Ranks and Presentations for Finite Transformation Semigroups

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by

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CENTRAL UNIVERSITY OF KASHMIR
KNOWLEDGE IS POWER

DEPARTMENT OF MATHEMATICS

CERTIFICATE

This is to certify that the work contained in this report entitled "**Ranks and Presentations for Finite Transformation Semigroups**" submitted by **Bilal Ahmad Kalas** (1919CUKmr42) to the Department of Mathematics, School of Physical and Chemical Sciences, Central University of Kashmir, Ganderbal, Tulmulla towards the requirement of the course **MTH-1001** Project of the semester-X for the Integrated B. Sc.-M.Sc. Mathematics programme has been carried out by him/her under my supervision.

Date: Signature

Place: **Dr. Aftab Hussain Shah**

CANDIDATE'S DECLARATION

I hereby declare that the dissertation work entitled "**Ranks and Presentations for Finite Transformation Semigroups**" submitted in partial fulfilment of the requirements for the degree of Integrated B.Sc. M.Sc. in Mathematics at the Department of Mathematics, Central University of Kashmir, is an original record of work carried out by me under the supervision of **Dr. Aftab Hussain Shah**. This work has not been submitted for the award of any other degree or qualification. I further declare that I have properly acknowledged and credited all sources of information and research where they have been cited in the text, and have complied with the university's requirements for project work as part of the Integrated B.Sc. M.Sc. in Mathematics.

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Preface

Semigroup theory is a relatively young but essential branch of algebra, with applications across various mathematical fields. As a distinct area of study, it evolved over the past sixty years, focusing on the structural properties and behavior of semigroups. Among these, transformation semigroups provide compelling examples due to their natural occurrence in mathematics, where transformations of sets are ubiquitous. The associativity of composition in these transformations ensures that any set of transformations, closed under this operation, forms a semigroup.

Three classical series of finite transformation semigroups stand out: the full transformation semigroup $\mathcal{T}_n$, the symmetric inverse semigroup $\mathcal{IS}_n$, and the partial transformation semigroup $\mathcal{PT}_n$. These semigroups serve as foundational examples and have been extensively studied due to their universality and structural properties. In recent years, particular attention has been given to semigroups of order-preserving transformations, including the semigroup $O_{n,p}$, which plays a significant role in this dissertation. The semigroup $O_{n,p}$ consists of all order-preserving transformations on a finite chain with only one fixed point. This semigroup exhibits fascinating structural and algebraic properties, making it an ideal object of study. The focus of this dissertation is to explore the structural properties, ranks, and presentations of $O_{n,p}$, contributing to the broader understanding of transformation semigroups.

This work is organized into four chapters, each addressing different aspects of $O_{n,p}$ and its related concepts.

The first chapter lays the foundation by introducing the basic concepts and results related to transformation semigroups. We discuss the full transformation semigroup $\mathcal{T}_n$ and the partial transformation semigroup $\mathcal{PT}_n$, providing a comprehensive overview of their definitions, structures, and properties. In addition, we also briefly discuss congruences, ideals, and Green's Relations.

In Chapter 2, we focus on the semigroup of order-preserving transformations $O_{n,p}$, nilpotent semigroups, and relationship between the two. We explore its structural properties. This chapter also delves into the combinatorial aspects of $O_{n,p}$, particularly how its elements can be expressed and manipulated. The theoretical results are supported by theorems that illustrate the unique characteristics of $O_{n,p}$ within the broader context of transformation semigroups.

Chapter 3 is dedicated to the rank of the semigroup $O_{n,p}$, with particular focus on the case when $p > 1$, which represents the minimum number of elements required to generate the semigroup. We investigate the rank of $O_{n,p}$ in various contexts, developing a series of results that characterize the generating sets and their relationship to the semigroup's rank.

Chapter 4 deals with the presentation of the semigroup $O_{n,1}$, providing a concise way to describe the semigroup in terms of generators and relations. We systematically identify the generating sets for $O_{n,1}$ and explore the defining relations among them. The presentation developed here offers a unified framework for understanding the structure of $O_{n,1}$.

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Contents

CHAPTER 1

Finite Transformation Semigroups

Introduction

This chapter introduces the study of finite transformation semigroups, focusing on transformations and partial transformations of finite sets. A transformation is defined as a function from a finite set to itself, while a partial transformation maps a subset of the set to set itself. Both types of transformations form semigroups under function composition, a key operation in this context.

Additionally, we discuss identity and zero elements, which play crucial roles in understanding the structure of transformation semigroups. The chapter further explores binary relations, equivalence relations, and congruences, which help classify and analyze transformations within a semigroup. These concepts, building upon foundational work presented in [8] and [11], lay the groundwork for the deeper exploration of finite semigroups in the subsequent chapters.

1.1 Basic Definitions

The principal objects of interest in this work are finite sets and transformations of finite sets. Let M be a finite set, say $M = \{m_1, m_2, \ldots, m_n\}$, where n is a non-negative integer. The Transformation of M is an array of the following form:

$$\alpha = \begin{pmatrix} m_1 & m_2 & \ldots & m_n \\ k_1 & k_2 & \ldots & k_n \end{pmatrix}, \tag{1.1}$$

where all $k_i \in M$. If $x \in M$, say $x = m_i$, the element k_i will be called the value of the transformation α at the element x and will be denoted by $x\alpha$. The fact that α is a transformation of M is usually written as $\alpha : M \longrightarrow M$. As the nature of elements of M is not important for us, instead of M we shall usually consider the set $\mathbf{N} = X_n = \mathbf{N}_n = \{1, 2, \ldots, n\}$.

Definition 1.1.1. Let $A = \{l_1, l_2, \ldots, l_k\}$ be a subset of M. Then, the transformation $\alpha : A \longrightarrow M$ is called as the *Partial Transformation* of M. Note that A can be empty. Again, the element α can be written in the following tabular form:

$$\alpha = \begin{pmatrix} l_1 & l_2 & \ldots & l_k \\ l_1\alpha & l_2\alpha & \ldots & l_k\alpha \end{pmatrix}. \tag{1.2}$$

Abusing notation, we may also write $\alpha : M \longrightarrow M$ for a partial transformation, having in mind that such α is only defined on some elements from M. Note that the order of elements in the first row of arrays (1.1) and (1.2) is not important.

With each (partial) transformation α as above we associate the following standard notions:

- The *domain* of α: $dom(\alpha) = A$.
- The *codomain* of α: $\overline{dom}(\alpha) = M \setminus A$.
- The *image* of α: $im(\alpha) = \{x\alpha : x \in A\}$.
- The *kernel* of α: $\ker(\alpha) = \{(x, y) : x\alpha = y\alpha \text{ for all } x, y \in M\}$.

The word *range*, which is also frequently used in the literature, is a synonym of the word *image*.

Definition 1.1.2. If $dom(\alpha) = M$, the transformation α is called $full$ or $total$.

The set of all total transformations of M is denoted by $\mathcal{T}(M)$, and the set of all partial transformations of M is denoted by $\mathcal{PT}(M)$. Obviously, $\mathcal{T}(M) \subset \mathcal{PT}(M)$. To simplify our notation we set $\mathcal{T}_n = \mathcal{T}(\mathbf{N})$ and $\mathcal{PT}_n = \mathcal{PT}(\mathbf{N})$.
Sometimes it is convenient to use a slightly modified version of (1.2) for some $\alpha \in \mathcal{PT}_n$. In the case of $\mathcal{PT}_n$ it is natural to form the first row of the array for α by simply listing all the elements from $\mathbf{N}$ in their natural order. Then, to define α completely, one needs a special symbol to indicate that some element x belongs to $\overline{dom}(\alpha)$. We shall use the symbol $\varnothing$. In other words, $x\alpha = \varnothing$ means that $x \in \overline{dom}(\alpha)$. Thus the element α can be written in the following form:

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ k_1 & k_2 & \dots & k_n \end{pmatrix}. \tag{1.3}$$

where $k_i = i\alpha$ if $i \in dom(\alpha)$ and $k_i = \varnothing$ if $i \in \overline{dom}(\alpha)$.

Example 1.1.3. Here is the list of all partial transformation on X_2:

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 \\ \varnothing & \varnothing \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & \varnothing \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & \varnothing \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \varnothing & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \varnothing & 2 \end{pmatrix}.$$

The first row of this list consists of total transformations and hence lists all elements in $\mathcal{T}_2$.

Proposition 1.1.4. *The set $\mathcal{T}_n$ contains n^n elements and the set $\mathcal{PT}_n$ contains $(n+1)^n$ elements.*

Proof. If α is any transformation on the set $\mathbf{N}$, then α can be uniquely defined by the array (1.3), where each $k_i \in \mathbf{N} \cup \{\varnothing\}$.
Now, if $\alpha \in \mathcal{T}_n$ then $k_i = i\alpha \in \mathbf{N}\ \forall\, i \in \mathbf{N}$. Since the choices of k_i's are independent and, hence, each k_i has n-independent choices. Consequently, by product rule,

$$|\mathcal{T}_n| = n^n.$$

Again, if $\alpha \in \mathcal{PT}_n$, then each k_i can be independently chosen from the set $N \cup \{\emptyset\}$ of $(n+1)$ elements. Similarly, by product rule,

$$|\mathcal{PT}_n| = (n+1)^n.$$

□

Definition 1.1.5. The cardinality $|im(\alpha)|$ of the image of a partial transformation $\alpha \in \mathcal{PT}_n$ is called the *rank* of this partial transformation and is denoted by $rank(\alpha)$. Thus $rank(\alpha)$ equals the number of different elements in the second row of array (1.3).

The number $def(\alpha) = n - rank(\alpha)$ is called the defect of the partial transformation α.

Definition 1.1.6. A partial transformation $\alpha \in \mathcal{PT}_n$ is called

- *Surjective* if $im(\alpha) = \mathbf{N}$,
- *Injective* if $x \neq y$ implies $x\alpha \neq y\alpha \ \forall \ x, y \in dom(\alpha)$,
- *Bijective* if α is both surjective and injective.

If α is given by (1.2), then surjectivity means that the second row of array (1.2) contains all elements of $\mathbf{N}$; and injectivity means that all elements in the second row of array (1.2) are different. Bijective transformations on $\mathbf{N}$ are also called permutations of $\mathbf{N}$.

Proposition 1.1.7. *Let $\alpha \in \mathcal{T}_n$. Then the following conditions are equivalent:*

a) α is surjective

b) α is injective

c) α is bijective

Proof. By the definition of a bijective transformation, it is enough to show that the conditions (a) and (b) are equivalent.
Let α be given by (1.3). Suppose α is injective, then by definition of injectivity, second row of (1.3) gives n different elements of $\mathbf{N}$ namely $1\alpha, 2\alpha, \ldots, n\alpha$. But $\mathbf{N}$ contains exactly n elements.
Hence, $\mathbf{N} = \{1\alpha, 2\alpha, \ldots, n\alpha\}$. That is, every $x \in \mathbf{N}$ has a pre-image in $\mathbf{N}$ and thus α is surjective.

Conversely, let α be surjective (i.e., $im(\alpha) = \mathbf{N}$). Then, the second row of (1.3) contains all the n-elements of $\mathbf{N}$. But since second row of (1.3) contains exactly n-elements, they all must be different. This implies that α injective. $\square$

1.2 Composition of Transformations

Let X and Y be two sets. A mapping from X to Y is an array of the form:

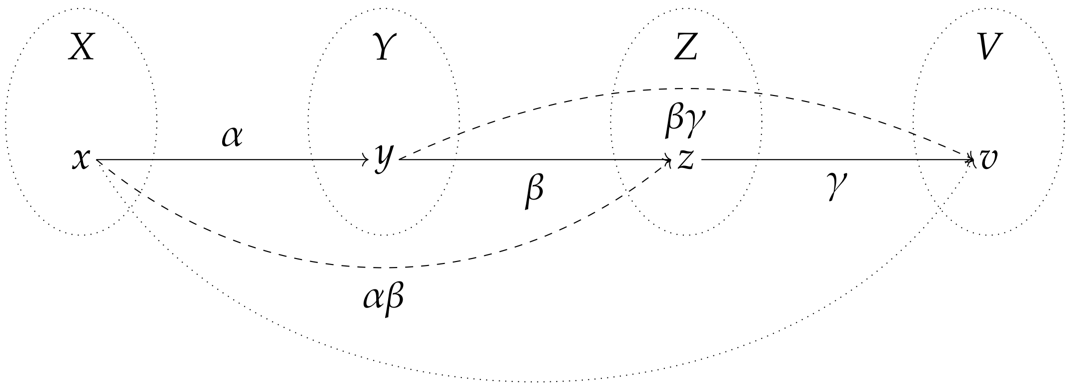
$$f = \begin{pmatrix} x \\ xf \end{pmatrix}_{x \in X},$$

where all $xf \in Y$. This is usually denoted by $f : X \longrightarrow Y$. The element xf is called the value of the mapping f at the element x. A transformation, as defined in Section 1.1, is just a mapping from a set to itself (generally called as self-mapping). Let now X, Y, Y', Z be sets such that $Y \subset Y'$ and let $f : X \longrightarrow Y$ and $g : Y' \longrightarrow Z$ be two mappings. In this situation, we can define the *product* or the *composition* fg (also denoted as $f \circ g$) of f and g by the following rule: The composition fg is the mapping from X to Z such that for all $x \in X$ we have $x(fg) = xf(g)$. In particular, we can always compose two total transformations of the same set and the result will be a total transformation of this set.

The above definition admits a straightforward generalization to partial mappings. A partial mapping from X to Y is a mapping $\alpha : X' \longrightarrow Y$, where $X' \subset X$. In this case, we say that the partial mapping α is defined on elements from X'. Again, a partial transformation, as defined in Section 1.1, is a partial mapping from a set to itself. One usually abuses notation and writes $\alpha : X \longrightarrow Y$ just emphasizing that α is a partial mapping. Let $\alpha : X \longrightarrow Y$ and $\beta : Y \longrightarrow Z$ be two partial mappings. We define their product or composition $\alpha\beta$ as the partial mapping, defined on all those $x \in X$ for which α and β are defined on the elements x and $x\alpha$, respectively; on such x the value of $\alpha\beta$ is given by $x(\alpha\beta) = x\alpha(\beta)$. In particular, we can always compose two partial transformations of the same set and the result will be another partial transformation of this set. We also note that the definition of the composition of total transformations is just a special case of that of partial transformations.

Proposition 1.2.1. *The composition of (partial) mappings is associative, that is, if α, β, and γ are partial mappings, then the composition $(\alpha\beta)\gamma$ is defined if and only if the composition $\alpha(\beta\gamma)$ is defined, and if they both are defined, we have $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.*

Proof. Follows immediately from the following picture:



$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

That is,

$$(\alpha\beta)\gamma = \alpha(\beta\gamma) \ \forall \ \alpha, \beta, \gamma \in \mathcal{PT}_n$$

$\square$

Associativity of the composition of partial transformations naturally leads to the notion of a semigroup.

Definition 1.2.2. Let S be a nonempty set, and let $\cdot : S \times S \longrightarrow S$ be a binary operation on S. Then $(S, \cdot)$ is called a *semigroup* provided that $\cdot$ is associative, that is, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c, \in S$. To simplify the notation, in the case when the operation $\cdot$ is clear from the context one usually writes S for $(S, \cdot)$. Furthermore, one usually writes ab instead of $a \cdot b$.

Lemma 1.2.3. *Let $(S, \cdot)$ be a semigroup. Then, the value of the product $a_1 a_2 \dots a_n$, where all $a_i \in S$, does not depend on the way of computing it (i.e., of putting brackets into this product).*

Proof. We use induction on n. For $n = 2$, the rusult is trivially true. Also, for $n = 3$ the result follows by associativity.

Assume that the result is true upto $n-1$. Then the value of the product $a_1 a_2 \dots a_{n-1}$ is independent of arrangement of brackets in it.

Consider the product $a_1 a_2 \cdots a_n$. We can group it as $(a_1 a_2 \dots a_{n-1}) a_n$ or $a_1(a_2 \dots a_n)$. By the inductive hypothesis, both groupings yield the same value.

Consequently, by principle of mathematical induction, the desired conclusion follows. □

Let $(S, \cdot)$ be a semigroup. From Lemma 1.2.3, it follows that for every $a \in S$ we have a well-defined element $a^k = \underbrace{a \cdot a \cdots a}_{k-times}$.

The number of elements in S is called the *cardinality* of S and is denoted by $|S|$.

By Proposition 1.2.1, in both $\mathcal{T}_n$ and $\mathcal{PT}_n$ the composition of (partial) transformations is an associative operation. Hence we have the following Proposition:

Proposition 1.2.4. *Both $\mathcal{T}_n$ and $\mathcal{PT}_n$ are semigroups with respect to the composition of (partial) transformations.* ■

Definition 1.2.5. The semigroup $\mathcal{T}_n$ is called the *full transformation semigroup* on the set $\mathbf{N}$ or the *symmetric semigroup* of all transformations of $\mathbf{N}$.

The semigroup $\mathcal{PT}_n$ is called the semigroup of all *partial transformations* on $\mathbf{N}$.

Definition 1.2.6. A non-empty subset T of a semigroup $(S, \cdot)$ is called a *subsemigroup* of S provided that T is closed with respect to $\cdot$ (that is, $a \cdot b \in T$ as soon as $a, b \in T$). Obviously, in this case, T itself is a semigroup with respect to the restriction of the operation $\cdot$ to T. The fact that T is a subsemigroup of S is usually denoted by $T < S$.

Lemma 1.2.7. *Show that for arbitrary $\alpha, \beta \in \mathcal{PT}_n$,*

(a) $dom(\alpha\beta) \subset dom(\alpha)$.

(b) $im(\alpha\beta) \subset im(\beta)$.

(c) $rank(\alpha\beta) \leq min(rank(\alpha), rank(\beta))$.

Proof. Let's consider an arbitrary element $x \in dom(\alpha\beta)$. This means that $\alpha\beta$ is defined at x. By the definition of composition of partial transformations, this implies that α is defined at x. Since α is defined at x, we have $x \in dom(\alpha)$. Hence, $dom(\alpha\beta) \subset dom(\alpha)$. Which proves (a).

