

MAT257 Notes

Jad Elkhaleq Ghalayini

November 23 2018

1 Manifolds with boundary

Recall the following definitions:

Definition 1. A half space $\mathbb{H}^k \subset \mathbb{R}^k$ is given by

$$\{x = (x_1, \dots, x_k) : x_k \geq 0\}$$

Definition 2. A subset $M \subseteq \mathbb{R}^n$ is a k -dimensional C^r manifold with boundary if for every $a \in M$, there is an open neighborhood U of a in \mathbb{R}^n , an open subset V of \mathbb{R}^n and a C^r diffeomorphism $h : U \rightarrow V$ such that either

(1) The usual condition for a manifold,

$$h(M \cap U) = V \cap (\mathbb{R}^k \times \{\mathbf{0} \in \mathbb{R}^{n-k}\})$$

(2) The manifold looks like a half space around a , i.e.

$$h(M \cap U) = V \cap (\mathbb{H}^k \times \{\mathbf{0} \in \mathbb{R}^{n-k}\}) = \{y = (y_1, \dots, y_n) : y_1, \dots, y_k \geq 0 \wedge y_{k+1} = \dots = y_n = 0\}$$

One thing you could ask is whether both conditions (1), (2) could be satisfied by the same point a . The answer is no. To show why, suppose $h_1 : U_1 \rightarrow V_1$ satisfies (1) and $h_2 : U_2 \rightarrow V_2$ satisfies (2) for some a . How can we get a contradiction? If we take

$$h_2 \circ h_1^{-1} : V_1 \rightarrow V_2$$

we get a C^r diffeomorphism in an open set containing a taking an open subset of \mathbb{R}^k onto an open subset of \mathbb{H}^k which is *not* open in \mathbb{R}^k . What we're contradicting here is the inverse function theorem, which says there would be open neighborhoods $h_1(a)$ and $h_2(a)$ which are mapped together with an inverse. Alternatively, we could use the topological arguments that diffeomorphisms, which are homeomorphisms, are open maps.

Once we've established this result, we can really distinguish these two kinds of points. That means, precisely, that we can make the following definitions

Definition 3. The boundary of M , written ∂M , is the set of points satisfying condition (2) in definition 2

We're going to be interested in manifolds because they generalize the idea of intervals with endpoints in the real line. In fact, an interval with endpoints is a one dimensional manifold with boundary.