2 Let T be the time from birth until death of a randomly selected member of a proportion. Assume that T has a uniform distribution on $[0, 100]$. Find $E[T]$ and $\text{Var}[T]$.

If T is uniformaly distributed on $[0, 100]$:

$$\begin{aligned} E[T] &= \frac{100+0}{2} = 50 \\ \text{Var}[X] &= \frac{(100-0)^2}{12} = 833.33. \end{aligned}$$

4 On a large construction site the lengths of pieces of lumber are rounded off to the nearest centimeter. Let X be the rounding error random variable (the actual length of a piece of lumber minus the rounded-off value). Suppose that X is uniformally distributed over $[-0.50, 0.50]$. Find

(a) $P(-0.10 \leq X \leq 0.20)$.
Since X is uniform on $[-50, 50]$, $F_X(x) = \frac{x-(-0.50)}{0.50-(-0.50)} = x + 0.50$.
Then $P(-0.10 \leq X \leq 0.20) = F_X(0.2) - F_X(-0.10) = 0.70 - 0.40 = 0.30$.

(b) $\text{Var}[X] = \frac{[0.50-(-0.50)]^2}{12} = \frac{1}{12}$.

9 Tests on a certain machine part have determined that the mean time until failure of this part is 500 hours. Assume that the time T until failure of this part is exponentially distributed.

(a) What is the probability that one of these parts will fail within 300 hours.
T has an exponential distribution with mean $\frac{1}{\lambda} = 500$, or $\lambda = 0.002$.

$$\begin{aligned} P(T \leq t) &= F_T(t) = 1 - e^{-0.002t} \\ P(T > t) &= S_T(t) = e^{-0.002t}, \quad \text{so} \\ P(T \leq 300) &= 1 - e^{-0.002(300)} = 0.4512. \end{aligned}$$

(b) What is the probability that one of these parts will still be working after 900 hours.

$$P(T > 900) = e^{-0.002(900)} = 0.1653.$$

10 If T has an exponential distribution with parameter λ, what is the median of T?
To find the median of the exponential distribution we solve

$$F_T(m) = 1 - e^{-\lambda m} = 0.50 \quad \Rightarrow \quad e^{-\lambda m} = 0.50$$

Then $-\lambda m = \ln(0.5) = -\ln(2)$, and the median is $m = \frac{\ln(2)}{\lambda}$.

11 For a certain population the time until death random variable T has an exponential distribution with mean 60 years.

(a) What is the probability that a member of this population will die by age 50?
For the exponential distribution with mwan 60, $\lambda = \frac{1}{60}$.
$P(T \leq 50) = 1 - e^{-\frac{50}{60}} = 0.5654$.

(b) What is the probability that a member of this population will live to be 100?
$P(T > 100) = e^{-\frac{100}{60}} = 0.1889$.

12 If T is uniformly distributed over $[a, b]$, what is its failure rate?
For the uniform distribution 0n $[a, b]$, $f(t) = \frac{1}{b-a}$ and $S_X(t) = \frac{b-t}{b-a}$.
The failure rate is $\lambda(t) = \frac{f(t)}{S_X(t)} = \frac{1}{b-t}$.

18 Let T be a random variable whose distribution is exponential with parameter λ. Show that $P(T \geq a+b \mid T \geq a) = P(T \geq b)$.
If T ia an exponential random variable with parameter λ, then

$$\begin{aligned} P(T \geq a+b \mid T \geq a) &= \frac{P(T \geq a+b)}{P(T \geq a} \\ &= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} \\ &= e^{-\lambda b} = P(T \geq b). \end{aligned}$$

20 Use the following two equations

$$\begin{aligned} \int_0^\infty x^n\ e^{-ax}\ dx &= \frac{\Gamma(n+1)}{a^{(n+1)}}, \text{ for } a > 0, \text{ and } n > -1 \\ \Gamma(n) &= (n-1).\Gamma(n-1), \end{aligned}$$

show that the mean of the gamma distribution with parameters α and β is $\frac{\alpha}{\beta}$.
For the gamma distribution $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\ x^{\alpha-1}\ e^{-\beta x}$ and $\int_0^\infty x^n\ e^{-ax}\ dx = \frac{\Gamma(n+1)}{a^{(n+1)}}$.

$$\begin{aligned} E[X] &= \int_0^\infty xf(x)\ dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha\ e^{-\beta x}\ dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} . \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\alpha}{\beta}. \end{aligned}$$

24 A gamma distribution has a mean of 18 and a variance of 27. What are α and β for this distribution?
Let T have a gamma distribution with a mean of 18 and a variance of 27.

$$\begin{aligned} \frac{E[T]}{\text{Var}[T]} &= \frac{\frac{\alpha}{\beta}}{\frac{\alpha}{\beta^2}} = \beta = \frac{18}{27} = \frac{2}{3} \\ \alpha &= \beta E[T] = \frac{2}{3}.\ 18 = 12. \end{aligned}$$

25 A gamma distribution has parameters $\alpha = 2$ and $\beta = 3$. Find

(a) $F_X(x)$
$f(x) = \frac{3^2}{\Gamma(2)}\ x^{2-1}\ e^{-3x}$.

$$\begin{aligned} F_X(x) &= 9 \int_0^x te^{-3t}\ dt \ \ \text{(integration by parts)} \\ &= 9\left(\frac{-te^{-3t}}{3} - \frac{e^{-3t}}{9} \right)_0^x = 1 - e^{-3x}(3x+1) \end{aligned}$$

(b) $P(0 \leq X \leq 3) = F_X(3) = 1 - e^{-9}(10) = 0.9988$.
(c) $P(1 \leq X \leq 2) = F_X(2) - F_X(1) = [1 - e^{-6}(7)] - [1 - e^{-3}(4)] = 0.1818$.