Undetermined Coefficients Exs.

1.3 #5 $\quad y'' - y = x^2 - x + 2$

Soln: for y_c : $y^2 - y = 0 \qquad$ C.E $\quad r^2 - 1 = 0 \qquad r = \pm 1$

$y_c = c_1 \cosh x + c_2 \sinh x$

$$\left.\begin{aligned} y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}\right\} \Rightarrow 2A - Ax^2 - Bx - C = x^2 - x + 2$$

$$-Ax^2 - Bx + 2A - C = x^2 - x + 2$$

$$\Rightarrow A = -1 \quad B = 1 \quad -2 - C = 2 \quad C = -4$$

$$y = y_p + y_c \quad \Rightarrow \quad \boxed{y = c_1 \cosh x + c_2 \sinh x - x^2 + x - 4}$$

#6 $\quad y'' - 2y' - 8y = 4 + 4x - 8x^2$

Soln: for y_c: $y^2 - 2y' - 8y = 0 \qquad$ C.E. $\quad r^2 - 2r - 8 = 0 \qquad r_1 = 4 \quad r_2 = -2$

$y_c = c_1 e^{4x} + c_2 e^{-2x}$

$$\left.\begin{aligned} y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}\right\} \Rightarrow 2A - 4Ax - 2B - 8Ax^2 - 8Bx - 8C = 4 + 4x - 8x^2$$

$$\left.\begin{aligned} 2A - 2B - 8C &= 4 \\ -4A - 8B &= 4 \\ -8A &= -8 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 1 \\ B &= -1 \\ C &= 0 \end{aligned}$$

$$y_p = x^2 - x \quad \Rightarrow \quad \boxed{y = c_1 e^{4x} + c_2 e^{-2x} + x^2 - x}$$

#7 $\quad y'' - 25y = 30 e^{-5x}$

Soln: for y_c : $y^2 - 25y = 0 \qquad$ C.E. $\quad r^2 - 25 = 0 \qquad r = \pm 5$

$y_c = c_1 \cosh 5x + c_2 \sinh 5x \qquad$ we have duplication so $y = Ae^{-5x}$ fails

$y_p = Axe^{-5x}, \quad y_p' = A(-5xe^{-5x} + e^{-5x}), \quad y_p'' = A(25xe^{-5x} - 10e^{-5x}) \Rightarrow$

$A(25xe^{-5x} - 10e^{-5x} - 25xe^{-5x}) = 30e^{-5x} \quad \Rightarrow \quad A = -3 \qquad y_p = -3xe^{-5x}$

$$\boxed{y = c_1 \cosh 5x + c_2 \sinh 5x - 3xe^{-5x}}$$

#8 $\quad 4y'' + y = 8\cos\frac{x}{2}$

Soln: for y_c: $\quad 4y'' + y = 0 \qquad$ C.E. $\quad 4r^2 + 1 = 0 \qquad r = \pm\frac{1}{2}i$

$$y_c = c_1 \cos\frac{x}{2} + c_2 \sin\frac{x}{2}$$

$y_p \sim \cos\frac{x}{2}$ or $y_p \sim \sin\frac{x}{2}$ will fail

let $\quad y_p = x\left(A\cos\frac{x}{2} + B\sin\frac{x}{2}\right)$

$$y_p' = x\left(-\frac{A}{2}\sin\frac{x}{2} + \frac{B}{2}\cos\frac{x}{2}\right) + A\cos\frac{x}{2} + B\sin\frac{x}{2}$$

$$y_p'' = x\left(-\frac{A}{4}\cos\frac{x}{2} \;\; \frac{B}{4}\sin\frac{x}{2}\right) + \left(-A\sin\frac{x}{2} + B\cos\frac{x}{2}\right)$$

after subbing: $\quad -4A\sin\frac{x}{2} + 4B\cos\frac{x}{2} = 8\cos\frac{x}{2} \implies A = 0 \quad B = 2$

$$y_p = 2x\sin\frac{x}{2}$$

$$y = c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2} + 2x\sin\frac{x}{2}$$

