MAT257 Notes

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1 Multivariate Taylor Series

1.1 Review of Single Variable Taylor Series

Recall the definition of the n^{th} degree Taylor polynomial of f centered at a

$$P_{n,a}(f) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

We define $R_{n,a}$ to be the "remainder", the difference between $P_{n,a}$ and f. We can write this remainder in "Lagrange form" as

$$\frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$$

Clearly,

$$\lim_{x \to a} \frac{R_{n,a}(x)}{(x-a)^n} = 0$$

We can now define the Taylor series of f at a to be

$$T_a f = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

1.1.1 Why these coefficients?

Suppose

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

Note

$$\frac{d^n}{dx^n}(x-a)^k = \begin{cases} 0 & \text{if } k < n \\ k! & \text{if } n = k \\ k(k-1)...(k-n+1)(x-a)^{k-n} & \text{if } k > n \end{cases}$$

1.2 Multivariable Case

For example: What's your favorite polynomial? Zero? I'm sure you meant something more like

$$f(x,y,z) = a_{12,8,2}x^{12}y^8z^2 + a_{0,22,0}y^{22} + a_{0,0,1}z + a_{20,20,20}x^{20}y^{20}z^{20}$$

How to do spot $a_{12,8,2}$? Apply:

$$\left.\frac{\partial^{22} f}{\partial x^2 \partial y^8 \partial z^2}\right|_{(x,y,z)=(0,0,0)} = a_{12,8,2}12!8!2!$$

In general, if f is a sum of terms like

$$a_{\alpha_1,\ldots,\alpha_n} x^{\alpha_1} \ldots . x^{\alpha_n}$$

Then

$$a_{\alpha_1,\ldots,\alpha_n} = \left. \frac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \right|_{(x_1,\ldots,x_n)=(0,\ldots,0)} \frac{1}{\alpha_1! \ldots \alpha_n!}$$

1.3 Multi-index notation

We're mathematicians here, and what do mathematicians do, we generalize. But we generalize in such a way such that the general case looks like the specialized case so we don't have to remember different notation when teaching first-years and when doing research. Multi-index notation is a prime example. Compare:

- Let $f : \mathbb{R} \to \mathbb{R}$. Then

$$f = \sum_{k \in \mathbb{N}} \frac{1}{k!} f^{(k)}(a)(x-a)^k$$

- Let $f : \mathbb{R}^n \to \mathbb{R}$. Then

$$f = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial x^\alpha}(a)(x-a)^\alpha$$

To make this to work, all we have to do is define, for

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$$

the notation

$$\alpha! = \alpha_1! \ldots \alpha_n!$$

$$|\alpha| = \alpha_1 + \ldots + \alpha_n$$

$$\partial x^\alpha = \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$$

$$(x-a)^\alpha = (x_1 - a_1)^{\alpha_1} \ldots (x_n - a_n)^{\alpha_n}$$

1.3.1 Examples

1. $$\frac{1}{6!y!} 12x^6y^7 = \frac{12}{(6,7)!}(x,y)^{6,7}$$

2. $$x^5 + x^4y + x^3y^2 + \ldots + y^5 = \sum_{\beta+\gamma=5} (x,y)^{(\beta,\gamma)} = \sum_{|\alpha|=5} (x,y)^\alpha$$

   Note that we're assuming here β, γ are non-negative. On good days you can just do what we did above, but if it worries you, write it down.

3. $$a_{0,0} + a_{1,0}x + a_{0,1}y + a_{1,1}xy + a_{2,0}x^2 + a_{0,2}y^2 = \sum_{|\alpha| \le 2} a_\alpha (x,y)^\alpha$$

2 Multivariable Taylor Series

Theorem 1. *If $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^{k+1}$ then*

$$f(a+h) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{x^\alpha}(a) h^\alpha + \sum \_|\alpha| = k+1 \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(a + \theta h) h^\alpha$$

where $\theta \in (0, 1)$

Proof. The key idea is

$$F(t) = f(a+th)(\text{Old Taylor } + \text{ Chain Rule})$$

More rigorously, we know from single variable calculus that

$$F(t) = F(0) + \frac{F'(0)}{1!} t + ... + \frac{F^{(k)}(0)}{k!} t^k + \frac{F^{k+1}(\theta)}{(k+1)!} t^{k+1}$$

Set $t = 1$ to get $F(1) = f(a+h)$. We know $F(0) = f(a)$. We have

$$\left.\frac{dF}{dt}\right|_{t=0} = \left.\frac{d}{dt} f(a+th)\right|_{t=0} = \sum_{i=1}^n \left.\frac{\partial f}{\partial x}(a+th)\right|_{t=0} \frac{\partial x_i}{\partial t} = \sum_{i=1}^n \left[\left.\frac{\partial f}{\partial x_i}(a+th)\right|_{t=0} \right] h_i = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a) h_i$$

We have

$$\left.\frac{d^2 F}{dt}\right|_{t=0} = \sum_{i=1}^n \left[\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}(a) h_j \right] h_i$$

By the magical process of ...,

$$\left.\frac{d^k F}{dt^k}\right|_{t=0} = \sum_{i_1, ..., i_k = 1}^n \frac{\partial^k f}{\partial x_{i_1} ... \partial x_{i_k}}(a) h_{i_1} ... h_{i_k}$$

Now all we need is some combinatorics, combined with the equality of mixed partials (for $\mathcal{C}^{k+1}$ functions), to complete the proof. Recall the *multinomial coefficient*

$$\binom{|\alpha|}{\alpha_1, ..., \alpha_n} = \frac{|\alpha|!}{\alpha_1! ... \alpha_n!}$$

is the number of ways of taking $|\alpha|$ things and making k groups of size $\alpha_1, ..., \alpha_n$. We can use this to rewrite the above as

$$F^{(S)}(0) = \sum_{|\alpha| = S} \frac{S!}{\alpha_1! ... \alpha_S!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(a) h^\alpha$$

So plugging this into our original expression for the Taylor series of F, where we divide the S^{th} term by $\frac{1}{S!}$, we have

$$F(t) = F(0) + \sum_{i=1}^k \sum_{|\alpha| = i} \frac{1}{\alpha_1! ... \alpha_k!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(a) h^\alpha = f(a) + \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(a) h^\alpha$$

as desired.

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