Now if $y \in im(\alpha\beta)$, then there exists $z \in \mathbf{N}$ such that $z\alpha\beta = y$.

$$\begin{aligned} & or\ \ k\beta = y \text{ where } k = z\alpha \\ \Longrightarrow\ & y \in im(\beta) \\ \Longrightarrow\ & im(\alpha\beta) \subset im(\beta). \end{aligned}$$

Which proves (b). Also, above gives,

$$rank(\alpha\beta) \leq rank(\beta).$$

Finally, since $\alpha\beta$ can only be defined on those elements of $dom(\alpha)$ that are mapped by α into $dom(\beta)$. Thus, $|im(\alpha\beta)|$ is limited by both $|im(\alpha)|$

(since we are first applying α) and $|im(\beta)|$ (as β is applied afterwards). Consequently,

$$rank(\alpha\beta) \leq rank(\alpha).$$

And, therefore,

$$rank(\alpha\beta) \leq min\{rank(\alpha), rank(\beta)\}, \text{ as required.}$$

□

1.3

Identity Element

An element e of a semigroup S is called a *left* or a *right identity* provided that $ea = a$, or $ae = a$, respectively, for all $a \in S$. An element x, which is a left and a right identity at the same time, is called a *two-sided identity* or simply an *identity*.

If S contains some left identity e_l and some right identity e_r we have $e_l = e_l \cdot e_r = e_r$ and hence these two elements coincide. Hence in this case S contains a unique identity element, which is, moreover, a two-sided identity. However, a semigroup may contain many different left identities or many different right identities. It is possible for a semigroup to contain neither left nor right identities. An example of such a semigroup is the semigroup $(\mathbb{N}, +)$. Another example is the semigroup $(\{2, 3, 4, ...\}, \cdot)$ with respect to the ordinary multiplication.

Definition 1.3.1. A semigroup which contains a two-sided identity element is called a *monoid*.

Example 1.3.2. Let S be a nonempty set. For $a, b \in S$, define $a \cdot b = b$. Then, each element in $(S, \cdot)$ is a left identity. Similarly, if we define $a * b = a$ for all $a, b \in S$, then each element in $(S, *)$ is a right identity.

Proposition 1.3.3. *Each semigroup can be extended to a monoid by adding at most one element.*

Proof. Let $(S, \cdot)$ be a semigroup. If S contains an identity, we have nothing to prove. If S does not contain any identity element, consider the set $S^1 = S \cup \{1\}$, where $1 \notin S$. Define the binary operation $*$ on S^1 as follows: For $a, b \in S^1$ set

$$a * b = \begin{cases} a \cdot b, & a, b \in S; \\ a, & b = 1; \\ b, & a = 1. \end{cases}$$

A direct calculation shows that $*$ is associative, hence S^1 is a semigroup. Furthermore, from the definition of $\cdot$ we have that 1 is the identity element in S^1. Moreover, the restriction of the operation $*$ to S coincides with the original operation $\cdot$. Hence S is a subsemigroup of S_1. □

Denote by $\varepsilon_n : \mathbf{N} \longrightarrow \mathbf{N}$ the identity transformation

$$\varepsilon_n = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}.$$

If n is clear from the context we shall sometimes write ε instead of ε_n.

Proposition 1.3.4. *The transformation ε_n is the (two-sided) identity element in both $\mathcal{T}$ and $\mathcal{PT}_n$. In particular, both $\mathcal{T}_n$ and $\mathcal{PT}_n$ are monoids.*

Proof. Let α be any (partial) transformation given by

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ 1\alpha & 2\alpha & \dots & n\alpha \end{pmatrix}.$$

Then, clearly $\alpha\varepsilon_n = \alpha = \varepsilon_n\alpha$. Consequently , the result follows. □

Definition 1.3.5. Let S be a monoid with the identity element 1. An element $a \in S$ is called *invertible* or a *unit* provided that there exists $b \in S$ such that $ab = ba = 1$.

Remark 1.3.6. If $a \in S$ be such that there exists $b \in S$ with $ab = 1 = ba$ then such a b is unique. To see this, let b' be such that $ab' = 1 = b'a$. Then, we have

$$b = b \cdot 1 = b(ab') = (ba)b' = 1 \cdot b' = b'.$$

□

The element b is called the *inverse* of a and is denoted by a^{-1}. Note that if b is the inverse of a, then a is the inverse of b. In other words, $(a^{-1})^{-1} = a$. The set of all invertible elements of the monoid S is denoted by S^*. Note that $1 \in S^*$ since $1 \cdot 1 = 1$. In particular, S^* is not empty.

The above terminology and notation deserve some explanation. Usually the operation in an abstract semigroup is thought of as a multiplication. Since the element 1 is the identity element in such multiplicative semigroups as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, it is natural to denote the identity element of an abstract semigroup by the same symbol 1. This also justifies the notions "unit" and "inverse." However, there are many semigroups where the operation is not the multiplication, for example the semigroup $(\mathbb{Z}, +)$. The identity element in this semigroup is the number 0 and not the number 1. And the inverse of the number $n \in Z$ is the number $-n$ and not the number n^{-1} (note that the latter one is not always defined, and when it is defined, it is not an integer in general).

Definition 1.3.7. A monoid in which each element has an inverse is called a *group*.

Proposition 1.3.8. *Let S be a monoid with the identity element 1. Then S^* is a group.*

Proof. Obviously, if $a \in S^*$, then $a^{-1} \in S^*$ as well. If $a, b \in S^*$, then we have

$$ab \cdot b^{-1}a^{-1} = a \cdot bb^{-1} \cdot a^{-1} = a \cdot 1 \cdot a^{-1} = aa^{-1} = 1.$$

Analogously one shows that $b^{-1}a^{-1} \cdot ab = 1$ and hence $b^{-1}a^{-1} = (ab)^{-1}$. In particular, $ab \in S^*$. Thus S^* is a submonoid of S and each element of S^* has an inverse in S^*. The claim follows. □

Proposition 1.3.9. *Let $\alpha \in \mathcal{T}_n$ or $\alpha \in \mathcal{PT}_n$. Then α is invertible if and only if α is a permutation on* **N**.

Proof. Assume that α is invertible and β is a (partial) transformation such that $\alpha\beta = \beta\alpha = \varepsilon$. Note that $dom(\varepsilon) = \mathbf{N}$. Hence Lemma 1.2.7 (a) implies $dom(\alpha) = \mathbf{N}$. Further, if $x, y \in \mathbf{N}$ are such that $x \neq y$, then $x\varepsilon \neq y\varepsilon$. If $x\alpha = y\alpha$, we would get $x\varepsilon = x\alpha\beta = y\alpha\beta = y\varepsilon$, a contradiction. This means that $x\alpha \neq y\alpha$. Hence α is everywhere defined and injective and thus is a permutation by Proposition 1.1.7.
Conversely, if

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}.$$

is a permutation, the element

$$\alpha = \begin{pmatrix} i_1 & i_2 & \dots & i_n \\ 1 & 2 & \dots & n \end{pmatrix}.$$

is a permutation as well and a direct computation shows that $\alpha\beta = \beta\alpha = \varepsilon$, that is, α is invertible. □

Remark 1.3.10. *The group $\mathcal{T}_n^* = \mathcal{PT}_n^*$ of all permutations on* $\mathbf{N}$ *is called the symmetric group on* $\mathbf{N}$ *and is denoted by* $\mathcal{S}_n$.

1.4 Zero Element

An element 0 of a semigroup S is called a *left* or a *right zero* provided that $0a = 0$, or $a0 = 0$, respectively, for all $a \in S$. An element 0 which at the same time is a left and a right zero, is called a *two − sided zero* or simply a *zero*.

Remark 1.4.1. If S contains some left zero 0_l and some right zero 0_r, we have $0_l = 0_l \cdot 0_r = 0_r$ and hence these two elements coincide. Hence in this case S contains a *unique* zero element, which is, moreover, a two-sided zero.

Proposition 1.4.2. *Each semigroup can be extended to a semigroup with zero by adding at most one element.*

Proof. Let $(S, \cdot)$ be a semigroup. If S contains a zero, we have nothing to prove. Otherwise, consider the set $S^0 = S \cup \{0\}$, where $0 \notin S$. Define the binary operation $*$ on S^0 as follows: For $a, b \in S^0$ set

$$a * b = \begin{cases} a \cdot b, & a, b \in S; \\ 0, & b = 0; \\ 0, & a = 0. \end{cases}$$

A direct calculation shows that $*$ is associative, hence S^0 is a semigroup. Furthermore, from the definition of $*$ we have that 0 is the zero element in S^0 and that the restriction of $*$ to S coincides with $\cdot$. Hence S is a subsemigroup of S^0. □

Notation 1.4.3. Denote by $\mathbf{0}_n$ the partial transformation $\mathbf{0}_n : \varnothing \longrightarrow \mathbf{N}$ on $\mathbf{N}$. We have $dom(\mathbf{0}_n) = \emptyset$. If n is clear from the context, we shall usually write simply $\mathbf{0}$ instead of $\mathbf{0}_n$.

Proposition 1.4.4. *$\mathbf{0}_n$ is the zero element of the semigroup $\mathcal{PT}_n$.*

Proof. The equalities $\alpha\mathbf{0}_n = \mathbf{0}_n\alpha = \mathbf{0}_n$; $\alpha \in \mathcal{PT}_n$, are obvious. □

Definition 1.4.5. For $a \in \mathbf{N}$ define the total transformation $0_a : \mathbf{N} \longrightarrow \mathbf{N}$ via $x \mapsto a$ for all $x \in \mathbf{N}$. The transformation 0_a is called the *constant* transformation.

Proposition 1.4.6. *For $n > 1$ the semigroup $\mathcal{T}_n$ does not contain any left zeros. $\alpha \in \mathcal{T}_n$ is a right zero if and only if $\alpha = 0_a$ for some $a \in \mathbf{N}$. In particular, the semigroup $\mathcal{T}_n$ contains exactly n right zeros.*

Proof. From the obvious equality $\beta 0_a = 0_a$ for any $\beta \in \mathcal{T}_n$ we obtain that each constant transformation on $\mathcal{T}_n$ is a right zero. Hence $\mathcal{T}_n$ contains at least n different right zeros. In particular, for $n > 1$ the semigroup $\mathcal{T}_n$ cannot contain any left zero.

Note that constant transformations are the only transformations of rank 1. Let $\alpha \in \mathcal{T}_n$ be a transformation of rank at least 2. Then for any 0_a the rank of $0_a\alpha$ is 1 by Lemma 1.2.7 (c) and hence the equality $0_a\alpha = \alpha$ is not possible. This implies that α is not a right zero of $\mathcal{T}_n$.

□

1.5 Isomorphism of Semigroups

Let $(S, \cdot)$ and $(T, *)$ be two semigroups. The semigroups S and T are said to be *isomorphic*, denoted by $S \cong T$, provided that there exists a bijection

$\phi : S \longrightarrow T$ such that

$$a\phi * b\phi = (a \cdot b)\phi \; for\ all\ a, b \in S \tag{1.4}$$

The bijection ϕ is called an *isomorphism* from S to T. Note that if $\phi : S \longrightarrow T$ is an isomorphism, then $\phi^{-1} : T \longrightarrow S$ is an isomorphism as well.

Example 1.5.1. It is handy to present small semigroups using their multiplication tables, which is also called *Cayley tables*. Such table is a square matrix with $|S|$ rows and $|S|$ columns, which are indexed by the elements of S. At the intersection of the a^{th} row and the b^{th} column, a, $b \in S$, one writes the product ab. For example, the semigroup $T_2 = \{\varepsilon, (12), 0_1, 0_2\}$ has the following Cayley table:

$\cdot$	ε	(12)	0_1	0_2
ε	ε	(12)	0_1	0_2
(12)	(12)	ε	0_2	0_1
0_1	0_1	0_1	0_1	0_1
0_2	0_2	0_2	0_2	0_2

(1.5)

The following set of 2×2 matrices is also a semigroup with respect to the usual matrix multiplication

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

To simplify the notation we denote

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

The Cayley table for S has the following form

$\cdot$	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	B	B	B
C	C	C	C	C

(1.6)

It is easy to check that the bijection

$$\phi = \begin{pmatrix} \varepsilon & (12) & 0_1 & 0_2 \\ E & A & B & C \end{pmatrix}$$

transforms the Cayley table (1.5) to the Cayley table (1.6) and hence is an isomorphism from $\mathcal{T}_2$ to S.

The primary importance of the semigroup $\mathcal{T}_n$ is revealed by the following statement.

Theorem 1.5.2. *(Cayley's Theorem) Each finite semigroup S of cardinality n is isomorphic to a subsemigroup of either $\mathcal{T}_n$ or $\mathcal{T}_{n+1}$.*

Proof. First we assume that S contains the identity element 1. Let $S = \{a_1 = 1, a_2, \cdots, a_n\}$ and define the mapping $\phi : S \longrightarrow \mathcal{T}_n$ as follows

$$a\phi = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix},$$

where for $k = 1, \dots, n$, the number i_k is uniquely determined by the equality $aa_k = a_{i_k}$.

Since $a_k \cdot 1 = a_k$, we have $1(a_i\phi) = i$ for all $i = 1, \dots, n$. In particular,

$a_i\phi \neq a_j\phi$ if $i \neq j$ and hence the mapping ϕ is injective. The mapping $\phi : S \longrightarrow S\phi$ is therefore bijective.

Further, the equality $(ab)a_k = a(ba_k)$ implies the equality $(k)(ab)\phi = ((k)(b)\phi)a\phi$ for all $k = 1, \dots, n$ and $a, b \in S$. This means that $(ab)\phi = a\phi \cdot b\phi$ for all $a, b \in S$ and thus ϕ is an isomorphism from S to the subsemigroup Sϕ of $\mathcal{T}_n$.

If S does not contain any identity element, we can consider S as a subsemigroup of the semigroup S^1 and $|S^1| = n + 1$. By the above, S^1 is isomorphic to a subsemigroup of $\mathcal{T}_{n+1}$, and the claim follows. □

1.6 Regular and Inverse Elements

An element a of a semigroup S is called *regular* provided that there exists $b \in S$ such that $aba = a$. Elements $a, b \in S$ form a pair of inverse elements provided that $aba = a$ and $bab = b$. Set

$$V_S(a) = \{b \in S : a \text{ and } b \text{ is a pair of inverse elements}\}.$$

If a and b is a pair of *inverse elements*, then one says that a is an inverse of b and b is an inverse of a. This might be slightly confusing, since in such generality the inverse element of a given element is not uniquely defined. Note that this notion extends the notion of an inverse element for invertible elements: if $a \in S$ is invertible, then $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Hence a and a^{-1} is a pair of inverse elements.

Lemma 1.6.1. *Let $a \in S$ be invertible. Show that $V_S(a) = \{a^{-1}\}$.*

Proof. If possible, assume $b \in V_S(a)$ then

$$aba = a$$

$$\text{or } aba \cdot a^{-1} = a \cdot a^{-1}$$
$$\Longrightarrow ab = 1, \text{ the identity element of S.}$$

Therefore, by Remark 1.3.6, $b = a^{-1}$ as required. □

Remark 1.6.2. If a has at least one inverse (i.e., $V_S(a) \neq \emptyset$), then the element a is obviously regular. The converse is also true.

Proposition 1.6.3. *Let $a \in S$ be regular and $b \in S$ be such that $aba = a$. Then a and $c = bab$ is a pair of inverse elements.*

Proof. Follows from the following computation

$$aca = a \cdot bab \cdot a = aba \cdot ba = aba = a$$
$$cac = bab \cdot a \cdot bab = b \cdot aba \cdot bab = b \cdot aba \cdot b = bab = c.$$

□

Definition 1.6.4. The semigroup S is called *regular* provided that every element of S is regular. Obviously, each group is a regular semigroup.

Theorem 1.6.5. *The semigroups $\mathcal{T}_n$ and $\mathcal{PT}_n$ are regular.*

Proof. Let $\alpha \in \mathcal{PT}_n$. Let us construct the element $\beta \in \mathcal{T}_n$ as follows: For each $x \in im(\alpha)$ take some $y \in \mathbf{N}$ such that $y\alpha = x$ and set $x\beta = y$. For $x \notin im(\alpha)$ define $x\beta = 1$. A direct calculation shows that $\alpha\beta\alpha = \alpha$. It follows that both $\mathcal{T}_n$ and $\mathcal{PT}_n$ are regular semigroups. □

Theorem 1.6.6. *Let $\alpha, \beta \in \mathcal{PT}_n$. Then the element β is inverse to the element α if and only if the following conditions are satisfied:*

(a) $dom(\beta) \supset im(\alpha)$

(b) $a\beta \in \{x \in \mathbf{N} : x\alpha = a\}$ for all $a \in im(\alpha)$.

(c) $im(\beta) = (im(\alpha))\beta$

Proof. Let $im(\alpha) = \{a_1, a_2, \dots, a_k\}$. For $i = 1, \dots, k$ set $B_i = \{x \in \mathbf{N} : x\alpha = a_i\}$. Then the equality $\alpha\beta\alpha = \alpha$ is equivalent to the following condition:

$$a_i\beta \in B_i \textit{ for all } i = 1, 2, \dots, k. \tag{1.7}$$

Assume now that the condition 1.7 is satisfied. Then for all i we have $a_i\beta\alpha = a_i$ and $a_i(\beta\alpha\beta) = a_i\beta$. Set $B = \{y\beta : y \in im(\alpha)\}$. From the previous equality we have $b(\alpha\beta) = b$ for all $b \in B$. As $im(\beta\alpha\beta) \subset B$, the equality $\beta\alpha\beta = \beta$ requires $im(\beta) \subset B$. However, the latter condition is also sufficient. Indeed, as $y\beta \in B$ for any $y \in dom(\beta)$, we have $y(\beta\alpha\beta) = y\beta(\alpha\beta) = y\beta$ and thus $\beta\alpha\beta = \beta$. This completes the proof. □

Theorem 1.6.6 is also true for the semigroup $\mathcal{T}_n$, just in this case the condition (a) is superfluous since it is automatically satisfied.

Corollary 1.6.7. *Let* $\alpha \in \mathcal{PT}_n$, $im(\alpha) = \{a_1, a_2, \dots, a_k\}$, $B_i = \{x \in \mathbf{N} : x\alpha = a_i\}$ *and* $m_i = |B_i|$. *Then*

(i) $|V_{\mathcal{PT}_n}(\alpha)| = m_1 m_2 \dots m_k \cdot (k+1)^{n-k}$;

(ii) if α *is total, then* $|V_{\mathcal{T}_n}(\alpha)| = m_1 m_2 \dots m_k \cdot k^{n-k}$.

Proof. For each $a_i \in im(\alpha)$ we have to choose $a_i\beta \in B_i$. As for different i these choices are independent, we have $m_1 m_2 \dots m_k$ different ways to define β on $im(\alpha)$.

