Math 4581 **Black-Scholes**

We now assume that the price of a stock at time t is X(t), the initial price is x_0, there is an interest rate of a, and that it costs \$c to buy an option to buy the stock at time T for \$K. We also assume that the stock price follows a geometric Brownian motion model.

Thus we have $X(t) = x_0 e^{\sigma B(t)+\mu t}$ and the present value of the stock is $e^{-at}X(t)$.

The value of the stock option at time T will be either $X(T) - K$ if $X(T) > K$ or 0, which is written as $(X(T) - K)^+$, so the present value of the option is $e^{-aT}(X(T) - K)^+$.

If we want want to make it so no arbitrage is possible then there needs to be a probability measure so that $E_p(e^{-aT}(X(T) - K)^+) = c$

We also need that the future value of the price we buy the stock for at time t is the same as the actual price at time t on average:

$E_p(X(t)) = x_0 e^{at}$.

We know from the last handout that $E_p(X(t)) = x_0 e^{\sigma^2 t/2+\mu t}$ so we get: $x_0 e^{\sigma^2 t/2+\mu t} = x_0 e^{at}$ or $\sigma^2/2 + \mu = a$

This can be solved in a few ways. One way is to use the revised model $X(t) = x_0 e^{\sigma B(t)+\mu' t}$ where $\mu' = a - \sigma^2/2$.

We now need to go back to the stock option equation using the revised $X(t) = x_0 e^{\sigma B(t)+\mu' t}$.

Instead of trying to find the distribution for this, let's use the distribution for $Y(t) = \sigma B(t) + \mu' t$.

Since B(t) is normal with a mean of 0 and a variance of t, Y(t) is normal and we find $E(Y) = \mu' t$ and $Var(Y) = \sigma^2 t$.

Thus we have $c = E_p(e^{-aT}(X(T) - K)^+) = E_p(e^{-aT}(x_0 e^y - K)^+) = \displaystyle\int e^{-aT}(x_0 e^y - K)^+ f_Y(y)\, dy$

$$= \int_{-\infty}^{\infty} e^{-aT}(x_0 e^y - K)^+ \frac{1}{\sqrt{2\pi T\sigma^2}} e^{(y-\mu' T)^2/2T\sigma^2}\, dy$$

$$= \int_{\ln(K/x_0)}^{\infty} e^{-aT}(x_0 e^y - K) \frac{1}{\sqrt{2\pi T\sigma^2}} e^{(y-\mu' T)^2/2T\sigma^2}\, dy = \ldots$$

$$\rightarrow c = x_0 \phi(\sigma\sqrt{T} + b) - K e^{-aT}\phi(b)$$

where $b = \dfrac{\ln(x_0/K) + (a - \sigma^2/2)T}{\sigma\sqrt{T}}$ and $\phi(t)$ is the standard normal cdf at t.

If we want to buy the option at time s instead of at time 0 then we can use the same equation where x_0 is replaced by x_s (price at time s) and t by $t - s$

Homework

1. The current price of a stock is \$50 and we assume it can be modeled by geometric Brownian motion with a drift parameter of \$.05 per year with a variance parameter of .6. If the interest rate is 3% and we want to sell an option to buy the stock for \$100 in 2 years:

a) what should the initial price of it be if we don't want an arbitrage opportunity?

b) what should the price be after one year if the stock price has risen to \$65 and we don't want an arbitrage opportunity?

2. Finish the calculation of the integral above.