3 Let $p(x,y) = \frac{xy+y}{27}$, for $x = 1,2,3$ and $y = 1,2$, be the joint probability for the random variables X and Y. Find $E[X]$ and $E[Y]$.

The values of $P(x,y)$ can be found by direct substitution, e.g.

$$P(1,1) = P(X=1, Y=1) = \frac{(1).(1)+1}{27} = \frac{2}{27}.$$

X / Y	1	2	3	$P(y)$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$\frac{9}{27} = \frac{1}{3}$
2	$\frac{4}{27}$	$\frac{6}{27}$	$\frac{8}{27}$	$\frac{18}{27} = \frac{2}{3}$
$P(x)$	$\frac{6}{27} = \frac{2}{9}$	$\frac{9}{27} = \frac{1}{3}$	$\frac{12}{27} = \frac{4}{9}$	

$$\begin{aligned}
P(X=1) &= P(1,1) + P(1,2) = \frac{2}{27} + \frac{4}{27} = \frac{2}{9} \\
P(Y=1) &= P(1,1) + P(2,1) + P(3,1) = \frac{2}{27} + \frac{3}{27} + \frac{4}{27} = \frac{1}{3} \\
E[X] &= \sum xP(x) = \frac{2}{9} + 2\left(\frac{3}{9}\right) + 3\left(\frac{4}{9}\right) = \frac{20}{9} \\
E[Y] &= \sum yP(y) = \frac{1}{3} + 2\left(\frac{2}{3}\right) = \frac{5}{3}.
\end{aligned}$$

8 Let $f(x,y) = 2x^2 + 3y$, for $0 \le y \le x \le 1$. Find $f_X(x)$ and $f_Y(y)$.

The density function $f(x,y) = 2x^2 + 3y$ is defined on the region bounded by the x-axis and the lines $y = x$ and $x = 1$.

$$\begin{aligned}
f_X(x) &= \int_0^x (2x^2 + 3y)\ dy = \left(2x^2y + \frac{3y^2}{2}\right)\Bigg|_{y=0}^{y=x} = 2x^3 + \frac{3x^2}{2}, \quad 0 \le x \le 1. \\
f_Y(y) &= \int_y^1 (2x^2 + 3y)\ dx = \left(\frac{2x^3}{3} + 3xy\right)\Bigg|_{x=y}^{x=1} = \frac{2}{3} + 3y - 3y^2 - \frac{2}{3}\ y^3, \quad 0 \le y \le 1.
\end{aligned}$$

11 For the joint density function $f(x,y) = \frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy$, for $0 \le x \le 1$ and $0 \le y \le 1$, find $P(X > Y)$.

If $f(x,y) = \frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy$, for $0 \le x \le 1$ and $0 \le y \le 1$, then

$$\begin{aligned} P(X > Y) &= \int_0^1 \int_0^x \left(\frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy\right) dydx = \int_0^1 \left(\frac{y}{4} + \frac{xy}{2} + \frac{y^2}{4} + \frac{xy^2}{2}\right)_{y=0}^{y=x} dx \\ &= \int_0^1 \left(\frac{x}{4} + \frac{3x^2}{4} + \frac{x^3}{2}\right) dx = \left(\frac{x^2}{8} + \frac{x^3}{4} + \frac{x^4}{8}\right)_0^1 = \frac{1}{2}. \end{aligned}$$

18 Let $p(x,y) = \frac{xy+y}{27}$, for $x = 1,2,3$ and $y = 1,2$, be the joint probability for the random variables X and Y. Find $E[X \mid Y = 1]$.

Using the data obtained above in Exercise 3:

$$E[X \mid Y = 1] = \sum x\ P(X = x \mid Y = 1) = 1\left(\frac{2}{9}\right) + 2\left(\frac{1}{3}\right) + 4\left(\frac{4}{9}\right) = \frac{20}{9}$$

22 If $f(x,y) = \begin{cases} 6\ x, & \text{for } 0 < x < y < 1 \\ 0 & \text{elsewhere,} \end{cases}$ find

(a) $f_Y(y) = \int_0^y f(x,y)\ dx = \int_0^y 6x\ dx = 3y^2$, for $0 < y < 1$.

(b) $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{6x}{3y^2} = \frac{2x}{y^2}$, for $0 < x < y < 1$.

