MAT257 Notes

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Let's begin with a $\mathcal{C}^r$ mapping $f : \mathbb{R}^n \to \mathbb{R}^p$. Well first of all, we know what the tangent mapping is, or rather we know what the derivative is. And we want to interpret this derivative as a linear mapping induced by f

$$f_{*a} : \mathbb{R}^n_a \to \mathbb{R}^p_{f(a)}, f_{*a}(v_a) = Df(a)(v)_{f(a)} \tag{1}$$

Now, by duality, we want to say that f_a induces in the other direction a mapping that we'll call f_a^* that takes alternating k-tensors on the tangent space $\mathbb{R}^p_{f(a)}$ to alternating k-tensors on the tangent space $\mathbb{R}^n_a$, i.e.

$$f_a^* : \Omega^k(\mathbb{R}^p_{f(a)}) \to \Omega^k(\mathbb{R}^n_a) \tag{2}$$

Let's try the case where $k = 1$. In this case, alternating 1-tensors are just 1-tensors, which are just elements of the dual space, i.e.

$$\Omega^1(\mathbb{R}^n_a) = \mathcal{T}^1(\mathbb{R}^n_a) = (\mathbb{R}^n_a)^* \tag{3}$$

So, if $T \in (\mathbb{R}^p_{f(a)})^*$, we can write

$$f_a^*(T)(v_a) = T(f_{*a}(v_a)) \tag{4}$$

In general, we do the exact same thing for $\omega \in \Omega^k(\mathbb{R}^p_{f(a)})$:

$$f_a^*(\omega)(v_{1,a}, ..., v_{k,a}) = \omega(f_{*a}(v_1), ..., f_{*a}(v_k)) \tag{5}$$

Now, we want to show that f induces a mapping f^* inducing a linear mapping whih takes, instead of pointwise objects as above, k-forms on $\mathbb{R}^p$ to k-forms on $\mathbb{R}^n$, with a k-form on $\mathbb{R}^p$ taking the form

$$\omega : b \in \mathbb{R}^p \mapsto \omega(b) \in \Omega^k(\mathbb{R}^p_b) \tag{6}$$

So we can write

$$(f^*\omega)(a) = f_a^*(\omega(f(a))) \tag{7}$$

which is the same as saying

$$(f^*\omega)(a)(v_{1,a}, ..., v_{k,a}) = \omega(f(a))(f_{*a}(v_{1,a}), ..., f_{*a}(v_{n,a})) \tag{8}$$

that is, just repeating what the definition is about. So this is the formal definition. I understand that everyone would like to forget this as soon as possible, so I'm going to try to help.

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be $\mathcal{C}^r$. Then*

$$f^*(dy_i) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j = df_i = d(y_i \circ f) \tag{9}$$

where

$$dy_i = \sum \frac{\partial y_i}{\partial x_j} dx_j \tag{10}$$

Recall also that

$$f_*\left(\frac{\partial}{\partial x_j}\right) = \sum \frac{\partial f_j}{\partial x_i} \frac{\partial}{\partial y_j} \tag{11}$$

Before we prove this, recall that equation 7 is true for every k, including $k = 0$. And what's a 0-form? It's just a function. So in this case, with a function (zero form) g,

$$f^*(g) = g \circ f \tag{12}$$

Let's get to the proof:

Proof. By definition,

$$f^*(dy_i)(a)(v_a) = dy_i(f(a))(f_{*a}v_a) \tag{13}$$

Now, what is the right hand side here? It's the matrix of partial derivatives of $f(a)$ applied to v_a. So if $v_a = (v_1, ..., v_n)$, then equation 13 is equal to

$$dy_i(f(a)) \left(\sum_{k=1}^{p} \mathbf{e}_{k,f(a)} \sum_{j=1}^{n} \frac{\partial f_k}{\partial x_j}(a) v_j \right) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(a) v_j = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(a) dx_j(a) v_a \tag{14}$$

Since this is true for any v_a, it follows that

$$f^*(dy_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j \tag{15}$$

as desired. $\square$

Let's do an example: assume gx is a $\mathcal{C}^r$ function on $\mathbb{R}^n$:

$$f^*(dg) = f^* \left(\sum_{i=1} \frac{\partial g}{\partial y_i} dy_i \right) = \sum_{i=1}^{p} \frac{\partial g}{\partial y_i} \circ f f^*(dy_i) = \sum_{i=1}^{n} \frac{\partial g}{\partial y_i} \circ f \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \tag{16}$$

Changing the order of the summation and using the chain rule, we obtain

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial g}{\partial y_i} \circ f \circ \frac{\partial f_i}{\partial x_j} \right) dx_j = \sum_{j=1}^{n} \frac{\partial (g \circ f)}{\partial x_j} dx_j = d(g \circ f) \tag{17}$$

Let's extend the proposition with some "trivial" facts:

Proposition 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be $\mathcal{C}^r$. Then*

1. $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$, *i.e.* f^* *is linear*
2. $f^*(g \cdot w) = (g \circ f) f^* \omega$
3. $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

Proof. Immediate from the definitions $\square$

Proposition 3. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^p$. Then*

$$f^*(g dx_1 \wedge ... \wedge dx_n) = g \circ f \det g' \cdot dx_1 \wedge ... \wedge dx_n \tag{18}$$

This is why the natural object of integration is not functions but rather differential forms. Because if we understand that the object we're integrating is a differential form, not a function, then the formula for change of variable is built into the definition of a differential form.

$$\int g dx_1 \wedge ... \wedge dx_n = \int g dx_1 ... dx_n \tag{19}$$

Let's get to the proof:

Proof. Consider

$$f^*(dx_1 \wedge ... \wedge dx_n)(a)(e_{1,a}, ..., e_{n,a}) \tag{20}$$

$\square$