MATH 4581:

Assignment # 8: SOLUTIONS

(1) $X(t) \sim GBM$, $X(0)=1$, $\mu=2$, $\sigma^2=3$

$\Rightarrow X(t) = X(0)\, e^{\mu t + \sigma B(t)} = e^{2t+\sqrt{3}B(t)}$.

$$R(3,1) = E[X(3)X(1)] = E\left[e^{6+\sqrt{3}B(3)+2+\sqrt{3}B(1)}\right]$$

$$= e^{8} E\left[e^{\sqrt{3}(B(3)-B(1))+2\sqrt{3}B(1)}\right]$$

$$= e^{8} E\left[e^{\sqrt{3}(B(3)-B(1))}\right] E\left[e^{2\sqrt{3}B(1)}\right]$$

$\because$ independent ↑

$B(3)-B(1) = \sqrt{2}Z$

$B(1) \sim Z$

$$\Rightarrow E\left[e^{\sqrt{3}(B(3)-B(1))}\right] = E\left[e^{\sqrt{3}\sqrt{2}Z}\right] = e^{\frac{6}{2}} = e^{3}$$

$$E\left[e^{2\sqrt{3}B(1)}\right] = E\left[e^{2\sqrt{3}Z}\right] = e^{\frac{12}{2}} = e^{6}$$

$$\Rightarrow R(3,1) = e^{8}\, e^{3}\, e^{6} = e^{17}$$

(2) $X(t) \sim GBM \quad \mu = -1, \quad \sigma^2 = 12.$

$X(0) = 50.$

a) $X(1) = X(0)\, e^{\mu + \sigma B(1)} = 50\, e^{-1 + \sqrt{12}\, Z}$

$$P(X(1) \geq 60) = P(50\, e^{\sqrt{12}\, Z - 1} \geq 60)$$

$$= P(\sqrt{12}\, Z - 1 \geq \ln(\tfrac{6}{5}))$$

$$= P(Z \geq \frac{1 + \ln(\frac{6}{5})}{\sqrt{12}})$$

$$= P(Z \geq 0.341) = F(-0.341)$$

$$= 0.37.$$

b) $X(1.5) = X(0)\, e^{(1.5)\mu + \sigma B(1.5)}$

$$= 50\, e^{-1.5 + \sqrt{12}\sqrt{1.5}\, Z}$$

$$\Rightarrow P(X(1.5) \geq 60) = P(50\, e^{\sqrt{18}\, Z - 1.5} \geq 60)$$

$$= P(Z \geq \frac{1.5 + \ln(\frac{6}{5})}{\sqrt{18}})$$

$$= P(Z \geq 0.397) = F(-0.397)$$

$$= 0.34$$

(3) $X(t) \sim GBM$ $\quad \mu = 0.03$

$\sigma^2 = 0.02$

$\alpha = 0.04$

$X(0) = 100$

$K = 120$

$T = 1.5$

a) Black-Scholes:

$$c = X(0)\,F(\sigma\sqrt{T} + b) - Ke^{-\alpha T} F(b)$$

$$b = \frac{\ln\left(\frac{100}{120}\right) + (0.03)(1.5)}{\sqrt{(0.02)(1.5)}} = -0.793$$

$$b + \sigma\sqrt{T} = -0.793 + \sqrt{(0.02)(1.5)} = -0.62$$

$$\Rightarrow c = 100\,F(-0.62) - 120\,e^{-(0.04)(1.5)}\,F(-0.79)$$

$$= 100\,(0.268) - 120\,(0.942)(0.2148)$$

$$= \$2.52$$

b) $T \to T-1, \quad X(0) \to X(1) = 140$

$T - 1 = 0.5$

$$c' = X(1)\, F(\sigma\sqrt{T-1} + b') - Ke^{-\alpha(T-1)} F(b')$$

$$b' = \frac{\ln \frac{X(1)}{K} + \mu(T-1)}{\sigma\sqrt{T-1}}$$

$$= \frac{\ln\left(\frac{140}{120}\right) + (0.03)(0.5)}{\sqrt{(0.02)(0.5)}} = 1.69$$

$$b' + \sigma\sqrt{T-1} = 1.69 + \sqrt{(0.02)(0.5)} = 1.79.$$

$$\Rightarrow c' = 140\, F(1.79) - 120\, e^{-(0.04)(0.5)} F(1.69)$$

$$= 140\,(0.9641) - 120\,(0.98)(0.9554)$$

$$= \$22.62$$