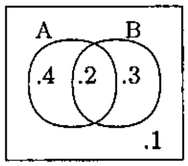
MATH 4681 Class Problem Set 1 Answers May 13, 2019

Name:

1. Suppose that $P(A \cap B) = 0.2$, $P(A) = 0.6$, and $P(B) = 0.5$. Find



(a) $P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B) = 1 - 0.2 = 0.8$

(b) $P(A' \cap B') = P((A \cup B)') = 0.1$

(c) $P(A' \cup B) = P((A \cap B')') = 1 - P(A \cap B') = 1 - 0.4 = 0.6$

(d) $P(A' \cap B) = 0.3$

2. For two events E and F we have the following probabilities:

$$P(F \cap E) = 0.07,\ P(E \cap F') = 0.33,\ P(E' \cap F') = 0.25.$$

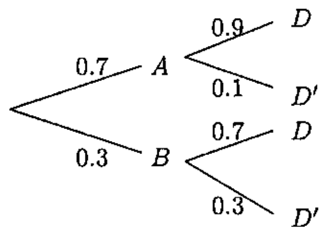
Use this information to answer the following questions:

(a) $P(F) = P(F \cap E) + P(F \cap E') = 0.42$

(b) $P(E|F) = \frac{P(F \cap E)}{P(F)} = \frac{0.07}{0.42} = \frac{7}{40} = 0.167.$

(c) $P(E|F') = \frac{P(F' \cap E)}{P(F')} = \frac{0.33}{0.58} = 0.569.$

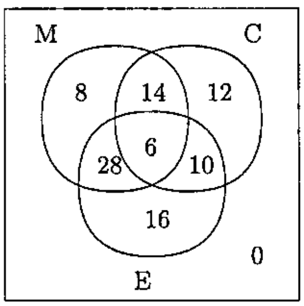
3. A building contractor buys 70% of his cement from supplier A and 30% from supplier B. A total of 90% of the bags from A arrive undamage, and 70% of bags from B arrive undamage. Find the probability that an undamage bag is from supplier B.



Let D : damage bag. Using Baye's formula, we get

$$\begin{aligned} P(B|D') &= \frac{P(B \cap D')}{P(D')} \\ &= \frac{(0.70)\ .\ (0.30)}{(0.70)\ .\ (0.30) + (0.90)\ .\ (0.70)} \\ &= \frac{0.21}{0.21 + 0.63} \\ &= \frac{21}{84} = 0.250. \end{aligned}$$

4. In a survey of 94 students, the following data was obtained. 60 took English, 56 took Math, 42 took Chemistry, 34 took English and Math, 20 took Math and Chemistry, 16 took English and Chemistry, 6 took all three subjects. Find the following proportions.



(a) Of those who took Math, the proportion who took neither English nor Chemistry,

$$P(E' \cap C'|M) = \frac{P(E' \cap C' \cap M)}{P(M)} = \frac{8}{56} = \frac{1}{7}.$$

(b) Of those who took English or Math, the proportion who also took Chemistry,

$$P(C|E \cup M) = \frac{P(C \cap (E \cup M))}{P(E \cup M)} = \frac{30}{82} = \frac{15}{41} = 0.366.$$

Name:

1. Let X be the random variable for the sum obtained by rolling two fair dice.

(a) What are the $P(x)$ and $F(x)$ functions for X?

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$F(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{36} & 2 \le x < 3 \\ \frac{3}{36} & 3 \le x < 4 \\ \frac{6}{36} & 4 \le x < 5 \\ \frac{10}{36} & 5 \le x < 6 \\ \frac{15}{36} & 6 \le x < 7 \\ \frac{21}{36} & 7 \le x < 8 \\ \frac{26}{36} & 8 \le x < 9 \\ \frac{30}{36} & 9 \le x < 10 \\ \frac{33}{36} & 10 \le x < 11 \\ \frac{35}{36} & 11 \le x < 12 \\ 1 & x \ge 12 \end{cases}$$

(b) What is Var$[X]$.?

