

# Covalence Typing Rules

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CCS Concepts: • **Mathematics of computing** → **Solvers**; • **Theory of computation** → **Type theory**; **Logic and verification**; **Higher order logic**; **Equational logic and rewriting**; • **Computing methodologies** → **Symbolic and algebraic manipulation**.

Additional Key Words and Phrases: E-graphs, extensional MLTT

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## 1 Syntax

$$a ::= x \mid \#i \mid \mathcal{U}_\ell \mid 0 \mid 1 \mid * \mid a = b \mid \Pi A. B \mid \Sigma A. B \mid f \ a \mid \lambda A. b \mid (a, b) \mid \pi_1 \ p \mid \pi_2 \ p \\ \mid \text{ite}(\varphi) \ a \ b \mid \|A\| \mid \epsilon A. \varphi \mid \mathbb{N} \mid 0 \mid s \ n \mid \text{rec}_{\mathbb{N}}\{C\} \ s \ z \\ \mid [a \in A] \mid \dagger$$

## 2 Sugar

$$\Pi x : A. B := \Pi A. B_x \quad \Sigma x : A. B := \Sigma A. B_x \quad \epsilon x : A. \varphi := \epsilon A. \varphi_x \\ \text{ite}(x : \varphi) \ a \ b := \text{ite}(\varphi) \ a_x \ b_x$$

$$2 := \mathcal{U}_0 \quad \perp := 0 \quad \top := 1 \quad \neg \varphi := (\varphi = \perp) \quad \varphi \wedge \psi := \Sigma \varphi. \psi^\uparrow \quad \varphi \vee \psi := \text{ite}(\varphi) \ \top \ \psi^\uparrow \\ \varphi \Rightarrow \psi := \Pi \varphi. \psi^\uparrow \quad \forall A. \varphi := \Pi A. \varphi \quad \exists A. \varphi := \|\Sigma A. \varphi\| \\ A \times B := \Sigma A. B^\uparrow \quad A + B := \Sigma \varphi : 2. \text{ite}(\varphi) \ A^\uparrow \ B^\uparrow \quad \iota_1 \ a := (\perp, a) \quad \iota_2 \ b := (\top, b)$$

## 3 Rules

### 3.1 Sugar

$$\frac{\Gamma \vdash a \equiv a : A}{\Gamma \vdash a : A} \text{has-ty-def} \quad \frac{\Gamma \vdash * : 1}{\Gamma \text{ ok}} \text{ok-def}$$

### 3.2 Well-formedness

$$\frac{}{\cdot \text{ ok}} \text{nil-ok} \quad \frac{\Gamma \text{ ok} \quad x \notin \text{dom}(\Gamma) \quad \Gamma \vdash A \equiv A : \mathcal{U}_\ell}{\Gamma, x : A \text{ ok}} \text{cons-ok}$$

