MTH 4581: Fall 2018: Prof. C. King

Homework 5

Reading: Grinstead and Snell, Chapters 11,12; notes on Poisson Process.

Due date: Thursday November 1

Problems:

1) Consider the Gambler's Ruin with states $\{0, 1, \ldots, N\}$, and with transition probabilities

$$\begin{aligned} &p_{k,k+1} = p, \quad p_{k,k-1} = q = 1 - p \qquad \text{for } k = 1, \ldots, N-1 \\ &p_{0,0} = p_{N,N} = 1 \end{aligned}$$

where $p \neq 1/2$. Let P_k be the probability that the Gambler's fortune reaches N before it reaches 0.

a) By conditioning on the first step derive a recursion equation for p_k in terms of p_{k+1} and p_{k-1} (your formula will involve p and q).

b) Find the possible values of a for which there is a solution of the recursion equation of the form $P_k = a^k$.

c) The most general solution of the recursion equation is a linear combination of the particular solutions you found in (**b**). Write down this general solution and use the boundary conditions at $k = 0$ and $k = N$ to determine the unknown constants.

2) In class we worked through the 'rat in the maze' problem, where there are 5 rooms, and where at each step the rat randomly goes through one of the available doors. Write down the transition matrix for this problem, and solve for the stationary vector by assuming that the chain is reversible. So write down the reversibility equations for each pair of states, and look for a solution. [Hint: the solution is quite similar to the 'knights on the chessboard' example from class].

3) Consider a random walk on the integers $\{0, 1, 2, 3, \ldots\}$ where at each step the walker moves one integer to the left with probability p, or one integer to the right with probability $q = 1 - p$; however if the walker is at 0 then it moves to 1 with probability 1. The chain is ergodic, so it has a stationary vector. Find the stationary vector by assuming that the chain is reversible, and solving the reversibility equations for each pair of neighboring states. Use your answer to compute the mean first return time to the state 0.

4) Suppose the number of calls a center receives is a Poisson process with a mean of 10 calls per minute.

a) Find the probability that there are 5 calls in the next minute.

b) Find the probability that there is one call in the next 15 seconds and then two calls in the next 20 seconds.

5) Suppose that on average ten people move into a city per week. If this is a Poisson process:

a) find the probability that 2 people move into the city in the next day.

b) find the probability that the time until the next arrival is more than 2 days.

c) find the expected time until the 100^{th} arrival.

d) estimate the probability that the 500^{th} arrival happens after more than one year.