MAT257 Notes

Jad Elkhaleq Ghalayini

January 18 2019

1 Change of Variable Theorem

Theorem 1. *Let $A \subset \mathbb{R}^n$ be open, $g : A \to \mathbb{R}^n$ be one to one, continuously differentiable and let, for all $x \in A$, $g'(x) \neq 0$. Then*

$$f : g(A) \to \mathbb{R} \tag{1}$$

is integrable if and only if

$$f \circ g |\det g'| \tag{2}$$

is integrable on A. In this case,

$$\int_{g(A)} f = \int_A f \circ g |\det g'| \tag{3}$$

Let's look at some examples, starting with polar coordinates: we use coordinates $r \in \mathbb{R}_0^+, \theta \in [0, 2\pi]$ and write

$$x = r\cos\theta, y = r\sin\theta \tag{4}$$

We have

$$D = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \implies \det D = r \tag{5}$$

So we can write

$$\iint_A f(x,y)dxdy = \iint_A f(r\cos\theta, r\sin\theta)rdrd\theta \tag{6}$$

We now examine a corollary of Theorem 1.

Corollary. *Let $A \subset C \subset \mathbb{R}^n$ where A is open, C is compact and Jordan-measurable and $C \setminus A$ has measure zero. If g is a continuously differentiable function from a neighborhood of C to $\mathbb{R}^n$ wich satisfies the conditions of theorem 1 on A, then*

$$f : g(C) \to \mathbb{R} \tag{7}$$

is integrable if and only if

$$f \circ g |\det g'| \tag{8}$$

is iintegrable on C, and in this case

$$\int_{g(C)} f = \int_C f \circ g |\det g'| \tag{9}$$

Lemma 1. *Assume $A \subset \mathbb{R}^n$ is open and $g : A \to \mathbb{R}^n$ is continuously differentiable. If $B \subset A$ has measure zero, then $g(B)$ has measure zero.*

Proof. Enough to prove that $g(B \cap C)$ has measure zero for any $C \subset A$ compact, since A has an exhaustion by countably many compact sets $C_1 \subset C_2 \subset ...$

To do so, remember that a countable intersection of measure 0 sets is measure 0. Using $\mathcal{C}_1$, which is more than uniformly continuous, we have that

$$\forall x \in C, \forall y \in U, |g(x) - g(y)| \leq c|x - y| \tag{10}$$

where U is some neighborhood of C. So g maps a ball of radius ϵ to a ball of radius $c\epsilon$. □

We now proceed to prove Corollary 1

Proof. $g(C) \setminus g(A) \subseteq g(C \setminus A)$, and so it is of measure zero. We hence have that

$$\int_{g(A)} f = \int_{g(C)} f \tag{11}$$

$$\int_A (f \circ g)|\det g'| = \int_C (f \circ g)|\det g'| \tag{12}$$

giving the desired equality by Theorem 1 $\square$

Let's move on to another example: what are called spherical coordinates. We use coordinates $r \in \mathbb{R}_0^+, \phi \in [0, 2\pi], \theta \in [0, \pi]$ where

$$x = r\cos\phi\sin\theta, y = r\sin\phi\sin\theta, z = r\cos]theta \tag{13}$$

We have... this is going to hurt...

$$D = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{pmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \implies \det D = r^2\sin\theta \tag{14}$$