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### 3.3 Congruence

$$\begin{array}{c}
\frac{\Gamma \text{ ok} \quad \Gamma(x) = A}{\Gamma \vdash x \equiv x : A} \text{var} \quad \frac{\Gamma \text{ ok}}{\Gamma \vdash \mathcal{U}_\ell \equiv \mathcal{U}_\ell : \mathcal{U}_{\ell+1}} \text{univ} \quad \frac{\Gamma \text{ ok}}{\Gamma \vdash \mathbf{0} \equiv \mathbf{0} : \mathcal{U}_\ell} \text{empty} \\
\frac{\Gamma \text{ ok}}{\Gamma \vdash \mathbf{1} \equiv \mathbf{1} : \mathcal{U}_\ell} \text{unit} \quad \frac{\Gamma \vdash a \equiv a' : A \quad \Gamma \vdash b \equiv b' : A}{\Gamma \vdash a = b \equiv a' = b' : \mathbf{2}} \text{eqn} \\
\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \forall x \notin L.\Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n \quad \text{imax}(m, n) \leq \ell}{\Gamma \vdash \Pi A.B \equiv \Pi A'.B' : \mathcal{U}_\ell} \text{pi} \\
\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \forall x \notin L.\Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \forall x \notin L.\Gamma, x : A \vdash b^x \equiv b'^x : B^x}{\Gamma \vdash \lambda A.b \equiv \lambda A'.b' : \mathcal{U}_\ell} \text{abs} \\
\frac{\Gamma \vdash A \equiv A : \mathcal{U}_m \quad \forall x \notin L.\Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \Gamma \vdash f \equiv f' : \Pi A.B \quad \Gamma \vdash a \equiv a' : A}{\Gamma \vdash f a \equiv f' a' : B^a} \text{app} \\
\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_m \quad \forall x \notin L.\Gamma, x : A \vdash B^x \equiv B'^x : \mathcal{U}_n \quad m \leq \ell \quad n \leq \ell}{\Gamma \vdash \Sigma A.B \equiv \Sigma A'.B' : \mathcal{U}_\ell} \text{sigma} \\
\frac{\Gamma \vdash A \equiv A : \mathcal{U}_m \quad \forall x \notin L.\Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \Gamma \vdash a \equiv a' : A \quad \Gamma \vdash b \equiv b' : B^a}{\Gamma \vdash (a, b) \equiv (a', b') : \Sigma A.B} \text{pair} \\
\frac{\Gamma \vdash A \equiv A : \mathcal{U}_m \quad \forall x \notin L.\Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \Gamma \vdash p \equiv p' : \Sigma A.B}{\Gamma \vdash \pi_1 p \equiv \pi_1 p' : A} \text{fst} \\
\frac{\Gamma \vdash A \equiv A : \mathcal{U}_m \quad \forall x \notin L.\Gamma, x : A \vdash B^x \equiv B^x : \mathcal{U}_n \quad \Gamma \vdash p \equiv p' : \Sigma A.B}{\Gamma \vdash \pi_2 p \equiv \pi_2 p' : B^{\pi_1 p}} \text{snd} \\
\frac{\Gamma \vdash \varphi \equiv \varphi' : \mathbf{2} \quad \forall x \notin L.\Gamma, x : \varphi \vdash a^x \equiv a'^x : A \quad \Gamma \vdash A \equiv A : \mathcal{U}_\ell \quad \forall x \notin L.\Gamma, x : \neg \varphi \vdash b^x \equiv b'^x : A}{\Gamma \vdash \text{ite}(\varphi) a b \equiv \text{ite}(\varphi) a' b' : A} \text{dite} \quad \frac{\Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash \|A\| \equiv \|A'\| : \mathbf{2}} \text{trunc} \\
\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_\ell \quad \Gamma \vdash \|A\| \equiv \top : \mathbf{2} \quad \forall x \notin L.\Gamma, x : A \vdash \varphi^x \equiv \varphi'^x : \mathbf{2}}{\Gamma \vdash \epsilon A.\varphi \equiv \epsilon A'.\varphi' : A} \text{choose} \\
\frac{\Gamma \text{ ok}}{\Gamma \vdash \mathbb{N} \equiv \mathbb{N} : \mathcal{U}_\ell} \text{nats} \quad \frac{\Gamma \text{ ok}}{\Gamma \vdash \mathbf{0} \equiv \mathbf{0} : \mathbb{N}} \text{zero} \quad \frac{\Gamma \vdash n \equiv n' : \mathbb{N}}{\Gamma \vdash s n \equiv s n' : \mathbb{N}} \text{succ} \\
\frac{\forall x \notin L.\Gamma, x : \mathbb{N} \vdash C^x \equiv C'^x : \mathcal{U}_\ell \quad \Gamma \vdash z \equiv z' : C^0 \quad \forall x \notin L.\Gamma, x : \mathbb{N} \vdash s \equiv s' : C^x \rightarrow C^{s x}}{\Gamma \vdash \text{rec}_{\mathbb{N}}\{C\} z s \equiv \text{rec}_{\mathbb{N}}\{C'\} z' s' : \Pi \mathbb{N}.C} \text{rec}_{\mathbb{N}} \\
\frac{\Gamma \vdash a \equiv a' : A \quad \Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash [a \in A] \equiv [a' \in A'] : \mathbf{2}} \text{has-ty}
\end{array}$$