After this we have to define β on the set $N \setminus im(\alpha)$. From Theorem 1.6.6 it follows that the restriction of β to $\mathbf{N} \setminus im(\alpha)$ is an arbitrary (partial in the case of $\mathcal{PT}_n$) mapping from $\mathbf{N} \setminus im(\alpha)$ to the set $\{y\beta : y \in im(\alpha)\}$. The latter set contains exactly k elements. Hence the restriction of β to $\mathbf{N} \setminus im(\alpha)$ can be chosen in k^{n-k} different ways for $\mathcal{T}_n$ and in $(k+1)^{n-k}$ different ways for $\mathcal{PT}_n$. □

Definition 1.6.8. A regular semigroup S is called an *inverse semigroup* provided that each $a \in S$ has a unique inverse element. This inverse element is usually denoted by a^{-1} (this can now be justified by the requirement that it is unique).

From Corollary 1.6.7 it follows that in the case of $n > 1$ the semigroups $\mathcal{T}_n$ and $\mathcal{PT}_n$ are not inverse semigroups.

Remark 1.6.9. Since the notion of an inverse element is a natural extension of the corresponding notion for invertible elements, inverse semigroups are very natural generalizations of groups. In some sense they form the class of semigroups, which is "closest" to groups.

1.7 Relations and Equivalences

Definition 1.7.1 (Binary Relation)**.** Let S be a non-empty set. Then any subset ρ of $S \times S$ is said to be a relation over S. In other words, a relation is a rule that is defined between two elements in S. Intuitively, if ρ is a relation over S, then the statement $a\rho b$ is either true or false for all $a, b \in S$.

Example 1.7.2. Here are some examples on binary relations;

1. The empty relation $\emptyset \subset S \times S$.
2. The universal relation $S \times S$.
3. The equality relation $1_S = \{(s, s) : s \in S\}$, also known as diagonal relation in which two elements are related iff they are equal.

Notation 1.7.3. We shall denote the set of all binary relations on S by $\mathcal{B}_S$.

Definition 1.7.4. Binary relation composition is defined on $\mathcal{B}_S$ by the rule that, for all $\rho, \sigma \in \mathcal{B}_S$,

$$\rho \circ \sigma = \{(x, y) \in S \times S : \exists\, z \in S \text{ such that } (x, z) \in \rho \;\&\; (z, y) \in \sigma\} \quad (1.8)$$

Proposition 1.7.5. $(\mathcal{B}_S, \circ)$ *is a semigroup.*

Proof. We only need to show that $\circ$ is associative. To see this, let $\rho, \sigma, \tau \in \mathcal{B}_S$, then

$(x, y) \in (\rho \circ \sigma) \circ \tau$
$\iff \exists\, z \in S$ *such that* $(x, z) \in \rho \circ \sigma$ *and* $(z, y) \in \tau$
$\iff \exists\, u \in S$ *such that* $(x, u) \in \rho$, $(u, z) \in \sigma$, *and* $(z, y) \in \tau$
$\iff (x, u) \in \rho$ *and* $(u, y) \in \sigma \circ \tau$
$\iff (x, y) \in \rho \circ (\sigma \circ \tau)$. □

Notation 1.7.6. If $\rho \in \mathcal{B}_S$, then we have following notations;

1. We shall write $\rho^2, \rho^3, \ldots$, instead of $\rho \circ \rho, \rho \circ \rho \circ \rho, \ldots$.
2. Domain of ρ: $dom\rho := \{s \in S : \exists\, t \in S \text{ with } (s, t) \in \rho\}$.
3. Image of ρ: $im\rho := \{t \in S : \exists\, s \in S \text{ with } (s, t) \in \rho\}$.
4. For each $s \in S$,
$$s\rho := \{t : (s, t) \in \rho\}.$$
5. If $A \subset S$, then
$$A\rho := \cup\{a\rho : a \in A\}.$$
6. $\rho^{-1} := \{(u, v) \in S \times S : (v, u) \in \rho\}$.

Observation 1.7.7.

(i) If $\rho \subset \sigma \implies \rho \circ \tau \subset \sigma \circ \tau$ *and* $\tau \circ \rho \subset \tau \circ \sigma$.

(ii) *If* $\rho \subset \sigma \implies dom\rho \subset dom\sigma$ *and* $im\rho \subset im\sigma$.

(iii) $(\rho^{-1})^{-1} = \rho$.

(iv) *If* $\rho \subset \sigma$ *then* $\rho^{-1} \subset \sigma^{-1}$.

Definition 1.7.8. A relation ρ on a set S is

$$reflexive \text{ if and only if } 1_S \subset \rho, \tag{1.9}$$

$$anti-symmetric \text{ if and only if } \rho \cap \rho^{-1} = 1_S, \tag{1.10}$$

$$transitive \text{ if and only if } \rho \circ \rho \subset \rho, \ and \tag{1.11}$$

$$symmetric \text{ if and only if } (\forall\ s,\ t \in S); (s,t) \in \rho \implies (t,s) \in \rho. \tag{1.12}$$

We define an *equivalence* ρ on a set S to be a relation that is reflexive, transitive, and symmetric.

It is easy to see that (1.12) can be written compactly as $\rho \subset \rho^{-1}$. Then, by Observation 1.7.7 (iii) and Observation 1.7.7 (iv), it then follows that $\rho^{-1} \subset \rho$; thus the symmetry condition can be equally well expressed as $\rho^{-1} = \rho$.
Again, if ρ is an equivalence, then by Observation 1.7.7 (i) we can deduce that

$$\rho = 1_S \circ \rho \subset \rho \circ \rho;$$

thus the transitivity condition can replaced by $\rho \circ \rho = \rho$.

Notation 1.7.9. If ρ is an equivalence on S, we shall sometimes write $s\,\rho\,t$ or $x \equiv y(mod\rho)$ as alternatives to $(s,t) \in \rho$. If there is no confusion about ρ then we shall simply write $s \equiv t$.

Remark 1.7.10. If ρ is an equivalence on S then, by Observation 1.7.7 (ii),

$$dom\rho \supset dom1_S = S, \quad im\rho \supset im1_S = S;$$

hence $dom\rho = im\rho = S$.

Proposition 1.7.11. *If $\phi : X \longrightarrow Y$ is a map, then $\phi \circ \phi^{-1}$ is an equivalence.*

Proof. The easiest way to see this is to note that

$$\begin{aligned}
\phi \circ \phi^{-1} &= \{(x, y) \in X \times X : (\exists z \in Y)\,(x, z) \in \phi,\ (y, z) \in \phi^{-1}\} \\
&= \{(x, y) \in X \times X : x\phi = z \text{ and } z\phi^{-1} = y\} \\
&= \{(x, y) \in X \times X : (x\phi)\phi^{-1} = y\} \\
&= \{(x, y) \in X \times X : x\phi = y\phi\}.
\end{aligned}$$

It is then clear that $\phi \circ \phi^{-1}$ is reflexive, symmetric, and transitive.
To see this,

$$\begin{aligned}
&(reflexive) \text{ since } x\phi = x\phi \implies (x, x) \in \phi \circ \phi^{-1}. \\
&(symmetric) \text{ if, for } x, y \in S, (x, y) \in \phi \circ \phi^{-1} \implies x\phi = y\phi \\
&\qquad\qquad \implies y\phi = x\phi \implies (y, x) \in \phi \circ \phi^{-1} \\
&(transitive) \text{ if } (x, y), (y, z) \in \phi \circ \phi^{-1} \implies x\phi = y\phi \text{ and } y\phi = z\phi \\
&\qquad\qquad \implies x\phi = z\phi \implies (x, z) \in \phi \circ \phi^{-1}.
\end{aligned}$$

□

Remark 1.7.12. We call the equivalence $\phi \circ \phi^{-1}$ the kernel of ϕ and write $\phi \circ \phi^{-1} = \ker \phi$.

1.8 Congruences

Let S be a semigroup. A relation ρ on S is said to be

- *left compatible* if

$$(\forall\ s, t, a \in S), (s, t) \in \rho \implies (as, at) \in \rho$$

- *right compatible* if

$$(\forall\ s,\ t,\ a\ \in\ S),\ (s,t)\ \in\ \rho\ \implies\ (sa,ta)\ \in\ \rho$$

- and *compatible* if

$$(\forall\ s,\ t,\ s^{'},\ t^{'}\ \in\ S),\ (s,t)\ \&\ (s^{'},t^{'})\ \in\ \rho\ \implies\ (ss^{'},tt^{'})\ \in\ \rho$$

Definition 1.8.1. A left (right) compatible equivalence is called a *left (right) congruence*. A compatible equivalence is called a *congruence*.

Proposition 1.8.2. *An equivalence relation ρ on a semigroup S is a congruence if and only if it is both left and right compatible.*

Proof. Suppose first that ρ is a congruence. If $(s,t) \in \rho$ and $a \in S$ then $(a,a) \in \rho$ by reflexivity and so $(as,at) \in \rho$ and $(sa,ta) \in \rho$. Thus ρ is both left and right compatible.

Conversely, suppose that ρ is both a left and a right congruence, and let (s,t), $(s^{'},t^{'}) \in \rho$. Then $(ss^{'},ts^{'}) \in \rho$ by right compatibility and $(ts^{'},tt^{'}) \in \rho$ by left compatibility. Hence $(ss^{'},tt^{'}) \in \rho$ by transitivity. Thus ρ is a congruence. □

Lemma 1.8.3. *Prove that the intersection of an arbitrary family of congruences on S is a congruence on S.*

Proof. Let Λ be the collection of all possible congruences on S.
If (s,t), $(s^{'},t^{'}) \in \cap_{\rho\in\Lambda}\rho$
$\implies$ (s,t), $(s^{'},t^{'}) \in \rho\ \ \forall\ \ \rho$
$\implies$ $(ss^{'},tt^{'}) \in \rho$
$\implies$ $(ss^{'},tt^{'}) \in \cap_{\rho\in\Lambda}\rho$
Hence, arbitrary intersection of congruences is again a congruence. □

Notation 1.8.4. Let S be a semigroup and ρ is any relation on S. Since arbitrary intersection of congruences is again a congruence, then we denote the intersection of all the congruences on S containing ρ by $\rho^\sharp$, read as rho-sharp. Here, $\rho^\sharp$ is said to be generated by ρ.

Note that $\rho^\sharp$ is the smallest congruence on S containing ρ being the intersection of all the congruences on S containing ρ.

1.9 Ideals

Notation 1.9.1. Let S be a semigroup. If $A, B \subset S$, we set

$$AB = \{ab : a \in A, b \in B\}.$$

Definition 1.9.2. A subset $I \subset S$ is called a *left ideal* provided that for all $a \in S$ and $b \in I$ we have $ab \in I$ (in other words, $SI \subset I$). Analogously, I is called a *right ideal* provided that $IS \subset I$. Left and right ideals are also called *one-sided ideals*. A subset $I \subset S$ is called a *two-sided ideal* or simply an *ideal* provided that it is both a left and a right ideal.

Let I be a (one-sided) ideal of S. Then we have $I \cdot I \subset I$ by definition and hence I is a subsemigroup of S. The converse is not true in the general case. For example, the group $\mathcal{S}_n$ is a subsemigroup of each of the semigroups $\mathcal{PT}_n$ and $\mathcal{T}_n$. However, for each (partial) transformation $\alpha \notin \mathcal{S}_n$, both sets $\alpha\mathcal{S}_n$ and $\mathcal{S}_n\alpha$ do not have any common elements with $\mathcal{S}_n$.

It is easy to see that both the intersection and the union of an arbitrary family of left (right, two-sided) ideals of S are in turn a left (resp. right, two-sided) ideal of S.

Notation 1.9.3. For each semigroup S, we denote the semigroup S by S^1 provided that S contains an identity element, and the semigroup constructed in Proposition 1.3.3 provided that S does not contain any identity element. Analogously, we define S^0. We also note that for $k > 1$, the notation S^k means something completely different, namely, $S^k = \{a_1 a_2 \cdots a_k : a_i \in S, i = 1, \ldots, k\}$.

Definition 1.9.4. A left (resp. right or two-sided) ideal I of S is called *principal* provided that there exists $a \in S$ such that $I = S^1 a$ (resp. $I = aS^1$, $I = S^1 a S^1$). The element a is called the *generator* of the ideal I. Note that $a \in S^1 a$, $a \in aS^1$, and $a \in S^1 a S^1$ by definition.

Proposition 1.9.5. *Each left (right or two-sided) ideal is a union of principal left (resp. right or two-sided) ideals.*

Proof. Since $a \in S^1 a$, if $I \subset S$ is a left ideal, we have $I = \bigcup_{a \in I} S^1 a$. For other ideals, the argument is similar. □

1.10 Green's Relations

In this section, we introduce several equivalence relations on semigroups, which play a central role in the structure theory. Let S be a semigroup. Elements $a, b \in S$ are called *$\mathcal{L}$-equivalent* provided that they generate the same principal left ideal. In other words, $a\mathcal{L}b$ if and only if $S^1 a = S^1 b$. Equivalence classes of the relation $\mathcal{L}$ are called $\mathcal{L}$-classes. For $a \in S$, the $\mathcal{L}$-class containing a will be denoted by $\mathcal{L}(a)$. In other words, $a\mathcal{L}b$ if and only if $a \in \mathcal{L}(b)$. The following easy but very useful fact follows immediately from the definition.

Proposition 1.10.1. *$a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $a = xb$ and $b = ya$.* □

In the dual way, we define the relation $\mathcal{R}$: Elements $a, b \in S$ are called *$\mathcal{R}$-equivalent* provided that they generate the same principal right ideal. In other words, $a\mathcal{R}b$ if and only if $aS^1 = bS^1$. Equivalence classes of the relation $\mathcal{R}$ are called $\mathcal{R}$-classes. For $a \in S$, the $\mathcal{R}$-class containing a will be denoted by $\mathcal{R}(a)$.

Proposition 1.10.2. *$a\mathcal{R}b$ if and only if there exist $x, y \in S^1$ such that $a = bx$ and $b = ay$.* ■

Another immediate property of $\mathcal{L}$ and $\mathcal{R}$ is as follows:

Proposition 1.10.3. *$\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence.* ■

Lemma 1.10.4. *The relations $\mathcal{L}$ and $\mathcal{R}$ commute, that is, $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.*

Proof. Let $(a, b) \in \mathcal{L} \circ \mathcal{R}$. Then there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$. By Propositions 1.10.1 and 1.10.2, there exist $x, y \in S^1$ such that $a = xc$, $b = cy$. Moreover, $c\mathcal{R}b$ implies $xc\mathcal{R}xb$, and $a\mathcal{L}c$ implies $ay\mathcal{L}cy$. But $xc = a$, $xb = xcy = ay$, and $cy = b$. Hence $a\mathcal{R}xcy$ and $xcy\mathcal{L}b$, which implies $(a, b) \in \mathcal{R} \circ \mathcal{L}$, and thus $\mathcal{L} \circ \mathcal{R} \subset \mathcal{R} \circ \mathcal{L}$.
Analogously one shows that $\mathcal{R} \circ \mathcal{L} \subset \mathcal{L} \circ \mathcal{R}$ and hence $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. □

Note that if ξ and η are two equivalences relations on X and $\xi \circ \eta = \eta \circ \xi$, then $\xi \circ \eta$ is again an equivalence relation. Moreover, this product is the minimum equivalence relation which contains both ξ and η.

The minimum equivalence relation on S which contains both $\mathcal{R}$ and $\mathcal{L}$ is denoted by $\mathcal{D}$ and is called the *$\mathcal{D}$-relation*. From Lemma 1.10.4 it follows that $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. All other notions and notation for $\mathcal{D}$ are similar to the ones used for $\mathcal{L}$. In particular, $\mathcal{D}(a)$ denotes the $\mathcal{D}$-class of an element a.

The intersection of two equivalence relations is always an equivalence relation. We define the *$\mathcal{H}$-relation* as the intersection of $\mathcal{R}$ and $\mathcal{L}$.

All other notions and notation for $\mathcal{H}$ are similar to the ones used for $\mathcal{L}$. In particular, $\mathcal{H}(a)$ denotes the $\mathcal{H}$-class of an element a.

Finally, we will say that the elements a and b are $\mathcal{J}$-equivalent provided that they generate the same principal two-sided ideal, that is, $S^1aS^1 = S^1bS^1$. All other notions and notation for $\mathcal{J}$ are similar to the ones used for $\mathcal{L}$. In particular, $\mathcal{J}(a)$ denotes the $\mathcal{J}$-class of an element a.

It is obvious that $\mathcal{R} \subset \mathcal{J}$ and $\mathcal{L} \subset \mathcal{J}$. In particular, it follows that $\mathcal{D} \subset \mathcal{J}$. Hence we have the following diagram depicting the introduced relations on S:

$$\begin{tikzcd}
 & \mathcal{L} \arrow[dr, hook] & & \\
\mathcal{H} \arrow[ur, hook] \arrow[dr, hook] & & \mathcal{D} \arrow[r, hook] & \mathcal{J} \\
 & \mathcal{R} \arrow[ur, hook] & &
\end{tikzcd}$$

The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ on the semigroup S are called *Green's relations* after J. A. Green, who introduced them in 1951.

CHAPTER 2

Order on Transformations and Nilpotent Semigroups

Introduction

This chapter introduces essential concepts crucial for understanding the ranks and presentations of finite transformation semigroups. We focus on order-preserving, order-increasing, and order-decreasing transformations within the context of finite chains, specifically .

We begin by defining partially and totally ordered sets, leading to the introduction of the semigroup of order-preserving transformations. Additionally, we explore the semigroup of order-decreasing transformations and define the Catalan monoid, which consists of transformations that are both order-preserving and order-decreasing.

This chapter also discusses nilpotent semigroups and introduces , the semigroup of order-preserving transformations with one fixed point, demonstrating its nilpotent nature. Key propositions and corollaries are presented to elucidate the properties of these semigroups.

Ultimately, this chapter establishes a foundational understanding of finite transformation semigroups, setting the stage for further exploration in subsequent chapters.

2.1 Basic Definitions

In this section we shall introduce basic concepts used in this dissertation. First we shall define ordered sets;

Definition 2.1.1. A binary relation ω on a set X (that is, a subset ω of $X \times X$) is said to be a (*partial*) *order* if

(1) it is reflexive,

(2) it is anti-symmetric, and

(3) it is transitive.

