

1. Basic ODEs

- (a) separable: $y' = \frac{F(x)}{G(y)}$
 $\Rightarrow \frac{dy}{dx} = \frac{F(x)}{G(y)} \Rightarrow \int G(y)dy = \int F(x)dx$
- (b) Linear: $y' + p(x)y = q(x)$
 Integrating Factor: $e^{\int p(x)dx}$
 $\Rightarrow (e^{\int p(x)dx}y)' = e^{\int p(x)dx}q(x)$
 $\Rightarrow e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx$
- (c) $ay'' + by' + cy = 0$ (constant coefficient)
 characteristic EQ: $ar^2 + br + c = 0$
 $\Delta > 0, y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
 $\Delta = 0, y = C_1 e^{rx} + C_2 x e^{rx}$
 $\Delta < 0, r = p \pm qi : y = e^{px}[c_1 \cos(qx) + c_2 \sin(qx)]$

2. Fourier Series

Given $F(x), x \in [-L, L]$ write $F(x)$ in a series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where a_n, b_n are constants.

- (a) Orthogonality Relations
 $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$
 $\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0 (m \neq n), L (m = n)$
 $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0 (m \neq n), L (m = n)$
- (b) $a_n = \frac{1}{L} \int_{-L}^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$
 $b_n = \frac{1}{L} \int_{-L}^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx, n \in [0, \infty], n \in \mathbb{Z}$
- (c) Convergence Statement of F.S.
 F.S. convergence to the "periodic extension" of $F(x)$ wherever $F(x)$ is continuous and to the average of $\frac{f(x^+) + f(x^-)}{2}$ at every point.
- (d) F.S.S and F.C.S of $F(x)$ on $[0, L]$:
 F.C.S = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), a_n = \frac{2}{L} \int_0^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$
 F.S.S = $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), a_n = \frac{2}{L} \int_0^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$
 F.C.S \rightarrow even extension, F.S.S \rightarrow odd extension

3. Sturm-Liouville Problem

- (a) Basic Examples of S-L BVP
 $f'' + \lambda f = 0, 0 \leq x \leq L$ and Boundary conditions
 Def: value λ for which the equation with the given boundary ends: has non-trivial solution is called eigen value, the corresponding solution is called eigen functions of the given S-L BVP.
 First, general solutions:
 $\lambda = 0, f(x) = ax + b$
 $\lambda > 0, f(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$
 $\lambda < 0, f(x) = C_1 \cosh(ax) + C_2 \sinh(ax), a^2 = -\lambda, a > 0$
 Then, impose the boundary in each case.
- i. Boundary COND: $f(0) = 0, f(L) = 0$
 e-values: $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 1, 2, 3$
 e-functions: $f_n \sim \sin\left(\frac{n\pi x}{L}\right)$

ii. Boundary COND: $f'(0) = 0, f'(L) = 0$

e-values: $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 0, 1, 2, 3$

e-functions: $f_n \sim \cos\left(\frac{n\pi x}{L}\right)$

iii. Boundary COND: $f(0) = 0, f'(L) = 0$

e-values: $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n = 1, 2, 3$

e-functions: $f_n \sim \sin\left(\frac{(2n-1)\pi}{2L}x\right)$

iv. Boundary COND: $f'(0) = 0, f(L) = 0$

e-values: $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, n = 1, 2, 3$

e-functions: $f_n \sim \cos\left(\frac{(2n-1)\pi}{2L}x\right)$

(b) Regular S-L Problems

EQ: $(pf')' + qf + \lambda \sigma f = 0, a < x < b$

Boundary: $k_1 f(a) + k_2 f'(a) = 0, k_3 f(b) + k_4 f'(b) = 0$

$$g(x) \sim \sum_{n=1}^{\infty} a_n f_n(x)$$

$$a_n = \frac{\int_a^b g(x) f_n(x) \sigma(x) dx}{\int_a^b \sigma(x) f_n^2(x) dx}$$

4. Method of Separation: Heat and Wave Equation

(a) Heat Equation

$\mu_t = k\mu_{xx}, \mu(x, t) = X(x)T(t)$ (usually $k = 1$)

Separation $\rightarrow X(x)T'(t) = kX''(x)T(t)$

Let $\frac{T'}{kT} = \frac{kX''}{X} = -\lambda$

We get $X'' + \lambda x = 0, T' + k\lambda T = 0$

For second EQ, $T \sim e^{-k\lambda t}$

For first EQ, we apply S-L BVP problem

$\mu(x, t) = \sum_{n=1}^{\infty} c_n f_n e^{-\lambda_n t}$

Usually, we find the bound. cond. in the 4 fourier series probs.

e.g. bond cond 1:

$\lambda_n = \left(\frac{n\pi}{L}\right)^2, f_n \sim \sin\left(\frac{n\pi x}{L}\right)$

$\mu(x, 0) = g(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

Fouries formal solution: $\mu(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{L^2} t}$

note : in bond cond 2, n starts from 0. calculate case 0 separately.