1st order eqns can also be solved like this for certain r.h.s.'s

#1 $\quad y' + 2y = 2x + e^{4x}$

for $y_c \quad y' + 2y = 0 \quad y = Ce^{-2x}$

let $\quad y_p = Ax + B + Ce^{4x} \implies y_p' = A + 4Ce^{4x}$

subbing $\quad A + 4Ce^{4x} + 2Ax + 2B + 2Ce^{4x} = 2x + e^{4x}$

$$\implies \left.\begin{aligned} 2A = 2 &\implies A = 1 \\ A + 2B = 0 &\implies B = -\tfrac{1}{2} \\ 6C = 1 &\implies C = \tfrac{1}{6} \end{aligned}\right\} \quad y_p = x - \frac{1}{2} + \frac{1}{6}e^{4x}$$

$$y = C_1 e^{-2x} + x - \frac{1}{2} + \frac{1}{6}e^{4x}$$

E-function Expansion Problems §7

7.1.1 #2

$$\begin{cases} u_t = u_{xx} + \left(3+\pi^2(3t-2)\right)\sin\pi x + (9\pi^2 t^2 + 2t)\sin 3\pi x \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) = -2\sin\pi x \end{cases}$$

e-function assumption: $u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin n\pi x$

after substitution: $c_n' + n^2\pi^2 c_n = q_n(t)$ $\quad q_n(t) =$ nth coeff of $q(x,t)$

$c_n(0) =$ nth coeff. of $f(x)$

to get $q_n(t)$ $\quad q(x,t) = \left(3+\pi^2(3t-2)\right)\sin\pi x + (9\pi^2 t^2 + 2t)\sin 3\pi x = \sum_{n=1}^{\infty} q_n(t)\sin n\pi x$

matching up $\Rightarrow$ $q_1(t) = 3 + \pi^2(3t-2)$

$q_3(t) = 9\pi^2 t^2 + 2t$

all other q_n's are 0

to get $c_n(0)$ $\quad f(x) = u(x,0) = -2\sin\pi x = \sum_{n=1}^{\infty} c_n(0)\sin n\pi x$

matching up $\Rightarrow$ $c_1(0) = -2$ all other $c_n(0)$ are 0.

o.d.e.'s

$n=1$	$n=3$
$c_1' + \pi^2 c_1 = 3 + \pi^2(3t-2)$	$c_3' + 9\pi^2 c_3 = 9\pi^2 t^2 + 2t$
$c_1(0) = -2$	$c_3(0) = 0$

solutions to o.d.e.'s by inspection or undetermined coeffs: $c_1 = 3t-2$; $c_3 = t^2$

all other $c_n \equiv 0$.

$$u(x,t) = (3t-2)\sin\pi x + t^2\sin 3\pi x$$

7.1.1 #8

$$\begin{cases} u_t = u_{xx} + \frac{1}{2}(x-1)t & 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) = x \end{cases}$$

Soln:

$$\left.\begin{aligned} q_n &= \frac{\int_0^1 \frac{1}{2}(x-1)t \sin n\pi x \, dx}{\int_0^1 \sin^2 n\pi x \, dx} = \frac{-t}{n\pi} \\ c_n(0) &= \frac{\int_0^1 x \sin n\pi x \, dx}{\int_0^1 \sin^2 n\pi x \, dx} = \frac{2}{n\pi}(-1)^{n+1} \end{aligned}\right\} \text{ after int. by parts}$$

The standard approach gives

$$c_n' + n^2\pi^2 c_n = \frac{-t}{n\pi} \qquad c_n(0) = \frac{2}{n\pi}(-1)^{n+1}$$

I. F. $e^{n^2\pi^2 t} \Rightarrow \left(e^{n^2\pi^2 t} \cdot c_n\right)' = \frac{-t}{n\pi} e^{n^2\pi^2 t}$

$$e^{n^2\pi^2 t} \cdot c_n = \frac{-t}{n\pi} e^{n^2\pi^2 t} + \frac{1}{n^5\pi^5} e^{n^2\pi^2 t} + m \qquad \text{after int.}$$

$$c_n = \frac{-t}{n^3\pi^3} + \frac{1}{n^5\pi^5} + m e^{-n^2\pi^2 t} \qquad \text{gen'l soln}$$

$$c_n(0) = \frac{2}{n\pi}(-1)^{n+1} \Rightarrow m = \frac{-1}{n^5\pi^5} + \frac{2}{n\pi}(-1)^{n+1}$$

$$c_n(t) = \frac{-t}{n^3\pi^3} + \frac{1}{n^5\pi^5} + \left(\frac{-1}{n^5\pi^5} + \frac{2}{n\pi}(-1)^{n+1}\right) e^{-n^2\pi^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin n\pi x$$

7.1.3 #1 $\quad u_t = u_{xx} + \sin\frac{3\pi x}{2} - 2\sin\frac{5\pi x}{2} \qquad 0<x<1$

$$u(0,t) = u_x(1,t) = 0$$

$$u(x,0) = \sin\tfrac{3}{2}\pi x = f(x)$$

e-function assumption $\quad u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin\frac{(2n-1)\pi x}{2}$

substitution $\quad \sum_{n=1}^{\infty} c_n'\sin\frac{(2n-1)\pi x}{2} = \sum_{n=1}^{\infty} -\frac{(2n-1)^2\pi^2}{4}c_n\sin\frac{(2n-1)\pi x}{2} + q(x,t)$

comparing to get o.d.e's $\quad c_n' + \frac{(2n-1)^2\pi^2}{4}c_n = q_n = n^{th}$ coeff. of $q(x,t)$