Traditionally, one writes $x \leq y$ rather than $(x, y) \in \omega$, where $x, y \in X$. We shall follow this convention, and write $x \geq y$, $x < y$, and $x > y$ to respectively mean $(y, x) \in \omega$, $(y, x) \in \omega$ but $x \neq y$, and $(x, y) \in \omega$ but $x \neq y$.
If a partial order satisfies extra property that

(4) $\forall\ x, y \in X$; *either* $x \leq y$ *or* $y \leq x$

then it is called a *total* order. We shall refer to $(X, \leq)$ as a *partially ordered set* if conditions (1), (2), and (3) are satisfied. And, if in addition (4) is also satisfied, then we refer to $(X, \leq)$ as a *totally* ordered set.

Notation. Let's denote the n-element finite chain by X_n. That is, $X_n = \{1, 2, \ldots, n\}$.

Definition 2.1.2 (Order-preserving Transformation). *A transformation α on X_n is said to be order preserving if*

$$x < y \implies x\alpha \leq y\alpha \text{ for all } x, y \in X_n$$

It is easy to see that composition of two order-preserving transformations is again an order-preserving transformation.

Notation 2.1.3. We shall use the symbol O_n to denote the semigroup of all order-preserving transformations on X_n.

Definition 2.1.4. Let $\alpha \in \mathcal{T}_n$ be any transformation on X_n then we say that α is

- *order increasing* if for all $x \in dom(\alpha)$; $x\alpha \geq x$;
- *order decreasing* if for all $x \in dom(\alpha)$; $x\alpha \leq x$.

Notation. We denote by D_n the set of all order-decreasing transformations on X_n.

Catalan Monoid: Let C_n be the set of all order-decreasing and order-preserving transformations. That is, $C_n = D_n \cap O_n$, is the semigroup of all order-decreasing and order-preserving transformations.

Since the identity full transformation ε_n is both order-decreasing and order-preserving full transformation on X_n, therefore $\varepsilon_n \in C_n$. Hence, C_n is a monoid and is called as *catalan monoid*.

Definition 2.1.5. The number $C_n = \frac{1}{n+1}\binom{2n}{n}$ is called the $n^{th}-catalan$ number. Set also $C_0 = 1$ (see [1]).

Also, see that

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{2n!}{n!\,(2n-n)!}$$

$$
\begin{aligned}
&= \frac{1}{n+1}\,\frac{2n!}{n!\,n!} = \frac{1}{n}\,\frac{2n!}{(n+1)!\,(n-1)!}\\
&= \frac{1}{n}\,\frac{2n!}{(n-1)!\,(2n-n+1)!}\\
&= \frac{1}{n}\,\frac{2n!}{(n-1)!\,(2n-(n-1))!}\\
&= \frac{1}{n}\binom{2n}{n-1}
\end{aligned}
$$

Therefore, the n^{th}-catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n}\binom{2n}{n-1}$.

The following result regarding cardiniality of catalan monoid C_n has been proved by Peter M. Higgins (see [9], Theorem 3.1).

Theorem 2.1.6. *Let C_n be the semigroup of all decreasing and order-preserving full transformations of X_n . Then $|C_n| = C_n$, the n^{th} – catalan number.* ■

Definition. A subset of $A \subset X_n$ is called *invariant* for a transformation $\alpha \in \mathcal{T}_n$ if $A\alpha = \{x\alpha : x \in A\} \subset A$.

Definition 2.1.7. Let α be any transformation on X_n and $x \in X_n$, then α is said to fix x if and only if $x\alpha = x$.

We define $Fix(\alpha) := \{x \in X_n : x\alpha = x\}$, to be the set of all the elements of X_n fixed by α.

Notation 2.1.8. Let $A \subset X_n$. We denote the semigroup of all order-preserving transformations which fix exactly the elements of A by $O_{n,A}$.

2.2 Basic Results

If $A = \{p\}$, then we write $O_{n,p}$ instead of $O_{n,A}$. That is, $O_{n,p}$ is the semigroup of all order-preserving transformations such that, for any $\alpha \in O_{n,p}$, we have $x\alpha = x$, if and only if, $x = p$.

In this section, we discuss some basic results for the transformation semigroup $O_{n,p}$. The below Proposition and succeeding Corollary is drawn from the work of Ayik G, Ayik H, and Koç M in [2].

Proposition 2.2.1. *Let* $p \in X_n$ *and let* $\alpha \in O_{n,p}$.

(i) If $1 \leq x < p$, *then* $x + 1 \leq x\alpha$.

(ii) If $p < x \leq n$, *then* $x\alpha \leq x - 1$.

Proof. (i) Let $\alpha \in O_{n,p}$. If $x = 1$, then it is clear that $1\alpha \neq 1$ as p is the only fixed point of α.
Thus, we have

$$2 \leq 1\alpha. \tag{2.1}$$

Consequently the result is true for $x = 1$.

Now, if $1 < x < p$, and if possible assume that $x\alpha < x + 1$. Then

$$x\alpha \leq x. \tag{2.2}$$

Also, since $x - 1 < x$, then by definition of $O_{n,p}$ and (2.2), we have

$$(x - 1)\alpha \leq x\alpha \leq x. \tag{2.3}$$

Again, from (2.2), $x\alpha \neq x$ because otherwise $x\,(< p)$ is a fixed point of α, which is not true as p is the only fixed point of α. Therefore,

$$x\alpha \leq x - 1. \tag{2.4}$$

Put $x = 2$ in (2.3) and using (2.4), we get

$$(2-1)\alpha \leq 2\alpha \leq 2-1 \implies 1\alpha \leq 2\alpha \leq 1 \implies 1\alpha \leq 1,$$

a contradiction to (2.1). Hence, our supposition is wrong. Consequently,

if $1 \leq x < p$, $x + 1 \leq x\alpha$.
(ii) Let $\alpha \in O_{n,p}$. If $x = n$, then it is clear that $n\alpha \neq n$ as $p < n$ is the one fixed point of α. Therefore,

$$n\alpha \leq n - 1$$

and consequently the result is true for $x = n$.
Now, if $p < x < n$ and if possible assume $x\alpha > x - 1$.

$$x\alpha \geq x.$$

But $x\alpha \neq x$, otherwise $x\,(> p)$ is another fixed point of α, a contradiction. Therefore,

$$x\alpha \geq x + 1. \tag{2.5}$$

Also, since $x + 1 > x$, then by definition of $O_{n,p}$,

$$(x + 1)\alpha \geq x\alpha$$
$$\text{or, by (2.5), } (x + 1)\alpha \geq x\alpha \geq x + 1.$$

Let $x = n - 1$, then we have

$$\begin{aligned} &(n - 1 + 1)\alpha \geq (n - 1)\alpha \geq n - 1 + 1 \\ \implies\ & n\alpha \geq n \\ \implies\ & n\alpha = n, \text{ a contradiction.} \end{aligned}$$

Hence, our supposition is wrong.
Consequently, if $p < x \leq n$, $x\alpha \leq n - 1$. □

Corollary 2.2.2. *If $\alpha \in O_{n,p}$, then*

(a) $(p - 1)\alpha = p$ whenever $p > 1$.

(b) $(p + 1)\alpha = p$ *whenever* $p < n$.

Proof. (a) If $p > 1$, then $1 \leq p - 1 < p$. Therefore, by proposition 2.2.1(i),

$$\begin{aligned} x + 1 &\leq x\alpha \\ \implies p - 1 + 1 &\leq (p-1)\alpha \\ \implies p &\leq (p-1)\alpha. \end{aligned}$$

But $p \nless (p - 1)\alpha$, otherwise since $p = p\alpha$ we get
$p\alpha < (p - 1)\alpha$ but $p > p - 1$ and $\alpha \in O_{n,p}$ is an order-preserving transformation. Hence, a contradiction. Therefore,

$$(p - 1)\alpha = p.$$

(b) If $p < n$, then $p < p + 1 \leq n$. Therefore, by proposition 2.2.1(b),

$$\begin{aligned} x\alpha &\leq x - 1 \\ \implies (p+1)\alpha &\leq p + 1 - 1 \\ \implies (p+1)\alpha &\leq p. \end{aligned}$$

But $(p+1)\alpha \nless p$, otherwise, since $p = p\alpha$, we get $(p + 1)\alpha < p = p\alpha$ but $p + 1 > p$ and $\alpha \in O_{n,p}$ is an order-preserving transformation. Hence, a contradiction. Therefore,

$$(p + 1)\alpha = p.$$

□

Definition. Let c_x be that constant map to x, for $x \in X_n$. That is,

$$c_x := \begin{pmatrix} 1 & 2 & \dots & n \\ x & x & \dots & x \end{pmatrix}. \tag{2.6}$$

Note that constant transformation has been defined in chapter 1 (see 1.4.5) and was denoted by 0_x. But in this chapter, we shall stick to above notation.

Remark 2.2.3. *c_1 is the zero element of the catalan monoid C_n.*
To see this, let $\alpha \in C_n$, then α is order-decreasing and order-preseving.
Let

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

then $i_k \le k$ for all $1 \le k \le n$ and $i_l \le i_m$ whenever $l \le m$.
Now,

$$\begin{aligned} \alpha c_1 &= \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \end{pmatrix} = c_1. \end{aligned}$$

Also,

$$\begin{aligned} c_1\alpha &= \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}. \end{aligned}$$

We claim that $i_1 = 1$. Otherwise if $i_1 > 1$, then we get a contradiction to the fact $i_k \le k$ for all $1 \le k \le n$.
Therefore, our claim that $i_1 = 1$ is established.
Thus,

$$c_1\alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \end{pmatrix} = c_1.$$

Consequently, c_1 is the zero element of C_n. □

Remark 2.2.4. *c_p is the zero element of $O_{n,p}$.*
To see this, let $\alpha \in O_{n,p}$. Then,

$$\begin{aligned} \alpha c_p &= \begin{pmatrix} 1 & 2 & \dots & p & \dots & n \\ 1\alpha & 2\alpha & \dots & p & \dots & n\alpha \end{pmatrix} \begin{pmatrix} 1 & 2 & \dots & p & \dots & n \\ p & p & \dots & p & \dots & p \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \dots & p & \dots & n \\ p & p & \dots & p & \dots & p \end{pmatrix} = c_p. \end{aligned}$$

Also

$$\begin{aligned} c_p\alpha &= \begin{pmatrix} 1 & 2 & \dots & p & \dots & n \\ p & p & \dots & p & \dots & p \end{pmatrix} \begin{pmatrix} 1 & 2 & \dots & p & \dots & n \\ 1\alpha & 2\alpha & \dots & p & n\alpha & \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \dots & p & \dots & n \\ p & p & \dots & p & \dots & p \end{pmatrix} = c_p. \end{aligned}$$

Since $\alpha \in O_{n,p}$ was arbitrarily chosen, c_p is the zero element of $O_{n,p}$. □

Lemma 2.2.5. *$O_{n,r}$ is isomorphic to $O_{n,n-r+1}$ for $r \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$, where $\lfloor \frac{n}{2} \rfloor$ is the greatest integer less or equal to $\frac{n}{2}$.*

Proof. To prove this, we proceed as follows:
Define a mapping $\varphi : O_{n,r} \longrightarrow O_{n,n-r+1}$ by

$$x(\alpha\varphi) = n - (n - x + 1)\alpha + 1 \text{ for all } x \in \{1, 2, \dots, n\}.$$

First, we show that φ maps $O_{n,r}$ into $O_{n,n-r+1}$. Let $\alpha \in O_{n,r}$, then

$$(n - r + 1)(\alpha\varphi) = n - (n - (n - r + 1) + 1)\alpha + 1) = n - r\alpha + 1 = n - r + 1.$$

Moreover, for $x < y \in X_n$, $-y < -x \implies n - y + 1 < n - x + 1$
$\implies -(n - x + 1) < -(n - y + 1)$. Thus, we have

$$\begin{aligned} x(\alpha\varphi) &= n - (n - x + 1)\alpha + 1 \\ &\leq n - (n - y + 1)\alpha + 1 \\ &= y(\alpha\varphi). \end{aligned}$$

Consequently, $\alpha\varphi$ is order-preserving fixing only $n - r + 1$ and, therefore, $\alpha\varphi \in O_{n,n-r+1}$.
On the other hand, for $\alpha \in O_{n,n-r+1}$, let $\overline{\alpha}$ be defined by the following rule;

$$x\overline{\alpha} = n - (n - x + 1)\alpha + 1$$

Then,

$$\begin{aligned} r\overline{\alpha} &= n - (n - r + 1)\alpha + 1 \\ &= n - (n - r + 1) + 1 \\ &= n - n + r - 1 + 1 \\ &= r. \end{aligned}$$

Further, it is a routine to check that $\overline{\alpha}$ is order-preserving. Consequently, $\overline{\alpha} \in O_{n,r}$. Also, $x(\bar{\alpha}\varphi) = n-(n-x+1)\bar{\alpha}+1 = n-(n-(n-(n-x+1)+1)\alpha+1)+1 = x\alpha$. So, we have $(\overline{\alpha})\varphi = \alpha$ and, therefore, φ is surjective.

Next, let $\alpha, \beta \in O_{n,r}$ with $\alpha\varphi = \beta\varphi$, then

$$\begin{aligned} & x(\alpha\varphi) = x(\beta\phi) \ \text{ for all } \ x \in X_n \\ \implies & n - (n - x + 1)\alpha + 1 = n - (n - x + 1)\beta + 1 \ \ \forall \ x \in X_n \\ \implies & (n - x + 1)\alpha = (n - x + 1)\beta \ \text{ for all } \ x \in X_n. \end{aligned}$$

Now, if $x = n$, $(n - n + 1)\alpha = (n - n + 1)\beta \implies 1\alpha = 1\beta$
and, if $x = 1$, $(n - 1 + 1)\alpha = (n - 1 + 1)\beta \implies n\alpha = n\beta$. Therefore, $\alpha = \beta$.
Hence, φ is injective.

Finally, let $\alpha, \beta \in O_{n,r}$. Then for $x \in X_n$, we have

$$\begin{aligned} x((\alpha\beta)\varphi) &= n - (n - x + 1)\alpha\beta + 1 \\ &= n - (n - n + (n - x + 1)\alpha - 1 + 1)\beta + 1 \\ &= n - (n - (n - (n - x + 1)\alpha + 1) + 1)\beta + 1 \\ &= (n - (n - x + 1)\alpha + 1)(\beta\varphi) \\ &= (x(\alpha\varphi))(\beta\varphi). \end{aligned}$$

This shows that $(\alpha\beta)\varphi = (\alpha\varphi)(\beta\varphi)$, which completes the proof. □

From above lemma, we can see that $O_{n,1}$ is isomorphic to $O_{n,n}$. Similarly, we shall show that $O_{n,2}$ and $O_{n-1,1}$ are isomorphic.

Lemma 2.2.6. *$O_{n,2}$ and $O_{n-1,1}$ are isomorphic.*

Proof. Let $\phi : O_{n,2} \longrightarrow O_{n-1,1}$ with $x(\alpha\phi) = (x + 1)\alpha - 1$ for all $x \in \{1, \ldots, n - 1\} = X_{n-1}$ and $\alpha \in O_{n,2}$.

Firstly, we show that ϕ maps into $O_{n-1,1}$. Let $\alpha \in O_{n,2}$. Then

$$1(\alpha\phi) = (1 + 1)\alpha - 1 = 2 - 1 = 1.$$

Moreover, for $x < y \in X_{n-1}$, we have

$$x(\alpha\phi) = (x + 1)\alpha - 1 \leq (y + 1)\alpha - 1 = y(\alpha\phi) \leq n - 1.$$

Further, we have

$$x(\alpha\phi) = (x + 1)\alpha - 1 < x + 1 - 1 = x$$

for all $x \in \{2, 3, 4, \ldots, n-1\}$. This shows that $\alpha\phi \in O_{n-1,1}$.

On the other hand, for $\alpha \in O_{n-1,1}$, let $\bar{\alpha} \in T_n$ with $1\bar{\alpha} = 2$ and $x\bar{\alpha} = (x-1)\alpha+1$ ($\leq n$) for all $x \in \{2, \ldots, n\}$. We have $2\bar{\alpha} = (2-1)\alpha+1 = 1+1 = 2$. It is routine to show that $\bar{\alpha}$ is order-preserving and $x\bar{\alpha} < x$ for $x \in \{3, \ldots, n\}$, so $\alpha \in O_{n,2}$.

Further, we have

$$x(\bar{\alpha}\phi) = (x+1)\bar{\alpha} - 1 = (x+1-1)\alpha + 1 - 1 = x\alpha$$

for $x \in X_{n-1}$, i.e., $\bar{\alpha}\phi = \alpha$, which proves that ϕ is surjective.

Next, we verify that ϕ is injective. Let $\alpha, \beta \in O_{n,2}$ with $\alpha\phi = \beta\phi$. For $x \in X_{n-1}$, the assumption $x\alpha\phi = x\beta\phi$ implies $(x+1)\alpha = (x+1)\beta$. Moreover, we have $1\alpha = 2 = 1\beta$, which completes the argument that $\alpha = \beta$.

Finally, let $\alpha, \beta \in O_{n,2}$. Then for $x \in X_{n-1}$, we have

$$\begin{aligned} x((\alpha\beta)\phi) &= (x+1)\alpha\beta - 1 = (((x+1)\alpha - 1) + 1)\beta - 1 \\ &= ((x+1)\alpha - 1)(\beta\phi) = (x(\alpha\phi))(\beta\phi). \end{aligned}$$

This shows that $(\alpha\beta)\phi = (\alpha\phi)(\beta\phi)$, which completes the proof. □

2.3 Nilpotent Semigroups

In this section, we introduce nilpotent semigroup. The definitions and results are partly drawn from the works in [2] and [8].

Definition 2.3.1. An element a of a semigroup S with the zero element 0 is called *nilpotent* or a *nil − element* provided that $a^k = 0$ for some $k \in \mathbb{N}$. The minimal k for which $a^k = 0$ is called the *nilpotency degree or nilpotency class* of the element a and is denoted by $nd(a)$.

The set of all nilpotent elements of S is denoted by $N(S)$.

Definition 2.3.2. The semigroup S with 0 is called as *nilpotent semigroup* if $S^k = \{0\}$ for some positive integer k. That is, the semigroup S is nilpotent if the product of k elements $a_1 a_2 \dots a_k = 0$ for all $a_i \in S$ for all $1 \leq i \leq k$.

Proposition 2.3.3. *Let S be a finite semigroup with the zero element 0. Then the following are equivalent:*

(a) S is nilpotent.

(b) Every element $a \in S$ is nilpotent.

