MAT257 Notes

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1 The Volume Element

We already talked about the volume element in the context of alternating tensors. Of course, we're going to apply that to the tangent space at every point of an oriented k-dimensional manifold $M \subseteq \mathbb{R}^n$ (with or without bound). Consider a particular orientation μ. If we look at the tangent space M_x at a point $x \in M$, it has an orientation μ_x and also has an inner product T_x given by the standard inner product in $\mathbb{R}^n$.

These two things, as we saw before, are what we need to talk about a volume element (at the point x), which is an alternating k-tensor on the tangent space

$$\omega(x) \in \Omega^k(M_x) \tag{1}$$

which is the unique element in the space of alternating k-tensors such that if $v_1, ..., v_k$ is a positively oriented orthonormal basis of M_x,

$$\omega(x)(v_1, ..., v_k) = 1 \tag{2}$$

In particular, for any $w_1, ..., w_k \in M_x$,

$$\omega(x)(w_1, ..., w_k) \tag{3}$$

is the *oriented* k-dimensional volume of the parallelopiped determined by $w_1, ..., w_k$, implying it's absolute value is the k-dimensional volume of the parallelopiped determined by $w_1, ..., w_k$. I didn't say these $w_1, ..., w_k$ were linearly independent, but if they weren't the k-dimensional volume would be zero. Also notice that if $w_1, ..., w_k$ are linearly independent, then

$$[w_1, ..., w_k] = [v_1, ..., v_k] \implies \omega(x)(w_1, ..., w_k) > 0 \tag{4}$$

For an example, note that det is the volume element of $\mathbb{R}^n$ (at every point) with the standard inner product and orientation. Note that, taken over the whole manifold, ω is by definition a (nonzero, and $\mathcal{C}^r$ if M is $\mathcal{C}^r$) k-form. We often call this form dV, but be careful: it's not necessarily the differential of some $(k-1)-$ form "V". In low dimensions,

- Where $k = 1$, dV is often written ds for <u>length</u>
- Where $k = 2$, dV is often written dA or dS for <u>surface area</u>

We're going to use these for specific versions of Stokes' Theorem.

2 The Volume of M

Definition 1. *Provided it exists (e.g., if M is compact), we define the <u>volume</u> of M to be*

$$\int_M dV \tag{5}$$

Let's look at some examples:

1. Let M be some n-dimensional submanifold of $\mathbb{R}^n$ with the standard orientation. Then

$$dV = dx_1 \wedge ... \wedge dx^n \tag{6}$$

   Hence we have that

$$\int_M dV = \int_M 1 \tag{7}$$

   giving us that the volume of M agrees with our old definition.

2. Another situation that we've looked at before is the case where M is a 1-dimensional oriented submanifold of $\mathbb{R}^n$ (a curve). Note that all 1-dimensional submanifolds are orientable.

   It doesn't make sense in general to say "let's compute the length of this", because the length may not even be finite. But let's compute the length of a finite piece. This means in general that we'll be looking at an oriented 1-cube in M

$$c : [0, 1] \to M \tag{8}$$

   Let's try to compute the length of $c([0, 1])$, i.e. just that piece. That means we're going to integrate

$$\int_{c([0,1])} ds = \int_c ds = \int_{[0,1]} c^*(ds) = \int_{[0,1]} \sqrt{\dot{c_1}^2 + ... + \dot{c_n}^2} \tag{9}$$

   Why? Say $c^*(ds) = f(t)dt$. That means that

$$f(t) = c^*(ds)(t)(e_{1,t}) = ds(c(t))(c_{*,t}(e_{1,t})) = ds(c(t))(c'(t)_{c(t)}) = |c'(t)| \tag{10}$$

3. Let's see how to compute k-dimensional volume (in fact, the integral of a k-form in general) on a k-dimensional manifold: let M be a k-dimensional oriented submanifold of $\mathbb{R}^n$ with orientation μ, and let ω be a k-form on M. We can write

$$\omega = \lambda dV \tag{11}$$

   where λ is a function on M. Let $\varphi : W \to M$ be a $\mathcal{C}^{r+1}$ one-to-one function of constant rank k (e.g. a coordinate chart). Provided the integral exists, then, we have

$$\int_{\varphi(W)} \omega = \int \omega\varphi^*(\omega) = \int_W \lambda \circ \varphi\varphi^* dV \tag{12}$$

   Say, for $x \in W$,

$$\varphi^* dV = h(x)dx_1 \wedge ... \wedge dx_k \tag{13}$$

   How do we compute $h(x)$. It's:

$$(\varphi^* dV)(x)(e_{1,x}, ..., e_{k,x}) \tag{14}$$