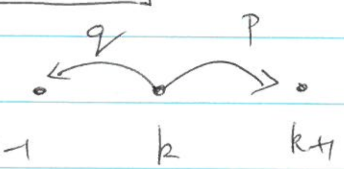
MATH 4581: Assignment #5

$\boxed{\text{SOLUTIONS}}$

1



a) $P_k = p\,P_{k+1} + q\,P_{k-1}$

b) $P_k = a^k \Rightarrow \quad a^k = p\,a^{k+1} + q\,a^{k-1}$

$$1 = pa + qa^{-1}$$

$$\Rightarrow \quad a = 1 \quad \text{or} \quad a = \frac{q}{p}.$$

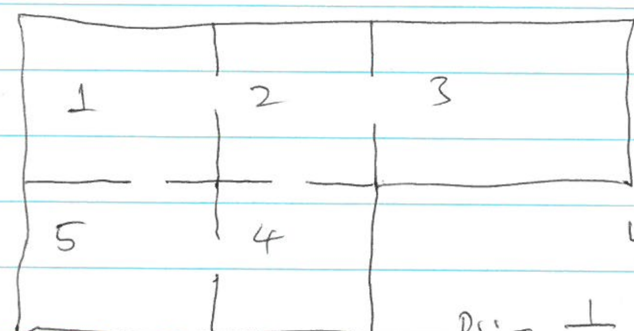
c) $P_k = A\,1^k + B\left(\frac{q}{p}\right)^k = A + B\left(\frac{q}{p}\right)^k$

B.C.: $P_0 = 0 = A + B$

$P_N = 1 = A + B\left(\frac{q}{p}\right)^N$

$$\Rightarrow \quad P_k = \frac{1-(q/p)^k}{1-(q/p)^N}.$$

2



$\cancel{w_i P_i}$

$$w_i P_{ij} = w_j P_{ji}$$

$$P_{ij} = \frac{1}{d(i)} = (\#\text{doors out of room } i)^{-1}$$

$d(1) = 2$ $\quad d(4) = 2$

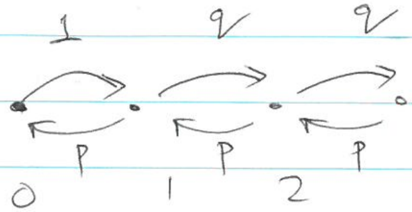
$d(2) = 3$ $\quad d(5) = 2$

$d(3) = 1$

$$w_i = c\,d(i) \Rightarrow c\,d(i)\frac{1}{d(i)} = c\,d(j)\frac{1}{d(j)} \checkmark$$

$$\Rightarrow \quad w = \frac{1}{10}(2, 3, 1, 2, 2)$$

3



$$w_i P_{ij} = w_j P_{ji} \Rightarrow \quad w_k\, q = w_{k+1}\, p \qquad k = 1, 2, 3, \ldots$$

$$w_0 = w_1\, p$$

$$\Rightarrow \quad w_{k+1} = \left(\frac{q}{p}\right) w_k = \left(\frac{q}{p}\right)^k w_1 = \left(\frac{q}{p}\right)^k \frac{1}{p}\, w_0$$

$$\sum_{k=0}^{\infty} w_k = w_0 + \sum_{k=1}^{\infty} \left(\frac{q}{p}\right)^k \frac{1}{p}\, w_0$$

$$= w_0 \left[1 + \frac{q}{p^2} \sum_{j=0}^{\infty} \left(q/p\right)^j \right]$$

$$= w_0 \left[1 + \frac{q}{p^2} \frac{1}{1 - q/p} \right] \qquad \text{if } q < p.$$

$$\Rightarrow \quad w_0 = \frac{p(p-q)}{p^2 - pq + q}.$$

Mean first return time at 0 is

$$\frac{1}{w_0} = \frac{p^2 - pq + q}{p(p-q)}$$

4 $\quad \lambda = 10 \quad min^{-1}$

a) $P(N(1) = 5) = \frac{\lambda^5}{5!} e^{-\lambda}$

b) $P\left(N\left(\tfrac{1}{4}\right) = 1,\ N\left(\tfrac{35}{60}\right) - N\left(\tfrac{1}{4}\right) = 2\right)$

$= P\left(N\left(\tfrac{1}{4}\right) = 1\right) \cdot P\left(N\left(\tfrac{20}{60}\right) = 2\right)$

$= \frac{\lambda/4}{1} e^{-\lambda/4} \cdot \frac{(\lambda/3)^2}{2} e^{-\lambda/3}$

5 $\quad \lambda = 10 \quad week^{-1}$

a) $P\left(N\left(\tfrac{1}{7}\right) = 2\right) = \frac{(\lambda/7)^2}{2!} e^{-\lambda/7}$

b) $P\left(T > \tfrac{2}{7}\right) = e^{-\lambda(2/7)}$

c) $E[T_{100}] = 100\ E[T] = \frac{100}{\lambda}$

d) $P(T_{500} > 52) = P\left(Z_{500} > \frac{52 - (500)\frac{1}{\lambda}}{\sqrt{500}\ \frac{1}{\lambda}}\right)$

$\mu = \frac{1}{\lambda}$

$\sigma = \frac{1}{\lambda}$

$\simeq P(Z > 0.89)$