$$\begin{aligned} E[X] &= \sum x\, p(x) \\ &= \frac{2.1 + 3.2 + 4.3 + 5.4 + 6.5 + 7.6 + 8.5 + 9.4 + 10.3 + 11.2 + 12}{36} = 7 \\ E[X^2] &= \sum x^2\, p(x) \\ &= \frac{2^2.1 + 3^2.2 + 4^2.3 + 5^2.4 + 6^2.5 + 7^2.6 + 8^2.5 + 9^2.4 + 10^2.3 + 11^2.2 + 12^2}{36} = 54.833 \\ \text{Var}[X] &= E[X^2] - E^2[X] \\ &= 54.833 - 49 = 5.833. \end{aligned}$$

(c) Using Chebychev's Theorem, what is the lower bound for the sum obtained for the probability that the value of X is within 2 standard deviations of the mean of X?

Lower bound is $1 - \left(\frac{1}{2}\right)^2 = 0.75$.

(d) What is the exact probability within this range?

Two standard deviations equal $2(5.833)^{\frac{1}{2}} = 4.8305$.

$$P(7 - 4.8305 < X < 7 + 4.8305) = P(3 < X < 11) = \frac{34}{36} = 0.9444.$$

2. A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

Claim Size	Probability	$xp(x)$	$x^2p(x)$
20	0.15	3	60
30	0.10	3	90
40	0.05	2	80
50	0.20	10	500
60	0.10	6	360
70	0.10	7	490
80	0.30	24	1920
Total	1	55	3500

What percentage of the claims are within one standard deviation of the mean claim size?

(A) 45% (B) 55% (C) 68% (D) 85% (E) 100%

Answer # 2: $\boxed{\text{A}}$

$$\begin{aligned} E[X] &= \sum x\, p(x) = 55 \\ E[X^2] &= \sum x^2\, p(x) = 3500 \\ \mathrm{Var}[X] &= E[X^2] - E^2[X] \\ &= 3500 - 55^2 = 475 \Rightarrow \sigma = \sqrt{\mathrm{Var}[X]} = 21.8. \end{aligned}$$

A value is within one standard deviation of the mean if it is in the interval $[\mu - \sigma, \mu + \sigma] = [33.2, 76.8]$. The values of x in this interval are 40, 50, 60 and 70. Thus the probability of being within one standard deviation of the mean is

$$P(40) + P(50) + P(60) + P(70) = 0.05 + 0.20 + 0.10 + 0.10 = 0.45.$$

$\square$

Name ______________________

1. Let X be a Poisson random variable with $E[X] = \ln 2$. Calculate $E[\cos(\pi X)]$.

   (A) 0 (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1 (E) 2 $\ln 2$ **Answer # 1:** $\boxed{\text{B}}$

   Given $E[X] = \ln 2 = \lambda$, $\Rightarrow$ $P[X = x] = \frac{e^{-\ln(2)}\,[\ln(2)]^x}{x!} = \frac{1}{2}\,\frac{[\ln(2)]^x}{x!}$ then

$$E[\cos(\pi X)] = \sum_{x=0}^{\infty} \cos(\pi x)\,\frac{1}{2}\,\frac{[\ln(2)]^x}{x!} = \frac{1}{2}\,\sum_{x=0}^{\infty}(-1)^x\,\frac{[\ln(2)]^x}{x!} = \frac{1}{2}\,\sum_{x=0}^{\infty}\frac{[-\ln(2)]^x}{x!} = \frac{1}{2}\,e^{-\ln(2)} = \frac{1}{4}.$$

2. A coin is twice as likely to turn up tails as heads. If the coin is tossed independently, what is the probability that the third head occurs on the fifth toss?

   (A) $\frac{8}{81}$ (B) $\frac{40}{243}$ (C) $\frac{16}{81}$ (D) $\frac{80}{243}$ (E) $\frac{3}{5}$ **Answer # 2:** $\boxed{\text{A}}$

   $P(H) = \frac{1}{3}$, $P(T) = \frac{2}{3}$.

$$\begin{aligned} P(3^{rd}\text{ head on }5^{th}\text{ toss}) &= P((2\text{ heads in the first 4 tosses}) \cap (\text{head on }5^{th}\text{ toss})) \\ &= P(2\text{ heads in the first 4 tosses}).P(\text{head on }5^{th}\text{ toss}) \\ &= \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2.\left(\frac{1}{3}\right) = \frac{8}{81}. \end{aligned}$$

   Note that the number of trials, X, that are tossed until the 3^{rd} head occurs can also be regarded as negative binomial distribution with $p = \frac{1}{3}$ and $r = 3$, and we are finding $P(X = 2)$.

