MAT257 Notes

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Extremum Problems

We're going to be considering functions, for open sets $U \subseteq \mathbb{R}^n$,

$$f : U \to \mathbb{R}$$

If f has a local maximum or minimum at a point a where

$$\frac{\partial f}{\partial x_i}(a)$$

exists, then it must be zero. This is true because the partial derivative is just the derivative along a one-dimensional subspace: a line. That is,

$$\frac{\partial f}{\partial x_i}(a) = g_i'(0) = 0, g_i(t) = f(a + t\mathbf{e}_i)$$

If f is differentiable at a, what is the direction of maximum increase (we could also ask for decrease) of f at a? It's the direction of the gradient, which I guess is what's called the gradient. Why is this? If we want to look at the increase of f in a particular direction, we have to look at the value of the directional derivative of f in that particular direction. We we want to maximize

$$D_{\mathbf{e}}f(a) = Df(a)(e) = \langle \operatorname{grad} f(a), \mathbf{e} \rangle$$

for some unit vector $\mathbf{e}$. In general, of course, what is this inner product? It's

$$|\mathbf{e}||\operatorname{grad} f(a)| \cos\theta = |\operatorname{grad} f(a)| \cos\theta$$

This, of course, is maximized when $\theta = 0$, i.e. $\mathbf{e}$ points in the direction of $\operatorname{grad} f(a)$.

What about the converse: does a point where the derivative is zero imply an extremum. Here, just like in first year, this is false, however, in this case, the converse is false *even* assuming $g_i''(0) \neq 0$. For example, consider

$$f(x, y) = x^2 - y^2$$

This graph is "saddle shaped". The second derivative with respect to both x and y are nonzero at the origin, the first partial derivatives are zero, and yet we have neither a local maximum or a minimum: in fact, we have a maximum in the y direction and a minimum in the x direction!

Let's do an example: consider an acute angled triangle with vetices $P_i = (x_i, y_i)$. Find a fourth point $P = (x, y)$ such that the sum of distances to each P_i is as small as possible. We have

$$f(x, y) = r_1 + r_2 + r_3$$

where

$$r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

One thing that's we can see is that there *is* a minimum, and it occurs on a closed disk whose boundary contains P_i. f is differentiable except at P_i, but the minimum can't occur at P_i, since the triangle is acute, implying we can take the intersection of an edge and the perpendicular from a vertex and get a smaller value for f. This isn't the only place where we'll use the fact that the triangle is acute angled, however.

So we know the maximum must occur at a critical point. At a critical point:

$$\frac{\partial f}{\partial x} = \frac{x - x_1}{r_1} + \frac{x - x_2}{r_2} + \frac{x - x_3}{r_3} = 0$$

$$\frac{\partial f}{\partial y} = \frac{y - y_1}{r_1} + \frac{y - y_2}{r_2} + \frac{y - y_3}{r_3} = 0$$

i.e. if

$$u_i = \left(\frac{x - x_i}{r_i}, \frac{y - y_i}{r_i}\right), u_1 + u_2 + u_3 = 0$$