MAT257 Notes

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We begin by recalling two definitions for a manifold which we discussed last time

Definition 1. *A set $M \subseteq \mathbb{R}^n$ is a $\mathcal{C}^r$ submanifold of $\mathbb{R}^n$ of dimension k if*

3. *For all $a \in M$, there is an open neighborhood U of a in $\mathbb{R}^n$, an open subset $V \subset \mathbb{R}^n$ and a $\mathcal{C}^r$ diffeomorphism $h : U \to V$ such that*

$$h(M \cap U) = V \cap (\mathbb{R}^k \times \{\mathbf{0}\})$$

4. *For all $a \in M$, there is an open neighborhood U of a in $\mathbb{R}^n$, an open $W \subset \mathbb{R}^n$ and a $\mathcal{C}^r$ mapping $\varphi : W \to \mathbb{R}^n$ such that*
   - *φ is a bijection*
   - *$\varphi(W) = M \cap U$*
   - *φ has rank k at every point of W*
   - *"$\varphi^{-1} : \varphi(W) \to W$ is continuous" i.e. for every open subset Ω of W,*

$$\varphi(\Omega) = \varphi(W) \cap \widetilde{U}$$

   where U is open in $\mathbb{R}^n$.

We'll now show that the fourth definition implies the third:

Proof. Say $a = \varphi(b)$ for some $b \in W$. We can assume

$$\frac{\partial(\varphi_1, ..., \varphi_k)}{\partial(y_1, ..., y_k)}$$

has rank k on W. Define $\psi : W \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by

$$(y, z) \mapsto \varphi(y) + (0, z)$$

Then we get the block matrix

$$\psi'(y, z) = \begin{pmatrix} \frac{\partial(\varphi_1, ..., \varphi_k)}{\partial(y_1, ..., y_k)} & 0 \\ * & I \end{pmatrix}$$

This shows that ψ has rank n for all $y \in W$, since it's determinant is nonzero. But that means that we can apply the inverse function theorem. So by the inverse function theorem, there are open neighborhoods V_1' of $(b, 0)$ and U_1' of $\psi(b, 0) = \varphi(b) = a$ such that $\psi : V_1' \to U_1'$ has a $\mathcal{C}^r$ inverse $\psi^{-1} : U_1' \to V_1'$.

We have that

$$\psi^{-1}(\varphi(y)) = (y, 0) \in V_1' = \varphi(W) \cap \widetilde{U}$$

where U is open in $\mathbb{R}^n$. Take $U_1 = U_1' \cap \widetilde{U}$ and $V_1 = \psi^{-1}(U_1)$. We have

$$M \cap U_1 = \{\varphi(y) : (y, 0) \in V_1\}$$

So

$$h = \psi^{-1}|_{U_1}$$

satisfies the conditions implied by (3) since

$$h(M \cap U_1) = \psi^{-1}(M \cap U_1) = \{(y, 0) : (y, 0) \in V_1\} = V_1 \cap (\mathbb{R}^k \times \{\mathbf{0}\})$$

$\square$

This is quite a delicate topological argument, and gi