MATH ~~4581~~ 4581: Final Review

SOLUTIONS

1 a) ANOVA: p-value $= 5.69 \times 10^{-6}$

Conclusion: reject H_0, evidence does not support hypothesis that means are all the same.

$s_p = 0.59$

b) $\bar{x}_{PTA} = 19.457$, $\bar{x}_{PTP} = 21.2$, $\bar{x}_P = 21.425$

contrast: $c = \frac{1}{2}\mu_{PTA} + \frac{1}{2}\mu_{PTP} - \mu_P$.

$$\sum \frac{a_i^2}{n_i} = \frac{(\frac{1}{2})^2}{7} + \frac{(\frac{1}{2})^2}{7} + \frac{(-1)^2}{4} = 0.321$$

$$\hat{c} = \frac{1}{2}\bar{x}_{PTA} + \frac{1}{2}\bar{x}_{PTP} - \bar{x}_P = -1.0965.$$

$$t = \frac{\hat{c}}{s_p\sqrt{\sum_i \frac{a_i^2}{n_i}}} = \frac{-1.0965}{(0.59)\sqrt{0.321}} = -3.28$$

$df = 23-4 = 19$; $\frac{\alpha}{2} = 0.05$

$t_{\frac{\alpha}{2}, df} = 2.093 < |t| \Rightarrow$ reject H_0

$\Rightarrow$ evidence does not support that PT groups are same as P group.

2 Regression: $y = 0.86625 + 1.045x$

a)

$s = 0.4084$

p-value $= 0.02259$

$\Rightarrow$ reject H_0 at 5% level

$\Rightarrow$ there is probably a linear relationship between X and Y.

b) $x = 23$; $\hat{y}(23) = 24.9$; $\bar{x} = 19.457$

$$\sum(x_i - \bar{x})^2 = 7(0.4806)^2 = 1.617$$

$df = 5 \Rightarrow t = 2.571$

$\Rightarrow$ 95% CI : $24.9 \pm (2.571)(0.4084)\sqrt{\frac{1}{7} + \frac{(23-19.457)^2}{1.617}}$

$= 24.9 \pm 2.95$

3 a) use $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $\lambda = 1.5$.

$P(X=0) = 0.223$	times 100	22.3
$P(X=1) = 0.335$	$\Rightarrow$	33.5
$P(X=2) = 0.251$		25.1
$P(\text{more}) = 1 - (\text{previous}) = 0.191$		19.1

b) Goodness of fit test:

$$\chi^2 = \sum \frac{(x_i - n\hat{p}_i)^2}{n\hat{p}_i} = 2.74$$

$df = 3 \Rightarrow$ critical value $\chi^2_c = 7.81$

$\alpha = 0.05$

p-value = 0.4334

since $\chi^2 < \chi^2_c \Rightarrow$ ~~reject~~ accept H_0, data fits distribution.

4 a) $w_1 = 0.5w_1 + 0.4w_2 + 0.3w_3$

$w_2 = 0.3w_1 + 0.4w_2 + 0.3w_3$

$w_3 = 0.2w_1 + 0.2w_2 + 0.4w_3$

$w_1 + w_2 + w_3 = 1$

$\Rightarrow w = \left(\frac{5}{12}, \frac{1}{3}, \frac{1}{4}\right)$

b) Make state 3 absorbing.

$$P' = \left(\begin{array}{cc|c} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ \hline 0 & 0 & 1 \end{array}\right) \qquad Q = \begin{pmatrix} 0.5 & 0.3 \\ 0.4 & 0.4 \end{pmatrix}$$

$$N = (I - Q)^{-1} = \frac{1}{9}\begin{pmatrix} 30 & 15 \\ 20 & 25 \end{pmatrix}$$

$$\Rightarrow E[\#\text{steps } 1 \to 3] = \frac{30}{9} + \frac{15}{9} = 5$$

5 a)

$$\begin{array}{c} \left(\begin{array}{cc|cc} 0.5 & 0.3 & 0.1 & 0.1 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} 1 \\ 4 \\ 2 \\ 3 \end{array} \qquad Q = \begin{pmatrix} 0.5 & 0.3 \\ 0.4 & 0.3 \end{pmatrix} \begin{array}{c} 1 \\ 4 \end{array}$$

$$R = \begin{pmatrix} 0.1 & 0.1 \\ 0.2 & 0.1 \end{pmatrix} \begin{array}{c} 2 \\ 3 \end{array}$$

b) $$N = (I - Q)^{-1} = \frac{1}{23}\begin{pmatrix} 70 & 30 \\ 40 & 50 \end{pmatrix}$$