(c) $E[X \mid Y = y] = \int_0^y xf(x \mid y)\ dx = \frac{1}{y^2} \int_0^y 2x^2\ dx = \frac{1}{y^2} \cdot \frac{2y^3}{3} = \frac{2y}{3}$.

(d) $E[X \mid Y = 0.5] = 2 \cdot \frac{\frac{1}{2}}{3} = \frac{1}{3}$.

25 For the joint density function $f(x,y) = \frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy$, for $0 \le x \le 1$ and $0 \le y \le 1$. Determine if the random variables X and Y are dependent or independent.

The joint density function is

$$f(x,y) = \frac{1}{4} + \frac{x}{2} + \frac{y}{2} + xy = \left(\frac{1}{2} + x\right) \cdot \left(\frac{1}{2} + y\right) = f(x)\ .\ f(y).$$

Hence X and Y are independent.

41 The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y) = \begin{cases} 2x & \text{for } \ 0 < x < 1, \quad x < y < x+1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional variance of Y given that $X = x$?

We can calculate the variance if we know the conditional distribution of Y given that $X = x$

$$f(y \mid X = x) = \frac{f(x,y)}{f(x)} = \frac{2x}{f(x)}, \quad \text{for } 0 < x < 1 \text{ and } x < y < x+1.$$

We need to find for $0 \le x \le 1$,

$$f(x) = \int_x^{x+1} 2x\ dx = 2xy\Big|_x^{x+1} = 2x(x+1) - 2x^2 = 2x.$$

This give us $f(y \mid X = x) = 1$ for $0 < x < 1$ and $x < y < x + 1$.

$$\begin{aligned}
E[Y \mid X = x] &= \int_x^{x+1} y \ dy = \left. \frac{y^2}{2} \right|_x^{x+1} = x + \frac{1}{2} \\
E[Y^2 \mid X = x] &= \int_x^{x+1} y^2 \ dy = \left. \frac{y^3}{3} \right|_x^{x+1} = x^2 + x + \frac{1}{3} \\
\mathrm{Var}[Y \mid X = x] &= x^2 + x + \frac{1}{3} - \left(x + \frac{1}{2} \right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.
\end{aligned}$$

46 An insurance policy is written to cover a loss X where X has density function

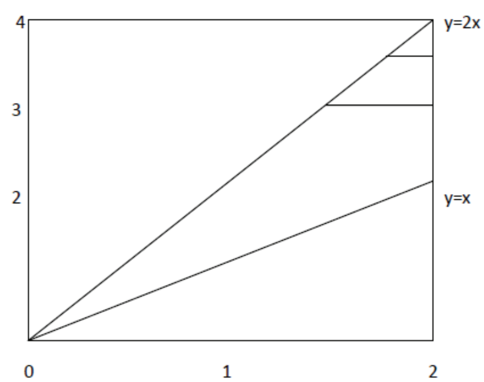
$$f(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

The time (in hours) to process a claim of size x, where $0 \le x \le 2$, is uniformly distributed on the interval from x to $2x$. Calculate the probability that a randomly chosen claim on this policy is processed in three hours or more.

We are asked to find $P(Y \ge 3) = \int\int_R f(x,y)\ dydx$, where R is the region filled in the graph bellow.

$$R = \{(x,y) \mid 3 \le y \le 2x \ \&\ 1.5 \le x \le 2\}$$

We can construct $f(x,y)$ since we are told that



(a) $f(x)$ is the density given in the problem above.

(b) $f(y \mid x) = \frac{1}{x}$ for $x \le y \le 2x$ from the statement " The time (in hours) to process the claim of size x, where $0 \le x \le 2$, is uniformly distributed on the interval from x to $2x$."

this till us that

$$f(x,y) = f(y \mid x).f(x) = \frac{3x}{8} \quad \text{for } 0 \le x \le 2 \text{ and } x \le y \le 2x.$$

$$\begin{aligned} P(Y \ge 3) &= \int\int_R f(x,y)\ dy\ dx = \int_{1.5}^{2}\int_{3}^{2x} \frac{3x}{8}\ dy\ dx = \int_{1.5}^{2} \frac{3xy}{8}\Big|_{3}^{2x}\ dx \\ &= \int_{1.5}^{2}\left(\frac{3x^3}{4} - \frac{9x}{8}\right)\ dx = \left(\frac{x^3}{4} - \frac{9x^2}{16}\right)\Big|_{1.5}^{2} = 0.172. \end{aligned}$$