SOLUTIONS §1

1.1 #1 $(x^2+1)\,y' = 2xy$

soln. $\displaystyle\int \frac{dy}{y} = \int \frac{2x}{x^2+1}\,dx$

$\ln|y| = \ln|x^2+1| + c$

$|y| = e^c\,|x^2+1|$

$\Rightarrow$ $y = m\,(x^2+1)$ all m

3) $(x-1)\,y' + 2y = x$

soln. $y' + \dfrac{2}{x-1}\,y = \dfrac{x}{x-1}$

lin. i.F. $e^{\int \frac{2}{x-1}dx} = e^{\ln (x-1)^2} = (x-1)^2$

$\left((x-1)^2 \cdot y\right)' = x\,(x-1)$

$(x-1)^2 \cdot y = \dfrac{x^3}{3} - \dfrac{x^2}{2} + c$

$y = \dfrac{1}{(x-1)^2}\left[\dfrac{x^3}{3} - \dfrac{x^2}{2} + c\right]$

1.2 #4 $2y'' - 5y' + 2y = 0$

soln. C.E. $2r^2 - 5r + 2 = 0$ $\Rightarrow$ $(2r-1)(r-2) = 0$ $\Rightarrow$ $\begin{cases} r = \frac{1}{2} \\ r = 2 \end{cases}$

$y = c_1 e^{\frac{1}{2}x} + c_2 e^{2x}$

#5 $4y'' + 4y' + y = 0$

soln. C.E. $4r^2 + 4r + 1 = 0$ $\Rightarrow$ $(2r+1)^2 = 0$ $\Rightarrow$ $r = -\frac{1}{2}$

$y = c_1 e^{-\frac{1}{2}x} + c_2 x\, e^{-\frac{1}{2}x}$

#8 $y'' - 6y' + 13y = 0$

soln. C.E. $r^2 - 6r + 13 = 0$ $r = \dfrac{6 \pm \sqrt{-16}}{2}$ $r = 3 \pm 2i$

$y = e^{3x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$

Fourier Series

Given $f(x)$ on $[-L, L]$

If we write $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

then the coefficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$n = 0, 1, 2, \ldots$ $\qquad$ $n = 1, 2, \ldots$

The right hand side of the expression for $f(x)$ is called the full Fourier Series of $f(x)$.

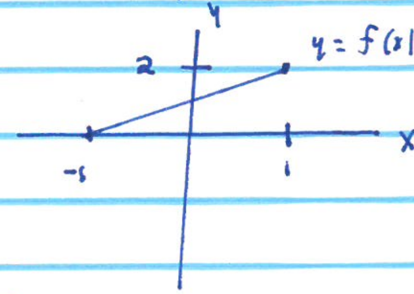
Convergence Theorem for Fourier Series

Assume $f(x)$ is p.w. differentiable on $[-L, L]$

The F.S. of $f(x)$ converges to the periodic extension of $f(x)$ wherever $f(x)$ is continuous and to the average $\frac{f(a^+) + f(a^-)}{2}$ in general.

Fourier Series Exercise

2.1 #5 Find and graph the F.S. of $f(x) = x+1, \quad -1 \le x \le 1$



Soln: $L = 1$

$$a_0 = \frac{1}{L}\int_{-1}^{1} f(x)\,dx = 2$$

$$a_n = \frac{1}{L}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{L}\,dx = \int_{-1}^{1}(x+1)\cos n\pi x\,dx$$

$$\boxed{\begin{array}{ll} u = x+1 & dv = \cos n\pi x\,dx \\ du = dx & v = \frac{1}{n\pi}\sin n\pi x \end{array}}$$

$$= \underbrace{uv\Big|_{-1}^{1}}_{0} - \frac{1}{n\pi}\int_{-1}^{1}\sin n\pi x\,dx = 0$$

