1. Moment Generating Function
   $M_X(t) = E(e^{tx})$
   Properties:

   (a) $M_x^{(n)}(0) = E(X^n)$
   (b) $M_x(t) = M_y(t) \Rightarrow XY$ has same distribution
   (c) $M_x(0) = 1$
   (d) $M_{ax+b}(t) = M_x(at)e^{bt}$
   (e) $M_{X+Y}(t) = M_X(t)M_Y(t)$ (X, Y independent)

2. Gama Function
   $\Gamma(n) = \int_0^\infty u^{n-1}e^{-u}du = (n-1)!$
   $\Gamma(n) = (n-1)\Gamma(n-1)$
   $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

3. Basic probability property

   (a) Conditional Probability and Bayles Theorem
   1. $P(A|B) = \frac{P(A\cap B)}{P(B)}$
   2. $P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A')P(A')}$

   (b) Expectation
   1. $E(X) = \sum xP_X(x)$ or $\int_{-\infty}^\infty xf_X(x)$
   2. $E(c) = c$
   3. $E(aX) = aE(X)$
   4. $E(X+Y) = E(X)+E(Y)$
   5. $E(X) = M_X'(0)$

   (c) Variance and Standard Deviation
   1. $VAR(X) = E(X^2) - E^2(X) = \sum[(x-E(X)]P_X(x) = E[(x-\mu)^2] = \sigma^2$
   2. $VAR(c) = 0$
   3. $VAR(aX) = a^2VAR(x)$
   4. $VAR(X \pm Y) = VAR(X) + VAR(Y) \pm 2COV(x,y)$, $COV(x,y) = E(XY) - E(X)E(Y)$

   (d) z-score
   $Z = \frac{x-\mu}{\sigma}$, measures the distance of x from expected value in standard units.

4. Discrete Distributions

   (a) Binomial Distribution
   Note as $X \sim B(n,p)$
   $P_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$
   $M_X(t) = (1-p+pe^t)^n, E(X) = np, VAR(X) = np(1-p)$

   (b) Hyper Geometric Distribution
   Note as $X \sim H(N,n,r)$, N:total size, n:total pick, r:size of special subgroup
   $P_X(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$
   $E(X) = n\frac{r}{N}, VAR(X) = n\frac{r}{N}(1-\frac{r}{N})(\frac{N-n}{N-1})$

   (c) Poisson Process
   $P_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, E(X) = \lambda, VAR(X) = \lambda$
   λ: average arrival in given time or space
   When $\lambda = np < 10$, poisson approximates Binomial.

   (d) Geometric Distribution
   i. X is the r.v of number of total trials (x includes the first success)
   $P_X(x) = (1-p)^{x-1}p, x = 1,2,3\ldots$
   $E(X) = 1/p, VAR(X) = \frac{1-p}{p^2}$
   ii. X is the r.v of number of failed trials (x excludes the first success)
   $P_X(x) = (1-p)^x p, x = 0,1,2,\ldots$
   $M_X(t) = \frac{p}{1-e^t(1-p)}, E(X) = \frac{1-p}{p}, VAR(X) = \frac{1-p}{p^2}$

   (e) Negative Binomial Distribution
   X is the r.v of number of trials need to observe the r^{th} success in a sequence of Bernoulli trails where p is the success probability.
   $P_X(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}, x = r, r+1, r+2\ldots$
   $M_X(t) = (\frac{p}{1-e^t(1-p)})^r, E(X) = \frac{r}{p}, VAR(X) = \frac{r(1-p)}{p^2}$
   Alternatively, X is the r.v of failures before the r^{th} success:
   $P_X(x) = \binom{x+r-1}{r-1}p^r(1-p)^x, x = 0,1,2\ldots$
   $M_X(t) = (\frac{1-p}{1-pe^t})^r, E(X) = \frac{r(1-p)}{p}, VAR(X) = \frac{r(1-p)}{p^2}$

