

MAT257 Notes

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Recall that if $\dim V = n$, then $\dim \Omega^n(V) = 1$. Furthermore, every $\omega \in \Omega^n(\mathbb{R}^n)$ is a multiple of \det . More generally, we have the following lemma:

Lemma 1. *If $\omega \in \Omega^n(V)$, and $v_1, \dots, v_n \in V$, let*

$$w_i = \sum_{j=1}^n a_{ij} v_j \quad (1)$$

Then, if $A = (a_{ij}) \in \mathbb{R}^{n \times n}$,

$$\omega(w_1, \dots, w_n) = \det(A) \omega(v_1, \dots, v_n) \quad (2)$$

Proof. Let's express things in terms of an n -form on \mathbb{R}^n . Define $\eta \in \Omega^n(\mathbb{R}^n)$ by equation 2 as on the columns of A (being a function taking in n vectors) as

$$\eta((a_{11}, \dots, a_{1n}), (a_{21}, \dots, a_{2n}), \dots, (a_{n1}, \dots, a_{nn})) = \omega\left(\sum a_{1j} v_j, \dots, \sum a_{nj} v_j\right) \quad (3)$$

We have that, for some $\lambda \in \mathbb{R}$

$$\eta = \lambda \det \implies \eta(e_1, \dots, e_n) = \lambda \det(e_1, \dots, e_n) = \lambda = \omega(v_1, \dots, v_n) \quad (4)$$

But this tells us that

$$\omega(w_1, \dots, w_n) = \omega(v_1, \dots, v_n) \det(a_j) \quad (5)$$

since the left hand side of equation 5 is equal to equation 3. One thing that's good to point out is the following: suppose v_1, \dots, v_n were linearly dependent. This would tell us that $\omega(v_1, \dots, v_n)$ and hence $\omega(w_1, \dots, w_n)$ would be zero. \square

In general, this shows us that if $\dim V = n$, any nonzero $\omega \in \Omega^n(V)$ divides the set of all bases of V into two groups:

- Those where $\omega(v_1, \dots, v_n) > 0$
- Those where $\omega(v_1, \dots, v_n) < 0$

Another way of saying the same thing without using differential forms is saying that (v_1, \dots, v_n) and (w_1, \dots, w_n) are in the same group if one can be transformed into the other with a matrix A having positive determinant.

1 Orientations of V

Definition 1. *Define an orientation of V to be a choice of one of these two groups. We can consider this as providing an equivalence relation: two bases are equivalent if they belong to the same group. We will write these as*

- $[v_1, \dots, v_n]$ to denote the equivalence class of v_1, \dots, v_n
- $-[v_1, \dots, v_n]$ to denote the equivalence class of the opposite group

Definition 2. The standard orientation of \mathbb{R}^n is $[e_1, \dots, e_n]$

For example, in \mathbb{R}^3 , the standard orientation is given by $[e_1, e_2, e_3]$, giving the physics right hand rule: if you put your index and middle fingers in the direction of e_1, e_2 , your thumb points in the direction of e_3 .

Suppose V has an inner product T . In this case, we can consider orthonormal bases with respect to T . So consider two different orthonormal bases v_1, \dots, v_n and w_1, \dots, w_n with respect to T . That is,

$$T(v_i, v_j) = T(w_i, w_j) = \delta_{ij} \quad (6)$$

So we can write one in terms of the other, i.e. find $A = (a_{ij})$ such that

$$w_i = \sum_{a_{ij}} v_j \quad (7)$$

where $\det A = \pm 1$. How do we see that? Well, let's compute:

$$T(w_i, w_j) = T\left(\sum_k a_{ik} v_k, \sum_\ell a_{j\ell} v_\ell\right) = \sum_{k,\ell} a_{ik} a_{j\ell} T(v_k, v_\ell) = \sum_{k,\ell} a_{ik} a_{j\ell} \delta_{k\ell} = \sum_k a_{ik} a_{jk} = \delta_{ij} \quad (8)$$

So everything is zero here unless $i = j, k = \ell$, in which this becomes

$$(AA^T)_{ij} = \delta_{ij} \implies AA^T = I \implies \det A^2 = 1 \implies \det A = \pm 1 \quad (9)$$

as desired. We can now obtain the following result

Lemma 2. Given inner product T and orientation μ for V , there is a unique $\omega \in \Omega^n(V)$ such that $\omega(v_1, \dots, v_n) = 1$ whenever v_1, \dots, v_n are an orthonormal basis and $[v_1, \dots, v_n] = \mu$.

The point is, if have an orthonormal basis v_1, \dots, v_n where $\omega(v_1, \dots, v_n) = 1$ and take *another* orthonormal basis w_1, \dots, w_n , then

$$\omega(w_1, \dots, w_n) = \pm \omega(v_1, \dots, v_n) = \pm 1 \quad (10)$$

which, of course, will be 1 if $(w_1, \dots, w_n) \in \mu$. This is what we're going to call volume. More precisely, we call ω the volume form on V . Of course, this is determined by an inner product and an orientation.

For example, the volume element (form) of \mathbb{R}^n determined by the standard inner product and the standard orientation is \det . We're usually going to use the word "form" to mean an alternating tensor at every point, but it doesn't matter, as alternating tensor and form can be used interchangeably.

One thing to mention before we leave this stuff is

1.1 The Cross Product

We have that

$$\det \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ w_1 & w_2 & w_3 \end{pmatrix} = \langle v_1 \times v_2, w \rangle \quad (11)$$

where $v_1 \times v_2$ denotes the cross product. It turns out you can do the same thing with $n - 1$ vectors in \mathbb{R}^n . Define $\varphi \in \Omega^1(V) = \mathcal{T}^1(V) = V^*$, i.e. let φ be a linear function from $\mathbb{R}^n \rightarrow \mathbb{R}$, where

$$\varphi(w) = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix} \quad (12)$$

Then there is a unique $z \in V$ such that $\varphi(w) = \langle z, w \rangle$. This is something you proved last year in linear algebra... I hope... So we'll call z the cross product of v_1, \dots, v_{n-1} , $v_1 \times v_2 \times \dots \times v_{n-1}$. Note that the cross product is an alternating multilinear form. It also satisfies the distributive property.

Wh do we want to use this? Well, we want to start proving the following:

$$\int_{\text{cube}} f(x_1, \dots, x_n) dx_1 \wedge \dots \wedge dx_n = \int_{\text{cube}} f dx_1 \dots dx_n \quad (13)$$

And we'll get to that next time...