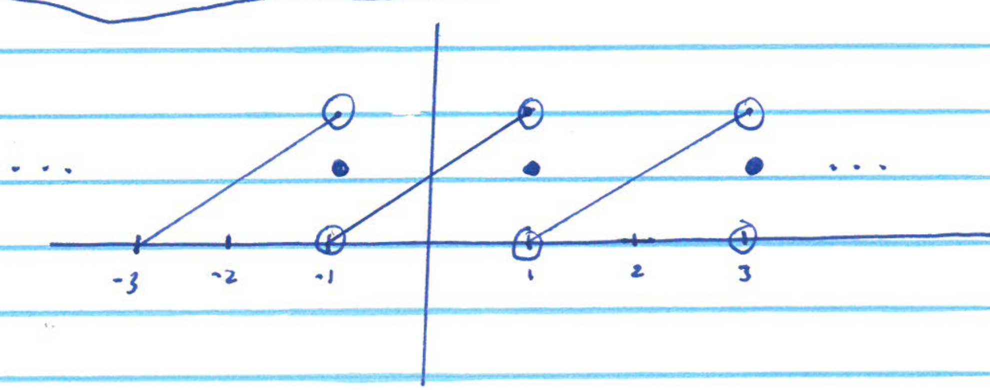
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \int_{-1}^{1}(x+1)\sin n\pi x\,dx$$

$$\boxed{\begin{array}{ll} u = x+1 & dv = \sin n\pi x\,dx \\ du = dx & v = -\frac{1}{n\pi}\cos n\pi x \end{array}}$$

$$= -\frac{(x+1)}{n\pi}\cos n\pi x\Big|_{-1}^{1} + \underbrace{\frac{1}{n\pi}\int_{-1}^{1}\cos n\pi x\,dx}_{0}$$

$$= -\frac{2}{n\pi}\cos n\pi$$

$$f(x) \sim 1 + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\cos n\pi\right)\sin n\pi x$$



A Full Fourier Series 2.1 #7

$$f(x) = \begin{cases} -1 & -2 \le x \le 0 \\ 2-x & 0 < x \le 2 \end{cases}$$ Construct and graph the F.S.

SOLN: $L = 2$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\left(\int_{-2}^{0} -dx + \int_0^2 (2-x)\,dx\right) = 0$$

$n \ge 1$ $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\left[\int_{-2}^{0} -\cos\frac{n\pi x}{2}\,dx + \int_0^2 (2-x)\cos\frac{n\pi x}{2}\,dx\right]$$

1st integral $-\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} = 0$;

parts $u = 2-x$, $du = -dx$, $dv = \cos\frac{n\pi x}{2}$, $v = \frac{2}{n\pi}\sin\frac{n\pi x}{2}$

2nd integral $$(2-x)\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^2 + \frac{2}{n\pi}\int_0^2 \sin\frac{n\pi x}{2}\,dx = -\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_0^2$$

$$= -\frac{4}{n^2\pi^2}\cos n\pi + \frac{4}{n^2\pi^2} \implies \boxed{a_n = \frac{2}{n^2\pi^2} - \frac{2}{n^2\pi^2}\cos n\pi}$$

$$b_n = \frac{1}{2}\left[\int_{-2}^{0} -\sin\frac{n\pi x}{2}\,dx + \int_0^2 (2-x)\sin\frac{n\pi x}{2}\,dx\right]$$

$u = 2-x$, $du = -dx$, $dv = \sin\frac{n\pi x}{2}$, $v = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}$

$$= \frac{1}{2}\left[\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{0} - \frac{2(2-x)}{n\pi}\cos\frac{n\pi x}{2}\Big|_0^2 - \underbrace{\frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\,dx}_{0}\right]$$

$$= \frac{1}{2}\left[\frac{2}{n\pi} - \frac{2}{n\pi}\cos n\pi + \frac{4}{n\pi}\right] = \boxed{\frac{3}{n\pi} - \frac{1}{n\pi}\cos n\pi}$$

$$f(x) = \sum_{n=1}^{\infty}\left[\left(\frac{2}{n^2\pi^2} - \frac{2}{n^2\pi^2}\cos n\pi\right)\cos\frac{n\pi x}{2} + \left(\frac{3}{n\pi} - \frac{1}{n\pi}\cos n\pi\right)\sin\frac{n\pi x}{2}\right]$$