MAT257 Notes

Jad Elkhaleq Ghalayini

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1 Vector Fields and Differential Forms

Recall the definitions of the tangent space to $\mathbb{R}^n$ at a, $\mathrm{T}\mathbb{R}^n_a$ or $\mathbb{R}^n_a$. We write the standard bases of $\mathbb{R}^n_a$ as

$$e_{1,a}, ..., e_{n,a} \tag{1}$$

and define the stadard inner product

$$\langle v_a, w_a \rangle_a = \langle v, w \rangle \tag{2}$$

Using the definitions from last time, we obtain a standard orientation of $\mathbb{R}^n_a$

$$[e_{1,a}, ..., e_{n,a}] \tag{3}$$

Recall the following definition:

Definition 1. *The tangent vector to a $\mathcal{C}^1$ curve $\gamma : [0,1] \to \mathbb{R}^n$ at $t \in [0,1]$ is given by*

$$\gamma'(t)_{\gamma(t)} = (\gamma_1'(t), ..., \gamma_n'(t))_{\gamma(t)} = \sum_{i=1}^n \gamma_i'(t) e_{i,\gamma(t)} \in \mathbb{R}^n_{\gamma(t)} \tag{4}$$

We write

$$\gamma'(t)_{\gamma(t)} = \gamma_{*t}(e_{1,t}) \tag{5}$$

where $e_{1,t} \in \mathbb{R}^1_t$, thinking of the derivative of γ $\mathrm{D}\,\gamma$ inducing a mapping

$$\gamma_{*t} : \mathbb{R}^1_t \to \mathbb{R}^n_{\gamma(t)}, \gamma_{*t}(ae_{1,t}) = \mathrm{D}\,\gamma(a+t)(e_1)_{\gamma(t)} \tag{6}$$

Consider now a function $\varphi : \mathbb{R}^n \to \mathbb{R}^p$, and consider the curve $(\varphi \circ \gamma) : [0,1] \to \mathbb{R}^p$. What's the tangent vector to $\varphi \circ \gamma$ at t? Well, by definition, it's

$$(\varphi \circ \gamma)_{*t}(e_{1,t}) = \mathrm{D}(\varphi \circ \gamma)(t)(e_1)_{\varphi(\gamma(t))} \tag{7}$$

By the Chain Rule, 7 simplifies to

$$\mathrm{D}\,\varphi(\gamma(t)) \circ \mathrm{D}\,\gamma(t)(e_1)_{\varphi(\gamma(t))} = \varphi_{*\gamma(t)}(\mathrm{D}\,\gamma(t)(e_1)_{\gamma(t)}) = \varphi_{*\gamma(t)} \circ \gamma_{*t}(e_{1,t}) \tag{8}$$

Writing this in words, the tangent vector to $\varphi \circ \gamma$ at t is equal to the linear mapping between tangent spaces $\varphi_{*\gamma(t)} : \mathbb{R}^n_{\gamma(t)} \to \mathbb{R}^p_{\varphi(\gamma(t))}$ applied to the tangent vector to γ at t.

We're now going to consider functions which, for every point in $\mathbb{R}^n$, give us a vector in the tangent space at that point. And that's what's called a *vector field*. Formally,

Definition 2. *A vector field F on $\mathbb{R}^n$ is a function such that for every point a, $F(a) \in \mathbb{R}^n_a$, i.e.*

$$F(a) = (F_1(a), ..., F_n(a))_a \tag{9}$$

This trivially gives the following definition

Definition 3. *A vector field F is $\mathcal{C}^r$ if each component F_i is $\mathcal{C}^r$.*

Now, why is it that we really want to think of these things as having values in different tangent spaces? Let's look at some examples:

1. In the plane, let's look at $F(x, y) = (x, y)_{(x,y)}$, i.e. the vector (x, y) "pointing out of" x, y. In terms of the basis,
$$xe_{1,(x,y)} + ye_{2,(x,y)} \tag{10}$$
We can imagine plotting this with every point having, coming out of it, the line between it and the origin, with a "source" at the origin. Imagining these to be the velocity vectors of a particle at each point, the points accelerate outwards from the origin with increasing rapidity.

2. Consider now $F(x, y) = (x, -y)$. If we trace a particle following the velocity vectors, this gives us a "saddle" shape.

3. Consider the following important example
$$F(x) = \operatorname{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), ..., \frac{\partial f}{\partial x_n}\right)_x \tag{11}$$
F is $\mathcal{C}^r$ if f is $\mathcal{C}^{r+1}$. Finding if such an f exists is an interesting problem in differential equations. We can think of it as finding a potential function whose gradient at each point is equal to the vector field.

We can apply operations on vectors pointwise to vector fields:
$$(F + G)(x) = F(x) + G(x) \tag{12}$$
$$\langle F, G\rangle (x) = \langle F(x), G(x)\rangle \tag{13}$$
$$(f \cdot F)(x) = f(x)F(x) \tag{14}$$
In $\mathbb{R}^n$ we can take the cross product
$$(F_1 \times ... \times F_{n-1})(x) = F_1(x) \times ... \times F_{n-1}(x) \tag{15}$$
There are several very important classical operations we're going to be interested in, chief of them being grad, div and curl.

Definition 4. *We define the* <u>*divergence of F*</u>*,* $\operatorname{div} F$*, as*
$$\operatorname{div} F = \sum_{i=1}^{n} \mathrm{D}_i\, F_i = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} \tag{16}$$
This is often considered to be the "inner product" of F with a certain <u>*operator*</u>*,*
$$\langle \nabla, F\rangle \tag{17}$$
where
$$\nabla = \sum \frac{\partial}{\partial x_i} e_i \tag{18}$$

Definition 5. *We define the* <u>*curl of F*</u> *(in $\mathbb{R}^3$),* $\operatorname{curl} F$*, as*
$$\operatorname{curl} F = \nabla \times F = (\mathrm{D}_2\, F_3 - \mathrm{D}_3\, F_2)e_1 + (\mathrm{D}_3\, F_1 - \mathrm{D}_1\, F_2)e_2 + (\mathrm{D}_1\, F_2 - \mathrm{D}_2\, F_1)e_3 \tag{19}$$

We're going to prove a very general version of Stokes' theorem, but we want to see what holds in these special cases. Maybe that's enough for today