Covalence Typing Rules

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CCS Concepts: • **Mathematics of computing → Solvers**; • **Theory of computation → Type theory**; **Logic and verification**; **Higher order logic**; **Equational logic and rewriting**; • **Computing methodologies → Symbolic and algebraic manipulation**.

Additional Key Words and Phrases: E-graphs, extensional MLTT

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1 Syntax

TODO 1: copy over from Lean

TODO 2: lore about capture-avoiding substitution and closure

2 Sugar

$$\Pi x : A.B := \Pi A.B_x \qquad \Sigma x : A.B := \Sigma A.B_x \qquad \epsilon x : A.\varphi := \epsilon A.\varphi_x$$
$$\mathsf{ite}(x : \varphi)\ a\ b := \mathsf{ite}(\varphi)\ a_x\ b_x$$

$$\mathbf{2} := \mathcal{U}_0 \qquad \bot := \mathbf{0} \qquad \top := \mathbf{1} \qquad \neg\varphi := (\varphi = \bot) \qquad \varphi \land \psi := \Sigma\varphi.\psi^{\uparrow} \qquad \varphi \lor \psi := \mathsf{ite}(\varphi)\ \top\ \psi^{\uparrow}$$
$$\varphi \Rightarrow \psi := \Pi\varphi.\psi^{\uparrow} \qquad \forall A.\varphi := \Pi A.\varphi \qquad \exists A.\varphi := \|\Sigma A.\varphi\|$$
$$A \times B := \Sigma A.B^{\uparrow} \qquad A + B := \Sigma\varphi : \mathbf{2}.\mathsf{ite}(\varphi)\ A^{\uparrow}\ B^{\uparrow} \qquad \iota_1\ a := (\bot, a) \qquad \iota_2\ b := (\top, b)$$

3 Rules

3.1 Sugar

$$\frac{\Gamma \vdash a \equiv a : A}{\Gamma \vdash a : A}\ \text{has-ty-def} \qquad \frac{\Gamma \vdash * : \mathbf{1}}{\Gamma\ \mathsf{ok}}\ \text{ok-def}$$

3.2 Well-formedness

$$\frac{}{\cdot\ \mathsf{ok}}\ \text{nil-ok} \qquad \frac{\Gamma\ \mathsf{ok} \quad x \notin \mathsf{dom}(\Gamma) \quad \Gamma \vdash A \equiv A : \mathcal{U}_\ell}{\Gamma, x : A\ \mathsf{ok}}\ \text{cons-ok}$$

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3.3 Congruence

$$\frac{\Gamma\ \mathsf{ok} \quad \Gamma(x) = A}{\Gamma \vdash x \equiv x : A}\,\textsf{var} \qquad \frac{\Gamma\ \mathsf{ok}}{\Gamma \vdash \mathcal{U}_\ell \equiv \mathcal{U}_\ell : \mathcal{U}_{\ell+1}}\,\textsf{univ} \qquad \frac{\Gamma\ \mathsf{ok}}{\Gamma \vdash \mathbf{0} \equiv \mathbf{0} : \mathcal{U}_\ell}\,\textsf{empty}$$

$$\frac{\Gamma\ \mathsf{ok}}{\Gamma \vdash \mathbf{1} \equiv \mathbf{1} : \mathcal{U}_\ell}\,\textsf{unit} \qquad \frac{\Gamma \vdash a \equiv a' : A \quad \Gamma \vdash b \equiv b' : A}{\Gamma \vdash a = b \equiv a' = b' : \mathbf{2}}\,\textsf{eqn}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \forall x \notin L. \Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n \quad \mathsf{imax}(m, n) \leq \ell}{\Gamma \vdash \Pi A.B \equiv \Pi A'.B' : \mathcal{U}_\ell}\,\textsf{pi}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \forall x \notin L. \Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \forall x \notin L. \Gamma, x : A \vdash b^x \equiv b'^x : B^x}{\Gamma \vdash \lambda A.b \equiv \lambda A'.b' : \mathcal{U}_\ell}\,\textsf{abs}$$

$$\frac{\Gamma \vdash A \equiv A : \mathcal{U}_m \quad \forall x \notin L. \Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \Gamma \vdash f \equiv f' : \Pi A.B \quad \Gamma \vdash a \equiv a' : A}{\Gamma \vdash f\ a \equiv f'\ a' : B^a}\,\textsf{app}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \forall x \notin L. \Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n \quad m \leq \ell \quad n \leq \ell}{\Gamma \vdash \Sigma A.B \equiv \Sigma A'.B' : \mathcal{U}_\ell}\,\textsf{sigma}$$

$$\frac{\Gamma \vdash A \equiv A : \mathcal{U}_m \quad \forall x \notin L. \Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \Gamma \vdash a \equiv a' : A \quad \Gamma \vdash b \equiv b' : B^a}{\Gamma \vdash (a, b) \equiv (a', b') : \Sigma A.B}\,\textsf{pair}$$

$$\frac{\Gamma \vdash A \equiv A : \mathcal{U}_m \quad \forall x \notin L. \Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \Gamma \vdash p \equiv p' : \Sigma A.B}{\Gamma \vdash \pi_1\ p \equiv \pi_1\ p' : A}\,\textsf{fst}$$

$$\frac{\Gamma \vdash A \equiv A : \mathcal{U}_m \quad \forall x \notin L. \Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \Gamma \vdash p \equiv p' : \Sigma A.B}{\Gamma \vdash \pi_2\ p \equiv \pi_2\ p' : B^{\pi_1\ p}}\,\textsf{snd}$$

