1. Moment Generating Function
   $M_X(t) = E(e^{tx})$
   continuous: $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_x(x)dx$
   discrete: $M_X(t) = \sum e^{tx} P(x)$
   e^{tx} expansion: $\sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$
   property:
   $M_X^{(n)}(0) = E(X^n)$
   $VAR[X] = E[X^2] - E^2[X] = M_x''(0) - [M_x'(0)]^2$
   $M_x(t) = M_y(t) \rightarrow XY$ has same distribution
   $M_{a+bX}(t) = e^{at} M_X(bt)$
   $M_{X+Y}(bt) = M_X(t) M_Y(t)$ $(X, Y$ independent)

2. Chi-Square
   (a) Goodness of fit test problem: Outcome vs Expected df = t-1. If expected is not clear, use maximum likelihood to maximize and df = Unknown - 1
   (b) Two way chi square test:(2 r.v indep. ?)
   calculate the probability from raw data
   assume two r.v. are independent, get the expected number to fill out the expected table
   $D = \sum \frac{(x_i - np_i)^2}{np_i}$, df = (row-1)(col-1)
   Compare with $\chi^2_{df,\alpha}$ or Use $P - value$
   Calculator: χ^2–test, input 2 matrices

3. ANOVA: test whether means are the same
   When ANOVA fails:
   (a) Contrast - split to two groups:
   $c = a_1 u_1 + a_2 u_2 + \ldots (\sum a_i = 0)$
   $H_0 : c = 0$, $\hat{c} = a_1 \overline{x_1} + a_2 \overline{x_2} + \ldots$
   MSE = ANOVA Error $\rightarrow$ MS
   $$t = \frac{\hat{c}}{Sxp\sqrt{\sum \frac{a_i^2}{n_i}}} \quad or \quad \frac{\hat{c}}{\sqrt{MSE \sum \frac{a_i^2}{n_i}}} \quad df = n - k$$
   n: **total** # of data, k: total # of groups
   compare with $t_{\frac{\alpha}{2},df}$
   (b) Bonferroni method
   comparing two group: $\alpha' = \frac{2\alpha}{k(k-1)}$, $df = n - k$
   $$t_{a,b} = \frac{\overline{x_a} - \overline{x_b}}{Sxp\sqrt{\frac{1}{n_a} + \frac{1}{n_b}}} \quad t_{\frac{\alpha'}{2},df}$$

4. Find Regression Curve for Y on X
   known: $f_{x,y}(x,y)$
   $f_x(x) = \int f_{x,y}(x,y)dy$
   $f_Y(y|x) = \frac{f_{x,y}(x,y)}{f_x(x)}$
   Regression Equation = $E(Y|X = x) = \int y f(y|x)dy$

5. Linear Regression
   (a) LinRegTTest: $y = \hat{\beta_0} + \hat{\beta_1} x$
   $\rightarrow r^2$ bigger is better fitting linear model
   (b) slope confidence Interval:
   $$\hat{\beta_0} \pm t_{n-2,\frac{\alpha}{2}} \frac{S}{\sqrt{\sum (x_i - \bar{x})^2}} \quad or \quad LinRegTInt$$
   (c) Test at the α% to see slope is t_0(usually 0)
   $$t = \frac{\hat{\beta_1} - t_0}{\frac{S}{\sqrt{\sum (x_i - \bar{x})^2}}} \quad t_{n-2,\frac{\alpha}{2}} \quad or \quad LinRegTTest$$
   (d) Forecasting, Find α% confidence Interval:
   Estimated mean:
   $$\bar{y} = \hat{y}(x = k) \pm S t_{n-2,\frac{\alpha}{2}} \sqrt{\frac{1}{n} + \frac{(x - \bar{x})^2}{\sum (x_i - \bar{x})^2}}$$
   Expected Value:
   $$\hat{y} = \hat{y}(x = k) \pm S t_{n-2,\frac{\alpha}{2}} \sqrt{1 + \frac{1}{n} + \frac{(x - \bar{x})^2}{\sum (x_i - \bar{x})^2}}$$
   (e) exponential model:
   $y = ab^x \ \rightarrow \ lny = lna + xlnb, \ \hat{\beta_0} \rightarrow lna \ \hat{\beta_1} \rightarrow lnb$
   (f) $\sum (x_i - \bar{x})^2 = (n-1) VAR_x^2$
   Use LIST$\rightarrow$variance or STAT$\rightarrow$Test$\rightarrow$T-test
   (g) r estimates correlation coefficient of (x,y)
   $\rho = \frac{COV[X,Y]}{STD[X]STD[Y]}$
   $STD[X] = \sqrt{E(X^2) - E^2(X)}$
   $E(X) = \int x f_x(x)dx, \ f_x(x) = \int f_{X,Y} dy$
   $COV[X,Y] = E[XY] - E[X]E[Y]$
   $E(XY) = \int \int xy f_{x,y} dxdy$

