MAT257: Analysis II Evening Lecture Notes

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Based on handwritten notes by Professor Edward Bierstone

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1 Topology of $\mathbb{R}^n$

1.1 Metric Spaces

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We have that

$$T(x) = T\left(\sum x_i e_i\right) = \sum x_i T(e_i)$$

Hence, $\exists c \in \mathbb{R}^+$ such that

$$|T(x)| \leq \sum |x_i||T(e_i)| \leq c \cdot \sum |x_i| \leq \sqrt{n}c|x|$$

with

$$|T(e_i) = |i^{th} \text{ column of } A| = \sqrt{\sum_{j=1}^{m} a_{ji}^2}$$

so, for example, we can choose

$$\forall i, |T(e_i)| \leq c = \sqrt{\sum a_{ij}^2}$$

Definition 1. *A <u>metric space</u> is a set X with a <u>distance function</u>*

$$d : X \times X \to \mathbb{R}$$

satisfying the following axioms, $\forall x, y, z \in X$,

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$
3. $d(x, y) \leq d(x, z) + d(z, y)$

$\mathbb{R}^n$ (with any of the norms above) is a metric space.

1.2 Open and Closed Sets

1.2.1 Definitions

In $\mathbb{R}^n$, the closed rectangles

$$[a_1, b_1] \times ... \times [a_n, b_n]$$

are higher order analogues of the closed interval $[a, b]$, as well as the closed ball around $a \in \mathbb{R}^n$ of radius $r \in \mathbb{R}^+$

$$\{x \in \mathbb{R}^n : |x - a| \leq r\}$$

Similarly, we define the open rectangle

$$(a_1, b_1) \times ... \times (a_n, b_n)$$

to be the higher order analogue of the open interval (a, b), and the open ball around $a \in \mathbb{R}^n$ of radius $r \in \mathbb{R}^+$

$$B(a, r) := \{x \in \mathbb{R}^n : |x - a| < r\}$$

Definition 2. *We say that $U \subset \mathbb{R}^n$ is open if*

$$\forall x \in U, \exists \text{ an open rectangle (ball) } A, x \in A \subset U$$

Note the definitions in terms of balls and rectangles are equivalent, since if there exists a rectangle, we can find a ball within it (since the rectangle is open in the ball definition) and if there exists a ball, we can find a rectangle within it (since the ball is open in the rectangle definition).

Definition 3. *We say that $C \subset \mathbb{R}^n$ is closed if $\mathbb{R}^n \setminus C$ is open*

Examples of closed sets include closed rectangles and any finite subset of $\mathbb{R}^n$. Note that the definitions for these terms based off open balls generalize readily to other metric spaces, and it is possible to define open sets in terms of closed sets instead of vice versa. For more information on this, I highly recommend the notes on topology by Ivan Khatchatourian at `http://www.math.toronto.edu/ivan/mat327`.

1.2.2 Properties

Proposition 1. *The union of an arbitrary collection $\mathcal{U}$ of open sets is open*

Proof. □

Proposition 2. *The intersection $U \cap V$ of any two open sets U, V is open*

Proof. □

Corollary. *The intersection of finitely many open sets is open*

Proof. Follows trivially from 2 by induction. □

2 Functions and continuity

3 Operations on functions