### 3.4 Conversion

$$\begin{array}{c}
\frac{\Gamma \vdash a \equiv a' : A \quad \Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash [a \in A] \equiv \top : 2} \text{has-ty}_\top \quad \frac{\Gamma \vdash a \equiv a' : A \quad \Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash [a \in A] \equiv \top : 2} \text{has-ty}_\top \\
\frac{\Gamma \vdash \lambda A.b : \Pi A.B \quad \Gamma \vdash a : A \quad \Gamma \vdash (\lambda A.b) a : C \quad \Gamma \vdash b^a : C}{\Gamma \vdash (\lambda A.b) a \equiv b^a : C} \beta_{\text{app}} \\
\frac{\Gamma \vdash (a, b) : \Sigma A.B \quad \Gamma \vdash \pi_1 (a, b) : A \quad \Gamma \vdash a : A}{\Gamma \vdash \pi_1 (a, b) \equiv a : A} \beta_{\pi_1} \\
\frac{\Gamma \vdash (a, b) : \Sigma A.B \quad \Gamma \vdash \pi_2 (a, b) : C \quad \Gamma \vdash b : C}{\Gamma \vdash \pi_2 (a, b) \equiv b : C} \beta_{\pi_2} \\
\frac{\Gamma \vdash A : \mathcal{U}_\ell \quad \forall x \notin L.\Gamma, x : \top \vdash a^x : A \quad \Gamma \vdash \text{ite}(\top) a b : A \quad \Gamma \vdash a^* : A}{\Gamma \vdash \text{ite}(\top) a b \equiv a : A} \beta_\top \\
\frac{\Gamma \vdash A : \mathcal{U}_\ell \quad \forall x \notin L.\Gamma, x : \neg \perp \vdash b^x : A \quad \Gamma \vdash \text{ite}(\perp) a b : A \quad \Gamma \vdash b^* : B}{\Gamma \vdash \text{ite}(\perp) a b \equiv b : A} \beta_\perp \\
\frac{\Gamma \vdash \text{rec}_{\mathbb{N}}\{C\} z s 0 : D \quad \Gamma \vdash z : D}{\Gamma \vdash \text{rec}_{\mathbb{N}}\{C\} z s 0 \equiv z : D} \beta_0 \\
\frac{\Gamma \vdash \text{rec}_{\mathbb{N}}\{C\} z s (s n) : D \quad \Gamma \vdash s^n (\text{rec}_{\mathbb{N}}\{C\} z s n) : D}{\Gamma \vdash \text{rec}_{\mathbb{N}}\{C\} z s (s n) \equiv (s n) : D} \beta_s
\end{array}$$

### 3.5 Choice and Extensionality

$$\begin{array}{c}
\frac{\Gamma \vdash A : \mathcal{U}_\ell \quad \Gamma \vdash \|A\| \equiv \top : 2 \quad \forall x \notin L.\Gamma, x : A \vdash \varphi^x : 2 \quad \Gamma \vdash \exists A.\varphi \equiv \top : 2 \quad \Gamma \vdash \varphi^{\epsilon A.\varphi} : 2}{\Gamma \vdash \varphi^{\epsilon A.\varphi} \equiv \top : 2} \text{choose-spec} \\
\frac{\Gamma \vdash A : \mathcal{U}_m \quad \Gamma \vdash f : \Pi A.B \quad \forall x \notin L.\Gamma, x : A \vdash B^x : \mathcal{U}_n \quad \Gamma \vdash g : \Pi A.B \quad \forall x \notin L.\Gamma \vdash f x \equiv g x : B^x}{\Gamma \vdash f \equiv g : \Pi A.B} \text{pi-ext} \\
\frac{\Gamma \vdash A : \mathcal{U}_m \quad \Gamma \vdash p : \Sigma A.B \quad \forall x \notin L.\Gamma, x : A \vdash B^x : \mathcal{U}_n \quad \Gamma \vdash q : \Sigma A.B \quad \Gamma \vdash \pi_1 p \equiv \pi_1 q : A \quad \Gamma \vdash \pi_2 p \equiv \pi_2 q : B^{\pi_1 p}}{\Gamma \vdash p \equiv q : \Sigma A.B} \text{sigma-ext} \\
\frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash a = b \equiv \top : 2} \text{eqn-rfl} \quad \frac{\Gamma \vdash a : A \quad \Gamma \vdash b : A \quad \Gamma \vdash a = b \equiv \top : 2}{\Gamma \vdash a \equiv b : A} \text{eqn-ext}
\end{array}$$

### 3.6 Equivalence and Casting

$$\begin{array}{c}
\frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A} \text{symm} \quad \frac{\Gamma \vdash a \equiv b : A \quad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A} \text{trans} \\
\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_\ell \quad \ell \leq \ell'}{\Gamma \vdash A \equiv A' : \mathcal{U}_{\ell'}} \text{cast}_{\leq} \quad \frac{\Gamma \vdash a \equiv a' : A \quad \Gamma \vdash A \equiv A' : \mathcal{U}_\ell}{\Gamma \vdash a \equiv a' : A'} \text{cast} \\
\frac{\Gamma \vdash a \equiv b : A \quad \Gamma \vdash a : B}{\Gamma \vdash a \equiv b : B} \text{transfer}
\end{array}$$