$$\frac{\begin{array}{c}\Gamma \vdash \varphi \equiv \varphi' : \mathbf{2} \quad \forall x \notin L. \Gamma, x : \varphi \vdash a^x \equiv a'^x : A \\ \Gamma \vdash A \equiv A : \mathcal{U}_\ell \quad \forall x \notin L. \Gamma, x : \neg\varphi \vdash b^x \equiv b'^x : A\end{array}}{\Gamma \vdash \mathsf{ite}(\varphi)\ a\ b \equiv \mathsf{ite}(\varphi)\ a'\ b' : A}\,\textsf{dite} \qquad \frac{\Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash \|A\| \equiv \|A'\| : \mathbf{2}}\,\textsf{trunc}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_\ell \quad \Gamma \vdash \|A\| \equiv \top : \mathbf{2} \quad \forall x \notin L. \Gamma, x : A \vdash \varphi^x \equiv \varphi'^x : \mathbf{2}}{\Gamma \vdash \epsilon A.\varphi \equiv \epsilon A'.\varphi' : A}\,\textsf{choose}$$

$$\frac{\Gamma\ \mathsf{ok}}{\Gamma \vdash \mathbb{N} \equiv \mathbb{N} : \mathcal{U}_\ell}\,\textsf{nats} \qquad \frac{\Gamma\ \mathsf{ok}}{\Gamma \vdash 0 \equiv 0 : \mathbb{N}}\,\textsf{zero} \qquad \frac{\Gamma \vdash n \equiv n' : \mathbb{N}}{\Gamma \vdash \mathsf{s}\ n \equiv \mathsf{s}\ n' : \mathbb{N}}\,\textsf{succ}$$

$$\frac{\forall x \notin L. \Gamma, x : \mathbb{N} \vdash C^x \equiv C'^x : \mathcal{U}_\ell \quad \Gamma \vdash z \equiv z' : C^0 \quad \forall x \notin L. \Gamma, x : \mathbb{N} \vdash s \equiv s' : C^x \to C^{\mathsf{s}\ x}}{\Gamma \vdash \mathsf{rec}_{\mathbb{N}}\{C\}\ z\ s \equiv \mathsf{rec}_{\mathbb{N}}\{C'\}\ z'\ s' : \Pi\mathbb{N}.C}\,\textsf{rec}_{\mathbb{N}}$$

$$\frac{\Gamma \vdash a \equiv a' : A \quad \Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash [a \in A] \equiv [a' \in A'] : \mathbf{2}}\,\textsf{has-ty}$$

3.4 Conversion

$$\frac{\Gamma \vdash a \equiv a' : A \quad \Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash [a \in A] \equiv \top : \mathbf{2}}\ \text{has-ty}_\top \qquad \frac{\Gamma \vdash a \equiv a' : A \quad \Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash [a \in A] \equiv \top : \mathbf{2}}\ \text{has-ty}_\top$$

$$\frac{\Gamma \vdash \lambda A.b : \Pi A.B \quad \Gamma \vdash a : A \quad \Gamma \vdash (\lambda A.b)\ a : C \quad \Gamma \vdash b^a : C}{\Gamma \vdash (\lambda A.b)\ a \equiv b^a : C}\ \beta_{\text{app}}$$

$$\frac{\Gamma \vdash (a, b) : \Sigma A.B \quad \Gamma \vdash \pi_1\ (a, b) : A \quad \Gamma \vdash a : A}{\Gamma \vdash \pi_1\ (a, b) \equiv a : A}\ \beta_{\pi_1}$$

$$\frac{\Gamma \vdash (a, b) : \Sigma A.B \quad \Gamma \vdash \pi_2\ (a, b) : C \quad \Gamma \vdash b : C}{\Gamma \vdash \pi_2\ (a, b) \equiv b : C}\ \beta_{\pi_2}$$

$$\frac{\Gamma \vdash A : \mathcal{U}_\ell \quad \forall x \notin L. \Gamma, x : \top \vdash a^x : A \quad \Gamma \vdash \mathsf{ite}(\top)\ a\ b : A \quad \Gamma \vdash a^* : A}{\Gamma \vdash \mathsf{ite}(\top)\ a\ b \equiv a : A}\ \beta_\top$$

$$\frac{\Gamma \vdash A : \mathcal{U}_\ell \quad \forall x \notin L. \Gamma, x : \neg\bot \vdash b^x : A \quad \Gamma \vdash \mathsf{ite}(\bot)\ a\ b : A \quad \Gamma \vdash b^* : B}{\Gamma \vdash \mathsf{ite}(\bot)\ a\ b \equiv b : A}\ \beta_\bot$$

$$\frac{\Gamma \vdash \mathsf{rec}_{\mathbb{N}}\{C\}\ z\ s\ 0 : D \quad \Gamma \vdash z : D}{\Gamma \vdash \mathsf{rec}_{\mathbb{N}}\{C\}\ z\ s\ 0 \equiv z : D}\ \beta_0$$

$$\frac{\Gamma \vdash \mathsf{rec}_{\mathbb{N}}\{C\}\ z\ s\ (\mathsf{s}\ n) : D \quad \Gamma \vdash s^n\ (\mathsf{rec}_{\mathbb{N}}\{C\}\ z\ s\ n) : D}{\Gamma \vdash \mathsf{rec}_{\mathbb{N}}\{C\}\ z\ s\ (\mathsf{s}\ n) \equiv (\mathsf{s}\ n) : D}\ \beta_{\mathsf{s}}$$

3.5 Choice and Extensionality

TODO 3: this

3.6 Equivalence and Casting

TODO 4: this