

1. Moment Generating Function

$$M_X(t) = E(e^{tx})$$

Properties:

- $M_X(t) = (1 - p + pe^t)^n$ for binomial(n,p)
- $M_X^{(n)}(0) = E(X^n)$
 $\Rightarrow \text{VAR}[X] = E[X^2] - E^2[X] = M_X''(0) - [M_X'(0)]^2$
- $M_X(t) = M_Y(t) \Rightarrow XY$ has same distribution
- $M_X(0) = 1$
- $M_{ax+b}(t) = M_X(at)e^{bt}$
- $M_{X+Y}(t) = M_X(t)M_Y(t)$ (X, Y independent)

2. Gama Function $\Gamma(n) = \int_0^\infty u^{n-1} e^{-u} du = (n-1)!$

3. Basic probability property

- Conditional Probability and Bayles Theorem
 - $P(A|B) = \frac{P(A \cap B)}{P(B)}$
 - $P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$
- Expectation
 - $E(X) = \sum x P_X(x)$ or $\int_{-\infty}^\infty x f_X(x)$
 - $E(c) = c$
 - $E(aX) = aE(X)$
 - $E(X + Y) = E(X) + E(Y)$
 - $E(X) = M_X'(0)$
- Variance and Standard Deviation
 - $\text{VAR}(X) = E(X^2) - E^2(X) = \sum [(x - E(X))]^2 P_X(x) = E[(x - \mu)^2] = \sigma^2$
 - $\text{VAR}(c) = 0$
 - $\text{VAR}(aX) = a^2 \text{VAR}(x)$
 - $\text{VAR}(X \pm Y) = \text{VAR}(X) + \text{VAR}(Y) \pm 2\text{COV}(x, y)$,
 $\text{COV}(x, y) = E(XY) - E(X)E(Y)$
- z-score
 $Z = \frac{x - \mu}{\sigma}$, measures the distance of x from expected value in standard units.

4. Discrete Distributions

- Binomial Distribution
 Note as $X \sim B(n, p)$
 $P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$
 $M_X(t) = (1 - p + pe^t)^n$, $E(X) = np$, $\text{VAR}(X) = np(1-p)$
- Hyper Geometric Distribution
 Note as $X \sim H(N, n, r)$, N:total size, n:total pick, r:size of special subgroup
 $P_X(x) = \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}}$
 $E(X) = n \frac{r}{N}$, $\text{VAR}(X) = n \frac{r}{N} (1 - \frac{r}{N}) (\frac{N-n}{N-1})$
- Poisson Process
 $P_X(x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $E(X) = \lambda$, $\text{VAR}(X) = \lambda$
 λ : average arrival in given time or space
 When $\lambda = np < 10$, poisson approximates Binomial.
- Geometric Distribution
 - X is the r.v of number of total trials (x includes the first success)
 $P_X(x) = (1-p)^{x-1} p$, $x = 1, 2, 3 \dots$
 $E(X) = 1/p$, $\text{VAR}(X) = \frac{1-p}{p^2}$

ii. X is the r.v of number of failed trials (x excludes the first success)

$$P_X(x) = (1-p)^x p, x = 0, 1, 2, \dots$$

$$M_X(t) = \frac{p}{1-e^t(1-p)}, E(X) = \frac{1-p}{p}, \text{VAR}(X) = \frac{1-p}{p^2}$$

(e) Negative Binomial Distribution

X is the r.v of number of trials need to observe the r^{th} success in a sequence of Bernoulli trails where p is the success probability.

$$P_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, x = r, r+1, r+2 \dots$$

$$M_X(t) = \left(\frac{p}{1-e^t(1-p)} \right)^r, E(X) = \frac{r}{p}, \text{VAR}(X) = \frac{r(1-p)}{p^2}$$

Alternatively, X is the r.v of failures before the r^{th} success:

$$P_X(x) = \binom{x+r-1}{r-1} p^r (1-p)^x, x = 0, 1, 2 \dots$$

$$M_X(t) = \left(\frac{1-p}{1-pe^t} \right)^r, E(X) = \frac{r(1-p)}{p}, \text{VAR}(X) = \frac{r(1-p)}{p^2}$$

5. Chebychev's Theorem

$$P(\mu - k\sigma \leq x \leq \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

6. Continuous Distributions

(a) Uniform(rectangle) Distribution

$$f_X(x) = \frac{1}{b-a}, (a \leq x \leq b); F_X(x) = \frac{x-a}{b-a}, a \leq x \leq b$$

$$E(X) = \frac{a+b}{2}, \text{VAR}(X) = \frac{(b-a)^2}{12}$$

(b) Exponential Distribution

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, (x \geq 0); F_X(x) = 1 - e^{-\frac{x}{\theta}}, x \geq 0$$

$$M_X(t) = \frac{1}{1-\theta t}, E(X) = \theta, \text{VAR}(X) = \theta^2$$

Note as $X \sim \exp(\theta)$, In terms of λ : $\lambda = \frac{1}{\theta}$

(c) Gamma Distribution

$$f_X(x) = \frac{1}{\Gamma(n)\theta^n} x^{n-1} e^{-\frac{x}{\theta}}, x \geq 0$$

$$M_X(t) = \frac{1}{(1-\theta t)^n}, E(X) = n\theta, \text{VAR}(X) = n\theta^2$$

In terms of α, β : $\alpha = n, \beta = \frac{1}{\theta}$

(d) Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, E(X) = \mu, \text{VAR}(X) = \sigma^2$$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

(e) Log-normal Distribution:

$$Y = e^X, X \sim N(\mu, \sigma^2), f_Y(y) = \frac{1}{\sigma y \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{\ln y - \mu}{\sigma} \right)^2}, y \geq 0$$

$$E(Y) = e^{\mu + \frac{\sigma^2}{2}}, \text{VAR}(Y) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$$

7. Central Limit Theorem

Let $\{X_1, X_2, \dots, X_n\}$ be the indep. r.v. with same distribution and mean μ and std. deviation σ . If n is large $n \geq 30$,
 $S = X_1 + X_2 + \dots + X_n \Rightarrow S \sim N(n\mu, n\sigma^2)$

OR $S' = \frac{X_1 + X_2 + \dots + X_n}{n} \Rightarrow S' \sim N(\mu, \frac{\sigma^2}{n})$

8. Finding CDF for $Y = g(X)$

If $g(X)$ is strictly increasing or decreasing

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|, h(y) = g^{-1}(x)$$