- Chapter 6

1 In a year, a policyholder with an insurance company has no claims with probability 0.69, one claim with probability 0.23, two claims with probability 0.07, and three claims with probability 0.01. If X is the random variable for the number of claims, find

(a) $E[500X + 50]500\ E[X] + 50 = 250$, Since

x	0	1	2	3
$p(x)$	0.69	0.23	0.07	0.01

$$E[X] = 0(0.69) + 1(0.23) + 2(0.07) + 3(0.01) = 0.40.$$

(b) $E[X^2] = 0(0.69) + 1(0.23) + 4(0.07) + 9(0.01) = 0.60.$

(c) $E[X^3] = 0(0.69) + 1(0.23) + 8(0.07) + 27(0.01) = 1.06.$

6 Use the moment generating function for Poisson distribution to verify that $E[X] = \text{Var}[X] = \lambda$.

For the Poisson random variable with rate λ,

$$\begin{aligned}
M_X(t) &= e^{\lambda(e^t-1)}, \\
M_X'(t) &= \lambda\ e^t\ e^{\lambda(e^t-1)}, \qquad M_X'(0) = \lambda = E[X] \\
M_X''(t) &= e^{\lambda(e^t-1)}\big(\lambda\ e^t + (\lambda\ e^t)^2\big), \qquad M_X''(0) = \lambda + \lambda^2 = E[X^2] \\
\text{Var}[X] &= E[X^2] - E^2[X] = \lambda.
\end{aligned}$$

9 Let X be a discrete random variable with $p = \frac{1}{n}$ for $x = 1, 2, \ldots, n$. (X is a discrete uniform random variable.)

(a) Show that the moment generating function for X is $M_X(t) = \frac{1}{n}\sum_{x=1}^{n} e^{xt}$.

For the discrete uniform random variable, $p(x) = \frac{1}{n}$, for $x = 1, 2, \ldots, n$.

$$M_X(t) = \sum_{x=1}^{n} e^{xt}\ p(t) = \frac{1}{n}\ \sum_{x=1}^{n} e^{xt}.$$

(b) $E[X]$ and $\text{Var}[X]$.

$$\begin{aligned}
M_X'(t) &= \frac{1}{n}\ \sum_{x=1}^{n} x\ e^{xt} \\
E[X] &= M_X'(0) = \frac{1}{n}\ \sum_{x=1}^{n} x = \frac{1}{n}\ \frac{n(n+1)}{2} = \frac{n+1}{2} \\
M_X''(t) &= \frac{1}{n}\ \sum_{x=1}^{n} x^2\ e^{xt} \\
E[X^2] &= M_X''(0) = \frac{1}{n}\ \sum_{x=1}^{n} x^2 = \frac{1}{n}\ \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6} \\
\text{Var}[X] &= E[X^2] - E^2[X] = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.
\end{aligned}$$

12 If X is a binomial random variable with $p = 0.6$ and $n = 8$, and if $Y = 3X + 4$, what is $M_Y(t)$?

For the binomial random variable with n trials and $P(s) = p$, $\quad M_X(t) = (q + pe^t)^n$.
If $n = 8$ and $p = 0.6$, then $M_X(t) = (0.4 + 0.6e^t)^8$.
If $Y = 3X + 4$, then $M_Y(t) = M_{3X+4}(t) = e^{4t}M_X(3t) = e^{4t}\ (0.4 + 0.6e^{3t})^8$.

- Chapter 7

12 The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ with probability density function f, where $f(x)$ is proportional to $(10 + x)^{-2}$. Calculate the probability that the lifetime of the machine part is less than 6.

Since $f(x)$ is proportional to $(10 + x)^{-2}$, $f(x) = k(10 + x)^{-2}$. We can find k using the fact that the total area bounded by the x-axis and the graph of $f(x)$ between $x = 0$ and $x = 40$ is 1.

$$\begin{aligned} 1 &= \int_0^{40} k(10+x)^{-2}\ dx = -k(10+x)^{-1}\Big|_0^{40} = -k(\frac{1}{50} - \frac{1}{10}) = k\frac{4}{50} \\ k &= \frac{50}{4} = 12.4 \ \Rightarrow \ f(x) = 12.5(10+x)^{-2}. \\ P(X < 6) &= \int_0^{6} 12.5(10+x)^{-2}\ dx = -12.5(10+x)^{-1}\Big|_0^{6} \\ &= -12.5(\frac{1}{16} - \frac{1}{10}) = 0.46875. \end{aligned}$$

13 An insurer's annual weather-related loss, X, is a random variable with density function

$$f(x) = \begin{cases} \frac{2.5\ (200)^{2.5}}{x^{3.5}} & \text{for } \ x > 200 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the difference between the 30^{th} and 70^{th} percentiles of X.

