MAT257 Notes

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1 Change of Variable

Let's begin with some examples

1. Find the volume of an ellipsoid
$$E = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\} \tag{1}$$
We can make a change of variable to "transform" this into a sphere. Specifically, we can use the transformation
$$g : \mathbb{R} \to \mathbb{R}^3, (x, y, z) \mapsto (x/a, y/bz/c) \tag{2}$$
to take the ellipsoid to the unit ball, where $(u, v, w) = g(x, y, z)$. So
$$\iiint_{B_1(\mathbf{0})} 1dudvdw = \iiint_E \frac{1}{abc}dxdydz = \frac{1}{abc}\iiint_E 1dxdydz = \frac{4}{3}\pi \implies V(E) = \frac{4}{3}\pi abc \tag{3}$$
Now what if we didn't know the volume of a ball of radius 1? We could figure it out using change of variables, using spherical coordinates:
$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2 \sin\theta dr d\theta d\phi = \int_0^1 r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{1}{3} \cdot 2 \cdot 2\pi = \frac{4\pi}{3} \tag{4}$$
Of course, the volume of the sphere of radius r can be treated as a special case of the ellipsoid where $a = b = c = r$, giving volume $\frac{4}{3}\pi r^3$.

2. What's the simplest kind of change of variable, save the identity? Of course, a linear change of variable. Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and suppose $A \subset \mathbb{R}^n$ is Jordan measurable and $B = T(A)$. Show that B is Jordan measurable and $V(B) = V(A)|\det T|$. We split this into two cases:

   (a) Assume T is not invertible, i.e. $\det T = 0$. Then the image I of T is a linear subspace of $\mathbb{R}^n$ of dimension less than N, and $B \subset I$. So all of B has n-dimensional measure zero, and hence $V(B) = 0 = V(A)|\det T|$ as desired.

   (b) Assume T is invertible, i.e. $\det T \neq 0$. Then B is Jordan measurable and now by the change of variable theorem
   $$V(B) = \int_B 1 = \int_A 1 \cdot |\det T| = |\det T| \int_A 1 = V(A)|\det T| \tag{5}$$
   as desired.

We're now going to prove the Change of Variables theorem, and it's going to take all week. Well, all the class time this week. Let's consider just one direction:

Theorem 1. *Let $A \subset \mathbb{R}^n$ be open and let $g : A \to \mathbb{R}^n$ be one to one and continuously differentiable such that $\det g'(x) \neq 0$. Then if $f : g(A) \to \mathbb{R}$ is integrable, $f \circ g|\det g'|$ is integrable on A and*

$$\int_{g(A)} f = \int_A (f \circ g)|\det g'| \tag{6}$$

Note that the converse, and hence the full theorem, follows, since

- As g is invertible f is locally bounded if and only if $f \circ g|\det g'|$ is locally bounded
- The discontinuities of f have measure zero if and only if the discontinuities of $f \cdot g|\det g'|$ have measure zero, since g is differentiable, using the lemma from last time.
- We can use g^{-1} to go from $g(A)$ to A.

We still have to show that if $F = f \circ g|\det g'|$ is integrable on A, then f is integrable on $g(A)$. All we have to do is apply the theorem with $F), g^{-1}$ instead of f, g. Then

$$F \circ g^{-1}|\det(g^{-1})'| = f \circ \cancel{g \circ g^{-1}}\cancel{|\det g' \circ g^{-1}|}\frac{1}{\cancel{|\det g' \circ g^{-1}|}} = f \tag{7}$$

as desired.

We can now begin to prove the theorem in a number of steps. We're going to start by looking at a Jordan-measurable open subset U of A with closure in A, e.g. an open rectangle with closure in A. The first thing that we're going to do is show that it's "good enough" to prove the theorem on such an open subset. How? With a partition of unity! That's going to simplify things a lot, but just one thing I want to note first is that if f is integrable on $g(A)$, then f is integrable on $g(U)$ and $f \circ g|\det g'|$ is integrable on U.