Covalence Design and Implementation

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Contents

1. Core Calculus

We begin by giving the core calculus for the `covalence` kernel, as formalized in Lean 4. This is a variant of extensional Martin-Löf Type Theory (MLTT) based on a single judgement, $\Gamma \vdash a \equiv b : A$, which reads "$a$ is equal to b at type A in context Γ." The rules and grammar given below correspond to those in the formalization, which is in the locally-nameless style [1].

1.1. Grammar

Terms a, A, φ consist of:

Variables x

Universes $\mathcal{U}_\ell$ parametrized by a *level* $\ell \in \mathbb{N}$

Unit and empty types $\mathbf{1}_\ell, *_\ell, \mathbf{0}_\ell$

Equations $a =_A b$

Dependent functions $\Pi_\ell A.B$, $\lambda_\ell A.(b : B)$

Applications $\mathsf{app}_{A.B}(f, a)$

Dependent pairs $\Sigma_\ell A.B$, $\mathsf{mk}_{\ell A.B}(a, b)$

Pair projections $\pi_{lA.B}(e)$, $\pi_{rA.B}(e)$

Dependent if-then-else $\mathsf{ite}(\varphi, A, a, b)$

Propositional truncations $\|A\|$

Choice terms $\varepsilon A.\varphi$

Natural numbers $\mathbb{N}$, 0, s

Iteration $\mathsf{rec}_{\mathbb{N}}(C, n, z, s)$

We introduce syntax sugar:

Propositions $\mathbf{2} = \mathcal{U}_0$

Booleans $\top \coloneqq \mathbf{1}_0$, $\bot \coloneqq \mathbf{0}_0$

Functions $A \to B \coloneqq \Pi A.B^{\uparrow}$

Negations $\neg A \coloneqq A \to \bot$

Existential quantifiers $\exists A.\varphi \coloneqq \|\Sigma A.\varphi\|$

1.2. Judgements

Judgement	Reading	Definition
$\Gamma \vdash a \equiv b : B$	"In Γ, a is equal to b at type A"	Primitive
Γ ok	"Γ is a well-formed context"	$\Gamma \vdash 0 \equiv 0 : \mathbb{N}$
$\Gamma \vdash a : A$	"In Γ, a has type A"	$\Gamma \vdash a \equiv a : A$

Judgement	Reading	Definition
$\Gamma \vdash A$	"In Γ, A is inhabited"	$\exists a. \Gamma \vdash a : A$

1.3. Rules

1.3.1. Context well-formedness

$$\frac{}{\cdot \vdash 0 \equiv 0 : \mathbb{N}}\text{nil-ok} \qquad \frac{\Gamma \vdash 0 \equiv 0 : \mathbb{N} \quad x \notin \Gamma \quad \Gamma \vdash A : \mathcal{U}_\ell}{\Gamma, x : A \vdash 0 \equiv 0 : \mathbb{N}}\text{cons-ok}$$

$$\frac{\Gamma \ \mathsf{ok}}{\Gamma \vdash \mathcal{U}_\ell : \mathcal{U}_{\ell+1}}\mathcal{U} \qquad \frac{\Gamma \ \mathsf{ok} \quad \Gamma(x) = A}{\Gamma \vdash x : A}\text{var}$$

