Wave by L.T. 9.2.1 #14

$u_{tt} = u_{xx} + e^{-x}$ $\quad u(0,t) = -e^{-t}$ $\quad u(x,0) = -2$ $\quad u_t(x,0) = 0$

$u_c(\infty, t) = 0$

Soln: $U = \mathcal{L}[u]$ $\quad U = U(x,s)$

$s^2 U - s u(x,0) - u_t(x,0) = U'' + \frac{e^{-x}}{s}$

$s^2 U + 2s = U'' + \frac{e^{-x}}{s}$ $\Rightarrow$ $U'' - s^2 U = 2s - \frac{1}{s} e^{-x}$

$U_c = k e^{-sx}$

Fn $2s \Rightarrow U_p = -\frac{2}{s}$

Fn $-\frac{1}{s} e^{-x}$ $\quad U_p = A e^{-x}$

$\Rightarrow A e^{-x} - s^2 A e^{-x} = -\frac{1}{s} e^{-x}$

$A(1-s^2) = -\frac{1}{s}$ $\Rightarrow A = \frac{1}{s(s^2-1)}$

$U_p = \frac{1}{s(s^2-1)} e^{-x}$ $\Leftarrow$

$U = k e^{-sx} - \frac{2}{s} + \frac{1}{s(s^2-1)} e^{-x}$

p.f.

$\frac{1}{s(s^2-1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$

$\Rightarrow A = -1 \quad B = \frac{1}{2} \quad C = \frac{1}{2}$

$U = k e^{-sx} - \frac{2}{s} + \left(-\frac{1}{s} + \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1}\right) e^{-x}$ $\Leftarrow$

$U(0,s) = \mathcal{L}[u(0,t)] = \mathcal{L}[-e^{-t}] = -\frac{1}{s+1}$ $\Rightarrow$

$-\frac{1}{s+1} = k - \frac{2}{s} - \frac{1}{s} + \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1}$ $\Rightarrow k = \frac{3}{s} - \frac{\frac{1}{2}}{s-1} - \frac{\frac{3}{2}}{s+1}$

$U = \left(\frac{3}{s} - \frac{\frac{1}{2}}{s-1} - \frac{\frac{3}{2}}{s+1}\right) e^{-sx} - \frac{2}{s} + \left(-\frac{1}{s} + \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1}\right) e^{-x}$

$u = \left(3 - \frac{1}{2} e^{t-x} - \frac{3}{2} e^{-(t-x)}\right) H(t-x) - 2 + \left(-1 + \frac{1}{2} e^{t} + \frac{1}{2} e^{-t}\right) e^{-x}$

Heat Problem

9.2.2 #6 $\quad u_t = k u_{xx} + q(x,t) \quad$ where $q(x,t) = -(2t+3)e^{-x}$, $k=2$

$u(0,t) = t+2$

$u(x,0) = 2e^{-x}$

Soln: $\quad U(x,s) = \mathcal{L}[u(x,t)] \quad \Rightarrow$

$$sU - u(x,0) = 2U'' - \left(\frac{2}{s^2} + \frac{3}{s}\right)e^{-x}$$

$$sU - 2e^{-x} = 2U'' - \left(\frac{2}{s^2} + \frac{3}{s}\right)e^{-x}$$

$$U'' - \frac{s}{2}U = \left(\frac{1}{s^2} + \frac{3}{2s} - 1\right)e^{-x}$$

$$U_c = a e^{-\sqrt{\frac{s}{2}}x}$$

$$U_p = Ae^{-x} \qquad U_p' = -Ae^{-x} \qquad U_p'' = Ae^{-x} \quad \Rightarrow$$

$$Ae^{-x} - \frac{s}{2}Ae^{-x} = \left(\frac{1}{s^2} + \frac{3}{2s} - 1\right)e^{-x}$$

$$A\left(1 - \frac{s}{2}\right)e^{-x} = \left(\frac{1}{s^2} + \frac{3}{2s} - 1\right)e^{-x}$$

$$A\left(\frac{2-s}{2}\right) = \frac{2 + 3s - 2s^2}{2s^2}$$

$$A = \frac{2}{s-2} \cdot \frac{2s^2 - 3s - 2}{2s^2} = \frac{(2s+1)(s-2)}{s^2(s-2)} = \frac{2}{s} + \frac{1}{s^2}$$

$$U = a e^{-\sqrt{\frac{s}{2}}x} + \left(\frac{2}{s} + \frac{1}{s^2}\right)e^{-x}$$

$$U(0,s) = \mathcal{L}[u(0,t)] = \frac{1}{s^2} + \frac{2}{s} \quad \Rightarrow$$

$$U = \left(\frac{2}{s} + \frac{1}{s^2}\right)e^{-x}$$

$$u = (2+t)e^{-x}$$