5. Chebychev's Theorem
   $P(\mu - k\sigma \le x \le \mu + k\sigma) \ge 1 - \frac{1}{k^2}$

6. Continuous Distributions

   (a) Uniform(rectangle) Distribution
   $f_X(x) = \frac{1}{b-a},\ (a \le x \le b); F_X(x) = \frac{x-a}{b-a}, a \le x \le b$
   $E(X) = \frac{a+b}{2}, VAR(X) = \frac{(b-a)^2}{12}$

   (b) Exponential Distribution
   $f_X(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, (x \ge 0); F_X(x) = 1 - e^{-\frac{x}{\theta}}, x \ge 0$
   $M_X(t) = \frac{1}{1-\theta t}, E(X) = \theta, VAR(X) = \theta^2$
   Note as $X \sim exp(\theta)$, In terms of λ: $\lambda = \frac{1}{\theta}$

   (c) Gamma Distribution
   $f_X(x) = \frac{1}{\Gamma(n)\theta^n}x^{n-1}e^{-\frac{x}{\theta}}, x \ge 0$
   $M_X(t) = \frac{1}{(1-\theta t)^n}, E(X) = n\theta, VAR(X) = n\theta^2$
   In terms of α, β : $\alpha = n, \beta = \frac{1}{\theta}$

   (d) Normal Distribution
   $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, E(X) = \mu, VAR(X) = \sigma^2$
   $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

   (e) Log-normal Distribution:
   $Y = e^X, X \sim N(\mu, \sigma^2),\ f_Y(y) = \frac{1}{\sigma y\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{lny-\mu}{\sigma})^2}, y \ge 0$
   $E(Y) = e^{\mu+\frac{\sigma^2}{2}}, VAR(Y) = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$

   (f) Pareto Distribution:
   $f_X(x) = \frac{k}{\beta}(\frac{\beta}{x})^{k+1}, F_X(x) = 1 - (\frac{\beta}{x})^k, k > 2, x \ge \beta > 0$
   $E(X) = \frac{k\beta^2}{k-1}, VAR(X) = \frac{k\beta^2}{k-2} - (\frac{k\beta}{k-1})^2$
   In terms of α, β: $\alpha = k, \beta = \beta$

   (g) Weibull Distribution:
   $f_X(x) = k\lambda x^{k-1}e^{-\lambda kx^k}, F_X(x) = 1 - e^{-\lambda kx^k}$
   $E(X) = \frac{\Gamma(1+\frac{1}{k})}{\lambda^{\frac{1}{k}}}, VAR(X) = \frac{1}{\lambda^{\frac{2}{k}}}[\Gamma(1+\frac{2}{k}) - \Gamma^2(1+\frac{1}{k})]$
   in terms of α, β:$k = \alpha, \lambda = \beta$
   Note: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

   (h) Beta Distribution
   $f_X(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}, x \in (0,1), a > 0, b > 0$
   $E(X) = \frac{a}{a+b}, VAR(X) = \frac{ab}{(a+b)^2(a+b+1)}$

(i) Chi-Square Distribution
$f_X(x) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}, x > 0$
$E(X) = k, VAR(X) = 2k, M_X(t) = \frac{1}{(1-2t)^{\frac{k}{2}}}$
Special case of Gamma Dist. of $\alpha = \frac{k}{2},\ \beta = 2$
$\chi^2 = \sum_{i=1}^{k} Z_i^2$(sum of k squared standard normal)
THEOREM:
i. $\chi^2(a) + \chi^2(b) = \chi^2(a+b)$
ii. If $X_1, X_2 \dots, X_k$ is a random sample from $N(\mu, \sigma^2)$
$Z = \sum_{i=1}^{k} (\frac{x_i - \mu}{\sigma})^2 \sim \chi^2(k)$ (mean)
$Z = \sum_{i=1}^{k} (\frac{x_i - \bar{x}}{\sigma})^2 \sim \chi^2(k-1)$ (sample mean)
iii. When $n \geq 30$, Normal approximates Chi Square
$\chi^2(k) \sim N(k, 2k)$