1.3.2. Congruence Rules

$$\frac{\Gamma \ \mathsf{ok}}{\Gamma \vdash \mathbf{1}_\ell : \mathcal{U}_\ell}\mathbf{1} \qquad \frac{\Gamma \ \mathsf{ok}}{\Gamma \vdash *_\ell : \mathbf{1}_\ell}* \qquad \frac{\Gamma \ \mathsf{ok}}{\Gamma \vdash \mathbf{0}_\ell : \mathcal{U}_\ell}\mathbf{0} \qquad \frac{\Gamma \ \mathsf{ok}}{\Gamma \vdash \mathbb{N} : \mathcal{U}_1}\mathbb{N} \qquad \frac{\Gamma \ \mathsf{ok}}{\Gamma \vdash \mathsf{s} : \mathbb{N} \to \mathbb{N}}\mathsf{s}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_\ell \quad \Gamma \vdash a \equiv a' : A \quad \Gamma \vdash b \equiv b' : A}{\Gamma \vdash (a =_A b) \equiv (a' =_{A'} b') : \mathbf{2}}\text{eqn}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n}{\Gamma \vdash \Pi_{\mathsf{imax}(m,n)} A.B \equiv \Pi_{\mathsf{imax}(m,n)} A'.B' : \mathcal{U}_{\mathsf{imax}(m,n)}}\Pi$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n \quad \Gamma \vdash f \equiv f' : \Pi A.B \quad \Gamma \vdash a \equiv a' : A}{\Gamma \vdash \mathsf{app}_{A.B}(f, a) \equiv \mathsf{app}_{A'.B'}(f', a') : B^a}\text{app}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n \quad \Gamma, x : A \vdash b^x \equiv b'^x : B^x}{\Gamma \vdash \lambda_{\mathsf{imax}(m,n)} A.(b : B) \equiv \lambda_\ell A'.(b' : B') : \Pi_{\mathsf{imax}(m,n)} A.B}\lambda$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n}{\Gamma \vdash \Sigma_{m \sqcup n} A.B \equiv \Sigma_{m \sqcup n} A'.B' : \mathcal{U}_{m \sqcup n}}\Sigma$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n \quad \Gamma \vdash a \equiv a' : A \quad \Gamma \vdash b \equiv b' : B^a}{\Gamma \vdash \mathsf{mk}_{\ell A.B}(a, b) \equiv \mathsf{mk}_{\ell A'.B'}(a', b') : \Sigma_{m \sqcup n} A.B}\text{mk}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n \quad \Gamma \vdash e \equiv e' : \Sigma_{m \sqcup n} A.B}{\Gamma \vdash \pi_{l A.B}(e) \equiv \pi_{l A'.B'}(e') : A}\pi_l$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n \quad \Gamma \vdash e \equiv e' : \Sigma_{m \sqcup n} A.B}{\Gamma \vdash \pi_{r A.B}(e) \equiv \pi_{r A'.B'}(e') : B^{\pi_{l A.B}(e)}}\pi_r$$

$$\frac{\begin{array}{c}\Gamma \vdash \varphi \equiv \varphi' : \mathbf{2} \quad \Gamma \vdash A \equiv A' : \mathcal{U}_\ell \quad \Gamma, x : \varphi \vdash a \equiv a' : A \quad x \notin \mathsf{fv}(a) \\ \Gamma, y : \neg\varphi \vdash b \equiv b' : A \quad y \notin \mathsf{fv}(b)\end{array}}{\Gamma \vdash \mathsf{ite}(\varphi, A, a, b) \equiv \mathsf{ite}(\varphi', A', a', b') : A}\text{dite}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash \|A\| \equiv \|A'\| : \mathbf{2}}\text{tr} \qquad \frac{\Gamma \vdash A \equiv A' : \mathcal{U}_\ell \quad \Gamma \vdash \|A\| \equiv \top : \mathbf{2} \quad \Gamma, x : A \vdash \varphi^x \equiv \varphi'^x : \mathbf{2}}{\Gamma \vdash \varepsilon A.\varphi \equiv \varepsilon A'.\varphi' : A}\varepsilon$$

$$\frac{\begin{array}{c}\Gamma, x : \mathbb{N} \vdash C^x \equiv C'^x : \mathcal{U}_\ell \quad \Gamma \vdash n \equiv n' : \mathbb{N} \quad \Gamma \vdash z \equiv z' : C^0 \\ \Gamma, x : \mathbb{N} \vdash s^x \equiv s'^x : C^x \to C^{\mathsf{app}_{\mathbb{N}.\mathbb{N}}(\mathsf{s}, x)}\end{array}}{\Gamma \vdash \mathsf{rec}_\mathbb{N}(C, n, z, s) \equiv \mathsf{rec}_\mathbb{N}(C', n', z', s') : C^n}\mathsf{rec}_\mathbb{N}$$

1.3.3. Equations

$$\frac{\Gamma \vdash A : \mathbf{2} \qquad \Gamma \vdash a : A}{\Gamma \vdash a \equiv *_0 : \mathbf{2}}\,!_* \qquad \frac{\Gamma \vdash a : \mathbf{0}_\ell}{\Gamma \vdash \top \equiv \bot : \mathbf{2}}\,\bot_{\mathbf{0}} \qquad \frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash (a =_A b) \equiv \top : \mathbf{2}}\,\top_=$$

$$\frac{\Gamma \vdash A : \mathcal{U}_m \qquad \Gamma, x : A \vdash B^x : \mathcal{U}_n \qquad \Gamma \vdash a : A \qquad \Gamma, x : A \vdash b^x : B^x}{\Gamma \vdash \mathsf{app}_{A.B}\left(\lambda_{\mathsf{imax}(m,n)} A.(b : B), a\right) \equiv b^a : B^a}\,\beta$$