Proof. Assume (a) holds. Let $nd(S) = k$ and $a \in S$ be arbitrary, then $\underbrace{aa\cdots a}_{k-times} = 0$. That is, $a^k = 0$.
Hence, a and consequently every element in S is nilpotent.

Conversely assume that (b) holds. Let $a_1, a_2, \dots, a_k$ be arbitrary elements of S such that $a_1 a_2 \cdots a_k \neq 0$.
For $i = 1, 2, \dots, k$ set $b_i = a_1 a_2 \cdots a_i$ and note that $b_i \neq 0$. Assume that $b_i = b_j$ for some $i < j$ and let $x = a_{i+1} a_{i+2} \cdots a_j$. Then, for all $m \in \mathbb{N}$, we have

$$b_i x^m = (b_i x) x^{m-1} = (b_j) x^{m-1} = b_i x^{m-1} = \dots = b_i$$

Hence, $x^m \neq 0$ for all $m \in \mathbb{N}$, which is a contradiction to (b) (because, $x = \prod_{k=i+1}^{j} a_k \in S$ and (b) holds).
Hence, all elements $b_i : 1 \leq i \leq k$ are different and non-zero. Thus, $k < |S|$.
In particular, $S^{|S|} = \{0\}$ and thus S is nilpotent. □

Remark 2.3.4. Above proposition says that a semigroup S is nilpotent if and only if $N(S) = S$.

Lemma 2.3.5. *$\alpha \in C_n$ is nilpotent if and only if $Fix(\alpha) = \{1\}$.*

Proof. Assume $\alpha \in C_n$ be nilpotent. Then, there exists $k \in \mathbb{N}$ such that $\alpha^k = c_1$, the zero element of C_n (see Remark 2.2.3).
Let $x \in Fix(\alpha)$. Then,

$$x\alpha = x.$$

Repeatedly applying α, we get

$$x\alpha^2 = x\alpha = x, x\alpha^3 = (x\alpha^2)\alpha = x\alpha = x, \ldots,$$
$$x\alpha^n = x \qquad n \in \mathbb{N} \tag{2.7}$$

But since $\alpha^k = c_1$, therefore

$$x\alpha^k = xc_1 = 1. \tag{2.8}$$

Thus, from (2.7), we get

$$x = x\alpha^k = 1.$$

That is,

$$x\alpha = x \text{ implies } x = 1$$

Consequently, since $x \in Fix(\alpha)$ was choosen arbitrary, $Fix(\alpha) = \{1\}$.

Conversely, assume that $Fix(\alpha) = \{1\}$. Let $x_i\,(\neq 1) \in X_n$ then $x_i\alpha \neq x_i$. Consider the sequence $x_i, x_i\alpha, x_i\alpha^2, \ldots$. Since X_n is finite, α is order-decreasing, and 1 is the only fixed point of α, therefore the above sequence must eventually reach 1. Thus, there exists $k_i \in \mathbb{N}$ such that $x_i\alpha^{k_i} = 1$. Since $x_i \in X_n$ was arbitrary chosen, there exists k_i for all $x_i \in X_n$ such that $x_i\alpha^{k_i} = 1$.
Let $k = max\{k_i : i = 1, 2, \ldots, |X_n| - 1\}$, then

$$x\alpha^k = 1 \text{ for all } x\,(\neq 1) \in X_n$$

Also, $1\alpha^k = 1$. Consequently, α is nilpotent. Hence, the result follows. □

Lemma 2.3.6. $O_{n,1} = N(C_n)$

Proof. Let $\alpha \in O_{n,1}$. First we show that $\alpha \in C_n$. To see this, we proceed as follows;
since $p = 1$, then by proposition 2.2.1(ii), we have

$$\begin{aligned} x\alpha &\leq x - 1 \text{ for all } 1 < x \leq n \\ \textit{i.e.}, \; x\alpha &\leq x - 1 < x \\ \Longrightarrow \; x\alpha &< x \text{ for all } x \in (1, n]. \end{aligned}$$

Therefore, α is order-decreasing and hence $\alpha \in C_n$.
Now, since $Fix(\alpha) = \{1\}$ and $\alpha \in C_n$, therefore by lemma 2.3.5, α is nilpotent. This gives,

$$\alpha \in N(C_n).$$

Thus, $O_{n,1} \subset N(C_n)$.

Conversely, let $\alpha \in N(C_n)$, then by lemma 2.3.5, $Fix(\alpha) = \{1\}$. Also, by definition of C_n, α is order-preserving and consequently $\alpha \in O_{n,1}$. Therefore, $N(C_n) \subset O_{n,1}$. Hence, $O_{n,1} = N(C_n)$, as required. □

Lemma 2.3.7. *$O_{n,p}$ is a nilpotent semigroup.*

Proof. Since $O_{n,1} = N(C_n)$ (see lemma 2.3.6), therefore it is enough to consider the case $p \geq 2$.
Let $\alpha \in O_{n,p}$, then by corollary 2.2.2,

$$(p-1)\alpha = (p+1)\alpha = p$$

and, therefore, $(p-1)\alpha^k = (p+1)\alpha^k = p$ for all $k \in \mathbb{N}$.

Now, for $1 \leq x \leq p-2$, $x\alpha^{p-1} = p$, since α is order-preserving, p is the only fixed point of α, and $x+1 \leq x\alpha$. So, there exists atleast one $y \in \{1, 2, \ldots, p-2\}$ such that $y\alpha = p-1$ or else every element is mapped to p (in that case, above is trivially true). Similarly, there exists atleast one $z \in \{1, 2, \ldots, y-1\}$ such that $z\alpha = y$ or else every element is mapped to $p-1$ (again, in thath case, the result follows). Therefore, we have to apply α atmost $p-2$ times such that

$$\begin{aligned} & x\alpha^{p-2} = p-1 \\ \Longrightarrow\ & x\alpha^{p-1} = (p-1)\alpha = p \\ \Longrightarrow\ & x\alpha^{p-1} = p \text{ for all } x \in \{1, 2, \ldots, p-2\}. \end{aligned}$$

Similarly, for $x \in \{p+2, \ldots, n\}$, using the fact that $x\alpha \leq x-1$, we can conclude that $x\alpha^{n-p} = p$.
Let $m = max\{p-1, n-p\}$, then

$$x\alpha^{m} = p \text{ for all } x \in \{1, 2, \ldots, n\}.$$

Consequently, $\alpha^m = c_p$.
Thus, $O_{n,p} \subset N(O_{n,p}) \subset O_{n,p}$. Hence,

$$O_{n,p} = N(O_{n,p}).$$

That is, $O_{n,p}$ is a nilpotent semigroup with zero element c_p. □

Proposition 2.3.8. *Let C_n be the semigroup of all decreasing and order-preserving full transformations of X_n. Further, let $N(C_n)$ be the set of all nilpotents in C_n. Then*

$$|N(C_n)| = C_{n-1} \text{ ; the } (n-1)^{th} \text{ Catalan number.}$$

Proof. The result will certainly follow if we establish a bijection from C_{n-1} onto $N(C_n)$. For all $\alpha \in C_{n-1}$, define a map $\theta : C_{n-1} \to N(C_n)$ by $\alpha \mapsto \alpha^{'}$ where

$$i\alpha^{'} = \max\{1, (i-1)\alpha\}, \quad (i = 1, 2, \ldots, n).$$

Now, since $n \notin \mathrm{Dom}\alpha = X_{n-1}$ and $i\alpha^{'} = (i-1)\alpha < i$ for all $i > 1$, it follows that $i\alpha^{'}$ has the same degrees of freedom as $(i-1)\alpha$ for all $i > 1$. Thus, since $1\alpha = 1\alpha^{'} = 1$, θ is a bijection onto $N(C_n)$ as required. □

Now, using Lemma 2.2.5, Lemma 2.3.6, and Proposition 2.3.8, we obtain the following result.

Lemma 2.3.9. $|O_{n,1}| = |O_{n,n}| = C_{n-1}$. ■

Lemma 2.3.10. $|O_{n,p}| = C_{p-1}C_{n-p}$.

Proof. For each $\alpha \in O_{n,p}$, we fix

$$\alpha_1 = \begin{pmatrix} 1 & 2 & \ldots & p-2 & p-1 & p \\ 1\alpha & 2\alpha & \ldots & (p-2)\alpha & p & p \end{pmatrix}. \tag{2.9}$$

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 3 & \ldots & n-(p-1) \\ 1 & 1 & (p+2)\alpha-(p-1) & \ldots & n\alpha-(p-1) \end{pmatrix}. \tag{2.10}$$

That is, in general,

$$k\alpha_2 = (p+k-1)\alpha-(p-1) \text{ for all } 2 \leq k \leq n-(p-1). \tag{2.11}$$

This gives,

$$n-(p-1)\alpha_2 = (p+n-p+1-1)\alpha-(p-1) = n\alpha-(p-1).$$

Now, by Proposition 2.2.1,

$$x+1 \leq x\alpha \text{ for all } 1 \leq x < p.$$

This gives, $x + 1 \leq x\alpha_1$ for all $1 \leq x < p$ *and* $p\alpha_1 = p$

and, therefore, $\alpha_1 \in O_{p,p}$.
Again, by proposition 2.2.1,

$$\alpha x \leq x - 1 \text{ for all } p < x \leq n$$
$$\Longrightarrow x\alpha - p \leq x - 1 - p \text{ for all } p < x \leq n$$
$$\textit{or } x\alpha - (p-1) \leq x - p \text{ for all } p < x \leq n.$$

Since $p < x, 1 < x - p \implies 2 < x - (p-1)$ or $2 < k$;
and $x \leq n \implies x - (p-1) \leq n - (p-1)$ or $k \leq n - (p-1)$;
where $k = x - (p-1)$.
Thus, $k\alpha_2 \leq k - 1$ for all $2 < k \leq x - (p-1)$ with $1\alpha_2 = 1$. Therefore, $\alpha_2 \in O_{n-p+1,1}$.

Now, consider the map

$$f : O_{n,p} \longrightarrow O_{p,p} \times O_{n-p+1,1}$$

defined by

$$\alpha f = (\alpha_1, \alpha_2).$$

That is, f maps each $\alpha \in O_{n,p}$ to the ordered pair (α_1, α_2).
Now, let $\alpha, \alpha' \in O_{n,p}$ then we can define α_1, α_2, α'_1, α'_2 as in equations (2.10) and (2.11) such that

$$\alpha f = (\alpha_1, \alpha_2) \text{ and } \alpha' f = (\alpha'_1, \alpha'_2).$$

If $\alpha f = \alpha' f$, then

$$(\alpha_1, \alpha_2) = (\alpha'_1, \alpha'_2)$$

$$\Longrightarrow \alpha_1 = \alpha_1^{'} \;\; and \;\; \alpha_2 = \alpha_2^{'}$$

That is,

$$x\alpha = x\alpha^{'} \text{ for all } x \in \{1, 2, 3, \ldots, p\} \tag{2.12}$$

and, also $k\alpha_2 = k\alpha_2^{'}$ for all $1 \leq k \leq n-(p-1)$. This implies, by (2.11), $(p+k-1)\alpha - (p-1) = (p+k-1)\alpha^{'} - (p-1)$ for all $3 \leq k \leq n-(p-1)$. That is, $(p+k-1)\alpha = (p+k-1)\alpha^{'}$ for all $3 \leq k \leq n-(p-1)$.

Let $x = p+k-1$, then

$$\begin{aligned} & 3 \leq k \leq n-(p-1) \\ \Longrightarrow\; & p+3 \leq p+k \leq n+1 \\ \Longrightarrow\; & p+2 \leq p+k-1 \leq n \\ \Longrightarrow\; & p+2 \leq x \leq n \end{aligned}$$

Thus, we obtain,

$$x\alpha = x\alpha^{'} \text{ for all } p+2 \leq x \leq n \tag{2.13}$$

Also,

$$(p+1)\alpha = p = (p+1)\alpha^{'} \quad (by\, corollary\ 2.2.2) \tag{2.14}$$

Therfore, by (2.12), (2.13), and (2.14), we have

$$x\alpha = x\alpha^{'} \text{ for all } x \in \{1, 2, \ldots, n\}$$

and, therefore, $\alpha = \alpha^{'}$

Consequently, f is injective.

Also, for every pair $(\alpha_1, \alpha_2) \in O_{p,p} \times O_{n-p+1,1}$ there exists $\alpha \in O_{n,p}$ such that

$$\alpha f = (\alpha_1, \alpha_2)$$

and hence f is a bijection. Thus,

$$|O_{n,p}| = |O_{p,p} \times O_{n-p+1,1}| = |O_{p,p}| \times |O_{n-p+1,1}| = C_{p-1} \times C_{n-p},$$

by Lemma 2.3.9. □

CHAPTER 3

Ranks for Transformation Semigroups

Introduction

In the study of algebraic structures, the concept of rank plays a pivotal role in understanding the generative characteristics of semigroups. This chapter focuses on the ranks of transformation semigroups, particularly the order-preserving transformation semigroups. The rank of a semigroup is defined as the minimum number of elements required to generate the entire semigroup, serving as a crucial metric for assessing its complexity and structural properties.

Transformation semigroups encompass various classes, including order-preserving and nilpotent transformations. Recent research has expanded our understanding of ranks within these contexts, revealing connections to broader mathematical fields.

We aim to analyze the rank of $O_{n,p}$ with an emphasis on the case when $p > 1$. We begin by establishing foundational definitions and results, gradually building towards a comprehensive examination of the rank, culminating in significant theorems and propositions that elucidate the relationships between the elements and their generating sets. This exploration not only enhances our understanding of transformation semigroups but also contributes to ongoing research in the algebraic domain, as noted in existing literature (see, for example, [7] by Gomes and Howie and [16] by Yağcı and Korkmaz).

3.1 Preliminaries

In this section, we introduce basic definitions related to rank of a semigroup. The concepts are drawn from the work presented in [8] and [14].

Definition 3.1.1. Let S be any semigroup and $A \subset S$. An element $s \in S$ is said to be *generated* by A provided that s can be written as a finite product of elements of A.

The set of all elements of S generated by A is denoted by $\langle A \rangle$.

A subset A of S is said to be a *generating system* or a *generating set* for S provided that $\langle A \rangle = S$.

If there exists a finite subset A of S such that $S = \langle A \rangle$, then S is called a *finitely generated semigroup*.

Definition 3.1.2. The *rank* of a finitely generated semigroup S is defined by

$$rank(S) = min\{|A| : \langle A \rangle = S\}.$$

An element a in $S \setminus S^2$ (if there exists) is called indecomposable. It is clear that every generating set must contain all indecomposable elements. In other words, if $S = \langle A \rangle$, then $S \setminus S^2 \subset A$. Therefore, if

$S = \langle S \setminus S^2 \rangle$, then $S \setminus S^2$ is the minimum generating set of S, and so $rank(S) = |S \setminus S^2|$.

Lemma 3.1.3. *Let S be a finite nilpotent semigroup. Then $A = S \setminus S^2$ is the unique minimal generating set for S.*

Proof. Let G be any generating set for S and let $x \in S \setminus S^2 = A$. Then, $x \neq x_1 x_2 \cdots x_n$; where $x_i \in X_n$, therefore, $x \in G$. Thus, $A \subset G$ and consequently A is contained in any generating set for S. Now, we only need to show that $S \subset \langle S \setminus S^2 \rangle = \langle A \rangle$. For that, let $x \in S$ be such that it doesn't belong to the subsemigroup of S generated by A; then $x \neq 0$. Further, since $x \notin A = S \setminus S^2$, we can write $x = x_1 x_2$, where atleast one of the x_1 or x_2 does not belong to subsemigroup of S generated by A. Without loss of generality, assume x_1 doesn't belong to the subsemigroup of S generated by A, then $x_1 \neq 0$.
Again, since $x_1 \notin A$, we can write $x_1 = x_3 x_4$, where atleast one of the x_3 and x_4 does not belong to subsemigroup generated by A. Repeating this process, we obtain

$$x = x_1 x_2 \cdots x_m; \text{ where } m \in \mathbb{N}.$$

Proceeding as above and since S is nilpotent, there exists some $k \in \mathbb{N}$ such that

$$x = x_1 x_2 \cdots x_k = 0,$$

which is a contradiction. Hence, $S = \langle S \setminus S^2 \rangle$, which completes the proof. □

3.2 Rank of $O_{n,1}$

We shall first calculate the rank of the nilpotent subsemigroup of C_n. Note that it has already been calculated by Yağcı M. and Korkmaz E. in [16]; we will only discuss it briefly here.

Since we have already shown in Chapter 2 that $N(C_1) = O_{1,1}$ (see Lemma 2.3.6) and for any $p \in X_n$, $x\alpha \leq x-1$ (by Lemma 2.3.6 and Proposition 2.2.1), we can conclude that for any $\alpha \in N(C_n)$, $i\alpha \leq i-1$ for all $i \geq 2$. Also, again from Lemma 2.3.6, $N(C_2) = \{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\}$.

From now on, in this section, we only consider the case $n \geq 3$.

Lemma 3.2.1. *For $n \geq 3$, if*

$$Y = \{\alpha \in N(C_n) : \text{ there exists at least one } i \geq 3 \text{ such that } i\alpha = i-1\},$$

then $Y = N(C_n) \setminus N(C_n)^2$, and so Y is the set of indecomposable elements of $N(C_n)$. Moreover, $3\alpha = 1$ for all $\alpha \in N(C_n)^2$.

Proof. Let $\beta, \gamma \in N(C_n)$. Then for all $i \geq 3$, we have

$$i(\beta\gamma) = (i\beta)\gamma \leq (i-1)\gamma \leq i-2.$$

Now it is clear that $Y = N(C_n) \setminus N(C_n)^2$, and so Y is the set of indecomposable elements of $N(C_n)$. Now, let $\alpha \in N(C_n)^2$, then

$$\begin{aligned} 3\alpha \leq 2 &\text{ but } 3\alpha \neq 2 \\ &\implies 3\alpha \leq 1 \\ &\implies 3\alpha = 1, \end{aligned}$$

and thus $3\alpha = 1$ for all $\alpha \in N(C_n)^2$. □

Theorem 3.2.2. *For $n \geq 3$, if*

$$Y = \{\alpha \in N(C_n) : \text{ there exists at least one } i \geq 3 \text{ such that } i\alpha = i - 1\},$$

then Y is the minimum generating set of $N(C_n)$. Moreover, $N(C_n) = Y \cup Y^2$.

