2 Suppose the amount of a single loss for an insurance policy has density function $f(x) = 0.001\ e^{-0.001x}$, for $x > 0$. If this policy has a \$300 per claim deductible and also has a payment cap of \$1500 per claim, what is the expected amount of a single claim for this policy?

Solution:

$$\begin{aligned} h(x) &= \begin{cases} 0 & 0 < x < 300 \\ x - 300 & 300 \le x \le 1800 \\ 1500 & x > 1800. \end{cases} \\ E[h(x)] &= \int_0^\infty h(x)\ 0.001\ e^{-0.001x}\ dx \\ &= \int_{300}^{1800} (x - 300)\ 0.001\ e^{-0.001x}\ dx + \int_{1800}^\infty 1500\ 0.001\ e^{-0.001x}\ dx \\ &= (-x - 700)e^{-0.001x}\Big|_{300}^{1800} - 1500e^{-0.001x}\Big|_{1800}^{\infty} = 575.52. \end{aligned}$$

4 Let X be the random variable which is uniformally distributed over the interval $[a, b]$. Find $M_X(t)$.

Solution: If X is uniformally distributed over $[a, b]$, then $f(x) = \frac{1}{b-a}$.

$$M_X(t) = E[e^{tX}] = \frac{1}{b-a}\int_a^b e^{tx}\ dx = \frac{e^{tx}}{t(b-a)}\Big|_a^b = \frac{e^{bt} - e^{at}}{t(b-a)}.$$

5 Find $E[X]$ for the random variable in the previous problem using its moment generating function.

Solution: For the moment generating function above,

$$M_X'(t) = \frac{t(be^{bt} - ae^{at}) - (e^{bt} - e^{at})}{t^2(b-a)}$$

As t approaches 0, the expression assumes an independent form, so we must use L'Hospital's rule

$$\begin{aligned} M_X'(0) &= \lim_{t\to 0} \frac{(be^{bt} - ae^{at}) + t(b^2e^{bt} - a^2e^{at}) - (be^{bt} - ae^{at})}{t^2(b-a)} \\ &= \lim_{t\to 0} \frac{b^2e^{bt} - a^2e^{at}}{2t(b-a)} = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2} = E[X]. \end{aligned}$$

6 Let X be the random variable whose density function is given by

$$f(x) = \begin{cases} 2(1-x) & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find $M_X(t)$.

Solution: If $f(x) = \begin{cases} 2(1-x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$, then

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \int_0^1 e^{tx}\ 2(1-x)\ dx \\ &= 2\int_0^1 e^{tx}\ dx - 2\int_0^1 xe^{tx}\ dx \\ &= \begin{cases} \frac{2e^t - 2t - 2}{t^2}, & \text{for } \ t \ne 0 \\ 1 & \text{if } t = 0. \end{cases} \end{aligned}$$

7 Let X be the random variable whose density function is given by

$$f(x) = \begin{cases} 2(1-x) & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find $E[X]$ using the moment generating function. (Note: the derivative of $M_X(t)$ is not defined at 0, but you can take the limit as t approaches 0 to find $E[X]$. This is a much more difficult way to find $E[X]$ than direct integration for this particular density function.)

Solution: For the moment generating function in the above problem

$$M_X'(t) = \frac{t^2(2e^t - 2) - 2t(2e^t - 2t + 2)}{t^4} = \frac{2te^t - 4e^t + 2t + 4}{t^3}.$$

As t approaches 0, the expression assumes an independent form, so we must use L'Hospital's rule.

$$M_X'(0) = \lim_{t\to 0} \frac{2te^t - 4e^t + 2t + 4}{t^3} = \lim_{t\to 0} \frac{2te^t}{6t} = \frac{1}{3} = E[X].$$

8 If the moment generating function of X is $\left(\frac{2}{2-t}\right)^5$, identify the random variable X.

Solution: The moment generating function for the gamma distribution with parameters α and β is

$$M_X(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha}.$$

Thus $\left(\frac{2}{2-t}\right)^5$ is the moment generating function for the gamma distribution with $\alpha = 5$ and $\beta = 2$.

11 Let X be a normal random variable with parameters μ and σ. Use the moment generating function for X to find $E[X^2]$. Then show that $\text{Var}[X] = \sigma^2$.

Solution: For the normal distribution, $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

$$\begin{aligned} M_X'(t) &= (\mu + \sigma^2 t)\ M_X(t) \\ M_X''(t) &= (\mu + \sigma^2 t)^2\ M_X(t) + \sigma^2\ M_X \\ M_X''(0) &= \mu^2 + \sigma^2 = E[X^2]. \quad (\text{Recall } M_X(0) = 1) \\ \text{Var}[X] &= E[X^2] - E[X]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2. \end{aligned}$$

12 Let X uniformally distributed over $[0,1]$ and $Y = e^X$. Find $F_Y(y)$ and $f_Y(y)$.

Solution: If X is uniformally distributed over $[0,1]$, then $F(x) = x,\ 0 \le x \le 1$. If $Y = e^X = g(X)$, then $X = \ln Y = h(Y)$. Since $0 \le x \le 1$, then $1 \le e^x \le e$.

$$\begin{aligned} F_Y(y) &= F_X(h(y)) = \ln y, \quad \text{for } \ 1 \le y < e \\ f(y) &= F_Y'(y) = \frac{1}{y} \ \text{ for } \ 1 \le y < e. \end{aligned}$$

13 Let X be a random variable with density function given by $f_X(x) = 3\,x^{-4}$, for $x \ge 1$ (Pareto with $\alpha = 3,\ \beta = 1$), and let $Y = \ln X$. Find $F_Y(y)$.

