

1. Basic ODEs

(a) separable: $y' = \frac{F(x)}{G(y)}$
 $\Rightarrow \frac{dy}{dx} = \frac{F(x)}{G(y)} \Rightarrow \int G(y)dy = \int F(x)dx$

(b) Linear: $y' + p(x)y = q(x)$
 Integrating Factor: $e^{\int p(x)dx}$
 $\Rightarrow (e^{\int p(x)dx}y)' = e^{\int p(x)dx}q(x)$
 $\Rightarrow e^{\int p(x)dx}y = \int e^{\int p(x)dx}q(x)dx$

(c) $ay'' + by' + cy = 0$ (constant coefficient)
 characteristic EQ: $ar^2 + br + c = 0$
 $\Delta > 0, y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
 $\Delta = 0, y = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$
 $\Delta > 0, r = p \pm qi : y = e^{ax}[c_1 \cos(bx) + c_2 \sin(bx)]$

ii. Boundary COND: $f'(0) = 0, f'(L) = 0$

e-values: $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 0, 1, 2, 3$

e-functions: $f_n \sim \cos(\frac{n\pi x}{L})$

iii. Boundary COND: $f(0) = 0, f'(L) = 0$

e-values: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2, n = 1, 2, 3$

e-functions: $f_n \sim \sin(\frac{(2n-1)\pi}{2L}x)$

iv. Boundary COND: $f'(0) = 0, f(L) = 0$

e-values: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2, n = 1, 2, 3$

e-functions: $f_n \sim \cos(\frac{(2n-1)\pi}{2L}x)$

(b) Regular S-L Problems EQ: $(pf')' + qf + \lambda \sigma f = 0, a < x < b$

Boundary: $k_1 f(a) + k_2 f'(a) = 0, k_3 f(b) + k_4 f'(b) = 0$

2. Fourier Series

Given $F(x), x \in [-L, L]$ write $F(x)$ in a series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$$

where a_n, b_n are constants.

(a) Orthogonality Relations

$$\int_{-L}^L \sin(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L}) dx = 0$$

$$\int_{-L}^L \cos(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L}) dx = 0 (m \neq n), L (m = n)$$

$$\int_{-L}^L \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) dx = 0 (m \neq n), L (m = n)$$

(b) $a_n = \frac{1}{L} \int_{-L}^L F(x) \cos(\frac{n\pi x}{L}) dx$

$$b_n = \frac{1}{L} \int_{-L}^L F(x) \sin(\frac{n\pi x}{L}) dx, n \in [0, \infty], n \in \mathbb{Z}$$

(c) Converge Statement of F.S.

F.S. convergence to the "periodic extension" of $F(x)$ wherever $F(x)$ is continuous and to the average of $\frac{f(x^+) + f(x^-)}{2}$ at every point.

(d) F.S.S and F.C.S of $F(x)$ on $[0, L]$:

$$\text{F.C.S} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, a_n = \frac{2}{L} \int_0^L F(x) \cos(\frac{n\pi x}{L}) dx$$

$$\text{F.S.S} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, a_n = \frac{2}{L} \int_0^L F(x) \cos(\frac{n\pi x}{L}) dx$$

F.C.S \rightarrow even extension, F.S.S \rightarrow odd extension

3. Sturm-Liouville Problem

(a) Basic Examples of S-L BVP

$$f'' + \lambda f = 0, 0 \leq x \leq L \text{ and Boundary conditions}$$

Def: value λ for which the equation with the given boundary ends: has non-trivial solution is called eigen value, the corresponding solution is called eigen functions of the given S-L BVP.

First, general solutions:

$$\lambda = 0, f(x) = \alpha x + \beta$$

$$\lambda > 0, f(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

$$\lambda < 0, f(x) = C_1 \cosh(ax) + C_2 \sinh(ax), a^2 = -\lambda, a > 0$$

Then, impose the boundary in each case.

i. Boundary COND: $f(0) = 0, f(L) = 0$

e-values: $\lambda_n = \frac{n^2 \pi^2}{L^2}, n = 1, 2, 3$

e-functions: $f_n \sim \sin(\frac{n\pi x}{L})$