6. Markov chains
   (a) Properties of Transition Matrix **P**
   Sum of Rows are 1, all terms$\in [0, 1]$
   $\mathbf{P}^n \rightarrow$ go through n transitions
   $\mathbf{P}^n \vec{v} \rightarrow$ the probability of getting each state given starting vector $\vec{v}$ after n transitions
   (b) Absorbing Markov Chains rewrite the Transition Matrix P to:
   $$\tilde{\mathbf{P}} = \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ 0 & \mathbf{I} \end{pmatrix}$$
   Fundamental Matrix $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}, (\mathbf{N} > 0)$
   Time to absorption:
   add up the corresponding row in **N**
   Absorption Probabilities: $\mathbf{B} = \mathbf{NR}$, find $P_{ij} = \mathbf{B}_{ij}$
   (c) Ergodic and Regular Markov Chains
   i. Ergodic:
   It is possible to go from every state to every state (not necessarily in one move)
   ii. Regular:
   It is possible to go to every state to every state after n transitions.$\rightarrow \exists k, s.t. \forall$ terms in $\mathbf{P}^k$ are non-zero
   (d) Stationary / Equilibrium vector
   For Regular Markov Cains, exsists
   $$\mathbf{wP} = \mathbf{w}, \quad \lim_{n\to\infty} \mathbf{P}^n = \mathbf{W}$$
   where **W**'s rows are all equal to **w**
   Calculate **w**: simply calculate wP = w, $\sum w_i = 1 \rightarrow$ $n + 1$ equations for n unknowns.
   (e) Meaning of $\mathbf{w}_j$

i. $\mathbf{w}_j$: Probability/fraction of time spent in j

ii. $\frac{1}{\mathbf{w}_j}$: first return time/revisit steps to state j

(f) Tricks:

i. mean time of state $i \to$ state j:
set state j to absorbing state, calculate $\mathbf{N}$ and find the time to get absorbed

ii. probability move from i to j in k steps: $\mathbf{P}^k_{ij}$

(g) Reversible Markov Chain: $\Leftrightarrow \mathbf{w}_i\mathbf{P}_{ij} = \mathbf{w}_j\mathbf{P}_{ji}$

7. Poisson Process: simple Poisson$\rightarrow$ possionpdf

(a) Basic Properties: λ = Rate of Arrivals
$N(t)$ = numbers of arrivals up to time t
$P(N(t)=k) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}$
$E(N(t)) = \lambda t$
$P(N(s)-N(e)=k) = P(N(s-t)=k)$
$E(N(s)-N(e)) = E(N(s-e)) = \lambda(s-e)$

(b) First arrival time(equal to next arrival time) T_1 is an Exponential Distribution
$f_{T_1}(t) = \lambda e^{-\lambda t}(t \geq 0)$
$F_{T_1}(t) = 1-e^{-\lambda t}(t \geq 0)$
$\Leftrightarrow P(T_1 < t) = 1-e^{-\lambda t} \Leftrightarrow P(T_1 > t) = e^{-\lambda t}$
$E(T_n) = nE(T_1)$

(c) Prob. that more than n arrivals comes in T time:
$\rightarrow n^{th}$ arrival is less than T time:
$P(T_n < T) \rightarrow$CLT: $P(Z_n < \frac{T-\mu n}{\sigma\sqrt{n}})$
Since T_n is Exponential Distribution, $\mu = \sigma = \frac{1}{\lambda}$

(d) Merging of Possion
If $N_1(t) \sim P.P$ rate $= \lambda_1$, $N_2(t) \sim P.P$ rate $= \lambda_2$
then $N(t) \sim P.P rate = \lambda_1 + \lambda_2$
P(first/next arrival is type 1)
$=P(N_1(t)=1, N_2(t)=0|N(t)=1) = \frac{\lambda_1}{\lambda_1+\lambda_2}$
P(first/next arrival is type 2)
$= P(N_2(t)=1, N_1(t)=0|N(t)=1) = \frac{\lambda_2}{\lambda_1+\lambda_2}$
Prob. that k of n arrivals are of type 1:
Binomial Distribution.
P(type 1) $= \frac{\lambda_1}{\lambda_1+\lambda_2}$

8. Discrete Finite Queues
Draw the chain; steady state prob.: each pair of arrows carry the same - $w_i\lambda = w_{i+1}\mu$
Little's Law:
$\lambda T = N \Rightarrow$(Arrival Rate)$E(T) = E(N)$
$E(T) \rightarrow$mean time; $E(N) \rightarrow$mean number
In discrete, arrival rate: $\sum_{i=0}^{i=n-1}\lambda W_i$

