**MTH 4581: Fall 2018: Prof. C. King**

**Notes on the 'No Arbitrage Theorem'**

Suppose that a random trial has $m$ possible outcomes, labeled $j = 1, \ldots, m$. Suppose also that there are $n$ wagers available, each of which produces a return (may be a gain or loss) based on the outcome. Let $(R_1, R_2, \ldots, R_n)$ denote the returns from these wagers. So each $R_i$ is a random variable, and its value depends on the outcome: if the outcome is $j$, then the return is $R_i(j)$. So the vector $(R_1(j), R_2(j), \ldots, R_n(j))$ describes the returns from all the wagers when the outcome of the trial is $j$.

We think of a wager as being a decision to buy one unit of stock, or the option to buy one unit of stock in the future, or some other financial instrument. The random trial is then the value of the stock at some future time. We can also buy multiple units, so for example if we spend $x$ amount on the first wager, then the return would be $xR_1$. We may also choose to wager by selling the unit of stock, which would correspond to spending the amount $-1$ on the wager, leading to the return $-R_1$. So in general we allow both positive and negative amounts to be spent on each wager.

Suppose now that we adopt a *strategy*, which is to spend amounts $x_1, \ldots, x_n$ on each of the $n$ available wagers. Then the total return will be the random variable

$$R = x_1 R_1 + \cdots + x_n R_n$$

So if the outcome is $j$, then the total return (positive if a gain, negative if a loss) is

$$R(j) = x_1 R_1(j) + \cdots + x_n R_n(j)$$

We say that there is an *arbitrage opportunity* if there is some strategy $x_1, \ldots, x_n$ such that $R(j) > 0$ for all $j = 1, \ldots, m$. That is, an arbitrage opportunity exists if it is possible to find a strategy which provides a positive return for every possible outcome.

The *'No Arbitrage Theorem'* provides a test for the existence of an arbitrage opportunity. Recall that there are $m$ possible outcomes, labeled $j = 1, \ldots, m$. Let $\vec{p} = (p_1, \ldots, p_m)$ be a probability distribution on the set of outcomes, so $0 \le p_j \le 1$ and $\sum_j p_j = 1$. We say that $\vec{p}$ is a *risk-free probability vector* if the expected return for every wager is zero. That is, for all $i = 1, \ldots, n$

$$\mathrm{E}[R_i] = \sum_{j=1}^{m} R_i(j)\, p_j = 0$$

Note that in some cases there may be a risk-free probability vector for the wagers, but in other cases there may not be one.

**Theorem**
There is no arbitrage opportunity if and only if there is a risk-free probability vector.