c) $$NR = \frac{1}{23}\begin{array}{cc} \begin{pmatrix} 13 & 10 \\ 14 & 9 \end{pmatrix} & \begin{array}{c} 1 \\ 4 \end{array} \\ \begin{array}{cc} 2 & \quad 3 \end{array} & \end{array}$$

$$(NR)_{42} = \frac{14}{23}.$$

6 a)
$$\begin{tikzcd} 0 \arrow[r, bend left, "3"] & 1 \arrow[l, bend left, "2"] \arrow[r, bend left, "3"] & 2 \arrow[l, bend left, "4"] \arrow[r, bend left, "3"] & 3 \arrow[l, bend left, "4"] & \cdots \end{tikzcd}$$

$$2p_1 = 3p_0 \qquad p_1 = \frac{3}{2}p_0$$

$$4p_2 = 3p_1 \qquad p_2 = \cancel{\frac{9}{8}p_0}\left(\frac{3}{2}\right)\left(\frac{3}{4}\right)p_0$$

$$4p_3 = 3p_2 \qquad p_3 = \left(\frac{3}{2}\right)\left(\frac{3}{4}\right)^2 p_0$$

$$\vdots \qquad\qquad \vdots$$

$$4p_{k+1} = 3p_k \qquad p_{k+1} = \left(\frac{3}{2}\right)\left(\frac{3}{4}\right)^k p_0$$

$$1 = p_0 + p_1 + p_2 + \cdots$$

$$= p_0\left(1 + \frac{3}{2} + \frac{3}{2}\left(\frac{3}{4}\right) + \frac{3}{2}\left(\frac{3}{4}\right)^2 + \cdots\right)$$

$$= p_0\left(1 + \frac{3}{2}\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right]\right)$$

$$= p_0\left(1 + \frac{3}{2}\,\frac{1}{1-3/4}\right) = 7p_0$$

$$\Rightarrow \quad p_0 = \frac{1}{7}, \qquad p_{k+1} = \left(\frac{3}{2}\right)\left(\frac{3}{4}\right)^k \frac{1}{7} \quad (k \geq 0)$$

$$\rho = \frac{3}{4}$$

b) $$E[N_q] = \frac{\rho}{(1-\rho)^2} P_2 = \frac{3/4}{(1-3/4)^2} P_2 = 1.93$$

$$E[W] = \frac{1}{\lambda} E[N_q] = 9/14$$

$$E[T] = E[W] + \frac{1}{\mu} = 8/7$$

$$E[N] = \lambda E[T] = \frac{24}{7}.$$

7 a)

$$\begin{tikzcd} 0 \arrow[r, bend left, "2"] & 1 \arrow[l, bend left, "1"] \arrow[r, bend left, "2"] & 2 \arrow[l, bend left, "2"] \arrow[r, bend left, "2"] & 3 \arrow[l, bend left, "3"] \arrow[r, bend left, "2"] & 4 \arrow[l, bend left, "3"] \end{tikzcd}$$

$$2p_0 = p_1 \qquad p_1 = 2p_0$$

$$2p_1 = 2p_2 \qquad p_2 = 2p_0$$

$$2p_2 = 3p_3 \qquad p_3 = \frac{4}{3} p_0$$

$$2p_3 = 3p_4 \qquad p_4 = \frac{8}{9} p_0$$

$$\Rightarrow \quad p_0 = \frac{9}{65}$$

$$p_1 = p_2 = \frac{18}{65}$$

$$p_3 = \frac{12}{65}$$

$$p_4 = \frac{8}{65}$$

b) $$E[N] = 0 \cdot p_0 + 1 \cdot p_1 + 2p_2 + 3p_3 + 4p_4 = \frac{122}{65}$$

$$E[N_q] = 1 \cdot p_4 = \frac{8}{65}$$

$$\lambda_a = 2p_0 + 2p_1 + 2p_2 + 2p_3 = 2(1-p_4) = \frac{114}{65}$$

$$E[T] = \frac{1}{\lambda_a} E[N] = \frac{61}{57}$$

$$E[W] = \frac{1}{\lambda_a} E[N_q] = \frac{4}{57}.$$

8) $$M_X(t) = E[e^{tX}] = \int_0^1 e^{tx}\, 2x\, dx$$

$$= 2 \frac{d}{dt} \int_0^1 e^{tx} dx$$

$$= 2 \frac{d}{dt}\left[\frac{1}{t}(e^t - 1)\right]$$

$$= \frac{2}{t^2}\left(1 - e^t + t e^t\right).$$

9) a) $$f_X(x) = \int_0^{2-x} \tfrac{1}{2} x (y+1)\, dy = \frac{x}{4}(x^2 - 8x + 8)$$

