Math 4581 **Stock Prices and Brownian Motion**

We want to be able to look at the buying and selling of options and stock. To do this, first we need to introduce some terminology from economics. For now, I'll break the ideas into two pieces that will be combined later.

Since we will be looking at prices over time, we will have to deal with inflation/interest. If we have something with an initial value of v with a constant interest rate of α, then the value at time t will be $ve^{\alpha t}$–this is called the **future value** of v. Conversely, if we have something with a value of v with a constant interest rate of α at time t, then the initial value will be $ve^{-\alpha t}$–this is called the **present value** of v.

Now let's look at the buying and selling of stock and stock options (which gives you the option to buy a stock at a set price in the future). To keep it simple for now, assume that there is no inflation.

Assume there are two options for a stock: you can buy or sell a share and you can buy or sell an option. We want to see what the price of options should be. To make this concrete, start with a simple example.

Example. Suppose the current price of a share of stock is \$100 and that it will change to either \$50 or \$200 in the future. We want to see what the price should be to buy a unit of an option to buy the stock for \$150 in the future, for now let it be \$c.

Now assume we will buy x shares of the stock and y units of options (we will allow these to be either positive or negative so we can buy and sell units). Look at the two possible cases:
if the future price is \$200 then the future worth will be $200x + (200 - 150)y = 200x + 50y$
if the future price is \$50 then the future worth will be $50x$ (since the options are worthless)

To keep it simple, let y be the value that makes the future worth equal for both cases:
$200x + 50y = 50x \to y = -3x$ which gives the future worth is $50x$ for both cases.

Our cost to buy the x shares and y options is $100x + cy = 100x - 3cx$, so our gain is:
$50x - (100x - 3cx) = 3cx - 50x = x(3c - 50)$

Since x can be either positive or negate, notice this says we will always have a positive gain unless $3c-50 = 0$ or $c = \dfrac{50}{3}$

Def.: A betting scheme that always makes money is called an **arbitrage**. The above calculations show that there will always be arbitrage unless the options are priced correctly.

Now generalize the example. Suppose a situation has:

m possible outcomes $S = \{1, 2, \ldots, m\}$ (it might be better to think of the values as $\{s_1, s_2, \ldots, s_m\}$);
n possible wagers;
possible returns of $r_i(j)$ = amount of return on wager i with the outcome j.

If we bet x_i on wager i then our return will be $\displaystyle\sum_{i=1}^{n} x_i r_i(j)$ with outcome j.

Arbitrage Theorem. Exactly one of the following is true:

1. there exists a probability vector $\vec{p} = \langle p_1, p_2, \ldots p_m \rangle$ for which $\displaystyle\sum_{j=1}^{m} p_j r_i(j) = 0$ for all i

2. there is a wager $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$ so that $\displaystyle\sum_{i=1}^{n} x_i r_i(j) > 0$ for all j.

Notice this says there exists such a vector $\vec{p}$ (for which $E_{\vec{p}}(r_i(X)) = 0$ for all i where X is the outcome) or there is a bet that will make money no matter the outcome.

We go back to the example where we let $p = P$(the price in the future is \$200), so P(price=\$50)=$1 - p$.

We have two ways of betting here: buying shares or buying options.
First look at the case where we buy shares of stock. The possible returns are either \$100 or −\$50 (the first case is where we buy at \$100 and the price goes up to \$200).

$\rightarrow E_{\vec{p}}(return) = 100p + (-50)(1-p) = 150p - 50$ which is zero if $p = \dfrac{1}{3}$

If we buy options, then our possible returns are either $\$50 - c$ or $-\$c$

$\rightarrow E_{\vec{p}}(return) = (50-c)p + (-c)(1-p) = 50p - c$ which is zero if $p = \dfrac{1}{3}$ and $c = \dfrac{50}{3}$. Thus either there exists such a $\vec{p}$ (if $c = \dfrac{50}{3}$) or there is a betting scheme that always makes money (if c is any other number).

Homework

1. What happens if the numbers in the example change to: current price=\$10; possible future prices of either \$5 or \$50; you have an option to buy for \$20 in the future?

2.Suppose the current price of a share of stock is \$100 and that it will change to either \$50, \$175 or \$200 in the future. Let \$c be the price to buy a unit of an option to buy the stock for \$150 in the future.
a) Set up the equations for equations for the gains for buying stock and buying options if $p_1 = P(\$50)$, $p_2 = P(\$175)$, and $p_3 = P(\$200)$.
b) Set the two equations in (a) equal to zero and solve for p_1 and p_2. The solutions will be in terms of c.
c) Find for what values of c an arbitrage opportunity exists.

We will now want to introduce some uncertainty in terms of price and allow there to be inflation. One possible model is:
$X(t) = x_0 e^{\mu t + \sigma B(t)}$
where x_0 is the initial stock price, μ is the inflation rate, B(t) is a random variable with a mean of zero, and σ is variance parameter.
One of the random variables used for B(t) is Brownian motion.

