MAT257 Notes

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1 Stokes' Theorem

We now want to try to prove the general form of Stokes' theorem:

Theorem 1 (Stokes' Theorem)**.** *Let M be a compact oriented k-dimensional manifold with boundary (which is at least $\mathcal{C}^2$) and ω be a $(k-1)$-form on M (which is at least $\mathcal{C}^1$). Then*

$$\int_M d\omega = \int_{\partial M} \omega \tag{1}$$

where ∂M has induced orientation

The problem is, the integrals in equation 1 are yet to be defined. So that's what we're going to be working up to today. Consider a singular p-cube $c : [0,1]^k \to M$ in M. If ω is a p-form on M, then we define

$$\int_c \omega = \int_{[0,1]^p} c^*\omega \tag{2}$$

Integrals over p-chains are defined as before. In the case that $p = k$ (e.g. in theorem) we'll assume that our k-cubes $c : [0,1]^k \to M$ satisfy the following condition: there is a coordinate chart $\xi : W \to M$ such that $[0,1]^k \subset W$ and

$$c = \xi|_{[0,1]^k} \tag{3}$$

As a mapping, c is orientation preserving if and only if ξ is orientation preserving.

Lemma 1. *Let M be an oriented k-dimensional manifold (with or without boundary), and let $c_1, c_2 : [0,1]^k \to M$ be orientation-preserving singular k-cubes, with the above assumption holding. If ω is a k-form on M such that $\omega = 0$ outside $c_1([0,1]^k) \cap c_2([0,1]^k)$ then*

$$\int_{c_1} \omega = \int_{c_2} \omega \tag{4}$$

Proof. By definition, we have that

$$\int_{c_1} \omega = \int_{[0,1]^k} c_1^*\omega \tag{5}$$

Using the fact that

$$c_1 = c_2 \circ (c_2^{-1} \circ c_1) \tag{6}$$

Of course, ordinarily we can't talk about inverses when we're talking about cubes, but c_1, c_2 have inverses since they are assumed to be the restrictions of a diffeomorphism. If you want, you can even write that

$$c_1^*\omega = \xi^*\omega \tag{7}$$

Plugging equation 6 into equation 5, we obtain

$$\int_{[0,1]^k} (c_2^{-1} \circ c_1)^*(c_2^*\omega) \tag{8}$$

We have that

$$c_2^* \omega = f(\omega) dx_1 \wedge ... \wedge dx_k \tag{9}$$

Let $g = c_2^{-1} \circ c_1$. We then have that

$$(c_1^{-1} \circ c_2)^* (c_2^* \omega) = g^* (f dx_1 \wedge ... \wedge dx_k) = (f \circ g) \det g' dx_1 \wedge ... \wedge dx_k \tag{10}$$

Suince g is orientation preserving, we get that $\deg g' = |\det g'|$, implying the above is equal to

$$(f \circ g) |\det g'| dx_1 \wedge ... \wedge dx_k = \int_{c_1} \omega \tag{11}$$

as desired. TODO: check

□

We now want to define the integral of a form over a manifold:

Definition 1. *Let M be an oriented k-dimensional manifold and ω be a k-form on M.*

1. *If there is an orientation preserving k-cube c on M such that $\omega = 0$ outside $c([0,1]^j)$ we define*

$$\int_M \omega \int_c \omega \tag{12}$$

   which is independent of the choice of c by Lemma 1 as long as ω vanishes outside it.

2. *In the general case, there exists an open cover $\mathcal{O}$ of $M \subset \mathbb{R}^n$ such that for all $U \in \mathcal{O}$, thee is an orientation preserving k-cube c_U in M such that*

$$U \cap M \subset c_U([0,1]^k) \tag{13}$$

   Now, let's let Φ be a $\mathcal{C}^2$ partition of unity subordinate to $\mathcal{O}$. We define

$$\int_M \omega = \sum_{\varphi \in \Phi} \int_M \varphi \circ \omega \tag{14}$$

   If M is compact, this sum is finite, but in general, the definition still works out.