Proof. By Lemma 3.2.1, it suffices to show that $\alpha \in \langle Y \rangle$ for all $\alpha \in N(C_n)^2$. For any $\alpha \in N(C_n)^2$, if

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 1 & a_2 & \dots & a_p \end{pmatrix}; \textit{where } A_j \subset X_n \text{ such that } A_j\alpha = \begin{cases} 1 : j = 1 \\ a_j : 2 \leq j \leq p, \end{cases}$$

it follows from Lemma 3.2.1 that $3 \in A_1$ and that $a_p \leq n - 2$. For every $2 \leq i \leq p$, let $b_i = \max A_{i-1}$, $b_{p+1} = n$, $B_1 = \{1, \dots, b_2 - 1\}$ and $B_i = \{b_i, \dots, b_{i+1} - 1\}$. Then consider the transformations

$$\beta = \begin{pmatrix} \{1,2\} & A_1 \setminus \{1,2\} & A_2 & \cdots & A_p \\ 1 & 2 & b_2 & \cdots & b_p \end{pmatrix} \text{ and}$$

$$\gamma = \begin{pmatrix} B_1 & B_2 & \cdots & B_p & n \\ 1 & a_2 & \cdots & a_p & n-1 \end{pmatrix}$$

It is clear that $\beta \in N(C_n)$. From Lemma 3.2.1, since $a_i \leq (\min A_i) - 2$, it follows that $a_i \leq (\min B_i) - 1$ for all $2 \leq i \leq p$. Now it is clear that $\gamma \in N(C_n)$ and that $\alpha = \beta\gamma$. Moreover, since $3 \in A_1 \setminus \{1,2\}$, $3\beta = 2$ and $n\gamma = n - 1$, clearly $\beta, \gamma \in Y$, and so $\alpha \in \langle Y \rangle$ and $N(C_n) = Y \cup Y^2$, as required. □

Theorem 3.2.3. *For $n \geq 3$, rank$(N(C_n)) = C_{n-1} - C_{n-2}$.*

Proof. By Theorem 3.2.2, it suffices to determine the cardinality of $Y = N(C_n) \setminus N(C_n)^2$. Also, from Proposition 2.3.8, we have $|N(C_n)| = C_{n-1}$. Since $|N(C_n) \setminus N(C_n)^2| = |N(C_n)| - |N(C_n)^2|$, it suffices to determine the cardinality of $N(C_n)^2$. Recall that if α is any element in $N(C_n)^2$, then $i\alpha \leq i-2$ for all $i \geq 3$. For any $\alpha \in N(C_n)^2$, we define

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ 1 & a_2 & \cdots & a_p \end{pmatrix} \text{ and}$$
$$\beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_p \\ 1 & a_2 & \cdots & a_p \end{pmatrix}$$

where for every $2 \leq i \leq p$, $A_1 = \{1, 2, 3, \ldots, b_1\}$, $A_i = \{b_{i-1}+1, \ldots, b_i\}$, $B_1 = \{1, 2, \ldots, b_1 - 1\}$ and $B_i = \{b_{i-1}, \ldots, b_i - 1\}$ and $b_1 > 3$, $b_p = n$. Thus, if we consider the function

$$f : N(C_n)^2 \to N(C_{n-1})$$

defined by the rule $\alpha f = \beta$, then it is easy to check that f is a well-defined bijection. Therefore, $|N(C_n)^2| = |N(C_{n-1})| = C_{n-2}$ and so

$$\text{rank}(N(C_n)) = C_{n-1} - C_{n-2}, \text{ as required.}$$

□

Thus, using Theorem 3.2.3, Lemma 2.3.6, and Lemma 2.2.5, we obtain

Proposition 3.2.4. *$rank(O_{n,1}) = rank(O_{n,n}) = C_{n-1} - C_{n-2}$.* ■

3.3 Rank of $O_{n,p}$

As already seen, the rank of $O_{n,1}$ as well as the rank of $O_{n,n}$ is $C_{n-1} - C_{n-2}$. Since $O_{n,2}$ is isomorphic to both $O_{n-1,1}$ and $O_{n,n-1}$, we can conclude that

$$\text{rank}(O_{n,2}) = \text{rank}(O_{n,n-1}) = C_{n-2} - C_{n-3}.$$

Since $O_{n,p}$ is a nilpotent semigroup, we have

$$\operatorname{rank}(O_{n,p}) = |O_{n,p} \setminus O_{n,p}^2| \ \text{(see Lemma 3.1.3). Now, let}$$

$$G_{n,p} = \{\alpha \in O_{n,p} : |x\alpha - x| = 1 \text{ for at least one } x \in X_n \setminus \{p-1, p, p+1\}\}$$

and we will show that $O_{n,p} \setminus O_{n,p}^2 = G_{n,p}$. For this, we show first that

$$O_{n,p}^2 \cap G_{n,p} = \emptyset.$$

Lemma 3.3.1. *If $\alpha, \beta \in O_{n,p}$, then $\alpha\beta \notin G_{n,p}$.*

Proof. Assume that there are $\alpha, \beta \in O_{n,p}$ such that $\alpha\beta \in G_{n,p}$. Then there is $x \in X_n \setminus \{p-1, p, p+1\}$ such that $|x\alpha\beta - x| = 1$.

If $x < p-1$, then $x\alpha \geq x+1$ by Proposition 2.2.1, where $x+1 < p$. This implies $x\alpha\beta \geq (x+1)\beta \geq x+2$. Thus, $x\alpha\beta - x \geq x+2-x = 2$, a contradiction.

If $x > p+1$, then $x\alpha \leq x-1$ by Proposition 2.2.1, where $x-1 > p$. Thus, $x\alpha\beta \leq (x-1)\beta \leq x-2$, i.e., $x\alpha\beta - x \leq x-2-x = -2$, again a contradiction. □

Proposition 3.3.2. $O_{n,p} \setminus O_{n,p}^2 = G_{n,p}$.

Proof. By Lemma 3.3.1, we can conclude that $O_{n,p} \setminus O_{n,p}^2 \subset G_{n,p}$. In order to show the converse inclusion, we verify that $\alpha \in O_{n,p}^2$ for all $\alpha \in O_{n,p} \setminus G_{n,p}$. Let $\alpha \in O_{n,p} \setminus G_{n,p}$. We define $\alpha^* : X_n \to X_n$ by

$$x\alpha^* = \begin{cases} x\alpha - 1 & \text{for } x \leq p-2, \\ x\alpha + 1 & \text{for } x \geq p+2, \\ p & \text{otherwise.} \end{cases}$$

Since $\alpha \notin G_{n,p}$ implies $|x - x\alpha| \geq 2$ for all $x \in X_n \setminus \{p-1, p, p+1\}$, it is easy to see that $\alpha^* \in \mathcal{O}_{n,p}$. We show that $\alpha = \alpha^*\gamma$, where

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & \dots & p-2 & p-1 & p & p+1 & p+2 & \dots & n \\ 2 & 3 & 4 & \dots & p-1 & p & p & p & p+1 & \dots & n-1 \end{pmatrix}.$$

It is clear that $\gamma \in \mathcal{O}_{n,p}$. Let $x \in X_n$. Suppose that $x \le p-2$. We have $x\alpha^*\gamma = (x\alpha - 1)\gamma$, where $x\alpha \le p$ since α is order-preserving. If $x\alpha = p$, then $(x\alpha-1)\gamma = (p-1)\gamma = p = x\alpha$. If $x\alpha < p$, then $(x\alpha-1)\gamma = (x\alpha-1)+1 = x\alpha$. Thus,

$$x\alpha^*\gamma = x\alpha.$$

If $x \in \{p-1, p, p+1\}$, then

$$x\alpha = p = p\gamma = x\alpha^*\gamma.$$

Suppose that $x \ge p+2$. Then $x\alpha^*\gamma = (x\alpha+1)\gamma$. Since α is order-preserving, we have $x\alpha \ge p$. If $x\alpha = p$, then $x\alpha^*\gamma = (x\alpha+1)\gamma = (p+1)\gamma = p = x\alpha$. If $x\alpha > p$, then $x\alpha + 1 > p+1$ and, therefore,

$$x\alpha^*\gamma = (x\alpha+1)\gamma = (x\alpha+1-1) = x\alpha.$$

□

In general, Proposition 3.3.2 shows the following:

Observation 3.3.3. *$\alpha \in \mathcal{O}^2_{n,p}$ if and only if $|x\alpha - x| \ge 2$ for all $x \in X_n \setminus \{p-1, p, p+1\}$.*

Lemma 3.3.4. $|\mathcal{O}^2_{n,p}| = |\mathcal{O}_{n-2,p-1}|$.

Proof. Let $\alpha \in \mathcal{O}^2_{n,p}$. We define $\alpha^* \in \mathcal{T}_{n-2}$ by

$$x\alpha^* = \begin{cases} x\alpha - 1 & \text{for } x \le p-1, \\ (x+2)\alpha - 1 & \text{for } p \le x \le n-2. \end{cases}$$

Clearly, $x\alpha^* \leq n-1$ for all $x \in X_{n-2}$. Since $|x\alpha - x| \geq 2$ for all $x \in X_n \setminus \{p-1, p, p+1\}$, we have $|x\alpha^* - x| \geq 1$ for all $x \in \{1, \ldots, p-2, p, \ldots, n-2\}$. Moreover, $(p-1)\alpha = p$ implies $(p-1)\alpha^* = (p-1)\alpha - 1 = p-1$. Obviously, α^* is order-preserving. Hence, we can conclude that $\alpha^* \in O_{n-2,p-1}$. We define a mapping $\phi : O^2_{n,p} \to O_{n-2,p-1}$ by $\alpha\phi = \alpha^*$. It is easy to verify that ϕ is a bijection, i.e., $O^2_{n,p} = |O_{n-2,p-1}|$.

□

Now we are able to calculate the rank of nilpotent semigroup $O_{n,p}$.

Theorem 3.3.5. *rank*$(O_{n,p}) = C_{p-1}C_{n-p} - C_{p-2}C_{n-p-1}$

Proof. Note that $\text{rank}(O_{n,p}) = |O_{n,p} \setminus O^2_{n,p}| = |O_{n,p}| - |O^2_{n,p}| = |O_{n,p}| - |O_{n-2,p-1}|$, by Lemma 3.3.4.
Since $|O_{n,p}| = C_{p-1}C_{n-p}$ and $|O_{n-2,p-1}| = C_{p-2}C_{n-p-1}$ by Lemma 2.3.10, we get

$$\text{rank}(O_{n,p}) = |O_{n,p} \setminus O^2_{n,p}| = C_{p-1}C_{n-p} - C_{p-2}C_{n-p-1}.$$

□

CHAPTER 4

Presentations for Transformation Semigroups

Introduction

In semigroup theory, the concept of a *presentation* is pivotal for understanding the algebraic structure of semigroups. A presentation provides a framework where a semigroup can be described using a finite set of generators and defining relations, thereby offering insights into how elements interact within the semigroup. This approach not only simplifies the complexity inherent in the algebraic structure but also serves as a foundation for various applications in both theoretical and computational contexts.

Particularly for transformation semigroups, presentations play a crucial role in elucidating the relationships among transformations and enabling the classification of semigroups. By establishing a concise representation of transformations through generators and their corresponding relations, one can derive significant insights into the behavior and prop-

erties of the semigroup.
There has been various research on the presentations of finite transformation semigroups. A presentation for the transformation semigroups $\mathcal{IS}_n$, $\mathcal{T}_n$, and $\mathcal{PT}_n$ were provided in The Classical Finite Transformation Semigroups [8]. James East gave presentations for the semigroups $\mathcal{PT}_n \setminus \mathcal{S}_n$ and $\mathcal{PT}_n \setminus \mathcal{T}_n$ in [5]. Fernandes V.H., Gomes G.M.S., and Jesus M.M. presented the natural monoids on X_n in [6].

This chapter focuses on the presentation of $O_{n,1}$. We will systematically identify a set of generators and elucidate their defining relations, aiming to construct a coherent presentation for this semigroup.

4.1 Semigroup Presentations

In this section, the concepts are partly drawn from the work of Ganyushkin O and Mazorchuk V presented in Classical Finite Transformation Semigroups [8].

Definition 4.1.1. Let A be a non-empty set, which we will call the *alphabet*. Elements of A will be called *letters*. Any finite non-empty sequence $a_1a_2\cdots a_k$ of elements from A will be called a *word* or a *word over A*. The set of all words over A is denoted A^+.

If $u = a_1a_2\cdots a_k$ and $v = b_1b_2\cdots b_m$ are two words, we define their *product* or *concatenation* or *juxtaposition* as follows:

$$uv = a_1a_2\cdots a_kb_1b_2\cdots b_m.$$

It is obvious that this binary operation is associative, which turns A^+ into a semigroup. This semigroup is called the *free semigroup* with base A. Obviously, A is a generating system of A^+.

Let now S be a semigroup and A be a generating system for S. On the one hand, every element $s \in S$ can be written as a product $s = a_1a_2 \cdots a_k$ of some elements from A, however not uniquely in general. On the other hand, every product $a_1a_2 \cdots a_k$ can be considered as an element of A^+. If there exist two words $u = a_1a_2 \cdots a_k$ and $v = b_1b_2 \cdots b_m$ in A^+, which determine the same element $s \in S$, we say that the relation $u = v$ holds in S.

Notation 4.1.2. If ρ is a congruence on S, the equivalence class of $a \in S$ is denoted by $\overline{a}_\rho$ or simply by $\overline{a}$ if there is no confusion about ρ.

Let ρ be a congruence on a semigroup S. Consider the set S/ρ of all equivalence classes. For $a, b \in S$ set

$$\bar{a} \cdot \bar{b} = \overline{ab}. \tag{4.1}$$

Lemma 4.1.3. *The formula* (4.1) *defines a binary associative operation on S/ρ. In other words, $(S/\rho, \cdot)$, where $\cdot$ is defined by* (4.1)*, is a semigroup.*

Proof. First we have to show that (4.1) defines a well-defined binary operation on S/ρ. Let $a^{'} \in \bar{a}$ and $b^{'} \in \bar{b}$. Since ρ is compatible we have $ab \equiv ab^{'} \equiv a^{'}b^{'}$ and hence $\bar{a}\cdot\bar{b} = \bar{a}^{'}\cdot\bar{b}^{'}$, that is, the operation $\cdot$ is well-defined.

The associativity of $\cdot$ follows from the associativity of the multiplication in S:

$$(\bar{a} \cdot \bar{b}) \cdot \bar{c} = \overline{ab} \cdot \bar{c} = \overline{abc} = \bar{a} \cdot \overline{bc} = \bar{a} \cdot (\bar{b} \cdot \bar{c}).$$

□

Lemma 4.1.4. *Let $\varphi : S \longrightarrow T$ be a homomorphism of semigroups and π be the cannonical projection $\pi : S \longrightarrow S/Ker\varphi$. Then the mapping $\psi : S/Ker\varphi \longrightarrow T$*

defined via $\overline{a}\psi = a\varphi$ *is a monomorphism and* $\varphi = \pi \circ \psi$*, that is the diagram*

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\pi"'] & T \\ S/ker\varphi \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

commutes.

Moreover, if φ *is an epimorphism, then* ψ *is an isomorphism.*

Proof. First, we have to check that ψ is well-defined. For that let $\overline{a}, \overline{b} \in S/ker\phi$ such that $\overline{a} = \overline{b}$.

Without loss of generality, we may write $a \in \overline{b}$. Then, by definition of $\ker\varphi$, $a\varphi = b\varphi$.

Hence, $\overline{a}\psi = a\varphi = b\varphi = \overline{b}\psi$. Consequently, ψ is well-defined.

Now, if $\overline{a}\psi = \overline{b}\psi$, then $a\phi = b\phi$ and hence $a \in \overline{b}$. That is, $\overline{a} = \overline{b}$.

Therefore, ψ is injective.

The next step is to check that ψ is a homomorphism. Let $\bar{a}, \bar{b} \in S/\mathrm{Ker}(\varphi)$. Then

$$(\bar{a} \cdot \bar{b})\psi = (\overline{ab})\psi = (ab)\varphi = a\varphi b\varphi = \bar{a}\psi\bar{b}\psi.$$

Now for $a \in S$ we have $(a\pi)\psi = \bar{a}\psi = a\varphi$ and hence $\varphi = \pi\psi$.

Finally, if φ is surjective, then so is ψ since $\varphi = \pi\psi$. We already know that ψ is always injective. Hence ψ is bijective for surjective φ. This completes the proof. □

Proposition 4.1.5. *The binary relation*

$$\rho(S, A) = \{(u, v) \in A^+ \times A^+ : u = v \text{ is a relation in } S\} \tag{4.2}$$

is a congruence on A^+*, moreover* $A^+/\rho(S, A) \cong S$ *canonically.*

Proof. First we show that the binary relation in (4.2) is an equivalence.

Clearly, (4.2) is reflexive and symmetric as if $u \in A^+$ such that u determine some elements $s \in S$, then u and u determine the same element $s \in S$. This implies, $u = u$ is a relation in S and this gives, $(u, u) \in \rho(S, A)$. And if $(u, v) \in \rho(S, A)$, then $u = v$ is a relation in S. That is, u, v determine a same element $s \in S$. Equivalently, v, u determine a same element $s \in S$. This implies, $v = u$ is a relation in S, that is, $(v, u) \in \rho(S, A)$. Now, let $(u, v), (v, w) \in \rho(S, A)$, then u, v and v, w respectively determine the same elements $s_1 \, and \, s_2 \in S$.
If $v = a_1a_2 \cdots a_k$, then $s_1 = a_1a_2 \cdots a_k = s_2$. That is, $s_1 = s_2 = s$(say) (in S).This gives, u, w determine the same elements $s \in S$ and, therefore, $(u, w) \in \rho(S, A)$.

Further, let $(t, u), (v, w) \in \rho(S, A)$, then $t = u \, and \, v = w$ are relations in S. That is, $t, u \, and \, v, w$ respectively determine same elements $s_1, s_2 \in S$, which implies $t = s_1 = u$ and $v = s_2 = w$ in S.
Now, $tv = s_1s_2 = s \in S$ and $uw = s_1s_2 = s \in S$ (since S is a semigroup). This gives, $tv = s = uw$. That is, $tv \, and \, uw$ determine the same element $s \in S$. Therefore, $(tv, uw) \in \rho(S, A)$. Consequently, $\rho(S, A)$ is a congruence.