Brownian Motion

Start with the random variable D_i which moves left or right a distance of one with equal probability at time i. Notice this gives $E(D_i) = 0$ and $Var(D_i) = 1$. If we assume the individual steps are independent, then we get the position at time t is $X_n(t) = \sum_{i=1}^{n} D_i$ where n is the largest integer less than or equal to t. This is a random walk.

To get a continuous random process, let the time between movements and the distance traveled go to zero. For a fixed t, break the interval into n subintervals, assume it moves a distance of Δx, and again let $X_n(t)$ be the sum of the D_i.
$X_n(t) = \Delta x(D_1 + \ldots D_n)$, so $E(X_n(t) = 0$ and $Var(X_n(t) = (\Delta x)^2 n$
Notice that Δx and n have not been linked, so as we let $n \to \infty$, we can let $\Delta x = \dfrac{\sigma}{\sqrt{n}}$. Let:
$X(t) = \lim_{n\to\infty} X_n(t)$
This is a continuous random process with $E(X(t)) = 0$ and $Var(X(t)) = \sigma^2 t$. Notice that we can assume $X(t)$ has a normal distribution by the Central Limit Theorem since it's the infinite sum of independent, identically distributed random variables.
Also note that the independence of the steps means that $X(t)$ will have independent and stationary increments ($X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$ are independent and have the same distribution as long as $t_{i+1} - t_i$ is constant). It also gives us that $X(t_2) - X(t_1) = X(t_2 - t_1)$.
This random process is called Brownian motion because it is used to model the motion of particles. It is not a perfect model, so it's also called the Wiener process.

Now let's get the the joint distribution at times $t_1, t_2, \ldots, t_k$. Since the increments are independent:
$f_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1, x_2, \ldots, x_k) = f_{X(t_1)}(x_1) f_{X(t_2-t_1)}(x_2 - x_1) \cdots f_{X(t_k - t_{k-1})}(x_k - x_{k-1})$

$$\frac{\exp(-\frac{1}{2}(\frac{x_1^2}{\sigma^2 t_1}+\frac{(x_2-x_1)^2}{\sigma^2(t_2-t_1)}+\cdots\frac{(x_k-x_{k-1})^2}{\sigma^2(t_k-t_{k-1})})}{\sqrt{(2\pi\sigma^2)^k t_1(t_2-t_1)\cdots(t_k-t_{k-1})}}$$

It can be shown that $C_X(t_1,t_2)=\sigma^2\min(t_1,t_2)$.

Homework
3. $B(t)$ is Brownian motion with $\sigma=1$.
a) Find $P(B(4)>1)$
b) Find $P(B(4)>1, B(7)-B(4)<2)$
c) Find $P(B(4)>3|B(2)=1)$

Geometric Brownian Motion

If we assume that the percent changes in stock prices are independent and identically distributed, then it makes sense to use Brownian motion to model the price.
Here's the (very) rough reasoning:
let X_n be the price at time n, then $X_n=\dfrac{X_n}{X_{n-1}}\dfrac{X_{n-1}}{X_{n-2}}\cdots\dfrac{X_1}{X_0}X_0$
$\rightarrow \ln(X_n)=\displaystyle\sum_{i=1}^{n}\ln\left(\frac{X_i}{X_{i-1}}\right)+\ln(X_0)$
since the ratios are assumed to be independent, identically distributed we can assume the sum of the ratios is normal and, with some normalization, Brownian motion.

If we assume that B(t) is Brownian motion with $\sigma=1$, then $X(t)=\sigma B(t)+\mu t$ is called **Brownian motion with drift** (μ is called the drift coefficient and σ^2 is the variance parameter).

$X(t)$ is called **geometric Brownian motion** if $X(t)=x_0e^{\sigma B(t)+\mu t}$

Note: $E(e^{aZ})=e^{a^2/2}$

Proof: $E(e^{aZ})=\displaystyle\int_{-\infty}^{\infty}e^{ax}\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx=e^{a^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{(-x^2/2+ax-a^2/2)}\,dx$
$=e^{a^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(x-a)^2/2}\,dx=e^{a^2/2}$ (since the function inside the integral is a normal distribution with mean=a)

This gives $E(X(t))=x_0e^{\sigma^2t/2+\mu t}$ and, in a similar way, $Var(X(t)=x_0^2e^{\sigma^2t+2\mu t}(e^{\sigma^2 t}-1)$

Homework
4. Let $X(t)$ be the price of a stock at time t. If the current price of the stock is \$50 and we assume it can be modeled by geometric Brownian motion with a drift parameter of \$3 per year with a variance parameter of \$10, then find the probability that the price of the stock after two years is over \$60. Use the model and solve the inequality in terms of B(t).