mu(x, 0) = f(x)$$

$$f(x) = \sum_{n=1}^{\infty} b_{1,n} \sin\left(\frac{n\pi x}{L}\right)$$

$$b_{1,n} = \frac{\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx}$$

$$\begin{aligned} \text{ii. } \mu_t(x, 0) &= g(x) \\ g(x) &= \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_{2,n} \sin \frac{n\pi x}{L} \\ \frac{n\pi c}{L} b_{2,n} &= \frac{\int_0^L g(x) \sin \frac{n\pi x}{L} dx}{\int_0^L \sin^2 \frac{n\pi x}{L} dx} \end{aligned}$$

(c) d'Alembert's solution to the Wave Equation

Given  $\mu(x, 0) = F(x)$ ,  $\mu_t(x, 0) = 0$  we want to get  $\mu(x, t)$   
Do an odd extension on  $F(x)$ , then  $\mu(x, t) = \frac{F(x+ct) + F(x-ct)}{2}$ . Usually  $c$  is 1. Simply draw the graph of  $F(x+ct)$  (shift to left) and  $F(x-ct)$  (shift to right) and get the average.

## 5. Laplace Equation

Consider the equilibrium temperature in a uniform rectangle (Heat EQ)

$$\begin{cases} \mu_{xx}(x, y) + \mu_{yy}(x, y) = 0, & 0 < x < L, & 0 < y < K \\ \mu(0, y) = f_1(y), & \mu(L, y) = f_2(y), & 0 < y < K \\ \mu(x, 0) = g_1(x), & \mu(x, K) = g_2(x), & 0 < x < L \end{cases} \quad (1)$$

Where we might have first order partial derivatives on those  $\mu$ 's on 2nd or 3rd line.

Separate  $\rightarrow \mu = XY$ , then

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

we want to create a SL-BVP problem:

Determine  $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$  or  $-\lambda$

When  $f_1(y) = f_2(y) = 0$  we can form SL-BVP on  $Y$  and we choose  $-\lambda$ . Similarly, when  $g_1(x) = g_2(x) = 0$  we can form SL-BVP on  $X$  and we choose  $\lambda$

**Case1:**  $-\lambda: (X(0) \text{ or } X'(0) = 0, X(L) \text{ or } X'(L) = 0)$

$$\begin{cases} X'' + \lambda X = 0, & (SL - BVP) \\ \mu(x, 0) = f_1(y), & \mu(x, K) = f_2(y) \end{cases} \quad (2)$$

Get  $\lambda_n$  and  $X_n$  from SL-BVP.

$$\mu(x, y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi y}{L} + \beta_n \cosh \frac{n\pi y}{L}) X_n$$

For simplicity, we transform to:

$$\mu(x, y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi y}{L} + \beta_n \sinh \frac{n\pi(y-K)}{L}) X_n$$

Then we get  $\mu(x, 0)$ ,  $\mu(x, K)$  (as long as they fit the other 2 equations) and try to figure out  $\alpha_n$ ,  $\beta_n$  using match or SL-BVP. **Note** : in bond cond 2,  $n$  starts from 0. calc case 0 separately.

**Case2:**  $\lambda: (Y(0) \text{ or } Y'(0) = 0, Y(K) \text{ or } Y'(K) = 0)$

$$\begin{cases} Y'' + \lambda Y = 0, & (SL - BVP) \\ \mu(0, y) = g_1(y), & \mu(L, y) = g_2(y) \end{cases} \quad (3)$$

Get  $\lambda_n$  and  $Y_n$  from SL-BVP.

$$\mu(x, y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi x}{K} + \beta_n \cosh \frac{n\pi x}{K}) Y_n$$

For simplicity, we transform to:

$$\mu(x, y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi x}{K} + \beta_n \cosh \frac{n\pi(x-L)}{K}) Y_n$$

Then we get  $\mu(0, y)$ ,  $\mu(L, y)$  (as long as they fit the other 2 equations) and try to figure out  $\alpha_n$ ,  $\beta_n$  using match or SL-BVP. **Note** : in bond cond 2,  $n$  starts from 0. calc case 0 separately.

