

MAT257 Notes

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Let's begin where we left off last time

Theorem 1. *Given $A \subset \mathbb{R}^n$ and an open cover \mathcal{O} of A , there is a C^∞ partition of unity Φ subordinate to \mathcal{O} , i.e. a countable collection Φ of C^∞ functions on \mathbb{R}^n such that*

1. $\forall \varphi \in \Phi, 0 \leq \varphi \leq 1$
2. *Every $x \in A$ has a neighborhood V_x such that all but finitely many φ vanish on V_x*
3. $\forall x \in A, \sum_{\varphi \in \Phi} \varphi(x) = 1$
4. *For all $\varphi \in \Phi$, there is $U \in \mathcal{O}$ such that φ vanishes outside a compact subset of U .*

Before we try to construct such a partition, let's discuss why this is an interesting idea. We're going to use this for several things in this course:

- To extend the definition of the integral to more general regions
- To prove the change of variables theorem
- To extend the definition of the integral to manifolds
- To prove Stoke's theorem

So why? Suppose we wanted to extend the definition of the integral to manifolds. Take some shape in 3-space, with a boundary, which is a manifold. Right now, we can integrate over a rectangle. Once we prove the change of variables theorem, we'll be able to integrate over a space diffeomorphic to a rectangle, a sort of deformed rectangle. Now, if we have a coordinate chart, a diffeomorphism from \mathbb{R}^n to some part of the manifold, we could use this to extend the definition of the integral from the rectangle to at least some kind of "rectangular" piece of the manifold. The question remains: how can we extend the definition to the *entire* manifold? A reasonable way to try to do this is to try to cover the entire manifold by a grid of rectangles. If we add up the integrals on each of those pieces...

That was the way this was done classically: you'd take a manifold and divide it up by some kind of rectangular grid, and just add up the contributions of every piece. This classical approach, however, ran into a big obstacle, which was: how do you know that you can really cover every manifold with such a grid? How can you really divide up a really complex surface into rectangles? That is an extremely hard problem: it's true you can do it, but it's an extremely hard problem, and was solved way after the extension of integrals to manifolds.

So how can we avoid this issue? The way this was solved was, you do something a million times simpler. When you cover the manifold with rectangles, you don't care that they all fit together in a nice grid. Just, for every point, you take a rectangle containing that point, and we don't care whether they overlap or not. How do we exploit that? We use a partition of unity! Because what we'll do is, we'll pick a partition of unity such that every one of these bump functions sits in the interior of one of these rectangles. Now, if we want to integrate our function f over the manifold, and we have our partition of unity Φ , and we look at φf , it vanishes outside one of these rectangles, so we can just integrate over the rectangle and get the answer for the manifold. We can then write

$$\int f = \sum \int \varphi f \tag{1}$$

So the key is to take a function, and make it's support very small, and then work with those small pieces. And we're going to use this in the same way to prove Stoke's theorem. Stoke's theorem is like the FTC. What does the FTC say? It says that

$$\int_a^b f' = f(a) - f(b) \quad (2)$$

So the integral of f over the boundary of $[a, b]$ is the same as the integral of the derivative of f over the whole interval. So Stoke's theorem is that the integral of a function over the boundary of a manifold is the same as the integral of the derivative over the whole manifold.

We can prove this for rectangles. Then, if we want to prove it for manifolds, again we need to try to cover the manifolds with a grid like this. If we can do that, we can prove Stoke's theorem for surfaces that we know can be covered by a grid. But we can do it for general manifolds in a really cheap way, again by multiplying with partitions of unity. So that's whay we're going to do in the rest of the course. But first, we have to prove Theorem 1, because that's what we're going to use to glue together little pieces in a very nice but cheap way.

Proof. The way we're going to do this is by considering a number of cases building up to the general case.

1. Assume A is compact. That means finitely many $U \in \mathcal{O}$ cover A . Call them U_1, \dots, U_q . In this case we're going to construct a *finite* (not countable) partition of unity subordinate to $\{U_1, \dots, U_q\}$.

For each $i \in \{1, \dots, q\}$, the first thing that we're going to do is show that we can find a compact set $D_i \subset U_i$ such that the interiors of the D_i cover A . How?

For all $x \in A$, let D_x be a closed ball with centre x lying inside some U_i . Since A is compact, finitely many of the interiors of the D_x cover A . Number them D_{x_1}, \dots, D_{x_p} . For each $i \in \{1, \dots, q\}$, let

$$D_i = \bigcup_{D_{x_j} \subset U_i} D_{x_j} \quad (3)$$

Let ψ_i be a \mathcal{C}^∞ function $0 \leq \psi_i \leq 1$ such that $\psi_i > 0$ on D_i , and is 0 outside a compact subset of U_i . Notice if we do this, then, since the interior of the D_i 's cover's A ,

$$\forall x \in U \supseteq A, \sum_{i=1}^q \psi_i(x) > 0 \quad (4)$$

where U is an open neighborhood of A . SO we define our partition of unity as

$$\Phi = \{\varphi_1, \dots, \varphi_q\}, \varphi_i = \frac{\psi_i}{\sum_{i=1}^q \psi_i} \quad (5)$$

2. Now assume $A = \bigcup_{i \in \mathbb{N}} A_i$ where each A_i is compact and a subset of the interior of A_{i+1} . Let $B_i = A_i \setminus \text{Int } A_{i-1}$. Let \mathcal{O}_i be covers of B_i by the open sets

$$U \cap (\text{Int } A_{i+1} \setminus A_{i-2}) \quad (6)$$

for each $U \in \mathcal{O}$. By Case 1 there is a finite partition of unity Φ_i for B_i subordinate to \mathcal{O}_i . Consider

$$\sigma = \sum_i \sum_{\psi_i \in \Phi_i} \psi_i \quad (7)$$

This is a finite sum in some neighborhood of every $x \in A$. Take for a partition

$$\varphi = \frac{\psi}{\sigma} \quad (8)$$

for each $\psi \in \Phi_i$, for every i .

3. Assume A is open. Then $A = \bigcup_{i \in \mathbb{N}} A_i$ as in Case 2. How can we get such a representation? Try

$$A_i = \{x : |x| \leq i, d(x, \mathbb{R}^n \setminus A) > i^{-1}\} \quad (9)$$

4. Let A be arbitrary. Apply Case 3 to $\bigcup_{U \in \mathcal{O}} U$. A partition of unity subordinate to \mathcal{O} is also a partition of unity for A subordinate to \mathcal{O}

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