MAT257 Notes

Jad Elkhaleq Ghalayini

September 16 2018

Compactness

Definition 1. *A subset X of $\mathbb{R}^n$ is* <u>compact</u> *if every open covering of X has a finite subcover.*

We're later going to prove a deep theorem about compactness:

Theorem 1. *A subset X of $\mathbb{R}^n$ is compact if and only if X is closed and bounded*

We won't do this right now, because it'll involve some work, but we'll use something called the Heine-Borel Theorem: $[0, 1]$ is compact. We're going to prove this exactly the same way as we proved the "Three Hard Theorems" from first year calculus. It's a good exercise to try this.

Today, we'll do half of this theorem, the easy part: compact *implies* closed and bounded.

Lemma 1. *If $X \subset \mathbb{R}^n$ is compact then X is closed and bounded.*

Proof.
- X is closed: to say that X is closed is of course the same thing as saying $\mathbb{R}^n \setminus X$ is open. To show this, we take any point in $\mathbb{R}^n \setminus X$, and show there's some ball centered at that point which is a subset of $\mathbb{R}^n \setminus X$.

  So let $x \in \mathbb{R}^n \setminus X$. We want to show that $\exists\delta > 0$, $B(a, \delta) \subset \mathbb{R}^n \setminus X$ for some $\delta > 0$. Let's just look <u>at all possible</u> balls centered at A. Or we might as well just consider, for some $k \in \mathbb{N}$, the closed ball $\overline{B(a, k^{-1})}$. Let's look at the complement of this closed ball,

  $$U_k = \mathbb{R}^n \setminus \overline{B(a, k^{-1})}$$

  We have that

  $$\bigcup_k U_k = \mathbb{R}^n \setminus \{a\}$$

  So $\{U_k\}$ is an open cover of X. But X is compact, so this open cover has a finite subcover. This fact tells us that $\exists k$, $X \subset U_k$. But U_k is the complement of the closed ball of radius $\frac{1}{k}$, meaning the open ball

  $$B(a, k^{-1}) \subseteq \mathbb{R}^n \setminus X$$

  We can clearly see that this implies $\mathbb{R}^n \setminus X$ is open, implying X is closed.

- X is bounded: consider a cover of X by all open balls of radius 1 in $\mathbb{R}^n$.

$\square$

Theorem 2. *If $X \subset \mathbb{R}^m$ is compact and $f : X \to \mathbb{R}^n$ is continuous then $f(X)$ is pact*

Proof. Let $\mathcal{O}$ be an open cover of $f(X)$. For every $U \in \mathcal{O}$, $f^{-1}(U) = X \cap V_U$ where V_U is open in $\mathbb{R}^m$. So

$$\{V_U : U \in \mathcal{O}\}$$

is an open cover of X. Since X is compact, there is a finite subcover

$$V_{U_1}, ..., V_{U_k}$$

So $U_1, ..., U_k$ cover $f(X)$. $\square$

Theorem 3 (Extreme Value Theorem)**.** *A continuous function $f : X \to \mathbb{R}$ on a nonempty compact subset X of $\mathbb{R}^n$ attains a maximum and minimum*

Proof. Let $M = \sup\{f(x) : x \in X\}$. $M < \infty$ since $f(X)$ is bounded. What if $M \notin f(X)$? Since $f(X)$ is closed, we can draw a small interval around M which is not in $f(X)$, contradicting the fact that $M = \text{lub} f(X)$.

The proof for minima is analogous. □

Definition 2. *The ϵ-neighborhood of X is given by*

$$\bigcup_{x \in X} B(x, \epsilon) = \{y \in \mathbb{R}^n : d(y, X) < \epsilon\}$$

Theorem 4 (ϵ-neighborhood theorem)**.** *If $X \subset U \subset \mathbb{R}^n$ where X is compact and U is open, then there is $\epsilon > 0$ such that the ϵ-neighborhood of X lies in U.*

Proof. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, \mathbb{R}^n \setminus U)$. We showed this is a continuous function, which is strictly positive (since U is open). Hence, by the Extreme Value Theorem, it has a positive minimum. Take ϵ to be this minimum. □