

## 1. Moment Generating Function

$$M_X(t) = E(e^{tx})$$

Properties:

- (a)  $M_X(t) = (1 - p + pe^t)^n$  for binomial(n,p)
- (b)  $M_X^{(n)}(0) = E(X^n)$   
 $\Rightarrow \text{VAR}[X] = E[X^2] - E^2[X] = M_X''(0) - [M_X'(0)]^2$
- (c)  $M_X(t) = M_Y(t) \Rightarrow XY$  has same distribution
- (d)  $M_X(0) = 1$
- (e)  $M_{ax+b}(t) = M_X(at)e^{bt}$
- (f)  $M_{X+Y}(t) = M_X(t)M_Y(t)$  ( $X, Y$  independent)

## 2. Gamma Function $\Gamma(n) = \int_0^\infty u^{n-1} e^{-u} du = (n-1)!$

## 3. Basic probability property

- (a) Conditional Probability and Bayes Theorem
  - 1.  $P(A|B) = \frac{P(A \cap B)}{P(B)}$
  - 2.  $P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$
- (b) Expectation
  - 1.  $E(X) = \sum xP_X(x)$  or  $\int_{-\infty}^\infty x f_X(x)$
  - 2.  $E(c) = c$
  - 3.  $E(aX) = aE(X)$
  - 4.  $E(X + Y) = E(X) + E(Y)$
  - 5.  $E(X) = M_X'(0)$
- (c) Variance and Standard Deviation
  - 1.  $\text{VAR}(X) = E(X^2) - E^2(X) = \sum [(x - E(X))]^2 P_X(x) = E[(x - \mu)^2] = \sigma^2$
  - 2.  $\text{VAR}(c) = 0$
  - 3.  $\text{VAR}(aX) = a^2 \text{VAR}(x)$
  - 4.  $\text{VAR}(X \pm Y) = \text{VAR}(X) + \text{VAR}(Y) \pm 2\text{COV}(x, y)$ ,  
 $\text{COV}(x, y) = E(XY) - E(X)E(Y)$
- (d) z-score  
 $Z = \frac{x - \mu}{\sigma}$ , measures the distance of x from expected value in standard units.

## 4. Discrete Distributions

- (a) Binomial Distribution  
 Note as  $X \sim B(n, p)$   
 $P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$   
 $M_X(t) = (1 - p + pe^t)^n$   
 $E(X) = np, \text{VAR}(X) = np(1-p)$
- (b) Hyper Geometric Distribution  
 Note as  $X \sim H(N, n, r)$ , N:total size, n:total pick, r:size of special subgroup  
 $P_X(x) = \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}}$   
 $E(X) = n \frac{r}{N}$   
 $\text{VAR}(X) = n \frac{r}{N} (1 - \frac{r}{N}) (\frac{N-n}{N-1})$
- (c) Poisson Process  
 $P_X(x) = \frac{e^{-\lambda} \lambda^x}{x!}$   
 $\lambda$ : average arrival in given time or space  
 $E(X) = \lambda, \text{VAR}(X) = \lambda$   
 When  $\lambda = np < 10$ , poisson approximates Binomial.
- (d) Geometric Distribution

i. X is the r.v of number of total trials (x includes the first success)

$$P_X(x) = (1-p)^{x-1} p, x = 1, 2, 3 \dots$$

$$E(X) = 1/p, \text{VAR}(X) = \frac{1-p}{p^2}$$

ii. X is the r.v of number of failed trials (x excludes the first success)

$$P_X(x) = (1-p)^x p, x = 0, 1, 2, \dots$$

$$M_X(t) = \frac{p}{1 - e^t(1-p)}$$

$$E(X) = \frac{1-p}{p}, \text{VAR}(X) = \frac{1-p}{p^2}$$

## (e) Negative Binomial Distribution

X is the r.v of number of trials need to observe the  $r^{th}$  success in a sequence of Bernoulli trials where p is the success probability.

$$P_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, x = r, r+1, r+2 \dots$$

$$M_X(t) = \left( \frac{p}{1 - e^t(1-p)} \right)^r$$

$$E(X) = \frac{r}{p}, \text{VAR}(X) = \frac{r(1-p)}{p^2}$$

Alternatively, X is the r.v of failures before the  $r^{th}$  success:

$$P_X(x) = \binom{x+r-1}{r-1} p^r (1-p)^x, x = 0, 1, 2 \dots$$

$$M_X(t) = \left( \frac{1-p}{1 - pe^t} \right)^r$$

$$E(X) = \frac{r(1-p)}{p}, \text{VAR}(X) = \frac{r(1-p)}{p^2}$$

## 5. Chebychev's Theorem

$$P(\mu - k\sigma \leq x \leq \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

## 6. Continuous Distributions

### (a) Uniform(rectangle) Distribution

$$f_X(x) = \frac{1}{b-a}, a \leq x \leq b$$

$$F_X(x) = \frac{x-a}{b-a}, a \leq x \leq b$$

$$E(X) = \frac{a+b}{2}, \text{VAR}(X) = \frac{(b-a)^2}{12}$$

### (b) Exponential Distribution

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, x \geq 0$$

$$F_X(x) = 1 - e^{-\frac{x}{\theta}}, x \geq 0$$

$$M_X(t) = \frac{1}{1 - \theta t}$$

$$E(X) = \theta, \text{VAR}(X) = \theta^2$$

Note as  $X \sim \exp(\theta)$

### (c) Gamma Distribution

$$f_X(x) = \frac{1}{\Gamma(n)\theta^n} x^{n-1} e^{-\frac{x}{\theta}}, x \geq 0$$

$$M_X(t) = \frac{1}{(1 - \theta t)^n}$$

$$E(X) = n\theta, \text{VAR}(X) = n\theta^2$$

Note as  $X \sim \Gamma(n, \theta)$

Exponential dist. is a special case of Gamma dist where  $n = 1$ . Gamma Distribution can be viewed as a sum of Exponential dists.

$$X \sim \Gamma(n, \theta) \Leftrightarrow X = \sum_{i=1}^n X_i, X_i \sim \exp(\theta)$$