MAT257 Notes

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April 2019

This document is a collection of notes for the course MAT257: Analysis II, as taught by Professor Edward Bierstone in 2018 at the University of Toronto. The notes are a combination of notes I made in class (which can be found in their original form in the `notes` folder in this repository) and scans of handwritten notes which Professor Bierstone has generously given me the permission to use.

1 Introduction

TODO: this

2 Differentiation

TODO: this

3 Integration

3.1 The (Riemann) Integral Over a Rectangle

TODO: this

3.2 Integrals Over More General Bounded Sets

TODO: this

3.3 Fubini's Theorem

TODO: rewrite

We are now going to talk about Fubini's theorem, which is about how to integrate over a high-dimensional rectangle by repeatedly performing individual integrals. Let's start with an example. Suppose we want to integrate over a rectangle

$$A = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \tag{1}$$

and let's suppose, to make it easy, we have a continuous, non-negative function f, defined on this rectangle, say

$$z = f(x, y) \geq 0 \tag{2}$$

The idea is that if we fix a point on the x-axis, say x, we can consider the "slice" in the y-direction determined by this point. We could find, for example, the area of that slice, and it's reasonable to expect that the integral of f on the rectangle, we could obtain by integrating the area of that slice along the length of the rectangle.

The idea then is to try to find the area of such a slice, which would be in this case, for some x,

$$h(x) = \int_c^d g_x(y)dy = \int_c^d f(x, y)dy \tag{3}$$

where $g_x(y) = f(x,y)$ fixing x. It's reasonable to expect that

$$\int_A f = \int_a^b h = \int_a^b \left(\int_c^d f(x,y)dy \right) dx \tag{4}$$

So that's the idea of Fubini's theorem: that we should be able to integrate a function over a rectangle by repeated one-dimensional integrals. Of course, we're not interested in integrating only continuous functions, we want to look at more general, integrable functions, but if you think about that, supposing f is integrable, this could run into a problem: one of the functions g_x might not be integrable on $[c,d]$. After all, it's set of discontinuities could be $x_0 \times [c,d]$ for some x_0, so $\int_c^d g_{x_0}$ makes no sense!

So we'll have to formulate something maybe a little bit more technical, but it's to capture that problem. Suppose we just have $f : A \to \mathbb{R}$ bounded. The function may or may not be integrable, meaning the supremum of lower sums is equal to the infimum of lower sums. Whether the function is integrable or not, we can still look at the supermum of lower sums and the infimum of lower sums, which is what we'll do.

We'll define the <u>lower</u> and <u>upper integrals</u> of f on A, $\mathrm{L}\int_A f$ and $\mathrm{U}\int_A f$ respectively, to be the supremum of all the lower sums $L(f,\mathcal{P})$ and infimum of all the upper sums $U(f,\mathcal{P})$ respectively. We can now write down our theorem as the previous formula, but taking into account that we don't know that the function g_x mentioned before, we just replace it by the lower or upper integral:

Theorem 1. *Suppose $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are closed rectangles and $f : A \times B \to \mathbb{R}$ is integrable. For all $x \in A$, define*

$$g_x : B \to \mathbb{R}, g_x(y) = f(x,y) \tag{5}$$

Set

$$\mathcal{L}(x) = \mathrm{L}\int_B g_x = \mathrm{L}\int_B f(x,y)dy, \mathcal{U}(x) = \mathrm{U}\int_B g_x = \mathrm{U}\int_B f(x,y)dy \tag{6}$$

Then $\mathcal{L}(x), \mathcal{U}(x)$ are integrable on A and

$$\int_{A\times B} f = \int_A \mathcal{L} = \int_A \mathcal{U} = \int_A \left(\mathrm{L}\int_B f(x,y)dy \right) dx = \int_A \left(\mathrm{U}\int_B f(x,y)dy \right) dx \tag{7}$$

Before we prove this, some remarks:

1. We also have
$$\int_{A\times B} f = \int_B \left(\mathrm{L}\int_A (f(x,y)dx) \right) dy = \int_B \left(\mathrm{U}\int_A (f(x,y)dx) \right) dy \tag{8}$$
2. If g_x is integrable on B for every $x \in A$, then
$$\int_{A\times B} f = \int_A \left(\int_B f(x,y)dy \right) dx \tag{9}$$
3. If
$$A = [a_1,b_1] \times [a_2,b_2] \times ... \times [a_n,b_n] \tag{10}$$
we can apply Fubini's theorem repeatedly to obtain
$$\int_A f = \int_{a_n}^{b_n} \left(... \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x_1,...,x_n)dx_1 \right) dx_2 ... \right) dx_n \tag{11}$$
if f is "sufficiently nice" (otherwise we'll have to sprinkle in some L's or U's).