7. Central Limit Theorem
Let $\{X_1, X_2, \dots, X_n\}$ be the indep. r.v. with same distribution and mean μ and std. deviation σ. If n is large $n \geq 30$,
$S = X_1 + X_2 + \dots + X_n \Rightarrow S \sim N(n\mu, n\sigma^2)$
OR $S' = \frac{X_1 + X_2 + \dots + X_n}{n} \Rightarrow S \sim N(\mu, \frac{\sigma^2}{n})$

8. Finding PDF for $Y = g(X)$
If $g(X)$ is strictly increasing or decreasing
$f_Y(y) = f_X(h(y)) \cdot |h'(y)|, h(y) = g^{-1}(x)$

9. Multivariable Distributions

(a) Marginal Distribution
$p_X(x) = \sum_y p(x,y),\ p_Y(y) = \sum_x p(x,y)$
$f_X(x) = \int_{-\infty}^{\infty} p(x,y)dy,\ f_Y(y) = \int_{-\infty}^{\infty} p(x,y)dx$

(b) Conditional Distribution
$P(X = x|Y = y) = \frac{p(x,y)}{p_Y(y)},\ P(Y = y|X = x) = \frac{p(x,y)}{p_X(x)}$
$f(x|Y = y) = f(x|y) = \frac{f(x,y)}{f_Y(y)}$
$f(y|X = x) = f(y|x) = \frac{f(x,y)}{f_X(x)}$

(c) Conditional Expected Value
$E(y|X = x) = \sum_y y \cdot p(y|x),\ E(x|Y = y) = \sum_x x \cdot p(x|y)$
$E(y|X = x) = \int_{-\infty}^{\infty} y \cdot f(y|x)dy$
$E(x|Y = y) = \int_{-\infty}^{\infty} x \cdot f(x|y)dx$

(d) Independence
$p(x,y) = p_X(x) \cdot p_Y(y)$
$f(x|y) = f_X(x),\ f(y|x) = f_Y(y)$

(e) Covariance
$COV(x,y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - E(X)E(Y)$
Properties:
i. $COV(X,Y) = COV(Y,X)$
ii. $COV(X,X) = VAR(X)$
iii. $COV(X,K) = 0, K$ is constant
iv. $COV(aX, bY) = ab \cdot COV(X,Y)$
v. $COV(X, Y+Z) = COV(X,Y) + COV(X,Z)$
vi. If X, Y are independent, $COV(X,Y) = 0$
vii. $VAR(X \pm Y) = VAR(X) + VAR(Y) \pm 2COV(x,y)$

(f) Correlation Coefficient ρ

$$\rho = \rho_{X,Y} = \frac{COV(X,Y)}{\sigma_X \cdot \sigma_Y}, \in [-1, 1]$$

$|\rho_{X,Y}|$ close to 1 : more related
$\rho > 0$ positively related, $\rho < 0$ negatively related

(g) Moment Generating Function of Joint Distribution
$M_{X,Y}(t_1, t_2) = E(e^{t_1 X + t_2 Y})$
$E(X) = \frac{d}{dt_1} M_{X,Y}(t1, t2)|_{t_1 = t_2 = 0}, E(Y) = \frac{d}{dt_2} M_{X,Y}(t1, t2)|_{t_1 = t_2 = 0}$
$E(X^m Y^n) = \frac{\partial^{m+n}}{\partial^m t_1 \partial^n t_2} M_{X,Y}(t_1, t_2)|_{t_1 = t_2 = 0}$

(h) Multinomial Distribution
Counting Partitions:
$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$
Suppose that a random experiment has K mutually exclusive outcomes $E_1, E_2, \dots E_k$, with $P(E_i) = p_i, i \in [1, k]$. Suppose you repeat this experiment in n independent trails. Then:
$P(X_1 = E_1, \dots, X_k = E_k) = \binom{n}{n_1, n_2, \dots, n_k} p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$