Finally, define a mapping $\varphi : A^+ \longrightarrow S$ sending the word $a_1a_2 \cdots a_k$ to the product $a_1a_2 \cdots a_k$. Clearly, φ is well-defined and obviously a homomorphism of semigroups. It is surjective since A generates S.
Further,

$$\begin{aligned}
\ker \varphi &= \{(u, v) \in A^+ \times A^+ : u\varphi = v\varphi\} \\
&= \{(u, v) \in A^+ \times A^+ : (u_1u_2 \cdots u_k)\varphi = (v_1v_2 \cdots v_m)\varphi, \\
&\qquad\qquad where \ u = u_1u_2 \cdots u_k \, and \ v = v_1v_2 \cdots v_m\} \\
&= \{(u, v) \in A^+ \times A^+ : u_1u_2 \cdots u_k = v_1v_2 \cdots v_m \, in \, S\} \\
&= \{(u, v) \in A^+ \times A^+ : u, v \text{ determine same the same element in } S\}
\end{aligned}$$

$$= \{(u, v) \in A^+ \times A^+ : u = v \text{ is a relation in S}\}$$
$$= \rho(S, A).$$

That is,
$$\rho(S, A) = \ker \varphi.$$

Hence, $\rho(S, A)$ is a congruence on A^+. Since we know, by lemma 4.1.4, that
$$A^+/\ker \varphi \cong S \ for \ \varphi \ surjective$$

Therefore, we conclude that
$$A^+/\rho(S, A) \cong S.$$

Which completes the proof. □

Fixing some representatives in all classes of $\rho(S, A)$ determines a canonical form for each element of S with respect to the generating system A. Sometimes, it is convenient to identify the elements from S with their canonical forms. Such identification might be useful if the following two natural conditions are satisfied:

- There exists a constructable description for all canonical forms.
- If the canonical forms of some elements $g, h \in S$ are known, then one can determine the canonical form of the element gh.

Let $u = v$ be a relation in $S = \langle A \rangle$. We will say that some congruence ρ on A^+ contains the relation $u = v$ provided that $(u, v) \in \rho$. In particular, the uniform congruence $\omega_{A^+} = A^+ \times A^+$ contains all possible relations. As the intersection of an arbitrary family of congruences is a congruence, for each set Σ of relations, there exists a unique minimal congruence ρ_Σ

on A^+, which contains all relations from Σ. If $\rho_\Sigma = \rho(S, A)$, the set Σ is called a *set or a system of defining relations* for S with respect to A. In this case, one says that S is a semigroup generated by A with the system Σ of defining relations. This is denoted by $S = \langle A|\Sigma\rangle$. The pair $\langle A|\Sigma\rangle$ is called a **presentation** of S. Note that a system of defining relations for S with respect to A is not unique in general.

Another way to define semigroup presentation is using Lemma 4.1.4 as given below.

Definition 4.1.6. Let S be a semigroup and A be generating system for S. If $\varphi : A^+ \longrightarrow S$ be defined as in Proposition 4.1.5, then φ is well-defined surjective homomorphism of semigroups and, therefore, by Lemma 4.1.4

$$S \cong A^+/\ker\varphi$$

If $(u, v) \in \ker\varphi$, where u and v are words in A^+ then $u\varphi = v\varphi$.
As a result, $u = v$ or $u \equiv v$ is called a *relation* in S.
We define a Presentation for S in terms of generators $A = \{a_1, a_2, a_3, \ldots\}$ and relation $\rho = \{u_i = v_i : i \in I\} \subset A^+ \times A^+$ as

$$S = \langle a_1, a_2, \ldots \mid u_i = u_i : i \in I\rangle \text{ if } \rho^\sharp = \ker\varphi$$

If $A = \{a_1, a_2, \ldots, a_n\}$ is finite and $\rho = \{(x_1, y_1), (x_2, y_2, \ldots, (x_{m}, , y_m)\} \subset A^+ \times A^+$ such that $\rho^\sharp = \ker\varphi$, then we say that S has a *finite presentation*

$$\langle a_1, a_2, \ldots, a_n \mid x_1 = y_1, x_2 = y_2, \ldots, x_m = y_m\rangle$$

Equivalently, we say that S is *finitely presented*.

Let Σ be a subset of $A^+ \times A^+$. A pair $(u, v) \in A^+ \times A^+$ will be called a Σ-pair provided that $u = v$ or there exist decompositions $u = su_1t$ and

$v = sv_1t$ (where $s, t, u_1, v_1 \in A^+$ or either s or t or both are empty), such that either (u_1, v_1) or (v_1, u_1) belongs to Σ.

Proposition 4.1.7. *Congruence ρ_Σ coincides with the set of all pairs (u, v) for which there exists a finite collection $w_1, w_2, \dots, w_k \in A^+$ such that each of the pairs (u, w_1), (w_1, w_2), $\dots$, (w_{k-1}, w_k) and (w_k, v) is a Σ-pair.*

Proof. Let Ω denote the set of all pairs (u, v), which satisfy the condition of the proposition. Obviously, each Σ-pair is contained in ρ_Σ. As each congruence is an equivalence relation, in particular, is transitive, we get $\Omega \subset \rho_\Sigma$.

To complete the proof, it is enough to show that Ω is a congruence. Let $(u, v) \in \Omega$ and $(u, w_1), (w_1, w_2), \dots, (w_{k-1}, w_k), (w_k, v)$ be the corresponding collection of Σ-pairs. Then for every $w \in A^+$, we have that the pairs (wu, ww_1), $(ww_1, ww_2), \dots, (ww_{k-1}, wwk)$, (ww_k, wv) are Σ-pairs as well. Hence, $(wu, wv) \in \Omega$ and Ω is left compatible. Analogously, one shows that Ω is right compatible. This completes the proof. □

A relation $u = v$ is said to *follow from* Σ provided that $(u, v) \in \rho_\Sigma$.

Remark 4.1.8. Assume that both $u = v$ and $v = w$ follow from Σ. Then, obviously, $u = w$ follows from Σ as well.

4.2 Presentation for $O_{n,1}$

The goal of this section is to give a presentation for $O_{n,1}$. Let A_n be the set of all mappings $g \in \{0, 1, \dots, n-1\}^{X_n}$ (*i.e.*, $g : X_n \longrightarrow \{0, 1, \dots, n-1\}$) with $1g = 0$, $xg \geq 1$ for $x \in \{2, \dots, n\}$, where $lg = 1$ for some $l \in \{3, \dots, n\}$, and either $(x+1)g \leq xg$ or $(x+1)g = xg + 1$ for all $x \in \{1, \dots, n-1\}$.

Lemma 4.2.1. *Let $g \in A_n$. Then $x - xg \geq 1$ for all $x \in X_n$.*

Proof. We have $1 - 1g = 1 - 0 = 1$. Suppose $x - xg \geq 1$ for some $x \in X_n$, and we will show that $(x+1) - (x+1)g \geq 1$. We have $(x+1)g \leq xg$ or $(x+1)g = xg + 1$.
If $(x+1)g \leq xg$, then

$$1 \leq x - xg < x + 1 - xg \leq (x+1) - (x+1)g.$$

If $(x+1)g = xg + 1$, then

$$(x+1) - xg = (x+1) - (x+1)g + 1,$$

thus,

$$1 \leq x - xg = (x+1) - (x+1)g.$$

□

For each $g \in A_n$, let $\alpha_g \in \mathcal{T}_n$ with

$$x\alpha_g = x - xg \text{ for all } x \in X_n.$$

Lemma 4.2.2. $G_{n,1} = \{\alpha_g : g \in A_n\}$.

Proof. Let $g \in A_n$, and we will show that $\alpha_g \in G_{n,1}$. Clearly, $1\alpha_g = 1 - 1g = 1 - 0 = 1$ and $|x - x\alpha_g| = xg \geq 1$ for all $x \in \{2, 3, \ldots, n\}$. Note that there is $l \in \{3, \ldots, n\}$ such that $lg = 1$. This provides $l\alpha_g = l - lg = l - 1$, thus $|l - l\alpha_g| = 1$.

Next, we show that $\alpha_g \in O_{n,1}$, that is, α_g is order-preserving. Let $x < y \in X_n$. Then $y = x + k$ for some $k \in X_n$. Firstly, we verify that $x\alpha_g \leq (x+1)\alpha_g$. We have $xg \geq (x+1)g$ or $(x+1)g = xg+1$. If $xg \geq (x+1)g$, then

$$x\alpha_g = x - xg \le x - (x+1)g < (x+1) - (x+1)g = (x+1)\alpha_g.$$

If $(x+1)g = xg + 1$, then

$$x\alpha_g = x - xg = x - (x+1)g + 1 = (x+1) - (x+1)g = (x+1)\alpha_g.$$

By the same arguments, we obtain

$$x\alpha_g \le (x+1)\alpha_g \le (x+2)\alpha_g \le \cdots \le (x+k)\alpha_g,$$

That is, $x\alpha_g \le y\alpha_g$. Therefore, $\alpha_g \in G_{n,1}$. Altogether, we have shown that

$$\{\alpha_g : g \in A_n\} \subset G_{n,1}.$$

For the converse inclusion, let $\beta \in G_{n,1}$, and we define $g_\beta \in \{0, 1, \ldots, n-1\}^{X_n}$ by

$$xg_\beta = x - x\beta \text{ for all } x \in X_n.$$

We will show that $g_\beta \in A_n$. Clearly, $1g_\beta = 1 - 1\beta = 1 - 1 = 0$ and $xg_\beta = x - x\beta \ge 1$ for all $x \in \{2, \ldots, n\}$ by Proposition 2.2.1. Further, there is $p \in \{3, \ldots, n\}$ such that $1 = p - p\beta = pg_\beta$.

Let $x \in \{1, \ldots, n-1\}$ with $(x+1)g_\beta > xg_\beta$. This implies

$$(x+1) - (x+1)\beta > x - x\beta,$$

and so

$$1 - (x+1)\beta > -x\beta,$$

that is, $(x+1)\beta - 1 < x\beta$. Since β is order-preserving, $(x+1)\beta - 1 < x\beta$ is only possible if $(x+1)\beta = x\beta$, i.e., $(x+1) - (x+1)\beta = x - x\beta + 1$. This shows $(x+1)g_\beta = xg_\beta + 1$. Consequently, $g_\beta \in A_n$.

Finally, since $x\alpha_{g_\beta} = x - xg_\beta = x - (x - x\beta) = x\beta$, for all $x \in X_n$, we can conclude that $\beta = \alpha_{g_\beta} \in \{\alpha_g : g \in A_n\}$.

Consequently, we have shown the converse inclusion $G_{n,1} \subset \{\alpha_g : g \in A_n\}$, which completes the proof. □

Together with Proposition 3.3.2, we obtain:

Corollary 4.2.3. *$\{\alpha_g : g \in A_n\}$ is a generating set of $O_{n,1}$.* ■

For $g, h \in A_n$, let $g \oplus h \in \{0, 1, 2, \ldots, n-1\}^{X_n}$ with $1(g \oplus h) = 0, 2(g \oplus h) = 1$ and $x(g \oplus h) = (x - xg)h + xg - 1$ for $x \in \{3, 4, \ldots, n\}$. Note that $2g = 1$ for all $g \in A_n$.

Lemma 4.2.4. *Let $g, h \in A_n$. Then $g \oplus h \in A_n$.*

Proof. For $x \in \{3, 4, \ldots, n\}$, we have $xg \geq 1$. If $xg = 1$, then $x - xg \geq 2$; thus, $(x - xg)h \geq 1$. This provides $(x - xg)h + xg - 1 \geq 1$. If $xg > 1$, then $xg - 1 \geq 1$; thus $(x - xg)h + xg - 1 \geq 1$. We take $x = 3$. Then $3g \in \{1, 2\}$ by Lemma 4.2.1. If $3g = 1$, then $3(g \oplus h) = (3-1)h + 1 - 1 = 2h = 1$. If $3g = 2$, then $3(g \oplus h) = (3-2)h + 2 - 1 = 1h + 2 - 1 = 1$.

Let $x \in \{2, 3, 4, \ldots, n-1\}$ such that $(x+1)(g \oplus h) > x(g \oplus h)$. We show that

$$(x+1)(g \oplus h) = x(g \oplus h) + 1.$$

If $xg > (x+1)g$, then $xg = r + (x+1)g$ for some $r \in \{1, 2, \ldots, n\}$. Then

$$(x+1)(g \oplus h) = (x + 1 - (x+1)g)h + (x+1)g - 1$$

and

$$x(g \oplus h) = (x - xg)h + xg - 1 = (x - (r + (x+1)g))h + r + (x+1)g - 1,$$

that is
$$(x+1-(x+1)g)h > (x-(r+(x+1)g))h + r.$$
Thus,
$$(x+1-(x+1)g)h \geq (x-(r+(x+1)g))h + r + 1.$$

On the other hand, we can calculate that
$$(x+1-(x+1)g)h \leq (x+1-(x+1)g-(r+1))h + (r+1).$$
This implies
$$(x+1-(x+1)g)h = (x-r-(x+1)g)h + r + 1.$$
We get
$$\begin{aligned}(x+1-(x+1)g)h + (x+1)g - 1 &= (x-r-(x+1)g)h + r + (x+1)g \\ &= (x-r-(xg-r))h + r + xg - r \\ &= (x-xg)h + xg - 1 + 1 = x(g \oplus h) + 1.\end{aligned}$$
This implies
$$(x+1)(g \oplus h) = x(g \oplus h) + 1.$$
If $xg = (x+1)g$, then
$$(x+1)(g \oplus h) = (x+1-xg)h + xg - 1 > (x-xg)h + xg - 1 = x(g \oplus h).$$
This gives $(x+1-xg)h > (x-xg)h$, that is, $(x+1-xg)h = (x-xg)h + 1$. This provides
$$(x+1)(g \oplus h) = x(g \oplus h) + 1.$$

If $xg < (x+1)g$, that is $(x+1)g = xg + 1$, then we can calculate

$$\begin{aligned}(x+1)(g \oplus h) &= (x + 1 - (x+1)g)h + (x+1)g - 1\\ &= (x + 1 - xg - 1)h + xg + 1 - 1\\ &= ((x - xg)h + xg - 1) + 1\\ &= x(g \oplus h) + 1.\end{aligned}$$

□

Let us put $\omega \in \{0, 1, 2, \dots, n-1\}^{X_n}$ with $1\omega = 0$ and $x\omega = 1$ for $x \in \{2, 3, 4, \dots, n\}$. Clearly, $\omega \in A_n$. It is useful to determine the product of any transformation in $G_{n,1}$ with

$$\alpha_w = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 2 & \dots & n-1 \end{pmatrix}.$$

Let us define a mapping $f : X_n \cup \{0\} \to X_n$ by $f(0) = 1$ and $f(x) = x$ for $x \in X_n$. In this case, we write the argument to the right of the mapping for convenience.

Lemma 4.2.5. *Let $g \in A_n$. Then*

$$\alpha_g\alpha_w = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & f(2-3g) & f(3-4g) & \dots & f((n-1)-ng) \end{pmatrix}.$$

Proof. We have $1\alpha_g\alpha_w = 1\alpha_w = 1$, $2\alpha_g\alpha_w = 1\alpha_w = 1$, and

$$x\alpha_g\alpha_w = \begin{cases} x - xg - 1 & \text{if } x - xg > 1;\\ 1 & \text{if } x - xg = 1.\end{cases}$$

That is, $x\alpha_g\alpha_w = f(x - xg - 1),\ \textit{for all } x \in \{3, 4, 5, \cdots, n\}$ □

For $g \in A_n$, let $\hat{g} \in \{0,1,2,\ldots,n-1\}^{X_n}$ with $1\hat{g} = 0$, $2\hat{g} = 1$, and $x\hat{g} = x-1-f(f(x-1-xg)-1)$ for $x \in \{3,4,\ldots,n\}$.

Lemma 4.2.6. *$\hat{g} \in A_n$ for all $g \in A_n$.*

Proof. Let $g \in A_n$. We have $1\hat{g} = 0$ and $2\hat{g} = 1$. If $x \geq 3$, then we obtain

$$x\hat{g} = x-1-f(f(x-1-xg)-1) = x-1-x+1+xg+1 = xg+1 \geq 1,$$

whenever $f(x-1-xg) \geq 2$, and

$$x\hat{g} = x-1-1 \geq 3-2 = 1,$$

whenever $f(x-1-xg) = 1$. Further, $3g \in \{1,2\}$ implies $f(2-3g) = 1$ and

$$3\hat{g} = 2-f(f(2-3g)-1) = 2-f(1-1) = 2-1 = 1,$$

that is, $2\hat{g} = 3\hat{g}$.

Let $x \in \{3,\ldots,n-1\}$ with $(x+1)\hat{g} > x\hat{g}$. Since $\alpha_{\hat{g}}\alpha_w =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(f(2-3g)-1) & f(f(3-4g)-1) & \cdots & f(f((n-1)-ng)-1) \end{pmatrix}$$
$\in O_{n,1}$,

we obtain

$$\begin{aligned} f(f(x-1-xg)-1) &\leq f(f(x-(x+1)g)-1), \\ or -f(f(x-(x+1)g)-1) &\leq -f(f(x-1-xg)-1); \end{aligned}$$

thus,

$$(x+1)\hat{g} = x-f(f(x-(x+1)g)-1) \leq x-f(f(x-1-xg)-1)$$

$$= x - 1 - f(f(x - 1 - xg) - 1) + 1 = x\hat{g} + 1.$$

On the other hand, $x\hat{g} < (x+1)\hat{g}$ implies $x\hat{g} + 1 \leq (x+1)\hat{g}$. Therefore, we have

$$x\hat{g} + 1 = (x+1)\hat{g}.$$

□

Now we define an alphabet set $Y_n = \{x_g : g \in A_n\}$ and define two sets of relations on Y_n^+. Let

$$R_1 = \{x_g x_\omega^2 \approx x_{\hat{g}} x_\omega : g \in A_n\} \text{ and}$$
$$R_2 = \{x_g x_h \approx x_{g \oplus h} x_\omega : g, h \in A_n\}.$$

It is easy to see that $|R_1| = |A_n|$ and $|R_2| = |A_n|^2$. Since $R_1 \cap R_2 = \emptyset$, we have

$$\begin{aligned} |R_1 \cup R_2| = |A_n| + |A_n|^2 = (1 + |A_n|) \cdot |A_n| &= (1 + |G_{n,1}|)|G_{n,1}| \\ &= (1 + C_{n-1} - C_{n-2})(C_{n-1} - C_{n-2}) \end{aligned}$$

using Lemma 4.2.2 and Proposition 3.2.2.