Now here's an application to $\int_C f$ when f is integrable on a Jordan-measurable set $C \subset \mathbb{R}^n$. What we're going to use is Fubini's theorem in a rectangle with the formula

$$\int_C f = \int_A f\chi_C \tag{12}$$

where $A \supseteq C$ is a closed rectangle. Let's do some quick examples:

1. Let's say we were integrating on the region

$$C = [-1,1] \times [-1,1] \setminus \{x \in \mathbb{R}^2, x_1^2 + x_2^2 < 1\} \tag{13}$$

i.e. the unit square with the unit circle removed from it. We have

$$\int_C f = \int_{[-1,1]^2} f\chi_C \tag{14}$$

We can write

$$\chi_C = \begin{cases} 1 & \text{if } -1 \leq y \leq -\sqrt{1-x^2} \text{ or } \sqrt{1-x^2} \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

We can write

$$\int_{-1}^{1} f(x,y)\chi_C(x,y)dy = \int_{-1}^{-\sqrt{1-x^2}} f(x,y)dy + \int_{\sqrt{1-x^2}}^{1} f(x,y)dy \tag{16}$$

i.e. the limits of integration correspond to the boundary of C. So we get

$$\int_C f = \int_{-1}^{1} \left(\int_{-1}^{-\sqrt{1-x^2}} f(x,y)dy + \int_{\sqrt{1-x^2}}^{1} f(x,y)dy \right) dx \tag{17}$$

2. Say we want to integrate over a triangle C, under the line from $(0,0)$ to (a,a). We have

$$\int_C f = \int_0^a \left(\int_y^a f(x,y)dx \right) dy \tag{18}$$

We could have done it the other way, integrating first with respect to y and then with respect to x, and get

$$\int_C f = \int_0^a \left(\int_0^x f(x,y)dy \right) dx \tag{19}$$

This shows the form might not be the same in different directions. And it could be important to exploit the difference. For example, what if our function $f(x,y)$ only depends on one of the variables, say y (i.e. f is *independent* of x). Then

$$\int_0^a \left(\int_0^x f(y)dy \right) dx \tag{20}$$

doesn't simplify very much, but

$$\int_0^a \left(\int_y^a f(y)dx \right) dy = \int_0^a (a-y)f(y)dy \tag{21}$$

So this double integral reduces to a single integral with respect to y.

TODO: proof

TODO: polish Before we move on, we will give some more examples of computing integrals, using Fubini's theorem and the change of variables theorems as tools.

Let us begin by recalling some facts about the regions on whuch we can integrate. What are the kinds of region on which we integrate? One of the kinds of regions would be the region between the graphs of two functions that are defined on a Jordan-measurable set. Specifically, if we have a subset $C \subset \mathbb{R}^n$ that is Jordan-measurable, and two integrable functions $\varphi(x) \leq \psi(x)$ defined on C, we're interested in integrating over the region bounded by the graphs of the two functions Let's call this region S, that is, define

$$S = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : x \in C, y \in [\varphi(x), \psi(x)]\} \tag{22}$$

We have that S is a Jordan-measurable set. We want to be able to integrate a continuous function on S. If $f(x,y)$ is a bounded continuous on S, then we can compute using Fubini's theorem that

$$\int_S f = \int_C \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy \right) dx \tag{23}$$

We went through the proof of this before, but I want to use it to compute some examples.

(a) Consider

$$\int_0^2 \int_{y/2}^1 ye^{-x^3} dxdy \tag{24}$$

There's no hope in proceeding naively, since e^{-x^3} doesn't even have an elementary primitive, i.e. you cannot write down it's integral with only elementary functions. But let's consider what region we're integrating along: the area under the line between $(0,0)$ and $(2,1)$. So using the above observation, we can rewrite the integral to be

$$\int_0^1 \left(\int_0^{2x} ye^{-x^3} dy \right) dx = \int_0^1 \frac{y^2}{2} e^{-x^3} \Big|_0^{2x} dx = 2\int_0^1 x^2 e^{-x^3} dx = -\frac{2}{3} e^{-x^3} \Big|_0^1 = \frac{2}{3}(1-e^{-1}) \tag{25}$$

Here, the change of variable was a useful thing to do, because it enabled us to write down an original integral, which we couldn't do in the original form.