Solution: If $f(x) = 3x^{-4}$, then $F_X(x) = 1 - x^{-3}$ for $x \ge 1$.
If $y = g(x) = \ln x$, then $x = h(y) = e^y$ for $y \ge 0$.
$F_Y(y) = F_X(h(y)) = 1 - (e^y)^{-3} = 1 - e^{-3y}$, for $y \ge 0$.

21 An auto insurance company issues a comprehensive policy with a \$200 deductible. Last year 90 percent of the policyholders filed no claims (either no damage or damage less than the deductible). For the 10 percent who filed claims, the claim amount had a Pareto distribution with $\alpha = 3$ and $\beta = 200$. If X is a random variable of the amount paid by the insurer, what is $F(x)$, for $x \geq 0$?

Solution: Let S be the event that a claim is filed, $P(S) = 0.10$. Let Y be the random variable for the amount of the claim filed. Y has a Pareto distribution with $\alpha = 3$ and $\beta = 200$.

$$P(Y \leq y) = 1 - \left(\frac{200}{y}\right)^3 \qquad \text{for } y > 200.$$

Let X be the random variable of the amount paid, $X = y - 200$.

$$P(X \leq x \mid S) = 1 - \left(\frac{200}{200+x}\right)^3 \qquad \text{for } x > 0.$$

Case 1: $x = 0$

No claim is filed, so $F(0) = 0.90$.

Case 2: $x > 0$

$$P(X \leq x) = P(X = 0) + P(0 \leq X \leq x \mid S)\ P(S) = 0.90 + \left[1 - \left(\frac{200}{200+x}\right)^3\right].$$

29 A piece of equipment is being insured against early failure. The time from purchase until failure of the equipment is exponentially distributed with mean 10 years. The insurance will pay an amount x if the equipment fails during the first year, and it will pay $0.5x$ if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made. At what level must x be set if the expected payment made under this insurance is to be 1000?

Solution: Let T be the random variable for the lifetime (time to failure) of the piece of equipment and Y the random variable for the amount paid by the insurance.

$$Y = \begin{cases} x, & 0 \leq t \leq 1 \\ 0.5x & 1 < t \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

The density function for T is $f(t) = \frac{e^{-\frac{t}{10}}}{10}$. We can find $E[Y]$, but it will have the unknown amount x in it.

$$\begin{aligned} E[Y] &= \int_0^1 x\ \frac{e^{-\frac{t}{10}}}{10}\ dt + \int_1^3 0.5x\ \frac{e^{-\frac{t}{10}}}{10}\ dt \\ &= -xe^{-\frac{t}{10}}\Big|_0^1 - 0.5xe^{-\frac{t}{10}}\Big|_1^3 = 0.17717x. \end{aligned}$$

We need this expected payment to equal 1000. $\Rightarrow$ $0.17717x = 1000$ $\Rightarrow$ $x = 5644.30$.

30 A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Determine $E[X]$.

Solution: The density function for T is $f(t) = \frac{e^{-\frac{t}{3}}}{3}$. The random variable $X = \max(T, 2)$ can be written as

$$X = \begin{cases} 2 & 0 \leq t \leq 2 \\ t & 2 \leq t \end{cases}$$

$$\begin{aligned} E[X] &= \int_0^2 2\ \frac{e^{-\frac{t}{3}}}{3}\ dt + \int_2^\infty t\ \frac{e^{-\frac{t}{3}}}{3}\ dt \\ &= -2e^{-\frac{t}{3}}\Big|_0^2 + (-te^{-\frac{t}{3}} - 3e^{-\frac{t}{3}})\Big|_2^\infty = 2 + 3e^{-\frac{2}{3}} \end{aligned}$$

35 The time, T, that a manufacturing system is out of operation has cumulative distribution function

$$F(t) = \begin{cases} 1 - \left(\frac{2}{t}\right)^2 & \text{for } \ t > 2 \\ 0 & \text{otherwise.} \end{cases}$$

The resulting cost to the company is $Y = T^2$. Determine the density function of Y, for $y > 4$.

Solution: We will find the cumulative distribution function for Y.

$$F_Y(y) = P(Y \leq y) = P(T^2 \leq y) = P(T \leq \sqrt{y}) = 1 - \left(\frac{2}{\sqrt{y}}\right)^2 = 1 - \frac{4}{y}.$$

Then the density function for Y is

$$f(y) = F'_Y(y) = \frac{4}{y^2}.$$