## 6. Method of Eigenfunction Expansion

Consider

$$PDE : \mu_t = k \mu_{xx}(x, t) + q(x, t), 0 < x < L, t > 0$$

$$BCs : \mu(0, t) = 0, \mu(L, t) = 0, t > 0$$

$$IC : \mu(x, 0) = f(x), 0 < x < L$$

(a) BCs  $\Rightarrow$  Eigenfunction:  $X_n(x)$ ,  $\lambda_n$

$$\Rightarrow \mu(x, t) = \sum_{n=1}^{\infty} C_n(t) X_n(x)$$

$$\text{Write } q(x, t) = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

(b) PDE

$$\Rightarrow \sum_{n=1}^{\infty} C'_n(t) X_n(x) = \sum_{n=1}^{\infty} C_n(t) X''_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} C'_n(t) X_n(x) = - \sum_{n=1}^{\infty} C_n(t) \lambda_n X_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C'_n(t) + C_n(t) \lambda_n] X_n(x) = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C'_n(t) + C_n(t) \lambda_n] = \sum_{n=1}^{\infty} q_n(t)$$

Form may vary ( $\mu$  is differentiated in other ways)

(c) IC  $\Rightarrow \mu(x, 0) = f(x) = \sum_{n=1}^{\infty} C_n(0) X_n(x)$  (SL-BVP)

$$\text{either match up or } C_n(0) = \frac{\int_0^L f(x) X_n(x) dx}{\int_0^L X_n^2(x) dx}$$

(d) Then we solve functions  $C'_n(t) + C_n(t) \lambda_n = q_n(t)$  for each  $n$  with initial condition  $C_n(0)$ 's and  $q_n(0)$ 's

## 7. Laplace Transform

$$\text{Def: } \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

$$\text{Def: } H_a(t) = H(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases} \quad (4)$$

$$\text{Def: } \text{erf}(x) = \frac{2}{\pi} \int_0^x e^{-u^2} du, \text{erfc}(x) = 1 - \text{erf}(x)$$

$f(t) = \mathcal{L}^{-1}F(t)$	$F(t) = \mathcal{L}f(s)$
$f^{(n)}(t)$ (nth derivative)	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$
$H(t-a)f(t-a)$	$e^{-as} F(s)$
$e^{at} f(t)$	$F(s-a)$
$(f * g)(t)$	$F(s)G(s)$
1	$\frac{1}{s} (s > 0)$
$t^n$ ( $n$ is positive integer)	$\frac{n!}{s^{n+1}} (s > 0)$
$e^{at}$	$\frac{1}{s-a} (s > a)$
$\sin(at)$	$\frac{a}{s^2+a^2} (s > 0)$
$\cos(at)$	$\frac{s}{s^2+a^2} (s > 0)$
$\sinh(at)$	$\frac{a}{s^2-a^2} (s >  a )$
$\cosh(at)$	$\frac{s}{s^2-a^2} (s >  a )$
$\delta(t-a) (a \geq 0)$	$e^{-as}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin bt$	$\frac{b}{((s-a)^2+b^2)}$
$e^{at} \cos bt$	$\frac{s-a}{((s-a)^2+b^2)}$
$\text{erfc}(\frac{a}{2\sqrt{t}})$	$\frac{1}{s} e^{-a\sqrt{s}}$

Example:

$$\mu_{tt}(x, t) = \mu_{xx}(x, t) + 1, \mu(x, 0) = \mu_t(x, 0) = -1, \mu(0, t) = t$$

$$\mathcal{L}\mu_{tt} = s^2 U - s\mu(x, 0) - \mu_t x, 0 = s^2 U + 1$$

$$\mathcal{L}\mu_{xx} + 1 = U_{xx} + \frac{1}{s}$$

$$\Rightarrow U'' - s^2 U = 1 - \frac{1}{s}$$

$$U_c = k_1 e^{sx} + k_2 e^{-sx}, k_1 = 0$$

$$U_p = \frac{1-\frac{1}{s}}{-s^2} = \frac{1}{s^3} - \frac{1}{s^2}$$

$$\mathcal{L}\mu(0, t) = \mathcal{L}t = \frac{1}{s} = U(0, s)$$

$$\Rightarrow k_2 + \frac{1}{s^3} - \frac{1}{s^2} = \frac{1}{s^2}$$

$$\Rightarrow U = (\frac{2}{s^2} - \frac{1}{s^3})e^{-sx} - \frac{1}{s^2} + \frac{1}{s^3}$$

$$\mu = \mathcal{L}^{-1}(U) = H(t-x)[2(t-x) - \frac{1}{2}(t-x)^2] - t + \frac{1}{2}t^2$$