3. A fair die is tossed until a 2 is obtained. If X is the number of trials required to obtain the first 2, what is the smallest value for x for which $P(X \le x) \ge \frac{1}{2}$?

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6 **Answer # 3:** $\boxed{\text{C}}$

$$\begin{aligned} P(X = k) &= \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right) \\ P(X \le x) &= \sum_{k=1}^{x} P(X = k) = \frac{1}{6}\left[\frac{1-(\frac{5}{6})^x}{1-\frac{5}{6}}\right] = 1 - \left(\frac{5}{6}\right)^x \ge \frac{1}{2} \\ \Rightarrow \left(\frac{5}{6}\right)^x &\le \frac{1}{2} \\ \Rightarrow x &\ge 4. \end{aligned}$$

4. A box contains 10 white and 15 black marbles. Let X denote the number of white marbles in a selection of 10 marbles selected at random without replacement. Find $\frac{\text{Var}[X]}{E[X]}$.

   (A) $\frac{1}{8}$ (B) $\frac{3}{16}$ (C) $\frac{2}{8}$ (D) $\frac{5}{16}$ (E) $\frac{3}{8}$ **Answer # 4:** $\boxed{\text{E}}$

   X has a geometric distribution with $N = 25$, $r = 10$ white marbles, and $n = 10$ marbles chosen. Then

$$\begin{aligned} E[X] &= n\frac{r}{N} = 10\,\frac{(10)}{25} = 4 \\ \text{Var}[X] &= n\frac{r}{N}\left(1-\frac{r}{N}\right)\frac{N-n}{N-1} = 4\left(\frac{15}{25}\right)\left(\frac{15}{24}\right) = \frac{3}{2} \\ \frac{\text{Var}[X]}{E[X]} &= \frac{3}{(2).(4)} = \frac{3}{8}. \end{aligned}$$

MATH 4681 Class Problem Set 4 Answers May 30, 2019

Name ____________________

1. If for a certain random variable X, $P(X < 500) = 0.5$ and $P(X > 650) = 0.0227$. Find σ_X.

(A) 150 (B) $\sqrt{75}$ (C) $\sqrt{150}$ (D) 75 (E) 15

Answer # 1: $\boxed{\text{D}}$

The normal distribution is symmetric about its mean, with $P(X < \mu) = 0.5$ for any random variable. Thus, for this normal X we have $\mu = 500$. Then

$$P(X > 650) = 0.0227 = P\left(\frac{X - 500}{\sigma} > \frac{150}{\sigma}\right).$$

Since $\frac{X-500}{\sigma}$ has a standard normal distribution, it follows from the table for the standard normal distribution that $\frac{150}{\sigma} = 2 \Rightarrow \sigma = 75$.

2. The random variable T has an exponential distribution such that $P(T \leq 2) = 2\ P(T > 4)$. Find Var[$T$].

(A) $\frac{2}{(\ln 2)^2}$ (B) $\frac{4}{(\ln 2)^2}$ (C) $\frac{2}{(\ln 2)}$ (D) 2 (E) $(\ln 2)^2$

Answer # 2: $\boxed{\text{B}}$

Suppose that T has mean $\frac{1}{\lambda}$. Then

$$P(T \leq 2) = F(2) = 1 - e^{-2\lambda} = 2P(T > 4) = 2[1 - P(T \leq 4)] = 2[1 - F(4)] = 2\ e^{-4\lambda}.$$

Set $x = e^{-2\lambda}$, we get $2x^2 + x - 1 = 0 \Rightarrow x = \frac{1}{2}, -1$ we ignore the negative root, so that $e^{-2\lambda} = \frac{1}{2} \Rightarrow \lambda = \frac{1}{2}\ln 2$. Then Var[$T$] $= \frac{1}{\lambda^2} = \frac{4}{(\ln 2)^2}$.