We write $\sim$ for the congruence on Y_n^+ generated by $R_1 \cup R_2$. Note that $\{\alpha_g : g \in A_n\}$ is a generating set of $O_{n,1}$ (see Corollary 4.2.3).
We define an epimorphism $\varphi : Y_n \to O_{n,1}$ by

$$x_g \varphi = \alpha_g \text{ for } g \in A_n.$$

We aim to show that $\ker \varphi = \sim$, so that $O_{n,1}$ has presentation $\langle Y_n | R_1 \cup R_2 \rangle$ via φ.

Lemma 4.2.7. *We have the inclusion $\sim \subset \ker \varphi$.*

Proof. It is sufficient to show that the relations in $R_1 \cup R_2$ hold as equations in $O_{n,1}$ when the variables are replaced by their images under φ. Note that

$$x_\omega\varphi = \alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & 2 & 3 & \dots & n-1 \end{pmatrix}.$$

Let $g \in A_n$, where $\hat{g} \in A_n$ by Lemma 4.2.6. Then

$$\alpha_g\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & f(2-3g) & \dots & f((n-1)-ng) \end{pmatrix}$$

by Lemma 4.2.5, and it is easy to verify that $\alpha_g\alpha_\omega = \alpha_h$ with $1h = 0$, $2h = 1$, and $xh = x - f(x-1-xg)$ for $x \in \{3, 4, \dots, n\}$.
Now, we have $(\alpha_g\alpha_\omega)\alpha_\omega = \alpha_h\alpha_\omega$

$$= \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & f(2-3+f(2-3g)) & \dots & f(n-1-n+f((n-1)-ng)) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & f(f(2-3g)-1) & f(f(3-4g)-1) & \dots & f(f((n-1)-ng)-1) \end{pmatrix}.$$

On the other hand, Lemma 4.2.5 gives $\alpha_{\hat{g}}\alpha_\omega =$

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & f(2-2+f(f(2-3g)-1)) & \dots & f((n-1)-(n-1)+f(f((n-1)-ng)-1)) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 1 & f(f(2-3g)-1) & f(f(3-4g)-1) & \dots & f(f((n-1)-ng)-1) \end{pmatrix},$$

using $f(f(x)) = f(x)$ for $x \in X_n$. Hence, $\alpha_g\alpha_\omega\alpha_\omega = \alpha_{\hat{g}}\alpha_\omega$ and we obtain

$$(x_g x_\omega^2)\varphi = x_g\varphi x_\omega\varphi x_\omega\varphi = \alpha_g\alpha_\omega\alpha_\omega = \alpha_{\hat{g}}\alpha_\omega = x_{\hat{g}}\varphi x_\omega\varphi = (x_{\hat{g}}x_\omega)\varphi.$$

Let $g, h \in A_n$, where $g \oplus h \in A_n$ by Lemma 4.2.4. Then we can easily calculate

$$\alpha_g\alpha_h = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 3-3g-(3-3g)h & \dots & n-ng-(n-ng)h \end{pmatrix}.$$

For $x \in \{3,4,5,\dots,n\}$, we have $x - xg - (x - xg)h \geq 1$ since $x - xg - (x - xg)h$ is in the image of $\alpha_g\alpha_h \in O_{n,1}$; thus, $f(x-xg-(x-xg)h) = x-xg-(x-xg)h$. Hence, $(x_{g\oplus h}x_\omega)\varphi = x_{g\oplus h}\varphi x_\omega\varphi = \alpha_{g\oplus h}\alpha_\omega =$

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 3-3g-(3-3g)h & \dots & n-ng-(n-ng)h \end{pmatrix}$$

$$= \alpha_g\alpha_h = x_g\varphi x_h\varphi = (x_g x_h)\varphi.$$ □

Now, we need to show the converse inclusion $\ker\varphi \subset \sim$

Lemma 4.2.8. *Let $v_1, v_2, \dots, v_r \in \{x_g : g \in A_n\}$ with $r \geq 2$. Then there exists $h \in A_n$ such that*

$$v_1v_2\dots v_r \sim x_h x_\omega.$$

Proof. We prove by induction on r. If $r = 2$, then the statement is satisfied by R_2. Suppose that for $v_1, v_2, \dots, v_r \in \{x_g : g \in A_n\}$, for some integer $r \geq 2$, there exists $h \in A_n$ such that

$$v_1v_2\dots v_r \sim x_h x_\omega.$$

Let $v_1, v_2, \dots, v_{r+1} \in \{x_g : g \in A_n\}$. Then $v_1(v_2v_3\dots v_rv_{r+1}) \sim v_1(x_hx_\omega)$ for some $h \in A_n$. Furthermore, $(v_1x_h)x_\omega \sim (x_{\hat{h}}x_\omega)x_\omega$ by R_2, for some $\hat{h} \in A_n$. By R_1, there exists $\widetilde{h} \in A_n$ such that

$$x_{\hat{h}}x_\omega x_\omega \sim x_{\widetilde{h}}x_\omega,$$

That is,

$$v_1v_2\dots v_rv_{r+1} \sim x_{\widetilde{h}}x_\omega.$$

□

We put now $\hat{A}_n = \{g \in A_n : |x - xg| > 1 \text{ for all } x \in \{3,4,5,\dots,n\}\}$.

Lemma 4.2.9. *For all $h \in A_n$, there exists $g \in \hat{A}_n$ such that $x_gx_\omega \sim x_hx_\omega$.*

Proof. Let $h \in A_n$. Suppose that $i - ih = 1$ for some $i \in \{3,4,5,\dots,n\}$. Let

$$Q = \{x \in \{3,4,5,\dots,n\} : x - xh = 1\} \neq \emptyset.$$

Then $xh \geq 2$ for all $x \in Q$. We define g as;

$$xg = \begin{cases} xh & \text{if } x \notin Q, \\ xh - 1 & \text{if } x \in Q. \end{cases}$$

We need to show that $g \in A_n$.

We have $1g = 1h = 0$ and for $x \in \{2,\dots,n\}$,

$$xg = \begin{cases} xh \neq 0 & \text{if } x \notin Q, \\ xh - 1 \geq 2 - 1 = 1 \neq 0 & \text{if } x \in Q. \end{cases}$$

Also, by definition of $h \in A_n$, there exists $l \in \{3,4,5,\dots,n\}$ such that $lh = 1$, i.e., $|l - lh| \neq 1$ and $l \notin Q$. Thus, $lg = lh = 1$. Now, let $x \in \{2,3,4,\dots,n-1\}$ such that $(x+1)g > xg$. We consider the following cases:
Case a: $xg = xh$. Assume that $(x+1)g = (x+1)h - 1$. Then

$$(x+1)h - 1 = (x+1)g > xg = xh.$$

This gives $(x+1)h > (x+1)h - 1 > xh$, that is, $(x+1)h = xh + 1$ (since

$h \in A_n$). However, this leads to

$$(x+1)h = (x+1)h - 1 + 1 > xh + 1 = (x+1)h, \text{ which is a contradiction.}$$

Hence,

$$(x+1)g = (x+1)h.$$

Thus, $(x+1)h > xh$, that is, $(x+1)h = xh + 1$; therefore

$$(x+1)g = (x+1)h = xh + 1 = xg + 1.$$

Case b: $(x+1)g = (x+1)h$ and $xg = xh - 1$, that is, $(x+1)h \geq xh$. This implies that $(x+1)h = xh$ or $(x+1)h = xh + 1$. If $(x+1)h = xh + 1$, then

$$1 = x - xh = x + 1 - xh - 1 = x + 1 - (x+1)h \geq 2,$$

which is a contradiction. Thus, we have $(x+1)h = xh$. This means that $(x+1)g = (x+1)h = xh$, and hence $(x+1)g = xg + 1$.
Case c: $xg = xh - 1$ and $(x+1)g = (x+1)h - 1$. Then $(x+1)g > xg$ implies $(x+1)h > xh$, that is, $(x+1)h = xh + 1$. Thus,

$$(x+1)h - 1 = xh - 1 + 1,$$

and hence $(x+1)g = xg + 1$.

Altogether, this shows that $g \in A_n$.

Let $x \in \{3, 4, 5, \dots, n\}$. If $x \notin Q$, then $x - xg = x - xh > 1$. If $x \in Q$, then $x - xg = x - xh + 1 = 2 > 1$. This means that $x - xg > 1$ for all $x \in \{3, 4, 5, \dots, n\}$. Therefore, $g \in \hat{A}_n$.

Next, we show that $x_g x_\omega \sim x_h x_\omega$. We have

$$x_h x_\omega \approx x_{h \oplus \omega} x_\omega \in R_2 \text{ and } x_g x_\omega \approx x_{g \oplus \omega} x_\omega \in R_2.$$

If $x \in Q$, then $x - xh = 1$ and $x - xg = x - xh + 1 = 2$. This gives

$$(x - xh)\omega + xh - 1 = 0 + xh - 1 = xh - 1 = 1 + xg - 1 = (x - xg)\omega + xg - 1.$$

If $x \notin Q$, then $xh = xg$ and

$$(x - xh)\omega + xh - 1 = (x - xg)\omega + xg - 1.$$

This provides $x_{h\oplus\omega}x_\omega = x_{g\oplus\omega}x_\omega$. Therefore, we obtain $x_h x_\omega \sim x_g x_\omega$ by transitivity. □

We define a function $\mu : \hat{A}_n \longrightarrow O_{n,1}$ by $g \mapsto \alpha_g\alpha_\omega$for all $g \in \hat{A}_n$.

Lemma 4.2.10. *μ is an injection from $\hat{A}_n$ into $O_{n,1} \setminus G_{n,1}$.*

Proof. We show that μ is a function from $\hat{A}_n$ to $O_{n,1} \setminus G_{n,1}$. For this, we take $g \in \hat{A}_n$ such that

$$g\mu = \alpha_g\alpha_\omega \in O_{n,1}.$$

This shows that $\alpha_g\alpha_\omega \notin G_{n,1}$ by Lemma 3.3.1.

Next, we show that μ is injective. For this, let $g_1, g_2 \in \hat{A}_n$ such that $g_1\mu = g_2\mu$. By Lemma 4.2.5, we have: $g_1\mu = \alpha_{g_1}\alpha_\omega =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(3 - 3g_1 - 1) & f(4 - 4g_1 - 1) & \cdots & f(n - ng_1 - 1) \end{pmatrix}$$

and $g_2\mu = \alpha_{g_2}\alpha_\omega =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(3 - 3g_2 - 1) & f(4 - 4g_2 - 1) & \cdots & f(n - ng_2 - 1) \end{pmatrix}.$$

For $x \in \{3, 4, 5, \ldots, n\}$, we have $x - xg_1 > 1$ and $x - xg_2 > 1$. This means

$$x - xg_1 - 1 = f(x - xg_1 - 1) = f(x - xg_2 - 1) = x - xg_2 - 1.$$

Hence, $xg_1 = xg_2$. Altogether, we have $xg_1 = xg_2$ for all $x \in X_n$, which means $g_1 = g_2$. □

Theorem 4.2.11. *The semigroup $O_{n,1}$ has the presentation $\langle Y_n \mid R_1 \cup R_2 \rangle$ via φ.*

Proof. By Lemma 4.2.7, we have $\sim \subset \ker(\varphi)$.
It remains to show that $\ker(\varphi) \subset \sim$. For this, let $(w_1, w_2) \in \ker(\varphi)$.

By Lemmas 4.2.8 and 4.2.9, there exist $g_1, g_2 \in \hat{A}_n$ such that $\omega_1 \sim x_{g_1}x_\omega$ and $\omega_2 \sim x_{g_2}x_\omega$. Since $\sim \subset \ker(\varphi)$, we can calculate:

$$(x_{g_1}x_\omega)\varphi = w_1\varphi = w_2\varphi = (x_{g_2}x_\omega)\varphi.$$

In particular, $g_1\mu = \alpha_{g_1}\alpha_\omega = x_{g_1}x_\omega\varphi$ and $g_2\mu = \alpha_{g_2}\alpha_\omega = x_{g_2}x_\omega\varphi$. This provides $g_1\mu = g_2\mu$.

Since μ is injective by Lemma 4.2.10, we obtain $g_1 = g_2$. This gives:

$$w_1 \sim x_{g_1}x_\omega = x_{g_2}x_\omega \sim w_2,$$

That is, $w_1 \sim w_2$.
Consequently, $ker(\varphi) \subset \sim$, as required. □

We want to illustrate Theorem 4.2.11 for $n = 4$. It is easy to verify that $O_{4,1}$ consists of the five transformations:

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix},$$

$$d = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let $g_1, g_2 \in \{0, 1, 2, 3\}^{X_4}$ with $1g_1 = 1g_2 = 0$, $2g_1 = 2g_2 = 3g_2 = 4g_1 = 1$, and $3g_1 = 4g_2 = 2$. It is easy to see that $A_4 = \{g_1, g_2, \omega\}$, i.e., $Y_4 = \{x_{g_1}, x_{g_2}, \omega\}$.

For convenience, we use a, b, and c for x_{g_1}, x_{g_2}, and ω, respectively. The relations in R_1 provide:

$$ac^2 \approx bc, \quad bc^2 \approx bc, \quad cc^2 \approx bc.$$

The relations in R_2 provide:

$$\begin{aligned} aa &\approx bc, & ba &\approx bc, & ca &\approx bc, \\ ab &\approx ac, & bb &\approx bc, & cb &\approx ac, \\ ac &\approx ac, & bc &\approx bc, & cc &\approx ac. \end{aligned}$$

Thus, we obtain the following presentation for $O_{4,1}$:

$$\begin{aligned} \langle \{a, b, c\} \mid \{&ac^2 \approx bc,\ bc^2 \approx bc,\ cc^2 \approx bc,\ aa \approx bc,\ ba \approx bc,\ ca \approx bc, \\ &ab \approx ac,\ bb \approx bc,\ cb \approx ac,\ ac \approx ac,\ bc \approx bc,\ cc \approx ac.\}\rangle. \end{aligned}$$

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List of Notations

α, β, γ	Transformations.
$x\alpha$	Value of α at x.
$x\alpha = \varnothing$	α is not defined at x.
$\alpha : M \to M$	α is a transformation on M.
$dom(\alpha)$	Domain of transformation α.
$\overline{dom}(\alpha)$	Codomain of α.
$im(\alpha)$	Image of α.
$rank(\alpha)$	The cardinality of $im(\alpha)$.
$def(\alpha)$	The defect of α.
$\mathbf{N}, \mathbf{N}_n, X_n$	The set $\{1, 2, \ldots, n\}$.
$a \cdot b$	The product of a and b.
ab	The product of a and b.
$T < S$	T is a subsemigroup of S.
$\mathcal{PT}_n$	The set of all partial transformations of $\mathbf{N}$.
$\mathcal{T}_n$	The set of all total transformations of $\mathbf{N}$.
ε	The identity transformation.
ε_n	The identity transformation of $\mathbf{N}$.

$\mathbb{N}$	The set of all natural numbers.
$\|S\|$	The cardinality of S.
$\mathcal{IS}_n$	The symmetric inverse semigroup on $\mathbf{N}$.
$f : X \to Y$	A mapping from X to Y.
0	The nowhere defined partial transformation.
0_n	The nowhere defined partial transformation of $\mathbf{N}$.
0_a	The constant transformation with image $\{a\}$.
S^0	S if $0 \in S$, and $S \cup \{0\}$ otherwise.
S^1	S if $1 \in S$, and $S \cup \{1\}$ otherwise.
S^*	The group of units of the semigroup S.
$V_a(S)$	The set of elements, inverse to $a \in S$.
a^{-1}	The inverse of a.
$S \cong T$	S is isomorphic to T.
AB	The product of the sets A and B.
$\rho \circ \sigma$	Product of binary relations ρ and σ.
$\rho^\sharp$	The congruence generated by the binary relation ρ.
$a\rho b$	The pair (a, b) belongs to the binary relation ρ.
$a \equiv b$	a is equivalent to b.
$a \equiv b(\rho)$	a is equivalent to b with respect to the equivalence relation ρ.
S/ρ	The quotient of the semigroup S modulo the congruence ρ.
$\mathcal{L}$	Green's relation $\mathcal{L}$.
$\mathcal{L}(a)$	The $\mathcal{L}$-class of a.
$\mathcal{R}$	Green's relation $\mathcal{R}$.
$\mathcal{R}(a)$	The $\mathcal{R}$-class of a.
$\mathcal{H}$	Green's relation $\mathcal{H}$.
$\mathcal{H}(a)$	The $\mathcal{H}$-class of a.
$\mathcal{D}$	Green's relation $\mathcal{D}$.

$\mathcal{D}(a)$	The $\mathcal{D}$-class of a.
$\mathcal{J}$	Green's relation $\mathcal{J}$.
$\mathcal{J}(a)$	The $\mathcal{J}$-class of a.
$\ker\varphi$	The kernel of a map φ.
D_n	The set of all order-decreasing transformations on **N**.
O_n	The set of all order-preserving transformations on **N**.
C_n	The set of all order-decreasing and preserving transformations on **N**.
C_n	The $n^{th} - catalan$ number.
$Fix(\alpha)$	The set of all the elements fixed by α.
$O_{n,A}$	The set of all order-preserving transformations on **N** fixing exactly the elements of $A \subset \mathbf{N}$.
$O_{n,p}$	The set of all order-preserving transformations on **N** fixing only $p \in \mathbf{N}$.
c_x	The constant transformation with image $\{x\}$.
$\mathbb{R}$	The set of all real numbers.
$\lfloor r \rfloor$	The greatest integer less or equal to $r \in \mathbb{R}$.
$nd(a)$	The nilpotency degree or nilpotency class of the element a.
$N(S)$	The set of nilpotent elements of S.
$\langle A \rangle$	The set of all elements of S generated by $A \subset S$.
$rank(S)$	Rank of the semigroup S.
$G_{n,p}$	The set of all $\alpha \in O_{n,p}$ such that $\|x\alpha - x\| = 1$ for atleast one $x \in X_n \setminus \{p-1, p+1\}$.
A^+	The set of all finite words over alphabet A.
$\bar{a}_\rho$	The equivalence class of a with respect to the equivalence relation ρ.
$\bar{a}$	The equivalence class of a.
$\rho(S, A)$	Kernel of the natural epimorphism $A^+ \to S$.
$\rho\Sigma$	The minimal congruence containing all relations from Σ.

$\langle A \mid \Sigma \rangle$	Semigroup generated by A with defining relations Σ.
$\langle A \mid \rho \rangle$	A Presentation of a semigroup S in terms of generating set A and relation ρ via the map $\varphi : A^+ \to S$ with $\rho^\sharp = \ker \varphi$.
$g \in X^Y$	A mapping from Y to X.
A_n	The set of all mappings $g \in \{0, 1, 2, \ldots, n-1\}^{X_n}$.