MAT257 Notes

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Continuity

Definition 1. *Consider a function*
$$f : A \to \mathbb{R}^n$$
f is <u>continuous</u> at A if for every $\epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta, x \in A$.

Theorem 1. *f is continuous if and only if for every open $U \subset \mathbb{R}^n$, $f^{-1}(U) = A \cap V$ where V is open in $\mathbb{R}^n$.*

Proof. $\implies$ consider U open in $\mathbb{R}^n$. Let $a \in f^{-1}(U)$. For some $\epsilon > 0$, $\exists$ a ball $B_{f(a)} = B(f(a), \epsilon) \subset U$ because U is open.

Since f is continuous at a, $\exists \delta > 0$ such that $f(x) \in B_{f(a)}$ for every $x \in A \cap B(a, \delta) = B_a$. We can define
$$V = \bigcup_{a \in f^{-1}(U)} B_a$$

$\impliedby$ Consider $a \in A$ and let $\epsilon > 0$. Let $U = B(f(a), \epsilon)$. Then $f^{-1}(U) = A \cap V$ for some open $V \subset \mathbb{R}^n$.
$$a \in V \implies \exists \delta > 0, B(a, \delta) \subset V$$
But everything which lies inside V gets mapped inside U which is that first ball that we started with. And this is the condition that we want.

$\square$

Corollary. *A composite of continuous functions is continuous*

Proof. Let
$$f : A \to \mathbb{R}^n, g : B \to \mathbb{R}^p$$
be continuous functions where $B \subset \mathbb{R}^n$ and $f(A) \subset B$.

Consider open $U \subset \mathbb{R}^p$. We know
$$\exists V \subset \mathbb{R}^n \text{ open}, g^{-1}(U) = B \cap V$$
So we can write
$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(B \cap V) = f^{-1}(V)$$
We know that
$$\exists W \subset \mathbb{R}^n \text{ open}, , f^{-1}(V) = A \cap W = (g \circ f)^{-1}(U)$$
But this is exactly what we wanted to show $\square$

Exercise 1. *Let $f = (f_1, ..., f_n)$ be a function from $\mathbb{R}$ to $\mathbb{R}^n$. Then f is continuous if and only if $\forall i \in \{1, ..., n\}$, f_i is continuous.*

Proof. $\Longleftarrow$ We need to estimate the norm $|f(x) - f(a)|$ with the differences between $f_i(x_i)$ and $f_i(a_i)$. We could use a variety of inequalities for this, including

$$|f(x) - f(a)| \leq \sum_{i=1}^{n} |f_i(x) - f_i(a)|$$

The full ϵ-δ argument is left as an exercise.

$\Longrightarrow$ We need to estimate the norm $|f_i(x) - f_i(a)| < |f(x) - f(a)|$. Alternatively, we can do this topologically. The full argument is left as an exercise.

$\square$

Exercise 2. *A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is <u>uniformly</u> continuous*

Definition 2. *There is a norm $M > 0$ such that $|T(x)| \leq M|x|$. Hence,*

$$|T(x) - T(y)| = |T(x - y)| \leq M|x - y|$$

Given ϵ, take $\delta = \frac{\epsilon}{M}$.

Exercise 3. *Are the following functions continuous?*

1. $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ *- No*
2. $f(x, y) = \frac{x^2 + 3xy + y^2}{x^2 + 4xy + y^2}$ *- No*
3. $f(x, y) = e^{-\frac{|x-y|}{x^2 - 2xy + y^2}} = e^{-\frac{1}{|x-y|}}$ *- Yes*

Proof. 1. f is 1 along the line $\{f(x, 0) : x \in \mathbb{R}\}$, but -1 along the line $\{f(0, y) : y \in \mathbb{R}\}$, both of which traverse the origin.

2. No for the same reason, but we can't check on the axes, since each axis is 1. Instead, we can check on the x axis and compare that with any line except the y axis, such as $y = x$, where the value is $\frac{5}{6}$

3. Composite of $z \mapsto e^{\frac{1}{|z|}}$ and $(x, y) \mapsto x - y$

$\square$

Exercise 4. *Let $X \subseteq \mathbb{R}^n$. We define the <u>distance function</u>*

$$d(x, X) = \inf_{a \in X} |x - a|$$

$\forall X \in \mathcal{P}(\mathbb{R}^n), d(x, X)$ *is uniformly continuous on* $\mathbb{R}^n$