MAT257 Notes

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The Implicit Function Theorem

In this course, we have three important theorems, of which this is one. Each is a generalization of something you've seen in first year calculus, but we'll see that these generalizations are very far-reaching and involve new techniques. Let's begin with a brief "plan" for the next month:

Plan

1. Today: the inverse function theorem
2. The inverse function theorem, which we'll see is equivalent.
3. Proof of the inverse function theorem (which will take a few lectures)
4. Implications of the implicit function theorem, in particular applications to extreme value problems, introducing the idea of Lagrange Multipliers. Afterwards, we'll talk about the idea of a differentiable manifold.

So this is what we're aiming to do in the next few weeks.

The Inverse Function Theorem

Recall: suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function in an open inverval containing a, and $f'(a) \neq 0$. Then f' is either greater than or lessthan zero in an open interval containing a. Therefore f is one-to-one, and so has an inverse defined on an open interval W containing $f(a)$. Moreover, f^{-1} is differentiable, and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$$

We're going to discuss a generalization of this idea to several variables.

Theorem 1 (Inverse Function Theorem)**.** *Let $f : U \to \mathbb{R}^n$ be a continuously differentiable ($\mathcal{C}^1$) on an open set $U \subset \mathbb{R}^n$. Let $a \in U$ be such that $\det f'(a) \neq 0$. Then there exist open sets $a \in V, f(a) \in W$ such that $f : V \to W$ with a continuous inverse $f^{-1} : W \to V$. Moreover, f^{-1} is differentiable on W and*

$$\forall y \in W, (f^{-1})'(y) = (f'(f^{-1}(y)))^{-1}$$

Remark: if we know already that f^{-1} is differentiable, the formula for it follows from the chain rule:

$$f(f^{-1})(y) = y \implies f'(f^{-1}(y))(f^{-1})'(y) = I \iff (f^{-1})'(y) = (f(f^{-1}(y)))^{-1}$$

Corollary. *1. f^{-1} is continuously differentiable ($\mathcal{C}^1$)*

2. If f is $\mathcal{C}^r$ then f^{-1} is also $\mathcal{C}^r$

Proof. 1. We know that since f' is continuous, and f^{-1} is continuous, $f' \circ f^{-1}$ is continuous. Now what about the inversion? What is the inverse of a matrix? It's given by a formula. So this is really a composite of 3 functions. So what about the formula for the inverse of a matrix? It follows from Cramer's rule that this formula is continuous, since the entries of the inverse matrix B of A are given as rational functions of the entries of A, that is,

$$b_{ij} = \frac{(-1)^{i+j} \det A^{ji}}{\det A}$$

where A^{ji} is the matrix obtained by deleting the j^{th} row and i^{th} column from A. So hence the derivative, being the composition of 3 continuous functions, must be continuous.

2. We proceed by induction on r. Assume that if f is $\mathcal{C}^{r-1}$, then f^{-1} is $\mathcal{C}^{r-1}$.

   If f is $\mathcal{C}^r$, then f is $\mathcal{C}^{r-1}$ implying that f^{-1} is $\mathcal{C}^{r-1}$ by the inductive hypotheis. So, using the formula

$$(f^{-1})' = (f'(f^{-1}))^{-1}$$

   is $\mathcal{C}^{r-1}$ implying that f^{-1} is $\mathcal{C}^r$.

□

This corollary is just to show that we could have stated the above theorem in a stronger way. Examples:

1. Continuity of f' cannot be removed from the hypotheses: consider $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x + x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   We have that

$$x \neq 0 \implies f'(x) = 1 + 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

   but the limit does not exist as $x \to 0$.

2. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x, y) = (e^x \cos y, e^x \sin y)$$

   We have

$$f'(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

   implying that

$$\det f'(x, y) = (e^x)^2 = e^{2x} \neq 0 \forall x \in \mathbb{R}$$

   But this function is not 1-1, since it is periodic in y. The inverse function theorem says that we can make some neighborhood around a point where f is one to one, but it *doesn't* say that it's *globally* one to one.

Remarks:

1. f may be invertible even though $f'(a) = 0$