Recall that the 100^{th} percentile of X is the number x_p defined by $F(x_p) = p$. We are asked to find the difference $x_{0.70} - x_{0.30}$. To begin, we must find an expression for $F(x)$, for $x \geq 200$,

$$\begin{aligned} F(x) &= \int_{200}^{x} \frac{2.5\ (200)^{2.5}}{u^{3.5}}\ du \\ &= 2.5(200)^{2.5}\ \frac{u^{-2.5}}{-2.5}\Big|_{200}^{x} \\ &= 2.5(200)^{2.5}\ \frac{x^{-2.5}}{-2.5} - 2.5(200)^{2.5}\ \frac{200^{-2.5}}{-2.5} \\ &= 1 - (\frac{200}{x})^{2.5}. \end{aligned}$$

Since we have to find two percentiles, we will derive a general formula for the 100^{th} percentile of X. The defining equation is

$$\begin{aligned} F(x_p) &= p = 1 - \left(\frac{200}{x}\right)^{2.5} \ \rightarrow \ 1 - p = \left(\frac{200}{x}\right)^{2.5} \ \rightarrow \\ (1-p)^{(\frac{1}{2.5})} &= \left(\frac{200}{x}\right) \ \rightarrow \ x_p = \frac{200}{(1-p)^{(\frac{1}{2.5})}} = \frac{200}{(1-p)^{0.4}}. \quad \text{Thus} \\ x_{0.70} - x_{0.30} &= \frac{200}{(0.25)^{0.4}} - \frac{200}{(0.75)^{0.4}} = 123.829. \end{aligned}$$

14 An insurance companys monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1 + x)^{-4}$, where $0 < x < \infty$. Determine the companys expected monthly claims.

Since $f(x)$ is proportional to $(1 + x)^{-4}$, $f(x) = k(1 + x)^{-4}$. We can find k using the fact that the total area bounded by the x-axis and the graph of $f(x)$ between $x = 0$ and $x = \infty$ is 1.

$$\begin{aligned} 1 &= \int_0^{\infty} k(1+x)^{-4}\ dx = k\frac{(1+x)^{-3}}{-3}\Big|_0^{\infty} \\ k &= 3 \ \Rightarrow \ f(x) = 3(10+x)^{-4}. \end{aligned}$$

Thus

$$E[X] = \int_0^\infty 3x\ (1+x)^{-4}\ dx.$$

Let $u = 1 + x$, then

$$\begin{aligned}
E[X] &= \int_0^\infty 3x\ (1+x)^{-4}\ dx = \int_1^\infty 3(u-1)\ u^{-4}\ du \\
&= 3\int_1^\infty (u^{-3} - u^{-4})\ du \\
&= 3\left[\frac{u^{-2}}{-2} - \frac{u^{-3}}{-3}\right]\Bigg|_1^\infty \\
&= 3\left[0 - (\frac{1}{-2} - \frac{1}{-3})\right]\Bigg|_1^\infty = \frac{1}{2}.
\end{aligned}$$

17 An insurance company insures a large number of homes. The insured value, X, of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} 3x^{-4} & \text{for } \ x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, what is the probability that it is insured for less than 2?

We want to find

$$P(X < 2 \mid X \geq 1.5) = \frac{P(X < 2\ \&\ X \geq 1.5)}{P(X \geq 1.5)} = \frac{F(2) - F(1.5)}{1 - F(1.5)}.$$

Since we need to find two expressions involving $F(x)$, we will calculate this in general.

$$F(x) = P(X \leq x) = \int_1^x 3u^{-4}\ du = -u^{-3}\Big|_1^x = 1 - x^{-3}.$$

It follows that

$$P(X < 2 \mid X \geq 1.5) = \frac{F(2) - F(1.5)}{1 - F(1.5)} = \frac{1.5^{-3} - 2^{-3}}{1.5^{-3}} = 0.57813.$$