MAT257 Notes

Jad Elkhaleq Ghalayini

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Examples:

1. Let
$$f(x, y) = \int_a^{x+y} g$$
where $g : \mathbb{R} \to \mathbb{R}$ is continuous. We compute $Df(c, d)$ as follows: We have that $f = q \circ s$, where
$$q(t) = \int_a^t g, s(x, y) = x + y$$
We have
$$Df(c, d) = g'(s(c, d))s'(c, d) = g(c + d)(c + d)$$
since $q'(t) = g(t)$.

2. Let
$$f(x, y) = \int_a^{x^y} g$$
We have
$$f = g \circ \xi, \xi(x, y) = x^y = e^{y \log x}$$
Hence,
$$\begin{gathered} D\xi(x, y) = x^y D(y \log x) = x^y((0, 1) \log x + (1/x, 0)y) = x^y(y/x, \log x) \\ \implies Df(c, d) = q'(\xi(c, d))\xi'(c, d) = g(c^d)c^d(d/c, \log c) \end{gathered}$$

Higher-order Derivatives

Let $f : U \to \mathbb{R}$ be a function where $U \subseteq \mathbb{R}^m$ and suppose
$$D_i f = \frac{\partial f}{\partial x_i} : U \to \mathbb{R}$$
exists for all i. So we could now consider
$$D_j(D_i f) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$
We'll write $D_{ij} f$ to denote the above. Notice i is applied *first*. We may also write $f_{x_i x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$. We have that
$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$$
if both mixed partials exist and are continuous in a neighborhood of a (proof uses $\int$).

In general, we can consider taking higher order partials as well, as in
$$\frac{\partial^{\alpha_1 + ... + \alpha_n} f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$
Of course we have to worry about the order, but the order is irrelevant if f is $\mathcal{C}^\infty$, i.e. that all partial derivatives of all orders exist (and are hence continuous).

Multi-index notation

In multi-index notation, $\alpha = (\alpha_1, ..., \alpha_m)$ is a vector of non-negative integers. We define the *total order of* α

$$|\alpha| = \alpha_1 + ... + \alpha_m$$

Furthermore, we define

$$x = (x_1, ..., x_m) \implies x^\alpha = x_1^{\alpha_1} ... x_m^{\alpha_m}$$

Once we look at Taylor's theorem, the notation

$$x! = x_1! ... x_m!$$

will also come in handy.

We can now perform another example:

3. Let

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Is this function differentiable at the origin? Yes: $f'(0,0) = (0,0)$. Let's check: we need to show that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \cancel{(0,0)\begin{pmatrix} x \\ y \end{pmatrix}}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2}\frac{1}{\sqrt{x^2+y^2}} = 0$$

We have that

$$\left| xy\frac{x^2-y^2}{x^2+y^2}\frac{1}{\sqrt{x^2+y^2}} \right| \leq \frac{|xy|}{\sqrt{x^2+y^2}} \to 0$$