$$f_Y(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2(y+1)}{x^2 - 8x + 8} \qquad 0<x<2, \quad 0<y<2-x.$$

b) $$\hat{y} = E[Y | X = x] = \int y\, f_Y(y|x)\, dy$$

$$= \int_0^{2-x} y \, \frac{2(y+1)}{x^2 - 8x + 8}\, dy.$$

(10) $F_Y(y) = P(Y \le y)$

$= P(10 - X^2 \le y)$

$= P(X \ge \sqrt{10-y}) + \cancel{P(X \le -\sqrt{10-y})}^{\,0}$

$= \int_{\sqrt{10-y}}^{3} \frac{1}{9} x^2 \, dx$

$= 1 - \frac{1}{27}(10-y)^{3/2}$, $1 \le y \le 10.$

(11) $X(t) \sim$ GBM $\mu = 0.025, \ \sigma^2 = 0.01, \ \alpha = 0.03.$

$K = 75, \quad X(0) = 50, \quad T = 2.$

$b = \dfrac{\ln \frac{50}{75} + (0.025)2}{\sqrt{(0.01)(2)}} = -2.514$

$b + \sigma\sqrt{T} = -2.514 + \sqrt{(0.01)2} = -2.373.$

$C = 50\, F(-2.373) - 75\, e^{-(0.03)(2)} F(-2.514)$

$= 50(1 - 0.9911) - 75(0.942)(1 - 0.9940)$

$= \$0.021 = 2¢$

12. a) W1:

o/c	cost	yield	R1
40	100	40	−60
300	100	300	200

W2:

o/c	cost	yield	R2
40	c	0	−c
300	c	140	140−c

$E[R1] = 0 = -60p + 200(1-p) \Rightarrow p = \frac{10}{13}$

$E[R2] = 0 = -cp + (140-c)(1-p) \Rightarrow c = \$32.31$

b) $R = xR_1 + yR_2$, $c = 10$.

o/c	R	
40	$-60x - 10y$	> 0
300	$200x + 130y$	> 0

$\Bigg\}$ $x = -1$, $y = 2$

13. a) $P(\text{next} = \text{spam}) = \frac{15}{40} = \frac{3}{8}$.

$\#\text{spam} \sim Bin(n, p)$ $\quad p = 3/8$.

$\Rightarrow$ if $n=5$, $P(\#\text{spam} = 3) = \binom{5}{3}\left(\frac{3}{8}\right)^3\left(\frac{5}{8}\right)^2 = 0.206$

b) $\lambda = 40$, $t = \frac{1}{24}$, $N(t) = \#$ emails to time t.

$P(N(t) = 3) = \frac{(\lambda t)^3}{3!} e^{-\lambda t} = \frac{(40(\frac{1}{24}))^3}{6} e^{-40(\frac{1}{24})} = 0.146$

14) $X(t) = t - 2A$

a) $F_{X(t)}(x) = P(X(t) \le x) = P(t - 2A \le x)$

$$= P\left(A \ge \frac{t-x}{2}\right).$$

$$= \int_{\frac{t-x}{2}}^{2} \frac{3}{8} a^2 \, da.$$

$$= 1 - \frac{1}{64}(t-x)^3 \quad , t-4 < x < t$$

b) $E[X(t)] = E[t - 2A] = t - 2E[A]$

$$= t - 2\int_0^2 a \frac{3}{8} a^2 \, da$$

$$= t - 3$$

c) $R(t,s) = E[X(t)X(s)]$

$$= E[(t-2A)(s-2A)]$$

$$= \int_0^2 (t-2a)(s-2a) \frac{3}{8} a^2 \, da$$

15) a) $M_X(t) = E[e^{tX}] = \int_4^\infty e^{tx} \frac{1}{3} e^{-\frac{1}{3}(x-4)} dx$ $\quad \begin{cases} u = x-4 \\ du = dx \end{cases}$

$$= \int_0^\infty e^{t(u+4)} \frac{1}{3} e^{-\frac{1}{3}u} du$$

$$= \frac{1}{3} e^{4t} \int_0^\infty e^{-u(\frac{1}{3}-t)} du$$

$$= \frac{1}{3} e^{4t} \frac{1}{\frac{1}{3}-t} = \frac{e^{4t}}{1-3t} \quad \left(t < \frac{1}{3}\right)$$