(b) Consider

$$\int_2^4 \int_{4/x}^{\frac{20-4x}{8-x}} (y-4) dydx \tag{26}$$

So here, again, should we just go ahead and do it as it's written? Well then we'll get $\frac{y^2}{2} - 4y$ and we'll substitute those things in and we'll just get a terrible mess. But what's the region we're integrating over? $\frac{4}{x}$ is like a hyperbola, which is 2 when x is 2 and 1 when x is 4. The other function on top, what does it look like? It's also a hyperbola: it's a constant plus something over $8 - x$, so it'll open down. And this is the region we're integrating on: the area between a hyperbola open up and another opening down.

If we change the order of integration, we'll be integrating on the outside with respect to y, which goes from 1 to 2, and we'll be integrating on the inside with respect to x with x going from $4/y$ to... let's solve:

$$y = \frac{20-4x}{8-x} = 4 + \frac{12}{x-8} \implies x - 8 = \frac{12}{y-4} \iff x = 8 + \frac{12}{y-4} \tag{27}$$

Hence we can write the above as

$$\int_1^2 \left(\int_{4/y}^{8+12/(y-4)} (y-4) dx \right) dy = \int_1^2 (y-4)x \Big|_{4/y}^{8+12/(y-4)} dy = \int_1^2 (y-4)\left(8 + \frac{12}{y-4} - \frac{4}{y}\right) dy \tag{28}$$

which is all stuff that's very easy to integrate

(c) Let's see how to integrate a simple function z over a region of 3-space

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \le z^2 \wedge x^2 + y^2 + z^2 \le 1\} \tag{29}$$

So, what is this region? The second equation says we're inside the closed ball of radius 1 centered at the origin. The first equation is the region inside two cones, opening up and down from the origin (kind of like an hour glass). So we want the intersection of these two regions. In terms of the theorem that we wrote down at the beginning, the set C is like the discs inside the circles formed by the intersections of the top and bottom of the cones and circles, and we're integrating on the region between the semicircles above and below the plane, and the cones above and below the plane. So what's this going to be? Well, by symmetry, it's going to be *zero*.

Let's make it a little harder. Let's consider only the top, i.e.

$$S^+ = \{(x,y,z) \in S : z \ge 0\}$$

So, rewriting our integral to be over the disc C, we obtain

$$\begin{aligned}
&\int\int_{x^2+y^2\leq\frac{1}{2}}\left(\int_{\sqrt{x^2+y^2}}^{\sqrt{1-(x^2+y^2)}} zdz\right)dxdy \\
&= \int\int_{x^2+y^2\leq\frac{1}{2}}\frac{1}{2}(1-(x^2+y^2)-(x^2+y^2))dxdy \\
&= \frac{1}{2}\int\int_{x^2+y^2\leq\frac{1}{2}} dxdy - \int\int_{x^2+y^2-\frac{1}{2}}(x^2+y^2)dxdy = \frac{1}{2}\pi\frac{1}{2} = \frac{\pi}{4}
\end{aligned} \tag{30}$$

How do we justify that last, "magical" step? Well, the prettiest way to do so is to change to polar coordinates, but we haven't justified this quite yet, which is the point of this example. You'll remember from first year caclulus that we can write

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \tag{31}$$

This is what we're going to be doing in several variables. But first: polar coordinates? That means writing

$$x = r\cos\theta, y = r\cos\theta, \theta \in [0, 2\pi], r \in [0, \infty) \tag{32}$$

This gives that

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \implies \det\frac{\partial(x,y)}{\partial(r,\theta)} = r \implies dxdy = drd\theta \tag{33}$$

"generalizing" equation 31. On the other hand, we have that

$$x^2 + y^2 = r^2 \tag{34}$$

Since we have that, for this disc, θ ranges over the whole interval $[0, 2\pi]$ whereas r ranges over $[0, 1/\sqrt{2}]$, we can hence rewrite the above integral as

$$\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} r^2 \cdot rdrd\theta = \int_0^{2\pi} d\theta \int_0^{\frac{1}{\sqrt{2}}} r^3dr = 2\pi\frac{r^4}{4}\Big|_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{8} \tag{35}$$

3.4 Partitions of Unity

TODO: this

3.5 Change of Variables

TODO: this

3.6 Parametrically Defined Curves

TODO: this

4 Manifolds

4.1 What is a manifold?

TODO: this

4.2 Functions Between Manifolds

TODO: this

4.3 Manifolds with Boundary

TODO: this

4.4 Multilinear Algebra

TODO: this

4.5 Vector Fields and Differential Forms

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4.6 The Differential Operator

TODO: this

5 Integration on Manifolds

TODO: this

5.1 Integration of Parametrized Curves

TODO: this

5.2 Integral of a k-form over a k-cube

TODO: this

5.3 Integration of Differential Forms on Manifolds

TODO: this

5.4 Manifolds with Boundary

TODO: this

5.5 Stoke's Theorem on Manifolds

TODO: this