$$\frac{\Gamma \vdash A : \mathcal{U}_m \qquad \Gamma, x : A \vdash B^x : \mathcal{U}_n \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : B^a}{\Gamma \vdash \pi^l_{l A.B}(\mathsf{mk}_{m \sqcup n A.B}(a, b)) \equiv a : A}\,\beta_{\pi_l}$$

$$\frac{\Gamma \vdash A : \mathcal{U}_m \qquad \Gamma, x : A \vdash B^x : \mathcal{U}_n \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : B^a}{\Gamma \vdash \pi_{r A.B}(\mathsf{mk}_{m \sqcup n A.B}(a, b)) \equiv b : B^a}\,\beta_{\pi_r}$$

$$\frac{\Gamma \vdash \varphi \equiv \top : \mathbf{2} \qquad \Gamma \vdash A : \mathcal{U}_\ell \qquad \Gamma, x : \varphi \vdash a : A \qquad x \notin \mathsf{fv}(a) \qquad \Gamma, y : \neg\varphi \vdash b : A \qquad y \notin \mathsf{fv}(b)}{\Gamma \vdash \mathsf{ite}(\varphi, A, a, b) \equiv a : A}\,\beta_\top$$

$$\frac{\Gamma \vdash \varphi \equiv \bot : \mathbf{2} \qquad \Gamma \vdash A : \mathcal{U}_\ell \qquad \Gamma, x : \varphi \vdash a : A \qquad x \notin \mathsf{fv}(a) \qquad \Gamma, y : \neg\varphi \vdash b : A \qquad y \notin \mathsf{fv}(b)}{\Gamma \vdash \mathsf{ite}(\varphi, A, a, b) \equiv b : A}\,\beta_\bot$$

$$\frac{\Gamma, x : \mathbb{N} \vdash C^x : \mathcal{U}_\ell \qquad \Gamma \vdash z : C^0 \qquad \Gamma, x : \mathbb{N} \vdash s^x : C^x \to C^{\mathsf{app}_{\mathbb{N}.\mathbb{N}}(\mathsf{s}, x)}}{\Gamma \vdash \mathsf{rec}_\mathbb{N}(C, 0, z, s) \equiv z : C^0}\,\mathsf{rec}_\mathbb{N}\text{–}\mathsf{0}$$

$$\frac{\Gamma, x : \mathbb{N} \vdash C^x : \mathcal{U}_\ell \qquad \Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash z : C^0 \qquad \Gamma, x : \mathbb{N} \vdash s^x : C^x \to C^{\mathsf{app}_{\mathbb{N}.\mathbb{N}}(\mathsf{s}, x)}}{\Gamma \vdash \mathsf{rec}_\mathbb{N}(C, \mathsf{app}_{\mathbb{N}.\mathbb{N}}(\mathsf{s}, n), z, s) \equiv \mathsf{app}_{C^n . C^{\mathsf{app}_{\mathbb{N}.\mathbb{N}}(\mathsf{s}, n)}}\left(s, \mathsf{rec}_\mathbb{N}(C, n, z, s)\right) : C^{\mathsf{app}_{\mathbb{N}.\mathbb{N}}(\mathsf{s}, n)}}\,\mathsf{rec}_\mathbb{N}\text{-}\mathsf{s}$$

1.3.4. Axioms

$$\frac{\Gamma \vdash a =_A b}{\Gamma \vdash a \equiv b : A}\,\text{ext} \qquad \frac{\Gamma \vdash A : \mathbf{2} \qquad \Gamma \vdash B : \mathbf{2} \qquad \Gamma \vdash A \to B \qquad \Gamma \vdash B \to A}{\Gamma \vdash A \equiv B : \mathbf{2}}\,\text{propext}$$

$$\frac{\Gamma \vdash A : \mathcal{U}_m \qquad \Gamma, x : A \vdash B^x : \mathcal{U}_n \qquad \Gamma \vdash e : \Sigma_{m \sqcup n} A.B}{\Gamma \vdash \mathsf{mk}_{m \sqcup n A.B}\left(\pi^l_{m \sqcup n.A.B}(e), \pi^r_{m \sqcup n.A.B}(e)\right) \equiv e : \Sigma_{m \sqcup n} A.B}\,\text{ext}_\Sigma$$

1.3.5. Closure

$$\frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A}\,\text{trans} \qquad \frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A}\,\text{symm}$$

$$\frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash A \equiv B : \mathcal{U}_\ell}{\Gamma \vdash a \equiv b : B}\,\text{cast}$$

Bibliography

[1] A. Charguéraud, "The Locally Nameless Representation," *J. Autom. Reason.*, vol. 49, no. 3, pp. 363–408, 2012, doi: 10.1007/S10817-011-9225-2.