MATH 4581: Homework 4

SOLUTIONS

(1). p.422, #7:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & 0 \end{pmatrix} \qquad R = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

$$I - Q = \begin{pmatrix} 1 & -\frac{3}{4} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{3}{4} & 1 \end{pmatrix} \Rightarrow N = (I-Q)^{-1} = \begin{pmatrix} \frac{5}{2} & 3 & \frac{3}{2} \\ 2 & 4 & 2 \\ \frac{3}{2} & 3 & 5/2 \end{pmatrix}.$$

$$NR = \begin{pmatrix} \frac{5}{2} & 3 & 3/2 \\ 2 & 4 & 2 \\ \frac{3}{2} & 3 & 5/2 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{3}{8} & 5/8 \end{pmatrix}$$

(2) $$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N = (I-Q)^{-1} = \begin{pmatrix} 1 & \frac{\cancel{3}\,1}{4} & \frac{1}{\cancel{}3} & \cancel{3}\,1/2 \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{E}$[#steps to reach 5 | start at 1]

$= N_{11} + N_{12} + N_{13} + N_{14} = \frac{27}{4}$, $\frac{25}{12}$

(3) $g_k = \mathbb{E}[\text{\# steps to reach } 0 \text{ or } N \mid X_0 = k]$

$= \mathbb{E}[\text{\# steps to } 0, N \mid X_0 = k, X_1 = k+1]\, \frac{1}{2}$

$+ \mathbb{E}[\text{\# steps to } 0, N \mid X_0 = k, X_1 = k-1]\, \frac{1}{2}$

$= (g_{k+1} + 1)\frac{1}{2} + (g_{k-1} + 1)\frac{1}{2}$

$\Rightarrow \boxed{g_k = \frac{1}{2} g_{k+1} + \frac{1}{2} g_{k-1} + 1}$ $k = 1, 2, \ldots, N-1$

$g_0 = g_N = 0.$

Try $g_k = A + Bk + Ck^2$

$\Rightarrow A + Bk + Ck^2 = \frac{1}{2}(A + B(k+1) + C(k+1)^2) + \frac{1}{2}(A + B(k-1) + C(k-1)^2) + 1$

$\Leftrightarrow A + Bk + Ck^2 = A + Bk + Ck^2 + C + 1.$

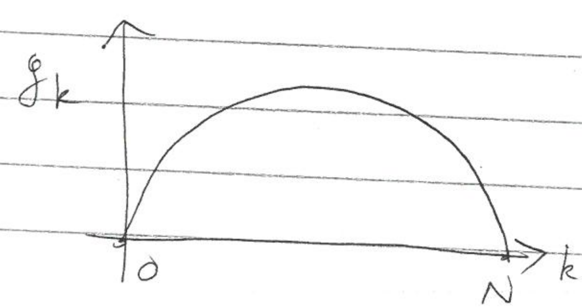
$\Leftrightarrow \boxed{C = -1}$.

So $g_k = A + Bk - k^2$ $\rightarrow k = 0 \Rightarrow 0 = A$

$k = N \Rightarrow 0 = A + BN - N^2$

$\Rightarrow B = N.$

$\Rightarrow \boxed{g_k = k(N-k)}$



4 p.442 #3.

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}.$$

a) P is absorbing if $a=0$ or $b=0$. $\Leftrightarrow ab=0$

b) P is ergodic if $a\neq 0$ and $b \neq 0$ $\Leftrightarrow ab > 0$.

c) P is regular if $0<a<1$ or $0<b<1$, $ab>0$

$\Leftrightarrow 0 < ab < 1.$

5 a) $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow \frac{3}{4}w_1 + \frac{1}{2}w_2 = w_1 \Rightarrow \frac{1}{2}w_2 = \frac{1}{4}w_1$

$\Rightarrow w_1 = 2w_2$

$w_1 + w_2 = 1 \Rightarrow w = \left(\frac{2}{3}, \frac{1}{3}\right).$

b) $P = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$ column sums $=1$

$\Rightarrow w = \left(\frac{1}{2}\ \frac{1}{2}\right).$

c) $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

$w_1 = \frac{3}{4}w_1 + \frac{1}{4}w_3$

$w_3 = \frac{1}{3}w_2 + \frac{1}{2}w_3$

$\Rightarrow \boxed{w_1 = w_3}$

$\boxed{w_2 = \frac{3}{2}w_3}$

$w_1 + w_2 + w_3 = 1 \Rightarrow w_3 = \frac{2}{7}$

$\Rightarrow w = \left(\frac{2}{7}\ \frac{3}{7}\ \frac{2}{7}\right).$

6) $X_n = (R_1 + \cdots + R_n) \mod 5$

a) $P = \begin{pmatrix} \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{6} \\ \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$

Matrix is regular.

All column sums $= 1$

$\Rightarrow W = \left(\frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5}\right)$

Regular $\Rightarrow \lim_{n\to\infty} P(X_n = 1) = w_1 = \frac{1}{5}$

Mean ~~first return~~ time at 4 $= \frac{1}{w_4} = 5$