(b) Wave Equation

$\mu_{tt} = C^2 \mu_{xx}, \mu(x, t) = X(x)T(t)$, usually $C = 1$

$\mu(0, t) = 0, \mu(L, t) = 0$

Separation $\rightarrow XT'' = C^2 X''T$

Let $\frac{T''}{C^2 T} = \frac{X''}{X} = -\lambda$

We get $X'' + \lambda x = 0, T'' = -\lambda C^2 T$

$\mu(x, t) = \sum_{n=1}^{\infty} (b_{1,n} \cos\left(\frac{n\pi ct}{L}\right) + b_{2,n} \sin\left(\frac{n\pi ct}{L}\right)) \sin\left(\frac{n\pi x}{L}\right)$

i. $\mu(x, 0) = f(x)$

$f(x) = \sum_{n=1}^{\infty} b_{1,n} \sin\left(\frac{n\pi x}{L}\right)$

$b_{1,n} = \frac{\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx}$

ii. $\mu_t(x, 0) = g(x)$

$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_{2,n} \sin\left(\frac{n\pi x}{L}\right)$

$\frac{n\pi c}{L} b_{2,n} = \frac{\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx}$

(c) d'Alembert's solution to the Wave Equation

Given $\mu(x, 0) = F(x), \mu_t(x, 0) = 0$ we want to get $\mu(x, t)$

Do an odd extension on $F(x)$, then $\mu(x, t) =$

$\frac{F(x+ct)+F(x-ct)}{2}$. Usually c is 1. Simply draw the graph of $F(x+ct)$ (shift to left) and $F(x-ct)$ (shift to right) and get the average.

5. Laplace Equation

Consider the equilibrium temperature in a uniform rectangle(Heat EQ)

$$\begin{cases} \mu_{xx}(x,y) + \mu_{yy}(x,y) = 0, & 0 < x < L, & 0 < y < K \\ \mu(0,y) = f_1(y), & \mu(L,y) = f_2(y), & 0 < y < K \\ \mu(x,0) = g_1(x), & \mu(x,K) = g_2(x), & 0 < x < L \end{cases} \quad (1)$$

Where we might have first order partial derivatives on those μ 's on 2nd or 3rd line.

Separate $\rightarrow \mu = XY$, then

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

we want to create a SL-BVP problem:

$$\text{Determine } \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \text{ or } -\lambda$$

When $f_1(y) = f_2(y) = 0$ we can form SL-BVP on Y and we choose $-\lambda$. Similarly, when $g_1(x) = g_2(x) = 0$ we can form SL-BVP on X and we choose λ

Case1: $-\lambda: (X(0) \text{ or } X'(0) = 0, X(L) \text{ or } X'(L) = 0)$

$$\begin{cases} X'' + \lambda X = 0, & (SL - BVP) \\ \mu(x,0) = f_1(y), & \mu(x,K) = f_2(y) \end{cases} \quad (2)$$

Get λ_n and X_n from SL-BVP.

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi y}{L} + \beta_n \cosh \frac{n\pi y}{L}) X_n$$

For simplicity, we transform to:

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi y}{L} + \beta_n \sinh \frac{n\pi(y-K)}{L}) X_n$$

Then we get $\mu(x,0)$, $\mu(x,K)$ (as long as they fit the other 2 equations) and try to figure out α_n , β_n using match or SL-BVP. **Note** : in bond cond 2, n starts from 0. calc case 0 separately.

Case2: $\lambda: (Y(0) \text{ or } Y'(0) = 0, Y(K) \text{ or } Y'(K) = 0)$

$$\begin{cases} Y'' + \lambda Y = 0, & (SL - BVP) \\ \mu(0,y) = g_1(y), & \mu(L,y) = g_2(y) \end{cases} \quad (3)$$

Get λ_n and Y_n from SL-BVP.

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi x}{K} + \beta_n \cosh \frac{n\pi x}{K}) Y_n$$

For simplicity, we transform to:

$$\mu(x,y) = \sum_{n=1}^{\infty} (\alpha_n \sinh \frac{n\pi x}{K} + \beta_n \cosh \frac{n\pi(x-L)}{K}) Y_n$$

Then we get $\mu(0,y)$, $\mu(L,y)$ (as long as they fit the other 2 equations) and try to figure out α_n , β_n using match or SL-BVP. **Note** : in bond cond 2, n starts from 0. calc case 0 separately.

6. Method of Eigenfunction Expansion

Consider

$$PDE : \mu_t = k\mu_{xx}(x,t) + q(x,t), 0 < x < L, t > 0$$

$$BCs : \mu(0,t) = 0, \mu(L,t) = 0, t > 0$$

$$IC : \mu(x,0) = f(x), 0 < x < L$$

(a) BCs \Rightarrow Eigenfunction: $X_n(x), \lambda_n$

$$\Rightarrow \mu(x,t) = \sum_{n=1}^{\infty} C_n(t) X_n(x)$$

$$\text{Write } q(x,t) = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

(b) PDE

$$\Rightarrow \sum_{n=1}^{\infty} C'_n(t) X_n(x) = \sum_{n=1}^{\infty} C_n(t) X''_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} C'_n(t) X_n(x) = - \sum_{n=1}^{\infty} C_n(t) \lambda_n X_n(x) + \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C'_n(t) + C_n(t) \lambda_n] X_n(x) = \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} [C'_n(t) + C_n(t) \lambda_n] = \sum_{n=1}^{\infty} q_n(t)$$

Form may vary (μ is differentiated in other ways)

(c) IC $\Rightarrow \mu(x,0) = f(x) = \sum_{n=1}^{\infty} C_n(0) X_n(x)$ (SL-BVP)

$$\text{either match up or } C_n(0) = \frac{\int_0^L f(x) X_n(x) dx}{\int_0^L X_n^2(x) dx}$$

(d) Then we solve functions $C'_n(t) + C_n(t) \lambda_n = q_n(t)$ for each n with initial condition $C_n(0)$'s and $q_n(0)$'s