9. M/M/1
λ = arrival rate of customer
μ = service rate per customer
$\rho = \frac{\lambda}{\mu}$, $\lambda P_k = \mu P_{k+1} \rightarrow P_{k+1} = \rho P_k$
$P_k = \rho^k(1-\rho)$
number in system:$E(N) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$
number in queue:$E(N_q) = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$
Mean time in system:$E(T) = \frac{E(N)}{\lambda} = \frac{1}{\mu-\lambda} = \frac{1}{\mu} + E(T_q)$
Mean time in queue:$E(T_q) = \frac{E(T_q)}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$
Mean service time:$E(T) - E(T_q) = \frac{1}{\mu}$
Distribution of Time in System is equivalent to an Exponential Distribution where mean:$\frac{1}{\lambda^*} = \frac{1}{\mu-\lambda}$
T: time in system
PDF: $P(T<t) = F(X=x) = 1-e^{-\lambda^* x}, (x>0)$

10. M/M/C single queue, C servers
λ = arrival rate of customer
μ = single server's service rate per customer
$\rho = \frac{\lambda}{\mu}$

$$P_0 = (1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{3!} + \cdots + \frac{\rho^{c-1}}{(c-1)!} + \frac{\rho^c}{c!(1-\frac{\rho}{c})})^{-1}$$

$P_1 = \rho P_0,\ P_2 = \frac{\rho^2}{2}P_0,\ P_3 = \frac{\rho^3}{3!}P_0\ \dots$
$P_c = \frac{\rho^c}{c!}P_0,\ P_{c+k} = (\frac{\rho}{c})^k P_c$
Traffic density: $\frac{\rho}{c} = \frac{\lambda}{c\mu}$, stable if $\frac{\rho}{c} < 1$
number in system:$E(N) = \rho + P_c\frac{\frac{\rho}{c}}{(1-\frac{\rho}{c})^2} = \rho + E(N_q)$
number in queue:$E(N_q) = P_c\frac{\frac{\rho}{c}}{(1-\frac{\rho}{c})^2} =$
Mean time in system:$E(T) = \frac{1}{\lambda}E(N)$
Mean time in queue:$E(T_q) = \frac{1}{\lambda}E(N_q)$
Mean service time:$E(T) - E(T_q) = \frac{1}{\mu}$

11. M/M/C/k C servers, max customer in system k
λ = arrival rate of customer
μ = single server's service rate per customer
$P_i = \frac{\rho^i}{i!}P_0, i = 0,1,\dots,c$
$P_{c+p} = \frac{\rho^p}{c^p}P_c, p = 0,1,\dots,k-c$
$\sum P_i = 1,\ E(N) = \sum iP_i,\ E(N_q) = \sum(i-c)P_i(i>c)$
Effective arrive rate $\lambda_a = \lambda(1-P_k)$
$E(T) = \frac{E(N)}{\lambda_a}, E(T_q) = \frac{E(N_q)}{\lambda_a}$

12. Stochastic Process
mean: $m(t) = E[X(t)]$, autocorrelation: $R(t,s) = E[X(t)X(s)]$, autocovariance: $C(t,s) = R(t,s) - m(t)m(s)$
Stock and Option problem:(I outcomes, J wagers):
list outcome, cost, future value, return for each wager.
$E(R_j) = \sum_{i=1}^{I} r_{ji}p_i,\ R_i = \sum_{j=1}^{J} x_i r_{ji}$
No arbi. price: $E(R_i) = 0, \sum p_i = 1 \rightarrow c$(option price)
Find arbi. return: $\forall R_i > 0$(usually sell stock buy option)

13. Brownian Motion
B(t) $\Leftrightarrow$ normal dist. with $\mu = 0$,VAR = t; (B(t) = $\sqrt{t}\mathbf{Z}$
Process start over at every time: $B(t_2) - B(t_1) = B(t_2 - t_1)$
$B(t_2) - B(t_1), B(t_3) - B(t_2)$ are independent if no overlap.
BM with drift:
$X(t) = x_0 + \mu t + \sigma B(t)$, x_0: starting value, μt: drift term.

14. Geometric Brownian Motion
$X(t) = x_0 e^{\mu t + \sigma B(t)} \rightarrow$ convert to normal. $E(e^{a\mathbf{Z}}) = e^{\frac{a^2}{2}}$
Black-Scholes Formula:
α = interest rate (risk-free interest), $x(0)$ = current price (start price), K = Threshold of option (strike price), T = time to experience option (maturity), σ = (annualized) volatility, σ^2 = variance para. c = cost of option.
No arbitrage option: $\alpha = \mu + \frac{\sigma^2}{2}$, F: std. normal cdf
$C = X(0)F(b + \sigma\sqrt{T}) - Ke^{-\alpha T}F(b),\ b = \frac{(\alpha - \frac{\sigma^2}{2})T + ln(\frac{x(0)}{k})}{\sigma\sqrt{T}}$