b) $E_X(t) = M_X'(0)$

$$M_X'(t) = \frac{(1-3t)\,4e^{4t} - e^{4t}(-3)}{(1-3t)^2}$$

$$M_X'(0) = \frac{4+3}{1} = 7$$

16 a) state = # white marbles = $\{0, 1, 2, 3\}$

$$P = \begin{array}{cc} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

b) 2 Red = 1 White ; 1 Red = 2 White

$$\Rightarrow \quad (P^3)_{1,2} = 0.315$$

c) $$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

$$N = (I - Q)^{-1} = \begin{pmatrix} 5 & 9 & 6 \\ 3 & 9 & 6 \\ 2 & 6 & 6 \end{pmatrix}$$

$$\Rightarrow E[\#\text{steps } 0 \to 3] = 5 + 9 + 6 = 20.$$

17 a)

$$\begin{tikzcd} 0 \arrow[r, bend left, "6"] & 1 \arrow[l, bend left, "3"] \arrow[r, bend left, "6"] & 2 \arrow[l, bend left, "6"] \arrow[r, bend left, "6"] & 3 \arrow[l, bend left, "9"] \arrow[r, bend left, "6"] & 4 \arrow[l, bend left, "9"] & \cdots \end{tikzcd}$$

$\lambda = 6$

$\mu = 3$, $c = 3$

$\rho = 2$.

$6p_0 = 3p_1$ $\qquad$ $p_1 = 2p_0$

$6p_1 = 6p_2$ $\qquad$ $p_2 = p_1 = 2p_0$

$6p_2 = 9p_3$ $\qquad$ $p_3 = \frac{2}{3}p_2 = 2\left(\frac{2}{3}\right)p_0$

$6p_3 = 9p_4$ $\qquad$ $p_4 = \frac{2}{3}p_3 = 2\left(\frac{2}{3}\right)^2 p_0$

$\vdots$ $\qquad$ $\vdots$

$$\Rightarrow 1 = p_0\left(1 + 2 + 2 + 2\left(\tfrac{2}{3}\right) + 2\left(\tfrac{2}{3}\right)^2 + \cdots\right)$$

$$= p_0\left(1 + 2 + 2\left[1 + \tfrac{2}{3} + \left(\tfrac{2}{3}\right)^2 + \cdots\right]\right)$$

$$= p_0\left(1 + 2 + \frac{2}{1 - 2/3}\right)$$

$$\Rightarrow p_0 = \frac{1}{9}, \quad p_1 = \frac{2}{9}, \quad p_2 = \frac{2}{9}, \quad p_3 = \frac{4}{27}.$$

b) $P(W=0) = p_0 + p_1 + p_2 = \frac{5}{9}$.

$$\Rightarrow P(W < \tfrac{1}{2}) = \cancel{P(W=0) + 2P(W>0)}$$

$$= P(W < \tfrac{1}{2} \mid W = 0)\, P(W=0) + P(W < \tfrac{1}{2} \mid W > 0)\, P(W>0)$$

$$= 1.\, P(W=0) + P(W>0)\, F_W\left(\tfrac{1}{2} \mid W > 0\right)$$

$$= \frac{5}{9} + \frac{4}{9}\left(1 - e^{-(3)(3)\left(1-\frac{2}{3}\right)\left(\frac{1}{2}\right)}\right)$$

$$= 0.9008$$

18) a) $B(t) \sim N(0, \sigma^2 t)$.

$$B(1) \sim N(0, \sigma^2) \Rightarrow B(1) = \sigma Z = 2.5 Z.$$

$$P(B(1) < 3) = P(2.5 Z < 3) = P\left(Z < \frac{3}{2.5}\right) = 0.8849.$$

b) Independent increments

$$\Rightarrow P(B(4) - B(1) > 5,\ B(8) - B(4) < -4)$$

$$= P(B(4) - B(1) > 5)\, P(B(8) - B(4) < -4)$$

$$B(4) - B(1) \sim N(0, \sigma^2(4-1)) = N(0, 3\sigma^2)$$

$$B(8) - B(4) \sim N(0, \sigma^2(8-4)) = N(0, 4\sigma^2).$$

$$\Rightarrow B(4) - B(1) = \sqrt{3}\sigma Z = 2.5\sqrt{3} Z$$

$$B(8) - B(4) = 2\sigma Z = (2.5)(2) Z.$$

$$\Rightarrow P(B(4) - B(1) > 5\ \ B(8) - B(4) < -4)$$

$$= P(2.5\sqrt{3} Z > 5)\ P((2.5)(2) Z < -4)$$

$$= P\left(Z > \frac{5}{2.5\sqrt{3}}\right) P\left(Z < -\frac{4}{(2.5)(2)}\right)$$

$$= 0.0263$$