3. The random variable X has an exponential distribution with mean $\frac{1}{b}$. It is found that $M_X(-b^2) = 0.2$. Find b.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer # 3: $\boxed{\text{D}}$

$$M_X(t) = \frac{1}{1 - \frac{t}{b}} \Rightarrow M_X(-b^2) = \frac{1}{1+b} = 0.2 \Rightarrow b = 4.$$

4. If X has a normal distribution with mean 1 and variance 4. Find $P(X^2 - 2X \leq 8)$.

(A) 0.13 (B) 0.43 (C) 0.75 (D) 0.86 (E) 0.93

Answer # 4: $\boxed{\text{D}}$

Since $X \sim N(1,4)$, $Z = \frac{X-1}{2}$ has a standard normal distribution. Then

$$\begin{aligned} P(X^2 - 2X \leq 8) &= P(X^2 - 2X + 1 \leq 9) = P[(X-1)^2 \leq 9] \\ &= P(-3 \leq X - 1 \leq 3) = P(-1.5 \leq Z \leq 1.5) = 0.86. \end{aligned}$$

5. Let the random variable X have moment generating function $M_X(t) = e^{3t+t^2}$. What is $E[X^2]$?

(A) 1 (B) 2 (C) 3 (D) 9 (E) 11

Answer # 5: $\boxed{\text{E}}$

$\mu = 3,\ \sigma^2 = 2,\ \ E[X^2] = \sigma^2 + \mu^2 = 11.$

6. Let X_1, X_2, X_3 be a random sample from a normal distribution with mean $\mu \neq 0$ and variance $\sigma^2 = \frac{1}{24}$. What are the values of a and b, respectively, in order for $L = aX_1 + 4X_2 + bX_3$ to have a standard normal distribution?

(A) $a = -2,\ b = -2$ (B) $a = -2,\ b = 2$ (C) $a = -1,\ b = -3$
(D) $a = 2,\ b = 2$ (E) Cannot be determined from the information above

Answer # 6: $\boxed{\text{A}}$

$$\begin{aligned} L &= aX_1 + 4X_2 + bX_3 \\ E[L] &= aE[X_1] + 4E[X_2] + bE[X_3] = (a+4+b)\mu = 0 \\ &\Rightarrow a + 4 + b = 0 && (1) \\ \text{Var}[L] &= a^2\text{Var}[X_1] + 16\text{Var}[X_2] + b^2\text{Var}[X_3] = (a^2+16+b^2)\left(\frac{1}{24}\right) = 1 \\ &\Rightarrow a^2 + 16 + b^2 = 24 && (2) \\ (1)\&(2) &\Rightarrow a = b = -2. \end{aligned}$$

MATH 4681 Solutions: Class Problem Set 5 June 5, 2019

Name ____________________

Problems 1 to 3 relate to the following information. Three individuals are running a one kilometer race. The completion time for each individual is a random variable. X_i is the completion time, in minutes, for person i.
X_1 : uniform distribution on the interval $[2.9, 3.1]$
X_2 : uniform distribution on the interval $[2.7, 3.1]$
X_3 : uniform distribution on the interval $[2.9, 3.3]$
The three completion times are independent of one another.

1. Find the probability that the earliest completion time is less than 3 minutes.

(A) 0.89 (B) 0.91 (C) 0.94 (D) 0.96 (E) 0.98

Answer # 1: B

$$\begin{aligned}
f_{X_1}(t) &= \frac{1}{0.2} = 5 \text{ for } 2.9 \le t \le 3.1, \quad F_{X_1} = P(X_1 \le t) = 5(t-2.9) \text{ for } 2.9 \le t \le 3.1. \\
f_{X_2}(t) &= \frac{1}{0.4} = 2.5 \text{ for } 2.7 \le t \le 3.1, \quad F_{X_2} = P(X_2 \le t) = 2.5(t-2.7) \text{ for } 2.7 \le t \le 3.1. \\
f_{X_3}(t) &= \frac{1}{0.2} = 2.5 \text{ for } 2.9 \le t \le 3.3, \quad F_{X_3} = P(X_3 \le t) = 2.5(t-2.9) \text{ for } 2.9 \le t \le 3.3. \\
P(\min(X_1, X_2, X_3) < 3) &= 1 - P(\min(X_1, X_2, X_3) \ge 3) \\
&= 1 - P((X_1 \ge 3) \cap (X_2 \ge 3) \cap (X_3 \ge 3)) \\
&= 1 - [1 - F_{X_1}(3)].[1 - F_{X_2}(3)].[1 - F_{X_3}(3)] \\
&= 1 - [1 - 5(3-2.9)].[1 - 2.5(3-2.7)].[1 - 2.5(3-2.9)] = 0.90625.
\end{aligned}$$