$c_n(0) = n^{th}$ coeff of $f(x)$

for $q_n(t)$ $\quad \sin\frac{3}{2}\pi x - 2\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} q_n(t)\sin\frac{(2n-1)\pi x}{2}$

matching up : $n=2 \Rightarrow q_2(t) = 1$; $n=3 \Rightarrow q_3(t) = -2$ all the rest $=0$

for $c_n(0)$ $\quad f(x) = u(x,0) = \sin\frac{3}{2}\pi x = \sum_{n=1}^{\infty} c_n(0)\sin\frac{(2n-1)\pi x}{2}$

matching up : $n=2 \Rightarrow c_2(0) = 1$

o.d.e.s

$n=2$: $\quad c_2' + \frac{9\pi^2}{4}c_2 = 1$; $c_2(0) = 1$

$n=3$: $\quad c_3' + \frac{25}{4}\pi^2 c_3 = -2 \qquad c_3(0) = 0$

all other $c_n \equiv 0$

After solving the o.d.e.'s and substituting

$$u(x,t) = \left[\left(\frac{9\pi^2-4}{9\pi^2}\right)e^{-\frac{9\pi^2}{4}t} + \frac{4}{9\pi^2}\right]\sin\frac{3\pi x}{2} + \left[\frac{8}{25\pi^2}e^{-\frac{25\pi^2}{4}t} - \frac{8}{25\pi^2}\right]\sin\frac{5\pi x}{2}$$

$$u_{tt} = u_{xx} + t\sin\pi x \qquad 0<x<1$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin\pi x = f(x) \qquad u_t(x,0) = 2\sin\pi x + 4\sin 3\pi x = g(x)$$

Soln

e-function setup: $u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin n\pi x$

sub $\Rightarrow$ $c_n'' + n^2\pi^2 c_n = q_n(t) = n$th coeff of $t\sin\pi x$

$c_n(0) = n$th coeff of $f(x)$; $c_n'(0) = n$th coeff of $g(x)$

q_n $\quad q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin n\pi x \Rightarrow t\sin\pi x = \sum_{n=1}^{\infty} q_n(t)\sin n\pi x \Rightarrow$

$q_1(t) = t$ all others are 0

$c_n(0)$ $\quad \sin\pi x = \sum_{n=1}^{\infty} c_n(0)\sin n\pi x \Rightarrow c_1(0) = 1$ all others are 0

$c_n'(0)$ $\quad 2\sin\pi x + 4\sin 3\pi x = \sum_{n=1}^{\infty} c_n'(0)\sin n\pi x \Rightarrow c_1'(0) = 2 \quad c_3'(0) = 4$, others $= 0$

eqns

$n=1$: $c_1'' + \pi^2 c_1 = t, \quad c_1(0) = 1 \quad c_1'(0) = 2$

$n=3$: $c_3'' + 9\pi^2 c_3 = 0, \quad c_3(0) = 0 \quad c_3'(0) = 4$

solutions

$(c_1)_c = a\cos\pi t + b\sin\pi t$

$(c_1)_p = \frac{t}{\pi^2}$

$c_1 = a\cos\pi t + b\sin\pi t + \frac{t}{\pi^2}$

$c_1' = -a\pi\sin\pi + b\pi\cos\pi t + \frac{1}{\pi^2}$

$c_1(0) = 1 \Rightarrow a = 1, \quad c_1'(0) = 2 \Rightarrow b = \frac{2}{\pi} - \frac{1}{\pi^2}$

$c_1 = \cos\pi t + \left(\frac{2}{\pi} - \frac{1}{\pi^3}\right)\sin\pi t + \frac{t}{\pi^2}$

$c_3 = a\cos 3\pi t + b\sin 3\pi t$

$c_3' = -3\pi a\sin 3\pi t + 3\pi b\cos 3\pi t$

$c_3(0) = 0 \Rightarrow a = 0$

$c_3'(0) = 4 \Rightarrow b = \frac{4}{3\pi}$

$c_3 = \frac{4}{3\pi}\sin 3\pi t$

Assembling

$$u(x,t) = \left(\cos\pi t + \left(\frac{2}{\pi} - \frac{1}{\pi^3}\right)\sin\pi t + \frac{t}{\pi^2}\right)\sin\pi x + \frac{4}{3\pi}\sin 3\pi t\sin 3\pi x$$