MATH 4581: Assignment # 6

SOLUTIONS

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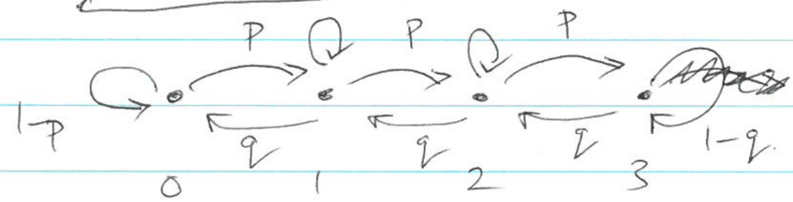

a) $w_0 P_{01} = w_1 P_{10} \Rightarrow p w_0 = q w_1$

$w_1 P_{12} = w_2 P_{21} \Rightarrow p w_1 = q w_2$

$w_2 P_{23} = w_3 P_{32} \Rightarrow p w_2 = q w_3$

$\Rightarrow \quad w_1 = (p/q) w_0, \quad w_2 = (p/q)^2 w_0, \quad w_3 = (p/q)^3 w_0$

$w_0 \left[1 + p/q + (p/q)^2 + (p/q)^3 \right] = 1.$

$\Rightarrow \quad w_0 = \left(1 + \frac{p}{q} + (p/q)^2 + (p/q)^3 \right)^{-1}$

b) i) $P(\text{empty}) = w_0$

ii) $P(\text{queue empty}) = w_0 + w_1$

iii) $P(\text{queue full}) = w_3$

iv) $E[N] = w_1 + 2w_2 + 3w_3$

v) $E[N_q] = w_2 + 2w_3$

vi) $\lambda_a = p w_0 + p w_1 + p w_2$

vii) $\lambda_d = q w_1 + q w_2 + q w_3$

c) i) $E[T] = \dfrac{E[N]}{\lambda} = \dfrac{w_1 + 2w_2 + 3w_3}{p(w_0 + w_1 + w_2)}$

ii) $E[T_q] = \dfrac{E[N_q]}{\lambda} = \dfrac{w_2 + 2w_3}{p(w_0+w_1+w_2)}$

iii) $E[S] = E[T] - E[T_q] = \dfrac{w_1+w_2+w_3}{p(w_0+w_1+w_2)} = \dfrac{1}{q}.$

d) $P(X_{n+2} = 3, X_{n+1} = 3, X_n = 3)$

$= P(X_{n+2} = 3 \mid X_{n+1} = 3)\, P(X_{n+1} = 3 \mid X_n = 3)\, P(X_n = 3)$

$= (1-q)^2 w_3.$

2. $\lambda_1 = 5 \text{ hour}^{-1}, \quad \lambda_2 = 15 \text{ hour}^{-1}, \quad \lambda = 20 \text{ hour}^{-1}$

i) $P(N(\tfrac{1}{12}) \geq 3) = 1 - P(N(\tfrac{1}{12})=0) - P(N(\tfrac{1}{12})=1) - P(N(\tfrac{1}{12})=2)$

$= 1 - e^{-\frac{\lambda}{12}} - \frac{\lambda}{12} e^{-\lambda/12} - \dfrac{(\lambda/12)^2}{2} e^{-\lambda/12}$

ii) $P(N_1(t) \geq 2 \mid N(t) = 10)$

$= 1 - P(N_1(t) = 0 \mid N(t) = 10) - P(N_1(t) = 1 \mid N(t) = 10)$

$= 1 - \left(\dfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^{10} - 10 \dfrac{\lambda_1}{\lambda_1+\lambda_2} \left(\dfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^9.$

3) $\lambda = 2 \text{ hour}^{-1}$, $\mu = \frac{60}{25} \text{ hour}^{-1}$; $\rho = \frac{\lambda}{\mu} = \frac{5}{6}$

$M/M/1$:

i) $\cancel{P_3 = (1-\rho)\rho^3 = \frac{1}{6}\left(\frac{5}{6}\right)^3}$ $P_4 = (1-\rho)\rho^4 = \frac{1}{6}\left(\frac{5}{6}\right)^4$

ii) $E[N_q] = \frac{\rho^2}{1-\rho} = \frac{(5/6)^2}{1/6}$

iii) $E[T_q] = \frac{1}{\lambda} E[N_q] = 3\,(5/6)^2$

iv) $P(T_q > 3) = \rho\, e^{-(\mu-\lambda)3}$

4). $M/M/3$: $\lambda = 5 \text{ hour}^{-1}$

$\mu = \frac{60}{33} = \frac{20}{11} \text{ hour}^{-1}$. $\rho = \frac{\lambda}{\mu} = \frac{11}{4}$.

a) $P_0 = \left(1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{6(1-\rho/3)}\right)^{-1}$; $P_1 = \rho P_0$, $P_2 = \frac{\rho^2}{2} P_0$, $P_3 = \frac{\rho^3}{6} P_0$

$= 0.02 \Rightarrow \mathbb{P}(\text{queue empty}) = P_0 + P_1 + P_2 + P_3$
$= 0.237$

ii) $E[N_q] = \frac{\rho/3}{(1-\rho/3)^2} \cdot \frac{\rho^3}{6} P_0 = 22\left(\frac{11}{4}\right)^3 P_0 = 9.15$

iii) $E[T_q] = \frac{1}{\lambda} E[N_q] = 1.83$ hour