2. Find the probability that the latest completion time is less than 3 minutes (nearst .01).

(A) 0.089 (B) 0.091 (C) 0.094 (D) 0.096 (E) 0.098

Answer # 2: C

$$\begin{aligned}
P(\max(X_1, X_2, X_3) < 3) &= P((X_1 < 3) \cap (X_2 < 3) \cap (X_3 < 3)) \\
&= F_{X_1}(3).F_{X_2}(3).F_{X_3}(3) \\
&= [5(3-2.9)].[2.5(3-2.7)].[2.5(3-2.9)] = 0.09375.
\end{aligned}$$

3. Find the expected latest completion time (nearst 0.1).

(A) 2.9 (B) 3.0 (C) 3.1 (D) 3.2 (E) 3.3

Answer # 3: C

$$\begin{aligned}
Y &= \max(X_1, X_2, X_3) < y \Rightarrow f_Y(y) = F_Y'(y), \text{ where} \\
F_Y(y) &= P(Y \le y) = P(\max(X_1, X_2, X_3) \le y) = P((X_1 \le y) \cap (X_2 \le y) \cap (X_3 \le y)) \\
&= P(X_1 \le y).P(X_2 \le y).P(X_3 \le y) \\
&= \begin{cases} 0 & \text{for } y < 2.9 \\ [5(y-2.9)].[2.5(y-2.7)].[2.5(y-2.9)] & \text{for } 2.9 \le y < 3.1 \\ 2.5(y-2.9) & \text{for } 3.1 \le y < 3.3 \\ 1 & \text{for } y \ge 3.3 \end{cases} \quad \text{then,} \\
f(y) &= F_Y'(y) = \begin{cases} 31.25(3y^2 - 17y + 24.07) & \text{for } 2.9 \le y < 3.1 \\ 2.5 & \text{for } 3.1 \le y < 3.3. \end{cases} \quad \text{Finally,} \\
E[Y] &= \int_{2.9}^{3.1} y.31.25(3y^2 - 17y + 24.07)\, dy + \int_{3.1}^{3.3} y.(2.5)\, dy = 3.12.
\end{aligned}$$

4. Average loss size per policy on a portfolio of policies is 100. Actuary 1 assumes that the distribution of loss size has an exponential distribution with a mean of 100, and Actuary 2 assumes that the distribution of loss size has a pdf of $f_2(x) = \frac{2\theta^2}{(x+\theta)^3}$, $x > 0$. If m_1 and m_2 represent the median loss sizes for the two distributions, find $\frac{m_1}{m_2}$.

(A) 0.6 (B) 1.0 (C) 1.3 (D) 1.7 (E) 2.0

Answer # 4: $\boxed{\text{D}}$

The cdf for distribution 1 is $F_1(x) = 1 - e^{-\frac{x}{100}}$. The median m_1 must satisfy

$$0.50 = F_1(m_1) = 1 - e^{-\frac{m_1}{100}} \Rightarrow m_1 = 69.3.$$

The cdf for distribution 2 is $F_2(x) = \int_0^x f_2(t)\,dt = \int_0^x \frac{2\theta^2}{(t+\theta)^3}\,dt = 1 - \frac{\theta^2}{(x+\theta)^2}$. The mean of distribution 2 is $100 = E[X_2] = \int_0^\infty x\,\frac{2\theta^2}{(x+\theta)^3}\,dx = \theta \Rightarrow \theta = 100.$

$$0.50 = F_2(m_2) = 1 - \frac{100^2}{(m_2+100)^2} \Rightarrow m_2 = 41.4.$$

Then $\frac{m_1}{m_2} = \frac{69.3}{41.4} = 1.67$.